Algebraic Cycles on Real Varieties and $\mathbb{Z}_2$-Equivariant Homotopy Theory

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Abstract of Dissertation

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In Chapter 1 of this dissertation we study spaces of algebraic cycles on a Real projective variety X. We consider these spaces with the $\mathbb{Z}_2$-action induced from the Real structure on X. When X is $\mathbb{P}^n_{\mathbb{C}}$ these spaces are products of classifying spaces for $\mathbb{Z}_2$-equivariant cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. We prove this by establishing an equivariant version of the classical Dold-Thom theorem. We also propose a version of Lawson homology for Real varieties and compute it in some examples. In Chapter 2 we relate $\mathbb{Z}_2$-equivariant cohomology with $\underline{\mathbb{Z}}$ coefficients to certain Galois-Grothendieck cohomology groups which are invariants frequently used in real algebraic geometry.

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CHAPTER 1

Cycles on Real Varieties

1. Introduction

The Chow varieties of a projective variety X are classical objects of algebraic geometry that are easy to define but rather difficult to deal with in general. A major breakthrough in the understanding of these objects was achieved by Lawson in the fundamental paper [**15**] where he computed the homotopy type of the space $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ of algebraic p-cycles on $\mathbb{P}^n_{\mathbb{C}}$. The computation rests on his algebraic suspension Theorem — the suspension of a variety X in $\mathbb{P}^n_{\mathbb{C}}$ is the subvariety of $\mathbb{P}^{n+1}_{\mathbb{C}}$ whose points lie on the lines joining the points of X and $(0 : \cdots : 0 : 1) \in \mathbb{P}^{n+1}_{\mathbb{C}} - \mathbb{P}^n_{\mathbb{C}}$. The suspension Theorem asserts that the suspension map

$$\Sigma\!\!\!/ : \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \to \mathcal{Z}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$$

is a homotopy equivalence. The results and techniques of Lawson's paper led to the definition by Friedlander and Lima-Filho of a homology theory for quasi-projective varieties: **Lawson homology**. For a projective variety X the Lawson homology groups of X are the homotopy groups $\pi_*(\mathcal{Z}_p(X))$. This theory has exact sequences, satisfies excision (in the appropriate sense) and comes equipped with a cycle map which takes values in the homology of X with integer coefficients. These features make it into a theory in which computations are often possible.

Now, $\mathbb{P}^n_{\mathbb{C}}$ has more structure; namely, it is a Real variety in the sense that it has an anti-holomorphic involution given by complex conjugation of its homogeneous coordinates. This involution carries algebraic varieties into algebraic varieties and gives a continuous $\mathbb{Z}_2$-action on the space of algebraic cycles $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$. It is natural to ask if Lawson's techniques can be applied to study $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ with this additional structure.

The first steps in this direction were taken by Lam who proved that Lawson's suspension Theorem holds equivariantly [**14**]. More accurately his work shows that the suspension map (which is easily seen to be equivariant) is a $\mathbb{Z}_2$-homotopy equivalence. This was then used to compute the homotopy type of the space of reduced cycles

$$\widetilde{\mathcal{Z}_{p,\mathbb{R}}(\mathbb{P}^n_{\mathbb{C}})} \stackrel{\text{def}}{=} \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})^{\mathbb{Z}_2}/\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})^{av},$$

where $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})^{av} = \{c + \tau \cdot c \mid c \in \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})\}$.

More recently Lawson, Lima-Filho and Michelsohn have computed the homotopy type of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})^{\mathbb{Z}_2}$; [**16**]. Moreover, using the suspension map $\Sigma\!\!\!/ : \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \to \mathcal{Z}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ and the natural inclusions $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \to \mathcal{Z}_p(\mathbb{P}^{n+1}_{\mathbb{C}})$ they considered the $\mathbb{Z}_2$-space of stabilized cycles

$$\mathcal{Z} \stackrel{\text{def}}{=} \lim_{n,p\to\infty} \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$$

and computed the ring structure induced on the homotopy groups of $\mathcal{Z}^{\mathbb{Z}_2}$ by the operation $\#$ of joining cycles — given varieties V, W in $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^m_{\mathbb{C}}$, respectively, their join is the subvariety of $\mathbb{P}^{n+m+1}_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{C}} \# \mathbb{P}^m_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^{n+m+2})$ whose points lie on the lines joining points of V and W in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$. This naturally extends to an operation on cycles which is equivariant.

We continue the study of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ with the additional structure given by the $\mathbb{Z}_2$-action and compute its full $\mathbb{Z}_2$-homotopy type. In the non-equivariant case Lawson used the suspension Theorem to reduce to the case of zero cycles. The Dold-Thom Theorem then gives

$$\pi_k \mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) = H_k(\mathbb{P}^n_{\mathbb{C}}; \mathbb{Z}).$$

This completely determines the homotopy type of $\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})$ because this space is a topological abelian group, and so, by a result of Moore, it is homotopy equivalent to a product of Eilenberg-Mac Lane spaces. Thus

$$\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \cong \mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}}) \cong K(\mathbb{Z},0) \times K(\mathbb{Z},2) \times \cdots \times K(\mathbb{Z}, 2(n-p)).$$

Actually this splitting can be made in a canonical way. For each $m \le n$ one constructs group homomorphisms

$$\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \to \mathcal{Z}_0(\mathbb{P}^m_{\mathbb{C}})/\mathcal{Z}_0(\mathbb{P}^{m-1}_{\mathbb{C}}) \cong AG(\mathbb{P}^m_{\mathbb{C}}/\mathbb{P}^{m-1}_{\mathbb{C}}) \cong K(\mathbb{Z}, 2m)$$

using the homeomorphisms $\mathbb{P}^m_{\mathbb{C}} \cong SP^m(\mathbb{P}^1_{\mathbb{C}})$. Here SP^m denotes the m-fold symmetric product and AG denotes zero cycles of degree zero. These maps assemble into a group homomorphism

$$\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \prod_{k=0}^{n} AG(S^{2k}) \tag{1.1}$$

which is a homotopy equivalence; see [**8**]. This splitting has the following important property. Denote by $i_{m,n}$ the inclusion $\mathcal{Z}_0(\mathbb{P}^m_{\mathbb{C}}) \subset \mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})$ induced by $\mathbb{P}^m_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$ and by $j_{m,n}$ the inclusion of $\prod_{k=0}^{m} AG(S^{2k})$ in $\prod_{k=0}^{n} AG(S^{2k})$ as the first m factors of the product. Then the diagram

$$\begin{array}{ccc} \mathcal{Z}_0(\mathbb{P}^m_{\mathbb{C}}) & \xrightarrow{i_{m,n}} & \mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \\ \downarrow & & \downarrow \\ \prod_{k=0}^{m} AG(S^{2k}) & \xrightarrow{j_{m,n}} & \prod_{k=0}^{n} AG(S^{2k}) \end{array}$$

commutes up to homotopy.

In the $\mathbb{Z}_2$-equivariant case one can still reduce to zero cycles by using Lam's result. The main obstacles are then the non existence of generalizations of the Theorems of Moore and Dold-Thom in a form that immediately allows the identification of the equivariant homotopy type of $\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})$. There are however some tools available: The group homomorphisms used to construct the splitting (1.1) are equivariant if one considers $\mathbb{P}^n_{\mathbb{C}}$ with the $\mathbb{Z}_2$-action given by complex conjugation. Now, as a $\mathbb{Z}_2$-space, the quotient $\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}}$ is the sphere $S^{n,n}$; where $S^{n,n} = \mathbb{R}^{n,n} \cup \{\infty\}$ and $\mathbb{R}^{n,n}$is $\mathbb{R}^{2n}$ with the $\mathbb{Z}_2$-action given of multiplication by -1 in the last n coordinates. In [**16**] it is proved that the equivalence (1.1) is a $\mathbb{Z}_2$-homotopy equivalence

$$\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \cong \prod_{k=0}^{n} AG(S^{k,k}).$$

Also, Lima-Filho has an equivariant version of the Dold-Thom Theorem (see [**23**]) which says that, for a finite group G and a G-CW complex X,

$$\pi_k(AG(X)^G) \cong \widetilde{H}^G_k(X; \underline{\mathbb{Z}}). \tag{1.2}$$

Here $\widetilde{H}^G_*(X; \underline{\mathbb{Z}})$ is the Bredon G-equivariant (reduced) homology of X with coefficients in the Mackey functor $\underline{\mathbb{Z}}$. The definition of Bredon homology is similar to homology with local coefficients: A G-CW complex is a space built up of spaces of the form $D^n \times G/H$ for $H \leq G$ — called equivariant cells. A (covariant) coefficient system M is a functor that assigns to each orbit space G/H an abelian group and to each G-map $f : G/H \to G/K$ a group homomorphism $M(G/H) \to M(G/K)$. An n-chain with coefficients in M is a weighted sum of cells $\sum_i w_i D^{n_i} \times G/H_i$ where the weights w_i elements of $M(G/H_i)$. The value of M on morphisms is used to define a differential on this chain complex. The functor $\underline{\mathbb{Z}}$ is a (covariant) coefficient system which assigns the group $\mathbb{Z}$ to all orbits G/H . There is a corresponding version of this for cohomology which uses contravariant systems instead. The functor $\underline{\mathbb{Z}}$ is also a contravariant functor, *i.e.* given a G-map $f : G/H \to G/K$ it also assigns a morphism $\underline{\mathbb{Z}}(G/K) \to \underline{\mathbb{Z}}(G/H)$. Moreover, there are some compatibility conditions between the contravariant and the covariant functors defined by $\underline{\mathbb{Z}}$. Functors with these properties are called Mackey functors. A precise definition will be given in Section 2.

Lima-Filho's result (1.2) brings us very close to our objective. For example, together with Spanier-Whitehead duality, (1.2) can be used to compute the homotopy groups $\pi_k(AG(S^{n,n})^{\mathbb{Z}_2})$ which were in computed in [**16**] by a direct method. Following Lima-Filho ([**23**]) we consider the equivariant prespectrum

$AG(S^\infty S^0)$ defined by the correspondence

$$V \mapsto AG(S^V)$$

where V runs over the finite dimensional G-submodules of a complete G-universe $\mathcal{UN}$. We denote by $\mathbb{Z}(\mathbb{S})$ the G-spectrum associated to this G-prespectrum. Our first result is the identification of the the homotopy type of $\mathbb{Z}(\mathbb{S})$:

PROPOSITION 1.1. *The G-spectrum $\mathbb{Z}(\mathbb{S})$ is an Eilenberg-Mac Lane G-spectrum $K(0, \underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ is the Mackey functor constant at $\mathbb{Z}$.*

Thus the G-spectrum $\mathbb{Z}(\mathbb{S})$ represents equivariant cohomology with coefficients in the Mackey functor $\underline{\mathbb{Z}}$.

For any based G-space X and any G-module W there is a G-map

$$\epsilon^W : AG(X) \to \Omega^W AG(S^W \wedge X)$$

obtained as the adjoint of the natural map

$$\sigma^W : S^W \wedge AG(X) \to AG(S^W \wedge X).$$

It is well-known that ϵ^W is a non-equivariant homotopy equivalence. We prove that ϵ^W is also an equivariant homotopy equivalence.

THEOREM 1.1. *For any based G-CW complex X the natural inclusion*

$$AG(X) \to \mathrm{colim}_W \Omega^W AG(S^W \wedge X)$$

is a G-homotopy equivalence. Here W runs through the finite dimensional G-submodules of a complete G-universe $\mathcal{UN}$.

In particular, $AG(X)$ is an equivariant infinite loop space. When we specialize to $X = S^V$ we get

COROLLARY 1.1. *The G-prespectrum $AG(S^\infty S^0)$ is an Ω-G-prespectrum, i.e. the maps $\epsilon^W : AG(S^V) \to \Omega^W AG(S^{V+W})$ are equivariant homotopy equivalences.*

The problem of identifying the homotopy type of $\mathbb{Z}(\mathbb{S})$ was first addressed by Lima-Filho in [**23**]. Our result, which is proved using mainly the techniques of [**23**] is a correction to Lima-Filho's original answer. We also prove the following $RO(G)^+$-graded version of Lima-Filho's result (1.2) :

COROLLARY 1.2. *If X is a G-CW-complex and V is a G-module then*

$$\pi_V(\mathcal{Z}_0(X)) \stackrel{def}{=} [S^V, \mathcal{Z}_0(X)]_G \cong H^G_V(X; \underline{\mathbb{Z}}).$$

The results above also identify the spaces $AG(S^{n,n})$ as classifying spaces for equivariant cohomology groups with $\underline{\mathbb{Z}}$ coefficients in dimension $\mathbb{R}^{n,n}$ (recall that $\mathbb{Z}_2$-equivariant cohomology is $RO(\mathbb{Z}_2)$-graded). Using these facts we see that $\mathcal{Z}$ is a weak product of classifying spaces for $\mathbb{Z}_2$-equivariant cohomology with $\underline{\mathbb{Z}}$ coefficients in the non-integral dimensions $\mathbb{R}^{1,1}, \mathbb{R}^{2,2}, \ldots, \mathbb{R}^{n,n}, \ldots$. This was conjectured by Daniel Dugger and it was this conjecture that led us to the identification of $\mathbb{Z}(\mathbb{S})$. These dimensions are precisely the ones in which the equivariant Chern classes for Real bundles live. As in the non-equivariant case, $\mathcal{Z}$ comes with a natural (equivariant) map $BU \to \mathcal{Z}$. The following Theorem summarizes the results concerning the space $\mathcal{Z}$ that we prove in Section 3.

THEOREM 1.2. *The space $\mathcal{Z}$ has the $\mathbb{Z}_2$-homotopy type of a weak product of classifying spaces*

$$\mathcal{Z} \cong \prod_{n=0}^{\infty} K(\underline{\mathbb{Z}}, \mathbb{R}^{n,n}).$$

The natural map $BU \to \mathcal{Z}$ classifies the total equivariant Chern class for Real vector bundles.

The product

$$\mathcal{Z} \wedge \mathcal{Z} \to \mathcal{Z}$$

induced by the join of varieties, classifies the cup product in $\mathbb{Z}_2$-equivariant cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.

This Theorem is the natural $\mathbb{Z}_2$-equivariant version of the main results of [**18**]; see Section 3 for the Definition of equivariant Chern classes for Real vector bundles.

Having obtained a complete description on the $\mathbb{Z}_2$-homotopy type of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ and of the effect of the join map we propose a definition of **Real Lawson homology** for Real quasi-projective varieties. For a projective Real variety X these are invariants defined in terms of equivariant homotopy groups of the $\mathbb{Z}_2$-spaces $\mathcal{Z}_p(X)$. Using the techniques developed by Friedlander and Lima-Filho, we establish the existence of exact sequences, excision and a cycle map which takes values in the $\mathbb{Z}_2$-equivariant homology of X with $\underline{\mathbb{Z}}$ coefficients. In the definition of the cycle map we make use of the $RO(\mathbb{Z}_2)^+$-graded version of Dold-Thom mentioned above. Our analysis of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ gives us, in particular, the Real Lawson homology groups of affine space $\mathbb{A}^n = \mathbb{P}^n_{\mathbb{C}} - \mathbb{P}^{n-1}_{\mathbb{C}}$, with its standard Real structure, and computes the effect of the cycle map. Excision and exact sequences then allow us to compute these invariants for a class of examples: Real varieties with a Real cell decomposition. This class includes, for example, the Grassmannians $G_p(\mathbb{C}^n)$ (with the Real structure induced by sending a complex plane to its complex conjugate plane) and $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$ with the product of the standard Real structures on $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^m_{\mathbb{C}}$.

By considering different Real structures we can produce lots of different examples of Real varieties for which the non-equivariant Lawson homology groups are known. Examples of this are $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ with the action $\tau \cdot (x, y) = (\bar{y}, \bar{x})$, the Real quadrics and the the Quaternionic spaces, $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$. These have a Real structure induced by multiplication of the homogeneous coordinates by the imaginary quaternion j. The Real Lawson homology groups of some of these examples are computed here and we expect that they should all be possible to compute. In the case of the Quaternionic spaces the homotopy groups of the Real cycles have been computed in [**17**] by a highly non-trivial technique. We expect that the spaces of cycles in $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$ should be related to some fundamental construction in $\mathbb{Z}_2$-homotopy theory, as is the case with the cycles in $\mathbb{P}^n_{\mathbb{C}}$ with the standard Real structure.

We point out here that the knowledge of the Real Lawson homology groups of a Real variety X completely determines, in particular, the homotopy type of the spaces of Real cycles on X, *i.e.*, the cycles fixed by the involution, $\mathcal{Z}_p(X)^{\mathbb{Z}_2}$.

2. The G-equivariant Dold Thom Theorem

We start by recalling the definitions of some topological functors used by Lima-Filho in [**23**]. For the most part we follow the notation of this paper.

DEFINITION 2.1. Let X be a topological space. We define the following topological monoids and groups:

1. The infinite symmetric product of X is
$$SP(X) = \{\underline{0}\} \amalg \left\{ \coprod_{n \geq 1} X^n/\Sigma_n \right\}$$
where Σ_n is the symmetric group on n letters; $\underline{0}$ is an extra point not in any of the spaces X^n/Σ_n. The spaces X^n/Σ_n are given the quotient topology and $SP(X)$ is endowed with the disjoint union topology. Note that $SP(X)$ is a topological monoid: If $x \in X^n, x' \in X^m$ then (x, x') represents another element $x + x' \in SP(X)$. If $x = \underline{0}$ then we set $x + x' = x' + x = x'$, for any $x' \in SP(X)$.

2. The group of zero cycles on X, $\mathcal{Z}_0(X)$ is the naive group completion of $SP(X)$. It is defined as the quotient of

$$SP(X) \times SP(X)$$

by the equivalence relation $(x, y) \sim (x', y')$ if $x + y' = y + x'$. The elements of $\mathcal{Z}_0(X)$ are formal sums of points, $\sum_i n_i z_i$ with $n_i \in \mathbb{Z}$ and $z_i \in X$

3. $AG(X)$ is the kernel of the augmentation homomorphism deg from $\mathcal{Z}_0(X)$ to $\mathbb{Z}$. This homomorphism is defined by $deg(\sum_i n_i x_i) = \sum_i n_i$, for $n_i \in \mathbb{Z}$ and $x_i \in X$. The elements of $AG(X)$ are the cycles of degree zero.

NOTE 1. The notation used here for the zero cycles of degree zero is not the same as in [**3**] where $AG(X, 0)$ is used instead of $AG(X)$. Our notation is taken from [**23**].

If G is a finite group and X is a G-space, these functors take values in the categories of G-topological monoids and G-topological groups. In [**23**] Lima-Filho proves the following equivariant version of the Dold-Thom theorem; [**3**].

THEOREM 2.1. *([**23**]) Let (X, A) be a G-CW pair (see Definition 6.1). Then there is a natural equivalence*

$$\pi_i(AG(X/A)^G) \cong H_i^G(X, A; \underline{\mathbb{Z}}).$$

where $H_i^G(\bullet; \underline{\mathbb{Z}})$ denotes Bredon homology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$ (Definition 2.4).

Observe that $\pi_i(AG(X/A)^G)$ can also be described as the equivariant homotopy classes of maps from S^i to $AG(X/A)$ where S^i is equipped with the trivial G-action. The G-space S^i is the one point compactification of the trivial

G-module $\mathbb{R}^i$. We now want to replace $\mathbb{R}^i$ by a general G-module. Before we can state our goal some more definitions are needed. We follow the conventions and notation of [**24**] for G-spectra and for all matters regarding equivariant homotopy theory.

A complete G-universe is an orthogonal G-module containing countably many copies of each irreducible G-representation. We fix a complete G-universe $\mathcal{UN}$ and all the G-modules we consider are assumed to be finite dimensional submodules of $\mathcal{UN}$. An indexing set $\mathcal{A}$ in $\mathcal{UN}$ is a collection of submodules of $\mathcal{UN}$ — called indexing spaces — which contains the zero submodule and is cofinal. For example, in the case of $G = \mathbb{Z}_2$ we let $\mathcal{UN}$ be the real $\mathbb{Z}_2$-module $\mathbb{C}^\infty$ with the action given by complex conjugation. Then $\{\mathbb{C}^n\}$ with $\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \cdots$ is an indexing set.

NOTATION . If V is a G-module then S^V denotes the one point compactification of V, $S^V = V \cup \{\infty\}$ and, for a G-space X, $\Sigma^V X \overset{\text{def}}{=} S^V \wedge X$.

DEFINITION 2.2. A G-prespectrum X indexed by $\mathcal{A}$ is a collection of G-spaces $X(V)$, where $V \in \mathcal{A}$ with structural G-maps

$$\Sigma^{W-V} X(V) \xrightarrow{\sigma_V^W} X(W).$$

where V is a submodule of W and $W - V$ is the the orthogonal complement of V in W. The maps σ_V^W are required to satisfy certain compatibility conditions; see [**24**, ChapterXII]. A G-spectrum is a G-prespectrum X such that the adjoints

$$X(V) \longrightarrow \Omega^{W-V} X(W)$$

of the structural maps are G-homeomorphisms.

NOTATION . The following notation will be used throughout.

1. If X and Y are pointed G-spaces, the set of pointed maps $X \to Y$ has a natural G-action given by conjugation. We denote this G-space by $F(X, Y)$. The set of equivariant homotopy classes is denoted by $[X, Y]_G$.
2. If X and Y are G-spaces, the set of stable G-equivariant homotopy classes of maps $X \to Y$ is denoted by $\{X, Y\}_G$. This is defined as the limit

$$\mathrm{colim}_W [\Sigma^W X, \Sigma^W Y]_G.$$

Here the limit is taken over the finite dimensional G-submodules of the universe $\mathcal{UN}$.
3. The suspension G-prespectrum and G-spectrum of a G-space X are denoted by $S^\infty X$ and $\Sigma^\infty X$, respectively. We recall that $\Sigma^\infty X = L(S^\infty X)$, where L is the left adjoint functor of the forgetful functor, ℓ, from the category $G\mathcal{S}$ of G-spectra to $G\mathcal{P}$, the category of G-prespectra. The sphere spectrum, $\Sigma^\infty S^0$ is also denoted by $\mathbb{S}$.

Associated to a G-spectrum E there are $RO(G)$-graded equivariant cohomology and homology theories defined on G-spaces by

$$E^\alpha(X) \stackrel{\text{def}}{=} \{S^{-\alpha} \wedge X, E\}_G = \mathrm{colim}_W [S^{-\alpha+W} \wedge X, E(W)]_G$$

and

$$E_\alpha(X) \stackrel{\text{def}}{=} \{S^\alpha, E \wedge X\}_G = \mathrm{colim}_W [S^{\alpha+W}, E(W) \wedge X]_G.$$

When α is the trivial representation $\mathbb{R}^n$ we write E^n and E_n instead of E^α, E_α. There exists an equivariant spectrum $H\underline{\mathbb{Z}}$ so that the groups $H\underline{\mathbb{Z}}_n(X)$ are the Bredon homology groups $\widetilde{H}^G_n(X; \underline{\mathbb{Z}})$; see [**24**]. It is now natural to ask if Theorem 2.1 has a generalization for general G-modules V. That is, we ask if there is an isomorphism

$$\pi_V(AG(X)) \cong H\underline{\mathbb{Z}}_V(X)$$

where $\pi_V(AG(X)) = [S^V, AG(X)]_G$. This is what we mean by the $RO(G)^+$ version of the equivariant Dold-Thom Theorem.

Our main goal in this section is to answer this question.

NOTATION . Since $H\underline{\mathbb{Z}}_*(X)$ is the extension of (reduced) Bredon homology with $\underline{\mathbb{Z}}$ coefficients to an $RO(G)$-graded theory we will denote it by $\widetilde{H}^G_*(X;\underline{\mathbb{Z}})$. The same convention will be used for equivariant cohomology.

CONSTRUCTION . We need the following construction introduced by Lima-Filho:

If X is a G-prespectrum we can apply the functor AG to the spaces $X(V)$ and the structural maps

$$\Sigma^W X(V) \longrightarrow X(V+W)$$

naturally extend to maps

$$\Sigma^W AG(X(V)) \longrightarrow AG(X(V+W))$$

defining a G-prespectrum $AG(X)$

Following Lima-Filho we define

DEFINITION 2.3. Let X be a G-prespectrum. Then

$$\mathbb{Z}(X) \stackrel{\text{def}}{=} L(AG(X)).$$

We also need the definitions of Mackey functor and Burnside category. See [**24**] for more details.

DEFINITION 2.4. The **Burnside category** $\mathcal{B}_G$ has as objects the orbit spaces G/H and the morphisms are

$$\mathcal{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G.$$

A **Mackey functor** is a contravariant functor from $\mathcal{B}_G$ to the category of abelian groups, $\mathcal{B}_G \to \mathcal{A}b$. Observe that $\mathcal{B}_G$ is an abelian category since the sets $\{G/H_+,\, G/K_+\}_G$ are abelian groups and the composition is bilinear.

We will now describe the morphisms in $\mathcal{B}_G$. A G-map $G/K_+ \to G/H_+$ induces a stable map $\Sigma^\infty G/K_+ \to \Sigma^\infty G/H_+$ thus giving a morphism in $\mathcal{B}_G$. Associated to every subgroup H of G there is a stable transfer map $\tau(G/H) : G/G_+ \to G/H_+$ defined as follows. Let V be a G-module in which G/H embeds equivariantly. Choose disjoint disks around the points of G/H so that they are permuted by the action of G. Collapsing the complement of the disks we get an equivariant map

$$\tau(G/H) : S^V \to S^V \wedge G/H_+.$$

The class of $\tau(G/H)$ in $\{S^0, S^0 \wedge G/H_+\}_G$ is independent of V and of the embedding $G/H \subset V$; see [**29**].

More generally there is a stable transfer map $G/H_+ \to G/K_+$ for every group inclusion $K \subset H$ which we now describe. First embedd H/K into a G-module V considered as an H-module by restriction of the action. Apply the Pontrjagin-Thom construction above to get a transfer map $\tau(H/K) : S^V \to S^V \wedge H/K_+$. Now extend this to a G-map

$$S^V \wedge G/H_+ \cong S^V \wedge_H G_+ \to (S^V \wedge H/K_+) \wedge_H G_+ \cong S^V \wedge G/K_+$$

A general G-map $f : G/K \to G/H$ is a composition of a conjugation isomorphism $c_g : G/K \to G/g^{-1}Kg$ and a projection induced by an inclusion $g^{-1}Kg \subset H$. Define $\tau(c_g)$ as the stable map induced by $c_{g^{-1}}$. Define $\tau(f) \in$

$\{G/H_+, G/K_+\}$ as the composition of $\tau(c_g)$ and the transfer of the projection $G//g^{-1}Kg \to G/H$. Every morphism of $\mathcal{B}_G$ is a composition of maps of the form f and $\tau(f)$; see [**24**, Chapter XIV§3].

Associated to any G-prespectrum there are naturally defined Mackey functors $\underline{\pi_n}X$, $n \in \mathbb{Z}$.

DEFINITION 2.5. Let X be a G-prespectrum. Define the Bredon homotopy groups of X

$$\underline{\pi_n}X(G/H) = \pi_n^H X \stackrel{\text{def}}{=} \operatorname{colim}_W [S^{W+n} \wedge G/H_+, X(W)]_G.$$

A specially important case of this is

DEFINITION 2.6. The Burnside ring Mackey functor $\underline{A}$ is defined by

$$\underline{A}(G/H) \stackrel{\text{def}}{=} \underline{\pi_0}\mathbb{S}(G/H) = \{G/H_+, S^0\}_G = \{S^0, S^0\}_H.$$

It is a fundamental result of Segal that, for a finite group G, $\{S^0, S^0\}_G$ is isomorphic to $B(G)$, the Burnside ring of G. The ring $B(G)$ is a classical object defined as the Grothendieck ring of isomorphism classes of finite G-sets with the operations of disjoint union and Cartesian product; see [**24**, Chapter XVII]. This isomorphism is the motivation for calling $\mathcal{B}_G$ the Burnside category. We briefly sketch the definition of the isomorphism $\chi : B(G) \to \{S^0, S^0\}_G$. On the finite G-sets G/H, $\chi(G/H)$ is the stable class defined by the composition

$$S^V \xrightarrow{\tau(G/H)} S^V \wedge G/H_+ \xrightarrow{pr} S^V$$

where $pr : S^V \wedge G/H_+ \to S^V$ is the projection onto the first factor. For a general finite G-set Y, $\chi(Y)$ is defined the a Pontrjagin-Thom construction associated to an equivariant embedding of Y in a G-module V. Restricting to H-fixed point sets we get a map $\chi(Y)^H : S^{V^H} \to S^{V^H}$ whose degree is the Euler characteristic of Y^H. For this reason the function χ is also called the Euler characteristic.

We will need the following result concerning the Mackey functor $\underline{A}$.

PROPOSITION 2.1 ([**29**, Section II.8]). *The group*

$$A(G) = \{S^0, S^0\}_G$$

is a free abelian group on generators $\chi(G/H)$ where H runs through a set of representatives of the conjugacy classes of subgroups of G and χ is the Euler characteristic defined above.

DEFINITION 2.7. The constant Mackey functor $\underline{\mathbb{Z}}$ is the unique Mackey functor, $\mathcal{M}$, that satisfies:

1. $\mathcal{M}(G/H) = \mathbb{Z}$ for all $H \leq G$.
2. On morphisms $f : G/H \to G/K$ of $\mathcal{B}_G$ induced by G-maps $G/H \to G/K$ $\mathcal{M}(f)$ is the identity homomorphism.

The homomorphism $\underline{\mathbb{Z}}(\tau)$ induced by the stable transfer map $G/H_+ \to G/K_+$ associated to an inclusion $K \subset H$ is multiplication by the order $|H/K|$. The functor $\underline{\mathbb{Z}}$ can also be defined as a quotient of $\underline{A}$ by another Mackey functor $\underline{I}$ whose value at G/H is the kernel of the map

$$\{G/H_+, S^0\}_G \longrightarrow \{G_+, S^0\}_G \cong \mathbb{Z}$$

induced by the G-map $G \to G/H$. See [**24**, Chapter IX] for the proofs of these facts.

There is an alternative description of $\underline{\mathbb{Z}}$ which is purely algebraic. The existence and uniqueness of this functor as well as the equivalence of the two definitions are proved in [**24**].

In the case $G = \mathbb{Z}_2$, $\underline{\mathbb{Z}}$ is the Mackey functor with the value $\mathbb{Z}$ on both elements $\mathbb{Z}_2$ and $\mathbb{Z}_2/\mathbb{Z}_2$ of $\mathcal{B}_{\mathbb{Z}_2}$. The homomorphisms $\mathbb{Z} \to \mathbb{Z}$ induced by the projection $\mathbb{Z}_2 \to \mathbb{Z}_2/\mathbb{Z}_2$ and its transfer are the identity and multiplication by two, respectively.

Before we can state our first result we need another definition.

DEFINITION 2.8 ([**24**]). Let $\mathcal{M}$ be a Mackey functor. A G-spectrum E is an Eilenberg-Mac Lane G-spectrum $K(\mathcal{M},0)$ if $\underline{\pi_n}(E) = 0$ for $n \neq 0$ and $\underline{\pi_0}(E) = \mathcal{M}$. This property determines E up to homotopy equivalence.

Given a Mackey functor $\mathcal{M}$ there exists an Eilenberg-Mac Lane G-spectrum $K(\mathcal{M},0)$ and this spectrum represents the $RO(G)$-graded equivariant cohomology theory with coefficients in $\mathcal{M}$.

In particular, the equivariant spectrum $H\underline{\mathbb{Z}}$ that represents equivariant cohomology with $\underline{\mathbb{Z}}$ coefficients is a $K(0,\underline{\mathbb{Z}})$ spectrum.

PROPOSITION 2.2. *For a finite group G, the G-spectrum $\mathbb{Z}(\mathbb{S})$ is an Eilenberg-Mac Lane spectrum $K(\underline{\mathbb{Z}},0)$. Thus , for $\alpha \in RO(G)$, we have*

$$(2.3) \qquad \pi_\alpha(\mathbb{Z}(\mathbb{S})) = \operatorname{colim}_W[S^\alpha \wedge S^W, AG(S^W)]_G \cong H_\alpha^G(pt;\underline{\mathbb{Z}})$$

Where, as before, $H_*^G(\bullet;\underline{\mathbb{Z}})$ denotes equivariant homology with coefficients in the Mackey functor $\underline{\mathbb{Z}}$.

PROOF. It is proved in [**23**] that, for a large enough representation V of G and for $n > 0$,

$$\pi_{V+n}(AG(S^V)) = 0;$$

see also Section 6. This implies that $\underline{\pi_n}(\mathbb{Z}(\mathbb{S})) = 0$ for $n > 0$. For $n < 0$, we have

$$\underline{\pi_n}^H(\mathbb{Z}(\mathbb{S})) = \operatorname{colim}_W[S^{W+n} \wedge G/H_+, AG(S^W)]_G \cong \operatorname{colim}_W[S^{W+n}, AG(S^W)]_H.$$

Let W be large enough so that it has $-n$ copies of the trivial H-representation and set $d = \dim W$. Since $AG(S^W)$ is d-connected and the complex S^{W+n} has dimension less than d, it follows that

$$[S^{W+n}, AG(S^W)]_H = 0$$

hence $\underline{\pi_n}^H(\mathbb{Z}(\mathbb{S})) = 0$.

We have to show that $\underline{\pi_0}(\mathbb{Z}(\mathbb{S})) = \underline{\mathbb{Z}}$. We start by observing that there is a morphism of Mackey functors

$$\underline{A} = \underline{\pi_0}\mathbb{S} \xrightarrow{\iota_*} \underline{\pi_0}(\mathbb{Z}(\mathbb{S}))$$

induced by the map $\iota : \mathbb{S} \to \mathbb{Z}(\mathbb{S})$ defined at the space level by

$$\iota_W : S^W \longrightarrow AG(S^W)$$

$$x \mapsto x - \infty$$

where, as usual $S^W = W \cup \{\infty\}$.

The map ι_* is surjective; this is implied by the stronger fact that for a G-module V containing a copy of the trivial G-representation and any $H \leq G$ the map

$$[S^V, S^V]_H \xrightarrow{\iota_{V*}} [S^V, AG(S^V)]_H$$

is onto. This is proved in Lemma 6.3.

Thus $\underline{\pi_0}(\mathbb{Z}(\mathbb{S})) = \underline{A}/\ker \iota_*$ and it suffices to identify $\ker \iota_*$ as the Mackey functor $\underline{I}$ mentioned after Definition 2.7. This amounts to showing that two elements in $\{S^0, S^0\}_H$ have the same image under ι_* if their image under the forgetful map $\epsilon : \{S^0, S^0\}_H \to \{S^0, S^0\}$ is the same. So the proposition will follow if we show that two elements of $\{S^0, S^0\}_G$ which are non-equivariantly homotopic have the same image under ι_*.

Recall from Proposition 2.1 that the elements $\chi(G/H)$ form a basis for the stable stem $\{S^0, S^0\}_G$. We now show that $\iota_*(\chi(G/H)$ is a multiple of the class $\iota_*(\chi(G/G))$ — which is the class induced by the identity map $S^0 \to S^0$ — and that completes the proof.

The composition

$$S^V \xrightarrow{\tau(G/H)} S^V \wedge G/H_+ \xrightarrow{pr} S^V \xrightarrow{\iota_V} AG(S^V)$$

factors as

$$S^V \xrightarrow{\tau(G/H)} S^V \wedge G/H_+ \xrightarrow{\iota_{G/H}} AG(S^V \wedge G/H_+) \xrightarrow{pr_*} AG(S^V)$$

where $\iota_{G/H} : S^V \wedge G/H_+ \to AG(S^V \wedge G/H_+)$ is given by

$$\sum_i (x_i \wedge g_i H) \mapsto \sum_i (x_i \wedge g_i H - \infty \wedge g_i H)).$$

Note that $AG(S^V \wedge G/H_+) \cong F(G/H_+, AG(S^V))$ and

$$[S^V, F(G/H_+, AG(S^V))]_G \cong [S^V, AG(S^V)]_H. \tag{2.4}$$

Arguing by induction on the subgroups of G we can assume that there is a G-module W so that the image of $\chi(G/H) \circ \iota_{G/H}$ in $[S^{V+W},\ AG(S^{V+W})]_H$ is a multiple of $\iota_*(\chi(H/H))$. Translating this back into $[S^{V+W}, F(G/H_+, AG(S^{V+W}))]_G$ via (2.4) we see that $\tau(G/H) \circ \iota_{G/H}$ is G-homotopic to a multiple of

$$x \mapsto \sum_{gH \in G/H} (x \wedge gH - \infty \wedge gH)$$

which projects to $|G/H| \iota_*(\chi(G/G))$.

□

Having identified $\mathbb{Z}(\mathbb{S})$ as a $K(\underline{\mathbb{Z}}, 0)$ G-spectrum it follows that, for any based G-space X and for $\alpha \in RO(G)$,

$$\pi_\alpha(\mathbb{Z}(\mathbb{S}) \wedge X) \cong \widetilde{H}^G_\alpha(X; \underline{\mathbb{Z}}).$$

NOTATION . Given based G-spaces X, Y there is a natural map

$$AG(X) \wedge X \to AG(X \wedge Y)$$

defined by

$$\left(\sum_i x_i\right) \wedge y \mapsto \sum_i (x_i \wedge y).$$

We use Lima-Filho's notation and denote this map by $R_{X,Y}$ or, just R_X whenever Y is understood.

We are now in a position to prove the $RO(G)^+$-graded version of Theorem 2.1.

THEOREM 2.2. *Let X be a based G-CW complex. Then the inclusion*

$$AG(X) \xrightarrow{\Phi} \mathbb{Z}(\Sigma^\infty X)(0) = \mathrm{colim}_W \Omega^W AG(S^W \wedge X)$$

is a G-homotopy equivalence. Moreover, for every G-CW-complex X, the map

$$R_X : (\mathbb{Z}(\mathbb{S}) \wedge X)(0) \to \mathbb{Z}(\Sigma^\infty X)(0)$$

is a G-homotopy equivalence. In particular, for any $V \in RO(G)^+$,

$$\pi_V(AG(X)) \cong \widetilde{H}_V(X; \underline{\mathbb{Z}}). \tag{2.5}$$

PROOF. Consider the functors, $\mathcal{H}_*^G$ defined on the category of pointed G-CW-complexes, with values in abelian groups, by

$$\mathcal{H}_n^G(X) \stackrel{\text{def}}{=} \mathrm{colim}_W [S^{W+n}, AG(S^W \wedge X)]_G = \underline{\pi_n^G}(\mathbb{Z}(\Sigma^\infty X)).$$

From Lima-Filho's equivariant version of the Dold-Thom Theorem, it follows that the functors $\mathcal{H}_*^G$ define a reduced equivariant homology theory, *i.e.*, they the following axioms:

Functoriality: It is clear that a map $f : X \to Y$ induces a homomorphism $f_* : \mathcal{H}_*^G(X) \to \mathcal{H}_*^G(Y)$.

Exact Sequences: By Lima-Filho's result, Theorem 2.1, and the fact that colimits preserve exact sequences, it follows that a G-cofibration $A \to X \to Y$ gives rise to an exact sequence

$$\mathcal{H}_k^G(A) \to \mathcal{H}_k^G(X) \to \mathcal{H}_k^G(Y).$$

Homotopy Invariance: It is is clear that G-homotopic maps maps $f, g : X \to Y$ induce the same maps on $f_*, g_* : \mathcal{H}_*^G(X) \to \mathcal{H}_*^G(Y)$.

Suspension Axiom: We have $\mathcal{H}_k^G(X) \cong \mathcal{H}_{k+1}^G(S^1 \wedge X)$.

By Proposition 2.2 we can compute the coefficients of the theory $\mathcal{H}_*^G$. In fact,

$$\begin{aligned}\mathcal{H}_*^G(G/H_+) &= \mathrm{colim}_W[S^W, AG(S^W \wedge G/H_+)]_G \\ &\cong \mathrm{colim}_W[S^W, F(G/H_+, AG(S^W))]_G \\ &\cong \mathrm{colim}_W[S^W \wedge G/H_+, AG(S^W)]_G = \underline{\pi_0}(\mathbb{Z}(\mathbb{S}))(G/H)\end{aligned}$$

and so $\mathcal{H}_*^G$ also satisfies

Dimension Axiom: Let $H \leq G$ and $k \neq 0$ then, by Proposition 2.2, we have $\mathcal{H}_k^G(G/H_+) = 0$.

Moreover,

$$\mathcal{H}_0^G(G/H_+) = \underline{\pi_0}(\mathbb{Z}(\mathbb{S}))(G/H) = \underline{\mathbb{Z}}(G/H).$$

We conclude that $\mathcal{H}_*^G(X) \cong \widetilde{H}_*^G(X; \underline{\mathbb{Z}})$. The natural map $\Phi_*^G : \pi_*(AG(X)^G) \to \mathcal{H}_*^G(X)$ is a transformation of homology theories and, therefore, a self transformation of (reduced) Bredon homology. It's clear that the transformation Φ_*^G an isomorphism when $X = G/H_+$, and so it is an isomorphism for any X. Since G is an arbitrary finite group, we have proved that the natural map

$$AG(X)^H \xrightarrow{\Phi^H} \mathrm{colim}_W \left\{\Omega^W AG(S^W \wedge X)\right\}^H$$

is a homotopy equivalence for all $H \leq G$. This implies that Φ is a G-homotopy equivalence. That completes the proof of the first part of the Theorem.

For the second part we observe that, by Proposition 2.2,

$$\widetilde{H}_n^G(X;\underline{\mathbb{Z}}) = \mathrm{colim}_W[S^{n+W}, AG(S^W) \wedge X]_G = \pi_n^G(\mathbb{Z}(\mathbb{S}) \wedge X).$$

There is a natural map $R_X : \mathbb{Z}(\mathbb{S}) \wedge X \to \mathbb{Z}(\Sigma^\infty X)$ induced by the space level map $R_X : AG(S^W) \wedge X \to AG(S^W \wedge X)$ and this defines another self transformation, R, of Bredon homology. We claim that it is also an isomorphism. As before, this is proved by showing that R is an isomorphism on the coefficients G/H_+. By definition, $R_{G/G+}$ is the identity. We now use the fact that $H_0^G(\bullet;\underline{\mathbb{Z}})$ is the Mackey functor $\underline{\mathbb{Z}}$: For any $H \leq G$, there is a commutative diagram

$$\begin{array}{ccc}
\underline{\mathbb{Z}}(G/H) = H_0^G(G/H;\underline{\mathbb{Z}}) & \xrightarrow{R_{G/H_+}} & H_0^G(G/H;\underline{\mathbb{Z}}) \\
{\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
\underline{\mathbb{Z}}(G/G) = H_0^G(G/G;\underline{\mathbb{Z}}) & \xrightarrow{R_{G/G_+}=\mathrm{id}} & H_0^G(G/G;\underline{\mathbb{Z}})
\end{array}$$

where the vertical maps are the contravariant homomorphisms induced by the G-map $G/H \to G/G$. It follows that R_{G/H_+} is an isomorphism and hence that, for any G-CW-complex X, the map

$$\mathbb{Z}(\mathbb{S}) \wedge X(0) \xrightarrow{R_X} \mathbb{Z}(\Sigma^\infty X)(0)$$

is a G-homotopy equivalence.

Together with the first part of the Theorem, this implies, that, for every G-module V,

$$\pi_V(AG(X)) \cong \mathrm{colim}_W[S^{V+W}, AG(S^W \wedge X)]_G \cong$$
$$\mathrm{colim}_W[S^{V+W}, AG(S^W) \wedge X]_G = \widetilde{H}_V^G(X;\underline{\mathbb{Z}}).$$

$\square$

NOTE 2. The fact that

$$R_X : (\mathbb{Z}(\mathbb{S}) \wedge X)(0) \to \mathbb{Z}(\Sigma^\infty X)(0)$$

is a G-homotopy equivalence is proved in [**23**] with more generality. In fact, Lima-Filho shows that for any G-spectrum X the map $R_X : \mathbb{Z}(\mathbb{S}) \wedge X \to \mathbb{Z}(X)$ is a G-homotopy equivalence. We chose not to use this result in the proof above so as to make the exposition more self-contained.

COROLLARY 2.1. *The G-prespectrum $AG(S^\infty S^0)$ is an Ω-G-prespectrum, i.e., the maps*

$$AG(S^V) \longrightarrow \Omega^W AG(S^{W+V})$$

are G-homotopy equivalences.

PROOF. The follows from Theorem 2.2 with $X = S^V$. □

3. The $\mathbb{Z}_2$-homotopy type of $\mathcal{Z}$ and Real Vector Bundles

Using the $\mathbb{Z}_2$-equivariant version of the Dold-Thom Theorem of the previous Section and the results of [**14**] and [**16**] it is now easy to describe the $\mathbb{Z}_2$-homotopy type of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ — the space of p-dimensional algebraic cycles on $\mathbb{P}^n_{\mathbb{C}}$. In the non-equivariant case (see [**18**]) the spaces $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ are classifying spaces for even dimensional cohomology with integer coefficients. When considered as a $\mathbb{Z}_2$-space $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ is still a classifying space for certain equivariant cohomology groups. However, $\mathbb{Z}_2$-equivariant cohomology is $RO(\mathbb{Z}_2)$-graded rather than $\mathbb{Z}$-graded. The cohomology groups that $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ classifies are the ones in which Chern classes of Real vector bundles live and these have non-integral dimensions.

We begin by collecting some definitions and notation introduced in [**15**]. From now on $\mathbb{P}^n_{\mathbb{C}}$ is considered as a $\mathbb{Z}_2$-space with the action of $\mathbb{Z}_2$ given by complex conjugation of its homogeneous coordinates. This action sends algebraic varieties to algebraic varieties and thus induces an action on $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$.

DEFINITION 3.1. Let X be a subvariety of $\mathbb{P}^n_{\mathbb{C}}$. The subset of $\mathbb{P}^{n+1}_{\mathbb{C}}$ consisting of all points of the lines joining points of X to the point $(0 : \cdots : 0 : 1)$ is a subvariety of $\mathbb{P}^{n+1}_{\mathbb{C}}$ denoted by $\not\Sigma X$. It is defined by the same equations as X but now considered as equations in $n+2$ variables. The operation $\not\Sigma$ increases dimension by one and keeps the codimension fixed.

More generally, given varieties $X \subset \mathbb{P}^n_{\mathbb{C}}$ and $Y \subset \mathbb{P}^m_{\mathbb{C}}$ the set of points of the lines joining points of X to points of Y — which we denote by $X \# Y$ — is a subvariety of $\mathbb{P}^{n+m+1}_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{C}} \# \mathbb{P}^m_{\mathbb{C}}$. Observe that $\dim X \# Y = \dim X + \dim Y + 1$ and $\not\Sigma X = X \# \mathbb{P}^0_{\mathbb{C}}$.

The importance of the operation $\not\Sigma$ for the computation of the $\mathbb{Z}_2$-homotopy type of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ is given by the following result of Lam. This result is an equivariant version of Lawson's **suspension Theorem** ; [**15**].

THEOREM 3.1. [**14**] *The suspension map*

$$\not\Sigma : \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \mathcal{Z}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$$

is a $\mathbb{Z}_2$-homotopy equivalence.

Theorem 3.1 reduces the computation of the homotopy type of $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ to the case of dimension zero cycles, but it actually does more than that. It allows us to define an *equivariant product* on $\mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}})$. First we need a definition.

DEFINITION 3.2. The space $\mathcal{Z}$ of **stabilized cycles** is

$$\mathcal{Z} = \lim_{n,p \to \infty} \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$$

where the limit is defined *w.r.t.* the suspension map and the natural inclusions $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \subset \mathcal{Z}_p(\mathbb{P}^{n+1}_{\mathbb{C}})$.

Observe that, by Theorem 3.1, $\mathcal{Z}$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}})$. We are ready to defined our product.

DEFINITION 3.3. Let $\sigma : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{2n+2}$ be the "shuffle" isomorphism defined by $\sigma(z, w) = (z_0, w_0, \dots, z_n, w_n)$. Consider the compositions

$$\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \wedge \mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{\#} \mathcal{Z}_1(\mathbb{P}^{2n+1}_{\mathbb{C}}) \xrightarrow{\sigma_*} \mathcal{Z}_1(\mathbb{P}^{2n+1}_{\mathbb{C}}) \xrightarrow{\Sigma^{-1}} \mathcal{Z}_0(\mathbb{P}^{2n}_{\mathbb{C}}).$$

The maps $\Sigma^{-1} : \mathcal{Z}_1(\mathbb{P}^{2n+1}_{\mathbb{C}}) \to \mathcal{Z}_0(\mathbb{P}^{2n}_{\mathbb{C}})$ can be chosen so that the compositions above are compatible with the inclusions $\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \to \mathcal{Z}_0(\mathbb{P}^{n+1}_{\mathbb{C}})$. Thus they define a product $\mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}}) \wedge \mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}}) \to \mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}})$. Using the $\mathbb{Z}_2$-homotopy equivalence $\mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}}) \cong \mathcal{Z}$ this defines $\mu : \mathcal{Z} \wedge \mathcal{Z} \to \mathcal{Z}$. It is clear that μ is equivariant.

The product μ plays a central role in Lawson homology. We will come back to it after computing the $\mathbb{Z}_2$-homotopy type of $\mathcal{Z}$.

In the non-equivariant case the Dold-Thom Theorem computes the homotopy type of $\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}})$ since this space is a topological group and a Theorem of Moore says that every such space is product of Eilenberg-Mac Lane spaces. In the equivariant world there is no version Moore's Theorem so the homotopy groups of a topological G-group do not determine its equivariant homotopy type. However in the case at hand we have the following result.

NOTATION . Let $\mathcal{V}$ denote the $\mathbb{Z}_2$-module $\mathbb{R}^{1,1}$. Equivalently, $\mathcal{V}$ is the real vector space $\mathbb{C}$ with $\mathbb{Z}_2$ acting by complex conjugation. The one dimensional sign representation of $\mathbb{Z}_2$ will be denoted by $\mathcal{U}$.

Given a $\mathbb{Z}_2$-module V, we define $|V^*| : \{\mathbb{Z}_2, \{0\}\} \to \mathbb{Z}$ by

$$|V^*|(H) \stackrel{\text{def}}{=} |V^H| = \dim V^H \qquad\qquad H = \{0\} \text{ or } H = \mathbb{Z}_2$$

THEOREM 3.2. [**16**] *There is a map*

$$\mathcal{Z}_0(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \prod_{k=0}^{n} AG(S^{k\mathcal{V}})$$

which is a $\mathbb{Z}_2$-homotopy equivalence.

Our equivariant version of the Dold-Thom Theorem now computes the $\mathbb{Z}_2$-homotopy type of $\mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}})$ but before we can state the result another definition is needed.

DEFINITION 3.4. Let V be a $\mathbb{Z}_2$-module with at least one copy of the trivial representation. An **equivariant Eilenberg-Mac Lane space** $K(\underline{\mathbb{Z}}, V)$ is a $\mathbb{Z}_2$-space X satisfying

1. X is $|V^*| - 1$ connected, *i.e.*,

$$\pi_k(X^H) = 0, \qquad \text{for } k < \dim V^H,\ H = \mathbb{Z}_2 \text{ or } H = \{0\}.$$

2. $\underline{\pi}_{k+V}(X) = \begin{cases} 0 & \text{if } k > 0, \\ \underline{\mathbb{Z}} & \text{if } k = 0. \end{cases}$

A $K(\underline{\mathbb{Z}}, V)$ space is characterized up to $\mathbb{Z}_2$-homotopy equivalence by these properties; see [**24**, Chapter XI,§5].

NOTATION . From now on we will use the notation $H^*(\bullet; \underline{\mathbb{Z}})$ for $\mathbb{Z}_2$-equivariant cohomology with $\underline{\mathbb{Z}}$ coefficients. In order to avoid confusion with the singular cohomology groups with $\mathbb{Z}$ coefficients will be denoted by $H^*_{sing}(\bullet; \mathbb{Z})$. Similar conventions hold for homology groups.

THEOREM 3.3. *The space of stabilized cycles, $\mathcal{Z}$, is $\mathbb{Z}_2$-homotopy equivalent to the following* **weak product** *of* **equivariant Eilenberg-Mac Lane spaces**

$$(3.6) \qquad \mathcal{Z} \xrightarrow{\cong} \prod_{k=0}^{\infty} K(\underline{\mathbb{Z}}, k\mathcal{V}).$$

Thus, for a pointed $\mathbb{Z}_2$-space X,

$$[X, \mathcal{Z}_0(\mathbb{P}^\infty_{\mathbb{C}})]_{\mathbb{Z}_2} \cong \bigoplus_{k=0}^{\infty} \widetilde{H}^{k\mathcal{V}}(X; \underline{\mathbb{Z}}).$$

PROOF. This is an immediate consequence of Theorems 3.2 and 3.1, Proposition 2.2 and Corollary 2.1.

□

We now analyze the equivariant product μ. In **[18]** it is proved that, non-equivariantly, μ classifies the cup product in singular cohomology with integer coefficients. There is a notion of cup product in Bredon cohomology with $\underline{\mathbb{Z}}$ coefficients. The existence of this product makes the equivariant spectrum $H\underline{\mathbb{Z}}$ into a ring spectrum (see **[24]**) and so the equivariant cohomology theory $H\underline{\mathbb{Z}}^*$ has a product which we also call cup product. In the next Proposition we show that μ classifies the cup product in equivariant cohomology.

NOTE 3. In the proof of the next proposition we will need to following fact: Let G be a finite group and let $\{X_\alpha\}$ be a family of G-spaces. Then $AG(\bigvee_\alpha X_\alpha)$ is G-homeomorphic to the weak product $\prod_\alpha AG(X_\alpha)$ If i_α denotes the inclusion $X_\alpha \subset \bigvee_\alpha X_\alpha$ the homeomorphism is given by $\oplus_\alpha i_{\alpha *}$; see **[23]** for a proof.

PROPOSITION 3.1. *Under the equivalence of Theorem 3.2 the map*

$$\mu : \mathcal{Z} \wedge \mathcal{Z} \to \mathcal{Z}$$

is $\mathbb{Z}_2$-homotopic to the biadditive extension

$$\left(\prod_{k=0}^{\infty} AG(S^{k\mathcal{V}})\right) \wedge \left(\prod_{k=0}^{\infty} AG(S^{k\mathcal{V}})\right) \xrightarrow{\wedge} \prod_{k=0}^{\infty} AG(S^{k\mathcal{V}}).$$

of the smash product of spheres, $S^{k\mathcal{V}} \wedge S^{k'\mathcal{V}} \to S^{(k+k')\mathcal{V}}$. In particular, it classifies the cup product in $\mathbb{Z}_2$-equivariant cohomology with $\underline{\mathbb{Z}}$ coefficients.

PROOF. Consider $S^{k\mathcal{V}}$ included in $\mathcal{Z}$ by

$$S^{k\mathcal{V}} \ni x \mapsto x - \infty \in AG(S^{k\mathcal{V}}) \subset \mathcal{Z}.$$

We start by showing that, for every k, k', the restrictions $\mu, \wedge|S^{k\mathcal{V}} \wedge S^{k'\mathcal{V}}$ are $\mathbb{Z}_2$-homotopic. In **[18]** it is proved that these restrictions are non-equivariantly homotopic. We will see that the forgetful map

$$\Phi : [S^{n\mathcal{V}}, \mathcal{Z}]_{\mathbb{Z}_2} \longrightarrow [S^{n\mathcal{V}}, \mathcal{Z}] \tag{3.7}$$

is an isomorphism. Thus the fact that $\mu, \wedge|S^{k\mathcal{V}} \wedge S^{k'\mathcal{V}}$ are homotopic implies that they are also $\mathbb{Z}_2$-homotopic. Note that

$$[S^{n\mathcal{V}}, \mathcal{Z}]_{\mathbb{Z}_2} \cong \bigoplus_{k=0}^{\infty} \widetilde{H}^{k\mathcal{V}}(S^{n\mathcal{V}}; \underline{\mathbb{Z}}) \cong \bigoplus_{k=0}^{\infty} \widetilde{H}^{(k-n)\mathcal{V}}(pt; \underline{\mathbb{Z}}) \cong H^0(pt; \underline{\mathbb{Z}})$$

; see (3.9). The map Φ is the map induced in equivariant cohomology by the projection $p : S^{n\mathcal{V}} \wedge \mathbb{Z}_{2+} \to S^{n\mathcal{V}}$. All we are saying here is that for any $\mathbb{Z}_2$-spaces X, Y, the composition

$$[X, Y]_{\mathbb{Z}_2} \xrightarrow{p^*} [X \wedge \mathbb{Z}_{2+}, Y]_{\mathbb{Z}_2} \cong [X, Y].$$

is the map that forgets the $\mathbb{Z}_2$-action. The isomorphism (3.7) now follows from the fact that the $H^0(pt; \underline{\mathbb{Z}}) \to H^0(\mathbb{Z}_2; \underline{\mathbb{Z}})$ induced by $\mathbb{Z}_2 \to \mathbb{Z}_2/\mathbb{Z}_2$ is an isomorphism — Bredon cohomology with coefficients in $\underline{\mathbb{Z}}$ is just singular cohomology of the orbit space.

We now use the $\mathbb{Z}_2$-homeomorphism

$$\prod_{k=0}^{\infty} AG(S^{k\mathcal{V}}) \cong AG(\bigvee_{k=0}^{\infty} S^{k\mathcal{V}})$$

mentioned before. From what was said above we see that restrictions of μ and $\wedge$ to $\bigvee_{k=0}^{\infty} S^{k\mathcal{V}} \wedge \bigvee_{k=0}^{\infty} S^{k'\mathcal{V}}$ are $\mathbb{Z}_2$-homotopic. Let

$$H : \bigvee_{k=0}^{\infty} S^{k\mathcal{V}} \wedge \bigvee_{k=0}^{\infty} S^{k'\mathcal{V}} \wedge I_+ \to \mathcal{Z}$$

be an equivariant homotopy from the restriction of $\wedge$ to the restriction of μ. Extend H to an equivariant homotopy through biadditive maps

$$F : AG(\bigvee_{k=0}^{\infty} S^{k\mathcal{V}}) \times AG(\bigvee_{k=0}^{\infty} S^{k'\mathcal{V}}) \times I \to \mathcal{Z}$$

Since $\wedge$ is biadditive we have $F(\bullet, \bullet; 0) = \wedge$. Now, μ is biadditive up to $\mathbb{Z}_2$-homotopy, so $F(\bullet, \bullet; 1)$ is $\mathbb{Z}_2$-homotopic to μ. Since $F(\bullet, \bullet; t)$ is biadditive, F descends to

$$AG(\bigvee_{k=0}^{\infty} S^{k\mathcal{V}}) \wedge AG(\bigvee_{k=0}^{\infty} S^{k'\mathcal{V}}) \wedge I_+.$$

This completes the proof that $\wedge$ and μ are $\mathbb{Z}_2$-homotopic.

The statement regarding the cup product is a consequence of fact that the product $\wedge$ is the product induced on the $\mathbb{Z}_2$-Ω-prespectrum $V \mapsto AG(S^V)$ from the cup product in Bredon cohomology and this is proved in by Daniel Dugger in [**4**].

□

REMARK 3.1. The equivariant product μ on $\mathcal{Z}$ restricts to a product on the fixed point set $\mathcal{Z}_{\mathbb{R}} \stackrel{\text{def}}{=} \mathcal{Z}^{\mathbb{Z}_2}$. So the $\mathbb{Z}$-graded group of homotopy groups of $\mathcal{Z}_{\mathbb{R}}$ has a ring structure given by this product. The computation of this ring is one of the main results of [**16**]. In view of Proposition 3.1 this ring can be interpreted as a subring of the equivariant cohomology of a point: For $k > 0$, Theorem 3.3 gives,

(3.8)
$$\pi_k(\mathcal{Z}_{\mathbb{R}}) \cong \bigoplus_{n\geq 0}[S^k, AG(S^{n\mathcal{V}})]_{\mathbb{Z}_2} \cong \bigoplus_{n\geq 0} \widetilde{H}^{n\mathcal{V}}(S^k; \underline{\mathbb{Z}}) \cong \bigoplus_{n\geq 0} H^{(n-k)+n\mathcal{U}}(pt; \underline{\mathbb{Z}}).$$

where, S^k is equipped with the trivial $\mathbb{Z}_2$-action, as usual. The cohomology groups of a point are (see [**4**])

$$(3.9)\qquad H^{n+m\mathcal{U}}(pt;\underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}_2 & n \text{ even, } 0 \leq -n < m \\ \mathbb{Z} & n \text{ even, } -n = m \\ \mathbb{Z}_2 & n \text{ odd, } 1 < n \leq -m \\ 0 & \text{otherwise} \end{cases}$$

So we see from (3.8) that $\pi_*(\mathcal{Z}_{\mathbb{R}})$ is isomorphic as a group to the subring consisting of the $H^{p+q\mathcal{U}}(pt;\underline{\mathbb{Z}})$ such that $q \geq 0$. The fact that the product structure is the same follows from Proposition 3.1. From (3.9) we can also conclude that the homotopy type of $K(n\mathcal{V}, \underline{\mathbb{Z}})^{\mathbb{Z}_2} \cong AG(S^{n\mathcal{V}})^{\mathbb{Z}_2}$ is

(3.10)

$$\begin{cases} K(\mathbb{Z}, 2n) \times K(\mathbb{Z}_2, 2n-2) \times K(\mathbb{Z}_2, 2n-4) \cdots \times K(\mathbb{Z}_2, n) & n \text{ even,} \\ K(\mathbb{Z}_2, 2n-1) \times K(\mathbb{Z}_2, 2n-3) \times \cdots \times K(\mathbb{Z}_2, n) & n \text{ odd.} \end{cases}$$

NOTATION . We will denote the generators of

$$H^{n+k}(K(\mathbb{Z}_2, n+k); \mathbb{Z}_2) \text{ and of } H^{2n}(K(\mathbb{Z}, 2n); \mathbb{Z}),$$

in the decomposition (3.10) by $\iota_{k,n}$ and $\iota_{n,n}$, respectively. The classes $\iota_{k,n}$ are studied in [**16**].

REMARK 3.2. The classes $\iota_{k,n}$ have the following interpretation in terms of equivariant cohomology. Let X be a $\mathbb{Z}_2$-space with the trivial action. The equivariant cohomology groups $H^*(X;\underline{\mathbb{Z}})$ correspond to equivariant homotopy classes of maps

$$H^{n+n\mathcal{U}}(X;\underline{\mathbb{Z}}) \cong [X_+, AG(S^{n\mathcal{V}})]_{\mathbb{Z}_2} \cong [X_+, AG(S^{n\mathcal{V}})^{\mathbb{Z}_2}]$$

where the last equivalence follows from the fact that X has the trivial action. From 3.10 there is a further equivalence

$$H^{n+n\mathcal{U}}(X;\underline{\mathbb{Z}}) \cong \begin{cases} H^{2n}_{sing}(X;\mathbb{Z}) \oplus \bigoplus_{k=1}^{n/2} H^{2n-2k}_{sing}(X;\mathbb{Z}_2) & n \text{ even}, \\ \bigoplus_{k=0}^{(n-1)/2} H^{2n-2k-1}_{sing}(X;\mathbb{Z}_2) & n \text{ odd}, \end{cases}$$

Another way of writing this is

$$H^{n+n\mathcal{U}}(X;\underline{\mathbb{Z}}) \cong \bigoplus_{p+q=n} H^{p}_{sing}(X; H^{q+n\mathcal{U}}(pt;\underline{\mathbb{Z}})),$$

and, in general

$$H^{m+n\mathcal{U}}(X;\underline{\mathbb{Z}}) \cong \bigoplus_{p+q=m} H^{p}_{sing}(X; H^{q+n\mathcal{U}}(pt;\underline{\mathbb{Z}})).$$

The classes $\iota_{k,n}$ determine the restriction of the functor $X \mapsto H^{n\mathcal{V}}(X;\underline{\mathbb{Z}})$ to spaces X with trivial action. This functor will be analyzed more closely in Chapter 2.

One of the interesting features of the space $\mathcal{Z}$ is that the classifying space BU maps naturally into it, as follows. We have

$$BU = \lim_{n\to\infty} G_n(\mathbb{C}^{2n}).$$

Linear spaces in $\mathbb{C}^{2n}$ are degree one cycles on $\mathbb{P}^{2n-1}_{\mathbb{C}}$, thus BU maps to the space $\mathcal{Z}$ of stabilized cycles and this map is equivariant when BU is considered as a $\mathbb{Z}_2$-space under the action of complex conjugation of planes. With this action, BU is the classifying space for the reduced KR-theory of Atiyah ([**1**]). Its $\mathbb{Z}_2$-equivariant cohomology can be easily computed using the equivariant cell decomposition coming from the Schubert cells. Denoting by R the cohomology ring of a point $H^*(pt;\underline{\mathbb{Z}})$, we get

$$H^*(BU;\underline{\mathbb{Z}}) \cong R[\widetilde{c}_1, \dots, \widetilde{c}_n, \dots],$$

where the $\widetilde{c}_n$'s are classes of dimension $n\mathcal{V}$ whose images under the forgetful functor to non-equivariant cohomology are the Chern classes, c_n; see Lemma 5.1 for a similar computation. The classes $\widetilde{c}_n$ are universal characteristic classes for Real vector bundles. We call them equivariant Chern classes for Real vector bundles. We show that, as in the non-equivariant case, the map $P : BU \to \mathcal{Z}$ classifies the total equivariant Chern class:

PROPOSITION 3.2. *Let $\iota_{n\mathcal{V}}$ denote the universal $n\mathcal{V}$-dimensional class in $H^*(K(n\mathcal{V}, \underline{\mathbb{Z}}); \underline{\mathbb{Z}})$. Using the isomorphism (3.6), we consider $\iota_{n\mathcal{V}}$ has an element in the cohomology of $\mathcal{Z}$. Then*

$$P^*(\iota_{n\mathcal{V}}) = \widetilde{c}_n.$$

PROOF. The proof goes exactly as in the non-equivariant case. One observes that $BU(n) = \lim_k G_k(\mathbb{C}^{n+k})$ maps to

$$\lim_{k\to\infty} \mathcal{Z}_k(\mathbb{P}_{\mathbb{C}}^{n+k}) \cong \prod_{k=0}^{n} AG(S^{k\mathcal{V}}),$$

where the limit is defined using the map $\not{\Sigma} : \mathcal{Z}_k(\mathbb{P}_{\mathbb{C}}^{n+k}) \to \mathcal{Z}_{k+1}(\mathbb{P}_{\mathbb{C}}^{n+k+1})$. Thus $P^*(\iota_{n\mathcal{V}})|BU(n-1) = 0$ and so $P^*(\iota_{n\mathcal{V}}) = \lambda\widetilde{c}_n$, for some $\lambda \in \mathbb{C}$. Let ϕ denote the forgetful functor from equivariant cohomology to singular cohomology. Since $\phi\iota_{n\mathcal{V}} = \iota_{2n}$ — the generator of $H^{2n}(AG(S^{2n}; \mathbb{Z}))$ — and by [18] $P^*(\iota_{2n}) = c_n$ we conclude that $\lambda = 1$ and the result follows. $\square$

4. A version of Lawson Homology for Real Varieties

In this section we propose a definition of Lawson homology for Real algebraic varieties. This definition is a natural equivariant generalization of Lawson homology for projective varieties and we check that all the basic properties which make Lawson homology computable go over to the Real case.

We start by recalling the definition of Real variety.

DEFINITION 4.1. A Real quasi-projective algebraic variety U is a quasi-projective variety with an anti-holomorphic involution $\tau : U \to U$. The projective space $\mathbb{P}^n_{\mathbb{C}}$ with $\tau(x_0 : \cdots : x_n) = (\overline{x}_0 : \cdots : \overline{x}_n)$ is an example of a Real variety. Any Real quasi-projective variety has a Real embedding into a projective space, *i.e.*, there is an embedding $\phi : U \to \mathbb{P}^n_{\mathbb{C}}$ which is equivariant *w.r.t* the action induced by complex conjugation on $\mathbb{P}^n_{\mathbb{C}}$ (see [**26**], for example).

Recall also that, for a projective variety X, the Lawson homology groups of X, $L_pH_k(X)$ are

$$L_pH_k(X) \stackrel{\text{def}}{=} \pi_{k-2p}\mathcal{Z}_p(X) \qquad \text{for } k \geq 2p \text{ and } p \leq \dim X.$$

If X is a Real variety with involution $\tau : X \to X$ then the group of p-cycles $\mathcal{Z}_p(X)$ is naturally a $\mathbb{Z}_2$-space: Observe that, for a subvariety V, $\tau_* V$ is a subvariety and extend τ_* to all cycles by linearity. Suppose $f : (X', \tau') \to (X, \tau)$ is a Real isomorphism. It follows that f induces a $\mathbb{Z}_2$-equivariant homeomorphism $f_* : \mathcal{Z}_p(X') \to \mathcal{Z}_p(X)$ defined by

$$f_* \sum_i n_i V_i = \sum_i n_i f_*(V_i);$$

see [**6**]. Thus the equivariant homeomorphism type of $\mathcal{Z}_p(X)$, equipped with the action τ_* is an invariant of the Real structure on X. From here onwards the groups $\mathcal{Z}_p(X)$ are always considered as $\mathbb{Z}_2$-spaces with this action.

In trying to define invariants of the Real structure it is natural to look at spheres with $\mathbb{Z}_2$-actions and consider equivariant homotopy classes. We are therefore naturally led to the following definition.

DEFINITION 4.2. Let X be a Real projective variety. The **Real Lawson homology** groups of X, are the groups

$$L_p H\!\!\!R_\alpha(X) \stackrel{\text{def}}{=} \pi_{\alpha - p\mathcal{V}}\mathcal{Z}_p(X) \stackrel{\text{def}}{=} [S^{\alpha - p\mathcal{V}}, \mathcal{Z}_p(X)]_{\mathbb{Z}_2}$$

where p is an integer such that $0 \leq p \leq \dim X$, α is a real orthogonal representation of $\mathbb{Z}_2$ containing $p\mathcal{V}$ and $\alpha - p\mathcal{V}$ denotes the orthogonal complement of $p\mathcal{V}$ in α. Recall that $\mathcal{V}$ denotes the $\mathbb{Z}_2$-module $\mathbb{C}$ with the $\mathbb{Z}_2$-action defined by complex conjugation.

The previous observations imply that these are invariants of the Real structure on X.

REMARK 4.1. Our version of the equivariant Dold-Thom theorem shows that for cycles of dimension zero the the Real Lawson homology groups of a Real projective variety X are $\mathbb{Z}_2$-equivariant homology groups of X with coefficients in the Mackey functor $\underline{\mathbb{Z}}$, *i.e.*

$$L_0 H\!R_\alpha(X) \cong H_\alpha(X; \underline{\mathbb{Z}}).$$

If we expect these invariants to be computable we need relative groups and exact sequences for pairs. Following Lima-Filho's definition in the non-equivariant case, we define

DEFINITION 4.3. Let (X, X') be a Real pair, *i.e.* X' is a Real subvariety of X. The group of relative p-cycles is the quotient

$$\mathcal{Z}_p(X, X') \stackrel{\text{def}}{=} \frac{\mathcal{Z}_p(X)}{\mathcal{Z}_p(X')},$$

with the quotient topology. Note that $\mathcal{Z}_p(X, X')$ is a $\mathbb{Z}_2$-space with the action induced from $\mathcal{Z}_p(X)$.

The Real Lawson homology groups of the pair (X, X') are

$$L_p H\!R_\alpha(X, X') \stackrel{\text{def}}{=} \pi_{\alpha - p\mathcal{V}} \mathcal{Z}_p(X, X')$$

where, as above, α is a real $\mathbb{Z}_2$-submodule of $\mathbb{C}^\infty$ containing $p\mathcal{V}$ and $0 \leq p \leq \dim X$.

The next result is the main step in showing the existence of long exact sequences in Real Lawson homology. The proof is a simple generalization to the equivariant context of [**20**, Theo.3.1].

PROPOSITION 4.1. *The short exact sequence of topological groups*

$$0 \longrightarrow \mathcal{Z}_p(X') \longrightarrow \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_p(X, X') \longrightarrow 0 \tag{4.11}$$

is an equivariant fibration sequence.

PROOF. From [**20**, Theo.3.1] we know that (4.11) is a non-equivariant fibration. It remains to show that

$$0 \longrightarrow \mathcal{Z}_p(X')^{\mathbb{Z}_2} \longrightarrow \mathcal{Z}_p(X)^{\mathbb{Z}_2} \longrightarrow \mathcal{Z}_p(X, X')^{\mathbb{Z}_2} \longrightarrow 0$$

is a fibration. In fact, a stronger result is true: The sequence above is a principal fibration sequence. The proof is essentially the same as in the non-equivariant case: We observe that $\mathcal{Z}_p(X)^{\mathbb{Z}_2}$ is the naive group completion of free monoid $\mathcal{C}_p(X)^{\mathbb{Z}_2} = \{\amalg_d \mathcal{C}_{p,d}(X)\}^{\mathbb{Z}_2}$. The Real structure on X induces a Real structure on the Chow varieties $\mathcal{C}_{p,\leq d}(X)$. The fixed point set $\mathcal{C}_{p,\leq d}(X)^{\mathbb{Z}_2}$ is the set of Real points of $\mathcal{C}_{p,\leq d}(X)$. In particular, $(\mathcal{C}_{p,\leq d}(X)^{\mathbb{Z}_2}, \mathcal{C}_{p,\leq d}(X')^{\mathbb{Z}_2})$ is a pair of algebraic sets. By a classical result of [**11**] this pair can be triangulated. The result now follows from [**21**]; see Section 6 for the details. □

PROPOSITION 4.2. *Let (X, X', X'') be a Real triple. Then the short exact sequence of topological groups*

$$0 \longrightarrow \mathcal{Z}_p(X', X'') \longrightarrow \mathcal{Z}_p(X, X'') \longrightarrow \mathcal{Z}_p(X, X') \longrightarrow 0$$

is an equivariant fibration sequence.

As a consequence, there is a long exact sequence of Real Lawson homology groups

$$\rightarrow L_p H\!R_\alpha(X', X'') \rightarrow L_p H\!R_\alpha(X, X'') \rightarrow L_p H\!R_\alpha(X, X') \rightarrow L_p H\!R_{\alpha-1}(X', X'') \rightarrow$$

PROOF. As in Proposition 4.1 we only need to show that the exact sequence of topological groups

$$0 \longrightarrow \mathcal{Z}_p(X', X'')^{\mathbb{Z}_2} \longrightarrow \mathcal{Z}_p(X, X'')^{\mathbb{Z}_2} \longrightarrow \mathcal{Z}_p(X, X')^{\mathbb{Z}_2} \longrightarrow 0 \tag{4.12}$$

is a fibration sequence. Just as in the non-equivariant case ([**20**, Prop.3.1]) , this follows from Proposition 4.1 in a standard fashion. Since (4.12) is a sequence of topological groups the result will follow if we can show that (4.12) has a local cross-section at zero (see [**27**]). By the proof of Proposition 4.1 there is a neighborhood U of zero in $\mathcal{Z}_p(X, X')^{\mathbb{Z}_2}$ and a section $s : U \to \mathcal{Z}_p(X)^{\mathbb{Z}_2}$ to the projection $\pi_1 : \mathcal{Z}_p(X)^{\mathbb{Z}_2} \to \mathcal{Z}_p(X, X')^{\mathbb{Z}_2}$. Composing s with the projection $\pi_2 : \mathcal{Z}_p(X)^{\mathbb{Z}_2} \to \mathcal{Z}_p(X, X'')^{\mathbb{Z}_2}$ we get the desired section. $\square$

Finally, we recall a fundamental result of Lima-Filho that provides a definition of Lawson homology for quasi-projective varieties. This also gives a localization sequence which makes Lawson homology into a theory in which computations are often possible.

THEOREM 4.1. [**20**, Theo.4.3] *A relative isomorphism* $\Psi : (X, X') \to (Y, Y')$ *induces an isomorphism of topological groups:*

$$\Psi_* : \mathcal{Z}_p(X, X') \to \mathcal{Z}_p(Y, Y')$$

for all $p \geq 0$.

REMARK 4.2. Our observation here is that, if $\Psi : (X, X') \to (Y, Y')$ is a relative Real isomorphism of Real pairs then $\Psi_* : \mathcal{Z}_p(X, X') \to \mathcal{Z}_p(Y, Y')$ is an equivariant homeomorphism.

This shows that Lima-Filho's definition of Lawson homology for quasi-projective varieties also applies to Real quasi-projective varieties:

DEFINITION 4.4. Let U be a Real quasi-projective variety. The **group of p-cycles** on U is the Real group

$$\mathcal{Z}_p(U) \stackrel{\text{def}}{=} \mathcal{Z}_p(X, X')$$

where (X, X') is a Real pair such that $X - X'$ is isomorphic to U as Real varieties. Such a pair is called a Real compactification of U. The group $\mathcal{Z}_p(U)$ is considered as a $\mathbb{Z}_2$-space with the action induced from the action on $\mathcal{Z}_p(X)$.

The Real Lawson homology groups of U are defined as the groups of the pair (X, X'):

$$L_p I\!\!R_\alpha(U) \stackrel{\text{def}}{=} \pi_{\alpha - p\mathcal{V}} \mathcal{Z}_p(U)$$

where, as before $0 \leq p \leq \dim U$ and α is a representation containing $p\mathcal{V}$.

REMARK 4.3. Theorem 4.1 then shows that this definition is independent of the compactification (X, X'). In fact, suppose (X, X') and (Y, Y') are two Real compactifications of the Real quasi-projective variety U so that there is an isomorphism $\phi : X - X' \to Y - Y'$ of Real quasi-projective varieties. Let $\Gamma \subset X \times Y$ be the closure of the graph $\text{Graph}(\phi)$, where $X \times Y$ is endowed with the product Real structure. Set $\Gamma' = \Gamma - \text{Graph}(\phi)$. Let π_1 and π_2 denote the projections on the first and second factors, respectively. Then $\pi_1 : (\Gamma, \Gamma') \to (X, X')$ and and $\pi_2 : (\Gamma, \Gamma') \to (Y, Y')$ are relative Real isomorphisms. From Theorem 4.1 it follows that π_{1*} and π_{2*} are equivariant homeomorphisms.

The long exact sequence for triple now gives the localization sequence for Real Lawson homology: Let V be a Real closed subset of a Real quasi-projective variety U and let $n, m \geq p$. Then there is a long exact sequence of

Real Lawson homology groups

$$(4.13) \quad \cdots \to L_p I\!R_{n+m\mathcal{U}}(V) \to L_p I\!R_{n+m\mathcal{U}}(U) \to L_p I\!R_{n+m\mathcal{U}}(U-V) \\ \to L_p I\!R_{n-1+m\mathcal{U}}(V) \to \cdots$$

ending at

$$\cdots \to L_p I\!R_{p+m\mathcal{U}}(V) \to L_p I\!R_{p+m\mathcal{U}}(U) \to L_p I\!R_{p+m\mathcal{U}}(U-V).$$

As a consequence we can now prove the Real version of the "homotopy property" for Lawson homology.

PROPOSITION 4.3. *Let U be a Real quasi-projective variety and let $E \xrightarrow{\pi} U$ be a Real algebraic vector bundle of rank k. Then the flat pull-back of cycles*

$$\pi^* : \mathcal{Z}_p(U) \longrightarrow \mathcal{Z}_{p+k}(E)$$

is an equivariant homotopy equivalence.

PROOF. The proof goes exactly as in the non-equivariant context ([**7**]): It suffices to show that the map induced by π^* on the Bredon homotopy groups is an isomorphism. Using localization and the 5-lemma we can reduce to the case where E is trivial. At this point one can use induction on k to reduce to the case of $k = 1$. Then one can further reduce to the case where U has a projective closure $\overline{U}$ such that $E \to U$ is the restriction to U of $\mathcal{O}(1)|\overline{U} \to \overline{U}$. The result now follows from the suspension Theorem. □

Following Friedlander and Gabber we can now define the intersection with a Real effective Cartier divisor. Let U be a Real quasi-projective variety and let D be a Real Cartier divisor. By this we mean that D is defined by the vanishing of a Real section, s_D, of a Real line bundle, $L_D \xrightarrow{\pi} U$. The inclusion of D in U is denoted by i_D. Also let V be the complement of $|D|$ in U.

The composition

$$res \circ s_{D*} : \mathcal{Z}_p(U) \to \mathcal{Z}_p(L_D) \to \mathcal{Z}_p(L_D|V)$$

is equivariantly homotopic to the zero map: The homotopy is defined as multiplication by t in the fibres of L_D, for $t \in [1, \infty[$ and as the zero map for $t = \infty$. Thus, since by Proposition 4.2 the sequence

$$0 \longrightarrow \mathcal{Z}_p(i_D^* L_D) \longrightarrow \mathcal{Z}_p(L_D) \longrightarrow \mathcal{Z}_p(L_D|V) \longrightarrow 0$$

is an equivariant fibration sequence we get an equivariant map

$$\sigma_D : \mathcal{Z}_p(U) \to \mathcal{Z}_p(i_D^* L_D)$$

well defined up to equivariant homotopy.

DEFINITION 4.5. The intersection with D is defined as the composition

$$i_D^! \stackrel{\text{def}}{=} (\pi^*)^{-1} \circ \sigma_D : \mathcal{Z}_p(U) \to \mathcal{Z}_p(i_D^* L_D) \to \mathcal{Z}_{p-1}(|D|)$$

4.1. The s-map. The last feature of Lawson homology that we must reinterpret in the Real case is the s map defined by Friedlander and Gabber in [7]. If U is a Real quasi-projective variety, s maps $\mathcal{Z}_p(U) \wedge \mathbb{P}^1_{\mathbb{C}} \to \mathcal{Z}_{p-1}(U)$ as follows. Consider the composition

$$\mathcal{Z}_p(U) \wedge AG(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\omega} \mathcal{Z}_p(U \times \mathbb{P}^1_{\mathbb{C}}) \xrightarrow{i_U^!} \mathcal{Z}_{p-1}(U)$$

where $\omega(V, t) = V \times \{t\}$ and $i_U^!$ denotes intersection with $U \times \{\infty\}$ which is a divisor in $U \times \mathbb{P}^1_{\mathbb{C}}$. This map is clearly equivariant for the diagonal action on $\mathcal{Z}_p(U) \times AG(\mathbb{P}^1_{\mathbb{C}})$ where $\mathbb{P}^1_{\mathbb{C}}$ is equipped with the action induced by complex conjugation. Consider $\mathbb{P}^1_{\mathbb{C}}$ embedded in $AG(\mathbb{P}^1_{\mathbb{C}})$ by $t \mapsto t - \infty$. Restricting $i_U^! \circ \omega$ to $\mathcal{Z}_p(U) \wedge \mathbb{P}^1_{\mathbb{C}}$ and taking the adjoint we get a map

$$s : \mathcal{Z}_p(U) \to \Omega^{\mathcal{V}} \mathcal{Z}_{p-1}(U)$$

which induces a map in Real Lawson homology groups

$$L_p\mathit{RR}_\alpha(U) \xrightarrow{s_*} L_{p-1}\mathit{RR}_\alpha(U). \tag{4.14}$$

We usually abuse notation and denote this map by s as well.

In [**7**] it is also proved that in Lawson homology, the induced map $s_* : L_pH_n(U) \to L_{p-1}H_n(U)$ can also defined by a different construction. Consider $\mathbb{P}^1_{\mathbb{C}}$ embedded in $AG(\mathbb{P}^1_{\mathbb{C}})$, as above, by mapping $t \in \mathbb{P}^1_{\mathbb{C}}$ to $t - \infty$. The adjoint of the composition

$$\mathcal{Z}_p(U) \wedge \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\#} \mathcal{Z}_{p+1}(U\#\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\Sigma^{-2}} \mathcal{Z}_{p-1}(U)$$

is another a map $\mathcal{Z}_p(U) \xrightarrow{s'} \Omega^2\mathcal{Z}_{p-1}(U)$, which is homotopic to s. Therefore the map s_* can also be realized in the following way: the inclusion of $\mathbb{P}^1_{\mathbb{C}}$ in $AG(\mathbb{P}^1_{\mathbb{C}})$ is a generator x for $\pi_2 AG(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$. Joining with x gives a map

$$\pi_{n-2p}\mathcal{Z}_p(U) \to \pi_{n+2-2p}\mathcal{Z}_{p+1}(U\#\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\Sigma^{-2}} \pi_{n+2-2p}\mathcal{Z}_{p-1}(U)$$

which is the induced map $s_* : L_pH_n(U) \to L_{p-1}H_n(U)$ in Lawson homology.

All this works equivariantly but now x is seen as the generator of the group $\pi_{\mathcal{V}} AG(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$, so joining with it gives the map

$$\pi_{\alpha-p\mathcal{V}}\mathcal{Z}_p(U) \to \pi_{\alpha+\mathcal{V}-p\mathcal{V}}\mathcal{Z}_{p+1}(U\#\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\Sigma^{-2}} \pi_{\alpha+\mathcal{V}-p\mathcal{V}}\mathcal{Z}_{p-1}(U)$$

of (4.14).

Now, p iterations of the map s give the Real version of the cycle map:

$$L_p\mathit{RR}_\alpha(U) \xrightarrow{s^p} L_0\mathit{RR}_\alpha(U).$$

and this is the motivation for the indexing in Real Lawson homology. If U is a closed variety, this last group is isomorphic to $H_\alpha(U;\underline{\mathbb{Z}})$. We think of the elements of $L_p\mathit{RR}_\alpha(U)$ as having algebraic dimension p and homological dimension α.

REMARK 4.4. The following observation is often useful. Suppose the cycle map $s : \mathcal{Z}_p(X) \to \Omega^{\mathcal{V}}\mathcal{Z}_{p-1}(X)$ is an equivariant homotopy equivalence. From the definition of s it is clear that the diagram

$$\begin{tikzcd} \mathcal{Z}_p(X) \arrow[r, "s"] \arrow[d, "\Sigma\!\!\!/"'] & \Omega^{\mathcal{V}}\mathcal{Z}_{p-1}(X) \arrow[d, "\Omega^{\mathcal{V}}\Sigma\!\!\!/"'] \\ \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X) \arrow[r, "s"] & \Omega^{\mathcal{V}}\mathcal{Z}_p(\Sigma\!\!\!/ X) \end{tikzcd}$$

is commutative, so $s : \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X) \to \Omega^{\mathcal{V}}\mathcal{Z}_p(\Sigma\!\!\!/ X)$ is also an equivariant homotopy equivalence.

To see this we use the definition of the adjoint of s given by joining with a generator of $\pi_{\mathcal{V}}AG(\mathbb{P}^1_{\mathbb{C}})$. The result follows from the commutativity of the diagram

$$\begin{tikzcd} \mathcal{Z}_p(X) \times AG(\mathbb{P}^1_{\mathbb{C}}) \arrow[r, "\#"] \arrow[d, "\Sigma\!\!\!/ \times id"'] & \mathcal{Z}_{p+1}(\Sigma\!\!\!/^{\,2} X) \arrow[r, "\Sigma\!\!\!/^{\,-2}"] \arrow[d, "\Sigma\!\!\!/"'] & \mathcal{Z}_{p-1}(X) \arrow[d, "\Sigma\!\!\!/"'] \\ \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X) \times AG(\mathbb{P}^1_{\mathbb{C}}) \arrow[r, "\#"] & \mathcal{Z}_{p+2}(\Sigma\!\!\!/^{\,3} X) \arrow[r, "\Sigma\!\!\!/^{\,-2}"] & \mathcal{Z}_p(\Sigma\!\!\!/ X) \end{tikzcd}$$

up to $\mathbb{Z}_2$-homotopy.

5. Examples and Computations

In this section we compute the Real Lawson homology groups for some Real varieties. The main tools in these computations are the localization sequence (4.13) and the cycle map s^p. We start with the fundamental example of affine space $\mathbb{A}^n$ with its standard Real structure.

EXAMPLE 1 (The Real Lawson homology of affine space $\mathbb{A}^n$). Let $\mathbb{A}^n$ have the Real structure given by complex conjugation of its coordinates. We call this Real structure on $\mathbb{A}^n$ the standard Real structure. By definition,

$$\mathcal{Z}_p(\mathbb{A}^n) = \frac{\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})}{\mathcal{Z}_p(\mathbb{P}^{n-1}_{\mathbb{C}})}.$$

The suspension Theorem gives $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \cong \mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}})$ and from Theorem 3.3

$$\mathcal{Z}_p(\mathbb{A}^n) = \frac{\mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}})}{\mathcal{Z}_0(\mathbb{P}^{n-1-p}_{\mathbb{C}})} \cong K((n-p)\mathcal{V}, \underline{\mathbb{Z}}).$$

Here we used the following important property of the equivalence in Theorem 3.2: Under this equivalence the inclusion $\mathcal{Z}_0(\mathbb{P}^{n-1-p}_{\mathbb{C}}) \subset \mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}})$ corresponds to the inclusion

$$\prod_{k=0}^{n-p-1} AG(S^{k\mathcal{V}}) \to \prod_{k=0}^{n-p} AG(S^{k\mathcal{V}})$$

as a factor; see [8]. Thus, for $0 \leq p \leq n$ and α containing $p\mathcal{V}$,

$$L_p H\!\!\!\!R_\alpha(\mathbb{A}^n) \cong \pi_{\alpha - p\mathcal{V}} K((n-p)\mathcal{V}, \underline{\mathbb{Z}}) \cong H^{n\mathcal{V}-\alpha}(pt; \underline{\mathbb{Z}}).$$

Moreover we see that the cycle map s^p gives an equivariant homotopy equivalence

$$s^p : \mathcal{Z}_p(\mathbb{A}^n) \longrightarrow \Omega^{p\mathcal{V}} \mathcal{Z}_0(\mathbb{A}^n).$$

Since s will play a central role in the examples to follow we will try to explain this carefully.

Recall the description of s on $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$ as the adjoint of the restriction of the composition

$$\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \wedge AG(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\#} \mathcal{Z}_{p+1}(\mathbb{P}^{n+2}_{\mathbb{C}}) \xrightarrow{\Sigma^{-2}} \mathcal{Z}_{p-1}(\mathbb{P}^n_{\mathbb{C}}) \tag{5.15}$$

to $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \wedge \mathbb{P}^1_{\mathbb{C}}$. Where we consider $\mathbb{P}^1_{\mathbb{C}}$ embedded in $AG(\mathbb{P}^1_{\mathbb{C}})$ by the map $t \mapsto t - \infty$. Recall also that we have a complete description of the action of the join and suspension maps on the cycles of $\mathbb{P}^n_{\mathbb{C}}$: The suspension Theorem identifies $\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}})$, $\mathcal{Z}_{p-1}(\mathbb{P}^n_{\mathbb{C}})$, canonically, with $\mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}})$ and $\mathcal{Z}_0(\mathbb{P}^{n+1-p}_{\mathbb{C}})$, respectively.

The commutativity of the diagram (up to equivariant homotopy)

$$\begin{tikzcd}
\mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \times \mathbb{P}^1_{\mathbb{C}} \arrow[r, "\#"] \arrow[d, "\not{\Sigma}^{-p}\times \mathrm{id}"'] & \mathcal{Z}_{p+1}(\mathbb{P}^{n+2}_{\mathbb{C}}) \arrow[r, "\not{\Sigma}^{-2}"] \arrow[d, "\not{\Sigma}^{-p}"] & \mathcal{Z}_{p-1}(\mathbb{P}^n_{\mathbb{C}}) \arrow[d, "\not{\Sigma}^{-(p-1)}"] \\
\mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}}) \times \mathbb{P}^1_{\mathbb{C}} \arrow[r, "\#"] & \mathcal{Z}_1(\mathbb{P}^{n+2-p}_{\mathbb{C}}) \arrow[r, "\not{\Sigma}^{-1}"] & \mathcal{Z}_0(\mathbb{P}^{n+1-p}_{\mathbb{C}})
\end{tikzcd}$$

shows that the, under the above identifications, the map (5.15) is identified with the product μ (see Proposition 3.1) restricted to $\mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}}) \wedge \mathbb{P}^1_{\mathbb{C}}$.

By Theorem 3.2

$$\mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}}) \cong \prod_{k=0}^{n-p} AG(S^{k\mathcal{V}})$$

and by Proposition 3.1, μ restricted to $\mathcal{Z}_0(\mathbb{P}^{n-p}_{\mathbb{C}}) \wedge \mathbb{P}^1_{\mathbb{C}}$ is identified with the map

$$\prod_{k=0}^{n-p} AG(S^{k\mathcal{V}}) \wedge \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\wedge} \prod_{k=1}^{n-p+1} AG(S^{k\mathcal{V}})$$

induced by the smash map $S^{k\mathcal{V}} \wedge S^{\mathcal{V}} \xrightarrow{\wedge} S^{(k+1)\mathcal{V}}$ (recall that $\mathbb{P}^1_{\mathbb{C}} \cong S^{\mathcal{V}}$). It follows that this map is one of the structural maps of the Ω-$\mathbb{Z}_2$-prespectrum $W \to AG(S^W)$ hence its adjoint is an equivariant homotopy equivalence

$$\prod_{k=0}^{n-p} AG(S^{k\mathcal{V}}) \cong \Omega^{S^{\mathcal{V}}} \prod_{k=1}^{n-p+1} AG(S^{k\mathcal{V}}) \tag{5.16}$$

This concludes the analysis of the map $s : \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}}) \to \Omega^{\mathcal{V}}\mathcal{Z}_{p-1}(\mathbb{P}^n_{\mathbb{C}})$. To obtain the result for the affine space $\mathbb{A}^n$ we observe that s is natural, so $s : \mathcal{Z}_p(\mathbb{A}^n) \to \Omega^{\mathcal{V}}\mathcal{Z}_{p-1}(\mathbb{A}^n)$ is obtained by passing to the quotient in (5.16) and we get that, for $\mathbb{A}^n$, s is the adjoint of

$$\mathcal{Z}_p(\mathbb{A}^n) \wedge S^{\mathcal{V}} \cong AG(S^{(n-p)\mathcal{V}}) \wedge S^{\mathcal{V}} \xrightarrow{\wedge} AG(S^{(n+1-p)\mathcal{V}}) \cong \mathcal{Z}_{p-1}(\mathbb{A}^n)$$

which is an equivariant homotopy equivalence, as desired.

The following summarizes our conclusions regarding the Real Lawson homology of affine space $\mathbb{A}^n$.

THEOREM 5.1. *The space $\mathcal{Z}_p(\mathbb{A}^n)$ is an equivariant Eilenberg-Mac Lane space of type $K(\underline{\mathbb{Z}}, (n-p)\mathcal{V})$, for every $0 \le p \le n$. Moreover, cycle map*

$$s^p : \mathcal{Z}_p(\mathbb{A}^n) \longrightarrow \Omega^{p\mathcal{V}} \mathcal{Z}_0(\mathbb{A}^n)$$

is an equivariant homotopy equivalence.

We will now make use of the cycle map and the localization sequence to prove the following general result about the Real Lawson homology of Real varieties with a Real cell decomposition. The next definition is an adaptation to the Real case of [**22**, Definition 5.3]

DEFINITION 5.1. Let (X, Y) be a pair of Real projective varieties. We say that X is a Real algebraic cellular extension of Y if there is a filtration

$$X = X_n \supset X_{n-1} \supset \cdots X_0 \supset X_{-1} = Y$$

by Real projective subvarieties X_i such that $X_i - X_{i-1}$ is a union of affine spaces $\mathbb{A}^{n_{i,j}}$. If $Y = \varnothing$ we say that X has a Real cell decomposition.

PROPOSITION 5.1. *Let X be a Real variety with a Real cell decomposition, then the cycle map*

$$s^p : \mathcal{Z}_p(X) \longrightarrow \Omega^{p\mathcal{V}} \mathcal{Z}_0(X)$$

is an equivariant homotopy equivalence. In particular, the cycle map induces an isomorphism

$$L_p I\!R_\alpha(X) \cong H_\alpha(X; \underline{\mathbb{Z}}).$$

Proof. The result is proved by induction using the localization sequence and the fact that, by Example 1, it holds for affine spaces: Assume that

$$s^p : \mathcal{Z}_p(X_{i-1}) \longrightarrow \Omega^{p\mathcal{V}} \mathcal{Z}_0(X_{i-1})$$

is an equivariant homotopy equivalence. Applying the localization sequence and the cycle map s^p we get a map of long exact sequences

$$\begin{array}{ccccccc}
L_p H\!R_{k+p\mathcal{V}}(X_i) & \longrightarrow & L_p H\!R_{k+p\mathcal{V}}(X_i - X_{i-1}) & \longrightarrow & L_p H\!R_{k+p\mathcal{V}-1}(X_{i-1}) & \longrightarrow \\
\downarrow{\scriptstyle s^p} & & \downarrow{\scriptstyle s^p} & & \downarrow{\scriptstyle s^p} & \\
L_0 H\!R_{k+p\mathcal{V}}(X_i) & \longrightarrow & L_0 H\!R_{k+p\mathcal{V}}(X_i - X_{i-1}) & \longrightarrow & L_0 H\!R_{k+p\mathcal{V}-1}(X_{i-1}) & \longrightarrow
\end{array}$$

ending at

$$\begin{array}{ccccccc}
L_p H\!R_{p\mathcal{V}}(X_{i-1}) & \longrightarrow & L_p H\!R_{p\mathcal{V}}(X_i) & \longrightarrow & L_p H\!R_{p\mathcal{V}}(X_i - X_{i-1}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle s^p} & & \downarrow{\scriptstyle s^p} & & \downarrow{\scriptstyle s^p} & & \\
L_0 H\!R_{p\mathcal{V}}(X_{i-1}) & \longrightarrow & L_0 H\!R_{p\mathcal{V}}(X_i) & \longrightarrow & L_0 H\!R_{p\mathcal{V}}(X_i - X_{i-1}) & \longrightarrow & 0.
\end{array}$$

Exactness at the last group of the bottom row follows from the fact that, since X_{i-1} has a Real cell decomposition, $H_{m\mathcal{V}-1}(X_{i-1}; \underline{\mathbb{Z}}) = 0$, for all $m \in \mathbb{Z}$; see Lemma 5.1.

By the assumptions and the 5-Lemma it follows that

$$s^p_* : L_p H\!R_{k+p\mathcal{V}}(X_i) \longrightarrow L_0 H\!R_{k+p\mathcal{V}}(X_i)$$

is an isomorphism for all $k \geq 0$. Translating this into homotopy groups, it means that

$$s^p_* : \pi_k^{\mathbb{Z}_2}\left(\mathcal{Z}_p(X_i)\right) \longrightarrow \pi_k^{\mathbb{Z}_2}\left(\Omega^{p\mathcal{V}} \mathcal{Z}_0(X_i)\right)$$

is an isomorphism for all $k \geq 0$. Since we already know that s^p is a non-equivariant homotopy equivalence (see **[22]**). This implies that $s^p : \mathcal{Z}_p(X_i) \to \Omega^{p\mathcal{V}} \mathcal{Z}_0(X_i)$ is an equivariant homotopy equivalence. □

LEMMA 5.1. *Let X be a Real variety with a Real cell decomposition*

$$X = X_n \supset X_{n-1} \supset \cdots X_0 \supset X_{-1} = \varnothing$$

such that $X_i - X_{i-1}$ is a union of affine spaces $\mathbb{A}^{n_{i,j}}$. Let R denote the cohomology ring of a point, $H^*(pt; \underline{\mathbb{Z}})$. Then $H_*(X; \underline{\mathbb{Z}})$ is an R-free module. Each cell $\mathbb{A}^{n_{i,j}}$ gives rise to a generator $x_{i,j}$ in dimension $n_{i,j} \cdot \mathcal{V}$.

PROOF. The Real cell decomposition gives X an equivariant cell decomposition with cells of type $D^{n_{i,j}\mathcal{V}}$. The proof is by induction on the cells: By Definition 5.1, X_0 is a disjoint union of points fixed by the action, so the result holds. Assume it also holds for X_{i-1} and consider the cofibration sequence

$$X_{i-1+} \longrightarrow X_{i+} \longrightarrow \bigvee_j S^{n_{i,j}\mathcal{V}}.$$

There is a long exact sequence

$$\to H_\alpha(X_i; \underline{\mathbb{Z}}) \to \bigoplus_j \widetilde{H}_\alpha(S^{n_{i,j}\mathcal{V}}; \underline{\mathbb{Z}}) \xrightarrow{\delta} H_{\alpha-1}(X_{i-1}; \underline{\mathbb{Z}}) \to \tag{5.17}$$

Observe that this is an exact sequence of R-modules and, by assumption, the homology of X_{i-1} is free on generators $x_{k,j}$, $k < i$, of dimensions $n_{k,j} \cdot \mathcal{V}$. Also

$$\widetilde{H}_*(S^{n_{i,j}\mathcal{V}}; \underline{\mathbb{Z}}) \cong H^{n_{i,j}\mathcal{V}-*}(pt; \underline{\mathbb{Z}}) = R^{n_{i,j}\mathcal{V}-*}$$

so, in particular, this R-module is free and generated by an element $x_{i,j}$ in dimension $n_{i,j}\mathcal{V}$ ($x_{i,j}$ is sent to the identity element in R by the isomorphism above). The connecting homomorphism δ in the sequence (5.17) is determined by the image of the generators $x_{i,j}$. But the induction hypothesis implies that this image is zero because

$$H^{m\mathcal{V}-1}(pt; \underline{\mathbb{Z}}) = 0$$

for all $m \in \mathbb{Z}$; see (3.9). This completes the proof. $\square$

The following are examples of Real varieties with a Real cell decomposition.

EXAMPLE 2 (The Grassmannians $G^q(\mathbb{C}^{n+1})$). The variety $G^q(\mathbb{C}^{n+1})$ has a Real structure given by the action induced by complex conjugation in $\mathbb{C}^{n+1}$. The Schubert cells give a Real cell decomposition for $G^q(\mathbb{C}^{n+1})$.

EXAMPLE 3 (Products of varieties with Real cell decompositions). Real varieties with a Real cell decomposition form a class which is closed under products. So, for example, $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$ has a Real cell decomposition and we have that the group $L_pH\!R_\alpha(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}})$ is isomorphic to the α degree part of the $RO(\mathbb{Z}_2)$-graded module $R[x,y]/(x^n, y^m)$ where R is the cohomology of a point and x, y have degree $\mathcal{V}$.

EXAMPLE 4 (Quadrics with signature zero). Any Real smooth quadric in $\mathbb{P}^{n-1}_{\mathbb{C}}$ is equivalent to a quadric of the form

$$\mathcal{Q}_{n,k} \stackrel{\text{def}}{=} \left\{(x_1 : \dots : x_n) \in \mathbb{P}^{n-1}_{\mathbb{C}} | x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2 = 0\right\}$$

where $k \leq n/2$.

We consider the case, $n = 2k$, *i.e.* the quadratic form defining the quadric has signature zero. We will show that the cycle map is an isomorphism. From now on we use homogeneous coordinates $(X : Y) = (x_1 : \dots : x_n : y_1 : \dots : y_n)$ for the points of $\mathbb{P}^{2n-1}_{\mathbb{C}}$. In these coordinates the quadratic form is $X^T X - Y^T Y$. The point $p_0 = (X_0 : Y_0) = (0 : \dots : 0 : 1 : 0 : \dots : 0 : 1)$ is a real point of $\mathcal{Q}_{2n,n}$ and the tangent plane to $\mathcal{Q}_{2n,n}$ through p_0 is

$$H = \left\{(X : Y) \in \mathbb{P}^{2n-1}_{\mathbb{C}} | (X : Y) \cdot (X_0 : -Y_0) = 0\right\}$$

and

$$\mathcal{Q}_{2n,n} \cap H = \left\{(X : Y) \in \mathbb{P}^{2n-1}_{\mathbb{C}} | x_n - y_n = 0 \text{ and } X^T X - Y^T Y = 0\right\}.$$

Using coordinates $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$ and $t = x_n + y_n$ for H we see that the quadric $\mathcal{Q}_{2n,n} \cap H$ is given by the equation $x_1^2 + \dots + x_{n-1}^2 - y_1^2 - \dots - y_{n-1}^2 = 0$.

Let H' be the Real hyperplane given by the equation $t = x_n + y_n = 0$. The intersection $\mathcal{Q}_{2n,n} \cap H \cap H'$ is a quadric $\mathcal{Q}_{2n-2,n-1}$ and we have

$$\mathcal{Q}_{2n,n} \cap H = \mathcal{Q}_{2n-2,n-1} \# p_0.$$

Thus $\mathcal{Q}_{2n,n} \cap H \cong \not\Sigma\, \mathcal{Q}_{2n-2,n-1}$. Assume that the cycle map is an isomorphism $L_p I\!R_\alpha(\mathcal{Q}_{2n-2,n-1}) \to H_\alpha(\mathcal{Q}_{2n-2,n-1}; \underline{\mathbb{Z}})$. From Remark 4.4 it follows that the same holds for $\not\Sigma\, \mathcal{Q}_{2n-2,n-1}$.

It is also easy to see that, if $\pi : \mathbb{P}_{\mathbb{C}}^{2n-1} - p_0 \to H'$ is the projection on H' centered at p_0, then $\pi | \mathcal{Q}_{2n,n} - \mathcal{Q}_{2n,n} \cap H$ is an isomorphism onto $H' - H \cap H' \cong \mathbb{A}^{2n-2}$. Since everything is Real, this is a Real isomorphism. It is easy to check that $\mathcal{Q}_{4,2} \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ with the standard Real structure, hence, by Example 3 the cycle map is an isomorphism in this case. Using induction on n, the sequence for the Real pair $(\mathcal{Q}_{2n,n}, \mathcal{Q}_{2n,n} \cap H)$ and the five Lemma, it follows that the cycle map

$$L_p I\!R_\alpha(\mathcal{Q}_{2n,n}) \longrightarrow H_\alpha(\mathcal{Q}_{2n,n}; \underline{\mathbb{Z}})$$

is an equivariant homotopy equivalence. This reduces the computation of Real Lawson homology to the computation of the equivariant homology of $\mathcal{Q}_{2n,n}$.

Remark 5.1. The computation of the Real Lawson homology groups gives us, in particular, the homotopy type of the spaces of Real cycles, *i.e.* the cycles fixed by the involution. This is an immediate consequence of that fact that these spaces are topological groups and hence products of Eilenberg-Mac Lane spaces and so their homotopy type is completely determined by the homotopy groups. The homotopy groups $\pi_k(\mathcal{Z}_p(X)^{\mathbb{Z}_2})$ are the Real Lawson homology groups $L_p I\!R_{k+p\mathcal{V}}(X)$.

The next example is a very simple case — albeit somewhat artificial — in which the cycle map is an isomorphism but the variety doesn't have a Real cell decomposion.

EXAMPLE 5. Let $U \subset \mathbb{P}^n_{\mathbb{C}}$ be a quasi-projective variety such that the intersection $U \cap \overline{U}$ is empty. Then $U \amalg \overline{U}$ is a Real variety with the involution induced by complex conjugation. Assume that the non-equivariant cycle map $s : \mathcal{Z}_p(U) \to \Omega^2 \mathcal{Z}_{p-1}(U)$ is a homotopy equivalence. We will see that $s : \mathcal{Z}_p(U \amalg \overline{U}) \to \Omega^{\mathcal{V}} \mathcal{Z}_{p-1}(U \amalg \overline{U})$ is an equivariant homotopy equivalence.

We have $\mathcal{Z}_p(U \amalg \overline{U}) \cong \mathcal{Z}_p(U) \oplus \mathcal{Z}_p(\overline{U})$ and, under this isomorphism, the involution is given by

$$\tau \cdot (C_1, C_2) = (\overline{C_2}, \overline{C_1}).$$

Recall that, for pointed $\mathbb{Z}_2$-spaces S and X, $F(S, X)$ denotes the space of based maps with the conjugation action: $f \mapsto \tau \circ f \circ \tau$. It follows that, if we consider $\mathcal{Z}_p(U)$ equipped with the trivial $\mathbb{Z}_2$-action, then

$$\mathcal{Z}_p(U \amalg \overline{U}) \cong F(\mathbb{Z}_{2+}, \mathcal{Z}_p(U)).$$

In particular we see that there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_p(U \amalg \overline{U})^{\mathbb{Z}_2} & \xrightarrow{\cong} & \mathcal{Z}_p(U) \\
{\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow \\
\left\{\Omega^{\mathcal{V}} \mathcal{Z}_{p-1}(U \amalg \overline{U})\right\}^{\mathbb{Z}_2} & \xrightarrow{\cong} & \Omega^2 \mathcal{Z}_{p-1}(U)
\end{array}$$

where the left and right vertical arrows denote the equivariant and the non-equivariant cycle maps, respectively. The result now follows.

EXAMPLE 6. Consider the variety $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ with the Real structure given by

$$\tau \cdot (X, Y) = (\overline{Y}, \overline{X}) \qquad (X, Y) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}. \tag{5.18}$$

We will show that the cycle map is an equivariant homotopy equivalence. In the case $n = 0$ there is nothing to prove. Assume the result holds for $n - 1$.

We have

$$\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} = \mathbb{A}^n \times \mathbb{A}^n \cup \left\{\mathbb{A}^n \times \mathbb{P}^{n-1}_{\mathbb{C}} \cup \mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{A}^n\right\} \cup \mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^{n-1}_{\mathbb{C}}. \tag{5.19}$$

Note that $\mathbb{A}^n \times \mathbb{A}^n$ (with the action of 5.18) is a Real subvariety and it is actually isomorphic to $\mathbb{A}^{2n}$ with the standard Real structure. The isomorphism is

$$(X, Y) \mapsto (X + Y, \sqrt{-1}(X - Y)).$$

Also the second factor in the decomposition (5.19) can be written as the disjoint union $U \amalg \tau \cdot U$ where $U = \mathbb{A}^n \times \mathbb{P}^{n-1}_{\mathbb{C}}$. By Example 5 we know that the cycle map $s : \mathcal{Z}_p(U \amalg \tau \cdot U) \to \Omega^{\mathcal{V}} \mathcal{Z}_{p-1}(U \amalg \tau \cdot U)$ is a $\mathbb{Z}_2$-homotopy equivalence. Finally, by induction, the cycle map is also an equivalence in the case of the last factor, $\mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^{n-1}_{\mathbb{C}}$. By localization and the 5-lemma it follows that the cycle map $s : \mathcal{Z}_p(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}) \to \Omega^{\mathcal{V}} \mathcal{Z}_{p-1}(\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}})$ is an equivariant homotopy equivalence.

EXAMPLE 7 (The Quaternionic line). Let $X = \mathbb{P}_{\mathbb{C}}(\mathbb{H}^2)$ with the anti-holomorphic involution induced from multiplication by j. If one identifies $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^2)$ with the two sphere S^2, the involution is the antipodal map. In particular, there are no fixed points so X cannot have a Real cell decomposition. The cycle map is very simple in this case because there are no cycles above dimension 1 and $\mathcal{Z}_1(X) \cong \mathbb{Z}$.

The cycle map, in this case, sends $\mathcal{Z}_1(X)$ to $\Omega^{\mathcal{V}} \mathcal{Z}_0(X)$ and we know it is a non-equivariant homotopy equivalence. We want to show that the induced map $s_* : \pi_0 \mathcal{Z}_1(X) \to \pi_{\mathcal{V}} \mathcal{Z}_0(X)$ is an isomorphism. We start by computing the group $\pi_{\mathcal{V}} \mathcal{Z}_0(X)$ which, by the equivariant Dold-Thom Theorem, is $H_{\mathcal{V}}(X; \underline{\mathbb{Z}})$. To do this we use the identification of X with $S(3\mathcal{U})$ mentioned above. There is a cofibration sequence

$$S(3\mathcal{U})_+ \longrightarrow D(3\mathcal{U})_+ \longrightarrow S^{3\mathcal{U}}$$

where $D(3\mathcal{U})$ denotes the unit disk of the representation $3\mathcal{U}$. Note that $D(3\mathcal{U})_+ \cong S^0$, equivariantly. The associated long exact sequence in homology is

$$\to H_{V+1}(pt;\underline{\mathbb{Z}}) \to \widetilde{H}_{V+1}(S^{3\mathcal{U}};\underline{\mathbb{Z}}) \to H_V(S(3\mathcal{U});\underline{\mathbb{Z}}) \to H_V(pt;\underline{\mathbb{Z}}) \to$$

For V effective we have $H_V(pt;\underline{\mathbb{Z}}) = 0$ so, in particular,

$$H_{\mathcal{V}}(S(3\mathcal{U});\underline{\mathbb{Z}}) \cong \widetilde{H}_{1+\mathcal{V}}(S^{3\mathcal{U}};\underline{\mathbb{Z}}) \cong H_{2-2\mathcal{U}}(pt;\underline{\mathbb{Z}}). \tag{5.20}$$

This shows that $\pi_{\mathcal{V}}\mathcal{Z}_0(X) \cong \mathbb{Z}$. Let

$$\Phi : [S^{\mathcal{V}}, \mathcal{Z}_0(X)]_{\mathbb{Z}_2} \to [S^{\mathcal{V}}, \mathcal{Z}_0(X)]$$

be the forgetful map. Let $[X]$ denote the generator of $H_2^{sing}(X;\mathbb{Z})$. We know that the non-equivariant cycle map sends the cycle X to $[X]$. This means that $\Phi(s(X)) = [X]$. It follows that $s(X) \in H_{\mathcal{V}}(X;\underline{\mathbb{Z}})$ is a generator and Φ is an isomorphism. We conclude that $s : \mathcal{Z}_1(X) \to \Omega^{\mathcal{V}}\mathcal{Z}_0(X)$ is an equivariant homotopy equivalence.

Example 8 (Quadrics with signature 3). It is easy to check that $\mathbb{P}_{\mathbb{C}}(\mathbb{H})$ is isomorphic as a Real variety to the plane quadric $\mathcal{Q}_{3,0}$. From Example 4 it follows that the quadric of signature 3, $\mathcal{Q}_{2n-1,n-1}$, is obtained from $\mathcal{Q}_{3,0}$ by adding Real cells $\mathbb{A}^k$ and taking suspensions. Using exact sequences, the 5-lemma and the results of the previous examples, it follows that the cycle map

$$s^p : \mathcal{Z}_p(\mathcal{Q}_{2n-1,n-1}) \longrightarrow \Omega^{p\mathcal{V}}\mathcal{Z}_0(\mathcal{Q}_{2n-1,n-1})$$

is an equivariant homotopy equivalence.

Example 9 (Quadrics with signature 2). From Example 4 it follows that the signature 2 quadric, $\mathcal{Q}_{2n+2,n}$, is obtained from $\mathcal{Q}_{4,1}$ by adding Real cells $\mathbb{A}^k$ and taking suspensions. One can check that $\mathcal{Q}_{4,1} \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with the Real structure of Example 6. It follows that the cycle map is also an equivariant homotopy equivalence in this case.

6. Auxiliary Results

In this Section we give the details of the unproved assertions in the proof of Proposition 2.2. We begin by reviewing some results of [**23**] whose proofs we include here for completeness. Throughout this section G is a finite group, unless otherwise stated.

DEFINITION 6.1 ([**24**]). A G-CW complex X is the union of sub G-spaces

$$X_0 \subset X_1 \subset \cdots X_n \subset \cdots$$

such that X_0 is a disjoint union of orbits G/H and X_n is obtained from X_{n-1} by attaching G-cells $D^n \times G/H$ along attaching maps $S^{n-1} \times G/H \to X_{n-1}$. X_n is the n-skeleton of X. Note that the interior points of a cell $D^n \times G/H$ have isotropy group conjugated to H. A cell $D^n \times G/H$ is called an n-cell of type G/H.

If Λ_n is the set of n-cells of X and, for each $\alpha \in \Lambda_n$, α has type G/H_α, we have

$$X_n/X_{n-1} \cong \bigvee_{\alpha \in \Lambda_n} S^n \wedge G/H_{\alpha+}.$$

Let (X, A) be a G-CW pair. In [**23**] Lima-Filho shows that the sequence of topological G-groups

$$0 \longrightarrow AG(A) \longrightarrow AG(X) \longrightarrow AG(X/A) \longrightarrow 0 \tag{6.21}$$

is a G-fibration. Given a G-module V, and a subgroup K of G, define the functors

$$(X, A) \xrightarrow{h_k(X,A)} \pi_k\left(F(S^V, AG(X/A))^K\right) = [S^{V+k}, AG(X/A)]_K.$$

It is important to note the following property of the functor $AG(-)$, which is proved in [**23**]. For a finite G-set S and a G-space X there is a G-homeomorphism

$$F(S_+, AG(X)) \cong AG(X \wedge S_+).$$

From the fact that (6.21) is a G-fibration we get long exact sequences

$$\cdots \to h_k(X) \to h_k(X, A) \to h_{k-1}(A) \to \cdots$$

ending at

$$\cdots \to h_0(A) \to h_0(X) \to h_0(X, A)$$

In general we cannot assert exactness at the last group, but if we assume that V has at least one copy of the trivial representation then we can make sense of $h_{-1}(\cdot)$ and the sequence extends to

$$\cdots \to h_0(A) \to h_0(X) \to h_0(X, A) \to h_{-1}(A) \to h_{-1}(X) \to h_{-1}(X/A).$$

LEMMA 6.1 ([**23**]). *Let V be a G-module with at least one copy of the trivial representation and let X be a G-CW complex. Denote the p-skeleton of X by X_p, $p \geq 0$ and let Λ_p be the set of p-cells of X. Then there is a spectral sequence with E^1-term*

$$\bigoplus_{p \geq 0} \bigoplus_{\alpha \in \Lambda_p} h_k(S^p \wedge G/H_{\alpha +})$$

converging to $h_k(X)$.

PROOF. The properties of the functor h_* mentioned above allow us to apply the usual machinery to the bigraded group $h_{p+q}(X_p, X_{p-1})$ to produce a spectral sequence

$$h_{p+q}(X_p, X_{p-1}) \Longrightarrow h_{p+q}(X);$$

see [28], for example. Now

$$h_{p+q}(X_p, X_{p-1}) = \pi_{V+p+q} AG\left(\bigvee_{\alpha\in\Lambda_p} S^p \wedge G/H_{\alpha+} \wedge G/K_+\right) \cong$$

$$\pi_{V+p+q}\left(\bigoplus_{\alpha\in\Lambda_p} AG(S^p \wedge G/H_{\alpha+} \wedge G/K_+)\right) \cong \bigoplus_{\alpha\in\Lambda_p} h_{p+q}(S^p \wedge G/H_{\alpha+}).$$

□

We can now prove the following result which is used in the proof of Proposition 2.2.

LEMMA 6.2 ([23]). *Let V be a G-module with at least one copy of the trivial representation and let K be a subgroup of G. Then, for all $k > 0$,*

$$[S^{V+k}, AG(S^V)]_K = 0.$$

PROOF. We have to show that $h_k(S^V) = 0$. Let X_p denote the p-skeleton of S^V. Also let Λ_p be the set of p-cells of S^V. For $\alpha \in \Lambda_p$, we have

$$h_k(S^p \wedge G/H_{\alpha+}) = [S^{V+k} \wedge G/K_+, AG(S^p \wedge G/H_{\alpha+})]_G \cong$$

$$[S^{V+k} \wedge G/K_+, F(G/H_{\alpha+}, AG(S^p))]_G \cong [S^{V+k} \wedge G/K_+ \wedge G/H_{\alpha+}, AG(S^p)]_G \cong$$

$$[S^{V+k} \wedge G/K_+, AG(S^p)]_{H_\alpha}.$$

Write V as $V^{H_\alpha} \oplus V(H_\alpha)$, where $V(H_\alpha)$ has no copies of the trivial representation of H_α. Then

$$[S^{V+k} \wedge G/K_+, AG(S^p)]_{H_\alpha} \cong [S^{V^{H_\alpha}+k} \wedge (S^{V(H_\alpha)} \wedge G/K_+)/H_\alpha, AG(S^p)] \cong$$

$$\widetilde{H}^{p-|V^{H_\alpha}|-k}_{sing}((S^{V(H_\alpha)} \wedge G/K_+)/H_\alpha; \mathbb{Z}).$$

and his last group is zero since $p \leq |V^{H_\alpha}|$ — because the cell α is contained in $\bigcup_{g\in G}\{gS^{V^{H_\alpha}}g^{-1}\}$ — and $k > 0$. We conclude that $h_k(X_p, X_{p-1}) = 0$ and, by the previous Lemma, this implies $[S^{V+k}, AG(S^V)]_K = h_k(S^V) = 0$, as desired. $\square$

DEFINITION 6.2. Let G be a compact Lie group, U a G-module and let X, Y be G-spaces. For each subgroup H of G let $U(H)$ denote the orthogonal complement of U^H.

1. A map $f : X \to Y$ is a $|U^*|$-equivalence if, for every subgroup H, the map

$$(f^H)_* : \pi_m(X^H) \longrightarrow \pi_m(Y^H)$$

   is an isomorphism for $0 \leq m < |U^H|$ and an epimorphism for $m = |U^H|$.
2. A 0^*-**equivalence** $f : X \to Y$ is called a G-U-**equivalence** if, for every subgroup H, the map

$$f_* : \pi_{U(H)+m}(X) \longrightarrow \pi_{U(H)+m}(Y)$$

   is an isomorphism for $0 \leq m < |U^H|$ and an epimorphism for $m = |U^H|$.

It is well-known that a map $f : X \to Y$ is a $|U^*|$-equivalence if and only if it is a G-U-equivalence. This was proved by Waner for the case of G finite and generalized by Lewis to the case of G compact; see [**19**, Lemma 1.2]. We will use this result in the next Lemma.

LEMMA 6.3. *Let V be a G-module, the homomorphism*

$$\Phi : [S^V, S^V]_G \longrightarrow [S^V, AG(S^V)]_G$$

induced by the inclusion $\iota : S^V \hookrightarrow AG(S^V)$, defined as $x \mapsto x - \infty$, is surjective.

PROOF. The proof goes as follows: One shows that the inclusion ι is a $|V^*|$-equivalence. By the result of Waner mentioned above this implies that ι is a V-equivalence so, in particular, the induced map $\Phi : \pi_V(S^V) \to \pi_V(AG(S^V))$ is surjective.

We proceed to show that ι is a $|V^*|$-equivalence as claimed.

1. Observe that $AG(S^V)$ is $|V^*|-1$ connected: Let $\widetilde{H}_*^H(\bullet;\underline{\mathbb{Z}})$ denote reduced Bredon H-equivariant homology (where H is a subgroup of G) with coefficients in the Mackey functor $\underline{\mathbb{Z}}$. By Theorem 2.1

$$\pi_k(AG(S^V)^H) \cong \widetilde{H}_k^H(S^V;\underline{\mathbb{Z}}) \cong \widetilde{H}_k^H(S^{V^H} \wedge S^{V(H)};\underline{\mathbb{Z}}) \cong \widetilde{H}_{k-|V^H|}^H(S^{V(H)};\underline{\mathbb{Z}})$$

   and this last group is zero for $k < |V^H|$.

2. Moreover, with $n(H) = |V^H|$,

$$\pi_{n(H)}(S^{V^H}) \xrightarrow{\iota_*} \pi_{n(H)}(AG(S^{V^H}))$$

   is onto: For each subgroup, H, of G, write $V = V^H \oplus V(H)$, as before. The map $S^{V^H} \to AG(S^V)^H$ extends to an inclusion $AG(S^{V^H}) \to AG(S^V)^H$. It suffices to check that this inclusion induces a surjective map on $\pi_{n(H)}$. But the induced map on $\pi_{n(H)}$ corresponds by Theorem 2.1 to the map $\widetilde{H}_{n(H)}^H(S^{V^H};\underline{\mathbb{Z}}) \to \widetilde{H}_{n(H)}^H(S^V;\underline{\mathbb{Z}})$. Now observe that $S^V \cong S^{V^H} * S(V(H))$ — where $*$ denotes the unreduced join — so

$$S^V/S^{V^H} \cong S^{V^H+1} \wedge S(V(H))_+$$

   and this gives $\widetilde{H}_{n(H)}^H(S^V/S^{V^H};\underline{\mathbb{Z}}) = 0$. The Bredon H-homology exact sequence of the pair (V, V^H) shows that

$$\widetilde{H}_{n(H)}^H(S^{V^H};\underline{\mathbb{Z}}) \longrightarrow \widetilde{H}_{n(H)}^H(S^V;\underline{\mathbb{Z}})$$

   is onto, as required.

$\square$

7. Proof of Proposition 4.1

In this section we give the details of the proof of Proposition 4.1. The Proposition is mainly a consequence of Lima-Filho's result on completions and fibrations involving a certain class of abelian topological monoids.

We will need the following elementary result.

LEMMA 7.1. *Let (X, X') be a pair of Real algebraic varieties. The natural group homomorphism*

$$\frac{\mathcal{Z}_p(X)^{\mathbb{Z}_2}}{\mathcal{Z}_p(X')^{\mathbb{Z}_2}} \xrightarrow{\phi} \mathcal{Z}_p(X, X')^{\mathbb{Z}_2}$$

is a homeomorphism.

PROOF. It is clear that ϕ is closed and that it is induced by the continuous group homomorphism

$$\mathcal{Z}_p(X)^{\mathbb{Z}_2} \xrightarrow{\varphi} \mathcal{Z}_p(X, X')^{\mathbb{Z}_2}.$$

φ has kernel $\mathcal{Z}_p(X')^{\mathbb{Z}_2}$ so we need only show that it is surjective. Let c be an element of $\mathcal{Z}_p(X, X')^{\mathbb{Z}_2}$. Choose a representative $\sum_i n_i \cdot V_i$ for c so that none of the V_i's is contained in X'. Then, since $\mathcal{Z}_p(X')$ is invariant under the action of $\mathbb{Z}_2$, it follows that $\sum_i n_i \cdot V_i$ is a fixed element of $\mathcal{Z}_p(X)$ and

$$\varphi\left(\sum_i n_i \cdot V_i\right) = c.$$

□

Next we recall some definitions and one of the main results from [**21**].

DEFINITION 7.1 ([**21**]). An abelian topological monoid, C, with identity e, is c-**filtered** if it has the weak topology given by a compact filtration,

$$e = C_0 \subset C_1 \cdots \subset C_d \subset \cdots$$

by subspaces satisfying $C_d + C_{d'} \subset C_{d+d'}$.

From now on all monoids are assumed to be topological abelian monoids. Let C be a c-filtered monoid. Then $C \times C$ is also c-filtered. The filtration is

$$(C \times C)_d \overset{\text{def}}{=} \bigcup_{n+m \leq d} C_n \times C_m.$$

Suppose C' is a closed submonoid of C, then C' is c-filtered by $C'_d = C_d \cap C'$. In this context Lima-Filho makes the following definitions

$$\Delta_{C'} \stackrel{\text{def}}{=} \Delta + C' \times C' \subset C \times C,$$

and

$${}^{id}(C \times C)_d \stackrel{\text{def}}{=} \{(C \times C)_{d-1} + \Delta_{C'}\} \cap (C \times C)_d,$$

where Δ is the diagonal in $C \times C$.

DEFINITION 7.2 ([**21**]).
1. A pair of c-filtered monoids (C, C') is **properly c-filtered** if the filtration $\cdots \subset C_d \subset C_{d+1} \subset \cdots$ has the property that the inclusion ${}^{id}(C \times C)_d \subset (C \times C)_d$ is a cofibration.
2. The pair (C, C') is **free** if C is free and C' is freely generated by a subset of generators on C.

For a free monoid C the naive group completion, $\widetilde{C}$, is the quotient monoid

$$\widetilde{C} \stackrel{\text{def}}{=} C \times C/\Delta.$$

The topology of $\widetilde{C}$ is the quotient topology. Lima-Filho shows that if (C, C') is a **properly c-filtered** pair then $\widetilde{C'}$ is a closed subgroup of $\widetilde{C}$. Moreover his shows that $\widetilde{C}/\widetilde{C'}$ is filtered by cofibrations given by

$$Q_d \stackrel{\text{def}}{=} \pi(\widetilde{C}_d) = \pi \circ p((C \times C)_d)$$

where $p : C \times C \to \widetilde{C}$ and $\pi : \widetilde{C} \to \widetilde{C}/\widetilde{C'}$ are the projections. He proves the following result concerning the projection $\pi : \widetilde{C} \to \widetilde{C}/\widetilde{C'}$.

THEOREM 7.1 ([**21**]). *Given a properly* **c-filtered free** *pair of monoids* (C, C'), *there is a principal fibration* $\widetilde{C'} \to \widetilde{C} \to \widetilde{C}/\widetilde{C'}$, *where* $\widetilde{C}/\widetilde{C'}$ *is the topological group quotient of the naive group completions of* C *and* C', *respectively.*

The importance of ${}^{id}(C \times C)_d$ in the proof of the Theorem comes from the fact that

$$(C \times C, {}^{id}(C \times C)_d) \xrightarrow{\pi \circ p} (Q_d, Q_{d-1}) \tag{7.22}$$

is a relative isomorphism — in fact, Lima-Filho shows that it is a relative homeomorphism. This is used to construct a local cross-section s for the projection $\widetilde{C} \xrightarrow{\pi} \widetilde{C}/\widetilde{C'}$ inductively. In the induction step one assumes s has been defined over a neighborhood of zero in Q_{d-1} and then uses the relative homeomorphism (7.22) and the fact that ${}^{id}(C \times C)_d \subset (C \times C)_d$ is a cofibration to extend s over a neighborhood of zero in Q_d. By a result of Steenrod ([**27**]), the existence of s completes the proof of the Theorem.

We now apply Theorem 7.1 to the monoid of Real cycles on a Real algebraic variety: Let (X, X') be a pair of Real algebraic varieties. Recall that

$$\mathcal{C}_p(X) \stackrel{\text{def}}{=} \coprod_{d \geq 0} \mathcal{C}_{p,d}(X)$$

is a monoid under addition of cycles. It is endowed with the disjoint union topology; the algebraic sets $\mathcal{C}_{p,d}(X)$ are equipped with their analytic topology. This monoid is filtered by

$$\mathcal{C}_{p,\leq d}(X) \stackrel{\text{def}}{=} \coprod_{k \leq d} \mathcal{C}_{p,k}(X).$$

The Real structures on X and X' induce Real structures on the Chow varieties $\mathcal{C}_{p,d}(X)$, $\mathcal{C}_{p,d}(X')$. Set

$$C = \{\mathcal{C}_p(X)\}^{\mathbb{Z}_2} \qquad\qquad C' = \{\mathcal{C}_p(X')\}^{\mathbb{Z}_2}$$

Note that the naive group completions of C, C' are $\mathcal{Z}_p(X)^{\mathbb{Z}_2}$ and $\mathcal{Z}_p(X')^{\mathbb{Z}_2}$, respectively. It is also clear that the pair (C, C') is free.

The monoid C is filtered by

$$C_d = \{\mathcal{C}_{p,\leq d}(X)\}^{\mathbb{Z}_2}.$$

Observe that C_d and $C'_d = C_d \cap C'$ are the Real points of the Chow varieties $\mathcal{C}_{p,\leq d}(X)$ and $\mathcal{C}_{p,\leq d}(X')$ so, in particular, (C_d, C'_d) is a pair of algebraic sets.

C is c-filtered and we have

$$^{id}(C \times C)_d = ((C \times C)_{d-1} + \Delta + C' \times C') \cap (C \times C)_d.$$

Since the sum operation

$$\mathcal{C}_p(X) \times \mathcal{C}_p(X) \xrightarrow{+} \mathcal{C}_p(X)$$

is algebraic ([**6**]) it follows that $((C \times C)_d, {}^{id}(C \times C)_d)$ is also a pair of algebraic sets and hence can be triangulated (see [**11**]). We conclude that Theorem 7.1 applies and so $\widetilde{C'} \to \widetilde{C} \to \widetilde{C}/\widetilde{C'}$ is a principal fibration. But $\widetilde{C} = \mathcal{Z}_p(X)^{\mathbb{Z}_2}$, $\widetilde{C'} = \mathcal{Z}_p(X')^{\mathbb{Z}_2}$ and, by Lemma 7.1, $\widetilde{C}/\widetilde{C'} = \mathcal{Z}_p(X, X')^{\mathbb{Z}_2}$. Thus the exact sequence of topological groups

$$0 \longrightarrow \mathcal{Z}_p(X')^{\mathbb{Z}_2} \longrightarrow \mathcal{Z}_p(X)^{\mathbb{Z}_2} \longrightarrow \mathcal{Z}_p(X, X')^{\mathbb{Z}_2} \longrightarrow 0$$

is a principal fibration. Since the same holds for the sequence $\mathcal{Z}_p(X') \to \mathcal{Z}_p(X) \to \mathcal{Z}_p(X, X')$, and the maps are all equivariant, it follows that this sequence is actually a $\mathbb{Z}_2$-fibration sequence.

CHAPTER 2

Galois-Grothendieck cohomology

1. Introduction

In the previous Chapter we saw that the space $\mathcal{Z}$ of stabilized cycles is canonically a product of classifying spaces for certain $\mathbb{Z}_2$-equivariant cohomology groups. In this chapter we study the relation of these cohomology groups to **Galois-Grothendieck cohomology** groups which are invariants commonly used in Real algebraic geometry; see [**25**] and [**13**] , for example.

Recall that, for a G-space X, the Borel cohomology of X with coefficients in a ring Γ is just the cohomology of the Borel construction on X: $H^*(X \times_G EG; \Gamma)$. In geometry this is what is usually meant when one refers to equivariant cohomology. We will denote these groups by $\widehat{H}^*_G(X; \Gamma)$. For a finite group G, Galois-Grothendieck cohomology can be thought of as a generalization of Borel cohomology to non-constant coefficients. If $\mathcal{F}$ is a G-sheaf over X, then $\mathcal{F} \times_G EG$ is sheaf over the Borel construction $X \times_G EG$ and so we can define $H^*(X \times_G EG; \mathcal{F} \times_G EG)$. These are the Galois-Grothendieck cohomology groups of X with coefficients in $\mathcal{F}$. We will denote these groups by $\widehat{H}^*_G(X; \mathcal{F})$.

In [**12**] Kahn defined equivariant Chern classes for Real bundles in terms of Galois-Grothendieck cohomology. Here G is $\mathbb{Z}_2$ and the sheaf $\mathcal{F}$ is just the constant sheaf $\mathbb{Z}$ with the $\mathbb{Z}_2$-action of multiplication by $(-1)^n$ — denoted by $\mathbb{Z}(n)$. The universal n^{th} Real Chern class $\bar{c}_n$ is an element of $\widehat{H}^{2n}_{\mathbb{Z}_2}(BU; \mathbb{Z}(n))$. The $\mathbb{Z} \times \mathbb{Z}_2$-graded groups $\widehat{H}^*_{\mathbb{Z}_2}(-; \mathbb{Z}(*))$ will be denoted by $\mathbf{A}^{*,*}(-)$.

In Chapter 1 we identified a model for the Eilenberg-Mac Lane spectrum $H\underline{\mathbb{Z}}$ that represents $\mathbb{Z}_2$-equivariant cohomology with coefficients in the Mackey functor $\underline{\mathbb{Z}}$ — which is $RO(\mathbb{Z}_2)$-graded. In particular, we proved that $AG(S^{2n,n})$ is a classifying space the groups of the theory $H\underline{\mathbb{Z}}$ is the dimension $\mathbb{R}^{2n,n}$. Here we use the *motivic* notation, where, for $p \geq q$, $\mathbb{R}^{p,q}$ denotes the $\mathbb{Z}_2$-module $\mathbb{R}^p$ with the action of multiplication by -1 in the last q coordinates and $S^{p,q}$ is

$\mathbb{R}^{p,q} \cup \{\infty\}$. Also we will use the notation $H\underline{\mathbb{Z}}^{p,q}$ for the groups of the (reduced) theory $H\underline{\mathbb{Z}}^{*,*}$ in dimension $\mathbb{R}^{p,q}$. The reason for this change in notation is that the motivic notation is more compatible with the usual indexing of Galois-Grothendieck cohomology. As in Chapter 1, AG is the group of zero cycles.

It is proved in [4] that there is a ring isomorphism

$$H\underline{\mathbb{Z}}^{*,*}(BU) \cong R[\widetilde{c}_1, \ldots, \widetilde{c}_n, \ldots]$$

where $\widetilde{c}_n$ has dimension $(2n, n)$ and $R^{*,*}$ is the cohomology ring of a point. Since BU is a classifying space for (reduced) KR-theory the classes $\widetilde{c}_1, \widetilde{c}_2 \ldots$ are universal equivariant characteristic classes for Real vector bundles. We showed that the universal classes $\widetilde{c}_n$ can be defined by the following natural construction. The space $\mathcal{Z}$ of stabilized cycles is canonically a product

$$(1.23) \qquad \mathcal{Z} \cong \mathcal{Z}_0(\mathbb{P}^{\infty}_{\mathbb{C}}) \cong \prod_{n=0}^{\infty} AG(S^{2n,n})$$

and there exists a canonical $\mathbb{Z}_2$-equivariant map $P : BU \to \mathcal{Z}$. We proved that P classifies the total equivariant Chern class. That is, the class classified by the map $P : BU \to \mathcal{Z}$ is

$$1 + \widetilde{c}_1 + \widetilde{c}_2 + \cdots + \widetilde{c}_n + \cdots$$

At this point it is natural to try to relate the classes $\overline{c}_n$ defined by Kahn to the classes $\widetilde{c}_n$ defined in terms of the theory represented by $H\underline{\mathbb{Z}}$. This leads naturally to the following question. What is the relation between the theory represented by $H\underline{\mathbb{Z}}$ and the "theory" $\mathbf{A}^{*,*}$?

An answer to this question can be formulated in terms of a standard construction from equivariant homotopy theory. For based G-spaces X, Y the space of based maps $X \to Y$ is denoted by $F(X, Y)$; it has a G-action defined by conjugation. Given a G-spectrum k_G define

$$b(k_G) \stackrel{\text{def}}{=} F(EG_+, k_G).$$

We call $b(k_G)$ the **Borel analogue** of k_G. It is also a a G-spectrum. Let ϵ denote the map $EG \to *$. Then there is a transformation of G-spectra

$$\Psi = \epsilon^* : k_G \to F(EG_+, k_G) = b(k_G).$$

The following Theorem summarizes the main results of this Chapter

THEOREM 1.1. *For all p, q there exist natural transformations*

$$\Upsilon_{p,q} : b(H\underline{\mathbb{Z}})^{p,q}(X_+) \longrightarrow \mathbf{A}^{p,q}(X)$$

which are isomorphisms. Thus the groups of the theory $\mathbf{A}^{*,*}$ are groups of the Borel analogue of the equivariant cohomology theory $H\underline{\mathbb{Z}}^{*,*}$. The transformation Υ from $b(H\underline{\mathbb{Z}})^{*,*}$ to $\mathbf{A}^{*,*}$ preserves products. Moreover the Chern classes of these two theories are related by

$$(\Upsilon \circ \Psi)(\widetilde{c}_n) = \overline{c}_n.$$

We interpret this last statement as saying that the classes $\overline{c}_n$ of Kahn are the Borel type version of the classes $\widetilde{c}_n$ defined in terms of the theory $H\underline{\mathbb{Z}}$.

REMARK 1.1. The theory $\mathbf{A}^{*,*}(-)$ is $(\mathbb{Z} \times \mathbb{Z}_2)$-graded hence $\mathbf{A}^{p,q}(-)$ makes sense even for $q < 0$. This is implicitly used above. The equivariant cohomology theory $b(H\underline{\mathbb{Z}})^{*,*}$ is $(\mathbb{Z} \times \mathbb{Z})$-graded rather than is $(\mathbb{Z} \times \mathbb{Z}_2)$-graded; Theorem 1.1 shows that $b(H\underline{\mathbb{Z}})^{*,*}$ is periodic with period $(0, 2)$ so that it can be used to define a $(\mathbb{Z} \times \mathbb{Z}_2)$-graded theory. The theory $\mathbf{A}^{*,*}$ is exactly the $(\mathbb{Z} \times \mathbb{Z}_2)$-graded theory corresponding to $b(H\underline{\mathbb{Z}})^{*,*}$.

In general Borel cohomology classes are very crude invariants of group actions. In fact, if X and Y are G-spaces, a G-map $f : X \to Y$ which is an non-equivariant homotopy equivalence induces an equivariant homotopy equivalence

$$f \times \mathrm{id} : X \times EG \longrightarrow Y \times EG.$$

Therefore for any G-spectrum k_G the map $f \times \mathrm{id}$ gives an isomorphism

$$(f \times \mathrm{id})^* : b(k_G)^*(Y) \longrightarrow b(k_G)^*(X).$$

However, as a general principle, Borel type theories encode a lot of information about the fixed point set. It is therefore not very surprising that when restricted to spaces with trivial $\mathbb{Z}_2$-action both classes, $\widetilde{c}_n$ and $\overline{c}_n$ give the same information. We prove this by analyzing the restriction of the natural transformation $\Psi : H\underline{\mathbb{Z}} \to b(H\underline{\mathbb{Z}})$ to the fixed point set $H\underline{\mathbb{Z}}^{\mathbb{Z}_2}$.

Our result is the following

THEOREM 1.2. *Let Ψ denote the natural transformation from the theory $H\underline{\mathbb{Z}}^{*,*}$ to $b(H\underline{\mathbb{Z}})^{*,*}$ and let $p, q \geq 0$. Then, for any $\mathbb{Z}_2$-space X, Ψ includes $H\underline{\mathbb{Z}}^{p,q}(X^{\mathbb{Z}_2})$ naturally in $b(H\underline{\mathbb{Z}})^{p,q}(X^{\mathbb{Z}_2})$ as a direct summand.*

Since $\Upsilon_{p,q} : b(H\underline{\mathbb{Z}})^{p,q}(-) \to \mathbf{A}^{p,q}(-)$ is an isomorphism, we see that $\Upsilon \circ \Psi$ also includes $H\underline{\mathbb{Z}}^{p,q}(X^{\mathbb{Z}_2})$ naturally in $\mathbf{A}^{p,q}(X^{\mathbb{Z}_2})$ as a direct summand. Under the natural identification of $H\underline{\mathbb{Z}}^{2n,n}(BO)$ with a direct summand of $\mathbf{A}^{2n,n}(BO)$ we have

$$\widetilde{c}_n\big|_{BO} = \overline{c}_n\big|_{BO}$$

because $\overline{c}_n = (\Upsilon \circ \Psi)\widetilde{c}_n$.

The computation of the cohomology groups of a point, $R^{*,*} \stackrel{\text{def}}{=} H\underline{\mathbb{Z}}^{*,*}(S^0)$ shows that there is a natural isomorphism of functors between the restriction to spaces with trivial $\mathbb{Z}_2$-action of $X \mapsto H\underline{\mathbb{Z}}^{2n,n}(X)$ and the functor

$$X \mapsto \begin{cases} H^{2n}(X;\mathbb{Z}) \oplus \displaystyle\bigoplus_{k=1}^{n/2} H^{2n-2k}(X;\mathbb{Z}_2) & n \text{ even}, \\ \displaystyle\bigoplus_{k=0}^{(n-1)/2} H^{2n-2k-1}(X;\mathbb{Z}_2) & n \text{ odd}, \end{cases} \tag{1.24}$$

Using this isomorphism and Kahn's computation of $\overline{c}_n\big|_{BO}$ we obtain the following formula

$$\widetilde{c}_n\big|_{BO} = \begin{cases} p_{n/2} + Sq^{n-2}\omega_n + \cdots + Sq^2\omega_n + \omega_n & n \text{ even} \\ Sq^{n-1}\omega_n + \cdots + Sq^2\omega_n + \omega_n & n \text{ odd} \end{cases}$$

where p_k is the k^{th} Pontrjagin class and ω_k is the k^{th} Stiefel-Whitney class.

2. Galois-Grothendieck cohomology

2.1. Definitions. We start by recalling the definitions of Borel construction and Borel cohomology. Even though we only need these definitions in the case where the group is $\mathbb{Z}_2$ we give the definitions for an arbitrary finite group G since specializing to $\mathbb{Z}_2$ is not more elucidating.

DEFINITION 2.1. Let G be a finite group X be a G-space. The Borel construction of X is the quotient

$$X_G \overset{\text{def}}{=} X \times_G EG = \frac{X \times EG}{G}$$

where EG is a contractible free G-space. The Borel G-equivariant cohomology groups of X with coefficients in Γ are the groups $H^*(X_G;\Gamma)$. Borel cohomology is denoted by $\widehat{H}^*_G(X;\Gamma)$.

Observe that the natural projection $X_G \to BG$ is a fibration with fibre X. It is an important fact that a fixed point $x_0 \in X^G$ determines a section of this fibration. The section is defined by

$$s([t]) = [x_0, t] \qquad\qquad t \in EG$$

where $[t]$ denotes the class of t in $BG = EG/G$ and $[x_0, t]$ denotes the class of (x_0, t) in X_G.

Let X be a G-space and let $\mathcal{F}$ be a G-sheaf of abelian groups over X — that is, the sections of $\mathcal{F}$ over an open set U are G-abelian groups and the action is compatible with the restriction homomorphisms. In [**12**] the following construction is considered

CONSTRUCTION . Let $pr : X \times EG \to X$ be the first factor projection. Then

$$\mathcal{F}_G \stackrel{\text{def}}{=} pr^*(\mathcal{F})/G$$

— where we think of $\mathcal{F}$ as an étalé space — is a (non-equivariant) sheaf over X_G. The fact that G is finite is used to prove that $\mathcal{F}_G$ is a sheaf (more generally, this works if G is discrete).

A particular case of this construction which will be used throughout is the case where $G = \mathbb{Z}_2$ and $\mathcal{F}$ is the constant sheaf $\mathbb{Z}$ with the action of multiplication by $(-1)^n$. This equivariant sheaf is denoted by $\mathbb{Z}(n)$ and the associated non-equivariant sheaf is denoted by $\mathbb{Z}(n)_{\mathbb{Z}_2}$.

This construction can be used to define Galois-Grothendieck cohomology in terms of non-equivariant sheaf cohomology.

DEFINITION 2.2 ([**12**]). Let X be a G-space and let $\mathcal{F}$ be a G-sheaf of abelian groups over X. The Galois-Grothendieck G-equivariant cohomology groups of X with coefficients in $\mathcal{F}$ are defined as

$$\widehat{H}^*_G(X; \mathcal{F}) \stackrel{\text{def}}{=} H^*(X_G; \mathcal{F}_G).$$

There is a forgetful functor

$$j : \widehat{H}^*_G(X; \mathcal{F}) \longrightarrow H^*(X; \mathcal{F})$$

which is defined as restriction the fibre in $X \hookrightarrow X_G \to BG$.

This definition is equivalent to Grothendieck's original definition ([**10**]) of equivariant cohomology groups as the right derived functors of the functor of equivariant global sections.

NOTATION . For a $\mathbb{Z}_2$-space X let $\mathbf{A}^{*,*}(X)$ denote the direct sum

$$\bigoplus_{n,k} \widehat{H}^n_{\mathbb{Z}_2}(X;\, \mathbb{Z}(k)).$$

The pairing $\mathbb{Z}(k) \otimes \mathbb{Z}(k') \to \mathbb{Z}(k+k')$ induces a product on $\mathbf{A}^{*,*}(X)$ giving it a $(\mathbb{Z} \times \mathbb{Z}_2)$-graded ring structure. Also let $\mathcal{R}^{*,*} = \mathbf{A}^{*,*}(pt)$. We have $\mathcal{R}^{*,*} \cong \mathbb{Z}[\varepsilon]/(2\varepsilon)$ where ε has bidegree $(1,1)$ — see Section 5 for the computation the ring $\mathcal{R}^{*,*}$.

We can think of the groups $\mathbf{A}^{*,*}(X)$ as Borel cohomology groups with "twisted coefficients". More precisely, the groups $\mathbf{A}^{*,k}(X)$ are the cohomology groups of $X \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ with coefficients in the local system of groups defined by the locally constant sheaf $\mathbb{Z}(k) \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ over $X_{\mathbb{Z}_2}$.

2.2. The relation to Bredon cohomology. We will see that for a $\mathbb{Z}_2$-space X, the Galois-Grothendieck cohomology groups $\mathbf{A}^{*,*}(X)$ can also be defined as certain Bredon cohomology groups of $X \times E\mathbb{Z}_2$. First we need to review some facts about Bredon cohomology.

DEFINITION 2.3. Let G be a finite group. The orbit category of G is the category $\mathcal{O}_G$ whose objects are the orbit spaces G/H and whose morphisms are the G-maps $G/H \to G/K$. A contravariant coefficient system for G is a contravariant functor from $\mathcal{O}_G$ to the category of abelian groups.

Let X be a G-CW complex. There is a contravariant coefficient system

$$\underline{C}_n(X) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z}),$$

whose value on G/H is $H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z})$. Here X^n denotes the n^{th} skeleton of X. If $f : G/H \to G/K$ is a G-map and $f(eH) = gK$ then the map $x \mapsto gx$ sends X^K to X^H. The value of $\underline{C}_n(X)$ on the morphism f is the map induced on homology by this map. The connecting homomorphisms of the triples $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H)$ specify a morphism

$$d : \underline{C}_n(X) \to \underline{C}_{n-1}(X)$$

of contravariant coefficient systems satisfying $d^2 = 0$. Thus we have a chain complex $(\underline{C}_n(X), d)$ of contravariant coefficient systems. Then

$$C^n_G(X; M) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{O}_G}(\underline{C}_n(X), M), \text{with } \delta = \mathrm{Hom}_{\mathcal{O}_G}(d, \mathrm{id}),$$

is a cochain complex of abelian groups. The homology of this complex is the Bredon cohomology of X with coefficients in M, denoted by $H^*_G(X; M)$. In the case we are interested in M is the coefficient system $\underline{V}$ obtained from a G-module V as follows. The value of $\underline{V}$ on G/H is V^H. As before, if $f : G/H \to G/K$ is a G-map and $f(eH) = gK$ then the map $x \mapsto gx$ sends V^K to V^H. The value of $\underline{V}$ on the morphism f is this map.

If X is a free G-CW complex and V is a G-module then $C^n_G(X; \underline{V})$ is just the cochain complex of equivariant cochains with values in V.

We now return to the analysis of the groups $\mathbf{A}^{*,*}(X)$, for a $\mathbb{Z}_2$-space X.

Consider the usual cell decomposition of $E\mathbb{Z}_2$ with two cells in each dimension that are permuted by the $\mathbb{Z}_2$-action. Given an cell decomposition for X, give $X \times E\mathbb{Z}_2$ the product cell decomposition. Observe that there is a $1-1$ correspondence between equivariant cochains on $X \times E\mathbb{Z}_2$ with values in $\mathbb{Z}(k)$ and cochains in $\mathbb{Z}(k) \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ with values in $\mathbb{Z}(k) \times_{\mathbb{Z}_2} E\mathbb{Z}_2$. This last cochain complex computes the cohomology of $X \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ with coefficients in the local system of groups defined by the locally constant sheaf $\mathbb{Z}(k) \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ — see [**30**]. This proves

Proposition 2.1. *There is a natural isomorphism*

$$\mathbf{A}^{n,k}(X) \cong H^n_{\mathbb{Z}_2}(X \times E\mathbb{Z}_2; \underline{\mathbb{Z}(k)})$$

where $H^*_{\mathbb{Z}_2}(X \times E\mathbb{Z}_2; \underline{\mathbb{Z}}(k))$ denotes Bredon cohomology with coefficients in the contravariant coefficient system $\underline{\mathbb{Z}}(k)$ defined by the $\mathbb{Z}_2$-module $\mathbb{Z}(k)$.

Having identified the groups $\mathbf{A}^{*,*}(X)$ as certain Bredon cohomology groups we can now produce classifying spaces for them. We just need to give models for the equivariant Eilenberg-Mac Lane spaces corresponding to the coefficient systems $\underline{\mathbb{Z}}(k)$.

NOTATION . The group of zero cycles on the sphere S^n is denoted by $AG(S^n)$. We will use the notation $AG(S^n) \otimes \mathbb{Z}(k)$ to denote $AG(S^n)$ equipped with the action of multiplication by $(-1)^k$.

We denote the reduced groups of the "theory" $\mathbf{A}^{*,*}$ by $\widetilde{\mathbf{A}}^{*,*}$. For a based $\mathbb{Z}_2$-space (X, x_0) the elements of $\widetilde{\mathbf{A}}^{*,*}(X)$ are the classes in $\mathbf{A}^{*,*}(X)$ which restrict to zero on x_0.

PROPOSITION 2.2. *The space $AG(S^n) \otimes \mathbb{Z}(k)$ is a classifying space for (reduced) Bredon cohomology with $\underline{\mathbb{Z}}(k)$ coefficients. It follows that the spaces*

$$F(E\mathbb{Z}_{2+}, AG(S^n) \otimes \mathbb{Z}(k))$$

are classifying spaces for the Galois-Grothendiek (reduced) groups $\widetilde{\mathbf{A}}^{n,k}(-)$.

PROOF. Computing of the Bredon homotopy groups of $AG(S^n) \otimes \mathbb{Z}(k)$ we see that this space is an equivariant Eilenberg-Mac Lane space of type $K(\underline{\mathbb{Z}}(k), n)$. It follows that it classifies Bredon cohomology with $\underline{\mathbb{Z}}(k)$ coefficients in dimension n; see [**2**]. Since, for any $\mathbb{Z}_2$-space X,

$$\begin{aligned} [X_+, F(E\mathbb{Z}_{2+}, AG(S^n) \otimes \mathbb{Z}(k))]_{\mathbb{Z}_2} &\cong [(X \times E\mathbb{Z}_2)_+, AG(S^n) \otimes \mathbb{Z}(k)]_{\mathbb{Z}_2} \\ &\cong H^n_{\mathbb{Z}_2}(X \times E\mathbb{Z}_2; \underline{\mathbb{Z}}(k)), \end{aligned}$$

we see that $F(E\mathbb{Z}_{2+}, AG(S^n) \otimes \mathbb{Z}(k))$ is a classifying space for $\widetilde{\mathbf{A}}^{n,k}(-)$. □

NOTATION . From now on we will denote the classifying space

$$F(E\mathbb{Z}_{2+}, AG(S^p)\otimes \mathbb{Z}(q))$$

by $\mathbf{A}_{p,q}$.

REMARK 2.1. The pairings $\mathbb{Z}(q)\otimes\mathbb{Z}(q') \to \mathbb{Z}(q+q')$ and the smashing map $S^p \wedge S^{p'} \xrightarrow{\cong} S^{p+p'}$ induce a map

$$AG(S^p)\otimes \mathbb{Z}(q) \wedge AG(S^{p'})\otimes\mathbb{Z}(q') \xrightarrow{\overline{\wedge}} AG(S^{p+p'})\otimes\mathbb{Z}(q+q')$$

which represents the pairing

$$\widetilde{H}^*_{\mathbb{Z}_2}(\bullet;\mathbb{Z}(q))\otimes \widetilde{H}^*_{\mathbb{Z}_2}(\bullet;\mathbb{Z}(q')) \to \widetilde{H}^*_{\mathbb{Z}_2}(\bullet;\mathbb{Z}(q+q'))$$

in Bredon cohomology. This induces a map

$$\mathbf{A}_{p,q}\wedge \mathbf{A}_{p',q'} \xrightarrow{\overline{\wedge}_*} \mathbf{A}_{p+p',q+q'}$$

which represents the pairing of the groups $\mathbf{A}^{*,*}(-)$.

3. The theory $\mathbf{A}^{*,*}$ as a Borel type theory

In this section we prove that the functors $\mathbf{A}^{*,*}(-)$ are the cohomology groups of the theory represented by the Borel analogue of the $\mathbb{Z}_2$-spectrum $H\underline{\mathbb{Z}}$.

NOTATION . The following notation will be used throughout

1. For p, q non-negative integers such that $p \geq q$, $\mathbb{R}^{p,q}$ denotes the $\mathbb{Z}_2$-module $\mathbb{R}^{p-q}\times\mathbb{R}^q$ with the $\mathbb{Z}_2$-action of multiplication by -1 in the last q coordinates. As usual, $S^{p,q} = \mathbb{R}^{p,q}\cup\infty$.
2. The reduced cohomology groups of the theories represented by $H\underline{\mathbb{Z}}$ and $b(H\underline{\mathbb{Z}})$ in dimension $\mathbb{R}^{p,q}$ will be denoted by $H\underline{\mathbb{Z}}^{*,*}(-)$ and $b(H\underline{\mathbb{Z}})^{*,*}(-)$, respectively — see explanation below.

3. The spaces $AG(S^{p,q})$ and $F(E\mathbb{Z}_{2+}, AG(S^{p,q}))$ will be denoted by $H\underline{\mathbb{Z}}_{p,q}$ and $b(H\underline{\mathbb{Z}})_{p,q}$, respectively.

Recall from Chapter 1 that the correspondence

$$\mathbb{R}^{p,q} \mapsto H\underline{\mathbb{Z}}_{p,q}$$

defines an Ω-$\mathbb{Z}_2$-prespectrum $H\underline{\mathbb{Z}}$ that represents the $\mathbb{Z}_2$-equivariant cohomology theory $H_{\mathbb{Z}_2}(-;\underline{\mathbb{Z}})$ where $\underline{\mathbb{Z}}$ is Mackey functor constant at $\mathbb{Z}$. This theory is $(\mathbb{Z} \times \mathbb{Z})$-graded; the cohomology groups in dimension (p,q), $p,q \in \mathbb{Z}$, are denoted by $H^{p,q}_{\mathbb{Z}_2}(-;\underline{\mathbb{Z}})$.

Thus the spaces $H\underline{\mathbb{Z}}_{p,q}$ are classifying spaces for reduced cohomology groups $\widetilde{H}^{p,q}_{\mathbb{Z}_2}(-;\underline{\mathbb{Z}})$. Given a based $\mathbb{Z}_2$-space X we abbreviate $\widetilde{H}_{\mathbb{Z}_2}(X;\underline{\mathbb{Z}})$ by $H\underline{\mathbb{Z}}^{p,q}(X)$. The prespectrum $H\underline{\mathbb{Z}}$ has a product which is the biadditive extension of the map

$$S^{p,q} \wedge S^{p',q'} \xrightarrow{\cong} S^{p+p',q+q'}.$$

This product is denoted by $\widetilde{\wedge}$.

The *Borel analogue* of $H\underline{\mathbb{Z}}$ is the Ω-$\mathbb{Z}_2$-prespectrum $b(H\underline{\mathbb{Z}})$ defined by the correspondence

$$\mathbb{R}^{p,q} \mapsto b(H\underline{\mathbb{Z}})_{p,q}.$$

The theory represented by $b(H\underline{\mathbb{Z}})$ is also referred to as the *Borel analogue* of the cohomology theory $H\underline{\mathbb{Z}}^{*,*}(-)$. Its reduced groups are denoted by $b(H\underline{\mathbb{Z}})^{*,*}(X)$, for based $\mathbb{Z}_2$-spaces X. The spaces $b(H\underline{\mathbb{Z}})_{p,q}$ are classifying spaces for the functors $b(H\underline{\mathbb{Z}})^{p,q}(-)$. The product $\widetilde{\wedge}$ on $H\underline{\mathbb{Z}}$ induces a product on $b(H\underline{\mathbb{Z}})$ which we denote by $\widetilde{\wedge}_*$.

We will show that the groups $\mathbf{A}^{*,*}(-)$ are the cohomology groups of the Borel analogue theory $b(H\underline{\mathbb{Z}})^{*,*}(-)$ but first we need to establish facts about the groups $\mathbf{A}^{p,q}(S^{p,q})$. For this we use the fibration

$$\begin{tikzcd} S^{p,q} \arrow[r, hook, "i"] & S^{p,q}_{\mathbb{Z}_2} \arrow[d, "\pi"] \\ & B\mathbb{Z}_2 \end{tikzcd} \tag{3.25}$$

Associated to this fibration we have the Serre spectral sequence (see [**9**], for example)

$$E_2^{st} = H^s(B\mathbb{Z}_2;\, \mathcal{H}^t(S^{p,q};\, i^*\mathbb{Z}(q)_{\mathbb{Z}_2}) \Longrightarrow \mathbf{A}^{s+t,q}(S^{p,q}).$$

Here $\mathcal{H}^t(S^{p,q};\, i^*\mathbb{Z}(q)_{\mathbb{Z}_2}))$ the (locally constant) sheaf over $B\mathbb{Z}_2$ with sections

$$\mathcal{H}^t(S^{p,q};\, i^*\mathbb{Z}(q)_{\mathbb{Z}_2})(U) \stackrel{\text{def}}{=} H^t(\pi^{-1}(U); \mathbb{Z}(q)_{\mathbb{Z}_2})$$

over an open set U in $B\mathbb{Z}_2$. The stalk of this sheaf over a point $x \in B\mathbb{Z}_2$ is the group $H^t(\pi^{-1}(x); i^*\mathbb{Z}(q)_{\mathbb{Z}_2})$.

We claim that the sheaf $\mathcal{H}^p(S^{p,q};\, i^*\mathbb{Z}(q)_{\mathbb{Z}_2}))$ is actually the constant sheaf $\mathbb{Z}$. To see this, observe that $H^p(S^{p,q};\mathbb{Z}) \cong \mathbb{Z}$ and the action of $\mathbb{Z}_2$ induced from the action on $S^{p,q}$ is multiplication by $(-1)^q$. It follows that $\mathcal{H}^p(S^{p,q};\mathbb{Z})$ is the locally constant sheaf $\mathbb{Z}(q)_{\mathbb{Z}_2}$ and so

$$\mathcal{H}^p(S^{p,q};\, i^*\mathbb{Z}(q)_{\mathbb{Z}_2}) \cong \mathbb{Z}.$$

Note that the terms $E_2^{s,0}$ are the groups $\mathbf{A}^{s,q}(pt) \cong \mathcal{R}^{s,q}$. It is important to note that $\mathcal{R}^{*,*} \cong \mathbb{Z}[\varepsilon]/(2\varepsilon)$ where ε has bidegree $(1,1)$; this will be proved in Section 5. See figure 1 for a picture of the E_2 term in the case where p is even and q is odd.

Observe that, since $S^{p,q}$ has fixed points, the cohomology of the base injects in $\mathbf{A}^{*,q}(S^{p,q})$. In fact, a fixed point gives a section of the fibration (3.25) and so the map π^* has a left inverse. This implies that $E_2^{0,p}$ consists of universal cycles, *i.e.* they are killed by all differentials. Since these are obviously not

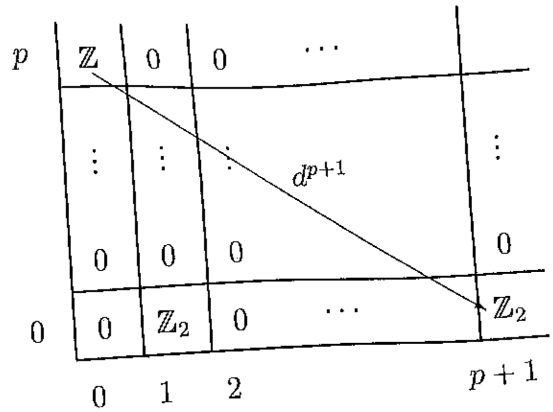

FIGURE 1. The E_2 term of the spectral sequence for the Galois-Grothendieck cohomology of $S^{p,q}$ in the case where p is even and q is odd.

in the image of any of the differentials, they survive to the term E_∞. Hence $\mathbf{A}^{p,q}(S^{p,q}) \cong \mathbb{Z}$ if $p-q$ is odd; if $p-q$ is even we get an exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbf{A}^{p,q}(S^{p,q}) \to \mathbb{Z} \to 0.$$

Thus, if $p-q$ is even

$$\mathbf{A}^{p,q}(S^{p,q}) \cong \mathbb{Z} \oplus \mathbb{Z}_2. \tag{3.26}$$

From now on we fix the splitting (3.26) by requiring that the generator of the summand $\mathbb{Z}$ be zero on the section σ_∞ of the fibration (3.25) defined by the fixed point $\infty \in S^{p,q}$ — see § 2. All this means is that the splitting (3.26) is the one obtained using the homomorphism σ_∞^*. We denote this generator by $\overline{\alpha}_{p,q}$. It is the unique p-class which is zero on the section mentioned above and whose restriction to the fibre is α_p, the generator of $H^p(S^{p,q};\mathbb{Z})$. In the case where $p-q$ is odd we also denote the generator of $\mathbf{A}^{p,q}(S^{p,q})$ by $\overline{\alpha}_{p,q}$.

In either case we have $\widetilde{\mathbf{A}}^{p,q}(S^{p,q}) \cong \mathbb{Z}$ and $\overline{\alpha}_{p,q}$ is a generator for this group. A classifying map $S^{p,q} \to \mathbf{A}_{p,q}$ extends to a group homomorphism

$$AG(S^{p,q}) = H\underline{\mathbb{Z}}_{p,q} \to \mathbf{A}_{p,q}$$

defining a class $\overline{\iota}_{p,q} \in \widetilde{\mathbf{A}}^{p,q}(H\underline{\mathbb{Z}}_{p,q})$ whose image under the forgetful map, j, is the generator ι_p of $\widetilde{H}^p(H\underline{\mathbb{Z}}_{p,q};\mathbb{Z}) \cong \mathbb{Z}$.

In summary we have shown

LEMMA 3.1. *Let σ_∞ be the section of* (3.25) *defined by the fixed point $\infty \in S^{p,q}$ and let α_p denote the generator of $H^p(S^{p,q};\mathbb{Z})$. There is a canonical isomorphism*

$$\mathbf{A}^{p,q}(S^{p,q}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & p-q \textit{ even} \\ \mathbb{Z} & p-q \textit{ odd.} \end{cases}$$

Under this isomorphism, the generator of the group $\mathbb{Z}$ is the only element $\overline{\alpha}_{p,q}$ of $\mathbf{A}^{p,q}(S^{p,q})$ such that $\overline{\alpha}_{p,q}\big|_{\sigma_\infty(B\mathbb{Z}_2)} = 0$ and whose fibre restriction is α_p.

In either case, $\widetilde{\mathbf{A}}^{p,q}(S^{p,q}) \cong \mathbb{Z}$ and $\overline{\alpha}_{p,q}$ denotes the generator of $\widetilde{\mathbf{A}}^{p,q}(S^{p,q})$. There exists a class $\overline{\iota}_{p,q} \in \widetilde{\mathbf{A}}^{p,q}(H\underline{\mathbb{Z}}_{p,q})$ such that $j(\overline{\iota}_{p,q})$ is the generator ι_p of $\widetilde{H}^p(H\underline{\mathbb{Z}}_{p,q};\mathbb{Z})$. Considering $S^{p,q}$ included in $H\underline{\mathbb{Z}}_{p,q}$ by the map $x \mapsto x - \infty$, we have $\overline{\iota}_{p,q}|_{S^{p,q}} = \overline{\alpha}_{p,q}$.

Using the classes $\overline{\iota}_{p,q}$ we will now construct a transformation Υ from the equivariant cohomology theory represented by $b(H\underline{\mathbb{Z}})$ — the Borel analogue of $H\underline{\mathbb{Z}}$ — to the "theory" $\widetilde{\mathbf{A}}^{*,*}$.

In the next Theorem we will need the following result. For any finite group G and any based G-spaces, X,Y and Z, a G-map $f : Y \to Z$ which is a non-equivariant homotopy equivalence induces a G-homotopy equivalence $f_* : F(X,Y) \to F(X,Z)$ — see [**24**].

THEOREM 3.1. *There are equivariant homotopy equivalences*

$$b(H\underline{\mathbb{Z}})_{p,q} \xrightarrow{\cong} \mathbf{A}_{p,q}.$$

This implies that there are natural transformations

$$\Upsilon_{p,q} : b(H\underline{\mathbb{Z}})^{p,q}(-) \longrightarrow \widetilde{\mathbf{A}}^{p,q}(-)$$

which are isomorphisms.

PROOF. By Proposition 2.2 the class $\bar{\iota}_{p,q}$ is classified by an equivariant map

$$\widetilde{g} : H\underline{\mathbb{Z}}_{p,q} \to \mathbf{A}_{p,q}$$

Let g be the corresponding map

$$H\underline{\mathbb{Z}}_{p,q} \wedge E\mathbb{Z}_{2+} \to AG(S^p) \otimes \mathbb{Z}(q)$$

— explicitly $g(x,t) = \widetilde{g}(x)(t)$. Recall that the fibre restriction of $\bar{\iota}_{p,q}$ is the generator of $\iota_p \in \widetilde{H}^p(H\underline{\mathbb{Z}}_{p,q};\mathbb{Z})$. This shows that g is a non-equivariant homotopy equivalence. It follows that

$$g_* : F(E\mathbb{Z}_{2+}, H\underline{\mathbb{Z}}_{p,q} \wedge E\mathbb{Z}_{2+}) \longrightarrow \mathbf{A}_{p,q}$$

is a $\mathbb{Z}_2$-homotopy equivalence. But there is a $\mathbb{Z}_2$-homotopy equivalence

$$F(E\mathbb{Z}_{2+}, H\underline{\mathbb{Z}}_{p,q} \wedge E\mathbb{Z}_{2+}) \xrightarrow{(\mathrm{id}\wedge\epsilon)_*} F(E\mathbb{Z}_{2+}, H\underline{\mathbb{Z}}_{p,q} \wedge S^0) \cong b(H\underline{\mathbb{Z}})_{p,q}$$

where ϵ is the map $E\mathbb{Z}_2 \to *$. The required equivariant homotopy equivalence is obtained by considering the composition

$$b(H\underline{\mathbb{Z}})_{p,q} \to F(E\mathbb{Z}_{2+}, H\underline{\mathbb{Z}}_{p,q} \wedge E\mathbb{Z}_{2+}) \xrightarrow{g_*} \mathbf{A}_{p,q}.$$

where the first map sends a function $F : E\mathbb{Z}_{2+} \to H\underline{\mathbb{Z}}_{p,q}$ to $F \wedge \mathrm{id}$ and id denotes the identity $E\mathbb{Z}_{2+} \to E\mathbb{Z}_{2+}$.

Since $b(H\underline{\mathbb{Z}})_{p,q}$ and $\mathbf{A}_{p,q}$ are classifying spaces for $b(H\underline{\mathbb{Z}})^{p,q}(-)$ and $\mathbf{A}^{p,q}(-)$, this equivariant homotopy equivalence defines a natural isomorphism $\Upsilon_{p,q}$ between these two functors. $\square$

REMARK 3.1. The class $\bar{\iota}_{p,q}$ is represented by the composition

$$H\underline{\mathbb{Z}}_{p,q} \xrightarrow{\epsilon^*} b(H\underline{\mathbb{Z}})_{p,q} \xrightarrow{\Upsilon_{p,q}} \mathbf{A}_{p,q}$$

where ϵ is the map $E\mathbb{Z}_2 \to *$ and we used the same letter to denote the transformation $\Upsilon_{p,q}$ and the $\mathbb{Z}_2$-homotopy equivalence used to define it. This is a consequence of the fact that the equivalence $\Upsilon_{p,q}$ is essentially defined by the map

$$\widetilde{g} : H\underline{\mathbb{Z}}_{p,q} \to \mathbf{A}_{p,q}$$

that classifies $\bar{\iota}_{p,q}$. It is now a matter unraveling the definition of the equivalence to see that $\Upsilon_{p,q} \circ \epsilon^* = \widetilde{g}$.

Recall that $b(H\underline{\mathbb{Z}})^{*,*}$ has a product induced from the product on $H\underline{\mathbb{Z}}^{*,*}$. Also $\mathbf{A}^{*,*}$ has a product coming from the pairing $\mathbb{Z}(q) \otimes \mathbb{Z}(q') \to \mathbb{Z}(q+q')$. We will now see that the transformation Υ of Theorem 3.1 is compatible with these products.

PROPOSITION 3.1. *The natural transformation*

$$\Upsilon : b(H\underline{\mathbb{Z}})^{*,*}(-) \to \mathbf{A}^{*,*}(-)$$

of Theorem 3.1 preserves products.

PROOF. Recall $\overline{\alpha}_{p,q}$ is the restriction of $\bar{\iota}_{p,q}$ to the sphere $S^{p,q} \subset AG(S^{p,q})$. Let $\widetilde{\alpha}_{p,q}$ denote the inclusion $S^{p,q} \subset AG(S^{p,q})$. The product $\overline{\alpha}_{p,q} \cdot \overline{\alpha}_{p',q'}$ is an element of $\widetilde{\mathbf{A}}^{p+p',q+q'}(S^{p+p',q+q'})$ whose image under the forgetful map j is the fundamental class of $S^{p+p',q+q'}$. It follows that

$$\overline{\alpha}_{p,q} \cdot \overline{\alpha}_{p',q'} = \overline{\alpha}_{p+p',q+q'}. \tag{3.27}$$

This implies that the diagram

$$\begin{array}{ccc} S^{p,q}\wedge S^{p',q'} & \xrightarrow{\widetilde{\alpha}_{p,q}\widetilde{\wedge}\widetilde{\alpha}_{p',q'}} & H\underline{\mathbb{Z}}_{p+p',q+q'} \\ \Big\downarrow{\scriptstyle \overline{\alpha}_{p,q}\wedge\overline{\alpha}_{p',q'}} & & \Big\downarrow{\scriptstyle \overline{\iota}_{p+p',q+q'}} \\ \mathbf{A}_{p,q}\wedge\mathbf{A}_{p',q'} & \xrightarrow{\overline{\wedge}_*} & \mathbf{A}_{p+p',q+q'} \end{array}$$

where $\overline{\wedge}_*$ is the product representing the paring in $\mathbf{A}^{*,*}$ — see Remark 2.1 — commutes up to equivariant homotopy. Since $H\underline{\mathbb{Z}}_{p,q} = AG(S^{p,q})$ and the maps $\widetilde{\wedge}$ and $\overline{\wedge}_*$ are biadditive it follows that the diagram

$$\begin{array}{ccc} H\underline{\mathbb{Z}}_{p,q}\wedge H\underline{\mathbb{Z}}_{p',q'} & \xrightarrow{\widetilde{\wedge}} & H\underline{\mathbb{Z}}_{p+p',q+q'} \\ \Big\downarrow{\scriptstyle \overline{\iota}_{p,q}\wedge\overline{\iota}_{p',q'}} & & \Big\downarrow{\scriptstyle \overline{\iota}_{p+p',q+q'}} \\ \mathbf{A}_{p,q}\wedge\mathbf{A}_{p',q'} & \xrightarrow{\overline{\wedge}_*} & \mathbf{A}_{p+p',q+q'} \end{array}$$

also commutes up to equivariant homotopy. This shows that the transformation Υ sends products to products. □

4. The relation between the Chern classes of the two theories

In the previous Section we related $\mathbf{A}^{*,*}$ to the equivariant cohomology theory represented by the equivariant spectrum $b(H\underline{\mathbb{Z}})$. Recall that the map $\epsilon : E\mathbb{Z}_2 \to *$ induces a map $\epsilon^* : H\underline{\mathbb{Z}} \to b(H\underline{\mathbb{Z}})$ and a corresponding transformation of equivariant cohomology theories

$$\Psi : H\underline{\mathbb{Z}}^{*,*} \to b(H\underline{\mathbb{Z}})^{*,*}.$$

We would like to say that the Chern classes $\overline{c}_n$ defined by Kahn are the Borel version of the classes $\widetilde{c}_n$ of the theory $H\underline{\mathbb{Z}}^{*,*}$, *i.e.* $\overline{c}_n = \Psi(\widetilde{c}_n)$. Unfortunately is not exactly true because the classes $\overline{c}_n$ live in $\mathbf{A}^{*,*}$ instead of $b(H\underline{\mathbb{Z}})^{*,*}$. To be precise we have to compose Ψ with the natural equivalence $\Upsilon_{2n,n}$, *i.e.*

$$\overline{c}_n = (\Upsilon \circ \Psi)\widetilde{c}_n.$$

Anyway, since Υ is just the projection to the $(\mathbb{Z}\times\mathbb{Z}_2)$-graded theory associated to $b(H\underline{\mathbb{Z}})^{*,*}$ we can still interpret this equality as saying that $\overline{c}_n$ is the Borel type version of $\widetilde{c}_n$.

Let us briefly review the definitions of the Chern classes in both theories.

A Real vector bundle over a $\mathbb{Z}_2$-space X is a complex vector bundle $E \xrightarrow{\pi} X$ with an anti-linear homomorphism $\lambda : E \to E$ such that $\lambda^2 = \mathrm{id}$ and so that π is equivariant *w.r.t.* the $\mathbb{Z}_2$-action defined by λ. This notion was first introduced by Atiyah in [**1**]. The space $BU(n)$ with the $\mathbb{Z}_2$-action induced by complex conjugation of planes is the classifying space for rank n Real vector bundles; see [**5**]. The classes $\widetilde{c}_n$ and $\overline{c}_n$ are classes in $H\underline{\mathbb{Z}}^{*,*}(BU)$ and $\mathbf{A}^{*,*}(BU)$ respectively. Hence they are universal characteristic classes for Real vector bundles in each of these theories.

The class $\widetilde{c}_n$ is an element of $H\underline{\mathbb{Z}}^{2n,n}(BU)$ whose image under the forgetful map $\Phi : H\underline{\mathbb{Z}}^{2n,n}(BU) \to H^{2n}(BU;\mathbb{Z})$ is the non-equivariant Chern class c_n. There is a ring isomorphism

$$H\underline{\mathbb{Z}}^{*,*}(BU) \cong R[\widetilde{c}_1,\ldots,\widetilde{c}_n,\ldots]$$

where and $R^{*,*}$ is the cohomology ring of a point. These facts are proved in [**4**].

The classes $\overline{c}_n$ are defined by the following result of Kahn.

THEOREM 4.1. ([**12**]) *There are classes $\overline{c}_i$ in $\mathbf{A}^{2i,i}(BU)$ such that the image by the forgetful map, $j(\overline{c}_i)$, is the i^{th} Chern class, c_i and*

$$\mathbf{A}(BU) \cong \mathcal{R}[\overline{c}_1,\ldots,\overline{c}_n,\ldots] \tag{4.28}$$

where $\mathcal{R}^{*,*} = \mathbf{A}^{*,*}(pt)$.

We can now prove

Proposition 4.1. *Up to composition with the natural equivalence* $\Upsilon_{2n,n}$, *the Chern classes* $\overline{c}_n \in \mathrm{A}^{2n,n}(BU)$ *are the Borel version of the Chern classes* $\widetilde{c}_n$ *of the theory* $H\underline{\mathbb{Z}}^{*,*}$. *That is*

$$\overline{c}_n = (\Upsilon \circ \Psi)(\widetilde{c}_n).$$

Proof. From Theorem 4.1 we see that the classes $\overline{c}_n$ are characterized by the following properties:

1. The fibre restriction, $j(\overline{c}_n)$ is the non-equivariant Chern class c_n.
2. $\overline{c}_n\big|_{BU(n-1)} = 0$
3. $\overline{c}_1$ restricts to zero on the sections of $BU(1)_{\mathbb{Z}_2} \to B\mathbb{Z}_2$ defined by the points of $\mathbb{P}^\infty_{\mathbb{R}} = BU(1)^{\mathbb{Z}_2}$.

Let $P_n : BU \to AG(S^{2n,n})$ be a classifying map for $\widetilde{c}_n$ and let $\widetilde{\iota}_{2n,n}$ denote the universal $H\underline{\mathbb{Z}}^{2n,n}$ class. Then $(\Upsilon \circ \Psi)(\widetilde{c}_n) = P_n^*(\Upsilon \circ \Psi)(\widetilde{\iota}_{2n,n})$. By Remark 3.1 we have $(\Upsilon \circ \Psi)(\widetilde{\iota}_{2n,n}) = \overline{\iota}_{2n,n}$, so

$$(\Upsilon \circ \Psi)(\widetilde{c}_n) = P_n^*(\overline{\iota}_{2n,n}).$$

Since $\overline{\iota}_{2,1}$ restricts to zero on the sections of $AG(S^{2,1})_{\mathbb{Z}_2} \to B\mathbb{Z}_2$ defined by the fixed points it follows that $P_1^*(\overline{\iota}_{2,1})$ also restricts to zero on the sections of $BU(1)_{\mathbb{Z}_2} \to B\mathbb{Z}_2$ defined by the points of $\mathbb{P}^\infty_{\mathbb{R}}$. From $\widetilde{c}_n\big|_{BU(n-1)} = 0$ we see that $(\Upsilon \circ \Psi)(\widetilde{c}_n)|_{BU(n-1)} = 0$. Finally, the image of $\widetilde{c}_n$ under the forgetful map from $H\underline{\mathbb{Z}}^{*,*}$ to integer cohomology is the non-equivariant Chern class, hence $j((\Upsilon \circ \Psi)(\widetilde{c}_n)) = c_n$. This completes the proof. □

5. The restriction of Ψ to spaces with trivial $\mathbb{Z}_2$-action

In this section we analyze the restriction of the transformation Ψ from $H\underline{\mathbb{Z}}^{*,*}$ to $b(H\underline{\mathbb{Z}})^{*,*}$ to spaces with trivial action. In terms of spectra this corresponds to the restriction of ϵ^* to the fixed point set:

$$H\underline{\mathbb{Z}}^{\mathbb{Z}_2} \xrightarrow{\epsilon^*} b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}.$$

The space level map is

$$H\underline{\mathbb{Z}}_{p,q} \xrightarrow{\epsilon^*} F(E\mathbb{Z}_{2+}, H\underline{\mathbb{Z}}_{p,q})^{\mathbb{Z}_2}. \tag{5.29}$$

This is an instance of a following general situation. For a G-space X, the set $F(EG_+, X)^G$ is called the homotopy fixed point set of X. The map $\epsilon : EG \to *$ induces $\epsilon^* : X \to F(EG_+, X)$. When restricted to X^G the map ϵ^* gives an inclusion of the fixed point set into the homotopy fixed point set. This inclusion has been the object of intense study in equivariant homotopy theory.

In (5.29) both spaces are topological abelian groups and hence they are products of Eilenberg-Mac Lane spaces. Also ϵ^* is a group homomorphism. This implies that its homotopy class of is completely determined by the induced homomorphism on the homotopy groups.

Let us compute the homotopy groups of $b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}_{p,q}$. These are cohomology groups of a point for the theory $\mathbf{A}^{*,*}$:

$$\pi_t b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}_{p,q} \cong b(H\underline{\mathbb{Z}})^{p,q}(S^{t,0}) \cong b(H\underline{\mathbb{Z}})^{p-t,q}(S^0) \cong \mathbf{A}^{p-t,q}(pt).$$

For q even we have

$$\mathbf{A}^{p,q}(pt) = H^p(B\mathbb{Z}_2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0, \\ \mathbb{Z}_2, & p > 0 \text{ and } p \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, as a ring we have, $\mathbf{A}^{*,0}(pt) \cong \mathbb{Z}[\theta]/(2\theta)$ where $\theta \in \mathbf{A}^{2,0}(pt)$.

In the case of q is odd we have $\mathbf{A}^{p,q}(pt) = H^p_{\mathbb{Z}_2}(E\mathbb{Z}_2; \underline{\mathbb{Z}}(1))$ where this last group is a Bredon cohomology group and $\underline{\mathbb{Z}}(1)$ is the coefficient system associated to the $\mathbb{Z}_2$-module $\mathbb{Z}(1)$ — see § 2.2. Thus we need to compute homology of the cochain complex $C^*_{\mathbb{Z}_2}(E\mathbb{Z}_2; \underline{\mathbb{Z}}(1))$.

Give $E\mathbb{Z}_2$ its usual cell decomposition with two cells in each dimension s which are permuted by the $\mathbb{Z}_2$-action. To indicated this fact we denote the cells by D^s and τD^s where τ is the generator of $\mathbb{Z}_2$. We have $\partial D^s = D^{s-1} + (-1)^s \tau D^{s-1}$ and $\partial\tau = \tau\partial$. Let $\{e_s^+, e_s^-\}$ be the basis dual to $\{D^s, \tau D^s\}$. It follows that there are no cocycles in even dimensions and, for odd dimensions s, the cocycles are $\mathbb{Z}\cdot\{e_s^+ - e_s^-\}$. Also, $\delta(e_{s-1}^+ - e_{s-1}^-) = 2(e_s^+ - e_s^-)$ so the homology of $C^*_{\mathbb{Z}_2}(E\mathbb{Z}_2; \mathbb{Z}(1))$ is zero in even dimensions and $\mathbb{Z}_2$ in odd dimensions. Thus we have

$$H^p_{\mathbb{Z}_2}(E\mathbb{Z}_2; \underline{\mathbb{Z}}(1)) = \begin{cases} \mathbb{Z}_2, & p > 0 \text{ and } p \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Let ε be the generator of $\mathbf{A}^{1,1}(pt)$. Then $\varepsilon^2 \in \mathbf{A}^{2,0}(pt)$ so either $\varepsilon^2 = 0$ or $\varepsilon^2 = \theta$. It is easy to see that the reduction $\mod 2$ of ε is the generator of $H^1(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and so $\varepsilon^2 = \theta$. We conclude that

$$\mathbf{A}^{*,*}(pt) \cong \mathbb{Z}[\varepsilon]/(2\varepsilon)$$

Notation . Recall that R denotes the cohomology ring of a point for the $\mathbb{Z}_2$-equivariant cohomology represented by $H\underline{\mathbb{Z}}$ and $\mathcal{R}$ denotes $\mathbf{A}^{*,*}(pt)$. We will denote the cohomology groups of a point in the theory $b(H\underline{\mathbb{Z}})^{*,*}$ by $b(R)$. The subrings of R and $b(R)$, consisting of elements in dimensions (p,q) with $q \geq 0$ will be denoted by R_+ and $b(R)_+$, respectively.

It is proved in **[4]** — see also **[16]** — that $R_+ \cong \mathbb{Z}[x,y]$ where x has degree $(0,2)$ and y has degree $(1,1)$. Our next objective is to compute the ring $b(R)_+$ and the ring homomorphisms

$$\Psi : R_+ \to b(R)_+ \quad \text{and} \quad \Upsilon : b(R)_+ \to \mathcal{R}.$$

Lemma 5.1. *Under the isomorphism $R_+ \cong \mathbb{Z}[x,y]/(2y)$ mentioned above we have: $\Psi(x)$ is a generator for $b(R)^{0,2} \cong \mathbb{Z}$ and $\Psi(y)$ is a generator for*

$b(R)^{1,1} \cong \mathbb{Z}_2$. *Hence*

$$\Upsilon \circ \Psi(x) = \pm 1 \in \mathcal{R}^{0,0} \quad \textit{and} \quad \Upsilon \circ \Psi(y) = \varepsilon \in \mathcal{R}^{1,1}.$$

PROOF. Observe that

$$\begin{aligned} R^{0,2} &\cong H\underline{\mathbb{Z}}^{4,2}(S^{4,0}) \cong \pi_4(H\underline{\mathbb{Z}}_{4,2}^{\mathbb{Z}_2}) \\ b(R)^{0,2} &\cong b(H\underline{\mathbb{Z}})^{4,2}(S^{4,0}) \cong \pi_4(b(H\underline{\mathbb{Z}})_{4,2}^{\mathbb{Z}_2}). \end{aligned}$$

Under these isomorphisms, the transformation $\Psi_{0,2}$ is the map induced on π_4 by the non-equivariant homotopy equivalence $\epsilon^* : H\underline{\mathbb{Z}}_{4,2} \to b(H\underline{\mathbb{Z}})_{4,2}$. In [**16**] it is proved that the inclusion

$$H\underline{\mathbb{Z}}_{4,2}^{\mathbb{Z}_2} \longrightarrow H\underline{\mathbb{Z}}_{4,2}$$

induces an isomorphism on π_4. It follows that $\Psi(x)$ is not divisible hence it is a generator for $b(R)^{0,2}$.

The proof of the assertion regarding $\Psi(y)$ is similar. Observe that

$$\begin{aligned} R^{1,1} &\cong H\underline{\mathbb{Z}}^{2,1}(S^{1,0}) \cong \pi_1(H\underline{\mathbb{Z}}_{2,1}^{\mathbb{Z}_2}) \\ b(R)^{1,1} &\cong b(H\underline{\mathbb{Z}})^{2,1}(S^{1,0}) \cong \pi_1(b(H\underline{\mathbb{Z}})_{2,1}^{\mathbb{Z}_2}). \end{aligned}$$

Now, $H\underline{\mathbb{Z}}_{1,1}$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathbb{P}_{\mathbb{C}}^{\infty}$ and so the inclusion $H\underline{\mathbb{Z}}_{1,1}^{\mathbb{Z}_2}$ in $H\underline{\mathbb{Z}}_{1,1}$ is not null-homotopic — in fact, this inclusion is homotopy equivalent the inclusion of $\mathbb{P}_{\mathbb{R}}^{\infty}$ in $\mathbb{P}_{\mathbb{C}}^{\infty}$. Again we use the fact that ϵ^* is a non-equivariant homotopy equivalence. This implies that

$$\epsilon^* : H\underline{\mathbb{Z}}_{2,1}^{\mathbb{Z}_2} \longrightarrow b(H\underline{\mathbb{Z}})_{2,1}^{\mathbb{Z}_2}$$

is not null-homotopic. Since both these spaces are $K(\mathbb{Z}_2, 1)$ spaces, it follows that ϵ^* is an isomorphism on π_1 and this means that $\Psi(y)$ is a generator for $b(R)^{1,1}$.

The last statement follows from the fact that $\Upsilon_{p,q}$ is an isomorphism for all $p, q \geq 0$. $\square$

This computes the ring homomorphism $R_+ \xrightarrow{\Psi} \mathcal{R}$. We will now use it to compute the ring $b(R)_+$. Recall that we already know this ring additively since

$$\Upsilon_{p,q} : b(R)^{p,q} \to \mathbf{A}^{p,q}(pt)$$

is an isomorphism, for all p, q.

LEMMA 5.2. *Let $\chi = \Psi(x)$, $\varepsilon_1 = \Psi(y)$ and let ε_2 be the generator of $b(R)^{2,0}$. Then there is a ring isomorphism*

$$b(R)_+ \cong \mathbb{Z}[\chi, \varepsilon_1, \varepsilon_2]/(2\varepsilon_1, 2\varepsilon_2, {\varepsilon_1}^2 - \chi\varepsilon_2).$$

PROOF. This follows immediately from the previous Lemma because Υ is a ring homomorphism and $\Upsilon_{p,q} : b(R)^{p,q} \to \mathcal{R}^{p,q}$ is an isomorphism. $\square$

In particular this shows

COROLLARY 5.1. *The transformation Ψ includes R_+ in $b(R)_+$ as a direct summand.*

Finally, we have the following description of the restriction of the transformation Ψ to spaces with trivial $\mathbb{Z}_2$-action and dimensions (p, q) with $q \geq 0$.

THEOREM 5.1. *Let Ψ denote the natural transformation from the theory $H\underline{\mathbb{Z}}^{*,*}$ to $b(H\underline{\mathbb{Z}})^{*,*}$ and let $q \geq 0$. Then, for any $\mathbb{Z}_2$-space X, Ψ includes $H\underline{\mathbb{Z}}^{p,q}(X^{\mathbb{Z}_2})$ naturally in $b(H\underline{\mathbb{Z}})^{p,q}(X^{\mathbb{Z}_2})$ as a direct summand.*

PROOF. Observe that, given a space X with trivial $\mathbb{Z}_2$-action, we have

$$H\underline{\mathbb{Z}}^{p,q}(X) \cong [X, H\underline{\mathbb{Z}}_{p,q}]_{\mathbb{Z}_2} \cong [X, H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{p,q}]$$

and

$$b(HZ)^{p,q}(X) \cong [X, b(H\underline{\mathbb{Z}})_{p,q}]_{\mathbb{Z}_2} \cong [X, b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}_{p,q}].$$

Now, both $H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{p,q}$ and $b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}_{p,q}$ are topological abelian groups and hence, they are products of Eilenberg-Mac Lane spaces. The transformation Ψ is the map induced from

$$\epsilon^* : H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{p,q} \to F(E\mathbb{Z}_{2+}, H\underline{\mathbb{Z}}_{p,q})^{\mathbb{Z}_2}$$

which is a group homomorphism. It follows that, Ψ is completely determined by the map induced by ϵ^* on homotopy groups. Our computation of the effect of Ψ on the groups of a point shows that ϵ^* includes the product of Eilenberg-Mac Lane spaces $H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{p,q}$ into $b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}_{p,q}$ as a direct summand. This completes the proof. $\square$

REMARK 5.1. Since, for any $p, q \geq 0$ the natural transformation $\Upsilon_{p,q} : b(H\underline{\mathbb{Z}})^{p,q}(-) \to \mathbf{A}^{p,q}(-)$ is an isomorphism, it also follows from the Theorem above that, for any $\mathbb{Z}_2$-space X, $\Upsilon \circ \Psi$ includes $H\underline{\mathbb{Z}}^{p,q}(X^{\mathbb{Z}_2})$ naturally in $\mathbf{A}^{p,q}(X^{\mathbb{Z}_2})$ as a direct summand.

To conclude we will give a formula for the restriction of the equivariant Chern classes $\widetilde{c}_n$ of the theory $H\underline{\mathbb{Z}}^{*,*}$. Recall that $\pi_t(H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{2n,n}) \cong R^{2n-t,n}$ so, from $R_+ \cong \mathbb{Z}[x, y]$, with $deg(x) = (0, 2)$ and $deg(y) = (1, 1)$, it follows that $H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{2n,n}$ has the homotopy type of

$$\begin{cases} K(\mathbb{Z}, 2n) \times K(\mathbb{Z}_2, 2n-2) \times \cdots \times K(\mathbb{Z}_2, n) & n \text{ even} \\ K(\mathbb{Z}_2, 2n-1) \times K(\mathbb{Z}_2, 2n-3) \times \cdots \times K(\mathbb{Z}_2, n) & n \text{ odd.} \end{cases} \tag{5.30}$$

This decomposition shows that there is a natural isomorphism of functors between the restriction to spaces with trivial $\mathbb{Z}_2$-action of $X \mapsto H\underline{\mathbb{Z}}^{2n,n}(X)$

and the functor

$$X \mapsto \begin{cases} H^{2n}(X;\mathbb{Z}) \oplus \bigoplus_{k=1}^{n/2} H^{2n-2k}(X;\mathbb{Z}_2) & n \text{ even}, \\ \bigoplus_{k=0}^{(n-1)/2} H^{2n-2k-1}(X;\mathbb{Z}_2) & n \text{ odd}, \end{cases} \tag{5.31}$$

Similarly, the computation of the groups $\mathbf{A}^{2n-*,n}(pt) \cong b(H\underline{\mathbb{Z}})^{2n-*,n}(S^0)$ shows that the topological abelian group $b(H\underline{\mathbb{Z}})^{\mathbb{Z}_2}_{2n,n}$ has the homotopy type of

$$\begin{cases} H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{2n,n} \times K(\mathbb{Z}_2, n-2) \times \cdots \times K(\mathbb{Z}_2, 0) & n \text{ even}, \\ H\underline{\mathbb{Z}}^{\mathbb{Z}_2}_{2n,n} \times K(\mathbb{Z}_2, n-2) \times \cdots \times K(\mathbb{Z}_2, 1) & n \text{ odd}, \end{cases} \tag{5.32}$$

and this shows that there is a natural isomorphism of functors between the restriction to spaces with trivial $\mathbb{Z}_2$-action of $X \mapsto \mathbf{A}^{2n,n}(X)$ and

$$X \mapsto \begin{cases} H^{2n}(X;\mathbb{Z}) \oplus \bigoplus_{k=1}^{n} H^{2n-2k}(X;\mathbb{Z}_2) & n \text{ even}, \\ \bigoplus_{k=0}^{n-1} H^{2n-2k-1}(X;\mathbb{Z}_2) & n \text{ odd}, \end{cases} \tag{5.33}$$

By Theorem 5.1, under the equivalences (5.31) and (5.33) the transformation $\Psi^{\mathbb{Z}_2}$ is inclusion as a direct summand.

We now recall Kahn's computation of $\overline{c}_n\big|_{BO}$.

THEOREM 5.2. ([**12**]) *The restriction of the equivariant Chern classes, $\overline{c}_n$, to BO is*

$$\begin{cases} p_{n/2} + \beta(\epsilon \cdot Sq^{n-2}\omega_n + \epsilon^3 \cdot Sq^{n-4}\omega_n + \cdots + \epsilon^{n-1} \cdot \omega_n), & n \textit{ even} \\ \beta'(Sq^{n-1}\omega_n + \epsilon^2 \cdot Sq^{n-3}\omega_n + \cdots + \epsilon^{n-1} \cdot \omega_n), & n \textit{ odd} \end{cases}$$

Where ω_n is the n^{th} Stiefel-Whitney class, $p_{n/2}$ is the Pontrjagin class and β, β' denote the coboundary maps associated to the sequences of $\mathbb{Z}_2$-groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}(1) \xrightarrow{\times 2} \mathbb{Z}(1) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

It is easy to see that in terms of the equivalence (5.33) this reads

$$\overline{c}_n\big|_{BO} = \begin{cases} p_{n/2} + Sq^{n-2}\omega_n + \cdots + Sq^2\omega_n + \omega_n & n \text{ even} \\ Sq^{n-1}\omega_n + \cdots + Sq^2\omega_n + \omega_n & n \text{ odd} \end{cases}$$

Since $\overline{c}_n = (\Upsilon \circ \Psi)\widetilde{c}_n$ we see that, under the isomorphism of functors (5.31), this is the formula for $\widetilde{c}_n\big|_{BO}$.

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