Hyperinvariant Subspaces and Structure Theory for K - tuples of Commuting Operators on Finite Dimensional Spaces

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Abstract of the Dissertation

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In this dissertation, we consider structure theory and similarity questions for a k-tuple of commuting nilpotent operators on a finite dimensional space V. To this end, we define a lattice $\mathcal{L}$ of subspaces and then a map $L : V \to \mathcal{L}$ which takes a vector onto the smallest element of $\mathcal{L}$ which contains it. We find that, in many cases, it is easier to work with the elements of the lattice $\mathcal{L}$ than the original vectors.

Our principal result asserts that a k-tuple of commuting nilpotents is determined up to similarity by its lattice of hyperinvariant subspaces $\mathcal{H}_{\mathcal{A}}$ together with a map from the matrix valued polynomials to $\mathcal{H}_{\mathcal{A}}$.

In the final chapter, we discuss a class of k-tuples for which there exists a structure theory strongly resembling Jordan theory.

For my parents

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Introduction

One begins the analysis of the structure of a single linear operator P on a finite dimensional vector space V by decomposing V into a direct sum of generalized eigenspaces for P. On each of these generalized eigenspaces, P acts as the sum of a scalar and a nilpotent. The study of structure theory for a single operator therefore reduces to the study of structure theory for nilpotents.

One would hope to begin the study of the structure of an ordered k-tuple of commuting operators $(T_1, \ldots, T_k)$ on a finite dimensional space V with an analogous result. [4]

Proposition 0.1 *Given operators $T_1, \ldots, T_k : V \to V$, $dim(V) = n$, such that $T_iT_j = T_jT_i$, for all i, j, then V can be expressed as the direct sum $V = \oplus V_{\lambda_{\mathbf{j}}}$, where $\lambda_{\mathbf{j}} = (\lambda_{1,j}, \ldots, \lambda_{k,j}) \in \mathbf{C}^k$, $V_{\lambda_{\mathbf{j}}}$ invariant under all the T_i and $(T_i - \lambda_{i,j}) |_{V_{\lambda_{\mathbf{j}}}}$ nilpotent.*

Further, there exists a basis in which the nilpotent part of all the T_i lies strictly above the diagonal and the diagonalizable part of all the T_i form diagonal matrices.

We will therefore restrict our attention to ordered k-tuples of commuting nilpotents $\mathbf{A} = (A_1, \ldots, A_k)$, where $A_1, \ldots, A_k : V \to V$ and $dim(V) = n < \infty$.

Ideally, a structure theory would answer many of the multivariable analogs of the questions answered by the elegant Jordan theory. In particular, we have the following goals in mind:

1. Classify ordered k-tuples of nilpotents up to joint similarity. i.e. Given two ordered k-tuples of commuting nilpotents $\mathbf{A} = (A_1, \ldots, A_k)$ and $\mathbf{A}' = (A'_1, \ldots, A'_k)$, where $A_i : V \to V$ and $A'_i : V' \to V'$, determine the existence of an invertible $S : V \to V'$ such that $A'_i = SA_iS^{-1}$, for all $i \in \{1, \ldots, k\}$.

2. Determine a direct sum decomposition of the space $V = V_1 \oplus \ldots \oplus V_k$, where V_i is both invariant for all the A_i and indecomposable under their joint action.

3. Clarify the action of a k-tuple of nilpotents on an indecomposable space V.

Such a structure theory is still beyond our reach. However, we develop machinery to aid the study of such k-tuples, in general, and analyze completely some important special cases. We begin by making some preliminary algebraic remarks.

The single variable analog of the second goal above, is the decomposition of a space V into the subspaces upon which the individual Jordan blocks act. We recall that this decomposition is not unique, but Jordan theory tells us that every decomposition is 'essentially' the same. One would hope for the same phenomenon in the multi-variable case. This is the essence of the following:

Theorem 0.2 (Krull-Schmidt) *If $\mathcal{S}$ is a set of operators on a finite dimensional space V,*

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_k = V'_1 \oplus V'_2 \oplus \ldots \oplus V'_l$$

are two decompositions of V into non-trivial subspaces invariant under the operators in $\mathcal{S}$ and indecomposable, then $k = l$, and if the V'_i are suitably ordered, then there exist invertible $S_i : V_i \to V'_i$, such that for each A in $\mathcal{S}$, $A |_{V_i} S_i = S_iA |_{V_i}$, $i = 1, \ldots, k$.

We also make note of a few basic differences between the single and multi-variable cases.

Proposition 0.3 *If $\mathcal{A}$ is the algebra generated by the operator A, $A : V \to V$, $dimV = n$, $A^k = 0$, then the following are equivalent:*

1. V is indecomposable under $\mathcal{A}$.

2. $\mathcal{A} = \mathcal{A}'$ (Where $\mathcal{A}'$ is the commutant of $\mathcal{A}$).

3. $\mathcal{A}$ has a cyclic vector.

For the algebra $\mathcal{A}$ generated by the k - tuple of commuting operators $\mathbf{A}$, the conditions are no longer equivalent. However we have

Proposition 0.4 *3) $\Rightarrow$ 2) $\Rightarrow$ 1).*

Neither of these implications is reversible.

We can determine the number of Jordan blocks for a single operator $A : V \to V$ by finding the codimension of the image of A in V. The above observation indicates that the determination of the number of indecomposable subspaces in a direct sum decomposition is more complicated in the multi variable case.

The problems of classifying k-tuples of commuting operators up to joint similarity, and the determination of the lattice of hyperinvariant subspaces for such a k-tuple are closely intertwined.

Fillmore, et al. [3] characterize the lattice of hyperinvariant subspaces $\mathcal{H}_{\mathcal{A}}$ for a single linear operator $A : V \to V$. They begin by decomposing V into a direct sum of generalized eigenspaces for A. This reduces the problem to the description of the lattice of hyperinvariant subspaces induced by a nilpotent operator. This simpler problem is solved by characterizing every hyperinvariant subspace as the span of some subset of the basis vectors in a given Jordan Basis.

In section 1, we parallel some of the results in [3] by developing a theory for a related class of lattices which include the lattices of hyperinvariant subspaces.

Clearly, the lattice structure of $\mathcal{H}_{\mathcal{A}}$ alone is insufficient to determine A up to similarity.

Example 0.1

Consider $A : V \to V$, where A is nilpotent of order 3 and $\dim V = 3$. The lattices of hyperinvariant subspaces for the operators A and A^2, which we will denote respectively by $\mathcal{H}_{\langle A \rangle}$, and $\mathcal{H}_{\langle A^2 \rangle}$, are identical (each is a chain of length three). These two operators are clearly not similar. We note that the operators A and A^2 induce lattice maps on $\mathcal{H}_{\langle A \rangle}$, and $\mathcal{H}_{\langle A^2 \rangle}$ respectively. Though the lattices are identical, these lattice maps are quite different. It is this observation which we exploit in chapter 1, and later, in a multivariable format, in chapter 3.

In the second chapter, we further develop the relationship between the lattice of hyperinvariant subspaces and similarity. Similarly, this time, in the multivariate case.

Given an algebra $\mathcal{A}$ generated by a k-tuple of operators $\mathbf{A}$ acting on a finite dimensional vector space V, we associate to each $x \in V$, the subspace $L_{\mathcal{H}_\mathcal{A}}(x)$, the smallest hyperinvariant subspace containing x. Clearly, the set of join-irreducible elements of $\mathcal{H}_\mathcal{A}$ is contained in the set $\{L_{\mathcal{H}_\mathcal{A}}(x) : x \in V\}$. We explore this containment in more depth. Further, using our single variable experience as a guide, we investigate the relationship between joint similarity, and the lattice maps induced by each operator in the k-tuple $\mathbf{A}$ on the lattice $\mathcal{H}_\mathcal{A}$. We find, that in the appropriate setting, we can obtain a complete set of joint similarity invariants.

We conclude the chapter with a characterization of the complete lattice of hyperinvariant subspaces for a k-tuple of operators $\mathbf{A}$, as well as a characterization of joint similarity in terms of this lattice and associated lattice mapping structures. As is to be expected, this characterization is not nearly so transparent as is the single variable case.

The final chapter presents a complete structure theory for a special class of k-tuples of commuting operators. This structure theory meets the goals outlined above for a good structure theory. Further, since our assumptions about these k-tuples are very strong, we end up with as structure theory which is purely combinatorial and strongly resembles Jordan theory.

1 An Induced Lattice of Subspaces

Consider a vector space V over the field $\mathbf{F}$ such that $\dim(V) = n$. The set of all subspaces of V can be partially ordered by inclusion, and forms a lattice S where the join and meet operations are defined as follows:

$$\wedge \stackrel{\text{def}}{=} \text{intersection,}$$
$$\vee \stackrel{\text{def}}{=} \text{linear span.}$$

This lattice is modular. That is, for any elements W_1, W_2, W_3 in S, such that $W_1 \geq W_3$, the following equality holds:

$$W_1 \wedge (W_2 \vee W_3) = (W_1 \wedge W_2) \vee W_3.$$

Clearly, any sublattice of a modular lattice is modular. Hence any lattice of subspaces is modular.

It is well known [1] that if a modular lattice has a finite maximal chain of elements containing, say, $q+1$ elements, then every chain contains at most $q+1$ elements. The number q is called the length or dimension of the lattice. We choose to use the word length to avoid an obvious confusion. Clearly, every lattice of subspaces of an n-dimensional vector space V has length bounded by n.

Given any lattice $\mathcal{L}$, we define the set $\mathcal{J}(\mathcal{L})$ to be the set of non-zero, join-irreducible elements of $\mathcal{L}$, and $\mathcal{J}_0(\mathcal{L})$ to be $\mathcal{J}(\mathcal{L}) \cup (0)$. For any U in $\mathcal{J}(\mathcal{L})$, define

$$\check{U} \equiv \bigvee_{\substack{U' \in \mathcal{J}(\mathcal{L}) \\ U' < U}} U'.$$

Clearly, $\check{U}$ is the unique maximal element of $\mathcal{L}$ strictly contained in U.

A lattice $\mathcal{L}$ is said to be distributive if for any W_1, W_2, W_3 in $\mathcal{L}$, the following equality holds:

$$W_1 \wedge (W_2 \vee W_3) = (W_1 \wedge W_2) \vee (W_1 \wedge W_3).$$

One notes that the distributive inequality implies the modular inequality. However, there exists a five-element lattice which is modular but which fails to be distributive. A distributive lattice of length q may be embedded in the lattice of subsets of a q-element set. As a consequence, any distributive lattice of length q must have no more than 2^q elements.

Distributivity is a most natural concept in the context of subspace lattices. Consider the following easy result, stated for convenience.

Proposition 1.1 *$\mathcal{L}$ is a lattice of subspaces of V containing both the zero subspace (0) and V, $dimV = n < \infty$. Then, the following are equivalent:*

1. $\mathcal{L}$ is distributive.

2. For all choices $(E_U)_{U \in \mathcal{J}(\mathcal{L})}$ such that $E_U \oplus \check{U} = U$, $\bigoplus_{U \in \mathcal{J}(\mathcal{L})} E_U = V$

3. There exists a direct sum decomposition $V = \bigoplus_{U \in \mathcal{J}(\mathcal{L})} E_U$ such that for each $W \in \mathcal{L}$, $W = \bigoplus_{\substack{U \in \mathcal{J}(\mathcal{L}) \\ U \leq W}} E_U$

Consider $A : V \to V$, $dimV = n \leq \infty$. If $\mathcal{L}$ is any lattice of subspaces of V closed under images and preimages of A, then we may consider A and A^{-1} to be lattice maps on $\mathcal{L}$. $A : \mathcal{L} \to \mathcal{L}$ and $A^{-1} : \mathcal{L} \to \mathcal{L}$ are respectively join and meet lattice homomorphisms. That is, for W_1 and W_2 in $\mathcal{L}$,

$$A(W_1 \vee W_2) = AW_1 \vee AW_2,$$

$$A^{-1}(W_1 \wedge W_2) = A^{-1}W_1 \wedge A^{-1}W_2.$$

Any meet or join lattice homomorphism is an order preserving lattice map [1].

The next proposition states conditions under which $A : \mathcal{L} \to \mathcal{L}$ is a meet lattice homomorphism and $A^{-1} : \mathcal{L} \to \mathcal{L}$ is a join lattice homomorphism.

Lemma 1.2 *If W is any subspace of V,*

$A^{-1}AW = W \vee A^{-1}(0)$,

$AA^{-1}W = W \wedge AV$.

Proof. Clear.

Proposition 1.3 *If* W_1 *and* W_2 *are subspaces of* V,
$A(W_1 \wedge W_2) = AW_1 \wedge AW_2 \Longleftrightarrow A^{-1}(0) \vee (W_1 \wedge W_2) = (A^{-1}(0) \vee W_1) \wedge (A^{-1}(0) \vee W_2)$,

and
$A^{-1}(W_1 \vee W_2) = A^{-1}W_1 \vee A^{-1}W_2 \Longleftrightarrow AV \wedge (W_1 \vee W_2) = (AV \wedge W_1) \vee (AV \wedge W_2)$.

Proof. In order to prove the first statement, we make use of the preceding lemma and the fact that A^{-1} is a meet-homomorphism on the lattice $\mathcal{S}$.

$A^{-1}A(W_1 \wedge W_2) = A^{-1}(0) \vee (W_1 \wedge W_2)$,

$A^{-1}(AW_1 \wedge AW_2) = A^{-1}AW_1 \wedge A^{-1}AW_2 = (A^{-1}(0) \vee W_1) \wedge (A^{-1}(0) \vee W_2)$

So that, if we assume that $A(W_1 \wedge W_2) = AW_1 \wedge AW_2$, we obtain $A^{-1}(0) \vee (W_1 \wedge W_2) = (A^{-1}(0) \vee W_1) \wedge (A^{-1}(0) \vee W_2)$. To get the converse assume that $A^{-1}(0) \vee (W_1 \wedge W_2) = (A^{-1}(0) \vee W_1) \wedge (A^{-1}(0) \vee W_2)$. We now have $A^{-1}A(W_1 \wedge W_2) = A^{-1}(AW_1 \wedge AW_2)$. Taking A of both sides and applying the previous lemma again, we obtain $A(W_1 \wedge W_2) \wedge AV = (AW_1 \wedge AW_2) \wedge AV$, or simply $A(W_1 \wedge W_2) = AW_1 \wedge AW_2$.

The other equivalence is handled similarly. ■

Corollary 1.4 *Assume* $\mathcal{L}$ *is a distributive lattice of subspaces of* V.

If $\mathcal{L}$ *is closed under images of* A*, and* $A^{-1}(0) \in \mathcal{L}$*, then* $A : \mathcal{L} \to \mathcal{L}$ *is a lattice homomorphism.*

If $\mathcal{L}$ *is closed under preimages of* A*, and* $AV \in \mathcal{L}$*, then* $A^{-1} : \mathcal{L} \to \mathcal{L}$ *is a lattice homomorphism.*

Proposition 1.5 *If* $\mathcal{L}$ *is a lattice of subspaces of* V *which is closed under both images and preimages of* A*, then* $A : \mathcal{J}(\mathcal{L}) \to \mathcal{J}_0(\mathcal{L})$*. That is, the lattice map on* $\mathcal{L}$ *induced by* A *maps the non-zero join-irreducible elements* $\mathcal{L}$ *into the join-irreducible elements of* $\mathcal{L}$.

Proof. Let U be an element of $\mathcal{J}(\mathcal{L})$. Consider any decomposition $AU = W_1 \vee W_2$. Since $W_1, W_2 \le AU \le AV$, then we have trivially $AV \wedge (W_1 \vee W_2) = (AV \wedge W_1) \vee (AV \wedge W_2)$. So that now

$$
\begin{aligned}
U &= (A^{-1}(0) \vee U) \wedge U \\
&= (A^{-1}AU) \wedge U && \text{(lemma 1.2)} \\
&= A^{-1}(W_1 \vee W_2) \wedge U \\
&= (A^{-1}W_1 \vee A^{-1}W_2) \wedge U \\
&= ((A^{-1}(W_1) \wedge U) \vee (A^{-1}(W_2) \wedge U) \vee A^{-1}(0)) \wedge U \\
&= (A^{-1}W_1 \wedge U) \vee (A^{-1}W_2 \wedge U) && \text{(modularity)}
\end{aligned}
$$

But since $U \in \mathcal{J}(\mathcal{L})$, either $A^{-1}W_1 \wedge U = U$ or $A^{-1}W_2 \wedge U = U$. And thus, either $W_1 = AU$ or $W_2 = AU$. ■

Lemma 1.6 *Let* $\mathcal{L}$ *be a sublattice of* $\mathcal{S}$ *which is closed under images of* A *and which contains* $A^{-1}(0)$*. If* $U_1, U_2 \in \mathcal{J}(\mathcal{L})$*, such that* $AU_1 = AU_2 \neq (0)$*, and either*

a) $U_1 \le U_2$

or

b) $\mathcal{L}$ *is distributive,*

then $U_1 = U_2$.

Proof. In both parts of this proof, we use the observation that $AU_1 = AU_2$ implies that $U_2 \subseteq A^{-1}(0) \vee U_1$.

a) Assume $U_1 \leq U_2$.

$$\begin{aligned} U_2 &= U_2 \wedge (A^{-1}(0) \vee U_1) \\ &= (U_2 \wedge A^{-1}(0)) \vee U_1 \quad \text{(modularity)}. \end{aligned}$$

Since U_2 is irreducible and $AU_2 \neq (0)$, $U_2 = U_1$.

b) Assume $\mathcal{L}$ is distributive.

$$\begin{aligned} U_2 &= U_2 \wedge (A^{-1}(0) \vee U_1) \\ &= (U_2 \wedge A^{-1}(0)) \vee (U_2 \wedge U_1) \quad \text{(distributivity)}. \end{aligned}$$

Again, by the irreducibility of U_2, and the fact that $AU_2 \neq (0)$, we obtain that $U_2 \leq U_1$. Similarly, we may show that $U_1 \leq U_2$. ■

Lemma 1.7 *If $U \in \mathcal{J}_0(\mathcal{L})$ and $U \leq AV$, then there exists $U' \in \mathcal{J}_0(\mathcal{L})$ such that $AU' = U$.*

Proof. Let $S = \{W \in \mathcal{L} | AW = U\}$. Note that $AA^{-1}U = U \wedge AV = U$. Hence S is not empty. Let U' be a minimum element of S. Assume $U' = W_1 \vee W_2$. Then $U = AU' = A(W_1 \vee W_2) = AW_1 \vee AW_2$. By the irreducibility of U, either $AW_1 = U$ or $AW_2 = U$. Since either W_1 or W_2 is in S, by the minimality of U', either $W_1 = U'$, or $W_2 = U'$. Hence, $U' \in \mathcal{J}_0(\mathcal{L})$. ■

Proposition 1.8 *Let $\mathcal{L}$ be closed under images of A and $U \in \mathcal{L}$ such that $AU \neq (0)$, E be a subspace of V such that $U = E \oplus \check{U}$. Then*

1. *$A\check{U} = (\check{AU})$,*
2. *$A\,|_E\colon E \to AE$ is an isomorphism,*
3. *$AU = AE \oplus A\check{U}$.*

Proof. To prove the first statement, we note that

$$A\check{U} = A \bigvee_{\substack{U' \in \mathcal{J}(\mathcal{L}) \\ U' < U}} U' = \bigvee_{\substack{U' \in \mathcal{J}(\mathcal{L}) \\ U' < U}} AU'$$

By the previous lemma, $AU' < AU$ for each $U' < U$. Hence $A\check{U} \leq (\check{AU})$. Now, consider $U'' \in \mathcal{J}(\mathcal{L})$, $U'' < AU$. Note that

$$\begin{aligned} A^{-1}(0) \vee (A^{-1}U'' \wedge U) &= (A^{-1}(0) \vee U) \wedge A^{-1}U'' \quad \text{(modularity)} \\ &= (A^{-1}(0) \vee A^{-1}U'') \wedge (A^{-1}(0) \vee U) \end{aligned}$$

So, by lemma 1.2 and proposition 1.3 $A(A^{-1}U'' \wedge U'') = AA^{-1}U'' \wedge AU = U'' \wedge AV \wedge AU = U''$. Hence $U'' \leq A\check{U}$.

For the second assertion, note that the definition of $\check{U}$ implies that either $U \wedge A^{-1}(0) \subseteq U$ or $U \wedge A^{-1}(0) = U$. Our assumption that $AU \neq (0)$ prohibits the latter. We deduce that $E \wedge A^{-1}(0) = (0)$ and the second assertion follows.

Lastly, we use the fact that A distributes over join to note that $AU = A(\check{U} \vee E) = A\check{U} \vee AE$. To obtain that $A\check{U}$ and AE are linearly independent, we compute

$$\begin{aligned}
& (A^{-1}(0) \vee \check{U}) \wedge (A^{-1}(0) \vee E) \\
= \; & A^{-1}(0) \vee ((A^{-1}(0) \vee \check{U}) \wedge E) \qquad \text{(modularity)} \\
= \; & A^{-1}(0) \vee (U \wedge (A^{-1}(0) \vee \check{U}) \wedge E) \\
= \; & A^{-1}(0) \vee (((U \wedge A^{-1}(0)) \vee \check{U}) \wedge E) \\
= \; & A^{-1}(0) \vee ((\check{U} \wedge A^{-1}(0)) \vee \check{U}) \wedge E) \qquad \text{(proof of 2.)} \\
= \; & A^{-1}(0) \vee (\check{U} \wedge E).
\end{aligned}$$

Hence, by proposition 1.3, $A\check{U} \wedge AE = A(\check{U} \wedge E) = A(0) = (0)$. The final assertion follows. ■

Proposition 1.9 *Let $\mathcal{L}$ be closed under images of A and containing $A^{-1}(0)$. There exist $\{E_U\}_{U \in \mathcal{J}_0(\mathcal{L})}$ such that for all $U \in \mathcal{J}_0(\mathcal{L})$,*

1. $U = \check{U} \oplus E_U$,

2. $AE_U = E_{AU}$,

3. $A|_{E_U}: E_U \to E_{AU}$ is an isomorphism, provided that $AU \neq (0)$.

Here, $E_{(0)}$ is defined to be (0).

Proof. We choose the E_U iteratively. Let $S \subseteq \mathcal{J}_0(\mathcal{L})$ for which an E_U has been chosen. If $S \neq \mathcal{J}_0(\mathcal{L})$, choose a $U \notin S$ such that $AU \in S$. If $AU = (0)$, choose E_U such that $U = \check{U} \oplus E_U$. Statement 2 holds automatically and statement 3 holds vacuously. If $AU \neq (0)$, choose E_U such that $AE_U = E_AU$, $E_U \leq U$ and $E_U \wedge A^{-1}(0) = (0)$. Clearly, both statements 2 and 3 hold. By the preceding proposition, $A\check{U} = (\check{AU})$. Hence $E_U \wedge \check{U} = (0)$. Further, $A(\check{U} \vee E_U) = A\check{U} \vee AE_U = AU$, and $U \wedge A^{-1}(0) = \check{U} \wedge A^{-1}(0)$. We deduce that $U = \check{U} \oplus E_U$. ■

Given a lattice $\mathcal{L}$, we note that a set $\{E_U\}$ satisfying property 1 in the preceding result generates a minimal complemented lattice of subspaces containing $\mathcal{L}$. We say that such a set properly complements $\mathcal{L}$ with respect to the operator A if it satisfies statements 2 and 3.

We recast the preceding results in a graph-theoretic milieu. We define a strict directed graph (to which we will refer simply as a directed graph), $\Gamma(\mathcal{V}, \mathcal{E})$ to consist of a vertex set $\mathcal{V}$, and an set of directed edges $\mathcal{E}$. The set $\mathcal{E}$ consists of ordered pairs of elements (x, y) where $x, y \in \mathcal{V}$, $x \neq y$. For a more standard formulation, see [2]. If $x, y \in \mathcal{V}$, the pair (x, y) may be visualized as a directed arc from x to y. The outdegree of a vertex is defined to be the number of directed edges leaving that vertex. Likewise, the indegree of a vertex is defined to be the number of edges entering the vertex.

Consider an operator $A : V \to V$ and lattice $\mathcal{L}$ containing the zero subspace (0), and closed under both images and preimages of A. Form a directed graph $\Gamma = \Gamma(\mathcal{J}(\mathcal{L}), \mathcal{E})$. As written, $\mathcal{J}(\mathcal{L})$ forms the vertex set of our graph. We take $(U_1, U_2) \in \mathcal{E}$ if $AU_1 = U_2$. Proposition 1.5 implies that the outdegree of a vertex $U \in \mathcal{J}(\mathcal{L})$ is 1 if $AU \neq (0)$ and 0 otherwise. Lemma 1.7 states that the indegree of any vertex $U \leq AV$ is non-zero. Lemma 1.6 states that the indegree of each vertex is at most 1 provided that $\mathcal{L}$ is distributive.

We define the connected components of a digraph to be equal to the connected components of the underlying graph. Denote the set of connected components of Γ by Π.

The last proposition has an important consequence.

Corollary 1.10 *Given a lattice $\mathcal{L}$ closed under images of A and containing* (0), *and given a set of subspaces $\{E_U\}_{U\in\mathcal{J}(\mathcal{L})}$, then for each $\pi \in \Pi$, there exists a collection of isomorphisms*

$$\theta_{U_1,U_2} : E_{U_1} \to E_{U_2},$$

where $U_1, U_2 \in \pi$, such that

1. $\theta_{U,AU} = A\,|_{E_U}$ *if* $AU \neq (0)$,
2. $\theta_{U_1,U_2} = \theta_{U_1,U_3}\theta_{U_3,U_2}$.

Given a collection of operators $\mathcal{T}$, denote by $\mathcal{L}_{\mathcal{T}}$, the unique smallest lattice of subspaces containing (0) and closed under both images and preimages of each $T \in \mathcal{T}$.

Proposition 1.11 *If $A : V \to V$, $dimV < \infty$, then $\mathcal{L}_A$ is distributive.*

Proof. Let $\mathcal{L}$ be the lattice generated by

$$\{A^k V : k \in \{1, \dots, n\}\} \cup \{A^{-k}(0) : k \in \{1, \dots, n\}\}.$$

That the union of two linearly ordered subsets of a modular lattice always generates a distributive sublattice is well known [1]. Hence, $\mathcal{L}$ is a distributive sublattice of $\mathcal{L}_A$. We note that both $A^{-1}(0)$ and $A(V)$ are in $\mathcal{L}$. Hence, by proposition 1.3,

$$A(W_1 \wedge W_2) = AW_1 \wedge AW_2,$$

$$A^{-1}(W_1 \vee W_2) = A^{-1}W_1 \vee A^{-1}W_2,$$

for any $W_1, W_2 \in \mathcal{L}$. $\mathcal{L}$ is therefore closed under both images and preimages of A. Hence $\mathcal{L}_A = \mathcal{L}$. ■

Note, that as a consequence of the above proof, we have the following.

Proposition 1.12 $\mathcal{J}(\mathcal{L}_A) \subseteq \{A^i V \wedge A^{-j}(0) : i, j \in \{1, \dots, n\}\}$

That is, every join-irreducible element of the lattice $\mathcal{L}_A$ can be realized as the intersection of the image of some power of A with the kernel of some (possibly different) power of A.

Proposition 1.13 *Given a nilpotent $A : V \to V$, there exists a direct sum decomposition*

$$V = \bigoplus_{U\in\mathcal{J}(\mathcal{L}_A)} E_U$$

such that for all $U \in \mathcal{J}_0(\mathcal{L}_A)$, *1.* $U = \check{U} \oplus E_U$,

2. $AE_U = E_{AU}$,

3. $A\,|_{E_U}: E_U \to E_{AU}$ *is an isomorphism provided that* $AU \neq (0)$.

4. $AE_{U_1} = AE_{U_2} \neq E_{(0)}$ *implies that* $U_1 = U_2$.

Here, $E_{(0)}$ *is defined as* (0).

Proof. For each $U \in \mathcal{J}(\mathcal{L}_A)$ for which no $U' \in \mathcal{J}(\mathcal{L}_A)$ exists such that $AU' = U$, choose E_U arbitrarily such that $U = \check{U} \oplus E_U$. Once an E_U is chosen, if $AU \neq (0)$, define $E_{AU} = AE_U$. Proposition 1.8 implies that 1. and 3. hold. Since, $\mathcal{L}$ is distributive, by proposition 1.1,

$$V = \bigoplus_{U \in \mathcal{J}(\mathcal{L}_A)} E_\lambda.$$

The last assertion follows from the distributivity of $\mathcal{L}_A$ and lemma 1.6. ■

The reader will recognize in the preceding result the traditional chain structure of Jordan form. In our result, each chain of E_U subspaces represents all Jordan chains of vectors of the same length. Hence, we have the following.

Corollary 1.14 *Every nilpotent operator* $A : V \to V$ *has a Jordan basis.*

2 Hyperinvariant Subspaces and Joint Similarity

Consider operators $A_1, \ldots, A_k : V \to V$, where $\dim V < \infty$ and $A_iA_j = A_jA_i$ for all $i, j \in \{1, \ldots, k\}$. Define $\mathcal{A} = \langle A_1, \ldots, A_k \rangle$, the algebra generated by $A_1, \ldots, A_k$. We denote by $\mathbf{A}$ the k-tuple of operators $(A_1, \ldots, A_k)$.

A subspace W of V is said to be hyperinvariant for the algebra $\mathcal{A}$ if it is invariant for the commutant algebra $\mathcal{A}'$ of $\mathcal{A}$. That is, $AW \subseteq W$ for all $A : V \to V$ such that $AA_i = A_iA$ for all $i \in \{1, \ldots, k\}$. Clearly, the set of all hyperinvariant subspaces for $\mathcal{A}$ forms a sublattice of $\mathcal{S}$.

We begin with the following easy result.

Lemma 2.1 *If* $W \in \mathcal{H}_\mathcal{A}$, *and* $p \in \mathbf{C}[z_1, \ldots, z_k]$ *then*

1. $p(\mathbf{A})W \in \mathcal{H}_\mathcal{A}$,

2. $p(\mathbf{A})^{-1}W \in \mathcal{H}_\mathcal{A}$.

As a consequence, we note that the lattice $\mathcal{H}_\mathcal{A}$ is closed under both images and preimages of A_i and contains both $A_i^{-1}(0)$ and A_iV for each $i \in \{1, \ldots, k\}$. Recall that in the preceeding chapter, we defined $\mathcal{L}_\mathcal{T}$ to be the smallest lattice of subspaces containing (0) and closed under images and preimages of each element of $\mathcal{T}$. Clearly, $\mathcal{L}_\mathbf{A} \subseteq \mathcal{L}_\mathcal{A}$. We have the following corollary to the lemma.

Corollary 2.2 $\mathcal{L}_{\mathbf{A}} \subseteq \mathcal{L}_{\mathcal{A}} \subseteq \mathcal{H}_{\mathcal{A}}$.

In [3], it is shown that for a single nilpotent operator A, the lattice $\mathcal{H}_{\mathcal{A}}$ is generated as a lattice by the kernels and images of the various powers of the operator A. The following is a consequence.

Proposition 2.3 *If* $\mathcal{A} = \langle A \rangle$, $\mathcal{L}_{\mathcal{A}} = \mathcal{H}_{\mathcal{A}}$.

Given a lattice $\mathcal{L}$ of subspaces of a vector space V, such that $V \in \mathcal{L}$, we define the natural map $L_{\mathcal{L}} : V \to \mathcal{L}$,

$$L_{\mathcal{L}}(v) = \bigwedge_{\substack{W \in \mathcal{L} \\ v \in W}} W.$$

Our goal is to use the lattices $\mathcal{L}_{\mathbf{A}}$ and $\mathcal{H}_{\mathcal{A}}$, and the associated maps $L_{\mathcal{L}_{\mathbf{A}}}$ and $L_{\mathcal{H}_{\mathcal{A}}}$ as tools to help understand the structure of various k-tuples of operators $\mathbf{A}$. We start with two simple observations:

$$L_{\mathcal{H}_{\mathcal{A}}}(x) = \{Ax : A \in \mathcal{A}'\},$$

and

$$AL_{\mathcal{H}_{\mathcal{A}}}(x) = L_{\mathcal{H}_{\mathcal{A}}}(Ax).$$

We introduce the following example, both to illustrate the above notions and to motivate a class of examples to be introduced in the next section.

Example 2.1

Consider an $n-$dimensional vector space V^n with basis $\mathcal{B} = \{e_i\}_{i=1}^n$ and the nilpotent $S_n : V^n \to V^n$ such that $S_n e_i = e_{i-1}$ for $i \in \{2, \ldots, n\}$, and $S_n e_1 = 0$. Let

$$V^{n,k} = \otimes_{i=1}^k V^n.$$

Define operators $A_i : V^{n,k} \to V^{n,k}$ for $i \in \{1, \ldots, k\}$ as follows:

$$A_i(x_1 \otimes x_2 \otimes \ldots \otimes x_k) = x_1 \otimes x_2 \otimes \ldots \otimes S_n x_i \otimes \ldots \otimes x_k$$

for $x_1 \otimes \ldots \otimes x_k \in V^{n,k}$. Finally, define the algebra $\mathcal{A}_{n,k}$ to be the algebra of operators generated by $A_1, \ldots, A_k$. We examine the subspaces $L_{\mathcal{H}_{\mathcal{A}}}(x)$ for $x \in V$ from several points of view.

Note that $c = e_n \otimes \ldots \otimes e_n$ is a cyclic vector for $\mathcal{A}_{n,k}$. Hence, by proposition 0.4, $\mathcal{A}' = \mathcal{A}$. Hence

$$L_{\mathcal{H}_{\mathcal{A}}}(x) = \{Ax : A \in \mathcal{A}\}.$$

Let $Z_x \subseteq \mathbf{C}[z_1, \ldots, z_k]$ be the annihilating ideal for the vector x, and $p_1, \ldots p_r \in \mathbf{C}[z_1, \ldots, z_k]$ be a basis for Z_x. Then

$$L_{\mathcal{H}_{\mathcal{A}}}(x) = p_1^{-1}(\mathbf{A})(0) \wedge \ldots \wedge p_r^{-1}(\mathbf{A})(0).$$

Lastly, if $p \in \mathbf{C}[z_1, \ldots, z_k]$ such that $p(\mathbf{A})(c) = x$, then

$$L_{\mathcal{H}_{\mathcal{A}}}(x) = p(\mathbf{A})V.$$

We define an equivalence relation $\equiv_{\mathcal{A}}$ induced by $\mathcal{A}$ on the vectors in V as follows. For $u, v \in V$, we write $u \equiv_{\mathcal{A}} v$ if there exists $C \in \mathcal{A}'$ such that $Cu = v$. Clearly, if $u \equiv_{\mathcal{A}} v$, then $L_{\mathcal{H}_{\mathcal{A}}}(u) = L_{\mathcal{H}_{\mathcal{A}}}(v)$. In order to prove the converse, we will need the following results.

Lemma 2.4 *If $A, B : V \to V$ and $dimV = n < \infty$, then there exists $p \in \mathbf{C}[z]$ such that $A + p(B)(1 - BA)$ is invertible.*

Proof. If m is the minimal polynomial for B, then write $m(z) = z^k m_1(z)$ where $m_1(0) \neq 0$. Decompose $V = V_0 \oplus V_1$ where $BV_i \subseteq V_i$, $(B\mid_{V_0})^k = 0$ and $m_1(B\mid_{V_1}) = 0$.

For all $\lambda \in \mathbf{C}$, define

$$r_\lambda(z) = \lambda \sum_{j=0}^{k} (-\lambda z)^j$$

and $q_\lambda(z) = \lambda z + 1$. By the Chinese Remainder Theorem, there exists a unique polynomial $p_\lambda \in \mathbf{C}[z]$ of minimal degree such that

$$z p_\lambda \equiv 1 (\mathrm{mod} m_1),$$

$$p_\lambda \equiv r_\lambda (\mathrm{mod} z^k).$$

Note that $p_\lambda(B)\mid_{V_0} = r_\lambda(B)\mid_{V_0}$, and $Bp_\lambda(B)\mid_{V_1} = I_{V_1}$.

Define $P : V \to V$ such that $PV_0 = V_0$, $PV_1 = 0$ and $P^2 = P$. Note that $P = p(B)$, for some $p \in \mathbf{C}[z]$ and hence commutes with B.

We compute as follows:

$$\begin{aligned}
& A + p_\lambda(B)(1 - BA) \\
= \;& P(A + p_\lambda(B)(1 - BA)) + (1 - P)(A + p_\lambda(B)(1 - BA)) \\
= \;& (1 - Bp_\lambda(B))PA + Pp_\lambda(B) + (1 - Bp_\lambda(B))(1 - P)A + (1 - P)p_\lambda(B) \\
= \;& (1 - Br_\lambda(B))PA + Pr_\lambda(B) + 0 + (1 - P)p_\lambda(B) \\
= \;& ((1 - Br_\lambda(B))PAP + Pr_\lambda(B)P) + ((1 - Br_\lambda(B))PA(1 - P)) + \\
& \quad (1 - P)p_\lambda(B)(1 - P).
\end{aligned}$$

The last line above gives a block upper-triangular representation of the operator $A + p_\lambda(B)(1 - BA)$ relative to the direct sum decomposition $V = PV \oplus (1 - P)V$. Since $Bp_\lambda(B)(1 - P) = 1 - P$ and hence $p_\lambda(B)\mid_{V_1}$ is invertible, it remains only to show that for an appropriate choice of λ, $((1 - Br_\lambda(B))PAP + Pr_\lambda(B)P)\mid_{V_0}$ is invertible.

We note the following:

$$r_\lambda(z)q_\lambda(z) = \lambda + (-\lambda)^{k+2} z^{k+1},$$

and

$$q_\lambda(z)(1 - z r_\lambda(z)) = 1 - (-\lambda z)^{k+2}.$$

Hence, substituting B for z, we obtain

$$\begin{aligned} & q_\lambda(B)((1 - Br_\lambda(B))PAP + Pr_\lambda(B)P) \\ = \; & (1 - (-\lambda B)^{k+2})PAP + P(\lambda + (-\lambda)^{k+2}B^{k+1})P \\ = \; & PAP + P\lambda P. \end{aligned}$$

This last expression is clearly invertible for all $\lambda \notin \sigma(PAP)$. Therefore, $((1 - Br_\lambda(B))PAP + Pr_\lambda(B)P)\,|_{V_0}$ is also invertible for these same values of λ. ■

Proposition 2.5 *If $A, B : V \to V$, $\dim V = n < \infty$, and $x, y \in V$ such that $Ax = y$ and $By = x$, then there exists an invertible $C \in \mathcal{A} = \langle A, B \rangle$ such that $Cx = y$.*

Proof. Given $A', B' \in \mathcal{A}$ such that $A'x = y$ and $B'y = x$, define $P = B'A'$. We claim that A' and B' may be chosen so that P has the following properties:

$$P^2 = P,$$

$$A'P = A',$$

and

$$PB' = B'.$$

In order to prove the claim, note that $BAx = x$. Define $P : V \to V$ to be the unique idempotent which has the following properties: $P(BA) = (BA)P$; $(PBA - 1)^n = 0$, and $(1 - P)BA - 1$ is invertible. That is, P is the idempotent induced by the primary decomposition of the operator BA onto the generalized eigenspace corresponding to the eigenvalue 1. We recall that P is a polynomial in BA and is hence in $\mathcal{A}$. Further note that $Px = x$.

Choose $A' = AP$ which gives $BA' = BAP$. The spectrum of BA' is contained in $\{0, 1\}$, and BA' is the zero operator on its generalized eigenspace associated to the eigenvalue (0). Hence, $(1 - BA')^n BA' = 0$. Further, since $(1 - BA')^n = 1 - P$, we have that

$$P = 1 - (1 - BA')^n = 1 - \sum_{j=0}^{n} \binom{n}{j} (-BA')^j = -\sum_{j=1}^{n} \binom{n}{j} (-BA')^j.$$

Define $B' = -PB\sum_{j=1}^{n} \binom{n}{j} (-BA')^{j-1}$. Note that A', B' and P meet the requirements outlined in the claim.

Given $p \in \mathbf{C}[z]$, consider the operator $C_p : V \to V$ defined as follows:

$$C_p = A' + Pp(B')(1 - P) + (1 - P).$$

We observe that $C_p x = A'x = y$. So it remains only to show that for appropriate choice of p, C_p is invertible.

Define $R_p = 1 - Pp(B')(1-P)$. Clearly, R_p is invertible. We compute R_pC_p and display the product in block form, relative to the decomposition $V = PV \oplus (1-P)V$.

$$R_pC_p = P(A' - p(B')(1-P)A')P + 0 + (1-P)A'P + (1-P).$$

Since R_pC_p has a block lower triangular form, we see that it is invertible if and only if $P(A' - p(B')(1-P)A')P\mid_{PV}$ is invertible.

Let $q \in \mathbf{C}[z]$ such that $qz - p \in \mathbf{C}$. Note that

$$\begin{aligned} & Pp(B')(1-P) \\ = \; & Pq(B')B'(1-P) \\ = \; & Pq(B')PB'(1-P) \\ = \; & q(B'P)B'(1-P). \end{aligned}$$

Hence

$$\begin{aligned} & P(A' - p(B')(1-P)A')P\mid_{PV} \\ = \; & (PA' - q(B'P)B'(1-P)A')\mid_{PV} \\ = \; & (PA' - q(B'P)(1_{PV} - (B'P)(PA')))\mid_{PV}. \end{aligned}$$

By the previous lemma, we may choose $p \in \mathbf{C}[z]$ to make C_p invertible. ∎

Corollary 2.6 *If $x, y \in V$, then $L_{\mathcal{H}_\mathcal{A}}x = L_{\mathcal{H}_\mathcal{A}}y$ if and only if $x \equiv_\mathcal{A} y$.*

Theorem 2.7 *Let $x \in V$. Then, $L_{\mathcal{H}_\mathcal{A}}x \in \mathcal{J}(\mathcal{H}_\mathcal{A})$ if and only if there exists a direct sum decomposition $V = V_1 \oplus V_2$ such that $\mathcal{A}V_i \subseteq V_i$, $x \in V_1$, and V_1 is indecomposable under the action of the algebra $\mathcal{A}$.*

Proof. ($\Rightarrow$) Assume $L_{\mathcal{H}_\mathcal{A}}x \in \mathcal{J}(\mathcal{H}_\mathcal{A})$. Choose a decomposition, $V = V_1' \oplus V_2'$ such that $\mathcal{A}V_i' \subseteq V_i'$, $x \in V_1'$, and $\dim V_1'$ is minimal with respect to these two properties. Assume $V_1' = V_1'' \oplus V_2''$, where $\mathcal{A}V_i'' \subseteq V_i''$. For $i \in \{1,2\}$, define $P_i : V \to V$ to be the idempotent such that $P_iV_i'' = V_i''$, and $P_i(V_{3-i}'' \oplus V_2') = (0)$. Clearly, $P_i \in \mathcal{A}'$. Hence, $P_ix \in V_i'' \wedge L_{\mathcal{H}_\mathcal{A}}x$. So, $L_{\mathcal{H}_\mathcal{A}}P_ix \subseteq L_{\mathcal{H}_\mathcal{A}}x$. Moreover, the fact that $x = P_1x + P_2x$ implies that $L_{\mathcal{H}_\mathcal{A}}x = L_{\mathcal{H}_\mathcal{A}}P_1x \vee L_{\mathcal{H}_\mathcal{A}}P_2x$.

By assumption, $L_{\mathcal{H}_\mathcal{A}}x$ is irreducible in the lattice $\mathcal{H}_\mathcal{A}$. Hence, there exists a $j \in \{1,2\}$ such that $L_{\mathcal{H}_\mathcal{A}}P_jx = L_{\mathcal{H}_\mathcal{A}}x$. By corollary 2.6, there exists an invertible operator $C \in \mathcal{A}'$ such that $CP_jx = x$. Note that $V = CV_1'' \oplus CV_2'' \oplus CV_2'$, $x \in CV_1''$, and that CV_1'', CV_2'', and CV_1' are all invariant under $\mathcal{A}$. By assumption, $\dim V_1'' = \dim V_1'$. Hence $V_1'' = V_1'$.

($\Leftarrow$) Assume we have the decomposition $V = V_1 \oplus V_2$ as in the proposition. Say

$$L_{\mathcal{H}_\mathcal{A}}x = \bigwedge_{i=1}^{r} U_i,$$

where $U_i \in \mathcal{J}(\mathcal{H}_\mathcal{A})$. For each i, choose $x_i \in U_i$ such that $\sum_{i=1}^r x_i = x$. Note that

$$L_{\mathcal{H}_\mathcal{A}}x_i \subseteq U_i,$$

and that

$$L_{\mathcal{H}_{\mathcal{A}}}x = \bigwedge_{i=1}^{r} L_{\mathcal{H}_{\mathcal{A}}}x_i.$$

The idempotent with range V_1 and kernel V_2 is an element of $\mathcal{A}'$. Hence, $L_{\mathcal{H}_{\mathcal{A}}}Px_i \subseteq L_{\mathcal{H}_{\mathcal{A}}}x_i$. Note that

$$\sum_{i=1}^{r} Px_i = P\sum_{i=1}^{r} x_i = Px = x,$$

and hence

$$L_{\mathcal{H}_{\mathcal{A}}}x = \bigwedge_{i=1}^{r} V_{P_{x_i}}.$$

For all $i \in \{1, \ldots, r\}$, there exists an operator $C_i \in \mathcal{A}'$ such that $C_i x = Px_i$. Note that

$$PC_iPx = Px_i \quad \text{for } i \in \{1, \ldots, r\},$$

and that both V_1 and V_2 are invariant under the operators PC_iP. Since V_1 is irreducible, and $PC_iP \in \mathcal{A}'$,

$$\text{card}\,(\sigma(PC_iP)\,|_{V_1}) = 1.$$

Further note that

$$\sum_{i=1}^{r} PC_iPx = x,$$

and hence that

$$1 \in \sigma(\sum_{i=1}^{r} PC_iP).$$

This further implies that

$$\text{trace} \sum_{i=1}^{r} PC_iP = \dim V_1.$$

So we have that $\text{trace}PC_jP \neq 0$ for some j. Then, $PC_jP + (1-P) : V \to V$ is invertible. Let $R = (PC_jP + (1-P))^{-1}$. $R(Px_j) = x$, and $R \in \mathcal{A}'$. Hence $L_{\mathcal{H}_{\mathcal{A}}}Px_j = U_j = L_{\mathcal{H}_{\mathcal{A}}}x$. The result follows. ∎

Corollary 2.8 *If the space V is indecomposable under the action of the algebra $\mathcal{A}$, then for all $x \in V$, $L_{\mathcal{H}_{\mathcal{A}}}x \in \mathcal{J}(\mathcal{H}_{\mathcal{A}})$.*

Based on our experience in the case when the algebra $\mathcal{A}$ is generated by a single operator $A : V \to V$, we may make the following conjectures for an algebra $\mathcal{A}$ generated by commuting nilpotents $A_1, \ldots, A_k : V \to V$:

1. $\mathcal{H}_{\mathcal{A}} = \mathcal{L}_{\mathcal{A}}$,
2. The lattice $\mathcal{H}_{\mathcal{A}}$, together with the induced lattice maps $A_i : \mathcal{J}(\mathcal{H}_{\mathcal{A}}) \to \mathcal{J}_0(\mathcal{H}_{\mathcal{A}})$ contain sufficient information to determine $A_1, \ldots, A_k$ up to joint similarity.

Unfortunately, neither of these assertions is true.

Example 2.2

Consider the shift $S_5 : V^5 \to V^5$ as defined in example 2.1. Define $A_1, A_2 : V \to V$ as follows: $A_1 = S_5^2$, $A_2 = S_5^3$. Let $\mathcal{A}$ be the algebra generated by A_1 and A_2.

Since both A_1 and A_2 are polynomials in S_5, the algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}_{5,1}$. Hence $\mathcal{A}'_{5,1}$ is a subalgebra of $\mathcal{A}'$. This further implies that the lattice $\mathcal{H}_{\mathcal{A}}$ is a sublattice of $\mathcal{H}_{\mathcal{A}_{5,1}}$. We know that the latter lattice is a chain of six elements:

$$\mathcal{H}_{\mathcal{A}_{5,1}} = \{U_i : U_0 = (0), U(i) = \langle e_1, \dots e_i \rangle, i \in \{1, \dots, 5\}\}.$$

It is not difficult to check that each of these six subspaces is an element of $\mathcal{H}_{\mathcal{A}}$. Note that $V_{e_i} = U_i$ for $i \in \{1, \dots, 5\}$. We further note that

$$A_j U_{e_i} = \begin{cases} U_{A_j(e_i)} & A_j(e_i) \neq 0 \\ U_0 & \text{otherwise.} \end{cases}$$

The lattice $\mathcal{H}_{\mathcal{A}}$ and the lattice maps thereupon induced by A_1 and A_2 are incapable of distinguishing the pairs $(A_1, \lambda A_2)$ for $\lambda \neq 0$. To disprove the second conjecture, we could observe that for different λ, the pairs $(A_1, \lambda A_2)$ are not jointly similar. In particular, we show that

$$(A_1, \lambda A_2) \sim (A_1, A_2)$$

implies that $\lambda = 1$. To see the last assertion consider the the 2×2 matrix valued polynomial operator in the variables A_1 and A_2, $F_a : V^5 \oplus V^5 \to V^5 \oplus V^5$ defined as follows:

$$F_a(x, y) = (aA_1^2 x + A_2 y, A_2 x + A_1 y).$$

If $a = 1$, then for each $x \in V^5$, there exists a $y \in V^5$ such that $(x, y) \in \ker F_a$. If $a \neq 1$, and if $x = e_5$, there is no $y \in V^5$ such that $(x, y) \in \ker R_a$. We have hence, found a similarity invarient which differs in the cases when $a = 1$ and $a \neq 1$. We have thus disproved the second conjecture.

To see that the first conjecture is also false, consider the operators $B_1, B_2 : V^5 \oplus V^5 \to V^5 \oplus V^5$ defined as follows: $B_1 = A_1 \oplus A_1$, $B_2 = A_2 \oplus 2A_2$.

By our comments above, we can immediately see that each element in $\mathcal{L}_1$ has even dimension. Define the operator $F : (V^5 \oplus V^5)^2 \to (V^5 \oplus V^5)^2$ (here $(V^5 \oplus V^5)^2 = (V^5 \oplus V^5) \oplus (V^5 \oplus V^5)$) as follows:

$$F(x, y) = (B_1^2 x + B_2 y, B_2 x + B_1 y)$$

where $x, y \in V^5 \oplus V^5$. Further define $P : (V^5 \oplus V^5)^2 \to V^5 \oplus V^5$,

$$P(x, y) = x.$$

Note that $P\ker F$ is a hyperinvariant subspace of $V^5 \oplus V^5$. By our earlier comments, $\dim P\ker F = 9$. Hence $P\ker F$ cannot be in $\mathcal{L}_1$.

This example suggests a new light in which to view the lattices $\mathcal{L}_{\mathbf{A}}$ and $\mathcal{L}_{\mathcal{A}}$. We define the set of matrix-valued functions with polynomial entries:

$$\mathcal{F} = \left\{ [p_{ij}]_{i,j \in \{1,\ldots r\}} : p_{ij} \in \mathbf{C}[z_1, \ldots, z_k], r \in \mathbf{Z}^+ \right\}.$$

If $F \in \mathcal{F}$, then $F(\mathbf{A}) : V^r \to V^r$. Further define $P : V^r \to V$ as follows:

$$P(x_1, \ldots, x_r) = x_1 \qquad x_i \in V.$$

Define

$$\mathcal{L}_* = \{ P \ker F(\mathbf{A}) : F \in \mathcal{F} \}.$$

Proposition 2.9 *$\mathcal{L}_*$ is a sublattice of $\mathcal{H}_{\mathcal{A}}$.*

Proof. We first show that $\mathcal{L}_*$ is closed under both meets and joins. Let $W_1, W_2 \in \mathcal{L}_*$. Then there exist $F, G \in \mathcal{F}$ such that $W_1 = P \ker F(\mathbf{A})$, and $W_2 = P \ker G(\mathbf{A})$. Without loss of generality, say that F and G are the same size, say r by r. Then

$$P \ker \begin{bmatrix} F_{11} & \ldots & F_{1r} & 0 & \ldots & 0 \\ \vdots & & & & & \\ F_{r1} & \ldots & F_{rr} & 0 & \ldots & 0 \\ G_{11} & \ldots & G_{1r} & 0 & \ldots & 0 \\ \vdots & & & & & \\ G_{r1} & \ldots & G_{rr} & 0 & \ldots & 0 \end{bmatrix} = W_1 \wedge W_2,$$

and

$$P \ker \begin{bmatrix} 1 & 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & F_{11} & F_{12} & \ldots & F_{1r} & 0 & 0 & \ldots & 0 \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & F_{r1} & F_{r2} & \ldots & F_{rr} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & G_{11} & G_{12} & \ldots & G_{1r} \\ \vdots & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \ldots & 0 & G_{r1} & G_{r2} & \ldots & G_{rr} \end{bmatrix} = W_1 \vee W_2.$$

Hence, $\mathcal{L}_*$ is a lattice. To see that this lattice is contained in the lattice of hyperinvariant subspaces, choose a vector $x \in P \ker F(\mathbf{A})$ for some $F \in \mathcal{F}$. There exist vectors $x_2, \ldots, x_r \in V$ such that $(x, x_2, \ldots, x_r) \in \ker F(\mathbf{A})$. Let $B \in \mathcal{A}'$ and define $G : V^r \to V^r$ as follows:

$$G(v_1, \ldots, v_r) = (Bv_1, \ldots, Bv_r).$$

Then

$$\begin{aligned} & F(Bx_1, Bx_2, \ldots, Bx_n) \\ = \quad & GF(x_1, x_2, \ldots, x_n) \\ = G(0, \ldots 0) & \\ = \quad & 0 \end{aligned}$$

Hence, $Bx_1 \in P \ker F(\mathbf{A})$. The result follows. ■

Proposition 2.10 *$\mathcal{L}_*$ is closed under images and preimages of $p(\mathbf{A})$ for all $p \in \mathbf{C}[z_1, \ldots, z_k]$.*

Proof. Let $W = P \ker F$ for some $F \in \mathcal{F}$. Then

$$P \ker \begin{bmatrix} p(\mathbf{A}) & 1 & 0 & \ldots & 0 \\ 0 & F_{11} & F_{12} & \ldots & F_{1r} \\ \vdots & & & & \\ 0 & F_{r1} & F_{r2} & \ldots & F_{rr} \end{bmatrix} = p^{-1}(\mathbf{A})W,$$

and

$$P \ker \begin{bmatrix} 1 & p(\mathbf{A}) & 0 & \ldots & 0 \\ 0 & F_{11} & F_{12} & \ldots & F_{1r} \\ \vdots & & & & \\ 0 & F_{r1} & F_{r2} & \ldots & F_{rr} \end{bmatrix} = p(\mathbf{A})W.$$

■

So we have the following

Corollary 2.11 $\mathcal{L}_{\mathbf{A}} \subseteq \mathcal{L}_{\mathcal{A}} \subseteq \mathcal{L}_* \subseteq \mathcal{H}_{\mathcal{A}}$.

The next theorem states that the last containment in the above corollary is actually equality, and gives necessary and sufficient conditions for two k-tuples of operators to be jointly similar.

Lemma 2.12 *If $\mathcal{S}$ is a subspace of linear operators on V, $\dim V < \infty$, and for all $x \in V$, there exists an $A_x \in \mathcal{S}$ such that $A_x x = x$, then there exists $A \in \mathcal{S}$ such that A is invertible.*

Proof. Choose $A \in \mathcal{S}$ such that $\dim(\ker A^n)$ is minimal. If A is not invertible, choose $x \in \ker A$, $A_x \in \mathcal{S}$ such that $A_x x = x$. Consider the matrix-valued function $A'(\epsilon) = (A + \epsilon A_x)$. The eigenvalues of A', including multiplicity, are continuous functions of ϵ. Also note that $\epsilon \in \sigma(A'(\epsilon))$. Hence, for sufficiently small ϵ, $\mathrm{Nul} A'(\epsilon)^n < \mathrm{Nul} A^n$. The contradiction establishes the lemma. ■

Theorem 2.13 *1. For all $F \in \mathcal{F}$, $P \ker F(A_1, \ldots, A_k) \in \mathcal{H}_{\mathcal{A}}$,*

2. For all $U \in \mathcal{H}_{\mathcal{A}}$, there exists an operator $F \in \mathcal{F}$ such that $U = P \ker F(A_1, \ldots, A_k)$,

3. The k-tuples $(A_1, \ldots, A_k)$ and $(A'_1, \ldots, A'_k)$ are jointly similar if and only if there exists an invertible map $S : V \to V'$ such that for all $F \in \mathcal{F}$, $SP \ker F(A_1, \ldots, A_k) = P' \ker F(A'_1, \ldots, A'_k)$.

Proof.

1. Let $U = P \ker F(A_1, \ldots, A_k)$. If $v \in U$, then there exist $v_2, \ldots, v_k \in V$ such that $F(A_1, \ldots, A_k)(v, v_2, \ldots, v_k) = 0$. Consider an operator $C \in \mathcal{A}'$. We compute

$$\begin{aligned} & F(A_1, \ldots, A_k)(Cv, Cv_2, \ldots, Cv_k) \\ = \; & diag(C, \ldots, C) F(A_1, \ldots, A_k)(v, v_2, \ldots, v_k) \\ = \; & 0 \end{aligned}$$

Hence $Cv \in U$ which in turn implies that $U \in \mathcal{H}_{\mathcal{A}}$.

2. Given $U \in \mathcal{H}_{\mathcal{A}}$, let

$$W = \bigwedge_{\substack{F \in \mathcal{F} \\ U \subseteq P \ker F(A_1, \ldots, A_k)}} P \ker F(A_1, \ldots, A_k).$$

For all $w \in W$, there exists $v_1 \in U$ such that $v_1 \in P \ker F(A_1, \ldots, A_k)$ implies that $w \in P \ker F(A_1, \ldots, A_k)$. Extend v_1 to a basis $\{v_1, v_2, \ldots, v_n\}$ of V.

For $r, s \in \{1, \ldots, n\}$, and $t \in \{1, \ldots, k\}$, define $a_{rst} \in \mathbf{C}$ such that

$$A_t x_i = \sum_{j=1}^{n} a_{jit} x_i.$$

Further define $F \in \mathcal{F}$ as follows:

$$F(z_1, \ldots, z_k) = \begin{bmatrix} z_1 & 0 & \ldots & 0 \\ 0 & z_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & z_1 \\ & \vdots & & \\ z_k & 0 & \ldots & 0 \\ 0 & z_k & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & z_k \end{bmatrix} - \begin{bmatrix} \\ [a_{ij1}]\, I_n \\ \\ \vdots \\ \\ [a_{ijk}]\, I_n \\ \\ \end{bmatrix}.$$

By assumption, there exist vectors $w_2, \ldots, w_n \in V$ such that

$$F(A_1, \ldots, A_k)(w_1, \ldots, w_n) = 0.$$

Define the operator $C : V \to V$ such that $C(v_i) = w_i$. Note that

$$A_t C v_i = A_t w_i = \sum_{j=1}^{n} a_{ijt} w_j = \sum_{j=1}^{n} a_{ijt} C v_j = C \sum_{j=1}^{n} a_{ijt} v_j = C A_t v_i.$$

Hence, $C \in \mathcal{A}'$ which implies $w \in U$, which in turn implies that $W = U$. This establishes the result.

3. ($\Rightarrow$) This direction is trivial.

($\Leftarrow$) Without loss of generality, assume that

$$P \ker F(A_1, \ldots, A_k) = P \ker F(A'_1, \ldots, A'_k)$$

for all $F \in \mathcal{F}$. Take $x_1 = x$ and extend to a basis $\{x_1, \ldots, x_n\}$ for V. Define a_{jit} and $F \in \mathcal{F}$ as in 2. Clearly, $(x, x_2, \ldots, x_n) \in \ker F(A_1, \ldots A_k)$. Hence, by definition, $x \in P \ker F(A_1, \ldots, A_k)$. By assumption, $x \in P \ker F(A'_1, \ldots, A'_k)$. So there exist $x'_2, \ldots, x'_n \in V$, such that $(x, x'_2, \ldots, x'_n) \in \ker F(A'_1, \ldots, A'_n)$.

Define the operator $X : V \to V$ as follows: $Xe_1 = x$, $Xe_i = x_i$ for $i \in \{2, \ldots, n\}$. Note that $A'_i X = X A_i$ for all $i \in \{1, \ldots, k\}$. Further note that x is an eigenvector for X with associated eigenvalue 1. By the lemma, there exists an invertible $T : V \to V$ such that $A'_i T = T A_i$. ■

If the vector space V is indecomposable under the action of the algebra $\mathcal{A}$, we can make several additional assertions.

Proposition 2.14 *If the vector space V is indecomposable under the action of the algebra $\mathcal{A}$, then the algebra $\mathcal{A}'$ may be diagonalized.*

Proof. Let D and N be the sets of diagonalizable and nilpotent elements of $\mathcal{A}'$ respectively. Since V is indecomposable under the action of $\mathcal{A}$, the set D consists of just the scalars. Consider elements B_1, B_2 in N. Note that $\text{Trace}(B_1 + B_2) = \text{Trace}B_1 + \text{Trace}B_2 = 0$. Since V is indecomposable, $\sigma(B_1 + B_2) = \{0\}$. Hence, N is a subspace. Further note that B_1B_2 is singular. Again, since V is indecomposable, B_1B_2 must be nilpotent. We now have that N is a nil algebra. Every finite dimensional nil algebra is nilpotent. The result follows. ■

Corollary 2.15 *If the vector space V is indecomposable under the action of the algebra $\mathcal{A}$, then the lattice $\mathcal{H}_{\mathcal{A}}$ contains a chain of $n+1$ subspaces.*

Recall, that given a lattice of subspaces $\mathcal{L}$, and a join-irreducible element $U \in \mathcal{L}$, we have defined $\check{U}$ to be the largest element of $\mathcal{L}$ properly contained in U.

Proposition 2.16 *If the vector space V is indecomposable under algebra $\mathcal{A}$, and $U \in \mathcal{J}(\mathcal{H}_{\mathcal{A}})$, then $\dim U/\check{U} = 1$.*

Proof. We wish to show that there is an operator $E \in \mathcal{A}'$ such that $\text{card}(\sigma(E)) \neq 1$. For, if so, then the generalized eigenspaces for E form a non-trivial decomposition of V into invariant subspaces of $\mathcal{A}$. Let $x, y \in U$ such that $x + \check{U}, y + \check{U}$ are linearly independent in $V/\check{U}$. There exists $C \in \mathcal{A}'$ such that $Cx = y$. If $\text{card}(\sigma(C)) \neq 1$, let $E = C$. Assume then, that $\sigma(C) = \lambda$. Let $z = (C - \lambda)x$. Note that $z \in U$, $z \notin \check{U}$. Hence, there exists $D \in \mathcal{A}'$ such that $Dz = x$. Note that $D(C-\lambda)x = x$ and that $D(C-\lambda)$ is not invertible. Therefore $\{0,1\} \subseteq \sigma(D(C-\lambda))$. Set $E = D(C - \lambda)$. Since $E \in \mathcal{A}'$, we have established the result. ■

3 D-Normal Form

Although, in chapter 2, we have given a complete set of joint-similarity invariants for a k-tuple of commuting operators, we can hardly claim to have a general structure theory. Our characterization of the lattice $\mathcal{H}_{\mathcal{A}}$, though concrete, is not transparent. We have not even offered a general solution to the problem of when the vector space V is indecomposable under the action of $\mathcal{A}$, though we have given several necessary conditions.

In this section, we develop a structure theory for a special class of k-tuples of commuting nilpotents $\mathbf{A} = (A_1, \ldots, A_k)$ where $A_1, \ldots, A_k : V \to V$. For such

k-tuples, there exists a basis $\mathcal{B}$ for V which clarifies the algebraic structure of the k-tuple in much the same way that a Jordan basis does for a single nilpotent. We state necessary and sufficient conditions for the existence of such a basis, and conclude the chapter by answering the previously posed structure theory problems for this special class.

Recall that we have defined the lattice $\mathcal{L}_{\mathbf{A}}$ to be the smallest non-empty lattice of subspaces closed under images and preimages of all the A_i. In general, this lattice is not distributive, nor even finite and the lattice maps on $\mathcal{L}_{\mathbf{A}}$ induced by A_i and A_i^{-1} may not be particularly nice.

Recall from the first chapter, that the distributivity of $\mathcal{L}_A$, in the case of a single linear operator A, had a number of nice consequences. Amongst them, the lattice maps on $\mathcal{L}_A$ induced by A_i and A_i^{-1} were both lattice homomorphisms. In this chapter, we begin by assuming that the lattice $\mathcal{L}_{\mathbf{A}}$ is distributive.

We may form the directed graphs $\Gamma_i = \Gamma_i(\mathcal{J}(\mathcal{L}_{\mathbf{A}}), \mathcal{E}_i)$ with vertex set $\mathcal{J}(\mathcal{L}_{\mathbf{A}})$. Given $U_1, U_2 \in \mathcal{J}(\mathcal{L}_{\mathbf{A}})$, let $(U_1, U_2) \in \mathcal{E}$ if $A_i U_1 = U_2$. Define the directed graph

$$\Gamma = \Gamma(\mathcal{J}(\mathcal{L}_{\mathbf{A}}), \bigcup_{i=1}^{k} \mathcal{E}_i).$$

As in chapter 1, denote the connected components of Γ by Π.

Define the set $\mathcal{J}_{A_i}(\mathcal{L}_{\mathbf{A}}) = \{U \in \mathcal{J}(\mathcal{L}_{\mathbf{A}}) : A_i U \neq (0)\}$, and the set $\mathcal{J}_{A_i^{-1}} = \{U \in \mathcal{J}(\mathcal{L}_{\mathbf{A}}) : U \leq A_i V\}$. By proposition 1.5, the outdegree of a vertex $U \in \mathcal{J}(\mathcal{L}_{\mathbf{A}})$ of Γ_i is equal to 1 if $U \in \mathcal{J}_{A_i}(\mathcal{L}_{\mathbf{A}})$ and equal to 0 otherwise. We further proved, in lemmas 1.6 and 1.7 that the distributivity of $\mathcal{L}_{\mathbf{A}}$ implies that the indegree of each vertex U in Γ_i is 1 if $U \in \mathcal{J}_{A_i^{-1}}$ and 0 otherwise.

We define the maps $\phi_{A_i} : \mathcal{J}_{A_i}(\mathcal{L}_{\mathbf{A}}) \to \mathcal{J}(\mathcal{L}_{\mathbf{A}})$ as follows:

$$\phi_{A_i}(U) = A_i U,$$

and the maps $\psi_{A_i} : \mathcal{J}_{A_i}(\mathcal{L}_{\mathbf{A}}) \times \mathcal{L}_{\mathbf{A}}$ thus:

$$\psi_{A_i}(U, W) = A_i W.$$

Further, define the maps

$$\Phi_{A_i} : U/W \to \phi_{A_i} U / \psi_{A_i}(U, W)$$

such that

$$\Phi_{A_i}(x + W) = A_i x + \psi_{A_i}(U, W).$$

Similarly, define the maps $\phi_{A_i^{-1}} : \mathcal{J}_{A_i^{-1}}(\mathcal{L}_{\mathbf{A}}) \to \mathcal{J}(\mathcal{L}_{\mathbf{A}})$

$$\phi_{A_i^{-1}}(U) = U',$$

the maps $\psi_{A_i^{-1}} : \mathcal{J}_{A_i^{-1}}(\mathcal{L}_{\mathbf{A}}) \times \mathcal{L}_{\mathbf{A}} \to \mathcal{L}_{\mathbf{A}}$.

$$\psi_{A_i^{-1}}(U, W) = A_i^{-1} W \wedge U',$$

and

$$\Phi_{A_i^{-1}}(x+W) = (A_i^{-1}x \wedge \phi_{A_i^{-1}}(U)) + \psi_{A_i}(U,W)$$

where $U' \in \mathcal{L}(\mathcal{J})$ and $A_iU' = U$.

Given $U_0 \in \mathcal{J}(\mathcal{L})$, say that $\Lambda = (C_1, \ldots, C_r)$ where

$$C_i \in \{A_1, \ldots, A_k, A_1^{-1}, \ldots, A_k^{-1}\}$$

is a U_0-admissible sequence, if $\phi_\Lambda U = \phi_{C_1} \ldots \phi_{C_r} U$ is defined. Given the U_0-admissible sequence Λ, define Λ^{-1} to be the sequence $(C_r^{-1}, \ldots C_1^{-1})$. If $\phi_\Lambda U_0 = U_1$, then Λ^{-1} is a U_1-admissible sequence and $\phi_\Lambda U_1 = U_0$. Say that Λ is a U_0-admissible loop if Λ is a U_0-admissible sequence, and $\phi_\Lambda U_0 = U_0$.

Even assuming that the lattice $\mathcal{L}_\mathbf{A}$ is distributive, it still takes fairly strong assumptions to give a multivariable analogue of proposition 1.9. To this end, we introduce the following definition.

Say that the lattice $\mathcal{L}$ is $\mathbf{A}$-good if, for all $U \in \mathcal{J}(\mathcal{L})$, and for all U-admissible loops Λ, there exists a subspace E_U such that $E_U \oplus \check{U} = U$ for which

$$\Phi_\Lambda(x) = \Phi_{C_1} \ldots \Phi_{C_r}(x) = x + \psi_\Lambda(U, (0)),$$

for all $x \in E_U$.

Example 3.1

Recall the terminology from example 2.1. The pair of operators $(S_2, 2S_2)$ generates a lattice $\mathcal{L}_{(S_2,2S_2)}$ which contains only three elements. However, $\mathcal{L}_{(S_2,2S_2)}$ is clearly not $(S_2, 2S_2)$-good.

Given the lattice $\mathcal{L}$, and a subspace E of V, define $\mathcal{L}_\mathbf{A}(\mathcal{L}, E)$ to be the smallest lattice containing $\mathcal{L}$ and E, and closed under images and preimages of each of the A_i.

Denote the connected components of $\Gamma_\mathcal{B}$ by $\Pi_\mathcal{B}$.

Recall that, given a lattice $\mathcal{L}$, we have defined $L_\mathcal{L} v$ to be the smallest element of $\mathcal{L}$ to contain v. We generalize this definition to subspaces.

Given a lattice $\mathcal{L}$, and a subspace $W' \subseteq V$, define

$$L_\mathcal{L}(W') = \bigvee_{\substack{W' \subseteq W \\ W \in \mathcal{L}}} W.$$

Lemma 3.1 *If the lattice $\mathcal{L}$ is distributive and $\mathbf{A}$-good, $U_0 \in \mathcal{J}(\mathcal{L})$, and $E \leq U_0$ such that $E \oplus \check{U}_0 = U_0$, then*

1. *The lattice $\mathcal{L}' = \mathcal{L}_\mathbf{A}(\mathcal{L}, E)$ is distributive and $\mathbf{A}$-good.*
2. *The function $L_\mathcal{L} |_{\mathcal{J}_0(\mathcal{L}')}: \mathcal{J}_0(\mathcal{L}') \to \mathcal{J}_0(\mathcal{L})$, and is one-to-one and onto,*
3. *For all $U' \in \mathcal{J}_0(\mathcal{L})$, $i \in \{1, \ldots, K\}$, $L_\mathcal{L} A_i U' = A_i L_\mathcal{L} U'$, and*
4. *We have the containment $U' \leq L_\mathcal{L} U'$.*

Proof. Let π_0 be the connected component of U_0 in the graph Γ. If $U_1 \in \pi_0$, and we have a U_0-admissible sequence Λ such that $\phi_\lambda U_0 = U_1$, then clearly $\psi_\Lambda(U_0, E) \vee \check{U}_1 = U_1$. We claim that, if $\phi_{\Lambda_i} U_0 = U_1$ for $i \in \{1, \ldots, s\}$, then

$$(\bigwedge_{i=1}^{s} \psi_{\Lambda_i}(U_0, E)) \vee \check{U}_1 = U_1.$$

To see this, note that if Λ_1 and Λ_2 are U_0-admissible sequences and $\phi_{\Lambda_1} U_0 = \phi_{\Lambda_2} U_0 = U_1$, then $\Lambda_2 \Lambda_1^{-1} \Lambda_1$ is a U_0-admissible sequence and $\phi_{\Lambda_2 \Lambda_1^{-1} \Lambda_1} U_0 = U_1$. Further note that under these same conditions,

$$\psi_{\Lambda_2 \Lambda_1^{-1} \Lambda_1}(U_0, E) = \psi_{\Lambda_1}(U_0, E) \vee \psi_{\Lambda_2}(U_0, E).$$

Hence,

$$\dim(\psi_{\lambda_1}(U_0, E) \vee \psi_{\lambda_2}(U_0, E)) = \dim((\psi_{\lambda_1}(U_0, E) \wedge \check{U}_1) \vee (\psi_{\lambda_2}(U_0, E) \wedge \check{U}_1)) + \dim E.$$

By application of the principle of inclusion-exclusion to the dimensions of the subspaces in question, we have proven the claim.

If $U_1 \in \mathcal{J}(\mathcal{L})$ and $U_1 \in \pi_0$, then define

$$U_1' = (\bigwedge_{\phi_\Lambda U_0 = U_1} \psi_\Lambda(U_0, E)).$$

If $U_1 \notin \pi_0$, define $U_1' = U_1$. Clearly, the elements U' form the join-irreducible elements of $\mathcal{L}'$. ∎

Proposition 3.2 *Given a k-tuple of operators* $\mathbf{A}$, *such that the lattice* $\mathcal{L}_{\mathbf{A}}$ *is* $\mathbf{A}$-*good and distributive, there exists a direct sum decomposition*

$$V = \bigoplus_{U \in \mathcal{J}(\mathcal{L})} E_U,$$

such that for each $U \in \mathcal{J}(\mathcal{L})$, $U = \check{U} \oplus E_U$, *and for each* $\pi \in \Pi$, *there exists a collection of isomorphisms*

$$\theta_{U_1,U_2} : E_{U_1} \to E_{U_2}$$

where $U_1, U_2 \in \pi$, *such that*

1. $\theta_{U,A_iU} = A_i |_{E_U}$ *if* $A_i U \neq (0)$,
2. $\theta_{U_1,U_2} = \theta_{U_1,U_3}\theta_{U_3,U_2}$.

Proof. We begin with $\mathcal{L} = \mathcal{L}_{\mathbf{A}}$ and iteratively apply the preceding lemma. Ultimately, one obtains a complemented, distributive, $\mathbf{A}$-good lattice $\mathcal{L}_*$, and a map $L_{\mathcal{L}_{\mathbf{A}}} : \mathcal{L}_* \to \mathcal{L}_{\mathbf{A}}$ whose restriction to $\mathcal{J}_0(\mathcal{L}_*)$ is one-to-one and onto. For each $U \in \mathcal{J}(\mathcal{L}_{\mathbf{A}})$, define E_U to be the unique element of $\mathcal{J}(\mathcal{L}_*)$ such that $L_{\mathcal{L}_{\mathbf{A}}}(E_U) = U$. By assumption, $\mathcal{L}_*$ is closed under the action of A_i, and A_I^{-1} for each $i \in \{1, \ldots, k\}$. Hence, the set $\mathcal{J}_0(\mathcal{L}_*)$ is invariant under images of each of the A_i. The last assertion follows directly from the $\mathbf{A}$-goodness of $\mathcal{L}_*$. ∎

Call two U-admissible sequences $\Lambda = (C_1, \ldots, C_r)$, and $\Lambda' = (C'_1, \ldots, C'_{r'})$ adjacent, where $r > r'$, if either

1. $r' = r$ and there exists a $j \in \{1, \ldots, r-1\}$ such that $C_j = C'_{j+1}$ and $C_{j+1} = C'_j$,

or

2. $r' = r - 2$, and there exists a $t \in \{1, \ldots, r'+1\}$ such that
$C_t = A_i$ and $C_{t+1} = A_i^{-1}$ or $C_t = A_i^{-1}$ and $C_{t+1} = A_i$ for some $i \in \{1, \ldots, k\}$, and
$C_j = C'_j$ for $j < t$ and
$C_j = C'_{j-2}$ for $j \geq t$.

Note that if Λ_1 and Λ_2 are adjacent, U_0-admissible sequences, then either $\psi_{\Lambda_1}(U_0, (0)) \subseteq \psi_{\Lambda_2}(U_0, (0))$, or $\psi_{\Lambda_2}(U_0, (0)) \subseteq \psi_{\Lambda_1}(U_0, (0))$. Further, if we assume the former, then

$$\Phi_{\Lambda_1}(x + (0)) + \psi_{\Lambda_2}(U_0, (0)) = \Phi_{\Lambda_2}(x + (0)).$$

Call $\mathcal{J}(\mathcal{L})$ doubly simply connected if for every $U_1, U_2 \in \mathcal{J}(\mathcal{L})$ such that $U_1 \geq U_2$, and for all loops Λ which are both U_1 and U_2 admissible, there exists a sequence of loops $\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_r = \Lambda_{Id}$ such that Λ_i is adjacent to Λ_{i+1}, and each Λ_i is both U_1 and U_2-admissible. Here, Λ_{Id} is the trivial loop.

The following easily proved result allows us to drop the assumption of **A**-goodness in some cases.

Proposition 3.3 *Given a k-tuple of operators $\mathbf{A}$, if $\mathcal{L}_\mathbf{A}$ is distributive, and $\mathcal{J}(\mathcal{L}_\mathbf{A})$ is doubly simply connected, then $\mathcal{L}$ is $\mathbf{A}$-good.*

Theorem 3.4 *Given a k-tuple of operators $\mathbf{A}$, if $\mathcal{L}_\mathbf{A}$ is distributive, and $\mathcal{J}(\mathcal{L}_\mathbf{A})$ is doubly simply connected, then there exists a direct sum decomposition*

$$V = \bigoplus_{U \in \mathcal{J}(\mathcal{L})} E_U,$$

such that

1. *For each $U \in \mathcal{J}(\mathcal{L})$, $U = \check{U} \oplus E_U$, and*
2. *If $A_i U \neq (0)$, then $A_i E_U = E_{A_i U}$.*

Proof. This follows immediately from the preceding proposition, and from proposition 3.2. ■

We note the following easily proven combinatorial result.

Proposition 3.5 *a) If $\mathcal{J}(\mathcal{L})$ is doubly simply connected, then there exists a map $K : \mathcal{J}(\mathcal{L}) \to \mathbf{Z}^k$ such that if $U \in \mathcal{J}(\mathcal{L})$, $A_i U \neq (0)$, then $K(A_i U) = K(U) - e_i$, where e_i is the ith coordinate vector.*

b) If $k = 2$ and such an $K : \mathcal{J}(\mathcal{L}) \to \mathbf{Z}^k$ exists, then $\mathcal{J}(\mathcal{L})$ is doubly simply connected.

Let us consider another pair of examples based on example 2.1.

Example 3.2

Define the set $\Omega = \{(0,3),(1,3),(2,3),(1,2),(2,2),(2,1),(3,1),(3,0)\}$. Consider the pair of operators $\mathbf{A} = (A_1, A_2)$, $A_1, A_2 : V_\Omega \to V_\Omega$ where $V_\Omega = \{e_\alpha\}_{\alpha\in\Omega}$. Define

$$A_1 e_\alpha = \begin{cases} e_{\alpha-(1,0)}, & \alpha-(1,0)\in\Omega \\ 0 & otherwise \end{cases}.$$

Define

$$A_2 e_\alpha = \begin{cases} e_{\alpha-(0,1)}, & \alpha-(0,1)\in\Omega \\ 0 & otherwise \end{cases}.$$

Define the set $\Omega' = \{(0,3),(1,3),(1,2),(2,2),(3,2),(2,1),(3,1),(3,0)\}$. Consider the pair of operators $\mathbf{A}' = (A_1', A_2')$, $A_1', A_2' : V_{\Omega'} \to V_{\Omega'}$ where $V_{\Omega'} = \{e_\alpha\}_{\alpha\in\Omega'}$. Define

$$A_1' e_\alpha = \begin{cases} e_{\alpha-(1,0)}, & \alpha-(1,0)\in\Omega' \\ 0 & otherwise \end{cases}.$$

Define

$$A_2 e_\alpha' = \begin{cases} e_{\alpha-(0,1)}, & \alpha-(0,1)\in\Omega' \\ 0 & otherwise \end{cases}.$$

We consider several of the structure questions outlined in the introduction applied to these two pairs of commuting operators. Namely, are the two pairs jointly similar (Clearly, all four of the operators are individually similar), and are the spaces $V_{\Omega'}$, and V_Ω irreducible under their respective algebras.

The answer to the first question may not be immediately obvious, as many of the obvious invariants are equal for the two pairs (eg. $\dim(\ker A^i B^j) = \dim(\ker A'^i B'^j)$ for all i and j). Upon initial consideration, the answer to the second question also may not be immediately obvious. The directed graphs $\check{\Gamma}$ and $\check{\Gamma}'$ formed by using the basis vectors as nodes, and inserting directed edges in the obvious way, are both connected. So it seems plausible that the spaces $V_{\Omega'}$, and V_Ω are both irreducible. However, one must show that our perception of irreducibility is not dependent on our particular choice of basis.

To clarify the situation, we consider the two induced lattices $\mathcal{L}_{\mathbf{A}}$ and $\mathcal{L}_{\mathbf{A}'}$. These lattices are both distributive (and, in fact, isomorphic as lattices). We may consider $A, B : \mathcal{L}_{\mathbf{A}} \to \mathcal{L}_{\mathbf{A}}$ and $A', B' : \mathcal{L}_{\mathbf{A}'} \to \mathcal{L}_{\mathbf{A}'}$ to be lattice maps. These maps are lattice morphisms on their respective lattices as detailed in chapter 1.

-The reader may easily verify the following: There exists a map $\beta : \Omega \to \mathcal{J}(\mathcal{L}_{\mathbf{A}})$ which is one-to-one and onto, and such that

1. $V_{e_\omega} = \beta(\omega)$ for $\omega \in \Omega$,
2. $A_i e_\omega = e_{\omega'}$ if and only if $A_i\beta(\omega) = \beta(\omega')$.

The map $\beta' : \Omega' \to \mathcal{J}(\mathcal{L}_0')$ with the corresponding properties also exists.

The directed graph structures of Γ and Γ' are respectively joint similarity invariants for their respective operator pairs. Since, these directed graph structures are inherited from the graphs $\check{\Gamma}$ and $\check{\Gamma}'$ respectively, we see that the two pairs are not jointly similar.

Assume that for the k-tuple of commuting operators $\mathbf{A} = (A_1, \dots, A_k)$ where $A_i : V \to V$, there exists a basis $\mathcal{B} = \{e_\lambda\}_{\lambda \in \Lambda}$ for V such that:

1. $A_i e_\lambda \in \mathcal{B} \cup \{0\}$, and

2. $A_i e_{\lambda_1} = A_i e_{\lambda_2} \neq 0$ implies that $\lambda_1 = \lambda_2$.

Note that in such a case, if W is an element of the lattice $\mathcal{L}_\mathbf{A}$, then there exists a subset $\Lambda_W \subseteq \Lambda$ such that

$$W = \mathrm{span}\{e_\lambda\}_{\lambda \in \Lambda_W}.$$

We form the digraph $\Gamma_\mathcal{B} = \Gamma_\mathcal{B}(\mathcal{B}, \mathcal{E}_\mathcal{B})$ where $(e_{\lambda_1}, e_{\lambda_2}) \in \mathcal{E}$ if $A_i e_{\lambda_1} = e_{\lambda_2}$ for some i.

Denote the connected components of $\Gamma_\mathcal{B}$ by$\Pi_\mathcal{B}$.

Say that a basis $\mathcal{B}$ of V is $\mathcal{D}$-normal for the k-tuple $\mathbf{A}$ if it has properties 1. and 2. above, and also satisfies the following:

3. Given $\pi \in \Pi$ and $e_{\lambda_1}, e_{\lambda_2} \in \pi$, such that $e_{\lambda_2} \in L_{\mathcal{L}_\mathbf{A}} e_{\lambda_1}$, and $e_{\lambda_1} \in L_{\mathcal{L}_\mathbf{A}} e_{\lambda_2}$, then $\lambda_1 = \lambda_2$.

The following result is now easy.

Proposition 3.6 *A k-tuple of operators has a $\mathcal{D}$-normal basis $\mathcal{B} = \{e_\lambda\}_{\lambda \in \Lambda}$, and a map $\beta : \mathcal{B} \to \mathcal{J}(\mathcal{L}_\mathbf{A})$ such that $\beta(A_i e_\lambda) = A_i \beta(e_\lambda)$ if and only if the lattice $\mathcal{L}_\mathbf{A}$ is distributive and $\mathbf{A}$-good.*

Proof. ($\Leftarrow$) This is a consequence of proposition 3.2.

($\Rightarrow$) Clear. ∎

As a consequence of proposition 3.5, we have the following.

Corollary 3.7 *If there exists a map $K : \mathcal{J}(\mathcal{L}) \to \mathbf{Z}^2$ such that if $U \in \mathcal{J}(\mathcal{L})$,$A_i U \neq (0)$, then $L(A_i U) = L(U) - e_i$, and there exists a direct sum decomposition*

$$V = \bigoplus_{U \in \mathcal{J}(\mathcal{L})} E_U,$$

such that

1. For each $U \in \mathcal{J}(\mathcal{L})$, $U = \check{U} \oplus E_U$, and

2. If $A_i U \neq (0)$, then $A_i E_U = E_{A_i U}$.

The reader may be wondering what obstructs such a result from holding in the case when $k = 3$. We introduce the following.

Example 3.3

Define the set

$$\Omega = \{(0,2,0), (0,3,0), (1,2,0), (0,2,1), (0,3,1), (1,2,1), (1,3,1), \\ (3,0,0), (2,1,0), (3,1,0), (2,0,1), (3,0,1), (2,1,1), (3,1,1)\}.$$

Define the fourteen dimensional vector space $V_\Omega = \{e_\omega\}_{\omega \in \Omega}$, and the operators $A_1, A_2 : V_\Omega \to V_\Omega$

$$A_1 e_\omega = \begin{cases} e_{\omega-(1,0,0)}, & \omega - (1,0,0) \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

$$A_2 e_\omega = \begin{cases} e_{\omega-(0,1,0)}, & \omega - (0,1,0) \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

For each $\lambda \in \mathbf{C}$, define $A_{2,\lambda} : V_\Omega \to V_\Omega$

$$A_{3,\lambda} e_\omega = \begin{cases} e_{\omega-(0,0,1)}, & \omega - (0,0,1) \in \Omega, \omega \neq (1,2,1) \\ e_{(1,2,0)} + \lambda e_{(3,0,0)}, & \omega = (1,2,1) \\ 0 & \textit{otherwise} \end{cases}.$$

The reader may verify that for each $\lambda \in \mathbf{C}$, $\mathcal{J}(\mathcal{L}_\mathbf{A}) = \{U_{e_\omega} : \omega \in \Omega\}$. Hence, both the lattice $\mathcal{J}(\mathcal{L}_i\mathbf{A})$ and the lattice maps induced by the three operators and their inverses are identical for each λ. However, the triple $A_1, A_2; A_{3,\lambda} : V_\Omega \to V_\Omega$ is $\mathbf{A}$-good if and only if $\lambda = 0$.

Now, using the machinery based on $\mathcal{L}_\mathbf{A}$, we may answer our original structure questions.

Proposition 3.8 *Consider the k-tuples of commuting operators $\mathbf{A} = (A_1, \ldots, A_k)$, $\mathbf{A}' = (A'_1, \ldots, A'_n)$ where $A_i : V \to V$ and $A'_i : V' \to V'$, have $\mathcal{D}$-normal bases $\mathcal{B} = \{e_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{B}' = \{e'_\lambda\}_{\lambda\in\Lambda'}$ respectively. Then $\mathbf{A}$ and $\mathbf{A}'$ are jointly similar if and only if there exists a one-to-one and onto map $\mu : \mathcal{B} \to \mathcal{B}'$ such that*

$$A'_i \mu e_\lambda = \begin{cases} \mu A_i e_\lambda & \textit{if } A_i e_\lambda \neq 0 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. The forward implication follows from proposition 3.6. The reverse implication is clear. ■

Proof. ($\Rightarrow$) Since both $\mathbf{A}$ and $\mathbf{A}'$ have $\mathcal{D}$-normal bases, the lattices $\mathcal{L}_\mathbf{A}$ and $\mathcal{L}_{\mathbf{A}'}$ are distributive, and respecitvely $\mathbf{A}$-good and $\mathbf{A}'$-good.

Proposition 3.9 *Assume that the k-tuple of commuting operators $\mathbf{A} = (A_1, \ldots, A_k)$ where $A_i : V \to V$ has a $\mathcal{D}$-normal basis $\mathcal{B}$. Then the space V is indecomposable under the joint action of the operators in $\mathbf{A}$ if and only if the underlying graph of the digraph $\Gamma_\mathcal{B}$ is connected.*

Proof. ($\Leftarrow$) Assume that we can decompose $V = V_1 \oplus V_2$, where V_i is invariant under the joint action of $\mathbf{A}$. Clearly each $W \in \mathcal{L}_\mathbf{A}$ can be decomposed $W = W_1 \oplus W_2$ where $W_i \subseteq V_i$. Also, by our assumption of indecomposibility, $U \in \mathcal{J}(\mathcal{L}_\mathbf{A})$ implies that $\dim U/\check{U} = 1$. For each $U \in \mathcal{J}(\mathcal{L}_\mathbf{A})$, we may choose an $x_U \in U$ such that $x_U \oplus \check{U} = U$ and $x_U \in V_1$ or $x_U \in V_2$. Without loss of generality, assume that for some U_0, $x_{U_0} \in V_1$. By connectedness, $x_U \in V_1$ for all U. We hence have a linearly independent set of vectors in V_1 of size n.

($\Rightarrow$) Clear. ■

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