Decompositions of Group Actions on Symmetric Tensors

A Dissertation Presented

by

Scott Raymond Greenleaf

to

The Graduate School

in Partial fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

August 2000

State University of New York
at Stony Brook

The Graduate School

Scott Raymond Greenleaf

We, the dissertation committee for the above candidate for the
Doctor of Philosophy degree,
hereby recommend acceptance of this dissertation.

Anthony W. Knapp

Anthony W. Knapp
Professor Emeritus of Mathematics
Dissertation Advisor

Amy E Ksir

Amy Ksir
James H. Simons Instructor of Mathematics
Chairperson of Dissertation Committee

L. Takhtajan

Leon Takhtajan
Professor of Mathematics

Folkert Tangerman
Visiting Assistant Professor
Department of Applied Mathematics and Statistics, SUNY
Outside Member

This dissertation is accepted by the Graduate School.

Lawrence Martin

Dean of the Graduate School

Abstract of the Dissertation

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For $G = SO(2m, 2n)$, $m \leq n$, impose the standard ordering on the roots relative to the standard compact Cartan subgroup. Let $\mathfrak{u} \cap \mathfrak{p}$ be the span of the root vectors for the positive noncompact roots, and let L be the compact subgroup built from the compact Cartan subgroup of G and from all linear combinations of compact simple roots. The decomposition of the symmetric algebra $S(\mathfrak{u} \cap \mathfrak{p})$ under L plays a role in cohomological induction, and the thesis studies just what this decomposition is. Although all the irreducible representations of L fall into an $(m+n)$-parameter family, it turns out that the highest weights of the irreducible representations of L that occur in $S(\mathfrak{u} \cap \mathfrak{p})$ fall into a $2m$-parameter family. A good upper bound is found for the multiplicities of these representations. The cases $m = 1$ and $m = 2$ were known earlier, as a consequence of work of W. Schmid and of B. Gross and N. Wallach.

For my Mother

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Many thanks to my advisor, Anthony Knapp, for all his help and patience.

Chapter 1: Introduction

Through work of E. Cartan [C] and Harish-Chandra [H-C], it has long been known that an irreducible Hermitian symmetric space G/K embeds canonically as a bounded symmetric domain in some $\mathbb{C}^n$. In a famous 1969 paper [S], Wilfried Schmid found how the space of holomorphic polynomials on the domain decomposes under the action of K. The set of highest weights is given by real-rank(G) parameters, Schmid determined which highest weights occur, and he showed that each irreducible representation of K occurs with multiplicity at most one. Already the number of parameters of the highest weights is a surprise, since the number of parameters needed to describe all representations of K is the complex rank of G, which is often larger than the real rank of G.

B. Gross and N. Wallach [G-W1,G-W2] discovered a related result for irreducible quaternionic symmetric spaces G/K. We state this result in a form different from that envisioned by the authors. For these spaces the group K splits as a commuting product $K = SU(2)L_{\text{ss}}$ of the three-dimensional compact semisimple group $SU(2)$ and the compact semisimple group L_{ss}. The intersection of these two subgroups is a circle, and we define L to be the product of this circle with L_{ss}. Relative to a positive system of roots compatible with the quaternionic structure, let $\mathfrak{u} \cap \mathfrak{p}$ be the space of linear combinations of root vectors of positive "noncompact" roots. Then L acts naturally on the complex vector space $\mathfrak{u} \cap \mathfrak{p}$, and the action of L extends to the complex symmetric algebra $\mathcal{S} = S(\mathfrak{u} \cap \mathfrak{p})$, which is dual to the space of holomorphic polynomials. Gross and Wallach found how $\mathcal{S}$ decomposes under L.

The set of highest weights is given by real-rank(G) parameters, and the authors determined which highest weights occur and what multiplicities they have. The multiplicities can be greater than one, but they found a substitute multiplicity-one theorem. Namely within $\mathcal{S}$, the algebra of elements invariant under L_{ss} is singly generated, with a generator f, say. It follows that $\mathcal{S}$ can be written as the tensor product of $\mathbb{C}[f]$ and the subspace $\mathcal{H}$ of all elements of $\mathcal{S}$ annihilated by the natural linear differential operator created from f. What Gross and Wallach proved is that $\mathcal{H}$ decomposes under L with multiplicity one, and they identified which highest weights occur.

This thesis deals with a more general setting that includes both the Schmid case and the Gross-Wallach case. We study an irreducible noncompact symmetric space G/K in which $\operatorname{rank} G = \operatorname{rank} K$. Borel and de Siebenthal [B-dS] proved that there is always a special positive system of roots generalizing the one in the above two examples—namely one in which there is at most one positive "noncompact root" and that root occurs with multiplicity at most two in the largest root. In this setting we take L to be the centralizer of the sum of all positive noncompact roots, and we again let $\mathfrak{u} \cap \mathfrak{p}$ be the space of linear combinations of root vectors of positive noncompact roots. Then L acts naturally on the complex vector space $\mathfrak{u}\cap\mathfrak{p}$, and the action of L extends to the complex symmetric algebra $\mathcal{S} = S(\mathfrak{u} \cap \mathfrak{p})$. The problem is to decompose $\mathcal{S}$ under the action of L.

In the Gross-Wallach setting, the authors showed that their results have implications for constructing "small", apparently fundamental, irreducible unitary representations of G, and we visualize that results in our more general setting will similarly find applications to the theory of irreducible unitary representations of our more general G.

The thesis is, however, limited to the structure theory. We shall, in fact, consider only the specific family $SO(2m, 2n)$ of groups, $m \leq n$. The case $m = 1$ is an instance of the Schmid theory, and the case $m = 2$ is an instance of the Gross-Wallach theory. Our main theorem is a specific statement to the effect that, in the decomposition of $\mathcal{S}$ under L, the set of highest weights of L that can occur is given by real-rank(G) parameters. We also obtain an upper bound for the multiplicities.

For the class of groups $SO(2m, 2n)$, the real rank is $2m$. The complex rank of G, which is the number of parameters needed to parametrize all irreducible representations of L, is $m + n$.

Actually we conjecture, for all groups G in the class of interest to us, that the number of parameters for highest weights of L needed to decompose $\mathcal{S}$ is always the real rank of G. But there are steps in our argument that we do not know how to push through without attempting case-by-case calculations.

Some remarks about techniques may help orient the reader. For his case, Schmid made use of earlier work of Koranyi and Wolf [K-W] concerning "Cayley transforms" of the bounded symmetric domain. A Cayley transform is constructed as a member c of the complexification $G^{\mathbb{C}}$ of G. For $SO(2, 2n)$, which is a little simpler than the general G that Koranyi-Wolf and Schmid study, Koranyi and Wolf identify a point e_+ on the boundary of the bounded symmetric domain such that the isotropy subgroup K_{iso} of K at this point equals the fixed-point group of the involution $\tau = c^2$ of K. From this identification they are able to identify the Bergman-Silov boundary of the bounded symmetric domain with K/K_{iso}. The simplest example occurs for $G = SU(1,1)$, in which case K is a circle and K_{iso} is a two-element subgroup. Schmid's work consists in identifying the decomposition of L^2 of this quotient of K and then, using character computations, in showing which representations of K actually correspond to polynomials. (For $SU(1,1)$, the analysis of the quotient would lead to a family of characters of K parametrized by the integers, and the character computations would eliminate the characters corresponding to the negative integers.)

In their case Gross and Wallach largely just translate some work done by others in algebraic geometry, particularly in the theory of prehomogeneous spaces of Sato and Kimura [S-K]. This theory identifies the invariant locus for them, and they apply "Luna's Slice Theorem" [L] to obtain an algebraic-geometric decomposition that mirrors Frobenius reciprocity. In any event, Cayley transforms and involutions play no role in the Gross-Wallach work.

For the groups $SO(2m, 2n)$ in this thesis, there is no bounded symmetric domain or analog from which we can get started. Our approach instead is to construct a special element e_+ in $\mathfrak{u} \cap \mathfrak{p}$ for which we can show that restriction of holomorphic polynomials on $\mathfrak{u} \cap \mathfrak{p}$ to the L orbit of e_+ is one-one. Then it follows that S embeds one-one in L-equivariant fashion into $L^2(L/L_{\text{iso}})$, where L_{iso} is the isotropy subgroup of L at e_+. The Cayley transform element c of $G^{\mathbb{C}}$ makes sense for our groups, and $\tau = c^2$ is an involution. But it is not true for $m > 1$ that the fixed group L_τ equals the isotropy group $L_{\rm iso}$. What does happen is that L_τ contains $L_{\rm iso}$ and the quotient space is a compact symmetric space. This much structure theory allows us to obtain our main theorem.

CHAPTER 2: SETTING AND NOTATION

The notation introduced in this chapter will be used throughout the remaining chapters.

For $m \leq n$, it is customary to let $SO(2m, 2n)$ be the identity component of the group of real matrices g such that $g^* I_{2m,2n} g = I_{2m,2n}$, where $I_{2m,2n}$ is the square diagonal matrix of size $2m + 2n$, with 1 in each of the first $2m$ entries along the diagonal, and -1 in each of the last $2n$ diagonal entries. The Lie algebra of this group may be regarded as the set

$$\{X \in \mathfrak{gl}(2m+2n, \mathbb{R}) \mid X^* I_{2m,2n} + I_{2m,2n} X = 0\}.$$

The complexification of this Lie algebra is an inconvenient set of matrices; the matrices are skew symmetric in the indices i and j if i and j are both $\leq 2m$ or both $> 2m$, and they are symmetric in the indices i and j otherwise.

For this reason we shall *redefine* $G = SO(2m, 2n)$ to be the conjugate of the above group by the diagonal matrix whose first $2m$ diagonal entries are 1 and whose last $2n$ diagonal entries are $i = \sqrt{-1}$. The members g of our new G still satisfy $g^* I_{2m,2n} g = I_{2m,2n}$. The Lie algebra $\mathfrak{g}_0$ of G is

$$\mathfrak{g}_0 = \left\{ X \in \mathfrak{gl}(2m+2n, \mathbb{C}) \;\middle|\; \begin{array}{l} X^* I_{2m,2n} + I_{2m,2n} X = 0, \\ X_{ij} \text{ is real if } i \text{ and } j \text{ are } \leq 2m, \\ X_{ij} \text{ is real if } i \text{ and } j \text{ are } > 2m, \\ X_{ij} \text{ is imaginary if } i \leq 2m \text{ and } j > 2m, \\ X_{ij} \text{ is imaginary if } j \leq 2m \text{ and } i > 2m \end{array} \right\}.$$

The complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is simply the set $\mathfrak{so}(2m+2n, \mathbb{C})$ of all skew-symmetric complex matrices of size $2m + 2n$.

The expression $\theta(X) = -X^*$ defines a Cartan involution on $\mathfrak{g}_0$, with associated Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. The analytic subgroup K of G with Lie algebra the subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$ is nothing other than $K = SO(2m) \times SO(2n)$, and it is a maximal compact subgroup of G.

The set of block diagonal matrices consisting of 2×2 blocks of the form $\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}$, with r real, is a Cartan subalgebra $\mathfrak{t}_0 \subseteq \mathfrak{g}_0$. Note that $\mathfrak{t}_0$ also serves as a Cartan subalgebra for $\mathfrak{k}_0$.

Let H be a member of the complexification $\mathfrak{t}$ of $\mathfrak{t}_0$. Let $\begin{pmatrix} 0 & ih_j \\ -ih_j & 0 \end{pmatrix}$ be the j^{th} diagonal block of H.

We define $e_j \in \mathfrak{t}^*$ to be the linear functional such that $e_j(H) = h_j$. The roots of $\mathfrak{g}$ are $\pm e_i \pm e_j$, with $1 \leq i < j \leq m+n$. We denote this set of roots by Δ.

If $\varphi = \sum_j a_j e_j$ is a member of $(i\mathfrak{t}_0)^*$, i.e., has each $a_j \in \mathbb{R}$, we say that a nonzero φ is *positive*, written $\varphi > 0$, if the first nonzero a_j is positive. This establishes an order for $(i\mathfrak{t}_0)^*$.

The simple roots for $\mathfrak{g}$ are the roots $e_1 - e_2$, $e_2 - e_3$, $\dots$, $e_{m+n-1} - e_{m+n}$, $e_{m+n-1} - e_{m+n}$.

We may write $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space associated to α. Each $\mathfrak{g}_\alpha$ is θ-stable; hence for each $\alpha \in \Delta$ we have either $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$, respectively. We say that α is a *compact root* when $\mathfrak{g} \subseteq \mathfrak{k}$. Otherwise, if $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$, we say that α is a *noncompact root.*

We define $\mathfrak{l} \subseteq \mathfrak{g}$ to be the space $\mathfrak{l} = \mathfrak{t} \oplus \sum_{\alpha\in\Delta'} \mathfrak{g}_\alpha$, where $\Delta' \subseteq \Delta$ is the subset of roots in the span of the compact simple roots of $\mathfrak{g}$. We define $\mathfrak{u} \subseteq \mathfrak{g}$ as the space

$\mathfrak{u} = \sum_{\alpha\in(\Delta-\Delta')_+} \mathfrak{g}_\alpha$, where $(\Delta - \Delta')_+$ consists of the positive roots in Δ not in Δ'. Likewise $\bar{\mathfrak{u}}$ is the space $\bar{\mathfrak{u}} = \sum_{\alpha\in(\Delta-\Delta')_-} \mathfrak{g}_\alpha$, where $(\Delta - \Delta')_-$ consists of the negative roots in Δ not in Δ'. We let $\mathfrak{l}_0 = \mathfrak{g}_0 \cap \mathfrak{l}$. Since the roots of $\mathfrak{l}$ are closed under negatives, it follows that $\mathfrak{l} = (\mathfrak{l}_0)^{\mathbb{C}}$. The subalgebra $\mathfrak{t}_0$ is a Cartan subalgebra for $\mathfrak{l}_0$. Let $L \subseteq G$ be the analytic subgroup associated to $\mathfrak{l}_0 \subseteq \mathfrak{g}_0$; L is known to be closed, hence compact.

For φ as above, we define $H_\varphi \in \mathfrak{t}$ to be the unique element such that $\varphi(H) = B(H, H_\varphi)$ for all $H \in \mathfrak{t}$. The existence and uniqueness of H_φ follows from the nondegeneracy of the restricted Killing form $B|_{\mathfrak{t}\times\mathfrak{t}}$.

The center of $\mathfrak{l}_0$ consists of the pure imaginary multiples of $H_{e_1+\cdots+e_m}$.

The commutator subalgebra of $\mathfrak{l}_0$ is

$$(\mathfrak{l}_0)_{ss} = \mathfrak{su}(m) \oplus \mathfrak{so}(2n),$$

the $\mathfrak{su}(m)$ being regarded as a subalgebra of the $so(2m)$ in the first $2m$ entries.

We say that a member φ of $\mathfrak{t}^*$ is *analytically integral* if φ is the differential of a multiplicative character of the torus $\exp \mathfrak{t}_0$.

Theorem of the Highest Weight. *Apart from equivalence the irreduicble finite-dimensional representations Φ of L stand in one-one correspondence with the dominant analytically integral linear functionals λ on $\mathfrak{t}$, the correspondence being that λ is the highest weight of Φ_λ.*

An *L-type* is an equivalence class of irreducible finite-dimensional representations of L.

Since $\mathfrak{l}_0 \subseteq \mathfrak{k}_0$, $[\mathfrak{l}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0$ and $[\mathfrak{l}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0$. Thus $\operatorname{Ad} L$ carries $\mathfrak{k}_0$ to itself and $\mathfrak{p}_0$ to itself. One also sees that $[\mathfrak{l}_0, \mathfrak{u}_0] \subseteq \mathfrak{u}_0$ and hence $(\operatorname{Ad} L)(\mathfrak{u}) \subseteq \mathfrak{u}$.

We are interested in the action of L via Ad on the symmetric tensors on $\mathfrak{u} \cap \mathfrak{p}$, denoted $S(\mathfrak{u} \cap \mathfrak{p})$. The space $S(\mathfrak{u} \cap \mathfrak{p})$ may also be regarded as the space of holomorphic polynomials on $\bar{\mathfrak{u}} \cap \mathfrak{p}$, denoted $\mathcal{P}(\bar{\mathfrak{u}} \cap \mathfrak{p})$.

We now define root vectors for roots $\alpha \in \Delta$, and give bracket relations. These are convenient to write down since $\mathfrak{g} = \mathfrak{so}(2m+2n, \mathbb{C})$. Let $\alpha = \pm e_i \pm e_j$ with $i < j$. Then

$$E_\alpha = \begin{array}{cc} \begin{array}{cc} i & j \end{array} & \\ \begin{pmatrix} 0 & X_\alpha \\ -X_\alpha^t & 0 \end{pmatrix} & \begin{array}{c} i \\ j \end{array} \end{array}$$

with

$$X_{e_i - e_j} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad X_{e_i + e_j} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$
$$X_{-e_i + e_j} = \frac{1}{2}\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}, \qquad X_{-e_i - e_j} = \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}.$$

The bracket relations are as follows, with $i < j < k$. We write $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$. The following table defines $c_{\alpha,\beta}$ whenever $\alpha + \beta > 0$ is a root. For

$\alpha+\beta<0$, $c_{-\alpha,-\beta}=-c_{\alpha,\beta}$.

Formula	α	β	$c_{\alpha,\beta}$
(1)	e_j-e_i	e_i+e_k	$+1$
(2)	e_j-e_i	e_i-e_k	$+1$
(3)	e_i-e_j	e_j+e_k	$+1$
(4)	e_i-e_j	e_j-e_k	$+1$
(5)	e_j-e_k	e_i-e_j	-1
(6)	e_j-e_k	e_i+e_k	$+1$
(7)	e_k-e_j	e_i+e_j	$+1$
(8)	e_k-e_j	e_i-e_k	-1
(9)	e_j+e_k	e_i-e_j	-1
(10)	e_j+e_k	e_i-e_k	$+1$
(11)	$-e_j-e_k$	e_i+e_j	$+1$
(12)	$-e_j-e_k$	e_i+e_k	-1
(13)	e_i+e_j	$-e_i+e_k$	$+1$
(14)	e_i+e_j	$-e_i-e_k$	$+1$

The vectors E_α have the property that $\overline{E}_\alpha=E_\alpha$ when α is noncompact and $\overline{E}_\alpha=-E_\alpha$ when α is compact. Here the bar denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$; this is different from entry-by-entry conjugation.

A maximal system of strongly orthogonal positive noncompact roots for $\mathfrak{g}$ is

$$\gamma_1=e_1+e_{m+1}$$

$$\gamma_2=e_1-e_{m+1}$$

$$\gamma_3=e_2+e_{m+2}$$

$$\gamma_4=e_2-e_{m+2}$$

$$\cdots$$

$$\gamma_{2m-1}=e_m+e_{2m}$$

$$\gamma_{2m}=e_m-e_{2m}\,.$$

We conclude this chapter with the statement of a weak form of the main theorem.

Theorem. *The highest weight of any irreducible representation of L occurring in $S(\mathfrak{u} \cap \mathfrak{p})$ is a rational linear combination of $\gamma_1, \ldots, \gamma_{2m}$, and the multiplicity of any L-type in $S(\mathfrak{u} \cap \mathfrak{p})$ is less than or equal to the degree of the L-type.*

Chapter 3: Geometry of $\mathfrak{u} \cap \mathfrak{p}$

Both $\mathfrak{u}$ and $\mathfrak{p}$ are invariant under the adjoint action of L on $\mathfrak{g}$. Thus the intersection $\mathfrak{u} \cap \mathfrak{p}$ is also invariant under the adjoint action of L. In this chapter we are interested in this action of L on $\mathfrak{u} \cap \mathfrak{p}$. In particular, we are interested in the orbit of the point

$$e_+ = E_{\gamma_1} + \cdots + E_{\gamma_{2m}} \in \mathfrak{u} \cap \mathfrak{p}.$$

In the following lemma, we find the isotropy subalgebra $\mathfrak{l}_{\text{iso}}$ of $\mathfrak{l}$ annihilating e_+.

Lemma 1. $\dim_{\mathbb{C}} \mathfrak{l} = \dim_{\mathbb{C}} \mathfrak{l}_{\text{iso}} + \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{p})$.

Proof. We have $\dim \mathfrak{l} = \dim(\mathfrak{sl}(m, C)) + \dim(\mathfrak{so}(2n, \mathbb{C})) + 1 = m^2 + 2n^2 - n$ and $\dim(\mathfrak{u} \cap \mathfrak{p}) = 2mn$. In order to find the space $\mathfrak{l}_{\text{iso}} \subseteq \mathfrak{l}$, we must find the cofficients d_α such that

$$\Big[\sum_{\alpha \in \Delta'} d_\alpha E_\alpha \, , \, e_+ \Big] = 0.$$

The left side of this equation may be expanded in terms of the root vectors for the noncompact positive roots other than $\gamma_1, \ldots, \gamma_{2m}$. Thus we obtain a system of $2mn - 2m$ equations with $|\Delta'| = m(m-1) + 2n(n-1)$ unknowns. For indices $1 \leq i < j \leq m$, we get equations

$$\begin{aligned}
d_{e_i - e_j} &= d_{e_{i+m} + e_{j+m}} - d_{e_{j+m} - e_{i+m}} \\
d_{e_i - e_j} &= d_{e_{i+m} - e_{j+m}} - d_{-e_{j+m} - e_{i+m}} \\
d_{e_j - e_i} &= -d_{e_{i+m} - e_{j+m}} - d_{e_{i+m} + e_{j+m}} \\
d_{e_j - e_i} &= -d_{-e_{i+m} - e_{j+m}} - d_{e_{j+m} + e_{i+m}} \, ,
\end{aligned}$$

and for indices $1 \leq i \leq m$ and $k \geq 2m+1$, we get the equations

$$d_{e_i+e_k} = d_{-e_i+e_k}$$

$$d_{e_i-e_k} = d_{-e_i-e_k}\,.$$

To arrive at these equations, we are making critical use of our bracket relations for the root vectors established in Chapter 2. Each equation is in an independent set of four equations, like the first set above, or of two equations, like the second set above. Each independent set of four yields a 2-dimensional subspace of $\mathfrak{l}_{\text{iso}}$, and each independent set of two equations yields a 2-dimensional subspace of $\mathfrak{l}_{\text{iso}}$.

A basis of $\mathfrak{l}_{\text{iso}}$ consists of the following elements:

$$E_{e_i-e_j} + \tfrac{1}{2}(E_{e_{i+m}+e_{j+m}} + E_{e_{i+m}-e_{j+m}}) - \tfrac{1}{2}(E_{-e_{i+m}-e_{j+m}} + E_{e_{j+m}-e_{i+m}})$$

$$E_{e_j-e_i} + \tfrac{1}{2}(E_{e_{i+m}+e_{j+m}} + E_{e_{i+m}-e_{j+m}}) - \tfrac{1}{2}(E_{-e_{i+m}-e_{j+m}} + E_{e_{j+m}-e_{i+m}})$$

$$\text{for } 1 \leq i < j \leq m,$$

$$E_{e_i+e_k} + E_{-e_i+e_k} \qquad \text{for } m+1 \leq i \leq 2m \text{ and } 2m+1 \leq k \leq m+n,$$

$$E_{e_i-e_k} + E_{-e_i-e_k} \qquad \text{for } m+1 \leq i \leq 2m \text{ and } 2m+1 \leq k \leq m+n,$$

$$E_{\pm e_k \pm e_\ell} \qquad \text{for } 2m+1 \leq k < \ell \leq m+n,$$

$$H_{e_k} \qquad \text{for } 2m+1 \leq k \leq m+n.$$

Thus $\dim \mathfrak{l}_{\text{iso}} = m^2 + 2n^2 - n - 2mn$. Comparison to $\dim \mathfrak{l}$ and $\dim(\mathfrak{u} \cap \mathfrak{p})$ gives us the desired result.

The complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is the Lie algebra of $G^{\mathbb{C}} = SO(2m+2n, \mathbb{C})$. Let $L^{\mathbb{C}}$ be the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{l}$.

Lemma 2. $(\mathrm{Ad}\, L^{\mathbb{C}})e_+$ *is an open subset of* $\mathfrak{u} \cap \mathfrak{p}$.

Proof. Under the map $L^{\mathbb{C}} \times e_+ \to \mathfrak{u}\cap\mathfrak{p}$ given by $(\ell, e_+) \mapsto \mathrm{Ad}(\ell)e_+$, the differential is the map $\mathfrak{l} \times e_+ \to \mathfrak{u} \cap \mathfrak{p}$ given by $(X, e_+) \mapsto [X, e_+]$. Lemma 1 shows that the differerential is onto. Hence the image of the original map contains an open neighborhood of e_+. Using translations within $L^{\mathbb{C}}$, we see that the image of the original map contains an open neighborhood of each of its points.

Theorem. *The restriction map from* $\mathcal{P}(\mathfrak{u} \cap \mathfrak{p})$ *to the* L *orbit of* e_+ *is one-one. Consequently if* L_1 *denotes the isotropy subgroup within* L *at the point* e_+*, then* $\mathcal{P}(\mathfrak{u} \cap \mathfrak{p})$ *embeds one-one into* $L^2(L/L_1)$ *in an* L*-equivariant fashion.*

Proof. Starting from P in $\mathcal{P}(\mathfrak{u}\cap\mathfrak{p})$, define $P^{\#}(\ell) = (\mathrm{Ad}\, L^{\mathbb{C}})e_+$ for $\ell \in L^{\mathbb{C}}$. Then $P^{\#}$ is a holomorphic function on $L^{\mathbb{C}}$, being the composition of holomorphic functions. By Lemma 2, $\mathrm{Ad}(L^{\mathbb{C}})e_+$ is an open subset of $\mathfrak{u} \cap \mathfrak{p}$, and it follows that the map $P \mapsto P^{\#}$ is one-one. On the other hand, restriction of $P^{\#}$ from $L^{\mathbb{C}}$ to L is one-one as we see by consideration of Taylor expansions in local coordinates about the identity of $L^{\mathbb{C}}$. This proves the first conclusion. The argument respects the action of L throughout, and thus the second statement follows.

It will be inconvenient to deal with the full isotropy subgroup L_1 of L at e_+. Instead we shall work with the identity component, which we denote L_{iso}. If we regard $L^2(L/L_1)$ as consisting of functions of L that are right invariant under L_1, then we can consider $L^2(L/L_1)$ to be a subset of $L^2(L/L_{\text{iso}})$, and it follows from the theorem that that $\mathcal{P}(\mathfrak{u}\cap\mathfrak{p})$ embeds one-one into $L^2(L/L_{\text{iso}})$ in an L-equivariant fashion. The range space for this embedding is stable under complex conjugation of functions; hence the L-type by L-type contradgredient mapping carries $L^2(L/L_{\text{iso}})$ into itself. The L-type by L-type contradgredient of $\mathcal{P}(\mathfrak{u} \cap \mathfrak{p})$ is just $S(\mathfrak{u} \cap \mathfrak{p})$, and we obtain the following corollary.

Corollary. *The symmetric algebra $S(\mathfrak{u} \cap \mathfrak{p})$ embeds one-one in L-equivariant fashion into $L^2(L/L_{\text{iso}})$.*

Since $L^2(L/L_{\text{iso}})$ can be regarded as a subspace of $L^2(L)$, it follows from the corollary that any L type in $S(\mathfrak{u} \cap \mathfrak{p})$ occurs at most as many times as its degree.

To proceed with the analysis, we first relate the Lie algebra $\mathfrak{l}_{\text{iso}}$ of the lemma to the connected Lie group L_{iso} defined above.

Let $(\mathfrak{l}_0)_{\text{iso}} = \mathfrak{l}_{\text{iso}} \cap \mathfrak{g}_0$.

Proposition. *The Lie algebra of L_{iso} is $(\mathfrak{l}_0)_{\text{iso}}$.*

Proof. Every element of L_{iso} fixes e_+, and hence every element of the Lie algebra of this group annihilates e_+. Thus the Lie algebra of L_{iso} is contained in $\mathfrak{l}_{\text{iso}}$, as well as $\mathfrak{g}_0$, and hence is contained in $(\mathfrak{l}_0)_{\text{iso}}$. Conversely suppose X is in $(\mathfrak{l}_0)_{\text{iso}}$. Then $[X, e_+] = 0$, and also X is in $\mathfrak{g}_0$ and $\mathfrak{l}$. These latter conditions say that X is in $\mathfrak{l}_0$. Hence $\exp tX$ is in L for all t, and $\operatorname{Ad}(\exp tX)e_+ = e_+$ for all t. Therefore $\exp tX$ is in L_{iso}, and X is in its Lie algebra.

As a consequence of the lemma, $\mathfrak{l}_{\text{iso}}$ is closed under the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Therefore $(\mathfrak{l}_0)_{\text{iso}}^{\mathbb{C}} = \mathfrak{l}_{\text{iso}}$. This is a special property of $\mathfrak{l}_{\text{iso}}$ that we have not used so far.

For each γ_j, define the *Cayley transform* as a map on $\mathfrak{g}$ by

$$c_{\gamma_j} = \mathrm{Ad}(\exp(\tfrac{\pi}{4} i(E_{\gamma_j} + \overline{E}_{\gamma_j}))).$$

Next define the map τ on $\mathfrak{g}$ by

$$\tau = c_{\gamma_1}^2 \cdot \ldots \cdot c_{\gamma_{2m}}^2 = \mathrm{Ad}(\exp(\tfrac{\pi}{2} i(e_+ + \bar{e}_+))).$$

Proposition. $\tau^2 = 1$ *on* $\mathfrak{g}$.

Proof. Consider some γ_j, $1 \leq j \leq 2m$. Corresponding to γ_j is a subgroup of $G^{\mathbb{C}} = SO(2m+2n, \mathbb{C})$ that is a homomorphic image of $SL(2,\mathbb{C})$. In $\mathfrak{g}$, c_{γ_j} acts in effect by $\mathrm{Ad}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then $c_{\gamma_j}^4$ acts as $\mathrm{Ad}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathfrak{g}$.

Consequently in $\mathfrak{g}$, $c_{\gamma_j}^4$ acts by $\mathrm{Ad}(\exp \pi i H'_{\gamma_j})$, where

$$H'_{\gamma_j} = \frac{2H_{\gamma_j}}{|\gamma_j|^2}.$$

For $\beta \in \Delta$, we have

$$c_{\gamma_j}^4 E_\beta = \mathrm{Ad}(\exp \pi i H'_{\gamma_j}) E_\beta = \exp(\mathrm{ad}(\pi i H'_{\gamma_j})) E_\beta.$$

Now

$$\begin{aligned} \mathrm{ad}(\pi i H'_{\gamma_j}) E_\beta &= \pi i [H'_{\gamma_j}, E_\beta] \\ &= \frac{2\pi i}{|\gamma_j|^2} [H_{\gamma_j}, E_\beta] \\ &= \frac{2\pi i}{|\gamma_j|^2} \beta(H_{\gamma_j}) E_\beta \\ &= \frac{2\pi i \langle \beta, \gamma_j \rangle}{|\gamma_j|^2} E_\beta. \end{aligned}$$

So $c_{\gamma_j}^4 E_\beta = e^{\pi i(2\langle\beta,\gamma_j\rangle/|\gamma_j|^2)} E_\beta = \pm E_\beta$, and therefore

$$\tau^2 E_\beta = \Big[\prod_j e^{\pi i(2\langle\beta,\gamma_j\rangle)/|\gamma|^2)}\Big] E_\beta,$$

where $|\gamma|^2$ denotes the common value of $|\gamma_j|^2$. The indices j for which $2\langle\beta,\gamma_j\rangle)/|\gamma|^2 = -1$ occur in pairs $e_k \pm e_\ell$ with $k < \ell$, and thus the number of indices j for which $2\langle\beta,\gamma_j\rangle)/|\gamma|^2 = -1$ is even. Thus $\tau^2(E_\beta) = E_\beta$ for all $\beta \in \Delta$.

Further, for any $H \in \mathfrak{t}$, we have $c_{\gamma_j}^4(H) = \mathrm{Ad}(\exp(\pi i H'_{\gamma_j}))H = H$ since $\mathfrak{t}$ is abelian. Thus $\tau^2 = 1$ on all of $\mathfrak{g}$.

Now we compute the effect of τ on root vectors in $\mathfrak{g}$. Expressing

$$\tau = \mathrm{Ad}(\exp(\tfrac{\pi i}{2}(E_{\gamma_1} + \overline{E}_{\gamma_1} + \cdots + E_{\gamma_m} + \overline{E}_{\gamma_m})))$$

and using the table in Chapter 2, we see that, for $i < j$,

$$\tau E_{\pm(e_i-e_j)} = \begin{cases} -E_{\mp(e_i-e_j)} & \text{for } 1 \le i < j \le m \\ -E_{\mp(e_i-e_j)} & \text{for } m+1 \le i < j \le 2m \\ +E_{\mp(e_i+e_j)} & \text{for } m+1 \le i \le 2m,\ 2m+1 \le j \le m+n \\ +E_{\pm(e_i-e_j)} & \text{for } 2m+1 \le i < j \le m+n \end{cases}$$

and

$$\tau E_{\pm(e_i+e_j)} = \begin{cases} -E_{\mp(e_i+e_j)} & \text{for } m+1 \le i < j \le 2m \\ +E_{\mp(e_i-e_j)} & \text{for } m+1 \le i \le 2m,\ 2m+1 \le j \le m+n \\ +E_{\pm(e_i+e_j)} & \text{for } 2m+1 \le i < j \le m+n. \end{cases}$$

Thus τ carries $\mathfrak{l}$ to itself.

Also τ carries $\mathfrak{k}_0 \oplus ip_0$ to itself because each c_{γ_j} has this property. Since $\mathfrak{l}_0 = \mathfrak{l} \cap (\mathfrak{k}_0 \oplus i\mathfrak{p}_0)$, τ carries $\mathfrak{l}_0$ to $\mathfrak{l}_0$.

We define $(\mathfrak{l}_0)_\tau$ to be the subalgebra of $\mathfrak{l}_0$ fixed by τ. We see, by the work above, that a basis for $\mathfrak{l}_\tau$ over $\mathbb{C}$ is

$$\begin{aligned}
E_{e_i-e_j} - E_{e_j-e_i} &\quad \text{for } 1 \le i < j \le m \\
E_{e_i-e_j} - E_{e_j-e_i} &\quad \text{for } m+1 \le i < j \le 2m \\
E_{e_i+e_j} - E_{-e_i-e_j} &\quad \text{for } m+1 \le i < j \le 2m \\
E_{e_i-e_j} + E_{-e_i-e_j} &\quad \text{for } m+1 \le i \le 2m,\ 2m+1 \le j \le m+n \\
E_{e_i+e_j} + E_{-e_i+e_j} &\quad \text{for } m+1 \le i \le 2m,\ 2m+1 \le j \le m+n \\
E_{\pm e_k \pm e_\ell} &\quad \text{for } 2m+1 \le k < \ell \le m+n \\
H_{e_k} &\quad \text{for } 2m+1 \le k \le m+n.
\end{aligned}$$

The first three members of the above list are in $(\mathfrak{l}_0)_\tau$ since $E_\alpha - E_{-\alpha}$ is in $\mathfrak{g}_0$ if α is compact. For the fourth and fifth members, we can group the terms in pairs consisting an element and its (independent) conjugate. For such a pair X and $\overline{X}$, the elements $X+\overline{X}$ and $i(X-\overline{X})$ span the complex 2-dimensional space spanned by X and $\overline{X}$, and thus $X+\overline{X}$ and $i(X-\overline{X})$ span over $\mathbb{R}$ the intersection of this complex 2-dimensional space with $\mathfrak{g}_0$. For the sixth members, the terms may similarly be grouped in pairs to give pairs of basis elements for $(\mathfrak{l}_0)_\tau$. Finally the elements iH_{e_k} are a basis of the intersection of $\mathfrak{g}_0$ with the complex span of $H_{e_{2m+1}}, \ldots, H_{e_{m+n}}$. In this way we obtain a basis of $(\mathfrak{l}_0)_\tau$ over $\mathbb{R}$.

Let $L_\tau \subseteq L$ be the analytic subgroup corresponding to $(\mathfrak{l}_0)_\tau$. Let $\mathfrak{l}_\tau = (\mathfrak{l}_0)_\tau^{\mathbb{C}}$. By comparing our basis of $\mathfrak{l}_\tau$ to the basis for $\mathfrak{l}_{\text{iso}}$ in Chapter 3, we observe that

$$\mathfrak{l}_{\text{iso}} \subseteq \mathfrak{l}_\tau \subseteq \mathfrak{l}.$$

Chapter 5: Proof of the Main Theorem

In Chapter 2, we stated a weak form of the main Theorem. Here we present the strong form of this Theorem, and provide the proof.

Theorem. *The highest weight of any irreducible representation of L occurring in $S(\mathfrak{u}\cap\mathfrak{p})$ is a rational linear combination of $\gamma_1, \ldots, \gamma_{2m}$, and the multiplicity of any L-type in $S(\mathfrak{u}\cap\mathfrak{p})$ is less than or equal to the multiplicity of the L-type in $L^2(L/L_{\text{iso}})$. The coefficients of this linear combination must be in $\frac{1}{2}\mathbb{Z}$.*

Proof. In Chapter 4, we established the inclusion

$$\mathfrak{l}_{\text{iso}} \subseteq \mathfrak{l}_{\tau} \subseteq \mathfrak{l}.$$

On the group level, we have

$$L_{\text{iso}} \subseteq L_{\tau} \subseteq L.$$

By the Corollary in Chapter 3, we may regard $S(\mathfrak{u}\cap\mathfrak{p})$ as a subset of $L^2(L/L_{\text{iso}})$. To prove our Theorem, we make use of the following identity

$$L^2(L/L_{\text{iso}}) = ind_{L_{\text{iso}}}^{L} 1.$$

But rather than examine these induced representations directly, we look at the equivalent two-step induction

$$ind_{L_\tau}^{L}(ind_{L_{\text{iso}}}^{L_\tau} 1).$$

We found bases for $\mathfrak{l}_{\text{iso}}$ and $\mathfrak{l}_\tau$ in Chapters 3 and 4, and we use these bases to observe that

$$\mathfrak{l}_0 = su(m) \oplus \mathbb{R}iH_{e_1+\ldots+e_m} \oplus so(2n),$$

$$(\mathfrak{l}_T)_0 = so(m) \oplus 0 \oplus (so(m) \oplus so(2n-m)),$$

and finally,

$$(\mathfrak{l}_{\text{iso}})_0 = \text{diag}(so(m) \oplus so(m)) \oplus so(2n-m).$$

The first induction, from L_{iso} to L_T, gives us the trivial representation in the third factor, and contragredient representations in the first two nonzero factors. In the next induction, we make use of a theorem of Gelbart [G], who establishes that there are only m paramaters in describing the irreducible representations of $SO(n)$ which appear in $L^2(SO(n)/SO(n-m))$. Through the use of Gelbart's result, and our expressions above, we establish that the L-types which occur in $S(\mathfrak{u} \cap \mathfrak{p})$ must have highest weights which are a linear combination of $\gamma_1, \ldots, \gamma_{2m}$.

The Theorem of the Highest Weight, presented in Chapter 2, puts some restriction on the coefficients of these combinations, through the integrality condition. A straightforward computation shows that this integrality condition forces the coefficients to lie in $\frac{1}{2}\mathbb{Z}$.

Finally, because $S(\mathfrak{u} \cap \mathfrak{p})$ can be regarded as a subspace of $L(L/L_{\text{iso}})$, the multiplicity of any L-type in $S(\mathfrak{u} \cap \mathfrak{p})$ must be less than or equal to its multiplicity in $L(L/L_{\text{iso}})$.

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