Circle Actions on Symplectic Manifolds

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Abstract of the Dissertation

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In this work we study different blow up constructions on symplectic orbifolds. Some of these may be used to describe the behavior of reduced spaces of a Hamiltonian circle action when passing a critical level of its Hamiltonian function containing critical points of signature $(2, 2k)$. We also describe this behavior for critical points of signature $(2k, 2d)$. We use these descriptions to generalize the Duistermaat-Heckman theorem to intervals of values of the Hamiltonian function containing critical values. Finally, we use localization formulas in equivariant cohomology to give conditions under which circle actions must be Hamiltonian.

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Chapter 1

Introduction

A circle action on a compact symplectic manifold M is symplectic if it preserves the symplectic form. In addition, it is Hamiltonian if its generating vector field X is Hamiltonian that is, if it satisfies $\iota_X\omega = dH$ where $H \in C^\infty(M)$ is the Hamiltonian function.

An obvious necessary condition for a circle action to be Hamiltonian is that it has fixed points, which correspond to the critical points of H. For Kähler manifolds and more generally for manifolds of Lefschetz type, this condition is also sufficient, (cf. [MD-S]). Moreover McDuff proved in [MD1] that this result also holds for four dimensional manifolds. However, this is not true for higher dimensions. In fact, McDuff constructed a six dimensional manifold with a symplectic circle action which has fixed points but is not Hamiltonian, (cf. [MD1]). Therefore we need more conditions either on the manifold or on the action to make sure the action is Hamiltonian for higher dimensions.

One possible conjecture is that a symplectic action with isolated fixed points must be Hamiltonian. This result has already been proved by Tolman and Weitsman, ([T-W]), in the case of a semi-free action that is, free outside the fixed point set. Moreover there are no known counterexamples, (the fixed point sets in McDuff's six-dimensional example are tori).

The argument used by Tolman and Weitsman uses integration in equivariant cohomology. Nevertheless, even though it generalizes to the case of non-semifree circle actions it fails to eliminate all non-Hamiltonian examples with a nonempty fixed point set. For example, as it is pointed out in ([T-W]), it does not rule out the existence of a symplectic six-manifold with a symplectic circle action with only two fixed points with the action having weights $(1, 1, -2)$ on the normal bundle to one fixed point and $(-1, -1, 2)$ on the normal bundle of the other.

Trying to obtain more information on this problem we first study in Chapter 3 non-semifree circle actions and describe the behavior of the reduced spaces $H^{-1}(\alpha)/S^1$ when passing a critical level. For this, it is necessary to generalize the usual blow up construction to orbifolds as these reduced spaces may now be orbifolds.

First, we define the blow up of an orbifold singularity x by considering the standard scalar circle action on a local Darboux uniformizing chart for x, $((B_\varepsilon^{2n}, \omega_o), \Gamma, \varphi)$. This action descends to a Hamiltonian action on a neighborhood of x and we can remove a neighborhood of this point of the form $H^{-1}([0,\varepsilon))$ and collapse the boundary of the resulting orbifold along the orbits of this circle action.

This construction can be made using other circle actions on the local Darboux uniformizing chart as long as they fix the origin and commute with the action of Γ. In the particular case of a conic singularity that is, one with a cyclic orbifold structure group we can for instance obtain interesting examples of *weighted* blow ups, (please refer to Section 3.4.1). Similarly to the manifold case, these special weighted blow ups are the connected sum of the orbifold with a weighted projective space with reverse orientation. Moreover, the exceptional divisors are also weighted projective spaces.

As an example of this type of blow up, the weighted blow up of a regular point corresponds to removing an ellipsoid around the point and collapsing the boundary of the resulting manifold along the orbits of the corresponding weighted circle action.

Just as the usual blow up of $\mathcal{O}(k+1)$ at a point of its zero section, (where $\mathcal{O}(t)$ is the bundle over $\mathbf{C}P^1$ with Euler class t), can be identified with the plumbing $E \# L_{can}$ where $E = \mathcal{O}(k)$ and $L_{can} \to \mathbf{C}P^{n-1}$ is the canonical line bundle, a similar result holds for these special cases of weighted blow up.

In fact, these can also be identified with a plumbing of two special orbibundles over a weighted projective space (cf. Section 3.4.2).

This result allows us in particular to determine how self-intersection numbers of spheres change with these weighted blow ups. Moreover as pointed out by McDuff in [MD2], it can be used in the context of almost complex structures in $S^2 \times S^2$, as we can see in Remark 3.4.8.

In addition to these blow up constructions on orbifolds, we also describe the blowing down process as well as the blow up along a symplectic suborbifold X. Here we consider a neighborhood of X in M of the form $P \times_{U(k)} \mathbf{C}^k$ where $P \to X$ is a principal $U(k)$-orbibundle. Moreover, we let the circle act on $P \times C^k$ by acting trivially on P and as the standard circle action on $\mathbf{C}^k$. This action will then descend to a a Hamiltonian action on the quotient $P \times_{U(k)} \mathbf{C}^k$ and so the blow up can be obtained by removing a smaller neighborhood of X of the form $H^{-1}([0,\varepsilon))$ and collapsing the boundary of the resulting orbifold, along the orbits of this new S^1 action.

Any circle action on $P \times \mathbf{C}^k$ that fixes P and descends to $P \times_{U(k)} \mathbf{C}^k$ can again be used to define a blow up of M along X.

Using these different blow up constructions, we obtain the following results, (Section 4.1 and 4.2) which continue the work of Guillemin and Sternberg in [G-S] and describe the behavior of the reduced spaces of a Hamiltonian circle action when passing through a critical level:

Theorem 1.0.1 *Let (M, ω) be a $(2n)$-symplectic manifold with a Hamiltonian S^1-action and let X be a critical submanifold with signature $(2, 2d)$, lying on the critical level $\mu = \lambda$. Then, on the normal directions to X, the reduced spaces $O_{\lambda+\varepsilon} = \mu^{-1}(\lambda+\varepsilon)/S^1$ are all diffeomorphic to the weighted $\mathbf{q}$-blow up of the reduced spaces $O_{\lambda-\varepsilon} = \mu^{-1}(\lambda-\varepsilon)/S^1$ at a point of order p, for suitable values of $\mathbf{q} \in (\mathbf{Z}^+)^d$ and $p \in \mathbf{Z}^+$ determined by the circle action.*

Theorem 1.0.2 *Let S^1 act on a symplectic manifold M^{2n} in a Hamiltonian fashion. Let μ be its Hamiltonian function and let X be a critical submanifold of signature $(2(d-k), 2k)$, lying on the critical level $\mu = \lambda$. Then on the normal directions to X, the reduced spaces $O_{\lambda+\varepsilon} = \mu^{-1}(\lambda+\varepsilon)/S^1$ can be obtained from the reduced spaces $O_{\lambda-\varepsilon} = \mu^{-1}(\lambda-\varepsilon)/S^1$ by a "singular blow down" of a copy of $\mathbf{C}P^{d-k-1}(\mathbf{p})$ followed by a "partial blow up", for suitable values of $\mathbf{p} \in (\mathbf{Z}^+)^{d-k}$ determined by μ. The exceptional divisor resulting from this blow up is $\mathbf{C}P^{k-1}(\mathbf{q})$ for $q \in (\mathbf{Z}^+)^{k-1}$ determined by μ.*

The "partial blow up" of a non-orbifold singularity of the critical reduced space used in this last case is obtained by removing a neighborhood of a singularity and then collapsing the boundary of the resulting orbifold along the fibres of a special fibration described in Section 4.2. The "singular blow down" will then be the opposite of this construction.

The Duistermaat-Heckman theorem, [D-H], allows us to compare the cohomology classes $[\omega_\varepsilon] \in H^2(O_\varepsilon, \mathbf{R})$ for different values of ε in the same component of the set of regular values of the Hamiltonian function H. In Section 4.3, using the above results, and based on a result of Brion and Procesi for circle actions on projective varieties, (cf. [B-P]), we generalize the Duistermaat-Heckman theorem to the case of an interval of values of the Hamiltonian function containing critical values.

In the case of an isolated critical point of signature $(2, 2(n-1))$ (or $(2(n-1), 2)$), this theorem generalizes to non-semifree circle actions, the similar result proved in [G-S] and in [Au]. In fact, if the fixed point has index 2 ($2(n-1)$) and S^1 acts with different weights on the normal bundle to the fixed point then only one of these weights is negative (positive). Let p be the absolute value of this weight. Then when passing such a critical point, the "rate of change" of the cohomology class of the symplectic form of the reduced spaces changes by $\frac{1}{p}\times$ the Poincare Dual of the homology class of the exceptional divisor resulting from the weighted p-blow up of the initial (final) reduced spaces.

Again like the equivariant cohomology methods used in [T-W] these results fail to rule out by themselves the possibility of existence of a non-Hamiltonian circle action with isolated fixed points in the non-semifree case.

On the second part of this work we use equivariant cohomology localization theorems to obtain additional information in the six dimensional case.

Here we study circle actions with only isolated fixed points. After showing that in the presence of non-trivial isotropy subgroups $\mathbf{Z}_k$, there are isotropy $\mathbf{Z}_k$-spheres through pairs of fixed points in the same connected component of $M^{\mathbf{Z}_k}$, we prove the following theorems:

Theorem 1.0.3 *If a circle action on a compact connected six dimensional symplectic manifold has only isolated fixed points and all the $\mathbf{Z}_k$-spheres have trivial normal bundles in M, then the action is necessarily Hamiltonian.*

Blowing up at fixed points of such actions will of course result in different Hamiltonian actions. Moreover examples of Hamiltonian circle actions on six dimensional manifolds with isolated fixed points satisfying the triviality condition on the normal bundles of isotropy spheres can be obtained from $S^2 \times S^2 \times S^2$ by considering different weighted diagonal S^1-actions.

If we assume instead that the circle action (with only isolated fixed points) satisfies the condition that the weights on the normal bundles to each fixed point are always $\pm p, \pm q, \pm r$ with p, q, r relatively prime and such that the largest of these numbers is different from the sum of the others, then the action is again Hamiltonian:

Theorem 1.0.4 *Let S^1 action act on a six-dimensional symplectic compact connected manifold. If the circle action has only isolated fixed points satisfying the condition that at the normal bundle to each fixed point the action weights are always $\pm p, \pm q, \pm r$, where $p, q, r > 2$ are relatively prime and the largest of these numbers is different to the sum of the others , then if $M^{S^1} \neq \emptyset$, the action must be Hamiltonian.*

Chapter 2

Preliminaries

This chapter contains the background needed for other chapters. First we review the usual definitions of orbifolds and orbibundles and then we describe some important results about weighted projective spaces wich we will need later.

2.1 Orbifolds and orbibundles

In this paragraph we recall the definitions of orbifold and orbibundle. Most of what is presented here can be found in [S1] and in [T2].

A differentiable n-orbifold M is basically a space locally modelled on $\mathbf{R}^n$ modulo a finite subgroup of the orthogonal group. These local coordinate systems are then glued toghether by diffeomorphismx. To give a formal definition we first need to define some additional structure:

Definition 2.1.1 *([S1]) Let M be a Hausdorff space. A C^∞* local uniformizing chart *for an open subset U_i of M, $(\widetilde{U}_i, \Gamma_i, \varphi_i)$, consists of a connected open set in $\mathbf{R}^n$, a finite group Γ_i of C^∞ automorphisms of $\widetilde{U}_i$ such that the set of its fixed points has at least codimension two and a continuous map φ_i from $\widetilde{U}_i$ to U_i such that $\varphi_i \circ \gamma = \varphi_i$ for all $\gamma \in \Gamma_i$ and which induces a homeomorphism from $\widetilde{U}_i/\Gamma_i$ onto U_i.*

After this we have,

Definition 2.1.2 *A C^∞* orbifold *M (*V-manifold *in Satake's terminology) consists of a Hausdorff topological space $|M|$ with a covering of open sets U_i closed under finite intersections such that to each U_i is associated a local uniformizing chart $(\widetilde{U}_i, \Gamma_i, \varphi_i)$, satisfying the following compatibility conditions:*

1. *For every point p of M there is at least one l.u.c. $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ with $p \in \varphi_i(\widetilde{U}_i) \cong U_i$.*

2. *If $(\widetilde{U}_i, \Gamma_i, \varphi_i)$ and $(\widetilde{U}_j, \Gamma_j, \varphi_j)$ are two l.u.c. such that $U_i \cong \varphi_i(\widetilde{U}_i) \subset \varphi_j(\widetilde{U}_j) \cong U_j$, then there is an injective homomorphism*

$$f_{ij} : \Gamma_i \to \Gamma_j$$

and a smooth open embedding $\lambda_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ equivariant with respect to f_{ij}, (i.e. $\lambda_{ij}(\gamma\tilde{x}) = f_{ij}(\gamma)\lambda_{ij}(\tilde{x})$),satisfying $\varphi_j \circ \lambda_{ij} = \varphi_i$.

$$\begin{array}{ccccc} & \widetilde{U}_i & \xrightarrow{\lambda_{ij}} & \widetilde{U}_j & \\ & \downarrow & & \downarrow & \\ & \widetilde{U}_i/\Gamma_i & \xrightarrow{\lambda_{ij}/\Gamma_i} & \widetilde{U}_j/\Gamma_j & \\ \varphi_i & \uparrow & & \uparrow & \varphi_j \\ & U_i & \hookrightarrow & U_j & \end{array}$$

Remark 2.1.3 *1. Each λ_{ij} is defined only up to composition with elements of Γ_i, and each f_{ij} is defined up to conjugation by elements of Γ_j.*

2. If $U_i \subset U_j \subset U_k$ and $\lambda_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ and $\lambda_{jk} : \widetilde{U}_j \to \widetilde{U}_k$ are the respective injections, then there is a $\gamma \in \Gamma_k$ such that $\gamma\lambda_{ik} = \lambda_{jk} \circ \lambda_{ij}$ and

$$\gamma \cdot f_{ik}(\tilde{\gamma}) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(\tilde{\gamma})$$

Example 2.1.4 *1. A typical example of an orbifold is provided by a properly discontinuous (e.g. finite) action of a group G on a manifold M. Then the topological orbit space $|M/G|$ with an atlas obtained by suitable restrictions of the quotient map, forms the orbifold M/G.*

2. Any closed manifold is an orbifold where G is the trivial group.

If p is a point on an orbifold M, $(\widetilde{U}, \Gamma, \varphi)$ is a l.u.s. for p and $\tilde{p} \in \widetilde{U}$ is such that $\varphi(\tilde{p}) = p$, then the isotropy subgroup of Γ at $\tilde{p}$, Γ_p is well-defined not depending up to isomorphism, on the choice of $\widetilde{U}$ or $\tilde{p}$ and is called the *(orbifold) structure group of p*. A point p is *regular* (or *generic*) if its orbifold structure group is trivial and is *singular* otherwise. The set of points with nontrivial structure group is called the *singular locus* of the orbifold.

The singular locus of a differentiable orbifold can then be described as follows. Let $(\widetilde{U}, \Gamma, \varphi)$ ba any l.u.s. There is a Riemannian metric on $\widetilde{U}$ which is Γ-invariant, (just consider any metric on$\widetilde{U}$ and average under Γ). For any point $\tilde{x} \in \widetilde{U}$ the exponencial map gives us a diffeomorphism from the ε-ball in the tangent space of $\tilde{x}$ to a small neighborhood of $\tilde{x}$. Moreover this map commutes with the action of the isotropy group of $\tilde{x}$, and so it descends to I giving us an isomorphism from a neighborhood of $\varphi(\tilde{x})$ in U and a neighborhood of the origin in the vector *orbi-sopace* $\mathbf{R}^n/\Gamma$, where Γ is a finite subgroup of the orthogonal group $O(n)$.

An *orbifold map* ψ from $(M_1, \mathcal{A}_1)$ into $(M_2, \mathcal{A}_2)$ where

$$\mathcal{A}_1 = \{(\widetilde{U}_i, \Gamma_i, \varphi_i)\}, \;\; \mathcal{A}_2 = \{(\widetilde{V}_i, \hat{\Gamma}_i, \hat{\varphi}_i)\},$$

is a system of mappings $\{\psi_{\widetilde{U}_i}\}$ such that $\psi_{\widetilde{U}_i} : \widetilde{U}_i \to \widetilde{V}_i$ is a C^∞ map and for any injection λ_{ij} : $\widetilde{U}_i \to \widetilde{U}_j$ there is another injection $\hat{\lambda}_{ij} : \widetilde{V}_i \to \widetilde{V}_j$ such that

$$\hat{\lambda}_{ij} \circ \psi_{\widetilde{U}_i} = \psi_{\widetilde{U}_j} \circ \lambda_{ij}.$$

There is also a concept of a (complex) *orbibundle* $\pi : E \to B$ over an orbifold B with generic fiber F and structure group G.

Definition 2.1.5 *A pair of atlases $\mathcal{A}, \mathcal{A}^*$, $\mathcal{A}$ being an atlas for B and $\mathcal{A}^*$ that of E, is called an atlas for an orbibundle (E, B, π, F, G) with fiber F and structure group G, if it satisfies the following conditions:*

1. *G is a Lie group operating on F as a C^∞-group of transformations and containing all the groups Γ_i.*

2. *There is a one-to-one correspondence $(\widetilde{U}_i, \Gamma_i, \varphi_i) \leftrightarrow (\widetilde{U}_i^*, \Gamma_i^*, \varphi_i^*)$ between $\mathcal{A}$ and $\mathcal{A}^*$ such that $\widetilde{U}_i^* = \widetilde{U}_i \times F$ (where $\Gamma_i^* = \Gamma_i$ acts by the diagonal action) and, denoting by $pr_{\widetilde{U}_i^*} : \widetilde{U}_i^* \to \widetilde{U}_i$ the projection on the first factor,*

$$\pi \circ \varphi_i^* = \varphi_i \circ pr_{\widetilde{U}_i^*}.$$

i.e. the following diagram commutes:

$$\begin{array}{ccccc} \widetilde{U}_i^* = & \widetilde{U}_i \times F & \xrightarrow{\varphi_i^*} & \widetilde{U}_i \times_{\Gamma_i} F & = \pi^{-1}(U_i) \\ & \downarrow pr_{\widetilde{U}_i^*} & & \downarrow \pi & \\ & \widetilde{U}_i & \xrightarrow{\varphi} & \widetilde{U}_i/\Gamma_i & = U_i \end{array}$$

3. *Let $((\widetilde{U}_i, \Gamma_i, \varphi_i), (\widetilde{U}_i^*, \Gamma_i^*, \varphi_i^*))$, $((\widetilde{U}_j, \Gamma_j, \varphi_j), (\widetilde{U}_j^*, \Gamma_j^*, \varphi_j^*))$ be two pairs of corresponding l.u.s. in $(\mathcal{A}, \mathcal{A}^*)$ and let $U_i \subset U_j$. Then,*

$$U_i^* \subset U_j^*$$

and there is a one-to-one correspondence $\lambda_{ij} \leftrightarrow \lambda_{ij}^*$ between embeddings $\lambda_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ and $\lambda_{ij}^* : \widetilde{U}_i^*, \to \widetilde{U}_j^*$ such that for $(\tilde{p}, f) \in \widetilde{U}_i^*$, we have

$$\lambda_{ij}^*(\tilde{p}, f) = (\lambda_{ij}(\tilde{p}), g_{\lambda_{ij}}(\tilde{p})f)$$

with $g_{\lambda_{ij}}(\tilde{p}) \in G$. (The mapping $g_{\lambda_{ij}} : \widetilde{U}_i \to G$ is a C^∞-map satisfying

$$\gamma g_{\lambda_{ik}}(\tilde{p}) = g_{\lambda_{jk}}(\lambda_{ij}(\tilde{p})) \cdot g_{\lambda_{ij}}(\tilde{p})$$

where γ is determined by $\gamma \cdot \lambda_{ik} = \lambda_{jk} \circ \lambda_{ij}$, for the embeddings $\widetilde{U}_i \overset{\lambda_{ij}}{\to} \widetilde{U}_j \overset{\lambda_{jk}}{\to} \widetilde{U}_k$. These maps are the orbifold analog to transition maps).

Example 2.1.6 *Consider the actions of μ_q on $\mathbf{C} \times \mathbf{C}^n$ and on $\mathbf{C}$ given respectively by:*

$$\xi \cdot (z, \mathbf{v}) = (\xi^a z, \xi^{w_1} v_1, \ldots, \xi^{w_n} v_n), \ \ \xi \in \mu_q \tag{2.1}$$

$$\xi \cdot v = (\xi^{w_1} v_1, \ldots, \xi^{w_n} v_n), \ \ \xi \in \mu_q \tag{2.2}$$

These define the orbibundle $E \equiv \mathbf{C} \times_{\mu_q(\tilde{\mathbf{w}})} \mathbf{C}^n \overset{\pi}{\longrightarrow} \mathbf{C}^n/\mu_q(\mathbf{w})$ given by the following commutative diagram:

$$\begin{array}{ccc} \mathbf{C} \times \mathbf{C}^n & \overset{\rho_j}{\longrightarrow} & \mathbf{C}^n \\ \tau(\tilde{\mathbf{w}}) \downarrow & & \downarrow \tau(\mathbf{w}) \\ \mathbf{C} \times_{\mu_{q_j}} \mathbf{C}^n & \overset{\rho_j'}{\longrightarrow} & \mathbf{C}^n/\mu_{q_j} \end{array}$$

where

1. *$\tau(\tilde{\mathbf{w}}) : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C} \times_{\mu_q(\tilde{\mathbf{w}})} \mathbf{C}^n$ and $\tau(\mathbf{w}) : \mathbf{C}^n \to \mathbf{C}^n/\mu_q(\mathbf{w})$ are respectively the quotient maps of the μ_q-actions on $\mathbf{C} \times \mathbf{C}^n$ and on $\mathbf{C}^n$ defined on (2.1) and on (2.2) with $\tilde{\mathbf{w}} = (a, \mathbf{w}) = (a, w_1, \ldots, w_n)$.*

2. *$\rho : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ is the projection on the last n coordinates.*

3. *$\rho' : \mathbf{C} \times_{\mu_q(\tilde{\mathbf{w}})} \mathbf{C}^n \to \mathbf{C}^n/\mu_q(\mathbf{w})$ is defined by*

$$\rho'(z, v_1, \ldots, v_n)_{q(\tilde{\mathbf{w}})} = (v_1, \ldots, v_n)_{q(\mathbf{w})}$$

Remark 2.1.7 *If q divides a, then E is a complex product bundle over $\mathbf{C}^n/\mu_q(\mathbf{w})$ in the usual sense, as now the action of μ_q on $\mathbf{C} \times \mathbf{C}^n$ described in (2.1) becomes*

$$\xi \cdot (z, \mathbf{v}) = (z, \xi^{w_1} v_1, \ldots, \xi^{w_n} v_n), \ \ \xi \in \mu_q$$

and so $\mathbf{C} \times_{\mu_q(\tilde{\mathbf{w}})} \mathbf{C}^n \cong \mathbf{C} \times (\mathbf{C}^n/\mu_q(\mathbf{w}))$.

The notion of principal bundle also extends to orbibundles i.e. a *principal orbibundle* is an orbibundle with $F = G$ and G acting on G as left multiplication.

In particular to a given F-orbibundle $E \overset{\pi}{\longrightarrow} B$ with structure group G, defined by the pair of atlases $(\mathcal{A}, \mathcal{A}^*)$, we can associate a principal orbibundle in the same way as it is done for manifolds. We can in fact form the principal G-orbibundle $P \overset{\pi_P}{\longrightarrow} B$ with fibre and structure group both equal to G, defined by the atlas $(\mathcal{A}, \mathcal{A}^*_P)$ where $\mathcal{A}^*_P$ is formed by the local uniformizing systems $(\widetilde{U}_i^* \equiv \widetilde{U}_i \times G, \Gamma^*_{i_P} \cong \Gamma_i, \varphi_{i_P})$, corresponding to the local uniformizing systems in $\mathcal{A}$, $(\widetilde{U}_i, \Gamma_i, \varphi_i)$, where $\pi_P \circ \varphi^*_{i_P} = \varphi_i \circ pr_{\widetilde{U}^*_{i_P}}$.

If we form the product bundle $P \times F \xrightarrow{\phi} P$ with fibre the orbifold F, structure group G and ϕ the projection on the first factor, we can define the principal map $\psi : P \times F \to E$ as follows,

$$\psi(p, f) = \varphi_i^*(\tilde{x}, u \cdot f)$$

with $\varphi_{i_P}(\tilde{x}, u) = p$. We can see easily that ψ is an orbifold map. Moreover, its fibers are diffeomorphic to G and so,

$$E \cong P \times_G F.$$

We can also define *differential forms* on orbifolds by considering a collection of forms on the local uniformizing systems which transform correctly under the overlapping maps. Having these, we can define a $2n$*-symplectic orbifold*, by defining a symplectic structure on a differentiable orbifold i.e. a differential form which in each local representation is a closed non-degenerate 2-form. In particular, if $((\widetilde{U}, \tilde{\omega}), \Gamma, \varphi)$ is a symplectic l.u.s, $\tilde{\omega}$ has to be Γ-invariant and so Γ is a subgroup of the group of symplectomorphisms of $\widetilde{U}$. Moreover, we can consider an invariant metric on $\widetilde{U}$ and, for any point $\tilde{x} \in \widetilde{U}$ the exponential map gives us a symplectomorphism ϕ from the ε ball $(B_\varepsilon^{2n}, \omega_o)$ to a small neighborhood of $\tilde{x}$ in $(\widetilde{U}, \tilde{\omega})$. Since the isotropy subgroup of $\tilde{x}$ preserves $\tilde{\omega}$, this symplectomorphism descends to U giving us a symplectomorphism between a neighborhood of the image of $\tilde{x}$ in (U, ω) and a neighborhood of the origin in the symplectic orbispace $(\mathbf{R}^{2n}, \omega_o)$, where Γ is a finite subroup of the unitary group

$$U(n) \cong O(2n, \mathbf{R}) \cap Sp(2n, \mathbf{R}).$$

The symplectic l.u.s. $((B_\varepsilon^{2n}, \omega_o), \varphi \circ \phi, \Gamma)$ is called a *normalized Darboux chart* for the symplectic orbifold.

As in the manifold case, we can define the *de Rham cohomology* of an orbifold M as well as the *de Rham cohomology with compact supports* of M. When M is orientable, we can also define the integral

$$\int_{M^{2n}} \eta$$

of a $2n$-form, as follows. If for a given l.u.s. $(\widetilde{U}, \Gamma, \varphi)$, *supp*$\eta$ is contained in $U = \varphi(\widetilde{U})$ then

$$\int_M \eta = \frac{1}{|\Gamma|} \int_{\widetilde{U}} \eta_{\widetilde{U}}$$

where $\eta_{\widetilde{U}}$ is the corresponding form on $\widetilde{U}$. In general, we can consider a partition of unity $\{f_i\}$ subordinate to a given covering of M and then,

$$\int_M \eta = \sum_i \int_M f_i M$$

Stokes' theorem also holds for orbifolds as well as the following theorem, (cf. [S2] for details):

Theorem 2.1.8 *([S2]) Let M be a $2n$-orbifold and let $k \in \mathbf{N}_o$ with $k \leq 2n$. Then the pairing*

$$\begin{array}{ccc} H^k(M,\mathbf{C}) \times H^{2n-k}(M,\mathbf{C}) & \to & \mathbf{C} \\ ([\mu],[\eta]) & \mapsto & \int_M \mu \wedge \eta \end{array}$$

is well defined and nondegenerate.

With this result we can identify $H^k(M)$ with the dual space $(H^{2n-k}(M))^*$ via $[\mu] \mapsto \int_M \mu \wedge \cdot$, which gives us Poincare Duality on orbifolds.

2.2 Weighted projective spaces

In the remaining paragraphs of this section we will describe weighted, (or twisted), projective spaces as they will play a fundamental role in sections 3.4 and 4. Part of what is presented here can be found in [Kaw] and in [A].

Throughout this paper, $\underline{a}(\mathbf{p})$, $a \in \mathbf{C}, \mathbf{p} \in (\mathbf{Z})^n$ will denote the $n \times n$ diagonal matrix

$$Diag(a^{p_1}, \dots, a^{p_n})$$

with diagonal entries $a^{p_i}, i = 1, \dots, n$. Moreover, μ_p, $p \in \mathbf{Z}$ will denote the set of p-roots of unity.

Definition 2.2.1 *Let $\mathbf{q} = (q_0, \dots, q_n)$ be a $(n+1)$-tuple of positive integers. The weighted (twisted) projective space of type $\mathbf{q}$, $\mathbf{C}P^n(\mathbf{q})$ is defined by:*

$$\mathbf{C}P^n(\mathbf{q}) = \{\mathbf{z} \in (\mathbf{C}^{n+1})^*\} \ / \ (\mathbf{z} \sim \underline{\lambda}(\mathbf{q}) \cdot \mathbf{z}, \ \lambda \in \mathbf{C}^*) \tag{2.3}$$

where $\underline{\lambda} = Diag\,(\lambda^{q_0}, \dots, \lambda^{q_n})$.

Remark 2.2.2 *1. The above $\mathbf{C}^*$-action is free iff $q_i = 1$ for every $i = 0, \dots, n$.*

2. If $\gcd(q_0, \dots, q_n) = d \neq 1$, then $\mathbf{C}P^n(\mathbf{q})$ is homeomorphic to $\mathbf{C}P^n(\frac{\mathbf{q}}{d})$ (by identifying λ^d with λ).

There are two maps $\varphi : \mathbf{C}P^n \to \mathbf{C}P^n(\mathbf{q})$ and $\psi : \mathbf{C}P^n(\mathbf{q}) \to \mathbf{C}P^n$ which will be very important in this section:

$$\varphi([z_0 : \dots : z_n]) = [z_0^{q_o} : \dots : z_n^{q_n}]_{\mathbf{q}} \tag{2.4}$$

$$\psi([z_0 : \dots : z_n]_{\mathbf{q}}) = [z_0^{m/q_0} : \dots : z_n^{m/q_n}] \tag{2.5}$$

with $m = lcm(q_0, \dots, q_n)$ and $[\cdot]_{\mathbf{q}}$ denoting a conjugacy class in $\mathbf{C}P^n(\mathbf{q})$.

Weighted projective spaces are, in general, conic orbifolds i.e. the orbifold singularities have cyclic structure groups. In particular, if all the q_i's are mutually prime, all the orbifold singularities are isolated. In fact, as it is usually done for projective spaces, we can consider the sets,

$$V_i = \{[\mathbf{z}]_{\mathbf{q}} \in \mathbf{C}P^n(\mathbf{q}) : z_i \neq 0\} \subset \mathbf{C}P^n(\mathbf{q}) \tag{2.6}$$

and the bijective maps ϕ_i from V_i to $\mathbf{C}^n/\mu_{q_i}$ given by

$$\phi_i([\mathbf{z}]_{\mathbf{q}}) = (\frac{z_0}{(z_i)^{q_0/q_i}}, \ldots, \frac{\hat{z}_i}{z_i}, \ldots, \frac{z_n}{(z_i)^{q_n/q_i}})_{q_i} \tag{2.7}$$

where $(z_i)^{1/q_i}$ is a q_i-root of z_i and $(.)_{q_i}$ denotes a μ_{q_i} conjugacy class in $\mathbf{C}^n/\mu_{q_i}(\hat{\mathbf{q}}^i)$ with μ_{q_i} acting on $\mathbf{C}^n$ by

$$\xi \cdot \mathbf{z} = \underline{\xi}(\hat{\mathbf{q}})\,\mathbf{z}, \;\; \xi \in \mu_{q_i}$$

with $\hat{\mathbf{q}} = (q_0, \ldots, \hat{q}_i, \ldots, q_n)$.

Then on $\phi_i(V_j \bigcap V_i) \subset \mathbf{C}^n/\mu_{q_i}$,

$$\phi_j \circ \phi_i^{-1}((z_1, \ldots, z_n)_{q_i}) = (\frac{z_1}{(z_j)^{q_0/q_i}}, \ldots, \frac{\hat{z}_j}{z_j}, \ldots, \frac{1}{(z_j)^{q_i/q_j}}, \ldots, \frac{z_n}{(z_j)^{q_n/q_j}})_{q_j}$$

and so, $\mathbf{C}P^n(\mathbf{q})$ has the structure of an orbifold.

These spaces are in fact symplectic orbifolds as they can be obtained from $(\mathbf{C}^{n+1}, \omega_0)$ by symplectic reduction, in the following way. First, again like usual projective spaces, $\mathbf{C}P^n(\mathbf{q})$ can be obtained from the quotient of $S^{2n+1} \subset (\mathbf{C}^{n+1})^*$ by an S^1-action, (the one obtained from the $\mathbf{C}^*$-action by restriction):

Proposition 2.2.3 *([A])*

$$\mathbf{C}P^n(\mathbf{q}) \cong \{\mathbf{z} \in S^{2n+1}\} / (\mathbf{z} \sim \underline{\lambda}(\mathbf{q})\,\mathbf{z}, \; \lambda \in S^1)$$

Proof: Consider the natural map $\chi : S^{2n+1}/S^1 \to \mathbf{C}P^n(\mathbf{q})$.

This map is injective: take $[\mathbf{z}]_{\mathbf{q}} = [\mathbf{z}']_{\mathbf{q}}$ such that $\mathbf{z}, \mathbf{z}' \in S^{2n+1}$ and $\mathbf{z}' = \underline{\lambda}(\mathbf{q})\,\mathbf{z}$, $\lambda \in \mathbf{C}^*$. Then, as $\sum |z_i|^2 = \sum |\lambda|^{2q_i} |z_i|^2 = 1$, we have $\lambda = 1$ and so $\mathbf{z} = \mathbf{z}'$.

Moreover, it is surjective: for any $[\mathbf{z}]_{\mathbf{q}} \in \mathbf{C}P^n(\mathbf{q})$, there is a $\mathbf{z}' \in S^{2n+1}$ such that $[\mathbf{z}']_{\mathbf{q}} = [\mathbf{z}]_{\mathbf{q}}$ i.e. $z_i' = \lambda^{q_i} z_i$, $\lambda \in \mathbf{C}^*$, $i = 1, \ldots, n$, (just make λ equal the solution of

$$\sum |\lambda|^{2q_i} |z_i|^2 = 1$$

which exists and is unique, as all the coefficients $|z_i|^2$ are nonnegative and $|z_j|^2 \neq 0$ for some j). For more details see [A]. □

Hence, if we consider the Hamiltonian S^1-action on $\mathbf{C}^{n+1}$ given by

$$\lambda \cdot \mathbf{z} = \underline{\lambda}(\mathbf{q})\,\mathbf{z}, \;\;\; \lambda \in S^1, \, \mathbf{z} \in \mathbf{C}^{n+1},$$

$\mathbf{C}P^n(\mathbf{q}) \cong H^{-1}(t)/S^1$, where $H : \mathbf{C}^{n+1} \to \mathbf{R}$ is the associated Hamiltonian function ($H(\mathbf{z}) = \sum q_i |z_i|^2$), and $t \neq 0$ is any of its regular values. We conclude then that $\mathbf{C}P^n(\mathbf{q})$ is a symplectic orbifold.

2.3 Orbibundles over weighted projective spaces

An important fact about $\mathbf{C}P^{n+1}(\mathbf{q})$ is that for any $\mathbf{e_i} = [(0,\ldots,1,\ldots,0)]_{\mathbf{q}}$, $E_i = \mathbf{C}P^{n+1}(\mathbf{q})\backslash\{\mathbf{e_i}\}$ projects over $\mathbf{C}P^n(\hat{\mathbf{q}}^{\mathbf{i}})$, with $\hat{\mathbf{q}}^{\mathbf{i}} = (q_0,\ldots,\hat{q}_i,\ldots,q_n)$. This projection is given by $\pi([z_0 : \cdots : z_{n+1}]_{\mathbf{q}}) = [z_0 : \cdots : \hat{z}_i : \cdots : z_{n+1}]_{\hat{\mathbf{q}}^{\mathbf{i}}}$, and the E_i's are orbibundles over $\mathbf{C}P^n(\hat{\mathbf{q}}^{\mathbf{i}})$:

Proposition 2.3.1 *Each $E_i = \mathbf{C}P^{n+1}\backslash\{\mathbf{e_i}\}$ is a line orbibundle over*

$$\mathbf{C}P^n(\hat{\mathbf{q}}^{\mathbf{i}})$$

with structure group S^1.

Proof: Consider the open covering of $\mathbf{C}P^n(\hat{\mathbf{q}}^{\mathbf{i}})$ formed by the sets V_j, $j \in J = \{0,\ldots,\hat{\imath},\ldots,n+1\}$ described in (2.6). Then, for each j we have the orbibundle

$$\mathbf{C} \times_{\mu_{q_j}(\tilde{\mathbf{q}})} \mathbf{C}^n \xrightarrow{\rho'_j} \mathbf{C}^n/\mu_{q_j}(\hat{\mathbf{q}}^{i,j})$$

defined in Example 2.1.6 with $\tilde{\mathbf{q}} = (q_i, \hat{\mathbf{q}}^{\mathbf{i},\mathbf{j}}) = (q_i, q_0, \ldots, \hat{q}_i, \ldots, \hat{q}_j, \ldots q_{n+1})$, and the commutative diagram:

$$\begin{array}{ccc} \mathbf{C} \times_{\mu_{q_j}(\tilde{\mathbf{q}})} \mathbf{C}^n & \xrightarrow{\rho'_j} & \mathbf{C}^n/\mu_{q_j}(\hat{\mathbf{q}}^{i,j}) \\ \chi_j \downarrow & & \downarrow \phi_j^{-1} \\ \pi^{-1}(V_j) & \xrightarrow{\pi} & V_j \end{array}$$

where

1. $\chi_j \equiv \tilde{\phi}_j^{-1} \circ P_j$ where P_j is the "permutation" map

$$P_j(z, w_0, \ldots, w_{n-1})_{q_j} = (w_0, \ldots, w_{i-1}, z, , w_i, \ldots, w_{n-1})_{q_j}$$

   and $\tilde{\phi}_j^{-1} : \mathbf{C}^{n+1}/\mu_{q_j} \to \pi^{-1}(V_j)$ is such that

$$\tilde{\phi}_j^{-1}((z_0, \ldots, z_n)_{q_j}) = [z_0 : \cdots : \underbrace{1}_{j} : \cdots : z_n]_q$$

2. ϕ_j is the map defined on (2.7)

3. $\rho'_j : \mathbf{C} \times_{\mu_{q_j}(\tilde{\mathbf{q}})} \mathbf{C}^n \to \mathbf{C}^n/\mu_{q_j}(\hat{\mathbf{q}}^{i,j})$ is defined by

$$\rho'_j(z_0, \ldots, z_n)_{q_j} = (z_1, \ldots, z_n)_{q_j}$$

□

Remark 2.3.2 *If each q_j divides q_i, then E_i is a complex line bundle over $\mathbf{C}P^n(\hat{\mathbf{q}}^{\mathbf{i}})$ as, in this case, the action of μ_{q_j} on $\mathbf{C} \times_{\mu_{q_j}(\tilde{\mathbf{q}})} \mathbf{C}^n$ described in (2.1) above, becomes*

$$\xi \cdot \mathbf{z} = (z_0, \underline{\xi}(\hat{\mathbf{q}}^{\mathbf{i},\mathbf{j}})\,(z_1, \ldots, z_n)^t), \;\; \xi \in \mu_{q_j}$$

and then,

$$\mathbf{C} \times_{\mu_{q_j}(\tilde{\mathbf{q}})} \mathbf{C}^n \equiv \mathbf{C} \times (\mathbf{C}^n / \mu_{q_j}(\tilde{\mathbf{q}})).$$

In addition, we have the following proposition:

Proposition 2.3.3 *([A]) When $q_i = m = lcm(q_j, j = 1, \cdots, \hat{i}, \cdots, n+1)$, $E_i \equiv \mathbf{C}P^{n+1}(\mathbf{q}) \backslash \{\mathbf{e}_i\}$ is isomorphic to the pullback bundle of the canonical line bundle L over $\mathbf{C}P^n$ by the map ψ defined in (2.5).*

Proof: L can be written as $\mathbf{C}P^{n+1} \backslash \{\mathbf{e}_i\}$ and so we have,

$$\begin{array}{ccccccc} E_i & \overset{\phi}{\to} & \psi^* L & \to & L & \equiv & \mathbf{C}P^{n+1}\backslash\{\mathbf{e}_i\} \\ & \pi \searrow & \downarrow \pi' & & \downarrow \tilde{\pi} & & \\ & & \mathbf{C}P^n(\hat{\mathbf{q}}^{\mathbf{i}}) & \overset{\psi}{\to} & \mathbf{C}P^n & & \end{array}$$

where

$$\psi^* L = \{([\mathbf{z}]_{\hat{\mathbf{q}}^{\mathbf{i}}}, v) \in \mathbf{C}P^n(\mathbf{q}) \times L : \psi([\mathbf{z}]_{\hat{\mathbf{q}}^{\mathbf{i}}}) = \tilde{\pi}(v)\}$$

and

$$\phi([z_0 : \cdots : z_{n+1}]_{\mathbf{q}}) = ([z_0 : \cdots : \hat{z}_i : \cdots : z_n]_{\hat{\mathbf{q}}^{\mathbf{i}}}, [z_0^{m/q_0} : \cdots : z_{n+1}^{m/q_{n+1}}])$$

is a bundle isomorphism □

2.4 Cohomology of weighted projective spaces

Let us consider again the map φ defined in (2.4). The fibers of this map are the orbits of the standard linear action of $\tilde{\mu}_{\mathbf{q}} = (\mu_{q_0} \times \cdots \times \mu_{q_n})/\mu_d$ on $\mathbf{C}P^n$:

$$(\xi_{q_0}, \ldots, \xi_{q_n}) \cdot [z_0 : \cdots : z_n] = [\xi_{q_0} z_0 : \cdots : \xi_{q_n} z_n], \;\; \xi_{q_i} \in \mu_{q_i}, \; i = 1, \ldots, n$$

where $d = gcd(q_0, \ldots, q_n)$ and μ_d is a subgroup of $\mu_{q_0} \times \cdots \times \mu_{q_n}$ via the diagonal. Therefore, we have the following results:

Proposition 2.4.1 *([Kaw]) φ induces a homeomorphism*

$$\mathbf{C}P^n / \tilde{\mu}_{\mathbf{q}} \cong \mathbf{C}P^n(\mathbf{q})$$

and

Corollary 2.4.2 *([Kaw]) φ induces an isomorphism on rational cohomology:*

$$\varphi^* : H^*(\mathbf{C}P^n(\mathbf{q}); \mathbf{Q}) \overset{\cong}{\to} H^*(\mathbf{C}P^n; \mathbf{Q})$$

Proof: First observe that the linear action of $\tilde{\mu}_{\mathbf{q}}$ on $\mathbf{C}P^n$ extends to the linear action of $T^n = (S^1)^n$, and so $H^*(\mathbf{C}P^n)$ is fixed by $\tilde{\mu}_{\mathbf{q}}$. The result therefore follows from the classical fact that, for a finite G-action on X,

$$H^*(X/G; \mathcal{S}) \cong H^*(X; \mathcal{S})^G \tag{2.8}$$

where the isomorphism is given by π^*, (with π the quotient map) and $\mathcal{S}$ is a ring containing $1/|G|$.

□

We have seen above that $E_i = \mathbf{C}P^n(\mathbf{q})\backslash\{\mathbf{e}_i\}$ is an orbibundle over $\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}})$. To compute the integer cohomology of $\mathbf{C}P^n(\mathbf{q})$ we need the following proposition,

Proposition 2.4.3 *([A],[Kaw]) $\mathbf{C}P^n(\mathbf{q})$ can be obtained from any of its subsets $\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}})$, by attaching a 2n- "cell" of the form $\mathbf{C}^n/\mu_{q_i}$ via the canonical map $f : \mathrm{L}(q_i; \hat{\mathbf{q}}^{\mathbf{i}}) \to \mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}})$, where $\mathrm{L}(q_i; \hat{\mathbf{q}}^{\mathbf{i}})$ is the lens complex S^{2n-1}/μ_{q_i} with μ_{q_i} acting on S^{2n-1} by,*

$$\xi \cdot \mathbf{z} = \underline{\xi}(\hat{\mathbf{q}}^{\mathbf{i}})\ \mathbf{z} \in S^{2n-1},\ \xi \in \mu_{q_i} \tag{2.9}$$

Proof:

$$\mathbf{C}P^n(\mathbf{q})\backslash\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}}) =$$
$$\{[z_0 : \cdots : \underbrace{1}_{i} : \cdots : z_n]_{\mathbf{q}} :\ (z_0, \ldots, \hat{z}_i, \ldots, z_n) \in \mathbf{C}^n\}$$

In addition, if $[z_0 : \cdots : 1 : \cdots : z_n]_{\mathbf{q}} = [z_0' : \cdots : 1 : \cdots : z_n']_{\mathbf{q}}$, then $z_j' = \xi^{q_j} z_j$, for some $\xi \in \mu_{q_i}$ and $j \neq i$. Therefore, $\mathbf{C}P^n(\mathbf{q})\backslash\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}}) \cong \mathbf{C}^n/\mu_{q_i}(\hat{\mathbf{q}^i})$. Moreover,

$$\partial(\mathbf{C}P^n(\mathbf{q})\backslash\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}})) = S^{2n-1}/\mu_{q_i} = \mathrm{L}(q_i; \hat{\mathbf{q}}^{\mathbf{i}})$$

and so,

$$\mathbf{C}P^n(\mathbf{q}) = \mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}}) \bigcup_f \mathrm{L}(q_i; \hat{\mathbf{q}}^{\mathbf{i}})$$

with, $f : \mathrm{L}(q_i; \hat{\mathbf{q}}^{\mathbf{i}}) \hookrightarrow \mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}})$, the inclusion map. □

Before computing the groups $H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z})$, we will first compute the groups $H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a)$ for $a > 0$, and then use the universal coefficient formula.

Theorem 2.4.4 *([A]) For any integer $a > 0$,*

$$H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a) = \begin{cases} \mathbf{Z}_a & \text{if } i = 2k,\ 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Proof: We may assume that $a = p^\alpha$ with p a prime integer different from 1 and $\alpha > 0$, (if not, we can decompose $\mathbf{Z}_a$). Moreover, as $\mathbf{C}P^n(d \cdot \mathbf{q}) = \mathbf{C}P^n(\mathbf{q})$ for any $d \in \mathbf{Z}^+$, we can also assume that $\gcd(q_0, \ldots, q_n) = 1$.

If $n = 0$, then $\mathbf{C}P^n(\mathbf{q})$ is just a point and the result follows trivially. Assuming the result true for $\mathbf{C}P^r(\mathbf{q})$ with any q of length $r + 1 \leq n$, we will prove it for $\mathbf{C}P^n(\mathbf{q})$ with q of length $n + 1$.

Trivially, $H^0(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a) \cong H^{2n}(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a) \cong \mathbf{Z}_a$. Moreover, there exists a $j \in \{0, \ldots, n\}$ such that p does not divide q_j.

As we have seen above,

$$\mathbf{C}P^n(\mathbf{q}) = \mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}}) \bigcup_f \mathrm{L}(q_i; \hat{\mathbf{q}}^{\mathbf{i}})$$

and so there is a cohomology exact sequence for the pair

$$(\mathbf{C}P^n(\mathbf{q}); \mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}})),$$

$$\begin{array}{llllll} \cdots & \to & H^i_c(\mathbf{C}^n/\mu_{q_j}; \mathbf{Z}_a) & \to & H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a) & \to \\ & \to & H^i(\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{i}}); \mathbf{Z}_a) & \to & H^{i+1}_c(\mathbf{C}^n/\mu_{q_j}; \mathbf{Z}_a) & \to \cdots \end{array}$$

Moreover by (2.8), $H^i_c(\mathbf{C}^n/\mu_{q_j}; \mathbf{Z}_{p^\alpha}) = [H^i_c(\mathbf{C}^n; \mathbf{Z}_{p^\alpha})]^{\mu_{q_j}}$ and as μ_{q_j} fixes $H^i_c(\mathbf{C}^n; \mathbf{Z}_{p^\alpha})$, we have

$$H^i(\mathbf{C}^n/\mu_{q_j}; \mathbf{Z}_{p^\alpha}) = \begin{cases} \mathbf{Z}_{p^\alpha} & \text{if } i = 2n \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{array}{cccccccc} 0 & \to & H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a) & \to & H^i(\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{j}}); \mathbf{Z}_a) & \to & 0 & i \neq 2n \\ & & & & \| & & & \\ & & & & \mathbf{Z}_a \text{or } 0 & & & \end{array}$$

and the result follows. For more details, see [A]. □

We can now prove the following result,

Theorem 2.4.5 *([A],[Kaw]):*

$$H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}) = \begin{cases} \mathbf{Z} & \textit{if } i = 2k \ 0 \leq k \leq n \\ 0 & \textit{otherwise} \end{cases}$$

Proof: The universal coefficient theorem for cohomology [B-T], tells us that

$$\begin{aligned} &H^i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}_a) \cong \\ &\qquad Hom(H_i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}), \mathbf{Z}_a) \oplus Ext(H_{i-1}(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}), \mathbf{Z}_a). \end{aligned}$$

For an odd i or $i > 2n$, we have for every a,

$$0 \cong H_i(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}) \otimes \mathbf{Z}_a$$

and so, $H_i(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) = 0$ for odd i or $i > 2n$.

For an even $i \leq 2n$,

$$\mathbf{Z}_a \cong (H_i(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) \otimes \mathbf{Z}_a) \oplus 0$$

Moreover $H_i(\mathbf{C}P^n(\mathbf{q});\mathbf{Z})$ is finitely generated and so, we can decompose it as

$$\mathbf{Z}^\beta \oplus \mathbf{Z}_{\beta_1} \oplus \cdots \oplus \mathbf{Z}_{\beta_s} \;\; \beta \geq 0, \beta_i \geq 1$$

Then,

$$\mathbf{Z}_a \cong (\mathbf{Z}^\beta \otimes \mathbf{Z}_a) \oplus (\mathbf{Z}_{\beta_1} \otimes \mathbf{Z}_a) \oplus \cdots \oplus (\mathbf{Z}_{\beta_s} \otimes \mathbf{Z}_a)$$

Consequently, choosing a prime to all the b_i's, we have $\beta = 1$ and $\beta_i = 1$ for $i = 1, \ldots, s$. Therefore,

$$H_i(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = 2k \;\; 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Moreover, also by a corollary of the universal coefficients theorem,

$$H^i(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) \cong F_q \oplus T_{q-1}$$

where $F_{\mathbf{q}}$ is the free part of $H_i(\mathbf{C}P^n(\mathbf{q});\mathbf{Z})$ and $T_{\mathbf{q}-1}$ is the torsion part of $H_{i-1}(\mathbf{C}P^n(\mathbf{q});\mathbf{Z})$, and the result follows. □

To determine the ring structure of $H^*(\mathbf{C}P^n(\mathbf{q});\mathbf{Z})$, we first need to define the following numbers:

$$l_k^{\mathbf{q}} = lcm\{\frac{q_{i_0} \cdots q_{i_k}}{\gcd(q_{i_0}, \ldots, q_{i_k})} \; : \; 0 \leq i_0 < \cdots < i_k \leq n\}$$

The minimum of these numbers is

$$l_n^{\mathbf{q}} = \frac{q_0 \cdots q_n}{\gcd(q_0, \ldots, q_n)}$$

and the maximum is $l_0^{\mathbf{q}} = lcm(q_0, \ldots, q_n)$

Let E be the set $\{p \text{ prime} : p \text{divides some } q_i, \;\, 0 \leq i \leq n\}$ If we decompose

$$q_i = \prod_{p \in E} p^{r_i} \;\; i = 0, \ldots, n$$

and for each p, $r_{i_0} \leq r_{i_1} \leq \cdots \leq r_{i_n}$, then

$$l^{\mathbf{q}_k} = \prod_{p \in E} p^{r_{i_{n-k+1}} + \cdots + r_{i_n}}$$

We are now able to state the following theorem:

Theorem 2.4.6 *([Kaw]) The induced map φ^* on cohomology, is the ring homomorphism*

$$\begin{array}{cccc} \varphi^* : & H^{2k}(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) & \longrightarrow & H^{2k}(\mathbf{C}P^n;\mathbf{Z}) \\ & \| & & \| \\ & \mathbf{Z} & \xrightarrow{l_k^{\mathbf{q}}} & \mathbf{Z} \end{array}$$

Remark 2.4.7 *If $h = c_1(L^*)$ is the usual generator of $H^2(\mathbf{C}P^n;\mathbf{Z})$, (where L is the canonical bundle over $\mathbf{C}P^n$) then we know that*

$$\{1, h, \ldots, h^n\}$$

is a **Z***-basis for $H^*(\mathbf{C}P^n;\mathbf{Z})$. Hence, for each k such that $0 \leq k \leq n$,there is a unique*

$$h_k \in H^{2k}(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) \tag{2.10}$$

such that

$$\varphi^*(h_k) = l_k^{\mathbf{q}} h^k$$

and $\{1, h_1, h_2, \ldots, k_k\}$, is a **Z***-basis of the abelian group $H^{2k}(\mathbf{C}P^n(\mathbf{q});\mathbf{Z})$.*

Proof:

Let $\mathbf{q} = (q_0, \ldots, q_n)$ and, for $k \in \{0, \ldots, n\}$, consider the set $I = \{i_0, \ldots, i_k\}$ with $0 \leq i_0 < \cdots < i_k \leq n$.

Let $\mathbf{C}P^k(I) = \mathbf{C}P^k(q_{i_0}, \ldots, q_{i_k})$ and denote by u_I the inclusion map

$$\mathbf{C}P^k(I) \overset{u_I}{\hookrightarrow} \mathbf{C}P^n(\mathbf{q})$$

defined by $z_i = 0$ for every $i \notin I$.

Let m_I be the integer defined by u_I^*,

$$\begin{array}{cccc} u_I^*: & H^{2k}(\mathbf{C}P^n(\mathbf{q});\mathbf{Z}) & \longrightarrow & H^{2k}(\mathbf{C}P^k(I);\mathbf{Z}) \\ & \| & & \| \\ & \mathbf{Z} & \xrightarrow{\cdot m_I} & \mathbf{Z} \end{array}$$

step1. The integer elements of $M = \{m_I : I \subset \{0, \ldots, n\}, |I| = k+1\}$ have *gcd* equal to 1: We will prove this step by induction on n. If $n = 0$, then $M = \{1\}$, ($\cdot m_I \equiv id$), and the statement is true. Let us assume now that $n \geq 1$ ant that the statement is true for $n-1$. If $k = n$, then again $M = \{1\}$. If $k \leq n-1$ and for a j such that $0 \leq j \leq n$ and $j \notin I = \{i_0, \ldots, i_k\}$, consider the following factorization

$$\begin{array}{ccc} \mathbf{C}P^k(I) & \xrightarrow{u_I} & \mathbf{C}P^n(\mathbf{q}) \\ u_{I,j} \searrow & & \nearrow u_j \\ & \mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{j}}) & \end{array}$$

where u_j is defined by $z_j = 0$ and $u_{I,j}$ is defined by $z_i = 0$ for every $i \notin I \cup \{j\}$. Then, $u_I^* = u_{I,j}^* \circ u_j^*$ i.e. $m_I = m_{I,j} \cdot m_j$. Hence,

$$M = \{m \in \mathbf{Z} \text{ for which there is an } I \subset \{0, \ldots, n\}, \text{with}$$
$$|I| = k+1 \text{and a } \; j \in \{0, \ldots, n\} \text{ such that } m = m_{I,j} m_j\}$$

However, by induction hypothesis, for each j, the elements of

$$M_j = \{m_{I,j} : I \subset \{1, \ldots, n\}, \; j \notin I, \; |I| = k+1\}$$

have *gcd* equal to 1. Hence, we just have to show that $\gcd(m_0, \ldots, m_n) = 1$.

Since $\mathbf{C}P^n(d \cdot \mathrm{q}) = \mathbf{C}P^n(\mathrm{q})$, we may assume that $\gcd(q_0, \ldots, q_n) = 1$. Moreover, we have established above the existence of the exact sequence with $\mathbf{Z}_a$ coefficients,

$$\begin{array}{ccccc} 0 \to H^{2k}(\mathbf{C}P^n(\mathbf{q})) & \overset{u_j^*}{\to} & H^{2k}(\mathbf{C}P^{n-1}(\hat{\mathbf{q}}^{\mathbf{j}})) & \to & H_c^{2k+1}(\mathbf{C}^n/\mu_{q_j}) \to 0 \\ \| & & \| & & \| \\ \mathbf{Z}_a & \overset{\cdot m_j}{\to} & \mathbf{Z}_a & & (q_j\text{-torsion}) \\ & & & & \| \\ & & & & 0 \text{ if } a \text{ prime to } q_j \end{array}$$

If a is prime to q_j, then $\cdot m_j$ is an automorphism of $\mathbf{Z}_a$ and then a does not divide m_j. We conclude then that $\gcd(m_0, \ldots, m_n) = 1$ and we have proved step 1.

step 2. Take $I = \{i_0, \ldots, i_k\}$ with $0 \le i_0 < \cdots < i_k \le n$. Then we have the commutative diagram:

$$\begin{array}{ccc} \mathbf{C}P^k & \hookrightarrow & \mathbf{C}P^n \\ \varphi_I \downarrow & & \downarrow \varphi \\ \mathbf{C}P^k(I) & \overset{u_I}{\hookrightarrow} & \mathbf{C}P^n(\mathrm{q}) \end{array}$$

where

$$\varphi_I([z_{i_0} : \cdots : z_{i_k}]) = [z_{i_0}^{q_0} : \cdots : z_{i_k}^{q_k}]_{\mathbf{q}_I}, \quad \mathbf{q}_I = (q_{i_0}, \ldots, q_{i_k})$$

This induces a commutative diagram in cohomology,

$$\begin{array}{ccc} H^{2k}(\mathbf{C}P^n(\mathbf{q}); \mathbf{Z}) & \overset{u_I^*}{\hookrightarrow} & H^{2k}(\mathbf{C}P^k(I); \mathbf{Z}) \\ \varphi^* \downarrow & & \downarrow (\varphi_I)^* \\ H^{2k}(\mathbf{C}P^n; \mathbf{Z}) & \hookrightarrow & H^{2k}(\mathbf{C}P^k; \mathbf{Z}) \end{array}$$

i.e.

$$\begin{array}{ccc} \mathbf{Z} & \overset{\cdot m_I}{\hookrightarrow} & \mathbf{Z} \\ \cdot a \downarrow & & \downarrow \cdot a_I \\ \mathbf{Z} & \overset{\equiv}{\hookrightarrow} & \mathbf{Z} \end{array}$$

The commutativity of the diagram implies $a = m_I \cdot a_I$ for every

$$I \subset \{0, \ldots, n\} \text{ with } |I| = k + 1$$

i.e. a is a common multiple of the a_I's. By step 1, we have,

$$a = lcm\{a_I : I \subset \{0, \ldots, n\},\ |I| = k + 1\}$$

On the other hand, as

$$\varphi_I : \mathbf{C}P^k \to \mathbf{C}P^k(I) \equiv \mathbf{C}P^k/\mu_{q_{i_0}} \times \cdots \times \mu_{q_{i_k}}$$

is the quotient map,

$$deg\, \varphi_I = \frac{|G|}{|H|} = \frac{q_{i_0} \cdots q_{i_k}}{\gcd(q_{i_0}, \ldots, q_{i_k})}$$

where $G = \mu_{q_{i_0}} \times \cdots \times \mu_{q_{i_k}}$ and

$$H = \{g \in G : \ g \cdot x = x \text{ for every } x \in \mathbf{C}P^k\}$$

then

$$a_I = deg\varphi_I = \frac{q_{i_0} \cdots q_{i_k}}{\gcd(q_{i_0}, \ldots, q_{i_k})}$$

and the result follows. □

As φ^* is a ring homomorphism, this theorem determines the ring structure of $H^*(\mathbf{C}P^n(\mathrm{q}); \mathbf{Z})$. In fact, if h_k is the generator of this ring defined in (2.10), then for $i + j \leq n$,

$$\varphi^*(h_j h_k) = \varphi^*(h_j)\varphi^*(h_k) = l_j l_k h^{k+j} = \frac{l_j l_k}{l_{k+j}} \varphi^*(h_{k+j})$$

and so,

$$h_j h_k = \begin{cases} \frac{l_j l_k}{l_{k+j}}\, h_{k+j} & \text{if } k + j \leq n \\ 0 & \text{otherwise} \end{cases}$$

We conclude then that

$$H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \frac{\mathbf{Z}[T_1, \ldots, T_n]}{< T_k T_j,\ k + j > n;\ \ T_k T_j - \frac{l_j l_k}{l_{k+j}} T_{k+j},\ k + j \leq n >}$$

where h_j corresponds to the class of T_j.

Chapter 3

Blowing up on symplectic orbifolds

We begin our discussion by defining the blow up of an orbifold singularity. First however, recall that the blow up of a symplectic $2n$-manifold at one of its points x is obtained by removing an embedded ball around this point and then collapsing the boundary of the resulting manifold along the fibers of the Hopf fibration which in turn, are the orbits of the standard scalar circle action on S^{2n-1}. Similar constructions can be made for orbifolds and will be the object of the following sections.

We will discuss in Section 3.1 the simplest form of blow up coming from the diagonal action of S^1 on a Darboux neighborhood. In fact, one can define a weighted blow up corresponding to any circle action that commutes with the action of the local structure group (cf. Section 3.4). However we will discuss this in detail only in the case of a conic singularity (i.e. when this local structure group is cyclic).

3.1 Blowing up an orbifold singularity

Let (M^{2n}, ω) be a $2n$-dimensional symplectic orbifold and $x \in M$, an orbifold singularity. Let $((\widetilde{U}\tilde{\omega}), \Gamma, \varphi)$ be a uniformizing symplectic chart around x and let $U \cong \widetilde{U}/\Gamma$.

We can then choose normalized Darboux coordinates $(z_1, \ldots, z_n)$ at $\tilde{x}$ such that $B_\varepsilon^{2n} = \{\mathbf{z} \in \mathbf{C}^n : \sum_{i=1}^n |z_i|^2 \leq \varepsilon\}$ is contained in $\widetilde{U}$ for a small enough ε.

Consider the standard scalar action of S^1 on B_ε with Hamiltonian function $\widetilde{H} : B_\varepsilon^{2n} \to \mathbf{R}$ given by

$$\widetilde{H}(\mathbf{z}) = \sum_{i=1}^{n} |z_i|^2.$$

$\widetilde{H}$ has a single critical point $\tilde{x}$ of index zero. Moreover S^1 acts freely on the regular level sets of $\widetilde{H}$ which are $(2n-1)$-Seifert manifolds with no singular fibres. They are in fact $(2n-1)$-spheres and the orbits of the S^1 action are the fibres of the Hopf fibration.

Γ is a finite subgroup of $U(n)$ therefore preserving $\widetilde{H}$. Moreover, the action of Γ commutes with the S^1-action and so $\widetilde{H}$ descends to a Hamiltonian function H for an induced circle action on U. The regular level sets of this new function H are the quotients of the regular level sets of $\widetilde{H}$ by the action of Γ.

$$H^{-1}(\alpha) \cong S^{2n-1}/\Gamma$$

We can then define neighborhoods of x in M by $N_\rho(x) = H^{-1}([0,\rho))$ for small values of ρ. The boundary of these neighborhoods is ω-compatible as $\iota(X_H)\omega(Y) = dH(Y) = 0$, for all Y tangent to $\partial N_\rho(x)$.

Let us then define the manifold with boundary $N_\varepsilon = N_{\delta+\varepsilon}(x)\backslash\{N_\delta(x)\}$, for fixed small $\varepsilon, \delta \geq 0$. As it was seen above, the boundary of N_ε is an S^1-orbifold where the S^1 action is defined by H. Taking the image, $L(\delta,\varepsilon)$, of N_ε under a map π which is symplectic on $intN_\varepsilon$ and which collapses each circle fibre of ∂N_ε to a point, we obtain a bundle, $L(\delta,\varepsilon) \to B$, over the orbifold

$$B = \pi(\partial N_\varepsilon) = \partial N_\varepsilon / S^1 \cong S^{2n-1}/(\Gamma \times S^1).$$

If we glue $M^-_{\delta,\varepsilon} = M\backslash N_{\varepsilon+\delta}(x)$ to $L^+(\delta,\varepsilon)$ along $S(\varepsilon+\delta) = \partial L(\delta,\varepsilon) \cong S^{2n-1}/\Gamma$, we obtain the blow up of M at x.

Definition 3.1.1 *Let (M^{2n},ω) be a $2n$-dimensional symplectic orbifold and let $x \in M$ be an orbifold singularity. Γ. The blow up of M at x is given by*

$$\widetilde{M_x} = M^-_{\delta,\varepsilon} \bigsqcup_{S(\varepsilon+\delta)} L(\delta,\varepsilon)$$

where $L(\delta,\varepsilon)$, is the total space of a Seifert orbibundle over a $(2n-2)$-orbifold and $S(\varepsilon+\delta)$ is the $(2n-1)$-manifold with the fixed point free S^1 action described above.

$L(\delta,\varepsilon)$ can also be obtained from symplectic reduction, in the following way: Consider the S^1-action on $N_\varepsilon \times \mathbf{C}$ given by

$$\lambda \cdot (\mathbf{z}, w) = (\lambda \cdot \mathbf{z}, \lambda^{-1} w), \;\; \lambda \in S^1$$

where $\lambda \cdot \mathbf{z}$ is the action on N_ε determined by H. This action is also Hamiltonian with Hamiltonian function equal to

$$\begin{array}{rccc} \mu: & N_\varepsilon \times \mathbf{C} & \longrightarrow & \mathbf{R} \\ & (\mathbf{z}, w) & \longmapsto & H_{|N_\varepsilon}(\mathbf{z}) - |w|^2 \end{array}$$

Now, $\mu^{-1}(\varepsilon) = \{(\mathbf{z}, w) : \delta + \varepsilon > H_{|N_\varepsilon}(\mathbf{z}) \geq \varepsilon, \, |w|^2 = H_{|N_\varepsilon}(\mathbf{z}) - \varepsilon\}$ and so,

$$L(\delta,\varepsilon) = \frac{\mu^{-1}(\varepsilon)}{S^1}$$

is the set $\{\mathbf{z} \in N_\delta : \delta + \varepsilon > H(\mathbf{z}) \geq \varepsilon\}$ with boundary collapsed along the fibers of the S^1 action determined by H, as desired.

The blow up of M at x is therefore obtained by removing a neighborhood of x and collapsing the boundary of the resulting orbifold, along the fibres of the circle action induced by the standard scalar S^1-action on a uniformizing chart around x. The exceptional divisor resulting from this blow up is $S^{2n-1}/(\Gamma \times S^1)$.

Remark 3.1.2 *Just like in the manifold case, this blow up is independent of the choice of δ. Nevertheless, different embedding of the "2n-cell" $B^{2n}_\varepsilon/\Gamma$ in M, may result in different (though diffeomorphic) blow ups.*

Example 3.1.3 *Consider the orbifold $\mathbf{C}P^2(k,1,1)$, the four dimensional weighted projective space (c.f.(2.3)) with an isolated orbifold singularity x of structure group μ_k. The blow-up of this space at this k-singularity is the k-Hirzebruch surface*

$$W_k = \{([a,b],[x,y,z]) : a^k y - b^k x = 0\} \subset \mathbf{C}P^1 \times \mathbf{C}P^2$$

In fact, a neighborhood of the blown up point can be modelled by $\mathbf{C}^2/\mu_k$ where the μ_k-action is given by:

$$\xi \cdot (z_1, z_2) = (\xi z_1, \xi z_2), \;\; \xi \in \mu_k$$

The exceptional divisor is therefore $\mathbf{C}P^1$.

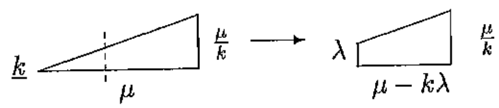

Figure 3.1: (a) $\mathbf{C}P^2(k,1,1)$; (b) $\widetilde{\mathbf{C}P^2}(k,1,1) \cong W_k$

The images of moment maps for torus actions on $\mathbf{C}P^2(k,1,1)$ and $W_k \cong (\widetilde{\mathbf{C}P^2(k,1,1)})_{x_k}$ are shown in Figures 3.1 (a) and (b). The preimage of a vertex is a point with structure group of order equal to its underlined label. The preimage of point in the interior of an edge is a circle and the preimage of a point on the interior of the polytope is a torus. Therefore, the edges of the image of the moment map represent intersecting spheres possibly with singularities. These images illustrate the fact that blowing up an isolated orbifold singularity of order k corresponds to removing a 4-"cell" of the form $\mathbf{C}^2/\mu_k$ and collapsing the resulting boundary along the orbits of a suitable S^1-action to form an exceptional $\mathbf{C}P^1$.

Example 3.1.4 *Let us consider now the space $\mathbf{C}^4/\Gamma$ where*

$$\Gamma = \{x, y : x^2 = (xy)^3 = y^5, \, x^4 = 1\}$$

is the binary icosahedral group of order 120.

The standard S^1-action on $\mathbf{C}^4$ induces an Hamiltonian action on $\mathbf{C}^4/\Gamma$ and the regular level sets of its Hamiltonian function are diffeomorphic to S^3/Γ, the Poincaré dodecahedral space, (c.f.

[T1]) obtained from a dodecahedron by gluing opposite faces. Γ *acts freely on* S^3 *and so* S^3/Γ *is a Seifert manifold with finite fundamental group* Γ.

The exceptional divisor obtained from the blow up of $\mathbf{C}^4/\Gamma$ *at the origin is the orbit space* $(S^3/\Gamma)/S^1$. *It is* S^2 *modulo the orientation-preserving symmetries of a dodecahedron, (c.f. [T2]). In particular, it is an orbifolod with underlying space* S^2, *having three isolated conic singularities with structure groups* μ_2, μ_3, *and* μ_5, *(c.f. Figure 3.2).*

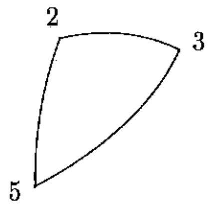

Figure 3.2: $S^2(2,3,5)$

Instead of the standard scalar S^1-action on a Darboux local uniformizer chart $(B_\varepsilon^{2n}, \Gamma, \varphi)$ we can also consider other Hamiltonian circle actions as long as they have a unique fixed point at the origin and they descend to the quotient $B_\varepsilon^{2n}/\Gamma$. We will see later some examples of different blow ups when we describe some special weighted blow ups of conic orbifold singularities.

3.2 Blowing down

The opposite process of blowing up in the symplectic manifold category is blowing down where we replace the exceptional divisor by a single point. Similarly, suppose we have $\Sigma \cong S^{2n-1}/(\Gamma \times S^1)$ embedded in a symplectic orbifold (M, ω) for suitable actions of S^1 and Γ, a finite subgroup of $U(n-1)$. If in addition a neighborhood $\mathcal{N}(\Sigma)$ of Σ is symplectomorphic to the bundle $L(\delta, \varepsilon)$ described above, where S^1 acts compatibly with the action on S^{2n-1}/Γ, then the blow down of M along Σ can be obtained by removing $L(\delta,\varepsilon)$ and gluing back $N_{\delta+\varepsilon}(x) = H^{-1}([0, \delta+\varepsilon))$ with H the Hamiltonian function associated to the S^1-action. This gluing is made along $S(\delta+\varepsilon) = \partial(L(\delta,\varepsilon)) \cong S^{2n-1}/\Gamma$. In this way we obtain an isolated orbifold singularity of structure group Γ.

By the orbifold neighborhood theorem , we know that a neighborhood of Σ, $\mathcal{N}(\Sigma)$, in M is symplectomorphic to a neighborhood of the zero section of its normal bundle. Therefore, $\mathcal{N}(\Sigma) \cong (S^{2n-1}/\Gamma) \times_{S^1} \mathbf{C}$ for an adequate circle action (cf. Section3.3). This action fixes Σ and commutes with the action of Γ. Hence, $\mathcal{N}(\Sigma)$ is the result of a blow up for this circle action which may not be the one described in the preceding section. We conclude then that such an orbifold Σ can always be blown down as long as we choose an adequate circle action.

3.3 Blowing up along a symplectic suborbifold

Let X be a compact symplectic suborbifold of (M, ω) of codimension $2k$. The symplectic neighborhood theorem easily extends to symplectic orbifolds (c.f. [L-T]) and so we have that a neighborhood of X in M is symplectomorphic to a neighborhood of the zero section in ν_X, the normal orbibundle of X.

$$N_M(X) \xrightarrow{\psi} N_{\nu_X}(z_0)$$

Again like in the manifold case, ([G-L-S]), we can assign a Hermitean form to ν_X and so $\nu_X \equiv P \times_{U(k)} \mathbf{C}^k$ where $P \xrightarrow{\pi_p} X$ is a principal $U(k)$-orbibundle, (c.f. Section 2.1). Consider now the Hamiltonian S^1-action on $P \times \mathbf{C}^k$ given by,

$$\lambda \cdot (z, \mathbf{w}) = (z, \underline{\lambda}(\mathbf{1}) \cdot \mathbf{w}) \;\; z \in P, \;\; \mathbf{w} \in \mathbf{C}^k. \tag{3.1}$$

The corresponding Hamiltonian function, $\widetilde{H} : P \times \mathbf{C}^k \to \mathbf{R}$ is

$$\widetilde{H}(z, \mathbf{w}) = \sum |w_i|^2.$$

P is a critical orbifold of $\widetilde{H}$ and S^1 acts freely on the regular level sets of $\widetilde{H}$, which are diffeomorphic to $P \times S^{2k-1}$. $U(k)$ preserves $\widetilde{H}$ and its action commutes with the S^1-action. Therefore, $\widetilde{H}$ descends to a Hamiltonian function H for an induced S^1-action on $P \times_{U(k)} \mathbf{C}^k$. The regular level sets of H are now $P \times_{U(k)} S^{2k-1}$.

We can again define neighborhoods of $P/U(k) \equiv X$ in ν_X by $N_\varepsilon(X) = H^{-1}([0, \varepsilon))$ for small enough values of ε such that $N_\varepsilon(X) \subset N_{\nu_X}(Z_0)$. Moreover, these neighborhoods have ω-compatible boundaries. Hence, if we consider $N_\varepsilon = N_{\delta+\varepsilon}(X) \setminus N_\varepsilon(X)$ for small fixed values of $\varepsilon, \delta \geq 0$, we can again reduce its boundary by collapsing each circle fibre to a point. We obtain in this way an orbifold $L(\delta, \varepsilon)$ with interior symplectomorphic to $int N_\varepsilon$ and which is an orbibundle over the orbifold

$$B \equiv (P \times_{U(k)} S^{2k-1})/S^1$$

Again, if we glue $M^-_{\delta,\varepsilon} = M \setminus \phi^{-1}(N_{\varepsilon+\delta}(X))$ to $\phi^{-1}(L(\delta, \varepsilon))^+$ along

$$S(\delta + \varepsilon) \equiv \phi^{-1}(\partial N_{\varepsilon+\delta}(X)) \cong P \times_{U(k)} S^{2k-1}$$

we obtain an orbifold which we will call the blow up of M along X.

Definition 3.3.1 *Let (M^{2n}, ω) be a $2n$-dimensional symplectic orbifold and let X be a suborbifold of M of codimension $2k$. The blow up of M along X is given by*

$$\widetilde{M}_X \equiv M^-_{\delta,\varepsilon} \bigsqcup_{S(\delta+\varepsilon)} \phi^{-1}(L(\delta, \varepsilon))^+$$

where $L(\delta, \varepsilon)$ is the total space of a Seifert orbibundle over a $(2k-2)$-orbifold and $S(\delta+\varepsilon)$ is the $(2k-1)$-manifold with the fixed point free action of S^1 described above.

Remark 3.3.2 *1. Again we could use different circle actions as long as they fix X and commute with the action of $U(k)$.*

2. Again the diffeomorphism class of these blow up constructions is independent of the choice of connection on the principal orbibundle P and of the symplectomorphism between $\mathcal{N}(X)$ and a neighborhood of the zero section on ν_X. However just as in the manifold case, (c.f. [MD-S]) it is not clear if there is an $\varepsilon > 0$ such that for a given circle action all the corresponding ε-blow ups of M along X are symplectomorphic.

3.4 Blowing up conic orbifold singularities

As we have mentioned before, we can obtain different blow ups by considering different compatible circle actions. We will study some of these in detail when the orbifold structure group of the singularity is cyclic. These special cases of weighted blow up that we describe, will play a fundamental role in the applications.

3.4.1 Weighted blow up

Instead of the standard scalar S^1-action on a Darboux uniformizing chart $(\widetilde{U}, \Gamma, \varphi)$ for x, we can consider the weighted circle actions given by

$$\lambda \cdot \mathbf{z} = (\lambda^{m_1} z_1, \ldots, \lambda^{m_n} z_n), \ \ \lambda \in S^1.$$

These will then descend to U and again we can remove a neighborhood of the form $H^{-1}([0, \varepsilon))$ and collapse the boundary of the resulting orbifold along the orbits of the circle action.

A special case of particular interest in the applications considered in this work occurs on the reduced spaces of a symplectic circle action when we pass a critical level containing a critical point of signature $(2, 2(n-1))$, (cf. Section 4.1). This special case occurs when we have a neighborhood of an order p conic singularity x modelled by $\mathbf{C}^n/\mu_p$ where μ_p acts by,

$$\xi \cdot \mathbf{z} = \underline{\xi}(\mathbf{q})\, \mathbf{z}, \ \ \xi = e^{\frac{2\pi i}{p}}, \ \ \mathbf{q} = (q_1, \ldots, q_n), \ 0 < q_i \leq p$$

and we choose a weighted circle action with weights $(\mathbf{m})$ where $m_i \cong q_i \pmod p$, for $1 \leq i \leq n$. In these cases it is possible to obtain a nice description of the blow up:

Lemma 3.4.1 *Let (M^{2n}, ω) be a $2n$-dimensional symplectic orbifold and let $x \in M$ be a conic orbifold singularity with structure group μ_p, the group of p-roots of unity. Let $(\widetilde{U}, \mu_p, \varphi)$ be a l.u.s. for x. If μ_p acts on $\widetilde{U}$ by*

$$\xi \cdot \mathbf{z} = \underline{\xi}(\mathbf{q}) \cdot \mathbf{z}, \ \ \xi \in \mu_p \ \ 0 < q_i \leq p, \ \xi_p \in \mu_p$$

then, for $\mathbf{m} = (m_1, \ldots, m_n)$*, with* $m_i \equiv q_i \pmod p$*, the weighted* $(\mathbf{m})$*-blow up of* (M, ω) *at* x *is given by:*

$$\widetilde{M}_x(\mathbf{m}) = M \# \overline{\mathbf{C}P}^n(p, \mathbf{m})$$

where $\overline{\mathbf{C}P}^n(p, \mathbf{m})$ *is the* $(p, m_1, \ldots, m_n)$*-weighted projective space with reverse orientation. Moreover, the exceptional divisor is symplectomorphic to* $\mathbf{C}P^{n-1}(\mathbf{m})$*.*

Proof: An action satisfying the above conditions descends to $U \cong \widetilde{U}/\mu_p$ and induces a new Hamiltonian S^1-action on this set. The regular level sets of its Hamiltonian function are now diffeomorphic to the lens complex (c.f. (2.9)),

$$\mathrm{L}(p; \mathbf{q}) \equiv S^{2n-1}/\text{"twisted } \mu_p\text{-action"} \tag{3.2}$$

$$\equiv S^{2n-1}/\,(\mathbf{z} \sim \underline{\xi}(\mathbf{q}) \cdot \mathbf{z},\ \xi \in \mu_p) \tag{3.3}$$

S^1 acts on these manifolds in the following way:

$$\lambda \cdot (\mathbf{z})_p^{'} = (\underline{\lambda}(\mathbf{m}) \cdot \mathbf{z})_p^{'} \tag{3.4}$$

where $(\cdot)_p^{'}$ denotes the μ_p-equivalence class. Hence the quotient spaces

$$H^{-1}(\alpha)/S^1 = S^{2n-1}/(\mu_p \times S^1) = (S^{2n-1}/S^1) = \mathbf{C}P^{n-1}(\mathbf{m})$$

are the weighted projective spaces $\mathbf{C}P^{n-1}(\mathbf{m})$, (cf. Proposition 2.2.3).

If we consider the space $(\mathbf{C}P^n(p, \mathbf{m}), \omega)$, and remove a small neighborhood $V(\delta)$ of the singularity of order p, such that

$$V(\delta) \equiv B_\delta^{2n}/\mu_p,$$

we obtain a space $E(\delta)$, which projects on $\mathbf{C}P^{n-1}(\mathbf{m})$

$$\begin{array}{ccc} E & \xrightarrow{\rho} & \mathbf{C}P^{n-1}(\mathbf{m}) \\ [z_0 : z_1 : \cdots : z_n] & \longmapsto & [z_1 : \cdots : z_n] \end{array}$$

E is the total space of an orbibundle over $\mathbf{C}P^{n-1}(\mathbf{m})$ (cf. Proposition 2.3.1) and, by Remark 2.3.2, if in particular each m_i divides p, it is a (complex) line bundle. Moreover, this space E is a symplectic orbifold with boundary,

$$S \equiv \partial E = S^{2n-1}/\mu_p = \mathrm{L}(p; \mathbf{q})$$

where μ_p acts on S^{2n-1} by (3.3).

We can now consider this orbifold with the opposite orientation, $\overline{E}$. It is also the total space of an orbibundle over $\mathbf{C}P^{n-1}(\mathbf{m})$ and it has boundary $\mathrm{L}(p; \mathbf{q})$. Both $\overline{E} \backslash \mathbf{C}P^{n-1}(\mathbf{m})$ and $E \backslash \mathbf{C}P^{n-1}(\mathbf{m})$ are diffeomorphic to $I \times \mathrm{L}(p; \mathbf{q})$ i.e. they are diffeomorphic to $\mathbf{C}^n \backslash \{0\}/\,\mu_p$. Moreover, by the symplectic neighborhhod theorem, $\overline{E}$ is symplectomorphic to $\pi(N_{\varepsilon+\delta}(x)\backslash N_{\varepsilon}(x)) \equiv L(\delta,\varepsilon)$ for adequate values of ε and δ, where π is the map defined in Section 3.1 and $N_\varepsilon(x) = H^{-1}([0,\varepsilon))$, for $\varepsilon > 0$.

Therefore, $\widetilde{M_x}$ is obtained by removing a neighborhood $N_{\varepsilon+\delta}(x)$ of x and gluing the resulting orbifold, $M\backslash N_{\delta+\varepsilon}(x)$, with $L(\delta,\varepsilon)$ along $S = \partial L(\delta,\varepsilon)$.

$$\begin{aligned}\widetilde{M_x}(\mathbf{m}) &= (M\backslash N_{\delta+\varepsilon}(x))^+ \bigsqcup_S L(\delta,\varepsilon)^- \\ &= (M\backslash N_{\delta+\varepsilon}(x))^+ \bigsqcup_S \overline{E}^{\,-} \\ &= M\#\overline{\mathbf{C}P}^n(p,\mathbf{m})\end{aligned}$$

and the exceptional divisor resulting from this blow up is $\mathbf{C}P^{n-1}(\mathbf{m})$.

Remark 3.4.2 *In the general case of a weighted blow up of a conic orbifold singularity, the exceptional divisors obtained are quotients of weighted projective spaces by finite cyclic groups.*

Example 3.4.3 *The weighted (n_1,n_2)-blow up of a regular point of a four dimensional manifold M, is obtained by removing an ellipsoid of equation $n_1|z_1|^2+n_2|z_2|^2 = C$ and reducing the boundary of the resulting manifold, along the orbits of the S^1-action*

$$\lambda \cdot (z_1,z_2) = (\lambda^{n_1}z_1, \lambda^{n_2}z_2)$$

the exceptional divisor obtained is $\mathbf{C}P^1(n_1,n_2)$, (c.f. Figure 3.3).

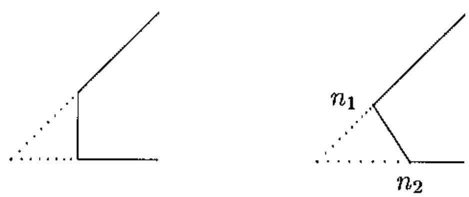

Figure 3.3: (a) $\widetilde{(\mathbf{C}^2)}$; (b) $\widetilde{(\mathbf{C}^2)}(n_1,n_2)$, $n_1 > n_2$

3.4.2 Weighted blow up as plumbing

It is a well known result that the blow up of $\mathcal{O}(k_1+1)\oplus\cdots\oplus\mathcal{O}(k_n+1)$ at a point on its zero section (where $\mathcal{O}(t)$ is the bundle over CP^1 with Euler class t), can be identified with the plumbing $E \# L_{can}$ where

$$E = \mathcal{O}(k_1)\oplus\cdots\oplus\mathcal{O}(k_n)$$

and $L_{can} \to \mathbf{C}P^{n-1}$ is the canonical line bundle.

Again in the case of the special weighted blow ups of conic singularities described above, a similar result holds. However, before we state it, we need to define the orbibundles $\mathcal{O}_{q,r}(m) \to \mathbf{C}P^1(q,r)$

obtained (when $m > 0$) from $\mathbf{C}P^2(q,r,m)$ as described in Proposition 2.3.1, by removing a small neighborhood of the point with orbifold structure group μ_m (for $m < 0$, $\mathcal{O}_{q,r}(m)$ is just $\mathcal{O}_{q,r}(-m)$ with opposite orientation).

Proposition 3.4.4 *The orbibundle $\mathcal{O}_{q,r}(m) \to \mathbf{C}P^1(q,r)$, $m > 0$, obtained from $\mathbf{C}P^2(q,r,m)$ by removing a small neighborhood of the point with orbifold structure group μ_m is given by,*

$$\begin{aligned} \mathcal{O}_{q,r}(m) &= \frac{D^+ \times \mathbf{C}}{\mu_q} \bigcup_{\alpha} \frac{D^- \times \mathbf{C}}{\mu_r} \\ &= \frac{S^3 \times \mathbf{C}}{(z_0, z_1, z_2) \sim ((\lambda^q z_0, \lambda^r z_1), \lambda^m z_2), \lambda \in S^1} \end{aligned}$$

where μ_q and μ_r act on $D^\pm \times \mathbf{C}$ by:

$$\xi \cdot (z_1, z_2) = (\xi^r z_1, \xi^m z_2), \quad \xi \in \mu_q$$
$$\xi \cdot (z_1, z_2) = (\xi^q z_1, \xi^m z_2), \quad \xi \in \mu_r$$

D^+, D^- are two discs with D^+ oriented positively and D^- negatively, and the gluing map α is given by

$$\begin{aligned} \alpha : \frac{\partial D^+ \times \mathbf{C}}{\mu_q} &\to \frac{\partial D^- \times \mathbf{C}}{\mu_r} \\ \left[e^{r\theta i}, y\right]_q &\mapsto \left[e^{-q\theta i}, e^{-m\theta i} y\right]_r \end{aligned}$$

Proof: Consider the covering of $\mathbf{C}P^1(q,r)$ by the sets

$$V_q = \{[z_1 : z_2]_{(q,r)} \in \mathbf{C}P^1(q,r) : \, z_1 \neq 0\} \cong \mathbf{C}/\mu_q$$

$$V_r = \{[z_1 : z_2]_{(q,r)} \in \mathbf{C}P^1(q,r) : \, z_2 \neq 0\} \cong \mathbf{C}/\mu_r$$

then, the orbibundle $\mathcal{O}_{q,r}(m) \to \mathbf{C}P^1(q,r)$, $m > 0$, is then defined , as in Proposition 2.3.1 by the l.u.s.

$$\{\mathbf{C}^2, \mu_q, \varphi_q\}$$
$$\{\mathbf{C}^2, \mu_r, \varphi_r\}$$

The gluing map α is then obtained by,

$$\begin{aligned} \alpha((e^{r\theta i}, y)_q) &= \tilde{\phi}_r \circ \tilde{\phi}_q^{-1}((e^{r\theta i}, y)_q) \\ &= \tilde{\phi}_r(\left[1 : e^{r\theta i} : y\right]_{(q,r)}) \\ &= \tilde{\phi}_r(\left[e^{-q\theta i} : 1 : e^{-m\theta i} y\right]_{(q,r)}) \\ &= (e^{-q\theta i}, e^{-m\theta i} y)_r \end{aligned}$$

where

$$\begin{array}{rcl} \pi_m : & CP^2(q,r,m) & \to & CP^1(q,r) \\ \tilde{\phi}_q : & \pi_m^{-1}(V_q) & \to & D^+ \times_{\mu_q} \mathbf{C} \\ \tilde{\phi}_r : & \pi_m^{-1}(V_r) & \to & D^- \times_{\mu_r} \mathbf{C} \end{array}$$

are the maps defined in Section 2.3 and $(.)_k$ denotes the μ_k-characteristic class, $(k = q, r)$. □

After the description of these bundles we can now state the following proposition:

Proposition 3.4.5 *Let x be an orbifold conic singularity with structure group μ_p lying on the zero section of the bundle*

$$\mathcal{O}_{p,a}(m_1) \oplus \mathcal{O}_{p,a}(m_2) \oplus \cdots \oplus \mathcal{O}_{p,a}(m_{n-1}) \to \mathbf{C}P^1(p,a)$$

Then, for $r = \pm a + k_0 p$, and $q_i = \pm m_i + k_i p, (i = 1, \cdots, n;\ k_i \in \mathbf{Z})$, the weighted $(r, q_1, \cdots, q_{n-1})$-blow up of this bundle at x can be identified with

$$E \# L$$

where

$$E = \mathcal{O}_{r,a}(\frac{m_1 r - a q_1}{p}), \oplus \cdots \oplus \mathcal{O}_{r,a}(\frac{m_{n-1} r - a q_{n-1}}{p}) \to \mathbf{C}P^1(r,a)$$

$L \to \mathbf{C}P^{n-1}(r, q_1, \cdots, q_{n-1})$ is the rank 2 orbibundle described below and the plumbing ($\#$) is made on a neighborhood of the fiber γ_r of E with multiplicity r where μ_r is the structure group of the intersection point of the exceptional divisor with the image of $\mathbf{C}P^1(p,a)$, after the blow up.

Remark 3.4.6 *1. $L \to \mathbf{C}P^{n-1}(r, q_1, \cdots, q_{n-1})$ is the orbibundle isomorphic to a neighborhood of the zero section of the normal orbibundle of $\mathbf{C}P^{n-1}(r, q_1, \cdots, q_{n-1})$ inside*

$$\mathbf{C}P^n(p, r, q_1, \cdots, q_{n-1}),$$

with opposite orientation.

2. By plumbing we mean the result of a surgery that matches the "horizontal sections" of E with the "vertical fibers" of L, in a neighborhood of the fiber γ_r of E, as we will see in the proof below.

Remark 3.4.7 *When we do not have any orbifold singularities, we have the usual line bundles over $\mathbf{C}P^1$ and the above formulas give us the known result that the Euler class of each line bundle $\mathcal{O}(m)$ decreases by 1 after the blow up, (we just have to make all the q_i's, p, a and r all equal to 1).*

Proof: The plumbing of E and L is obtained in the following way: Choose neighborhoods B_1, B_2 of the point of order r in $\mathbf{C}P^1(r,a)$ and in $\mathbf{C}P^{n-1}(r, \mathbf{q})$ and the orbibundles over them, $E_{|B_1}$, $L_{|B_2}$. Then there are natural identifications,

$$E_{|B_1} \cong D^2 \times_{\mu_r} D^{2(n-1)} \tag{3.5}$$

$$L_{|B_2} \cong D^{2(n-1)} \times_{\mu_r} D^2 \tag{3.6}$$

where μ_r acts on $D^2 \times D^{2(n-1)}$ and on $D^{2(n-1)} \times D^2$ by:

$$\xi \cdot (w, \mathbf{z}) = (\xi^a, \underline{\xi}(\tilde{\mathbf{m}}) \cdot \mathbf{z}), \;\; \xi \in \mu_r \tag{3.7}$$

$$\xi \cdot (\mathbf{z}, w) = (\underline{\xi}(\mathbf{q}) \cdot \mathbf{z}, \xi^{-p} w), \;\; \xi \in \mu_r \tag{3.8}$$

As $r = \pm a + k_0 p$ and $q_i = \pm m_i + k_i p$, for $i = 1, \cdots, n$,

$$\begin{aligned} m_i &\equiv \pm q_i \pmod{p} \\ a &\equiv \mp k_0 p \pmod{r} \end{aligned}$$

and then,

$$\begin{aligned} \tilde{m}_i &= \tfrac{m_i r - a q_i}{p} = \tfrac{1}{p}(\pm m_i a + m_i k_0 p \mp m_i a - a k_i p) \\ &= m_i k_0 - a k_i \equiv (\pm q_i \mp k_i p) k_0 \pm k_0 k_i p \pmod{r} \equiv \pm k_0 q \pmod{r}. \end{aligned}$$

Hence, the action of μ_r on $D^2 \times D^{2(n-1)}$ on (3.7) becomes

$$\xi \cdot (w, \mathbf{z}) = (\xi^{\mp k_0 p}, \underline{\xi}(\pm \mathbf{k_0 q}) \cdot \mathbf{z}), \;\; \xi \in \mu_r \tag{3.9}$$

$$= (\xi^{-p}, \underline{\xi}(\mathbf{q}) \cdot \mathbf{z}), \;\; \xi \in \mu_r \tag{3.10}$$

We can consider then the reflection

$$\begin{aligned} f : D^2 \times D^{2(n-1)} &\to D^{2(n-1)} \times D^2 \\ (w, \mathbf{z})_r &\mapsto (\mathbf{z}, w), \end{aligned}$$

and we can paste E and L together along $E_{|B_1}$ and $L_{|B_2}$ by the map f, obtaining the plumbing of E with L.

In Figure 3.4.2, we can see images of E, L and $E \# L$ by moment maps, in the four dimensional case:

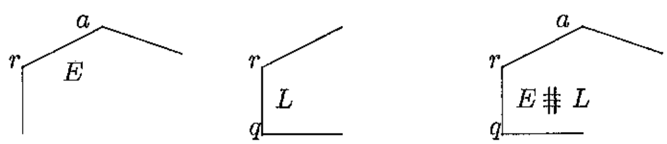

Figure 3.4: (a) E; (b) L; (c) $E \# L$;

Now that we have $E \# L$, we still have to show that the blow down of $F = \mathbf{C}P^{n-1}(r, \mathbf{q})$ on this space, is

$$\mathcal{O}_{p,a}(m_1) \oplus \cdots \oplus \mathcal{O}_{p,a}(m_{n-1})$$

A neighborhood of F, $N(F)$, has boundary $S(L) = S^{2n-1}/\mu_p$. Therefore the blow down of F is obtained by removing this neighborhood and replacing it by a $2n$-"cell" of the form B^{2n}/μ_p where μ_p acts on B^{2n} by

$$\xi \cdot \mathbf{z} = \underline{\xi}(\mathbf{q}, r) \cdot \mathbf{z}, \;\; \xi \in \mu_p$$

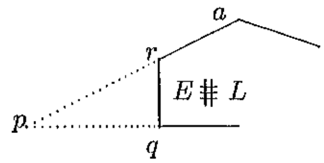

Figure 3.5: Blow down of $E \# L$ along $\mathbf{C}P^1(r,q)$

with the appropriate size. Note that the boundary of this cell is also S^{2n-1}/μ_p.

In this way, we obtain a space which fibres over

$$\mathbf{C}P^1(p,a) = \mathbf{C}/\mu_p \bigcup (\mathbf{C}P^1(r,a) \setminus N(x_r))$$

where $N(x_r) = N(F) \cap \mathbf{C}P^1(r,a)$. This orbibundle decomposes into a sum of rank 2 orbibundles over $\mathbf{C}P^1(p,a)$. This is best seen by looking at moment map pictures (see Remark 3.4.8 for a six-dimensional example).

Moreover, by construction, these line orbibundles have the following decomposition,

$$\frac{D_i^+ \times \mathbf{C}}{\mu_p} \bigcup_{\alpha} \frac{D_i^- \times \mathbf{C}}{\mu_a}$$

where μ_p acts on $D_i^+ \times \mathbf{C}$ by:

$$\begin{aligned} \xi \cdot (z_1, z_2) &= (\xi^r z_1, \xi^{q_i} z_2) \\ &= (\xi^{\pm a} z_1, \xi^{\pm m_i} z_2) \\ &= \left(\tilde{\xi}^a z_1, \tilde{\xi}^{m_i} z_2\right), \quad \xi, \tilde{\xi} \in \mu_p, \end{aligned}$$

μ_a acts on $D_i^- \times \mathbf{C}$ by:

$$\begin{aligned} \xi \cdot (z_1, z_2) &= (\xi^r z_1, \xi^{\tilde{m}_i} z_2) \\ &= (\xi^{k_0 p} z_1, \xi^{k_0 m_i} z_2) \\ &= \left(\hat{\xi}^p z_1, \hat{\xi}^{m_i} z_2\right), \quad \xi, \tilde{\xi}, \hat{\xi} \in \mu \end{aligned}$$

and the gluing map is given by,

$$\begin{aligned} \alpha : \frac{\partial D_i^+ \times \mathbf{C}}{\mu_p} &\to \frac{\partial D_i^- \times \mathbf{C}}{\mu_a} \\ \left[e^{a\theta i}, y\right]_p &\mapsto \left[e^{-p\theta i}, e^{-m_i \theta i} y\right]_a \end{aligned}$$

Consequently, by Proposition 3.4.4, each of these rank 2 orbibundles is isomorphic to $\mathcal{O}_{p,a}(m_i)$, for $1 \leq i \leq n$.

Again, the above holds because

$$\tilde{m}_i = \frac{m_i r}{p} - \frac{a q_i}{p} \pm \frac{m_i a}{p} \mp \frac{m_i a}{p} \equiv \frac{m_i}{p}(r \mp a) \pmod a \equiv k_0 m_i \pmod a$$

□

Remark 3.4.8 *This description of weighted blow up as plumbing was suggested by McDuff in [MD2] in the context of almost complex structures in* $X = S^2 \times S^2$*.If we consider* (X, ω^λ)*,with* $\omega^\lambda = (1+\lambda)\sigma_1 + \sigma_2$ *for some* $\lambda \geq 0$*, where the 2-form* σ_i *has total area 1 on the ith factor, then the space* $\mathcal{J}^\lambda$ *of all* C^∞ ω^λ*-compatible almost complex structures on X has a stratification which changes as* λ *passes each integer. In particular McDuff proves that for each* $m > k \geq 1$ *there is a neighborhood of the space*

$$\mathcal{J}_m^\lambda = \{J \in \mathcal{J}^\lambda : \text{there is a } J\text{-hol curve in class } A - mF\}$$

in $\overline{\mathcal{J}}_k^\lambda$ *that is fibered over* $\mathcal{J}_m^\lambda$ *with fiber equal to the cone* $C(\mathcal{L}_{m,k}^\lambda)$ *over a stratified space* $\mathcal{L}_{m,k}^\lambda$ *of dimension* $4(m-k)-1$*. Moreover, she illustrates this result by constructing the link* $\mathcal{L}_{2,0}^\lambda$*. For this she first constructs an auxiliary space* $\mathcal{L}_\mathcal{Z}$ *obtained by plumbing the unit sphere bundle of* $\mathcal{O}(-3)\oplus\mathcal{O}(-1)$ *with the singular circle bundle* $S(L_Y) \to Y$ *where this last bundle has total space* $L(Y) = S^5$ *and fibers equal to the orbits of the following* S^1*-action on* S^5*:*

$$\theta \cdot (x, y, z) = (e^{i\theta}x, e^{i\theta}y, e^{2i\theta}z), \quad x, y, z \in \mathbf{C}.$$

$\mathcal{L}_{2,0}^\lambda$ *is then obtained from* $\mathcal{L}_\mathcal{Z}$ *by collapsing the fibers over the exceptional divisor to a single fiber and so it can be identified with* S^5*. We can easily see then that this auxiliary space* $\mathcal{L}_\mathcal{Z}$ *is the sphere bundle of the weighted* $(1,2,1)$*-blow up of* $\mathcal{O}(-1)\oplus\mathcal{O}(0)$*.*

In Figure 3.4 we can see the image of a moment map for effective T^3*-actions on* $\mathcal{O}(-1)\oplus\mathcal{O}(0)$*, (cf. [L-T]) and on the corresponding blow up. Here we can see how the Euler classes of the line bundles over* $\mathbf{C}P^1$ *change with the blow up, (they can be determined by the difference in slopes of the lines on the same face of the polygon which "start" on the image, B, of* $\mathbf{C}P^1$*).*

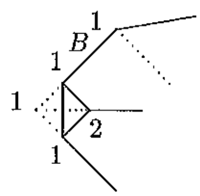

Figure 3.6: Weighted $(1,2,1)$-blow up of $\mathcal{O}(-1)\oplus\mathcal{O}(0)$

Example 3.4.9 *Consider the space* $O = S^2 \times_{\mu_2} S^2$ *obtained from* $S^2 \times S^2$ *by identifying* $([z_0 : z_1], [w_0 : w_1])$ *with* $([-z_0 : z_1], [-w_0 : w_1])$*. This space is an orbifold with four isolated singularities of order two:*

$$([1:0],[1:0]),\ (1:0],[0:1]),\ ([0:1],[1:0]),\ ([0:1],[0:1])$$

A neighborhood of each of these singularities can be modelled by

$$\mathbf{C}^2 / \{(z,w) \sim (-z,-w)\}$$

Moreover, the blow up of O, or in this case the weighted $(1,1)$-blow up, at one of these singularities will give us a $\mathbf{C}P^1$ as an exceptional divisor. In Figure 3.5 (a) we can see the image of a moment map for a suitable T^2-action on $S^2 \times_{\mu_2} S^2$. The labels on the vertices indicate the order of the singularities in their preimage. Moreover, the labels on the edges indicate the intersection numbers of the spheres on their preimage.

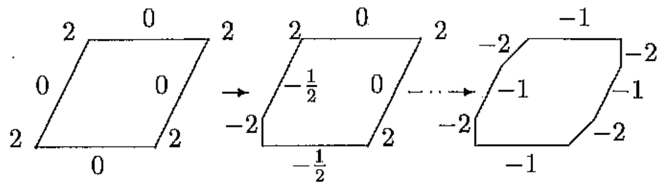

Figure 3.7: (a) $S^2 \times_{\mu_2} S^2$; (b) $S^2 \times_{\mu_2} S^2 \# \overline{\mathbf{C}P}^2(2,1,1)$ (c) $(S^2 \times_{\mu_2} S^2)\# 4\overline{\mathbf{C}P}^2(2,1,1)$

In Figure 3.5 (b) we can see the image of the blow up at one of the orbifold singularities. The intersection numbers are determined by Proposition 3.4.5. If we blow up all four orbifold singularities, we get the rational manifold represented in Figure 3.5 (c) , which is the blow up of $S^2 \times S^2$ at four points. Hence,

$$(S^2 \times_{\mu_2} S^2)\# 4\overline{\mathbf{C}P}^2(2,1,1) \cong (S^2 \times S^2)\# 4\overline{\mathbf{C}P}^2$$

3.4.3 Blowing down

If we have a weighted projective space $\Sigma \cong \mathbf{C}P^{n-1}(q_0,\dots,q_{n-1})$, symplectically embedded in a symplectic orbifold (M,ω), we have seen in Section 3.2 that a neighborhood of Σ, $\mathcal{N}(\Sigma)$, is symplectomorphic to a neighborhood of the zero section in $(S^{2n-1}/\Gamma) \times_{S^1} \mathbf{C}$. In particular, it is symplectomorphic to a neighborhood of the zero section of $S^{2n-1}/\times_{S^1} \mathbf{C}$ for the circle action on $S^{2n-1} \times \mathbf{C}$ given by,

$$\lambda \cdot (\mathbf{z}, w) = (\underline{\lambda}(\mathbf{q})\mathbf{z}, \lambda^p w),\ \ \lambda \in S^1,\ \ \mathbf{q} = (q_o,\dots,q_{n-1}).$$

Therefore, such a neighborhood is symplectomorphic to a neighborhood of $\mathbf{C}P^{n-1}(\mathbf{q})$ inside $\overline{\mathbf{C}P}^n(p,\mathbf{q})$ and so we can perform the weighted blow $(p,\mathbf{q})$-blow down of M along Σ.

In fact, we can choose δ and ε such that $\mathcal{N}(\Sigma)$ is symplectomorphic to $L(\delta,\varepsilon)$ described above and so we can remove $\mathcal{N}(\Sigma)$ and glue a "cell" $B^{2n}(\varepsilon)/\mu_p$ back in, where μ_p acts on $B^{2n}(\varepsilon)$ by:

$$\xi \cdot (z_0,\dots,z_{n-1}) = (\xi^{q_0} z_0,\dots,\xi^{q_{n-1}} z_{n-1}),\ \ \xi \in \mu_p.$$

Chapter 4

Applications

Continuing the work in [G-S] we can describe the behavior of the reduced spaces of an Hamiltonian S^1-action, which may not be quasi-free, when passing through a critical value of the moment map.

4.1 Passing through a critical level of signature $(2, 2n-2)$

In the case of a quasi-free action Guillemin and Sternberg prove that passing this type of critical point has the effect of blowing up the reduced spaces at a point, ([G-S]). Let us now see what happens if the action is not quasi-free.

Let (M, ω) be a symplectic manifold with a Hamiltonian S^1-action and let x be one of its critical pts with signature $(2, 2n-2)$. Then, a neighborhood of x can be identified with $\mathbf{C} \times \mathbf{C}^{n-1}$ with its standard symplectic structure and S^1-action

$$\lambda \cdot (\mathbf{z}) = \underline{\lambda}(-p, \mathbf{q}) \cdot \mathbf{z},$$

with moment map $\mu : M = \mathbf{C}^{n-1} \times \mathbf{C} \to \mathbf{R}$ given by:

$$\mu(\mathbf{z}) = -p|z_0|^2 + \sum_{i=1}^{n-1} q_i |z_i|^2.$$

We will see that

$$O^- \cong O_{-\varepsilon} \cong \mu^{-1}(-\varepsilon)/S^1 \cong \mathbf{C}^{n-1}/\mu_p$$

with the standard induced symplectic structure and

$$O^+ \cong O_\varepsilon \cong \mu^{-1}(\varepsilon)/S^1 \cong (\widetilde{\mathbf{C}^{n-1}/\mu_p})(\mathbf{q}),$$

is the weighted $\mathbf{q}$-blow up of $\mathbf{C}^{n-1}/\mu_p$ at the point of isotropy μ_p. The "exceptional divisor" resulting from this blow up is $\mathbf{C}P^{n-2}(\mathbf{q})$.

In fact, at $-\varepsilon$,

$$O_{-\varepsilon} = \{(z_0, \mathbf{z}) \in \mathbf{C} \times \mathbf{C}^{n-1} : p|z_0|^2 = \sum_{i=1}^{n-1} q_i |z_i|^2 + \varepsilon\}/S^1 \cong \mathbf{C}^{n-1}/\mu_p$$

where μ_p acts on $\mathbf{C}^{n-1}$ by:

$$\xi \cdot \mathbf{z} = \underline{\xi} \cdot \mathbf{z}, \;\; \xi \in \mu_p.$$

O^- is therefore an $(n-1)$-dimensional orbifold (with an isolated singularity of order p) with the standard induced symplectic form.

For $\varepsilon > 0$,

$$O_\varepsilon = \{(z_0, \mathbf{z}) \in \mathbf{C} \times \mathbf{C}^{n-1} : p|z_0|^2 + \varepsilon = \sum_{i=1}^{n-1} q_i |z_i|^2\} \,/\, S^1.$$

Making the following coordinate change,

$$\begin{cases} u_0 &= z_0 \\ u_i &= (\frac{q_i}{p|z_0|^2+\varepsilon})^{1/2} z_i \;\; i = 1, \cdots, n-1 \end{cases}$$

we get

$$\sum_{i=1}^{n-1} |u_i|^2 = \frac{\sum_{i=1}^{n-1} q_i |z_i|^2}{p|z_0|^2 + \varepsilon} = 1$$

and so,

$$O^+ = \mathbf{C} \times_{S^1} S^{2(n-1)-1}$$

where the S^1 action is given by:

$$\lambda \cdot (u_0, \mathbf{u}) = \underline{\lambda}(-p, \mathbf{q}) \cdot (u_0, \mathbf{u}), \;\; \lambda \in S^1.$$

Hence, O^+ is just the orbibundle $\overline{\mathbf{C}P}^{n-1}(p, \mathbf{q})$ we considered in Section 3.4.1 and so it is the weighted $\mathbf{q}$-blow up of O^- at the point of order p. In conclusion, we have proved the following:

Theorem 4.1.1 *Let (M, ω) be a $(2n)$-symplectic manifold with a Hamiltonian S^1-action and let x be a critical pt with signature $(2, 2n-2)$, lying on the critical level $\mu = \lambda$. Then, the reduced spaces $O_{\lambda+\varepsilon} = \mu^{-1}(\lambda+\varepsilon)/S^1$ are all diffeomorphic to the weighted $\mathbf{q}$-blow up of the reduced spaces $O_{\lambda-\varepsilon} = \mu^{-1}(\lambda-\varepsilon)/S^1$ at a point of order p, for suitable values of $\mathbf{q} \in (\mathbf{Z}^+)^{n-1}$ and $p \in \mathbf{Z}^+$*

Remark 4.1.2 *In the case where we have a critical submanifold of signature $(2k, 2)$, the same analysis holds in the normal directions to x.*

We can easily picture the result on Theorem 4.1.1 when $n = 3$. Take $(\mathbf{C}^3, \omega_0)$ with the S^1 action given by:

$$\lambda \cdot (z_0, z_1, z_2) = (\lambda^{-p} z_0, \lambda^{q_1} z_1, \lambda^{q_2} z_2), \; \lambda \in S^1$$

and moment map $\mu(z_0, z_1, z_2) = -p|z_0|^2 + q_1|z_1|^2 + q_2|z_2|^2$. First, we define an effective $S^1 \times S^1$-action on each reduced space $\mu^{-1}(\pm\varepsilon)/S^1$ in the following way:
Consider the standard symplectic action of $\mathbf{T}^3$ on $\mathbf{C}^3$ with moment map:

$$\mu_{\mathbf{T}^3}(z_0, z_1, z_2) = (|z_0|^2, |z_1|^2, |z_2|^2)$$

Define a linear projection $P : \mathbf{R}^3 \to \mathbf{R}^2$ by the matrix

$$\begin{pmatrix} m_1 & -p\beta m_3 & p\alpha m_2 \\ 0 & q_2/m_1 & -q_1/m_1 \end{pmatrix}$$

where $m_1 = \gcd(q_1, q_2)$, $m_2 = \gcd(p, q_1)$, $m_3 = \gcd(p, q_2)$ and $\alpha, \beta \in \mathbf{Z}^+$ are such that $\alpha m_2 q_2 - \beta m_3 q_1 = m_1$.

Remark 4.1.3 *As the S^1-action is effective,* $\gcd(p, q_1, q_2) = 1$. *Hence*

$$\gcd(m_3 q_1/m_1, m_2 q_2/m_1) = 1$$

and so α and β exist as defined.

P induces a map from $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ to $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. Its kernel is given by:

$$\begin{aligned} K &= \{[k] \in \mathbf{T}^3 : P(k) \in \mathbf{Z}^2\} \\ &= \{[-pt, q_1 t, q_2 t] \in \mathbf{T}^3 : t \in \mathbf{R}\} \\ &= \{(\lambda^{-p}, \lambda^{q_1}, \lambda^{q_2}),\ \lambda \in S^1\} \quad \text{with the identification } \lambda = e^{2\pi i t}. \end{aligned}$$

As K is a subgroup of $\mathbf{T}^3$, it acts on $\mathbf{C}^3$ with moment map

$$\mu_K = j^* \circ \mu_{\mathbf{T}^3} = -p|z_0|^2 + q_1|z_1|^2 + q_2|z_2|^2$$

where $j : K \to \mathbf{R}^3$ is the inclusion map. This action of K on $\mathbf{C}^3$ is our initial action of S^1 and so,

$$\mu_K^{-1}(\pm\varepsilon)/K \cong O_{\pm\varepsilon}.$$

Moreover, the action of $\mathbf{T}^3$ on $\mathbf{C}^3$ induces Hamiltonian actions of T^3 on both $O_{-\varepsilon}$ and O_ε and the moment map $\mu_{\mathbf{T}^3}|_{\mu_K^{-1}(\mp\varepsilon)}$ descends to the moment maps:

$$\begin{aligned} \tilde{\mu}_- : O_{-\varepsilon} &\to (\mathbf{R}^3)^* \\ \tilde{\mu}_+ : O_{+\varepsilon} &\to (\mathbf{R}^3)^* \end{aligned}$$

with images

$$\begin{aligned} \tilde{\mu}_\mp(O_{\mp\varepsilon}) &= \{(x, y, z) \in (\mathbf{R}^3)^* : x, y, z \geq 0 \wedge -px + q_1 y + q_2 z = \mp\varepsilon\} \\ &= P^t((\mathbf{R}^2)^*) + \begin{pmatrix} \mp\varepsilon/p & 0 & 0 \end{pmatrix}^t. \end{aligned}$$

Now the action of $\mathbf{T}^3$ on $O_{-\varepsilon}$ descends to a Hamiltonian action of $T^2 = \mathbf{T}^3/K$ on O^- and we have the following diagram:

$$\begin{array}{ccccc} (\mathbf{R}^2)^* & = & (Lie(T^2))^* & \stackrel{\psi_{-\varepsilon}}{\longrightarrow} & \tilde{\mu}_-(O_{-\varepsilon}) \\ & & \mu^-_{T^2} \nwarrow & & \nearrow \tilde{\mu}_- \\ & & & O_{-\varepsilon} & \end{array}$$

where $\psi_{-\varepsilon}(y) = P^t y + (\varepsilon/p, 0, 0)^t$. $\mu^-_{T^2} = \psi^{-1}_{-\varepsilon} \circ \tilde{\mu}_-$ is therefore a moment map for this action.

Moreover, this T^2-action on O_- is effective. To show this, it is enough to find a point $z \in O_{-\varepsilon}$ with trivial isotropy group or, equivalently, to find a point on $\mu_K^{-1}(-\varepsilon)$ with trivial isotropy group in $\mathbf{T}^3$, (any point $(x, y, z) \in \mu_K^{-1}(-\varepsilon)$ such that $x, y, z > 0$ will do).

We can easily see that $\mu_{T^2}(O_{-\varepsilon})$ is the following region of the plane:

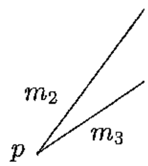

Figure 4.1: $\mu_{T^2}(O_{-\varepsilon})$

The labels m_i in the above Figure 4.1 are the orders of the orbifold structure groups of the points in the preimage of the respective edges and the label p is the order of the orbifold structure group of the point with image $(0, 0)$.

Similarly, the action of $\mathbf{T}^3$ on $O_{+\varepsilon}$ descends to a Hamiltonian action of T_2 on O_+ and we have the diagram

$$\begin{array}{ccccc} (\mathbf{R}^2)^* & = & (Lie(T^2))^* & \stackrel{\psi_{\varepsilon}}{\longrightarrow} & \tilde{\mu}_+(O_{+\varepsilon}) \\ & & \mu_{T^2} \nwarrow & & \nearrow \tilde{\mu}_+ \\ & & & O_{+\varepsilon} & \end{array}$$

where $\psi_\varepsilon(y) = P^t y + (-\varepsilon/p, 0, 0)^t$. Again this action on $O_{+\varepsilon}$ is effective, as any point $(x, y, z) \in \mu_K^{-1}(\varepsilon)$ such that $x, y, z > 0$ has trivial isotropy group in $\mathbf{T}^3$.

The image $\mu_{T^2}(O_{+\varepsilon})$ is the region of the plane in Figure 4.1 where the labels m_i indicate the order of the orbifold structure groups of the points on the pre-image of the labeled edges and the labels q_1, q_2 are the order of the orbifold structure groups of the points with image $(\varepsilon/pm_1, \alpha m_2 \varepsilon/q_1)$ and $(\varepsilon/pm_1, \beta m_3 \varepsilon/q_2)$ respectively. For more details on how to obtain effective torus actions on compact orbifolds and their moment map polytopes, please refer to [L-T].

We can now see from Figures 4.1 and 4.1, that the reduced spaces O_- for negative values of the moment map μ_K are all diffeomorphic to $\mathbf{C}^3/\mu_p$ and that for positive values of μ_K they are the weighted (q_1, q_2)-blow up of O_- at the point of orbifold structure group μ_p.

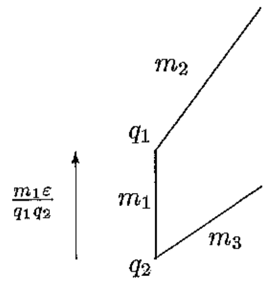

Figure 4.2: $\mu_{T^2}(O_{+\varepsilon})$

4.2 Passing through a critical level of signature $(2(n-k), 2k)$

In the case of a quasi-free circle action, Guillemin and Sternberg reduce this case to the previous one by proving that passing through a critical level of signature $(2(n-k), 2k)$ is the same as passing through two critical levels, one of signature $(2(n-k), 2)$ and the other of signature $(2, 2k)$. The trick used was to blow up the manifold at the critical point. This point then "becomes" two fixed manifolds. However, this may no longer hold when the action is not quasi-free. We will describe next how to proceed in the general case.

Let μ be a Hamiltonian function for an S^1-action on a symplectic manifold M and consider a critical point x of μ with signature $(2(n-k), 2k)$. A neighborhood of x in M can be identified with $\mathbf{C}^{n-k} \times \mathbf{C}^k$, where S^1 acts by

$$\lambda \cdot (\mathbf{z}, \mathbf{w}) = \underline{\lambda}(-\mathbf{p}, \mathbf{q}) \cdot (\mathbf{z}, \mathbf{w}),$$

with $\mathbf{p} = (p_1, \ldots, p_{n-k})$, $\mathbf{q} = (q_1, \ldots, q_k)$, and

$$\mu(\mathbf{z}, \mathbf{w}) = -\sum p_i |z_i|^2 + \sum q_i |w_i|^2.$$

We will see that

$$O^- \equiv O_{-\varepsilon} \cong \mu^{-1}(-\varepsilon)/S^1 \cong \mathcal{O}_{\mathbf{p}}(q_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{p}}(q_k)$$

where

$$\mathcal{O}_{\mathbf{p}}(q_i) \to \mathbf{C}P^{n-k}(\mathbf{p})$$

is the line orbibundle diffeomorphic to the normal orbibundle of $\mathbf{C}P^{n-k}(\mathbf{p})$ inside

$$\mathbf{C}P^{n-k+1}(\mathbf{p}, q_i),$$

with symplectic structure given by symplectic reduction.

Remark 4.2.1

$$\mathcal{O}_{\mathbf{p}}(q_i) = S^{2(n-k)+1} \times_{S^1} \mathbf{C}$$

where S^1 acts on $S^{2(n-k)+1} \times \mathbf{C}$ by

$$\lambda \cdot (\mathbf{z}, w) = \underline{\lambda}(\mathbf{p}, q_i) \cdot (\mathbf{z}, w)$$

On the other hand, we will see that

$$O^+ \equiv O_\varepsilon \cong \mu^{-1}(\varepsilon)/S^1 \cong \mathcal{O}_{\mathbf{q}}(-p_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{q}}(-p_{n-k}),$$

is the result of a blow down of O^- followed by a blow up.

In fact, at $-\varepsilon$,

$$\begin{aligned} O_{-\varepsilon} &= \{(\mathbf{z}, \mathbf{w}) \in \mathbf{C}^{n-k} \times \mathbf{C}^k : \sum p_i |z_i|^2 = \sum q_i |w_i|^2 + \varepsilon\}/S^1 \\ &= S^{2(n-k)-1} \times_{S^1} \mathbf{C}^k \\ &= \mathcal{O}_{\mathbf{p}}(q_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{p}}(q_k). \end{aligned}$$

O^- is therefore a $2(n-1)$-dimensional orbifold which is the total space of an orbibundle over $\mathbf{C}P^{n-k-1}(\mathbf{p})$. Moreover it is isomorphic to the normal orbibundle of

$$\mathbf{C}P^{n-k-1}(\mathbf{p})$$

inside $\mathbf{C}P^{n-1}(\mathbf{p}, \mathbf{q})$.

For $\varepsilon > 0$,

$$\begin{aligned} O_{\varepsilon} &= \{(\mathbf{z}, \mathbf{w}) \in \mathbf{C}^{n-k} \times \mathbf{C}^k : \varepsilon + \sum p_i |z_i|^2 = \sum q_i |w_i|^2\}/S^1 \\ &= \mathbf{C}^{n-k} \times_{S^1} S^{2k-1} \\ &= \mathcal{O}_{\mathbf{q}}(-p_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{q}}(-p_{n-k}). \end{aligned}$$

We now need to describe the critical level set $O_0 = \mu^{-1}(0)/S^1$. This space is a cone CX over a link X, where $X = S^{2(n-k)-1} \times_{S^1} S^{2k-1}$:

$$\begin{aligned} O_0 &= \{(\mathbf{z}, \mathbf{w}) \in \mathbf{C}^{n-k} \times \mathbf{C}^k : \sum p_i |z_i|^2 = \sum q_i |w_i|^2\}/S^1 \\ &= CX. \end{aligned}$$

S^1 acts on $S^{2(n-k)-1}$ by,

$$\lambda \cdot \mathbf{z} = \underline{\lambda}(-\mathbf{p})\mathbf{z}, \ \lambda \in S^1 \tag{4.1}$$

and so this sphere is a principal orbibundle over the orbit space

$$S^{2(n-k)-1}/S^1 \cong \mathbf{C}P^{n-k-1}(\mathbf{p}).$$

Therefore, the link X considered above is the associated orbibundle with fiber S^{2k-1}:

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^{2(n-k)-1} \times_{S^1} S^{2k-1} \\ & & \downarrow \\ & & \mathbf{C}P^{n-k-1}(\mathbf{p}). \end{array} \tag{4.2}$$

Similarly, S^1 acts on S^{2k-1} by,

$$\lambda \cdot \mathbf{w} = \underline{\lambda}(\mathrm{q})\mathbf{w},\ \lambda \in S^1 \tag{4.3}$$

and so this sphere is also a principal orbibundle but now over the orbit space

$$S^{2k-1}/S^1 \cong \mathbf{C}P^{k-1}(\mathbf{q}).$$

Therefore, the link X can also be described as the associated orbibundle to the principal orbibundle above but now with fiber $S^{2(n-k)-1}$:

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^{2(n-k)-1} \times_{S^1} S^{2k-1} \\ & & \downarrow \\ & & \mathbf{C}P^{k-1}(\mathbf{q}). \end{array} \tag{4.4}$$

On the other hand,

$$O_{-\varepsilon} = S^{2(n-k)-1} \times_{S^1} \mathbf{C}^k = \mathcal{O}_{\mathbf{p}}(q_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{p}}(q_k)$$

is the orbibundle over $\mathbf{C}P^{n-k-1}(\mathbf{p})$ with fiber $\mathbf{C}^k$, associated to the principal orbibundle described in (4.2) and

$$O_{\varepsilon} = \mathbf{C}^{n-k} \times_{S^1} S^{2k-1} = \mathcal{O}_{\mathbf{q}}(-p_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{q}}(-p_{n-k})$$

is the orbibundle over $\mathbf{C}P^{k-1}(\mathbf{q})$ with fiber $\mathbf{C}^{k-1}$, associated to the principal orbibundle described in (4.4).

As X fibers over both $\mathbf{C}P^{n-k-1}(\mathbf{p})$ and $\mathbf{C}P^{k-i}(\mathbf{q})$ there are two ways of "partially blowing CX up" at its singular point as we will see next. As the blown up point is not an orbifold point, the previous blow up definitions do not apply.

However, $O_{-\varepsilon}$ can be seen as a "partial blow up" of CX at its singular point, where a neighborhood of this point in CX is removed and replaced by a neighborhood of the zero section in $O_{-\varepsilon}$. The gluing involved is made along the boundary S of the resulting manifold, where

$$S = \partial(CX\backslash\{0\}) \cong S^{2(n-k)-1} \times_{S^1} S^{2k-1}.$$

The exceptional divisor resulting from this blow up is $\mathbf{C}P^{n-k-1}(\mathbf{p})$ and the "singular blow down" map corresponding to the opposite of this "partial blow up", is given by

$$\beta_{-\varepsilon} : O_{-\varepsilon} \to O_0$$

with

$$\beta_{-\varepsilon}([\mathbf{z}, \mathbf{w}]) = ([\frac{r}{\sqrt{r^2+\varepsilon}}\mathbf{z}, \mathbf{w}]) \in CX \tag{4.5}$$

where $r = \left(\sum q_i |w_i|^2\right)^{\frac{1}{2}}$ and $CX = O_0$.

Remark 4.2.2 *$\beta_{-\varepsilon}$ is well defined as*

$$\begin{aligned}\beta_{-\varepsilon}([\underline{\lambda}(-\mathbf{p},\mathbf{q})(\mathbf{z},\mathbf{w})]) &= [\underline{\lambda}(-\mathbf{p},\mathbf{q})(\frac{r}{\sqrt{r^2+\varepsilon}}\mathbf{z},\mathbf{w})] \\ &= [\frac{r}{\sqrt{r^2+\varepsilon}}\mathbf{z},\mathbf{w}].\end{aligned}$$

Moreover it maps $O_{-\varepsilon}\backslash\mathbf{C}P^{n-k-1}(\mathbf{p})$ injectively onto $O_0\backslash\{0\}$ and

$$\beta_{-\varepsilon}(\mathbf{C}P^{n-k-1}(\mathbf{p})) = \{0\}.$$

Alternatively, this "partial blow up" can be described as the space obtained by removing a neighborhood of the critical point in CX and then collapsing the boundary of the resulting orbifold along the fibres of the fibration described in (4.2).

Similarly, O_ε can be described as a "partial blow up" up CX at its singular point where a neighborhood of this point is now replaced by a neighborhood of the zero section of the orbibundle O_ε. The gluing is again made along a manifold $S \cong S^{2(n-k)-1} \times_{S^1} S^{2k-1}$, but now the "singular blow down" map

$$\beta_\varepsilon : O^+ \to O_0$$

is given by

$$\beta_\varepsilon([\mathbf{z},\mathbf{w}]) = [\mathbf{z}, \frac{s}{\sqrt{s^2+\varepsilon}}\mathbf{w}] \in CX = O_0 \tag{4.6}$$

where $s = \left(\sum p_i|z_i|^2\right)^{\frac{1}{2}}$.

Remark 4.2.3 *β_+ is also well defined, mapping $O^+\backslash\mathbf{C}P^{k-1}(\mathbf{q})$ injectively onto $O_0\backslash\{0\}$ with*

$$\beta_+(\mathbf{C}P^{k-1}(\mathbf{q})) = \{0\}.$$

We can now state the following result:

Theorem 4.2.4 *Let S^1 act on a symplectic manifold M^{2n} in a Hamiltonian fashion. Let μ be its Hamiltonian function and let x be a critical point of signature $(2(n-k),2k)$, lying on the critical level $\mu=\lambda$. Then the reduced spaces $O_{\lambda+\varepsilon} = \mu^{-1}(\lambda+\varepsilon)/S^1$ can be obtained from the reduced spaces $O_{\lambda-\varepsilon} = \mu^{-1}(\lambda-\varepsilon)/S^1$ by a "singular blow down" of a copy of $\mathbf{C}P^{n-k-1}(\mathbf{p})$ followed by a "partial blow up", for suitable values of $\mathbf{p} \in (\mathbf{Z}^+)^{n-k}$ determined by μ. The exceptional divisor resulting from this "partial blow up" is $\mathbf{C}P^{k-1}(\mathbf{q})$ for $q \in (\mathbf{Z}^+)^{k-1}$ determined by μ.*

Remark 4.2.5 *Again when we have a critical submanifold of signature $(2(d-k),2k)$ the same analysis holds in the normal directions to x.*

4.3 Duistermaat-Heckman with singularities

The Duistermaat-Heckman Theorem, [D-H] allows us to compare the cohomology classes $[\omega_\varepsilon] \in H^2(O_\varepsilon, \mathbf{R})$ for different values of ε in the same component of the set of regular values of the moment map μ. This piecewise affine function extends to a continuous function at critical points of μ.

In fact, let S^1 act on a $2n$-manifold in a Hamiltonian fashion and let x be a critical point of signature $(2(n-k), 2k)$ lying in a critical level $\mu = \lambda$ of the corresponding Hamiltonian function. Then, by the preceding section, we have the following commutative diagram:

$$\begin{array}{ccc} O^+ \times_{O_0} O^- & \xrightarrow{\phi^-} & O^- \\ \phi^+ \downarrow & \searrow \psi & \downarrow \beta_- \\ O^+ & \xrightarrow{\beta_+} & O_0 \end{array}$$

where β_- and β_+ are the maps defined in (4.5) and in (4.6), and

1.
$$\begin{aligned} O^- &\equiv \mu^{-1}(\lambda - \varepsilon)/S^1 \\ O^+ &\equiv \mu^{-1}(\lambda + \varepsilon)/S^1 \\ O_0 &\equiv \mu^{-1}(\lambda)/S^1 \end{aligned}$$

2.
$$O^+ \times_{O_0} O^- = \{(x,y) \in O^+ \times O^- : \beta_+(x) = \beta_-(y)\}$$
is the pullback orbibundle of O^- by β_+

3. μ is the Hamiltonian function for the S^1-action and $\lambda = \mu(x)$

4. ϕ^+, ϕ^- are the natural projections to O^+ and O^-.

In this situation we have the following theorem:

Theorem 4.3.1 *In $H^2(O^+ \times_{O_0} O^-; \mathbf{R})$,*

$$(\phi^+)^*([\omega_{\lambda+\varepsilon}])) - (\phi^-)^*([\omega^-_{\lambda+\varepsilon}])) = \varepsilon[\psi^{-1}(0)] = \varepsilon PD([\Sigma_+ . \Sigma_-])$$

where $[\omega^-_{\lambda+\varepsilon}]$ is the value at $\lambda+\varepsilon$ of the extension of the affine function on $t < \lambda$, $[\omega_t]$, to values of $t > \lambda$; Σ_+ and Σ_- are the singular locus of the blow down maps β_+ and β_- with $\beta_+(\Sigma_+) = \beta_-(\Sigma_-)$ and $PD([\Sigma_+, \Sigma_-])$ denotes the Poincare dual (cf. Theorem 2.1.8) of the homology class $[\Sigma_+, \Sigma_-] \in H_{2(n-1)-2}(O^+ \times_{O_0} O^-, \mathbf{R})$.

Proof: Let $\mu : M^{2n} \to \mathbf{R}$ be the Hamiltonian function associated to the S^1-action and consider the critical point x. As we have seen above, a neighborhood of x in M can be identified with $\mathbf{C}^{n-k} \times \mathbf{C}^k$, and the S^1-action becomes

$$\lambda \cdot (\mathbf{z}, \mathbf{w}) = \underline{\lambda}(-\mathbf{p}, \mathbf{q})$$

with Hamiltonian function $\mu(\mathbf{z},\mathbf{w}) = -\sum p_i|z_i|^2 + \sum q_i|w_i|^2$. We are then in the situation described in Section 4.2 and we can adapt the proof of Brion-Procesi (c.f. [B-P]) for a $\mathbf{C}^*$-action on a projective variety, in the following way:

By the Duistermaat-Heckman theorem, ([D-H]), all the regular quotients of $\mu^{-1}(\varepsilon)$ can be identified with the same orbifolds O^+ and all the regular quotients of $\mu^{-1}(-\varepsilon)$ can be identified with the same orbifolds O^- Moreover, O^+ is a blow up of O_0 at its singular point and $\mathbf{C}P^{k-1}(\mathbf{q})$ is the singular locus of the "blow down" map β_+. Again by the D-H theorem,

$$(\phi^+)^*([\omega_\varepsilon]) = (\beta^+ \circ \phi^+)^*([\omega_0]) - \varepsilon(\phi^+)^*(e_1^+)$$

where $e_1^+ \in H^2(O^+;\mathbf{Q})$ is the Euler class of the principal S^1-orbibundle $P^+ \to O^+$ with $P^+ \equiv \mu^{-1}(\varepsilon)/\Gamma$, and Γ is the subgroup of S^1 generated by the orbifold structure groups of the points in $\mu^{-1}(\varepsilon)/S^1$.

Similarly,

$$(\phi^-)^*([\omega_{-\varepsilon}]) = (\beta_- \circ \phi^-)^*([\omega_0]) + \varepsilon(\phi^-)^*(e_1^-)$$

and so,

$$(\phi^-)^*([\omega_\varepsilon^-]) = (\beta_- \circ \phi^-)^*([\omega_0]) - \varepsilon(\phi^-)^*(e_1^-).$$

Hence, as $\beta_+ \circ \phi^+ = \beta_- \circ \phi^- = \psi$,

$$(\phi^+)^*([\omega_\varepsilon]) - (\phi^-)^*([\omega_\varepsilon^-]) = -\varepsilon((\phi^+)^*(e_1^+) - (\phi^-)^*(e_1^-)). \tag{4.7}$$

Now the S^1-action on $\mathbf{C}^{d-k} \times \mathbf{C}^k$ extends to an action of $\mathbf{C}^*$ with the same fixed point and so, [B-P], gives us that the right hand side of (4.7) is equal to $-\varepsilon PD([\Sigma_+,\Sigma_-])$ and so we have,

$$(\phi^+)^*([\omega_\varepsilon]) - (\phi^-)^*([\omega_\varepsilon^-]) = -\varepsilon PD([\Sigma_+,\Sigma_-])$$

□

The following result first proved by McDuff in [MD1] comes as a corollary:

Corollary 4.3.2 *If M is 4-dimensional, then when passing through a critical value of μ on an isolated critical point of type $(-p,q)$*

$$e_1^+(O^+) - e_1^-(O^-) = \varepsilon \cdot \frac{1}{pq}$$

Proof: In this situation, $\Sigma_+ = \{x_q\}$ and $\Sigma_- = \{x_p\}$ are single points respectively in O^+ and O^- with orbifold structure group μ_q and μ_p.

Then,

$$\begin{aligned} e_1^+(O^+) - e_1^-(O^-) &= \left(\frac{d}{d\varepsilon}((\phi^+)^*([\omega_\varepsilon]) - (\phi^-)^*([\omega_\varepsilon^-]))\right)([O^+,O^-]) \\ &= PD([x_q,x_p])([O^+,O^-]) \\ &= (x_q \cdot O^+)(x_p \cdot O^-) \\ &= \frac{1}{pq} \end{aligned}$$

□

In the case of an isolated critical point of signature $(2, 2(n-1))$ or $(2(n-1), 2)$, we have another corollary which generalizes the similar result for the case of a semifree S^1-action proved in [G-S] and in [Au].

Corollary 4.3.3 *Let x is a critical point of signature $(2, 2(n-1))$, (resp. $(2(n-1), 2)$). Then S^1 acts on the normal bundle to this fixed point with different weights where only one of these is negative (positive). Let p be the absolute value of this weight. Then, $\beta^*([\omega_\varepsilon^-]) - [\omega_\varepsilon]$, (resp. $[\omega_\varepsilon^-] - \beta^*([\omega_\varepsilon])$), is a half line directed by*

$$\frac{1}{p}PD([\Sigma_+]) \in H^2(O^+, \mathbf{Q}),$$

where $\beta : O^+ \to O_-$, is the corresponding blowdown map between reduced spaces.

Proof: Let $[A] \in H_2(O^+, \mathbf{Q})$. Then,

$$[A, B] \in H_2(O^+ \times_{O_0} O^+; \mathbf{Q})$$

iff $(\beta_+)_*([A]) = (\beta_-)_*([B])$. Now

$$\begin{aligned}
(\beta^*([\omega_\varepsilon^-]) - [\omega_\varepsilon])([A]) &= (\phi^+)^*(\beta^*([\omega_\varepsilon^-]) - [\omega_\varepsilon])\,([A, B]) \\
&= ((\phi^-)^*([\omega_\varepsilon^-]) - (\phi^+)^*([\omega_\varepsilon]))\,([A, B]) \\
&= \varepsilon(PD([\Sigma_+, \Sigma_-])\,([A, B]) \\
&= \varepsilon(\Sigma_+ \cdot A)(\Sigma_- \cdot B) \\
&= \frac{1}{p}PD([\Sigma_+])([A])
\end{aligned}$$

□

This result can be easily seen in Figures 4.1 and 4.1 for the 6- dimensional case. From these figures, we can easily see that the Euler class e_1^+ of the principal S^1-orbibundle $P^+ \to O^+$, differs from $\beta^*(e_1^-)$, the pull back of the Euler class of $P^- \to O^-$ by a multiple of the $PD[\Sigma] \in H^2(O_+, \mathbf{R})$, where Σ is the exceptional divisor of β.

Hence,

$$\begin{aligned}
(\beta^*([w_\varepsilon^-]) - [\omega_{+\varepsilon}])\,([\Sigma]) &= \varepsilon k PD[\Sigma]([\Sigma]) \\
&= \varepsilon k(\Sigma \cdot \Sigma) \\
&= -\varepsilon k(\frac{p}{lcm(q_1, q_2)})
\end{aligned}$$

On the other hand, $[\omega_{+\varepsilon}](\Sigma) = \frac{\varepsilon}{lcm(q_1,q_2)}$ and $\beta^*([w_\varepsilon^-])(\Sigma) = [\omega_\varepsilon^-](\beta_*(\Sigma)) = 0$. In fact, as we can see from figure 6, we can define a quasi-free Hamiltonian S^1-action on a neighborhood $\mathcal{N}(\Sigma)$ of Σ

in $O_{+\varepsilon}$ having the orbifold singularities x_1, x_2 of order q_1, q_2 as fixed points and Σ as a gradient sphere. If $H_\varepsilon : \mathcal{N}(\Sigma) \to \mathbf{R}$ is the Hamiltonian function for this action, then, $\iota(X_\varepsilon)\omega_\varepsilon = dH_\varepsilon$ for X_ε the Hamiltonian vector field associated to H_ε. If we consider a path $\gamma(t)$, $(0 \leq t \leq 1)$, on $O_{+\varepsilon}$ from x_1 to x_2 and the map

$$\begin{aligned} \varphi : [0, 2\pi] \times [0,1] &\longrightarrow O_{+\varepsilon} \\ (s,t) &\mapsto e^{-is} \cdot \gamma(t) = \psi_s^{-1}(\gamma(t)) \end{aligned}$$

where $\psi_s : M \to M$ is such that $\frac{d\psi_s}{ds} = X_\varepsilon \circ \psi_\varepsilon$, then,

$$\begin{aligned} ([\omega_\varepsilon])(\Sigma) &= \frac{1}{2\pi}\int_\Sigma \omega_\varepsilon = \frac{1}{2\pi}\int_{[0,2\pi]\times[0,1]} \varphi^*\omega_\varepsilon \\ &= \frac{1}{2\pi}\int_{[0,2\pi]\times[0,1]} (\iota(X_\varepsilon)\omega_\varepsilon)(\dot\gamma(t)) \\ &= \frac{1}{2\pi}\int_{[0,2\pi]\times[0,1]} dH_\varepsilon(\dot\gamma(t)) \\ &= H_\varepsilon(x_2) - H_\varepsilon(x_1) = \frac{\varepsilon}{lcm(q_1,q_2)} \end{aligned}$$

and we have $\frac{\varepsilon}{lcm(q_1,q_2)} = \varepsilon k \frac{p}{lcm(q_1,q_2)}$.

We conclude then that $k = 1/p$ and so $\beta^*([w_\varepsilon^-]) - [\omega_\varepsilon]$ is a half line directed by $1/pPD[\Sigma]$. (this alternate proof of Corollary 4.3.3 for dimension 6 can be easily generalized for other dimensions).

Chapter 5

Equivariant cohomology and S^1-actions

5.1 Equivariant cohomology and the Cartan model

The *equivariant cohomology* of a G-manifold is defined as the ordinary cohomology of the space $M_G = M \times_G EG$ where EG is a contractible space on which G acts freely. Hence,

$$H^*_G(M) = H^*(M_G)$$

Moreover, $H^*_G(M)$ is a module over

$$H^*_G = H^*_G(pt) = H^*(BG)$$

where BG, the *classifying space*, is defined as

$$BG = EG/G.$$

5.1.1 The Cartan model

We will now present a DeRham version of equivariant cohomology, the Cartan model, for the simple case where $G = S^1$. For this we first need to define the spaces: $\Omega^k_{S^1}(M)$, the set of smooth k-forms on M which are invariant under the S^1-action and $\Omega^*_{S^1}(M)[u]$, the ring of polynomials in u with coefficients in $\Omega^k_{S^1}(M)$. u is a generator of degree two and so an element in $\Omega^k_{S^1}(M)[u]$ is the finite sum,

$$\alpha = \alpha_k + u\alpha_{k-2} + u^2\alpha_{k-4} + \dots$$

where $\alpha_j \in \Omega^j_{S^1}(M)$. In particular, $\Omega^*_{S^1}[u] = \Omega^*_{S^1}(pt)[u]$ is the ring of polynomials in u with constant coefficients.

On $\Omega^*_{S^1}(M)[u]$, we can define the differential operator,

$$d_X = d + u\iota(X)$$

where X is the vector field that generates the S^1-action. This degree one operator satisfies $d_X \circ d_X = 0$. Note that an element α of $\Omega^*_{S^1}(M)[u]$ is equivariantly closed, i.e. $d_X\alpha = 0$ iff α is closed in the usual sense and $\iota(X)\alpha = 0$ (α basic relative to X). Note that if $\alpha \in \Omega^*_{S^1}(M)[u]$ is closed but not basic, it may be possible to extend it to an equivariantly closed form by adding a polynomial $p \in \Omega^*_{S^1}(M)[u]$.

The equivariant DeRham complex $H^*_{DR}(\Omega^*_{S^1}(M)[u], d_X)$ can be naturally identified with the equivariant cohomology $H^*_{S^1}(M)$ of M. In particular, as d_X vanishes on $\Omega^*_{S^1}[u]$, $H^*_{S^1} = H^*_{S^1}(pt)$ is again the ring of polynomials in u with constant coefficients.

5.1.2 Equivariant characteristic classes in the Cartan model

Let $E \to M$ be a complex line bundle where S^1 acts compatibly with its action on M and consider an equivariant connection ∇ on E (i.e. the actions of ∇ and X commute). Note that we can obtain such a connection by taking an arbitrary connection on E and then averaging over S^1. If s is a generating smooth section of E over an open set U, then locally the action of X is described by a function $L(s)$ satisfying

$$Xs = L(s)s$$

while the connection ∇ is described by a 1-form $\theta(s)$ satisfying,

$$\nabla s = \theta(s)s.$$

As ∇ is equivariant, $L(s)$ and $\theta(s)$ satisfy,

$$dL(s) = \mathcal{L}_X\theta(s).$$

The ordinary Chern class of E is then locally represented by

$$-\frac{d\theta(s)}{2\pi i}.$$

This representation is independent of the generating section s and hence it is the restriction to an open set U of a global class $c_1(E, \nabla)$.

If we consider the form locally defined by

$$c_1^{S^1}(E, \nabla, s) = -\frac{1}{2\pi i}\left\{d\theta(s) - [L(s) - \iota(X)\theta(s)]u\right\}$$

we can easily see that it is independent of the choice of section, it is invariant under the S^1-action and it is S^1-closed, thus defining a global equivariant cohomology class.

Example 5.1.1 *Let S^1 act on a manifold M and consider a line bundle E over M with an S^1 action compatible with the action on M. Moreover, let F be a component of the fixed point set of S^1 on M. The action of S^1 on the fibers of $E_{|_F}$ is conjugate to the circle action on* $\mathbf{C}$,

$$z \mapsto e^{2\pi i \beta u} z$$

Hence if z is a generating section, $X = 2\pi i \beta z \partial_z$ and so $L(z) = 2\pi i \beta$. Moreover, $\nabla_X z = 0$ as X vanishes on F and so $\theta(z) = 0$. We have then,

$$c_1^{S^1}(E) = c_1 + \beta u$$

Example 5.1.2 *If F is again a component of the fixed point set, and ν_F is the normal bundle to F in M then S^1 acts on this bundle. By the splitting principle (cf. [B-T]) to obtain a polymomial identity for the equivariant Chern classes of a complex vector bundle it is enough to prove it under the assumption that the vector bundle can be decomposed equivariantly as a sum of line bundles. Let us assume then that the normal bundle to F in M decomposes equivariantly into complex line bundles*

$$\nu_F = L_1 \oplus \cdots \oplus L_m$$

on which the circle acts with weights $\beta_1, \ldots, \beta_m$. Hence,

$$c_1^{S^1}(\nu_F) = \sum_{i=1}^{m} (c_1(L_i) + \beta_i u)$$

and we can also consider the equivariant Euler class

$$e_1^{S^1}(\nu_F) = \prod_{i=1}^{m} (c_1(L_i) + \beta_i u).$$

since all the β_i are different from zero, we can write,

$$e_1^{S^1}(\nu_F) = \left(\prod_{i=1}^{n} \beta_i u \right) \prod_{i=1}^{m} (\frac{c_1(L_i)}{\beta_i u} + 1)$$

and so we may define the inverse of $e_1^{S^1}(\nu_F)$ in $\Omega^*_{S^1}(M)[u]$ (the ring of polynomials in u with coefficients in $\Omega^k_{S^1}(M)$),

$$\frac{1}{e_1^{S^1}(\nu_F)} = \prod_{i=1}^{n} \left(\frac{1}{\beta_i u} \sum_{k=0}^{\infty} (-1)^k \left(\frac{c_1(L_i)}{\beta_i u} \right)^k \right)$$

as only a finite number of terms contribute to this sum.

5.1.3 The localization theorem

We will see next that the integral of an equivariantly closed form α depends only on the restriction of α to the fixed point set M^{S^1}.

Theorem 5.1.3 *([B-V];[A-B]) Let M^{2n} be a compact manifold equipped with an action of S^1 and let $\mathcal{F}$ be the set of components of the fixed point set M^{S^1} of S^1 on M. Let $\alpha \in H^*_{S^1}(M, \mathbf{Z})$. Then,*

$$\int_M \alpha = \sum_{F \in \mathcal{F}} \int_F \frac{\alpha}{e_1^{S^1}(\nu_F)}.$$

Proof: We will present here the proof by Berline and Verne, ([B-V]), which uses the Cartan model for equivariant cohomology. Let $\alpha \in \Omega^*_{S^1}(M)[u]$, $\alpha = \alpha_{2n} + \alpha_{2n-2}u + \cdots + \alpha_0 u^{2n}$.

Step1 Let $F \in \mathcal{F}$ be a critical submanifold of M of codimension $2m$. a tubular neighborhood of F can be defined as in Section 3.3, by

$$U_\varepsilon(F) = P_F \times_{U(m)} \mathbf{C}^m$$

where P_F is a principal $U(m)$-bundle over F. We can choose coordinates,

$$w_1, \ldots, w_{2(n-m)}, x_1, \ldots, x_{2m}$$

on this neighborhood such that the vector field X, generated by the action takes the form,

$$X = \beta_1(x_2 \partial x_1 - x_1 \partial x_2) + \cdots + \beta_m(x_{2m}\partial x_{2m-1} - x_{2m-1}\partial x_{2m})$$

and so $e_1^{S^1}(\nu_F) = \frac{\beta_1 \cdots \beta_m}{(2\pi)^n} u^m$. Moreover, we can define a one form θ_F on this neighborhood of F by,

$$\theta_F = \beta_1^{-1}(x_2 dx_1 - x_1 dx_2) + \cdots + \beta_m^{-1}(x_{2m} dx_{2m-1} - x_{2m-1} dx_{2m}).$$

This form is S^1-invariant and $\iota(X)\theta_F = \theta_F(X) = \|x\|^2$. Using a S^1-invariant partition of unity subordinate to the covering of M by the S^1 invariant open sets U_F and $M \backslash M^{S^1}$, we can construct a one form θ such that $\mathcal{L}(X)\theta = 0$, $d_X\theta = d\theta + \|x\|^2 u$ is invertible outside M^{S^1} and $\theta = \theta_F$ on $U_\varepsilon(F)$. Now for any equivariant form $\alpha \in \Omega^*_{S^1}(M)[u]$ such that $d_X\alpha = 0$ we can easily prove that,

$$\alpha = d_X\left(\frac{\theta \wedge \alpha}{d_X\theta}\right)$$

on on $M \backslash M^{S^1}$

Step2 We can now prove the main statement:

Let $M_\varepsilon = M \backslash \bigcup_{F\in\mathcal{F}} U_\varepsilon(F)$, then

$$\begin{aligned} \int_M \alpha &= \lim_{\varepsilon\to\infty} \int_{M_\varepsilon} \alpha &&= \lim_{\varepsilon\to\infty} \int_{M_\varepsilon} d_X\left(\tfrac{\theta\wedge\alpha}{d_X\theta}\right) \\ &= \lim_{\varepsilon\to\infty} \int_{M_\varepsilon} d\left(\tfrac{\theta\wedge\alpha}{d_X\theta}\right) &&= \lim_{\varepsilon\to\infty} \sum_{F\in\mathcal{F}} \int_{\partial U_\varepsilon(F)} \left(\tfrac{\theta\wedge\alpha}{d_X\theta}\right) \end{aligned}$$

Rescaling the variable x, $\partial U_\varepsilon(F)$ we get

$$\int_{\partial U_\varepsilon(F)} \frac{\theta\wedge\alpha}{d_X\theta} = \int_{S^{2m-1}} \frac{\theta\wedge\alpha_\varepsilon}{d_X\theta}$$

where $\lim_{\varepsilon\to 0} \alpha_\varepsilon = \int_F \alpha$. Hence,

$$\int_M \alpha = \sum_{F\in\mathcal{F}} \int_F \alpha \int_{S^1} \frac{\theta}{d_X\theta}$$

Now,

$$\begin{aligned}
\int_{S^{2m-1}} \frac{\theta}{d_X\theta} &= \int_{S^{2m-1}} \frac{\theta}{d\theta + u} \\
&= \frac{1}{u}\int_{S^{2m-1}} \theta\left(-\frac{d\theta}{u}\right)^{m-1} \\
&= \frac{1}{u^m}\int_{B_1} (d\theta)^m \\
&= \frac{2^m\pi^m}{u^m\beta_1\cdots\beta_m} \\
&= \frac{1}{e_1^{S^1}(\nu_F)}
\end{aligned}$$

and the result follows. For more details please refer to [B-V].

□.

5.2 Isolated fixed points on 6-manifolds

In this section we extend the results of Tolman and Weitsman in [T-W] to special non-semifree circle actions with isolated fixed point. For this we first need to consider some facts about isotropy spheres.

5.2.1 Isotropy spheres

Let M be a compact symplectic six dimensional manifold equipped with an effective circle action with only isolated fixed points. Let $p \in M$ be one fixed point such that the absolute value of one of the weights of the circle action on the normal bundle to p is different from 1. Let k be this value and let E be the connected component of M^{μ_k} containing p, (recall that μ_k is the group of the k-roots of unity). If all other weights of the circle action on the normal bundle to p are not multiples of k, the tangent space to E at p is the two dimensional subspace of T_pM on which the circle acts with weight $\pm k$.

Consequently, E is also two dimensional. Moreover S^1/μ_k acts effectively on E with at least one fixed point. Therefore this circle action is Hamiltonian and as E is compact, the image $H(E)$ of its Hamiltonian function is a closed interval. Consequently, there is an additional fixed point lying on the other endpoint of the interval. Moreover the local normal forms for circle actions on surfaces allow us to construct an equivariant symplectomorphism between E and a sphere with the standard circle action (cf. [K] for details).

We conclude then that under the above assumptions, the connected component of M^{μ_k} through p is a sphere which contains only one additional fixed point.

If the condition on the weights of the circle action on the normal bundle for p is not satisfied i.e. if there is one other weight which is a multiple of k then the connected component of M^{μ_p} through p,

E is now four dimensional. Nevertheless there is still an embedded sphere through p which contains only one additional fixed point.

In fact, we now have a semifree S^1/μ_p-action on E with only isolated fixed points which extends to an action of the multiplicative group $\mathbf{C}^*$ on E. The fixed points of this new action are the same as the fixed points for the circle action, and the gradient flow on E is the flow generated by the vector field $-JX$, where X is the vector field generating the circle action and J is an almost complex structure on E preserved by the circle action. This gradient flow coincides with the gradient flow of the Hamiltonian function with respect to a compatible metric. We can therefore consider the gradient spheres in E (defined in [A] and [A-H]) as the closure of a non-trivial $\mathbf{C}^*$-orbit. The poles of these spheres are the limits at times ∞ and $-\infty$ of the gradient flow inside this orbit which are of course fixed points of the circle action on E. Again the circle acts on each of these spheres by standard rotation.

We conclude then that also in this case there is a sphere through p fixed by μ_k, which passes through an additional fixed point in M. We will call these spheres *isotropy spheres.*

5.2.2 Hamiltonian circle actions

Using the above considerations on isotropy spheres and some properties of circle actions on disc bundles over S^2, we prove the following theorem:

Proposition 5.2.1 *Let (M^6,ω) be a six dimensional compact, connected symplectic manifold equipped with a symplectic circle action with only isolated fixed points. If all the $\mathbf{Z}_k$-spheres ($k \neq 0$) Z have trivial normal bundles in M, (i.e. $c_1(\nu_Z) = e_1(\nu_Z) = 0$) and $M^{S^1} \neq 0$, then the circle action is necessarily Hamiltonian.*

Proof: We can assume without loss of generality that ω is rational. Then, if the action is not Hamiltonian, we have that $[\iota(X)\omega] \neq 0$ and so a multiple of ω admits a generalized moment map μ such that

$$\mu : M \longrightarrow S^1$$

with $\iota(X)\omega = \mu^*(d\theta)$. This map cannot have any local extremum. Consequently, the index of any of its critical points can only be two or four.

We need now a certain fact about disc bundles over the 2-sphere which is proved in [A-H]: Let the circle act on the 2-sphere by rotating it k times while fixing the north and south poles. Let $E \longrightarrow S^2$ be a complex line bundle to which the action lifts. The fiber over the north pole is acted upon by

$$\lambda : z \mapsto \lambda^{m_N} z$$

and the fibre over the south pole is acted upon by

$$\lambda : z \quad \mapsto \quad \lambda^{m_S} z$$

We then have,

$$m_N - m_S = -ek$$

where e is the Euler number of the bundle E.

The possibilities for pairing fixed points along a $\mathbf{Z}_k$-isotropy sphere passing through them can be seen in Figure 5.1.

Consider the $\mathbf{Z}_k$-sphere below in Figure 5.1 (a) and let E be its normal bundle on M. E decomposes as a sum of two complex line bundles where the complex structure is chosen to be compatible with the symplectic structure. The circle action on the fibers of these two bundles over the north and south poles is given by the respective weights, i, j and $-a, -b$, $(i, j, a, b \in \mathbf{Z}^+)$. By the above discussion we must have

$$i + a = -e_1 k \tag{5.1}$$

$$j + b = -e_2 k \tag{5.2}$$

which is impossible as by hypothesis $e_1 = e_2 = 0$.

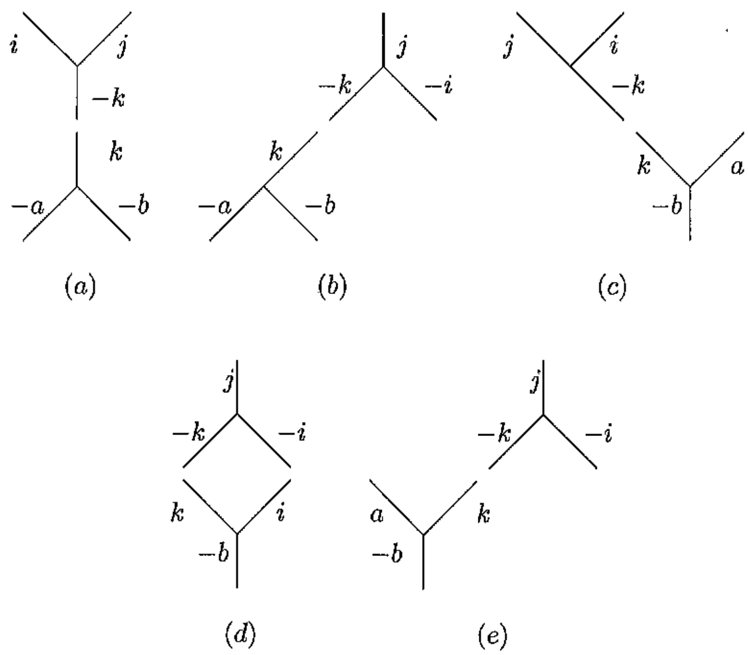

Figure 5.1: Pairings of fixed points along $\mathbf{Z}_k$-spheres

Similarly, the situations in Figure 5.1 (b), (c) and (d) are also impossible. For (e) to be possible, we need $j = a$ and $i = b$. We conclude then that (e) is the only possible situation for pairing fixed points along a $\mathbf{Z}_k$-sphere.

Each one of these spheres goes through two different fixed points. Consequently, we cannot have an index 2 fixed point with a negative weight $\alpha \neq -1$ or an index 4 fixed point with a positive weight $\alpha \neq 1$. In fact, if that was the case, the corresponding isotropy spheres could not pass through any other fixed point as situations (a) through (d) above are impossible. This fact and the equalities imposed by situation (e) above, determine that the only possibility for existence of isotropy spheres is the one described in Figure 5.2 below. Consequently, the set of fixed points has to be composed of points of the types represented in Figure 5.3.

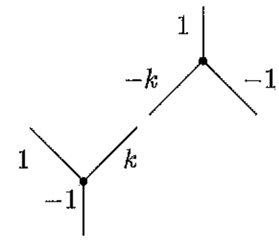

Figure 5.2: Pairing along a $\mathbf{Z}_k$-sphere

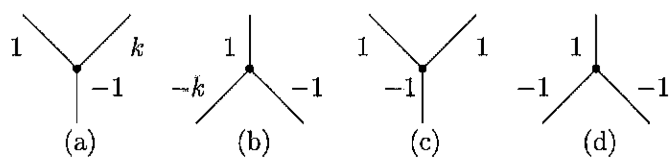

Figure 5.3: Possible kinds of fixed points

Now by the localization theorem for equivariant cohomology, (cf. Section 5.1.3)

$$\int_{M^6} \alpha = \sum_{F \in M^{S^1}} \frac{\alpha\,|_F}{e_1^{S^1}(\nu_F)}$$

for $\alpha \in H^*_{S^1}$. Let N_k, N_{-k}, N_1 and N_{-1} be the number of points of type (a), (b), (c) and (d). By Example 5.1.1, if F is a fixed point of type (a), (b), (c) or (d),then $e_1^{S^1}(\nu_F)$ is equal to $-ku^3$, ku^3, $-u^3$ or u^3. If in addition $\alpha = c_1^{S^1}(TM)$ then $\alpha\,|_F$ is equal to ku, $-ku$, u or $-u$.

As in this case $\int_M \alpha = 0$ for dimensional reasons, we have

$$0 = \sum_{F \in M^{S^1}} \frac{\alpha\,|_F}{e(\nu_F)} \quad = \quad -\frac{1}{u^2} \sum_{k=1}^{\infty} (N_k + N_{-k}) \tag{5.3}$$

then N_k, N_{-k}, N_1 and N_{-1} are all zero, contradicting the initial hypothesis that $M^{S^1} \neq 0$. We conclude then that the circle action is necessarily Hamiltonian and the result follows. □

Remark 5.2.2 *1. If the circle action can be obtained by a sequence of blow ups at fixed points from a circle action satisfying the above triviality conditions on the normal bundles of isotropy spheres then the action is also Hamiltonian.*

2. Examples of Hamiltonian circle actions on six dimensional manifolds with only isolated fixed points satisfying the triviality conditions on the normal bundles of isotropy spheres are diagonal weighted circle actions on $S^2 \times S^2 \times S^2$.

Using a similar argument we can also prove the following result:

Theorem 5.2.3 *Let* S^1 *action act on a six-dimensional symplectic compact connected manifold. If the circle action has only isolated fixed points satisfying the condition that at the normal bundle to each fixed point* S^1 *acts always with weights* $\pm n, \pm m, \pm k$, *where* $n > m > k > 2$ *are relatively prime and* $n \neq m + k$, *then if* $M^{S^1} \neq \emptyset$, *the action must be Hamiltonian.*

Proof: Again we can assume we have a generalized moment map on M which cannot have any local extremum.

Consider an index 4 fixed point x such that the circle acts on its normal bundle with weights $(p, -q, -r)$, $p, q, r \in \{n, m, k\}$ and $p \neq q \neq r$.

For this point to be paired with another fixed point along a $\mathbf{Z}_p$ sphere, (cf. Figure 5.4), we either need

$$q \equiv -r \pmod{p},$$

$2q \equiv 0 \pmod{p}$ or $2r \equiv 0 \pmod{p}$. These last two conditions are impossible as $p \neq 2$ and $\gcd(p, r) = \gcd(q, r) = 1$. Therefore (I) is the only case possible and then $q \equiv -r \pmod{p}$.

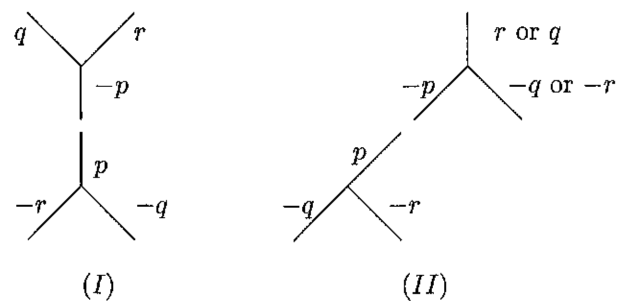

Figure 5.4: Pairings along the $\mathbf{Z}_p$-sphere

On the other hand, the possible pairings of this point with other fixed point along a $\mathbf{Z}_r$ sphere are described in Figure 5.5:

For (a) to be possible we again need $2p \equiv 0 \pmod{r}$ or $2q \equiv 0 \pmod{r}$, which we know is impossible. For (b) to be possible, we either need $2p \equiv 0 \pmod{r}$ or $p \equiv q \pmod{r}$ and $p \equiv r$

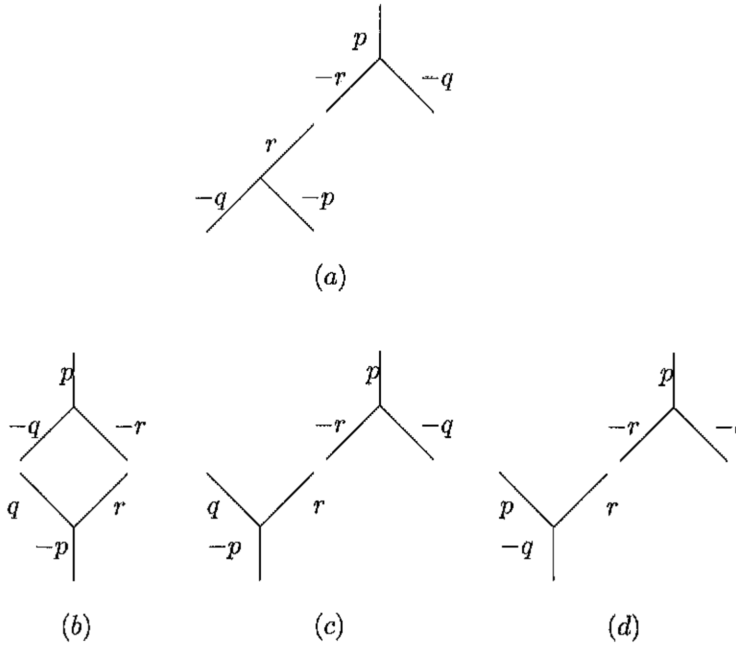

Figure 5.5: Pairings of fixed points along $\mathbf{Z}_r$-spheres

(mod q). The first we already know is impossible. The second is also impossible as we can see from the following lemma:

Lemma 5.2.4 *Let p, q, r be three integers greater than 2, relatively prime and such that*

$$\begin{aligned} p &\equiv q \pmod{r} \\ p &\equiv r \pmod{q} \\ q &\equiv -r \pmod{p}. \end{aligned} \tag{5.4}$$

Then necessarily $p = q + r$.

Proof: From the first two equations in (5.4) we have that

$$p = q + k_1 r = r + k_2 q$$

with $k_1, k_2 \in \mathbf{Z}$. Then,

$$(k_2 - 1)q = r(k_1 - 1)$$

and as q and r have no common divisor, this implies that

$$p = q + r \pmod{qr}. \tag{5.5}$$

From the third equation in (5.4) we have the additional condition

$$q = -r + tp$$

with $t \geq 1$. Therefore, as from (5.5), $p = q + r + k_3 qr$, we have that

$$p(1-t) = k_3 qr,$$

and so $k_3 \leq 0$. However, as $q + r \leq qr$ and $p > 0$, k_3 cannot be negative and so we have that $k_3 = 0$ and so $t = 1$ i.e.

$$p = q + r.$$

□

For (c) to be possible, we need $p \equiv q \pmod r$. However, (d) is always possible.

On the other hand, we cannot pair this point x with two other index-2-points along the $\mathbf{Z}_q$ and the $\mathbf{Z}_r$ spheres using pairings similar to the ones described in Figure 5.5 (c). In fact, if that was so, we would need $p \equiv q \pmod r$ and $p \equiv r \pmod q$ in addition to $q \equiv -r \pmod p$ and that we know, implies $p = q + r$.

Therefore we need at least one pairing of the type described in (d) either along a $\mathbf{Z}_q$ or a $\mathbf{Z}_r$ sphere. This fact implies the existence of a fixed point y of index 2 with negative weight β different from p.

To pair y with another fixed point along the corresponding $\mathbf{Z}_\beta$ sphere, we need $p \equiv -r \pmod q$, (if $\beta = q$) or $p \equiv -q \pmod r$ (if $\beta = r$). As we also need to pair x along the remaining isotropy spheres, we have additional conditions, which we can easily check cannot be simultaneously satisfied:

Case 1 $\beta = r$ and the pairing of x with an index 2 fixed point z along the $\mathbf{Z}_r$ sphere is of the type (d). As in this case z must also be paired by a pairing of type (I) in Figure 5.4, we have

$$\begin{aligned} q &= -r + k_1 p \\ p &= -r + k_2 q \\ p &= -q + k_3 r. \end{aligned}$$

Then,

$$q(1+k_2) = r(1+k_3) = p(1+k_1)$$

and

$$\begin{aligned} k_1 + 1 &= tqr \\ k_2 + 1 &= trp \\ k_3 + 1 &= tqp \end{aligned}$$

for $t \geq 1$. Hence $p = \frac{r+1}{tr-1}$ which is impossible as $p \geq 3$ and $t \geq 1$.

Case 2 $\beta = r$ and the pairing of x with an index 2 fixed point along the $\mathbf{Z}_r$ sphere is of the type (c). In this case we have

$$\begin{aligned} q &= -r + k_1 p \\ p &= q + k_2 r \\ p &= -q + k_3 r. \end{aligned}$$

Then,

$$2p \equiv (k_2 + k_3)r$$

which is impossible as $r \neq 2$ and $\gcd(p, r) = 1$.

Case 3 $\beta = q$ and the pairing of x with an index 2 fixed point along the $\mathbf{Z}_q$ sphere is of the type (d). The conditions on p, q and r determined in this case are the same as in Case 1.

Case 4 $\beta = q$ and the pairing of x with an index 2 fixed point along the $\mathbf{Z}_q$ sphere is of the type (c). In this case we have,

$$\begin{aligned} q &= -r + k_1 p \\ p &= -r + k_2 q \\ p &= r + k_3 q. \end{aligned}$$

Here we have,

$$2p = (k_2 + k_3)q$$

which again is impossible.

We conclude then that such an S^1 action has to be Hamiltonian.

□.

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