Transport by Random Stationary Flows

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Abstract of the Dissertation

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We consider transport properties for Gaussian, stationary, divergence free, random vector fields in $\mathbb{R}^d$, which are Markov in time. Using the discretization of the spectral measure of the field, we approximate the equation of motion in the random field, which is an ordinary differential equation with the random right hand side, by a system of stochastic differential equations. The infinitesimal generator of the system is a hypoelliptic operator, which couples the velocity mode space to physical space. We prove existence, uniqueness and a priori estimates, uniform in the number of points in the discretization, for the associated hypoelliptic equation.

We then use an analog of the harmonic coordinates to express the effective diffusivity of the approximating fields in terms of the solution of the PDE. The uniform a priori estimates allow for the transition back to the original velocity field, thus proving the existence of effective diffusivity.

We then consider two consequences of the above analysis. The first is a complete and rigorouos asymptotic expansion of diffusivity for vector fields with short time correlations. The second is a rigorous analysis of the divergence of diffusivity for generalized random fields with pure Kolmogorov spectrum.

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Chapter 1

Existence of Effective Diffusivity

1.1 Introduction

Consider the motion of a particle in the random velocity field $V(x,t)$, $x \in \mathbb{R}^d$, which is described by the system of random ordinary differential equations

$$\dot{X}_t = V(X_t, t) \ , \quad X_0 = x_0 \ . \tag{1.1}$$

The matrix of effective diffusivity is defined as

$$D^{ab} = \frac{1}{2} \lim_{t \to \infty} \frac{E\left((X_t^a - X_0^a)(X_t^b - X_0^b)\right)}{t} \ , \quad a, b = 1, ..., d \ , \tag{1.2}$$

where a and b are coordinate directions.

The problem of expressing the effective diffusivity, which is a Lagrangian characteristic of the flow, in terms of the correlation function of the flow vector field itself, which is Eulerian data, is an important question which has been discussed extensively in physical and mathematical publications. We refer to [16] for an introduction to this literature. Most of the results have been obtained in two cases. Either the time correlation scale of the random vector field is infinite, that is the field is time-independent, or on the contrary, the field has short time correlations. Some of the important recent results in the case of time independent vector fields are due to Fannjiang and Papanicolaou [7]. For the short time correlated vector fields the first results date back to Taylor [19]. Taylor's method gives the answer (on a physical level) for the main term of effective diffusivity in the short time correlated limit.

Recently Molchanov [17], and Carmona, Grishin and Molchanov [3] considered random vector fields with a finite number of spatial modes. It was shown by Molchanov [17] that for a class of vector fields with a finite number of spatial modes the effective diffusivity can be expressed in terms of the solution of a certain hypoelliptic partial differential equation, provided the solution of the PDE exists. This equation couples velocity mode space to physical space through (1.1). Through the coupling to physical space, it gives the influence of the velocity modes upon diffusive transport.

In the dissertation develop the required PDE existence theory. Moreover we avoid the non-physical assumption of a finite number of modes. As a result we obtain the existence of effective diffusivity under mild regularity hypothesis on the random velocity field. We also obtain its full asymptotics as the time correlation scale of the vector field tends to zero. The main term of the asymptotic expansion coincides with Taylor's answer, as is required from physical considerations.

Among the related results we would like to note a recent paper by Komorowski and Papanicolaou [14]. The existence of effective diffusivity is proved there for a Gaussian, stationary, incompressible velocity field under the assumption that the correlation function of the field has finite support in time. The main regularity assumption is that almost every realization of the velocity field is continuous in t and C^1 smooth in x. While the regularity assumptions in this dissertation are essentially the same, we study the fields which are Markov in time, rather than the ones with finite time correlations. Markov assumption implies that the correlation function is exponentially decreasing in time, and thus excludes the case considered in [14]. A different derivation of the existence of effective diffusivity for Markov in time vector fields was obtained independently of our work by Fannjiang and Komorowski [6].

The technique developed in the dissertation allows us to prove the following two consequences. The first is a complete, and rigorous asymptotic analysis of diffusivity in the short time correlated limit. The second is a rigorous analysis of the divergence of diffusivity for generalized random fields with pure Kolmogorov spectrum. To the author's knowledge, no comparable results have been obtained previously.

We assume a physical model of turbulence described by a Gaussian random field, which is stationary in time and space and Markov correlated in time. Following [17] and also using the ideas which succeeded in constructive quantum field theory [10], we use the discretization of the spectrum of the random field $V(x,t)$ in order to approximate the system of random ordinary differential equations (1.1) by a finite dimensional system of stochastic differential equations. Two types of cutoffs are needed to obtain a finite dimensional system. A finite volume (periodic) cutoff gives a discrete structure to mode space, and a truncation with a finite number of periodic modes gives a finite dimensional velocity space.

Thus, together with (1.1) we consider an auxiliary system

$$\dot{X}_t^n = V^n(X_t^n, t) \ , \quad X_0^n = x_0 \ . \tag{1.3}$$

From the Markov assumption governing the time correlations of the random velocity statistics, each Fourier mode in the random velocity field $V^n(x,t)$ is represented by a vector valued Ornstein-Uhlenbeck process. Thus (1.3) can be also viewed as a system of stochastic differential equations

$$dX_t^{n,a} = \sum_{i=1}^{n(d-1)} Y_t^{n,i} v_i^a(X_t^n) dt \ , \quad a = 1, ..., d \ , \tag{1.4}$$

$$dY_t^{n,i} = \sqrt{2\Omega_i} dW_t^i - \Omega_i Y_t^{n,i} dt \ , \quad i = 1, ..., n(d-1) \ , \tag{1.5}$$

where v_i are periodic with common period p, and $Y_t^{n,i}$ are independent Ornstein-Uhlenbeck processes. The Markov process (X_t^n, Y_t^n) is ergodic on

$\mathbb{T}_p^d \times \mathbb{R}^{n(d-1)}$. However, it appears to be impossible to prove directly sufficient mixing properties, which would allow one to apply the functional Central Limit Theorem to (X_t^n, Y_t^n).

Instead we use the harmonic coordinates method [18, 8] to approximate the process $X_t^{n,a}$ by a stochastic integral $M_t^{n,a}$. The mean square expectation $\frac{E(M_t^{n,a} M_t^{n,b})}{2t}$ can be calculated explicitly, and its limit as $t \to \infty$ is equal to $\lim_{t\to\infty} \frac{E(X_t^{n,a} X_t^{n,b})}{2t}$. The harmonic coordinates $u^a + x_a$, $a = 1, ...d$ are defined by the solutions u^a of hypoelliptic equations

$$M(u^a + x_a) = 0 \ ,$$

on $\mathbb{T}_p^d \times \mathbb{R}^{n(d-1)}$, where M is the infinitesimal generator of the system (1.4), (1.5). The existence and uniqueness of solutions to this PDE is one of the main technical results of the dissertation. Hormander's hypoellipticity principle [11] applied to the differential operator and its adjoint is a key element in the proof of existence and regularity. The effective diffusivity can be then expressed in terms of the harmonic coordinates.

In the dissertation we obtain a priori estimates for the operator M which are uniform in the number of modes n in the spectrum of the velocity field. These estimates allow us to perform the removal of the cutoffs and prove the existence of the effective diffusivity for the field $V(x,t)$.

In Section 1.2 of the dissertation we introduce assumptions on the random field $V(x,t)$ and formulate the theorems on existence of effective diffusivity and on its asymptotic expansion in the case of short time correlations.

In Section 1.3 we describe the discretization of the spectrum of the velocity field. We state the theorem that for finite fixed time the mean square displacement of a particle in the field $V^n(x,t)$ tends to the mean square displacement in the field $V(x,t)$ as the discretization gets finer, that is as $n \to \infty$. The proof of this theorem, being of purely technical nature, is given in the end of the chapter in Section 1.6.

In Section 1.4 we relate the diffusivity in the field $V^n(x,t)$ to the solution of a hypoelliptic PDE. This PDE couples the infinitesimal generator of the Ornstein-Uhlenbeck process, in each of the mode variables, to the transport operator in physical space.

Section 1.5 contains the proof of the main technical results, namely existence, uniqueness, and a priori estimates for the hypoelliptic PDE, which are uniform in the number of modes in the spectrum of the velocity field.

Section 2.1 is devoted to the full asymptotic expansion of the solution of the hypoelliptic PDE in the case of short time correlations. First we construct the series which satisfies the equation at a formal level. Then we use the estimates obtained in Section 1.5 to show that this series is the true asymptotic series for the solution. The asymptotic expansion for the solution of the hypoelliptic PDE provides in turn the asymptotic expansion for the effective diffusivity of the field $\frac{1}{\sqrt{\varepsilon}}V^n(x, \frac{t}{\varepsilon})$. We then justify the removal of the cut-offs in the expansion in order to get the asymptotics in ε of the effective diffusivity in the field $\frac{1}{\sqrt{\varepsilon}}V(x, \frac{t}{\varepsilon})$.

In Section 2.2 we calculate explicitly the first two terms of the expansion for the effective diffusivity.

In Section 3.1 we study the dependence of the effective diffusivity on the large wavenumber cutoff for velocity fields with pure Kolmogorov spectrum. Physically such a cutoff would be given by molecular viscosity in the Navier-Stokes equation generating the turbulent velocity field. In this picture our result is a bound on the effective diffusivity in terms of the flow Reynolds number.

1.2 Definitions, Assumptions, and Results

Throughout the rest of the dissertation we shall consider $x \in \mathbb{R}^3$. All the theorems, except in Section 3.1, remain valid as stated for $x \in \mathbb{R}^d$, $d \geq 2$. The proofs carry over to arbitrary $d \geq 2$ as well, so the only reason to consider $d = 3$ is notational simplicity. We did not take $d = 2$ as a model problem in order to avoid the temptation to consider the scalar stream function for the velocity field $V(x,t)$, something which may only be possible in 2 dimensions.

We shall consider the motion of a particle in a zero mean, Gaussian random vector field $V(x,t)$, $x \in \mathbb{R}^3$, which is stationary in x and t and Markov in time. Let $G^{ab}(x,t) = E\left(V^a(x,t)V^b(0,0)\right)$ be the correlation matrix of the field $V(x,t)$.

We shall also consider the motion of a particle in the random field $V_\varepsilon(x,t)$ given by the formula

$$V_\varepsilon(x,t) = \frac{1}{\sqrt{\varepsilon}} V(x, \frac{t}{\varepsilon}) \, . \tag{1.6}$$

The field V_ε has the same stationarity and Markov properties as V. The meaning of assumption (1.6) is that as $\varepsilon \to 0$ the field V_ε becomes short time correlated. The multiplicative factor $\frac{1}{\sqrt{\varepsilon}}$ in front of V ensures that the main term of effective diffusivity is of order one as $\varepsilon \to 0$.

The properties of $V(x,t)$ listed above imply a particular form for the correlation $G(x,t)$ and the spectral matrix

$$\widehat{G^{ab}}(k,t) = (2\pi)^{-3} \int e^{-ikx} G^{ab}(x,t) dx$$

of V. By stationarity, Gaussian and Markov properties there exists a matrix valued function $K(t,x)$ such that for $t_0 \le t_1 \le t_2$

$$E[V(x,t_1) | \sigma(V(\cdot,s), s \le t_0)] = [K(t_1 - t_0) * V(t_0)](x) \ ,$$

where the symbol $*$ denotes convolution in the space variables. From the Markov property

$$G(\cdot, t_2 - t_0) = K(t_2 - t_1) * K(t_1 - t_0) * G(\cdot, 0) = K(t_2 - t_0) * G(\cdot, 0) \ .$$

A Fourier transform in the space variables shows that

$$\widehat{G}(k, t_2 - t_0) = \widehat{K}(t_2 - t_1, k) \widehat{K}(t_1 - t_0, k) \widehat{G}(k, 0) =$$

$$\widehat{K}(t_2 - t_0, k) \widehat{G}(k, 0) \ . \tag{1.7}$$

Example Consider a divergence free vector field, whose finite dimensional distributions are invariant with respect to orthogonal transformations of space variables (isotropy), and time reversal. In this case the spectral matrix has the form

$$\widehat{G^{ab}}(k,t) = \phi(|k|,t)(\delta^{ab} - \frac{k^a k^b}{|k|^2}) \ , \tag{1.8}$$

where ϕ is scalar valued [1, 17]. Suppose ϕ is continuous in t. Then (1.8) and (1.7) imply that

$$\widehat{G^{ab}}(k,t) = \phi(|k|,0)\exp(-|t|\Omega(|k|)(\delta^{ab} - \frac{k^a k^b}{|k|^2}) \ ,$$

where Ω is scalar valued.

Formula (1.7) suggests the exponential in time behavior for $\widehat{G}(k,t)$. Namely,

$$\widehat{G}(k,t) = \exp(-|t|\Omega(k))M(k) \ , \tag{1.9}$$

for some matrices Ω and M. We shall not examine the conditions under which (1.7) implies (1.9). Instead we shall use (1.9) to define a class of vector fields, which will include the fields considered in the above Example. Namely, we shall assume that $\Omega(k)$ is a scalar, and the matrix $M(k)$ is symmetric. The condition $(M(k)k,k) = 0$ for all k is then equivalent to the divergence free property of the field.

Recall the Bochner theorem [9], which states that the Fourier transform of a positive definite continuous function is a positive measure, such that the measure of the whole space ($\mathbb{R}^3$ in our case) is finite. Assuming that $G(x,0)$ is continuous $M(k)$ is the Fourier transform of a positive definite matrix valued function $G(x,0)$, that is $\sum_{a,b=1}^{3} v^a v^b G^{ab}(x,0)$ is a positive definite continuous function for any vector $v \in \mathbb{R}^3$. Therefore, by Bochner theorem, for each a, b fixed, $M^{ab}(dk)$ is a real valued signed measure of finite variation on $\mathbb{R}^3$, and $M(k)$ is a positive matrix. The positivity means that for every vector $v \in \mathbb{R}^3$ and every measurable set A

$$\sum_{a,b=1}^{3} v^a v^b \int_A M^{ab}(dk) \geq 0 .$$

We shall denote the variation of $M^{ab}(dk)$ by $|M^{ab}|(dk)$.

Necessarily $\Omega(k) \geq 0$ on $\mathrm{supp} M(k)$, otherwise

$$G^{ab}(0,t) = \int \widehat{G^{ab}}(k,t)dk \to \infty \quad \text{as } t \to \infty ,$$

for some a, b, which would contradict the stationarity of V. We shall assume that $\Omega(k) > C > 0$ in order to exclude the time independent modes.

The classical solution of the equation of motion (1.1) exists whenever the function $V(x,t)$ on the RHS is continuous in (x,t) and Lipschitz continuous in x uniformly on any compact. This will hold for almost every realization of the random vector field $V(x,t)$ under certain smoothness conditions on its correlation function (we are allowed to take a modification of the random field $V(x,t)$). Let us summarize the above assumptions on the random transport:

Assumption A *$V(x,t)$ is a divergence free zero mean Gaussian field, stationary in x and t and Markov in time.*

Assumption B *The spectral matrix of the field V is given by (1.9), where $\Omega(k)$ is scalar, and the matrix $M(k)$ is symmetric. The measure $M^{ab}(dk)$ of* *the set* $\{k = 0\}$ *is equal to zero, that is*

$$\int_{\{0\}} M^{ab}(dk) = 0 \quad for \quad all \quad a, b. \tag{1.10}$$

There exists a constant $c > 0$*, such that*

$$\int_{\mathbb{R}^3} dM(k) > cI \ , \tag{1.11}$$

where I *is the identity matrix. There exists* $\delta > 0$*, such that*

$$\int (1 + |k|^{2+\delta})|M^{ab}|(dk) < \infty \tag{1.12}$$

for all a, b*. Moreover,* $\Omega(k) > C > 0$*;* $\Omega(k)$ *is Lipschitz continuous uniformly on any compact, and grows not faster than some power of* $|k|$ *as* $k \to \infty$*.*

Remark If condition (1.10) was omitted, we would need to introduce an additional mode corresponding to $k = 0$ in the discretization of the velocity spectrum, something we do not do for the sake of notational simplicity. Condition (1.11) could be avoided by considering an appropriate linear subspace of the k space. Condition (1.12) is essential, however, since it guarantees the differentiability of a typical realization of the field $V(x,t)$ in the space variables.

It will be shown that under the above assumptions there exists a modification of the vector field $V(x,t)$, such that the solutions to equation (1.1) exist for all t for almost all realizations of the vector field $V(x,t)$.

Let the initial data x_0 be a random variable. With $t \geq 0$ fixed, $X_t - X_0$ is the displacement in time t under the action of the random field $V(x,t)$.

Since $V(x,t)$ is stationary, the distribution of the vector $X_t - X_0$ does not depend on the initial data x_0, provided that x_0 is independent of the vector field $V(x,t)$.

Theorem 1.2.1. *Suppose Assumptions A and B hold. Then the effective diffusivity, which is defined by (1.2), exists and is finite.*

We denote by D_ε^{ab} the effective diffusivity in the vector field V_ε. In order to prove the asymptotic expansion of D_ε^{ab} through order ε^m we need stronger local regularity assumptions on the correlation function.

Assumption $\mathbf{C}_m$ *The spectral matrix of the field V is given by (1.9), where the matrix $M(k)$ satisfies the condition*

$$\int (1 + |k|^{2+4m})|M^{ab}|(dk) < \infty$$

for all a, b.

We can now formulate the theorem on asymptotic expansion of effective diffusivity.

Theorem 1.2.2. *Suppose Assumptions A, B, and C_m hold. Then there exist constant matrices $d_0^{ab}, ..., d_m^{ab}$, such that*

$$D_\varepsilon^{ab} = d_0^{ab} + d_1^{ab}\varepsilon + ... + d_m^{ab}\varepsilon^m + o(\varepsilon^m) \quad when \quad \varepsilon \to 0 \,. \tag{1.13}$$

1.3 Preliminary Considerations

The proofs of Theorems 1.2.1 and 1.2.2 are based on approximation of the vector field $V_\varepsilon(x,t)$ by the vector fields $V_\varepsilon^n(x,t)$, whose spectral matrices $\widehat{G^n_\varepsilon}(k,t)$ are supported on finite sets in k-space. Let us describe the construction of the field $V^n_\varepsilon(x,t)$.

Consider the partition of the cube $Q_m = \{|k^a| \le 2^{m-1},\ a = 1,..,3\}$ into $n = 2^{6m}$ cubes Δ_i, $i = 1,...,n$ of the size $\frac{1}{2^m}$. Let k_i be the center of Δ_i. Let α_i be the interior of Δ_i, β_i be the boundary of Δ_i excluding the edges, γ_i be the edges without the endpoints, and δ_i be the set which consists of six vertices of the cube Δ_i. Define

$$\Omega_i = \Omega(k_i), \quad \text{and}$$

$$N_i = \int\limits_{\alpha_i} M(dk) + \frac{1}{2}\int\limits_{\beta_i} M(dk) + \frac{1}{4}\int\limits_{\gamma_i} M(dk) + \frac{1}{6}\int\limits_{\delta_i} M(dk)\ . \tag{1.14}$$

It is important to note that Δ_i, k_i, Ω_i, α_i, β_i, γ_i, δ_i, and N_i depend on n. The first step in the transition from $M(k)$ to $M^n(k)$ consists of integrating $M(k)$ over each cube, and placing all the mass in the center. The four different integrals enter (1.14) with their specified factors because each side belongs to two different cubes, each edge - to four different cubes, and each vertex to six different cubes. The second step is designed to make the matrices M^n satisfy the condition $(M^n(k)k,k) = 0$. We define P_i to be the orthogonal projection on the subspace orthogonal to k_i. Define

$$M_i^n = P_i N_i P_i\ ;\ M^n(k) = \sum_{i=1}^{n} \delta(k_i) M_i^n, \quad \text{and}$$

$$\widehat{G^n}(k,t) = \exp(-|t|\Omega(k)) M^n(k)\ . \tag{1.15}$$

Then $V^n(x,t)$ is defined to be the real valued Gaussian random field whose spectral matrix is given by (1.15). $V_\varepsilon^n(x,t)$ is defined to be the real valued Gaussian random field whose spectral matrix is given by

$$\widehat{G_\varepsilon^n}(k,t) = \frac{1}{\varepsilon}\exp(-\frac{|t|}{\varepsilon}\Omega(k))M^n(k) \ .$$

From (1.12) it follows that there exists a constant $c > 0$, such that for all n, a, b

$$\int (1+|k|^q)|M^{n,ab}|(dk) < c \ , \quad \text{if} \quad 0 \le q \le 2+\delta \ , \tag{1.16}$$

and the integrals converge at infinity uniformly in n.

From (1.11) it follows that there exist constants n_0 and $C > 0$ such that for $n \ge n_0$ and any $x \in \mathbb{R}^3$

$$\sum_{i=1}^{n}\sum_{a,b=1}^{3} \frac{M_i^{n,ab}}{\Omega_i} x^a x^b \ge C|x|^2 \ . \tag{1.17}$$

Throughout the rest of the dissertation we shall only consider $n \ge n_0$.

The Fourier representation of the field $V_\varepsilon^n(x,t)$ in space variables is

$$V_\varepsilon^n(x,t) = \frac{1}{\sqrt{\varepsilon}}\int e^{ikx} Z(dk, \frac{t}{\varepsilon}) \ .$$

For t fixed $Z(k,t)$ is an orthogonal Gaussian measure, which depends on n. Since $\text{supp}\widehat{V^n}(k,0) \subset \{k_i\}$

$$V_\varepsilon^n(x,t) = \frac{1}{\sqrt{\varepsilon}}\sum_{i=1}^{n} e^{ik_i x} z(k_i, \frac{t}{\varepsilon}) \ , \tag{1.18}$$

where $z(k_i,t)$ are complex vector-valued Gaussian stationary processes. The normalization of $z(k_i,t)$ is fixed by (1.15) so that

$$E\left(z^a(k_i,t)\overline{z^b}(k_j,0)\right) = \delta_{ij} M_i^{n,ab}\exp(-|t|\Omega_i) \ .$$

The fact that $V(x,t)$ is a real valued field implies that together with the mode k_i the set $\{k_i\}$ also contains $-k_i$ with the same M_i^n and Ω_i. We shall write

$$\{k_i,\ i=1,...,n\} = \{k_i, -k_i,\ i=1,...,n/2\} \ .$$

Therefore we can write (1.18) as

$$V_\varepsilon^n(x,t) = \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{n/2} \left(A_{i1}(\frac{t}{\varepsilon}) \cos(k_i x) + A_{i2}(\frac{t}{\varepsilon}) \sin(k_i x) \right) \ . \tag{1.19}$$

Here $A_{il}(t), i=1,...,n/2;\ l=1,2$ are independent real vector-valued stationary Gaussian processes and

$$E\left(A_{il}^a(t) A_{i'l'}^b(0)\right) = 2\delta_{ii'}\delta_{ll'} M_i^{n,ab} \exp(-|t|\Omega_i) \ .$$

This implies that the $A_{il}(t)$ are independent vector valued Ornstein- Uhlenbeck processes with correlation scales Ω_i and variances $2M_i^{n,ab}$.

Recall that M_i^n is an orthogonal matrix, and k_i is its eigenvector with eigenvalue 0. Let e_i^1, e_i^2 and λ_i^1, λ_i^2 be the eigenvectors and eigenvalues of M_i^n in the subspace orthogonal to k_i. Note that

$$A_{il} = \sqrt{2}(\sqrt{\lambda_i^1} e_i^1 B_{il}^1 + \sqrt{\lambda_i^2} e_i^2 B_{il}^2) \ ,$$

where B_{il}^1 and B_{il}^2 are independent Ornstein Uhlenbeck scalar valued processes with correlation scales Ω_i and variances 1. We shall write

$$\{Y^i,\ i=1,...,2n\} = \{B_{il}^m,\ i=1,...,n/2;\ l=1,2;\ m=1,2\} \ .$$

We shall use the same notation Ω_i for the correlation scale of the process Y^i.

Thus (1.19) becomes

$$V_\varepsilon^n(x,t) = \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{2n} Y^i(\frac{t}{\varepsilon}) v_i(x) \ , \tag{1.20}$$

where the vectors $v_i(x),\ i = 1, ..., 2n$ are of the following form

$$\{v_i,\ i = 1, ..., 2n\} = \{\sqrt{2\lambda_i^1} e_i^1 \cos k_i x, \sqrt{2\lambda_i^1} e_i^1 \sin k_i x,$$

$$\sqrt{2\lambda_i^2} e_i^2 \cos k_i x, \sqrt{2\lambda_i^2} e_i^2 \sin k_i x,\ i = 1, ..., n/2\} \ . \tag{1.21}$$

Therefore the vector fields v_i are divergence free and infinitely smooth. By the definition of k_i the vector fields are periodic with common period $p = 2^{m+2}\pi$. From (1.17) it follows that $\mathrm{span}\{v_i(x)\} = \mathbb{R}^3$ for all x.

The fact that

$$(\frac{d}{dx})^{\alpha_1} \cos x (\frac{d}{dx})^{\alpha_2} \cos x + (\frac{d}{dx})^{\alpha_1} \sin x (\frac{d}{dx})^{\alpha_2} \sin x = 0, \tag{1.22}$$

if $\alpha_1 + \alpha_2$ is odd, the particular form (1.21) of the vector fields v_i, and the fact that $\Omega(k_j) = \Omega(-k_j)$ imply that

$$\sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla_x v_j^a = 0 \ . \tag{1.23}$$

The equation of motion for the particle in the vector field $V_\varepsilon^n(x,t)$ has the form

$$\dot{X}_t^n = V_\varepsilon^n(X_t^n, t) \ , \quad X_0^n = x_0^n \ . \tag{1.24}$$

As before we assume x_0^n to be independent of the random field $V_\varepsilon^n(x,t)$. Whenever the subscript ε is omitted from V_ε^n we shall imply that $\varepsilon = 1$ is being considered.

Provided that the following expectations exist we define the finite time displacement tensors

$$D^{n,ab}(t) = \frac{1}{2}E\left((X_t^{n,a} - X_0^{n,a})(X_t^{n,b} - X_0^{n,b})\right) ,$$

$$D^{ab}(t) = \frac{1}{2}E\left((X_t^{a} - X_0^{a})(X_t^{b} - X_0^{b})\right) ,$$

where X_t^n, and X_t are the solutions of (1.24) with $\varepsilon = 1$, and (1.1) respectively.

The proof of Theorem 1.2.1 is based on the following two theorems.

Theorem 1.3.1. *Suppose Assumptions A and B hold. Then for arbitrary $t \geq 0$ the displacement tensors $D^{n,ab}(t)$ and $D^{ab}(t)$ exist for some modification of the field $V(x,t)$, and for t fixed*

$$\lim_{n\to\infty} D^{n,ab}(t) = D^{ab}(t) . \tag{1.25}$$

Theorem 1.3.2. *Suppose Assumptions A and B hold. Then the following limit*

$$\lim_{t\to\infty} \frac{D^{n,ab}(t)}{t} , \qquad a,b = 1,...,3 \tag{1.26}$$

exists and is uniform in n.

Theorem 1.3.1 and Theorem 1.3.2 will be proved in Sections 1.6 and 1.4 respectively. Notice that the limit in (1.26) is the effective diffusivity for the vector field $V^n(x,t)$. We shall denote it by $D^{n,ab}$. The effective diffusivity for the vector field $V_\varepsilon^n(x,t)$ will be denoted by $D_\varepsilon^{n,ab}$.

Proof of Theorem 1.2.1 By Theorem 1.3.2

$$\lim_{t\to\infty} \frac{D^{n,ab}(t)}{t} = D^{n,ab}$$

uniformly in n. By Theorem 1.3.1

$$\lim_{n\to\infty} \frac{D^{n,ab}(t)}{t} = \frac{D^{ab}(t)}{t}$$

Therefore by the theorem on uniform convergence the following limits exist

$$D^{ab} = \lim_{t\to\infty} \frac{D^{ab}(t)}{t} = \lim_{n\to\infty} D^{n,ab} .$$

This completes the proof of Theorem 1.2.1.

1.4 Formulation of the Theorem on the Hypoelliptic Estimate and the Proof of Theorem 1.3.2

In this section we reduce the assertion of Theorem 1.3.2 to a hypoelliptic estimate for a PDE. In order to do that we shall employ an analog of the harmonic coordinates method of Freidlin [18, 8]. The method was originally used to reduce a problem on the large time asymptotics of a stochastic process with periodic drift to an elliptic problem, with the elliptic operator being the infinitesimal generator of the process. Our situation is more complicated, since the resulting equation is not elliptic, but only hypoelliptic, and we need to prove the existence, uniqueness, and a priori estimates for the solution. Besides, in order to prove the uniform limit in Theorem 1.3.2 we need to ensure that all the estimates are uniform in n, that is uniform in the dimension of the space on which the hypoelliptic operator is defined. The existence, uniqueness, and a priori estimates will be proved in Section 1.5. Since the same estimate will be used in the proof of Theorem 1.2.2 we preserve the ε-dependence throughout this section.

By (1.20), (1.24) the equation of motion in the vector field $V_\varepsilon^n(x,t)$ has the form

$$dX_t^{n,a} = \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^{2n} Y_{t/\varepsilon}^{n,i} v_i^a(X_t^n) dt \ , \quad a = 1,..,3 \ , \tag{1.27}$$

where the $Y_t^{n,i}$ are independent Ornstein-Uhlenbeck processes

$$dY_t^{n,i} = \sqrt{2\Omega_i} dW_t^i - \Omega_i Y_t^{n,i} dt \ , \quad i = 1, ..., 2n \ . \tag{1.28}$$

Here the superscript n is to indicate the dependence of the system on the number of modes in the spectrum of the velocity field.

We re-scale the time variable in (1.27) so that we can consider (1.27)-(1.28) as a system of stochastic differential equations. Thus define $\widetilde{X}_t^{n,a} = X_{\varepsilon t}^{n,a}$. In the new variables (1.27) takes the form

$$d\widetilde{X}_t^{n,a} = \sqrt{\varepsilon} \sum_{i=1}^{2n} Y_t^{n,i} v_i^a(\widetilde{X}_t^n) dt \ , \quad a = 1, ..., 3 \ . \tag{1.29}$$

The operator

$$M_\varepsilon = \sum_{i=1}^{2n} \Omega_i \left(\partial_{y_i}^2 - y_i \partial_{y_i}\right) + \sqrt{\varepsilon} \sum_{i=1}^{2n} y_i v_i(x) \nabla_x \tag{1.30}$$

is the infinitesimal generator of the system (1.28)-(1.29). Recall that p is the common period of the velocity modes defined in Section 1.3. Let

$$\eta(x,y) = p^{-3/2}\prod_{i=1}^{2n}(2\pi)^{-1/4}\exp(-\frac{y_i^2}{4}) \ .$$

As initial conditions for the system we take the distribution η^2, which is invariant for the process $(\widetilde{X}_t^n, Y_t^n)$ on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$. We shall repeatedly use the following elementary integrals

$$\int y_i\eta^2 dy = 0 \ , \quad \int y_i y_j \eta^2 dy = \frac{1}{p^3}\delta_{ij} \ . \tag{1.31}$$

Consider the equation

$$M_\varepsilon(\sqrt{\varepsilon}u^{n,a} + x_a) = 0 \tag{1.32}$$

for a function $u^{n,a}(x,y)$ defined on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$. The function $\sqrt{\varepsilon}u^{n,a} + x_a$ of (1.32) is the analog of the harmonic coordinate of [18, 8].

Theorem 1.4.1. *Suppose Assumptions A and B hold. Then equation (1.32) has a unique solution in the class of C^∞ functions which satisfy the relations*

$$\sum_{i=1}^{2n}\Omega_i\int\int\left((\partial_{y_i}u^{n,a})^2 + y_i^2(u^{n,a})^2 + (u^{n,a})^2\right)\eta^2 dxdy < \infty \ ,$$

$$\int\int u^{n,a}\eta^2 dxdy = 0 \ .$$

Moreover, there exists a constant C independent of n, ε, such that the solution of (1.32) satisfies

$$\int\int\left((u^{n,a})^2 + \sum_{i=1}^{2n}\Omega_i(\partial_{y_i}u^{n,a})^2\right)\eta^2 dxdy < C \ . \tag{1.33}$$

The proof of Theorem 1.4.1 is given as a consequence of the more general Theorems 1.5.1 and 1.5.8 in Section 1.5 below.

For the proof of Theorem 1.3.2 we shall need the following simple lemma

Lemma 1.4.2. *Let $f_t^{n,a}$ and $g_t^{n,a}$ be random variables, which depend on parameters n and t. Suppose*

$$E\left((f_t^{n,a} + g_t^{n,a})(f_t^{n,b} + g_t^{n,b})\right) = \phi^{ab}(n) \ . \tag{1.34}$$

Suppose there are constants C_1 and C_2, which do not depend on n and t, such that

$$tE(g_t^{n,a})^2 < C_1, \quad \text{and} \quad \phi^{ab}(n) < C_2 \ . \tag{1.35}$$

Then

$$\lim_{t\to\infty} E(f_t^{n,a} f_t^{n,b}) = \phi^{ab}(n) \ ,$$

and the limit is uniform in n.

Proof From (1.34) with $a = b$ it follows that

$$E(f_t^{n,a})^2 = \phi^{aa}(n) - E(g_t^{n,a})^2 - 2E(f_t^{n,a} g_t^{n,a}) \ . \tag{1.36}$$

From (1.36) and (1.35) we conclude that there exists a constant C_3 such that

$$E(f_t^{n,a})^2 < C_3 \quad \text{for all } n \text{ and } t \geq 1 \ . \tag{1.37}$$

From (1.34)

$$E(f_t^{n,a} f_t^{n,b}) - \phi^{ab}(n) = -E(g_t^{n,a} g_t^{n,b}) - E(f_t^{n,a} g_t^{n,b}) - E(f_t^{n,b} g_t^{n,a}) \ . \tag{1.38}$$

By the Schwartz inequality, (1.35), and (1.37) the RHS of (1.38) tends to zero uniformly in n as $t \to \infty$. This completes the proof of Lemma 1.4.2.

Proof of Theorem 1.3.2 We are here using the result of Theorem 1.4.1. Since the function $\sqrt{\varepsilon}u^{n,a} + x_a$ is smooth we can apply Ito's formula to $(\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_t^n, Y_t^n)$. By (1.28) and (1.29) we obtain

$$(\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_t^n, Y_t^n) = (\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_0^n, Y_0^n) +$$

$$\int_0^t M_\varepsilon(\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_s^n, Y_s^n)ds + \tag{1.39}$$

$$\sqrt{2}\sum_{i=1}^{2n} \sqrt{\Omega_i}\int_0^t \partial_{y_i}(\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_s^n, Y_s^n)dW_s^i \ .$$

Since $M_\varepsilon(\sqrt{\varepsilon}u^{n,a} + x_a) = 0$ the expression above can be rewritten as

$$(\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_t^n, Y_t^n) - (\sqrt{\varepsilon}u^{n,a} + x_a)(\widetilde{X}_0^n, Y_0^n) =$$

$$\sqrt{2\varepsilon}\sum_{i=1}^{2n} \sqrt{\Omega_i}\int_0^t \partial_{y_i}u^{n,a}(\widetilde{X}_s^n, Y_s^n)dW_s^i \ . \tag{1.40}$$

Similarly

$$(\sqrt{\varepsilon}u^{n,b} + x_b)(\widetilde{X}_t^n, Y_t^n) - (\sqrt{\varepsilon}u^{n,b} + x_b)(\widetilde{X}_0^n, Y_0^n) =$$

$$\sqrt{2\varepsilon}\sum_{i=1}^{2n} \sqrt{\Omega_i}\int_0^t \partial_{y_i}u^{n,b}(\widetilde{X}_s^n, Y_s^n)dW_s^i \ . \tag{1.41}$$

Since the measure η^2 is invariant for the process $(\widetilde{X}_t^n, Y_t^n)$, the expectation of the product of the right sides of (1.40) and (1.41) is equal to

$$2\varepsilon \sum_{i=1}^{2n} \Omega_i \int_0^t E[\partial_{y_i} u^{n,a}(\widetilde{X}_s^n, Y_s^n) \partial_{y_i} u^{n,b}(\widetilde{X}_s^n, Y_s^n)] ds =$$

$$2\varepsilon t \sum_{i=1}^{2n} \Omega_i \iint_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (\partial_{y_i} u^{n,a})(\partial_{y_i} u^{n,b}) \eta^2 dx dy,$$

which is finite by Theorem 1.4.1. Thus multiplying (1.40) by (1.41), dividing both sides by $2t\varepsilon$, and taking expectations of both sides of the resulting equality we obtain

$$\frac{1}{2t\varepsilon} E\Big[\Big(\widetilde{X}_t^{n,a} - \widetilde{X}_0^{n,a} + \sqrt{\varepsilon} u^{n,a}(\widetilde{X}_t^n, Y_t^n) - \sqrt{\varepsilon} u^{n,a}(\widetilde{X}_0^n, Y_0^n)\Big) \\ \Big(\widetilde{X}_t^{n,b} - \widetilde{X}_0^{n,b} + \sqrt{\varepsilon} u^{n,b}(\widetilde{X}_t^n, Y_t^n) - \sqrt{\varepsilon} u^{n,b}(\widetilde{X}_0^n, Y_0^n)\Big)\Big] = \\ \sum_{i=1}^{2n} \Omega_i \iint_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (\partial_{y_i} u^{n,a})(\partial_{y_i} u^{n,b}) \eta^2 dx dy \,. \tag{1.42}$$

Set

$$f_t^{n,a} = \frac{\widetilde{X}_t^{n,a} - \widetilde{X}_0^{n,a}}{\sqrt{2t\varepsilon}} \,, \; g_t^{n,a} = \frac{u^{n,a}(\widetilde{X}_t^n, Y_t^n) - u^{n,a}(\widetilde{X}_0^n, Y_0^n)}{\sqrt{2t}} \,, \text{ and}$$

$$\phi^{ab}(n) = \sum_{i=1}^{2n} \Omega_i \iint_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (\partial_{y_i} u^{n,a})(\partial_{y_i} u^{n,b}) \eta^2 dx dy \,.$$

With this choice of $f_t^{n,a}$, $g_t^{n,a}$ and $\phi^{ab}(n)$ equation (1.42) takes the form (1.34). Since the measure η^2 is invariant for the process $(\widetilde{X}_t^n, Y_t^n)$

$$E(u^{n,a}(\widetilde{X}_t^n, Y_t^n))^2 = E(u^{n,a}(\widetilde{X}_0^n, Y_0^n))^2 = \iint_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (u^{n,a})^2 \eta^2 dx dy \,, \tag{1.43}$$

which does not depend on t. By Theorem 1.4.1 the RHS of (1.43) is bounded uniformly in n. Therefore $tE(g_t^{n,a})^2 < C_1$. From Theorem 1.4.1 it also follows that $\phi^{ab}(n) < C_2$. Therefore we can apply Lemma 1.4.2 to conclude that the limit

$$\lim_{t\to\infty} E\left((f_t^{n,a})(f_t^{n,b})\right) = \frac{1}{2}\lim_{t\to\infty} \frac{E\left((\widetilde{X}_t^{n,a} - \widetilde{X}_0^{n,a})(\widetilde{X}_t^{n,b} - \widetilde{X}_0^{n,b})\right)}{t\varepsilon} =$$

$$\frac{1}{2}\lim_{t\to\infty} \frac{E\left((X_t^{n,a} - X_0^{n,a})(X_t^{n,b} - X_0^{n,b})\right)}{t} =$$

$$\sum_{i=1}^{2n} \Omega_i \iint\limits_{\mathbb{T}_p^3\times\mathbb{R}^{2n}} (\partial_{y_i} u^{n,a})(\partial_{y_i} u^{n,b})\eta^2 dxdy$$

exists uniformly in n. Put $\varepsilon = 1$ to obtain the statement of Theorem 1.3.2.

Corollary 1.4.3. *Suppose Assumptions A and B hold. Then the effective diffusivity $D_\varepsilon^{n,ab}$ is expressed in terms of the solution $u^{n,a}$ of equation (1.32) by the formula*

$$D_\varepsilon^{n,ab} = \sum_{i=1}^{2n} \Omega_i \iint\limits_{\mathbb{T}_p^3\times\mathbb{R}^{2n}} (\partial_{y_i} u^{n,a})(\partial_{y_i} u^{n,b})\eta^2 dxdy\ . \tag{1.44}$$

We make a change of variables so that M_ε becomes the sum of a formally self adjoint operator in the y variables (a simple harmonic oscillator) and a formally antisymmetric operator in the x variables. Thus let

$$u_{\text{new}}^{n,a} = u^{n,a}\eta\ , \tag{1.45}$$

and rename the unknown function by $u^{n,a}$ again. Then equation (1.32) becomes

$$\sum_{i=1}^{2n} \Omega_i \left(\partial_{y_i}^2 - \frac{y_i^2}{4} + \frac{1}{2} \right) u^{n,a} + \sqrt{\varepsilon} \sum_{i=1}^{2n} y_i v_i \nabla_x u^{n,a} = - \sum_{i=1}^{2n} y_i v_i^a(x) \eta(y) \ . \tag{1.46}$$

Note that the first term on the LHS of (1.46) is the simple harmonic oscillator and $\eta(y)$ is its eigenfunction with zero eigenvalue. We transform (1.44) by the above change of variables.

Corollary 1.4.4. *Suppose Assumptions A and B hold. Then the effective diffusivity $D_\varepsilon^{n,ab}$ is expressed in terms of the solution $u^{n,a}$ of equation (1.46) by the formula*

$$D_\varepsilon^{n,ab} = \frac{1}{2} \sum_{i=1}^{2n} \int\int_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (u^{n,a} v_i^b + u^{n,b} v_i^a) y_i \eta dx dy \ .$$

1.5 Existence, Uniqueness, and the Hypoelliptic Estimate

Let us introduce notation needed for the statement of the existence and uniqueness theorem. $\mathcal{S}^\perp$ is the space of functions on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$ which are infinitely smooth, orthogonal to η, and decay faster than any polynomial together with all their derivatives. That is $f \in \mathcal{S}^\perp$ if

$$\int\int f(x,y)\eta(y) dx dy = 0 \ , \quad \text{and} \quad \sup_{x,y} |Q(y) P_1(D_y) P_2(D_x) f| < \infty$$

for any polynomials P_1, P_2, and Q. $\mathcal{L}_2^\perp$ is the completion of $\mathcal{S}^\perp$ in $\mathcal{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n})$. Clearly

$$\mathcal{L}_2^\perp \oplus \{\text{const} \cdot \eta(y)\} = \mathcal{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n}) \ .$$

$|| \cdot ||$ is the usual norm of $\mathcal{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n})$. $\mathcal{H}^\perp$ is the completion of $\mathcal{S}^\perp$ in the harmonic oscillator inner product

$$(f, g)_{\mathcal{H}^\perp} = \sum_{i=1}^{2n} \Omega_i \int\int \left((\partial_{y_i} f)(\partial_{y_i} g) + y_i^2 fg + fg\right) dxdy \ .$$

$\mathcal{H}^{-1}(\mathbb{T}_p^3), \mathcal{L}_2(\mathbb{T}_p^3), \mathcal{H}^1(\mathbb{T}_p^3)$, and $\mathcal{H}^2(\mathbb{T}_p^3)$ are the usual Sobolev spaces of functions on the torus.

Write

$$Lu = \sum_{i=1}^{2n} \Omega_i(\partial_{y_i}^2 u - \frac{y_i^2}{4} u + \frac{1}{2} u) \ , \quad Au = \sum_{i=1}^{2n} y_i v_i(x) \nabla_x u \ ,$$

$$M_\varepsilon = L + \sqrt{\varepsilon} A \ .$$

Here M_ε is the transform by (1.45) of the expression (1.30), with change of notation to the new variables.

Theorem 1.5.1. *Suppose Assumptions A and B hold and $f \in \mathcal{L}_2^\perp$. Then the equation*

$$M_\varepsilon u = f \ . \tag{1.47}$$

has a unique weak solution u in the space $\mathcal{H}^\perp$. There is a constant C, which depends on n and ε, such that

$$||u||_{\mathcal{H}^\perp} \leq C||f|| \ . \tag{1.48}$$

If $f \in C^\infty$, then $u \in C^\infty$ also.

Remark *Theorem 1.5.1 implies that* 0 *is an isolated point of the spectrum of the operator* $M_\varepsilon : \mathcal{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n}) \to \mathcal{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n})$.

Indeed, 0 is in the resolvent set of $M_\varepsilon : \mathcal{L}_2^\perp \to \mathcal{L}_2^\perp$ by Theorem 1.5.1. Therefore there exists a δ neighborhood of 0 which belongs to the resolvent set of $M_\varepsilon : \mathcal{L}_2^\perp \to \mathcal{L}_2^\perp$. Let $f \in \mathcal{L}_2$, $f = f_1 + c\eta$ where $f_1 \in \mathcal{L}_2^\perp$. For λ such that $0 < |\lambda| < \delta$ the equation $(M_\varepsilon + \lambda E)u = f$ has the solution $u = (M_\varepsilon + \lambda E)^{-1} f_1 + \frac{c}{\lambda}\eta$.

The proof of Theorem 1.5.1 is based on a series of lemmas. First we obtain the a priori estimates.

Lemma 1.5.2. *Under the assumptions of Theorem 1.5.1, there is a constant* C *depending on* n *and* ε, *such that if* $u \in \mathcal{S}^\perp$ *is a solution of* $M_\varepsilon u = f$ *and* $\int f(x,y)\eta(y)dy = 0$ *for all* x, *then* u *satisfies the estimate*

$$||u||_{\mathcal{H}^\perp} \leq C||f|| \ . \tag{1.49}$$

Proof In what follows we shall denote by k_1, k_2, k_3, various constants depending only on n and ε. We represent u uniquely as a sum of two functions which are orthogonal in $\mathcal{L}_2(\mathbb{R}_y^{2n})$ for all x, that is

$$u(x,y) = w(x,y) + u_0(x)\eta(y) \ ,$$

where $\int w(x,y)\eta(y)dy = 0$ for all x, and $\int u_0(x)dx = 0$.

To prove (1.49) it is sufficient to estimate $||w||_{\mathcal{H}^\perp}$ and $||u_0||_{\mathcal{L}_2(\mathbb{T}_p^3)}$ separately. The equation $M_\varepsilon u = f$ can be written as

$$Lw + \sqrt{\varepsilon}Au_0(x)\eta(y) + \sqrt{\varepsilon}Aw = f \ . \tag{1.50}$$

In order to estimate the norms of w and u_0 we multiply (1.50) successively by w, η, and $y_j\eta$ and integrate in y. Multiplying by w and integrating in y, we obtain

$$\int wLwdy + \int \sqrt{\varepsilon} wAu_0(x)\eta(y)dy + \int \sqrt{\varepsilon} wAwdy = \int fwdy \ . \quad (1.51)$$

Multiplying (1.50) by η and integrating in y yields

$$\int \eta Lwdy + \int \sqrt{\varepsilon} Au_0(x)\eta^2(y)dy + \int \sqrt{\varepsilon} Aw\eta(y)dy = \int f\eta dy \ . \quad (1.52)$$

The RHS of (1.52) is equal to zero by assumption. Note that also $\int \eta Lwdy = \int wL\eta dy = 0$ since $L\eta = 0$, and $\int \sqrt{\varepsilon} Au_0(x)\eta^2(y)dy = 0$ since $\int y_i\eta^2(y)dy = 0$. Thus from (1.52) we obtain

$$\int \sqrt{\varepsilon} Aw\eta dy = 0 \ . \quad (1.53)$$

Note that

$$\int wAwdx = \sum_{i=1}^{2n} y_i \int wv_i(x)\nabla_x wdx = \frac{1}{2}\sum_{i=1}^{2n} y_i \int \operatorname{div}(v_i w^2)dx = 0 \ , \quad (1.54)$$

and thus the last term of the LHS of (1.51) vanishes after integration over x. By (1.53), since $A^* = -A$

$$\int\int \sqrt{\varepsilon} wAu_0(x)\eta(y)dxdy = -\int\int \sqrt{\varepsilon} u_0 Aw\eta(y)dxdy = 0 \ , \quad (1.55)$$

and thus the second term of the LHS of (1.51) vanishes after integration over x. Therefore

$$\int\int wLwdxdy = \int\int fwdxdy \ . \quad (1.56)$$

There exists a constant $k_1 > 0$, such that for $\phi \in \mathcal{S}(\mathbb{R}^n)$ the relation $\int \phi(y)\eta(y)dy = 0$ implies

$$\sum_{i=1}^{2n} \Omega_i \int \left((\partial_{y_i}\phi)^2 + y_i^2\phi^2 + \phi^2 \right) dy \leq -k_1 \int \phi L \phi dy \ . \tag{1.57}$$

Recall that $\int w\eta dy = 0$ for all x, and thus by (1.57),

$$||w||^2_{\mathcal{H}^\perp} \leq -k_1 \int \int wLwdxdy \ .$$

Together with (1.56) this implies that $||w||_{\mathcal{H}^\perp} \leq k_2||f||$.

Now we estimate the $\mathcal{L}_2$ norm of u_0. Multiplying (1.50) by $y_j\eta$ and integrating in y, we obtain

$$\int y_j\eta Lwdy + \int \sqrt{\varepsilon} A u_0 y_j \eta^2 dy + \int \sqrt{\varepsilon} A w y_j \eta dy = \int f y_j \eta dy \ . \tag{1.58}$$

Recall that $A = \sum_{i=1}^{2n} y_i v_i \nabla_x$. We evaluate

$$\int \sum_{i=1}^{2n} v_i \nabla_x u_0(x) y_i y_j \eta^2 dy = \frac{1}{p^3} v_j \nabla_x u_0(x)$$

with only the $i = j$ term contributing. By carrying the first and the third terms from the left to the right side of (1.58), we have the identity

$$\sqrt{\varepsilon} v_j \nabla_x u_0(x) = p^3 \left(\int f y_j \eta dy - \int y_j \eta L w dy - \int \sqrt{\varepsilon} y_j A w \eta dy \right) \ . \tag{1.59}$$

Applying the operator $\frac{1}{\Omega_j} v_j \nabla$ to both sides of (1.59), taking the sum $\sum_{j=1}^{2n}$, and dividing by $\sqrt{\varepsilon}$, we obtain due to (1.23)

$$\sum_{a,b=1}^{3} \sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j^a v_j^b u_{0x_a x_b} = \tag{1.60}$$

$$\frac{p^3}{\sqrt{\varepsilon}}\sum_{j=1}^{2n}\frac{1}{\Omega_j}v_j\nabla\Big(\int fy_j\eta dy-\int y_j\eta Lwdy-\sqrt{\varepsilon}\int y_jAw\eta dy\Big)\ .$$

For the following lemma it is important to note that $\mathbb{T}_p^3$ is a torus (a compact manifold), but not a cube with a boundary.

Lemma 1.5.3. *Suppose $u\in L_2(\mathbb{T}_p^3)$, $f\in H^{-2}(\mathbb{T}_p^3)$, and $\int_{\mathbb{T}_p^3} udx=0$. Let c be a constant, and A_{ab} a constant matrix, such that for any $x\in\mathbb{R}^3$*

$$\sum_{a,b=1}^{3}A_{ab}x^ax^b\geq c||x||^2\ .$$

Then there exists a constant c_1, which depends only on c, such that the equality

$$\sum_{a,b=1}^{3}A_{ab}u_{x_ax_b}(x)=f\qquad on\ \ \mathbb{T}_p^3$$

implies the estimate

$$||u||_{L_2(\mathbb{T}_p^3)}\leq c_1||f||_{H^{-2}(\mathbb{T}_p^3)}\ . \tag{1.61}$$

Proof In the case when $p=1$ the statement of the lemma is a standard a priori estimate from general elliptic theory [12]. If $p\neq 1$, then the change of variables

$$\widetilde{u}(x)=u(px)\ ,\quad \widetilde{f}(x)=p^3f(px)\ ,\quad x\in\mathbb{T}^3\ .$$

reduces the statement of the lemma to the case when $p=1$. This completes the proof of Lemma 1.5.3.

Note that by (1.21)

$$\sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j^a v_j^b = \sum_{j=1}^{n} \frac{M_j^{n,ab}}{\Omega_j} , \tag{1.62}$$

and thus by (1.17) we can apply the result of Lemma 1.5.3 to (1.60). We need to estimate the $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ norm of the RHS of (1.60). Since $v_j \nabla$ is a bounded operator from $\mathcal{H}^{-1}(\mathbb{T}_p^3)$ to $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ it is sufficient to estimate the $\mathcal{H}^{-1}(\mathbb{T}_p^3)$ norm of

$$\int f y_j \eta dy - \Omega_j \int w y_j \eta dy - \sqrt{\varepsilon} \int y_j A w \eta dy . \tag{1.63}$$

For the first term of (1.63)

$$\int \left(\int f y_j \eta dy \right)^2 dx \le \int dx \left(\int f^2 dy \int (y_j \eta)^2 dy \right) =$$

$$\frac{1}{p^3} \int \int f^2 dx dy = \frac{1}{p^3} ||f||^2 . \tag{1.64}$$

Thus we have the stronger $\mathcal{L}_2$ estimate of this term. For the second term

$$\int \left(\int y_j \eta L w dy \right)^2 dx = \int dx \left(\int w L(y_j \eta) dy \right)^2 \le$$

$$k_3 \int \int w^2 dx dy \le k_4 ||f||^2 . \tag{1.65}$$

Again we have an $\mathcal{L}_2$ estimate. Finally for the third term

$$|| \int y_j A w \eta dy ||_{\mathcal{H}^{-1}(\mathbb{T}_p^3)} \le$$

$$k_5 \sup_{i,j} || \int y_i y_j w \eta dy ||_{\mathcal{L}_2(\mathbb{T}_p^3)} \le k_6 ||w|| \le k_7 ||f|| . \tag{1.66}$$

Thus by Lemma 1.5.3 we conclude that $||u_0(x)||_{\mathcal{L}_2(\mathbb{T}^3_p)} \leq k_8||f||$. Combining this with the bound $||w||_{\mathcal{H}^\perp} \leq k_1||f||$ we see that there is the estimate

$$||u||_{\mathcal{H}^\perp} \leq k_9||f|| \, . \tag{1.67}$$

This completes the proof of Lemma 1.5.2.

Now we consider the general case when the right side f is not necessarily orthogonal to η for all x.

Lemma 1.5.4. *Under the assumptions of Theorem 1.5.1, let* $u \in \mathcal{S}^\perp$ *be a solution of* $M_\varepsilon u = f$. *Then the estimate (1.48) holds.*

Proof We represent uniquely f as a sum of two terms orthogonal in $\mathcal{L}_2(\mathbb{R}^{2n}_y)$ for all x, the first of which satisfies the hypothesis of Lemma 1.5.2. Thus

$$M_\varepsilon u = f = f_1(x,y) + g(x)\eta(y) \, , \tag{1.68}$$

where $\int f_1(x,y)\eta(y)dy = 0$ for all x and $\int g(x)dx = 0$. Write

$$u = u_1(x,y) + \sum_{i=1}^{2n} c_i(x) y_i \eta(y) \, , \tag{1.69}$$

where $c_i(x)$ are chosen below and $u_1(x,y)$ is defined by equation (1.69). We shall estimate the $\mathcal{H}^\perp$ norm of each of the terms in (1.69).

Take $G(x)$ such that $\Delta G(x) = g(x)$ and $\int G(x)dx = 0$. Such a function G exists and is unique in $\mathcal{H}^2(\mathbb{T}^3_p)$ by the general elliptic theory. By Lemma 1.5.3 $||G(x)||_{\mathcal{H}^2(\mathbb{T}^3_p)} \leq k_{10}||g||_{\mathcal{L}_2(\mathbb{T}^3_p)} \leq k_{10}||f||$.

We shall find $c_i(x)$ from the equation

$$\sqrt{\varepsilon}\sum_{i=1}^{2n} v_i(x)c_i(x) = \nabla G(x)\ . \tag{1.70}$$

Since $\{v_i\}$ are smooth and span $\mathbb{R}^3$, equation (1.70) can be solved with

$$||c_i(x)||_{\mathcal{H}^1(\mathbb{T}^3_p)} \le k_{11}||\nabla_x G||_{\mathcal{H}^1(\mathbb{T}^3_p)} \le k_{12}||g(x)||_{\mathcal{L}_2(\mathbb{T}^3_p)}\ . \tag{1.71}$$

This in particular implies that

$$||\sum_{i=1}^{2n} c_i(x)y_i\eta(y)||_{\mathcal{H}^\perp} \le k_{13}||g||_{\mathcal{L}_2(\mathbb{T}^3_p)}\ . \tag{1.72}$$

It remains to estimate $||u_1||_{\mathcal{H}^\perp}$. By (1.68) and (1.69) $u_1(x,y)$ satisfies

$$M_\varepsilon u_1 = f_1(x,y) + g(x)\eta(y) - \sum_{i=1}^{2n} L(c_i(x)y_i\eta(y)) -$$

$$\sum_{i=1}^{2n}\sqrt{\varepsilon}Ac_i(x)y_i\eta(y)\ . \tag{1.73}$$

Now we show that the RHS of (1.73) is orthogonal to η for all x. Multiplying the RHS by η and integrating in y we see that the first term $\int f_1(x,y)\eta(y)dy$ is equal to zero by the definition of f_1. The third term is equal to

$$-\sum_{i=1}^{2n}\int c_i(x)L\left(y_i\eta(y)\right)\eta(y)dy = -\sum_{i=1}^{2n}\int c_i(x)y_i\eta(y)L\eta(y)dy = 0\ ,$$

since $L\eta(y) = 0$. For the remaining two terms we obtain

$$\int g(x)\eta^2(y)dy - \sqrt{\varepsilon}\int\sum_{i=1}^{2n} Ac_i(x)y_i\eta^2(y)dy =$$

$$\frac{1}{p^3}\left(g(x)-\sqrt{\varepsilon}\sum_{i=1}^{2n} v_i \nabla c_i(x)\right) ,$$

which is seen to be equal to zero by applying div to both sides of (1.70). By (1.71) the $\mathcal{L}_2$ norm of the RHS of (1.73) is estimated by $k_{14}(||f_1||+||g||_{\mathcal{L}_2(\mathbb{T}^3_p)})$. Applying Lemma 1.5.2 to the equation (1.73) and using (1.69) and (1.72) we see that

$$||u||_{\mathcal{H}^\perp} \leq ||\sum_{i=1}^{2n} c_i(x) y_i \eta(y)||_{\mathcal{H}^\perp} + ||u_1||_{\mathcal{H}^\perp} \leq$$
$$k_{15}\left(||g||_{\mathcal{L}_2(\mathbb{T}^3_p)} + ||f_1||\right) \leq k_{16}||f|| . \tag{1.74}$$

This completes the proof of Lemma 1.5.4.

In the proof of the next lemma we shall use Hormander's hypoellipticity principle. For the proof of it we refer the reader to [11].

Theorem 1.5.5 ([11]). *Let $X_0, X_1, ..., X_r$ be first order differential operators in $\mathbb{R}^d$ with infinitely smooth coefficients, and let $g \in C^\infty(\mathbb{R}^d)$. If the Lie algebra generated by $\{X_i\}$ coincides with $\mathbb{R}^d$ in every point of the space then the operator $H = X_0+X_1^2+...+X_r^2+g$ is hypoelliptic. That is $Hu = f \in C^\infty$ implies that $u \in C^\infty$.*

Let M_ε^* be the operator formally adjoint to M_ε. Note that $M_\varepsilon^* u = Lu - \sqrt{\varepsilon} Au$ since $\mathrm{div} v_i = 0$.

Lemma 1.5.6. *Under the assumptions of Theorem 1.5.1, the equations $M_\varepsilon f = 0$ and $M_\varepsilon^* f = 0$, considered in the sense of distributions, each admit $f = 0$ as their unique solution in $L_2^\perp$.*

Proof Let $f \in L_2^\perp$ be a solution of the equation $M_\varepsilon^* f = 0$. Both M_ε and M_ε^* are hypoelliptic by Hormander's principle. Therefore f is a classical C^∞ solution of $M_\varepsilon^* f = 0$. To proceed we must understand the behavior of f at infinity.

Let $\chi_r(y)$ be cutoff functions chosen so that $\chi_r(y) = 1$ for $|y| \le r$, $\chi_r(y) = 0$ for $|y| \ge r+1$, and χ_r are bounded together with all their first and second derivatives uniformly in r. Recall that

$$\int \chi_r f A(\chi_r f) dx = \sum_{i=1}^{2n} y_i \int \chi_r f v_i(x) \nabla_x (\chi_r f) dx =$$

$$\frac{1}{2} \sum_{i=1}^{2n} y_i \int \mathrm{div}(v_i (\chi_r f)^2) dx = 0 \ .$$

Therefore, using integration by parts, we obtain

$$\int\int \chi_r f M_\varepsilon^* (\chi_r f) dx dy = \int \int \chi_n f L(\chi_n f) dx dy = \tag{1.75}$$

$$\sum_{i=1}^{2n} \Omega_i \int\int \left(-(\partial_{y_i}(\chi_r f))^2 - \frac{1}{4}(y_i \chi_r f)^2 + \frac{1}{2}(\chi_r f)^2 \right) dx dy \ .$$

On the other hand since $M_\varepsilon^* f = 0$, using the integration by parts again,

$$\int\int \chi_r f M_\varepsilon^* (\chi_r f) dx dy =$$

$$\sum_{i=1}^{2n} \Omega_i \int\int \left((\partial_{y_i}^2 \chi_r) \chi_r f^2 + 2(\partial_{y_i} \chi_r)(\partial_{y_i} f) \chi_r f \right) dx dy = \tag{1.76}$$

$$- \sum_{i=1}^{2n} \Omega_i \int\int f^2 (\partial_{y_i} \chi_r)^2 dx dy \ .$$

Since $f \in \mathcal{L}_2^\perp \subset \mathcal{L}_2$, the expression in the RHS of (1.76) is bounded uniformly in r. Thus the same is true for (1.75). Taking $r \to \infty$ in (1.75) and (1.76) we see that $\int\int(\partial_{y_i} f)^2 < \infty$, $\int\int(y_i f)^2 < \infty$ and

$$\sum_{i=1}^{2n} \Omega_i \int\int \left(-(\partial_{y_i} f)^2 - \frac{1}{4}(y_i f)^2 + \frac{1}{2}(f)^2\right) dx dy = 0 \ .$$

Thus, since the simple harmonic oscillator is a negative operator,

$$\sum_{i=1}^{2n} \Omega_i \int \left(-(\partial_{y_i} f)^2 - \frac{1}{4}(y_i f)^2 + \frac{1}{2}(f)^2\right) dy = 0$$

for almost all x. This implies $f(x,y) = c(x)\eta(y)$. Using $M_\varepsilon^* f = 0$ again we conclude that

$$-\sum_{i=1}^{2n} y_i v_i(x) \nabla_x c(x) \eta = 0 \ .$$

Therefore, $v_i(x)\nabla_x c(x) = 0$ for all i, and since $\text{span}\{v_i\} = \mathbb{R}^3$, we conclude that $c(x) = \text{const}$. Since $f \in \mathcal{L}_2^\perp$, we obtain $f = 0$. The equation $M_\varepsilon f = 0$ is treated analogously. This completes the proof of Lemma 1.5.6.

Consider M_ε as an (unbounded) operator from $\mathcal{H}^\perp$ to $\mathcal{L}_2^\perp$ with domain $\mathcal{S}^\perp$

Lemma 1.5.7. *Under the assumptions of Theorem 1.5.1, the closure of $M_\varepsilon \mathcal{S}^\perp$ coincides with $\mathcal{L}_2^\perp$.*

Proof Suppose the contrary. Then there exists $f \in \mathcal{L}_2^\perp$, such that $f \neq 0$, and $(f, M_\varepsilon \phi) = 0$ for all $\phi \in \mathcal{S}^\perp$, and thus for all $\phi \in C_0^\infty$. Then $M_\varepsilon^* f = 0$. By

Lemma 1.5.6 f must be equal to zero, and thus we arrive at the contradiction. This completes the proof of Lemma 1.5.7.

Proof of Theorem 1.5.1 By Lemma 1.5.7 there exists a sequence $u_k \in \mathcal{S}^\perp$, such that $M_\varepsilon u_k = f_k \to f$. Then $\{f_k\}$ is a Cauchy sequence in $\mathcal{L}_2^\perp$, and by Lemma 1.5.4 $\{u_k\}$ is a Cauchy sequence in $\mathcal{H}^\perp$. Let $u \in \mathcal{H}^\perp$ be the limit of the sequence u_k. Clearly u is a weak solution of $M_\varepsilon u = f$. The estimate (1.48) holds for any pair u_k, f_k, and therefore it holds for the limits u and f. Note that $f \in C^\infty$ implies $u \in C^\infty$ by Hormander's principle.

Finally, if u_1 and u_2 are two weak solutions of (1.47), then $u = u_1 - u_2$ satisfies $M_\varepsilon u = 0$. Then $u = 0$ by Lemma 1.5.6. This completes the proof of Theorem 1.5.1

Theorem 1.5.1 proves existence, uniqueness, and regularity for equation (1.46), and consequently for equation (1.32). However the estimate (1.48) does not imply (1.33) since the constant in the RHS of (1.48) depends on n and ε. The following theorem provides the estimate of the $\mathcal{L}_2$ norm of the solution, which is uniform in n and ε. This estimate will allow us to prove (1.33).

Theorem 1.5.8. *Suppose Assumptions A and B hold, and $f \in \mathcal{S}^\perp$ satisfies*

$$\int f\eta dy = 0 \quad for\ all\ x\ , \quad and \quad \sum_{j=1}^{2n} \int \frac{1}{\Omega_j} v_j \nabla_x f y_j \eta dy = 0 \quad for\ all\ x\ . \tag{1.77}$$

Then there exists a constant C, which does not depend on n or ε, such that *the solution u of (1.47), given by Theorem 1.5.1 satisfies*

$$||u|| \leq C||f|| \ . \tag{1.78}$$

Proof We represent $u \in \mathcal{H}^{\perp}$ uniquely as a sum of two functions which are orthogonal in $\mathcal{L}_2(\mathbb{R}_y^{2n})$ for all x, that is

$$u(x,y) = w(x,y) + u_0(x)\eta(y) \ , \tag{1.79}$$

where $\int u_0(x)dx = 0$, and $\int w(x,y)\eta(y)dy = 0$ as an element of $\mathcal{L}_2(\mathbb{T}_p^3)$. To prove (1.78) it is sufficient to estimate $||w||$ and $||u_0||_{\mathcal{L}_2(\mathbb{T}_p^3)}$ separately.

By Lemma 1.5.7 there exists a sequence $u^k \in \mathcal{S}^{\perp}$, such that

$$M_\varepsilon u^k = f^k \to f \ \text{ in } \ \mathcal{L}_2^{\perp}. \tag{1.80}$$

Then $\{f^k\}$ is a Cauchy sequence in $\mathcal{L}_2^{\perp}$, and by (1.48)

$$u^k \to u \ \text{ in } \ \mathcal{H}^{\perp} \ . \tag{1.81}$$

Let

$$u^k = w^k + u_0^k\eta$$

$$f^k = g^k + f_0^k\eta$$

as in (1.79). Note that by (1.80), (1.81), and since $\int f\eta dy = 0$

$$w^k \to w \ \text{ in } \mathcal{H}^{\perp} \ ; \ \ u_0^k \to u_0 \ \text{ in } \mathcal{L}_2(\mathbb{T}_p^3) \ ;$$

$$g^k \to f \ \text{ in } \mathcal{H}^{\perp} \ ; \ \ f_0^k \to 0 \ \text{ in } \mathcal{L}_2(\mathbb{T}_p^3) \ .$$

The equation $M_\varepsilon u^k = f^k$ can be written as

$$Lw^k + \sqrt{\varepsilon} A u_0^k(x)\eta(y) + \sqrt{\varepsilon} A w^k = g^k + f_0^k \eta \ . \tag{1.82}$$

In order to estimate the norms of w and u_0 we first derive three integral and differential relations satisfied by w^k and u_0^k. In order to do so we multiply (1.82) successively by η, w^k, and $y_j\eta$ and integrate in y. We could not perform this integration with w and u_0 replacing w^k and u_0^k in (1.82), since w may not belong to $S^\perp$, and thus the integral over y may not converge. Then we let $k \to \infty$ in order to obtain three corresponding relations on w and u_0. The first two of these are used to bound w, while the third gives the bound on u_0.

Note that (1.82) is the same as (1.50), except for the extra term $f_0^k\eta$ on the RHS of (1.82). Thus, multiplying (1.82) successively by w^k, η, and $y_j\eta$, and repeating the arguments which led to (1.53), (1.56), and (1.60) we obtain the following three relations

$$\sqrt{\varepsilon} \int A w^k \eta dy = \frac{1}{p^3} f_0^k(x) \ . \tag{1.83}$$

$$\int\int w^k L w^k dx dy = \int\int g^k w^k dx dy + \frac{1}{p^3} \int f_0^k(x) u_0^k(x) dx \ . \tag{1.84}$$

$$\sum_{a,b=1}^{3} \sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j^a v_j^b u^k_{0 x_a x_b} = \tag{1.85}$$

$$\frac{p^3}{\sqrt{\varepsilon}} \sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla \left(\int g^k y_j \eta dy - \int y_j \eta L w^k dy - \sqrt{\varepsilon} \int y_j A w^k \eta dy \right) .$$

In order to obtain the integral relations of the type (1.83) and (1.84) for the limits w and u_0, we consider $k \to \infty$ in those formulas. In (1.83) the LHS tends to $\sqrt{\varepsilon}\sum_{i=1}^{2n} v_i \nabla_x \int w y_i \eta dy$ in $\mathcal{H}^{-1}(\mathbb{T}_p^3)$, while the RHS tends to zero in $\mathcal{H}^{-1}(\mathbb{T}_p^3)$ since $f_0^k \to 0$. We conclude that

$$\sum_{i=1}^{2n} v_i \nabla \int w y_i \eta dy = 0 \; . \tag{1.86}$$

From (1.84) we conclude that

$$\int\int wLwdxdy = \int\int fwdxdy \; . \tag{1.87}$$

Formulas (1.86) and (1.87) are the first two relations for w. Using (1.87), we bound $||w||$. By the elementary properties of the simple harmonic oscillator there exists a constant $C_1 > 0$, such that for $w \in \mathcal{H}^{\perp}$

$$||w||^2 \leq -C_1 \int\int wLwdxdy \; .$$

By Schwartz inequality we conclude from (1.87) that

$$||w|| \leq C_2||f|| \; . \tag{1.88}$$

Next we estimate u_0. We start by deriving our third relation, an elliptic equation which u_0 satisfies. In (1.85) the LHS tends in $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ to $\sum_{a,b=1}^{3}\sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j^a v_j^b u_{0x_a x_b}$ as $k \to \infty$. The first term on the RHS tends in $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ to $\frac{p^3}{\sqrt{\varepsilon}} \sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla(\int f y_j \eta dy)$. The latter quantity vanishes by the assumptions of the theorem.

Since $\int y_j\eta L w^k dy = -\Omega_j \int y_j \eta w^k dy$, the second term on the RHS of (1.85) tends in $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ to

$\frac{p^3}{\sqrt{\varepsilon}} \sum_{j=1}^{2n} v_j \nabla \int w y_j \eta dy$, which is equal to zero by (1.86).

The last term on the RHS of (1.85) tends in $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ to

$p^3 \sum_{i,j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla (v_i \nabla (\int y_i y_j w \eta dy))$. Thus (1.85) yields

$$\sum_{a,b=1}^{3} \sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j^a v_j^b u_{0 x_a x_b} = p^3 \sum_{i,j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla (v_i \nabla (\int y_i y_j w \eta dy)) \,. \qquad (1.89)$$

Since v_i is divergence free the RHS of (1.89) can be written as

$$p^3 \sum_{i,j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla (\mathrm{div}(v_i \int y_i y_j w \eta dy)) \,.$$

We have the following inequality

$$|| \sum_{i,j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla \mathrm{div}(v_i \int y_i y_j w \eta dy) ||_{\mathcal{H}^{-2}(\mathbb{T}_p^3)} \leq$$

$$C_3 \sum_{i,j=1}^{2n} \frac{\sup_x |v_j| \sup_x |v_i|}{\Omega_j} || \int y_i y_j w \eta dy ||_{\mathcal{L}_2(\mathbb{T}_p^3)} \,. \qquad (1.90)$$

Using the Schwartz inequality and the fact that the Ω_j are uniformly bounded from below, we estimate the RHS of (1.90) as follows

$$\sum_{i,j=1}^{2n} \frac{\sup_x |v_j| \sup_x |v_i|}{\Omega_j} || \int y_i y_j w \eta dy ||_{\mathcal{L}_2(\mathbb{T}_p^3)} \leq$$

$$C_4 p^{-3/2} (\sum_{i,j=1}^{2n} (\sup_x |v_i|)^2 (\sup_x |v_j|)^2)^{1/2} (\sum_{i,j=1}^{2n} || \int p^{3/2} y_i y_j \eta w dy ||^2_{\mathcal{L}_2(\mathbb{T}_p^3)})^{1/2} \,. \qquad (1.91)$$

By (1.21) $(\sup_x |v_i|)^2 = 2\lambda$, where λ is one of the eigenvalues of the matrix M_i^n, and therefore by (1.16)

$$\sum_{i=1}^{2n} (\sup_x |v_i|)^2 < C_5 \ . \tag{1.92}$$

Therefore the factor $\sum_{i,j=1}^{2n} (\sup_x |v_i|)^2 (\sup_x |v_j|)^2$ is bounded uniformly in n. To estimate the second factor on the RHS of (1.91) we note that

$$\int p^{3/2} y_i y_j \eta w dy = \int p^{3/2} (y_i y_j - \delta_{ij}) \eta w dy \ ,$$

since $\int w\eta dy = 0$. The functions $p^{3/2}(y_i y_j - \delta_{ij})\eta$ form an orthogonal system in $\mathcal{L}_2(\mathbb{R}^{2n})$. Moreover,

$$||p^{3/2}(y_i y_j - \delta_{ij})\eta||^2_{\mathcal{L}_2(\mathbb{R}^{2n})} = 1 + \delta_{ij} \ .$$

Therefore

$$\sum_{i,j=1}^{2n} || \int p^{3/2} y_i y_j w \eta dy ||^2_{\mathcal{L}_2(\mathbb{T}_p^3)} = \int \sum_{i,j=1}^{2n} (\int p^{3/2} (y_i y_j - \delta_{ij}) w \eta dy)^2 dx \leq$$

$$2 \int \int w^2 dx dy \leq C_6 ||f||^2 \ . \tag{1.93}$$

The last inequality in (1.93) is due to (1.88).

From the chain of inequalities (1.90) - (1.93) we conclude that the $\mathcal{H}^{-2}(\mathbb{T}_p^3)$ norm of the RHS of (1.89) is estimated from above by $C_7 p^{3/2} ||f||$. In the view of (1.62) and (1.17) we can apply Lemma 1.5.3 to equation (1.89) and obtain

$$||u_0||_{\mathcal{L}_2(\mathbb{T}_p^3)} \leq C_8 p^{3/2} ||f|| \ .$$

Therefore

$$||u_0\eta|| \leq C_9||f||. \tag{1.94}$$

Combining (1.94) with (1.88) we obtain (1.78), which completes the proof of Theorem 1.5.8.

Corollary 1.5.9. *Theorem 1.4.1 is a consequence of Theorems 1.5.1 and 1.5.8.*

Proof With the change of variables (1.45) equation (1.32) takes the form (1.46). The existence and uniqueness result of Theorem 1.4.1 follows immediately from Theorem 1.5.1 applied to equation (1.46).

We transform the LHS of (1.33) by the change of variables (1.45). Thus the LHS of (1.33) is equal to

$$\iint\limits_{\mathbb{T}_p^3\times\mathbb{R}^{2n}} \left((u^{n,a})^2 + \sum_{i=1}^{2n} u^{n,a} v_i^a y_i \eta\right) dxdy \ , \tag{1.95}$$

where $u^{n,a}$ is the solution of equation (1.46).

We estimate the norm of the RHS of (1.46) as follows

$$||\sum_{i=1}^{2n} y_i v_i^a \eta||^2 = \int\int (\sum_{i=1}^{2n} y_i v_i^a \eta)^2 dxdy = \frac{1}{p^3}\sum_{i=1}^{2n}\int (v_i^a)^2 dx \leq$$

$$\sum_{i=1}^{2n} \sup_x (v_i^a)^2 \leq C_1 \ . \tag{1.96}$$

The last inequality in (1.96) is due to (1.92). From (1.96) and Schwartz inequality it follows that the expression in (1.95) is estimated from above by

$$C_2 \iint\limits_{\mathbb{T}_p^3\times\mathbb{R}^{2n}} (u^{n,a})^2 dx dy \ .$$

To complete the proof of Theorem 1.4.1 it remains to apply Theorem 1.5.8 to equation (1.46). In order to do that we need to show that $f = \sum_{i=1}^{2n} y_i v_i^a \eta$ satisfies (1.77). The first part, $\int f\eta dy = 0$, is trivial since $\int y_i \eta^2 dy = 0$. To show that $\sum_{j=1}^{2n} \int \frac{1}{\Omega_j} v_j \nabla_x f y_j \eta dy = 0$ we write

$$\sum_{j=1}^{2n} \int \frac{1}{\Omega_j} v_j \nabla_x f y_j \eta dy = \sum_{i,j=1}^{2n} \int \frac{1}{\Omega_j} v_j \nabla_x v_i^a y_i y_j \eta^2 dy =$$

$$\frac{1}{p^3} \sum_{j=1}^{2n} \frac{1}{\Omega_j} v_j \nabla_x v_j^a = 0 \ . \qquad (1.97)$$

The second equality in (1.97) is due to (1.31), and the last one is due to (1.23). This completes the proof of Corollary 1.5.9.

1.6 Proof of Theorem 1.3.1

In order to prove Theorem 1.3.1 we need a number of results from [13, 4, 2].

First we show that the RHS of (1.1) is smooth enough for some modification of the field $V(x,t)$, so that we can solve (1.1) for almost every realization of V. In Lemmas 1.6.1, 1.6.5, and Corollary 1.6.2 we state the general results on the regularity and on the behavior at infinity of a typical realization of a Gaussian field, whose correlation function is sufficiently regular. It follows from Lemma 1.6.6 that our vector field $V(x)$ indeed satisfies the assumptions of Corollary 1.6.2 and Lemma 1.6.5, and thus we can solve (1.1) for almost every realization of V.

Note that the Gaussian fields $V^n(x)$ are defined in Section 1.3 through their correlation matrices, and the underlining probability space for V^n may be different from the one for V. The argument which proves the convergence of the displacement tensors (1.25) is based Skorohod Theorem, which allows us to realize the random fields V and V^n on the same probability space.

Let $f(x,t)$ be a scalar, vector, or tensor valued function on $\mathbb{R}^4$. Let $K \subset \mathbb{R}^4$ be a compact set, and let $\alpha > 0$. We shall say that $f(x,t) \in \mathcal{H}^\alpha(K)$ if there exist positive constants C_1 and C_2 such that for $(x_1,t_1), (x_2,t_2) \in K$

$$|f(x_1,t_1) - f(s_2,t_2)| \leq C_1 |(x_1 - x_2, t_1 - t_2)|^\alpha ,$$

whenever $|(x_1 - x_2, t_1 - t_2)| \leq C_2$.

We shall say that $f(x,t) \in \mathcal{H}^\alpha$ if $f(x,t) \in \mathcal{H}^\alpha(K)$ for every compact set K.

Lemma 1.6.1. *([13]) Let ϕ be a Gaussian stationary random field (possibly vector or matrix valued), whose correlation tensor belongs to $\mathcal{H}^\alpha$. Then there is a modification of ϕ, whose almost every realization belongs to $\mathcal{H}^{\alpha/2-\varepsilon}$ for every $\varepsilon > 0$.*

Corollary 1.6.2. *Let $g^{ab}(x,t)$ be the correlation matrix of a Gaussian stationary vector field $v(x,t)$. Suppose that for every partial differential operator D_x of order not greater than two with constant coefficients the following holds*

$$D_x g^{ab}(x,t) \in \mathcal{H}^{\alpha} \ .$$

Then there exists a modification of v, such that almost every realization of the vector field belongs to $\mathcal{H}^{\alpha/2-\epsilon}$ together with the first order partial derivatives in x.

For a compact set $K \subset \mathbb{R}^4$ we define $C(K)$ to be the set of all continuous $\mathbb{R}^3$ valued functions on K. $C(K)$ is endowed with the usual Borel σ-algebra. It is assumed that $0 \in K$. We state several lemmas, which will be used in the proof of Theorem 1.3.1. Lemmas 1.6.3 and 1.6.4 are contained in Chapter 2 of Billingsley's book [2], while Lemma 1.6.5 follows from Lemma 4.5 and Proposition 2.5 of Collela and Lanford [4].

Lemma 1.6.3. *([2]) Let $v^n(x,t)$ and $v(x,t)$ be continuous random fields on a compact set $K \subset \mathbb{R}^4$. If finite dimensional distributions of v^n converge weakly to those of v and if the sequence v^n is tight on $C(K)$, then v^n converge weakly to v as measures on $C(K)$.*

Lemma 1.6.4. *([2]) The sequence v^n is tight on $C(K)$ if and only if these two conditions hold*

(a) For each positive η, there exists an a such that

$$P\{v^n : |v^n(0,0)| > a\} \leq \eta \ , \quad n \geq 1 \ . \tag{1.98}$$

(b) For each positive ε and η, there exist a δ, with $0 < \delta < 1$, and an integer n_0 such that

$$P\{v^n : \sup_{|(x_1-x_2,t_1-t_2)|<\delta} |v^n(x_1,t_1) - v^n(x_2,t_2)| \geq \varepsilon\} \leq \eta \ , \quad n \geq n_0 \ . \tag{1.99}$$

Lemma 1.6.5. *([4]) Suppose that $v(x,t)$ is a Gaussian stationary random vector field on $\mathbb{R}^4_{(x,t)}$ with continuous realizations, such that its correlation matrix $g^{ab}(x,t)$ is continuous. Suppose there exist constants $C_1, C_2, \alpha > 0$ such that*

(a) $|g^{ab}(0,0)| < C_1$

(b) $|g^{ab}(x,t) - g^{ab}(0,0)| \leq C_2|(x,t)|^\alpha$ *for* $|(x,t)| \leq \frac{1}{2}$.

Then for every $T_0, \gamma > 0$ there exist constants k_1 and k_2, which depend only on C_1, C_2, α, T_0 and γ such that

$$P\{v : \sup_{x > \varepsilon^{-\gamma}k_1; t \leq T_0} \frac{|v(x,t)|}{\sqrt{\log|x|}} > k_2\} < \varepsilon \quad \text{for all} \quad \varepsilon \leq 1 \ . \tag{1.100}$$

Moreover, for every compact set $K \subset \mathbb{R}^4$ and for each positive ε and η there exists a δ, which depends only on $C_1, C_2, \alpha, K, \varepsilon$ and η such that

$$P\{v : \sup_{|(x_1-x_2,t_1-t_2)|<\delta} |v(x_1,t_1) - v(x_2,t_2)| \geq \varepsilon\} \leq \eta \ .$$

We next show that the vector field $V(x,t)$ satisfies the assumptions of Corollary 1.6.2, and $V^n(x,t)$ satisfy the assumptions of Lemmas 1.6.4 and 1.6.5. Note that due to (1.20) almost all the realizations of $V^n(x,t)$ belong to $\mathcal{H}^\alpha$ together with the first order partial derivatives in x for some $\alpha > 0$.

Lemma 1.6.6. *Suppose Assumptions A and B hold. Then*

(a) The correlation matrix $G^{ab}(x,t)$ *of the vector field* $V(x,t)$ *satisfies the assumptions of Corollary 1.6.2 with some* $\alpha > 0$.

(b) For every compact set K *the vector fields* $V^n(x,t)$ *satisfy the assumptions of Lemma 1.6.4.*

(c) The correlation matrices G^{ab} *and* $G^{n,ab}$ *satisfy the assumptions of Lemma 1.6.5 with* C_1, C_2 *and* α *independent of* n.

(d) For every compact set K $G^{n,ab} \to G^{ab}$ *uniformly on* K.

Proof Note the following

$$|e^{-|t_1|\Omega(k)} - e^{-|t_2|\Omega(k)}| \leq |t_1 - t_2|^{\alpha}\Omega(k)^{\alpha} \ , \tag{1.101}$$

for all $t_1, t_2, k,$ and $0 \leq \alpha \leq 1$.

$$|e^{ikx_1} - e^{ikx_2}| \leq 2|x_1 - x_2|^{\alpha}k^{\alpha} \ , \text{ for all } x_1, x_2, k, \text{ and } 0 \leq \alpha \leq 1 \ . \tag{1.102}$$

We shall use (1.101) and (1.102) without reference. Note that if D_x is of order not greater than two, then

$$D_x G^{ab}(x,t) = \int p(k) e^{-|t|\Omega(k)} e^{ikx} M^{ab}(dk) \ ,$$

where $p(k)$ is a polynomial of degree not greater than two. Thus,

$$|D_x G^{ab}(x_1,t) - D_x G^{ab}(x_2,t)| =$$

$$|\int p(k) e^{-|t|\Omega(k)} (e^{ikx_1} - e^{ikx_2}) M^{ab}(dk)| \leq \tag{1.103}$$

$$2|x_1 - x_2|^\delta \int |p(k)|e^{-|t|\Omega(k)}|k|^\delta |M^{ab}|(dk) \ .$$

The integral on the RHS of (1.103) converges by Assumption B, and thus

$$|D_x G^{ab}(x_1,t) - D_x G^{ab}(x_2,t)| \leq c_1 |x_1 - x_2|^\delta \ . \tag{1.104}$$

Similarly

$$\begin{aligned} &|D_x G^{ab}(x,t_1) - D_x G^{ab}(x,t_2)| = \\ &|\int p(k)(e^{-|t_1|\Omega(k)} - e^{-|t_2|\Omega(k)})e^{ikx} M^{ab}(dk)| \leq \\ &|t_1 - t_2|^\gamma \int |p(k)|(\Omega(k))^\gamma |M^{ab}|(dk) \ . \end{aligned} \tag{1.105}$$

For γ small enough the integral on the RHS of (1.105) converges by Assumption B, and thus

$$|D_x G^{ab}(x,t_1) - D_x G^{ab}(x,t_2)| \leq c_2 |t_1 - t_2|^\gamma \ . \tag{1.106}$$

By (1.104) and (1.106)

$$|D_x G^{ab}(x_1,t_1) - D_x G^{ab}(x_2,t_2)| \leq c_3 |(x_1,t_1) - (x_2,t_2)|^\alpha \tag{1.107}$$

with $\alpha = \min\{\delta, \gamma\}$. Therefore the function $D_x G^{ab}$ belongs to $\mathcal{H}^\alpha$ with constants C_1 and C_2 independent of the compact set K. This completes the proof of part (a) of Lemma 1.6.6.

The arguments leading to (1.107) can be applied to $G^{n,ab}$ instead of G^{ab}. The only difference is that for $G^{n,ab}$ one needs to use (1.16) instead of (1.12). Thus

$$|G^{n,ab}(x_1,t_1) - G^{n,ab}(x_2,t_2)| \leq c_4 |(x_1,t_1) - (x_2,t_2)|^\alpha \ , \tag{1.108}$$

where c_4 does not depend on n. Recall that

$$G^{n,ab}(x,t) = \int e^{-|t|\Omega(k)} e^{ikx} M^{n,ab}(dk) \ .$$

Therefore, by (1.16)

$$|G^{n,ab}(0,0)| \le \int |M^{n,ab}|(dk) \le c_5 \ , \tag{1.109}$$

where c_5 does not depend on n. The assumptions of Lemma 1.6.5 are satisfied by (1.108) and (1.109). This completes the proof of part (c) of Lemma 1.6.6.

The relation (1.98) for the fields V^n follows from (1.109) by Chebyshev inequality. The relation (1.99) is a consequence of Lemma 1.6.5 applied to V^n. This completes the proof of part (b) of Lemma 1.6.6. It remains to prove part (d).

Let a compact $K \subset \mathbb{R}^4$ be given. We also fix a bounded set

$$M_N = \{1/N < |k^a| < N, a = 1, ..., 3\} \subset \mathbb{R}^3$$

in Fourier space. Recall from Section 1.3 that α_i, β_i, γ_i, and δ_i are the sets which consist of the interior, the faces, the edges, and the vertices of the cube Δ_i, respectively. We introduce the following notation, for any function $\phi(k)$ and measure μ

$$\int_{\Delta_i}' \phi(k)\mu(dk) = \int_{\alpha_i} \phi(k)\mu(dk) + \frac{1}{2}\int_{\beta_i} \phi(k)\mu(dk) +$$

$$\frac{1}{4}\int_{\gamma_i} \phi(k)\mu(dk) + \frac{1}{6}\int_{\delta_i} \phi(k)\mu(dk) \ .$$

Then,

$$|G^{ab}(x,t) - G^{n,ab}(x,t)| \leq$$

$$|\sum_{\{i,k_i \in M_N\}} \int_{\Delta_i}' e^{-|t|\Omega(k)} e^{ikx} (M^{ab}(dk) - M^{n,ab}(dk))| +$$

$$\int_{k \notin M_N} |e^{-|t|\Omega(k)} e^{ikx}||M^{ab}|(dk) + \tag{1.110}$$

$$\int_{k \notin M_N} |e^{-|t|\Omega(k)} e^{ikx}||M^{n,ab}|(dk) \ .$$

By (1.12), (1.10), and (1.16) the last two terms on the RHS of (1.110) can be made arbitrarily small uniformly in n, t and x by selecting N large enough. In order to estimate the first term we write

$$|\int_{\Delta_i}' e^{-|t|\Omega(k)} e^{ikx} (M^{ab}(dk) - M^{n,ab}(dk))| \leq$$

$$|\int_{\Delta_i}' (e^{-|t|\Omega(k)} - e^{-|t|\Omega(k_i)}) e^{ikx} M^{ab}(dk)| + \tag{1.111}$$

$$|\int_{\Delta_i}' e^{-|t|\Omega(k_i)} e^{ikx} (M^{ab}(dk) - M_i^{n,ab} \delta(k_i) dk)| \ .$$

The first term on the RHS of (1.111) is estimated from above by

$$c_1(N) \int_{\Delta_i}' |(\Omega(k) - \Omega(k_i))||M^{ab}|(dk) \ .$$

Since $\Omega(k)$ is Lipschitz continuous on M_N this is estimated from above by

$$c_2(N)|\Delta_i|\int_{\Delta_i}{}' |M^{ab}|(dk) \ , \tag{1.112}$$

where $|\Delta_i|$ is the length of the diagonal of Δ_i. The second term on the RHS of (1.111) is estimated from above by

$$|\int_{\Delta_i}{}' e^{ikx}(M^{ab}(dk) - N_i^{n,ab}\delta(k_i)dk)| + |N_i^{n,ab} - M_i^{n,ab}| \ . \tag{1.113}$$

The first term in (1.113) is estimated by

$$\sup_{k\in\Delta_i}\{|e^{ikx} - e^{ik_ix}|\}\int_{\Delta_i}{}' |M^{ab}|(dk) \leq c_3(K,N)|\Delta_i|\int_{\Delta_i}{}' |M^{ab}|(dk) \ .$$

From the definition of $N_i^{n,ab}$ and $M_i^{n,ab}$ it is readily seen that the second term in (1.113) is also estimated by

$$c_4(K,N)|\Delta_i|\int_{\Delta_i}{}' |M^{ab}|(dk) \ .$$

Thus the first term on the RHS of (1.110) is estimated from above by

$$c_5(K,N)\max_i|\Delta_i|\int |M^{ab}|(dk) \ . \tag{1.114}$$

Due to (1.12), since $\max_i|\Delta_i|$ tends to zero as $n \to \infty$, the quantity in (1.114) tends to zero as $n \to \infty$. Therefore

$$\lim_{n\to\infty}\sup_{(x,t)\in K}|G^{n,ab}(x,t) - G^{ab}(x,t)| = 0 \ .$$

This completes the proof of Lemma 1.6.6.

Proof of Theorem 1.3.1 Without loss of generality we may consider $X_0 = X_0^n = 0$. Let $t = T_0$ be fixed. By Lemma 1.6.5 there exist constants $k_1 > 1$

and k_2 such that (1.100) holds for $v = V$ and $v = V_n$ for every n. Let $R_\varepsilon = 2\varepsilon^{-\frac{1}{4}} k_1 e^{k_2 T_0}$ and $K_\varepsilon = \{|x| \leq R_\varepsilon\} \times [0, T_0]$.

The finite dimensional distributions of V^n converge weakly to those of V, since V^n and V are Gaussian, and the correlation functions converge pointwise (Lemma 1.6.6, part (d)). By Lemma 1.6.4 the sequence V^n is tight on $C(K_\varepsilon)$, and therefore, by Lemma 1.6.3 V^n converges weakly to V on $C(K_\varepsilon)$. By Skorohod Theorem ([5]) there exist vector fields W_ε^n and W_ε, which induce the same measures on $C(K_\varepsilon)$ as V^n and V, and which are defined on the same probability space Ω_ε with $W_\varepsilon^n \to W_\varepsilon$ in $C(K_\varepsilon)$ for almost all $\omega \in \Omega_\varepsilon$. By Corollary 1.6.2 the realizations of the vector fields W_ε^n and W_ε belong to $\mathcal{H}^\alpha(K_\varepsilon)$ together with their first order partial derivatives in x. We do not need to take the modifications since we know a priori that almost all the realizations belong to $C(K_\varepsilon)$.

Define $Y_{\varepsilon,s}$ and $Y_{\varepsilon,s}^n$ to be the solutions of the equations

$$\dot{Y}_{\varepsilon,s} = W_\varepsilon(Y_{\varepsilon,s}, s) \ , Y_{\varepsilon,0} = 0 \ . \tag{1.115}$$

$$\dot{Y}_{\varepsilon,s}^n = W_\varepsilon^n(Y_{\varepsilon,s}^n, s) \ , Y_{\varepsilon,0}^n = 0 \ . \tag{1.116}$$

Since (1.100) holds for $v = W_\varepsilon$, on a subset of Ω_ε of measure not less than $1 - \varepsilon$ the norm of $Y_{\varepsilon,s}$ is estimated as follows

$$|Y_{\varepsilon,s}| \leq y_s \ , s \leq T_0 \qquad \text{on } \Omega_\varepsilon^0 \ , \mu(\Omega_\varepsilon^0) \geq 1 - \varepsilon \ ,$$

where y_s is the solution of the equation

$$\dot{y}_s = k_2 \sqrt{\log y_s} \ , y_0 = \varepsilon^{-\frac{1}{4}} k_1 \ .$$

Clearly

$$\sup_{s\le T_0} y_s \le \frac{R_\varepsilon}{2} ,$$

and therefore

$$\sup_{s\le T_0} |Y_{\varepsilon,s}| \le \frac{R_\varepsilon}{2} . \tag{1.117}$$

This shows that for $\omega \in \Omega_\varepsilon^0$ equation (1.115) can be solved for $0 \le s \le T_0$, and that

$$P\{\sup_{s\le T_0} |Y_{\varepsilon,s}|^2 > \frac{R_\varepsilon^2}{4}\} \le \varepsilon . \tag{1.118}$$

Since Y_{ε,T_0} and X_{T_0} have the same distributions when they are restricted to the events $\{\sup_{s\le T_0} |Y_{\varepsilon,s}| < \frac{R_\varepsilon}{2}\}$ and $\{\sup_{s\le T_0} |X_s| < \frac{R_\varepsilon}{2}\}$ respectively, we conclude from (1.118) that

$$P\{\sup_{s\le T_0} |X_s|^2 > \frac{R_\varepsilon^2}{4}\} \le \varepsilon . \tag{1.119}$$

Therefore $|X_{T_0}|^2$ is integrable. Since (1.100) holds for $v = W_\varepsilon^n$ we can repeat the arguments leading to (1.119) to conclude that

$$P\{\sup_{s\le T_0} |X_s^n|^2 > \frac{R_\varepsilon^2}{4}\} \le \varepsilon ,$$

which shows that $|X_{T_0}^n|^2$ are uniformly integrable.

Since $X_0 = X_0^n = 0$, in order to complete the proof of Theorem 1.3.1 we need to show that the difference

$$E(X_t^{n,a} X_t^{n,b}) - E(X_t^a X_t^b) \tag{1.120}$$

tends to zero as $n \to \infty$. Let an arbitrary $\gamma > 0$ be given. Since $|X_{T_0}^n|^2$ and $|X_{T_0}|^2$ are uniformly integrable, we can select $\varepsilon > 0$ such that for every measurable set A, with $P(A) < 2\varepsilon$,

$$E|X_{T_0}^n|^2 \chi_A < \gamma \; ; \; E|X_{T_0}|^2 \chi_A < \gamma \, . \tag{1.121}$$

We next show that $\sup_{s \leq T_0} |Y_{\varepsilon,s}^n - Y_{\varepsilon,s}|$ tends to zero almost surely on Ω_ε^0. By (1.115) and (1.116)

$$|\dot{Y}_{\varepsilon,s}^n - \dot{Y}_{\varepsilon,s}| \leq |Y_{\varepsilon,s}^n - Y_{\varepsilon,s}| \sup_{(x,s') \in K_\varepsilon} |D_x W_\varepsilon(x,s')| +$$

$$\sup_{(x,s') \in K_\varepsilon} |W_\varepsilon(x,s') - W_\varepsilon^n(x,s')| \, , \tag{1.122}$$

whenever $\sup_{s' \leq s} |Y_{\varepsilon,s'}^n| < R_\varepsilon$. Since the second term on the RHS of (1.122) tends to zero as $n \to \infty$, and $\sup_{s \leq T_0} |Y_{\varepsilon,s}| \leq \frac{R_\varepsilon}{2}$ on Ω_ε^0 by (1.117), we conclude that $\sup_{s \leq T_0} |Y_{\varepsilon,s}^n| < R_\varepsilon$ on Ω_ε^0 for $n > n(\omega)$,
and $\sup_{s \leq T_0} |Y_{\varepsilon,s}^n - Y_{\varepsilon,s}| \to 0$ almost surely on Ω_ε^0. Therefore there exists a set $\Omega_\varepsilon^1 \subset \Omega_\varepsilon^0$ with $\mu(\Omega_\varepsilon^1) \geq 1 - 2\varepsilon$, such that

$$\sup_{s \leq T_0} |Y_{\varepsilon,s}^n - Y_{\varepsilon,s}| \to 0 \quad \text{uniformly on } \Omega_\varepsilon^1 \, .$$

Since for each n the variables $Y_{\varepsilon,s}^n - Y_{\varepsilon,s}$ and $X_s^n - X_s$ have the same distributions when restricted to the events $\{\sup_{s \leq T_0} |Y_{\varepsilon,s}^n|, |Y_{\varepsilon,s}| \leq R_\varepsilon\}$ and $\{\sup_{s \leq T_0} |X_s^n|, |X_s| \leq R_\varepsilon\}$ respectively, and since the constant γ in (1.121) was arbitrary, we conclude that (1.120) tends to zero as $n \to \infty$. This completes the proof of Theorem 1.3.1.

Chapter 2

Vector Fields with Short Time Correlations

2.1 Effective Diffusivity in Vector Fields with Short Time Correlations

This section is devoted to the proof of Theorem 1.2.2.

Let $\mathcal{P}$ be the space of functions on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$ which have the form

$$u(x, y) = p(x, y)\eta(y) \ ,$$

where $p(x, y)$ is a polynomial in y whose coefficients are infinitely smooth functions of x.

Let $\mathcal{P}^{\perp}$ be the subspace of $\mathcal{P}$ of functions which satisfy

$$\int u(x, y)\eta(y)dy = 0 \quad \text{for all } x \ .$$

We shall say that $u(x,y)$ is an odd (even) function if $u \in \mathcal{P}$ and the corresponding polynomial $p(x,y)$ contains only odd (even) powers of y. The next lemma follows from the general properties of the simple harmonic oscillator [10].

Lemma 2.1.1. *Suppose that $f(x,y) \in \mathcal{P}^{\perp}$. Then the equation*

$$Lu = f$$

has a unique solution $u \in \mathcal{P}^{\perp}$. If f is an odd (even) function, then u is also odd (even).

In the next theorem we state the asymptotic expansion for the solution of equation (1.46), which in turn provides the asymptotics for the effective diffusivity $D_{\varepsilon}^{n,ab}$ by Corollary 1.4.4.

Theorem 2.1.2. *Suppose Assumptions A, B, and C_m hold. Let $u^{n,a}$ be the solution of equation (1.46) given by Theorem 1.5.1. Then there exist functions $u_k^{n,a}(x,y) \in \mathcal{H}^{\perp}$ with $k = 0, 1, ..., 2m$, and constants $c(k)$ independent of n and ε, such that*

$$||u^{n,a} - (u_0^{n,a} + \varepsilon^{\frac{1}{2}} u_1^{n,a} + ... + \varepsilon^{\frac{k}{2}} u_k^{n,a})|| \leq c(k)\varepsilon^{\frac{k+1}{2}} \ . \tag{2.1}$$

Moreover, $u_{2k}^{n,a}$ is an odd function and $u_{2k-1}^{n,a}$ is an even function for all $k \leq 2m$.

Let us substitute the series $\sum_{k=0}^{\infty} \varepsilon^{\frac{k}{2}} u_k^{n,a}$ formally into the equation $M_\varepsilon u = f$. Equating the terms with the same powers of ε, we obtain

$$Lu_0^{n,a} = f \ , \tag{2.2}$$

$$Lu_{k+1}^{n,a} = -Au_k^{n,a} \ . \tag{2.3}$$

By Lemma 2.1.1 in order for the solutions of (2.2), (2.3) to exist in $\mathcal{P}^\perp$ it is enough to show that the right sides are in $\mathcal{P}^\perp$. Unfortunately, the fact that $u \in \mathcal{P}^\perp$ does not guarantee that $-Au \in \mathcal{P}^\perp$. We use the particular form of the vector fields v_i to describe the subspaces of $\mathcal{P}^\perp$ appropriate for solving (2.2), (2.3). By staying within these subspaces we verify $-Au_k^{n,a} \in \mathcal{P}^\perp$. Then we use (2.2), (2.3) as an inductive definition of the sequence $\{u_k^{n,a}\}$.

From (2.2), (2.3) it follows that $w_k = u^{n,a} - (u_0^{n,a} + ... + \varepsilon^{\frac{k}{2}} u_k^{n,a})$ satisfies

$$M_\varepsilon w_k = -\varepsilon^{\frac{k+1}{2}} Au_k^{n,a} \ . \tag{2.4}$$

We then estimate the RHS of (2.4) uniformly in n and employ Theorem 1.5.8 to obtain the desired estimate of w_k. These arguments will be made rigorous below.

Let us introduce notation needed for the proof of Theorem 2.1.2. To simplify the notation we consider $a = 1$ and drop the superscripts on the function u and the terms of the asymptotic expansion. Recall that the set $\{v_i, \ i = 1, ..., n\}$ can be written as follows:

$$\{v_i, \ i = 1, ..., 2n\} = \{\sqrt{2\lambda_i^1} e_i^1 \cos k_i x, \sqrt{2\lambda_i^1} e_i^1 \sin k_i x,$$

$$\sqrt{2\lambda_i^2} e_i^2 \cos k_i x, \sqrt{2\lambda_i^2} e_i^2 \sin k_i x, \ i = 1, ..., n/2\} \ . \tag{2.5}$$

On the set $\{i = 1, ..., 2n\}$ we introduce a reflection operation which interchanges sin and cos terms of the same wave length as follows: if $v_i =$

$\sqrt{2\lambda_i^m} e_i^m \cos k_i x$, then i' is the index for which $v_{i'} = \sqrt{2\lambda_i^m} e_i^m \sin k_i x$. Similarly, if $v_i = \sqrt{2\lambda_i^m} e_i^m \sin k_i x$, then $v_{i'} = \sqrt{2\lambda_i^m} e_i^m \cos k_i x$. Recall that

$$\Omega_i = \Omega_{i'} \ . \tag{2.6}$$

In what follows the constants $c(i_1, ..., i_k)$ are assumed to be invariant under this reflection operation:

$$c(i_1, ..., i_l, ..., i_k) = c(i_1, ..., i'_l, ..., i_k) \quad \text{for any } 1 \leq l \leq k \ . \tag{2.7}$$

We next describe the subspaces $\mathcal{R}^k$ of $\mathcal{P}^\perp$ appropriate for solving (2.2), (2.3). $\mathcal{R}^k$ will be defined as a linear space of functions of (n, x, y). Note that we include the dependence on n in the definition of the space $\mathcal{R}^k$. Thus we shall be solving (2.2), (2.3) for all n simultaneously. First we define the set of functions which span $\mathcal{R}^k$.

We shall say that a function $u(n; x, y)$ belongs to $\mathcal{T}^k$ if for some t and s, such that $t + 2s = k$

$$u = \sum_{i_1,...,i_k=1}^{2n} v_{i_k} \nabla(...\nabla(v_{i_2} \nabla v_{i_1}^1)...) y_{i_{r_1}} ... y_{i_{r_t}} \delta_{i_{l_1} i_{m_1}} ... \delta_{i_{l_s} i_{m_s}} c_n(i_1, ..., i_k) \eta \ , \tag{2.8}$$

where $(r_1, ..., r_t, l_1, ..., l_s, m_1, ..., m_s)$ is some permutation of $(1, ..., k)$, the constants $c_n(i_1, ..., i_k)$ satisfy (2.7) and

$$|c_n| \leq c \ , \tag{2.9}$$

with a constant c which may depend on u, but does not depend on n. Note that $\mathcal{T}^k$ is not a linear space since it is a set theoretic union over choices t, s, and the permutations, and thus is not closed under addition.

We shall say that a function $u(n,x,y)$ belongs to $\mathcal{R}^k$ if $u = u_1 + ... + u_d$, where $u_1, ..., u_d \in \mathcal{T}^k$. Thus $\mathcal{R}^k = \text{span}\{\mathcal{T}^k\}$. Note that the number d of functions of $\mathcal{T}^k$ which comprise an element of $\mathcal{R}^k$ can not depend on n, since n is already an argument in each of the functions $u, u_1, ..., u_d$. An element of $\mathcal{R}^k$ can combine terms of the form (2.8) with different t and s. If k is even, then all elements of $\mathcal{R}^k$ are even, if k is odd, then all elements of $\mathcal{R}^k$ are odd.

We prove several lemmas which are needed for the proof of Theorem 2.1.2. Lemmas 2.1.3, 2.1.4, and 2.1.5 imply that (2.2), (2.3) can be used as an inductive definition for the sequence $u_k^{n,a}$. Lemma 2.1.6 provides the estimate of the RHS of (2.4), which is uniform in n.

Lemma 2.1.3. *Suppose Assumptions A and B hold. If $u \in \mathcal{R}^k$, then $Au \in \mathcal{R}^{k+1}$, and $\sum_{j=1}^{2n} \frac{1}{\Omega_j} y_j v_j \nabla_x u \in \mathcal{R}^{k+1}$.*

Proof The statement follows from the fact that the same is true when $\mathcal{R}^k$ and $\mathcal{R}^{k+1}$ are replaced by $\mathcal{T}^k$ and $\mathcal{T}^{k+1}$.

Lemma 2.1.4. *Suppose Assumptions A and B hold. If $u \in \mathcal{R}^k$, then $u \in \mathcal{P}^\perp$.*

Proof Without loss of generality we can consider $u \in \mathcal{T}^k$ given by formula (2.8). If k is odd, then u is an odd function and $\int u\eta dy = 0$, that is $u \in \mathcal{P}^\perp$. Assuming now that k is even, $\int u\eta dy$ is equal to a sum of the terms, each of which has the following form:

$$\sum_{i_1,...,i_k=1}^{2n} v_{i_k} \nabla(...\nabla(v_{i_2} \nabla v_{i_1}^1)...) \delta_{i_{l_1} i_{m_1}} ... \delta_{i_{l_{k/2}} i_{m_{k/2}}} c_n(i_1, ..., i_k) \ , \qquad (2.10)$$

where $(l_1, ..., l_{k/2}, m_1, ..., m_{k/2})$ is a permutation of $(1, ..., k)$.

From (1.22) we derive the following: if $D_1^{\alpha_1}$ and $D_2^{\alpha_2}$ are (tensor valued) homogeneous partial differential operators of orders α_1 and α_2 respectively, then

$$D_1^{\alpha_1} v_i \otimes D_2^{\alpha_2} v_i + D_1^{\alpha_1} v_{i'} \otimes D_2^{\alpha_2} v_{i'} = 0 \ , \tag{2.11}$$

provided $\alpha_1 + \alpha_2$ is odd. Here product $\otimes$ denotes either a tensor product, or a convolution in one or more indices. From (2.11) we conclude that

$$\sum_{i=1}^{2n} D_1^{\alpha_1} v_i \otimes D_2^{\alpha_2} v_i c(i) = 0 \ , \tag{2.12}$$

if $\alpha_1 + \alpha_2$ is odd and $c(i) = c(i')$.

Distributing the derivatives in the expression $v_{i_k} \nabla(...\nabla(v_{i_2} \nabla v_{i_1}^1)...)$ we see that (2.10) is equal to a sum of the terms, each of which has the form

$$\sum_{i_1,...,i_k=1}^{2n} D_k^{\alpha_k} v_{i_k} \otimes ... \otimes D_1^{\alpha_1} v_{i_1} \delta_{i_{l_1} i_{m_1}} ... \delta_{i_{l_{k/2}} i_{m_{k/2}}} c_n(i_1, ..., i_k) \ . \tag{2.13}$$

Note that $\sum\limits_{j=1}^{k} \alpha_j = k - 1$ is odd. Therefore there exist l_p and m_p such that $\alpha_{l_p} + \alpha_{m_p}$ is odd. By (2.12)

$$\sum_{i_{l_p}, i_{m_p}=1}^{2n} D_k^{\alpha_k} v_{i_k} \otimes ... \otimes D_1^{\alpha_1} v_{i_1} \delta_{i_{l_1} i_{m_1}} ... \delta_{i_{l_p} i_{m_p}} ... \delta_{i_{l_{k/2}} i_{m_{k/2}}} c_n(i_1, ..., i_k) = 0 \ .$$

Therefore, the expression in (2.13) is equal to zero. Therefore, the expression in (2.10) is equal to zero. This completes the proof of Lemma 2.1.4.

Lemmas 2.1.1 and 2.1.4 imply that if $u \in \mathcal{R}^k$, then the equation

$$Lw = u \tag{2.14}$$

has the unique solution $w \in \mathcal{P}^{\perp}$.

Lemma 2.1.5. *Suppose Assumptions A and B hold. If $u \in \mathcal{R}^k$, then the solution w of (2.14) belongs to $\mathcal{R}^k$.*

Proof Without loss of generality we may consider $u \in \mathcal{T}^k$ given by formula (2.8). The proof will be by induction with step 2 on the number t of the y-factors in (2.8). Thus we introduce the following induction hypothesis:

(H_j) If $u \in \mathcal{T}^k$ is given by (2.8) and $t = j$, then the solution w of (2.14) belongs to $\mathcal{R}^k$.

If $t = j = 0$, then u has the form $u = u_0(x)\eta$. Since, by Lemma 2.1.4, $\int u\eta dy = 0$ for all x, we conclude that $u = 0$, and therefore $w = 0$. Thus H_0 holds.

If $t = j = 1$, then

$$u = \sum_{i_1,\ldots,i_k=1}^{2n} v_{i_k}\nabla(\ldots\nabla(v_{i_2}\nabla v_{i_1}^1)\ldots)y_{i_{r_1}}\delta_{i_{l_1}i_{m_1}}\ldots\delta_{i_{l_s}i_{m_s}}c_n(i_1,\ldots,i_k)\eta\ ,$$

where $(r_1, l_1, \ldots, l_s, m_1, \ldots, m_s)$ is a permutation of $(1, \ldots, k)$. Since $L(\frac{y_{i_{r_1}}}{\Omega_{i_{r_1}}}\eta) = -y_{i_{r_1}}\eta$, the solution of (2.14) is

$$w = -\sum_{i_1,\ldots,i_k=1}^{2n} v_{i_k}\nabla(\ldots\nabla(v_{i_2}\nabla v_{i_1}^1)\ldots)\frac{y_{i_{r_1}}}{\Omega_{i_{r_1}}}\delta_{i_{l_1}i_{m_1}}\ldots\delta_{i_{l_s}i_{m_s}}c_n(i_1,\ldots,i_k)\eta\ ,$$

which belongs to $\mathcal{T}^k$ since Ω_i are positive and uniformly bounded away from zero. Thus H_1 holds.

Assume a value of j for which $H_{j'}$ holds for any $j' \leq j$. We shall verify that H_{j+2} holds. Let $u \in \mathcal{T}^k$ be given by (2.8) with $t = j+2$. Define

$$w_0 = -\sum_{i_1,\dots,i_k=1}^{2n} v_{i_k}\nabla(\dots\nabla(v_{i_2}\nabla v_{i_1}^1)\dots)\frac{y_{i_{r_1}}\cdots y_{i_{r_t}}}{(\Omega_{i_{r_1}}+\dots+\Omega_{i_{r_t}})}\delta_{i_{l_1}i_{m_1}}\cdots$$

$$\dots\delta_{i_{l_s}i_{m_s}}c_n(i_1,\dots,i_k)\eta\ .$$

Since Ω_i are positive and uniformly bounded away from zero w_0 belongs to $\mathcal{T}^k$. Let $u_0 = u - Lw_0$. Then $u_0 \in \mathcal{R}^k$ and it can be represented as a sum of elements of $\mathcal{T}^k$, each of which has the form (2.8) with $t \le j$. Since

$$L(w - w_0) = u_0\ ,$$

and $H_{j'}$ holds for any $j' \le j$, we conclude that $w - w_0 \in \mathcal{R}^k$, and therefore $w \in \mathcal{R}^k$. Thus H_{j+2} holds. This completes the proof of Lemma 2.1.5.

Lemma 2.1.6. *Suppose Assumptions A, B, and C_m hold. Then for any function $u(n;x,y) \in \mathcal{R}^{2m+2}$ the $\mathcal{L}_2$ norms $||u(n;x,y)||$ are bounded uniformly in n.*

Proof Without loss of generality we can consider $u \in \mathcal{T}^{2m+2}$ given by formula (2.8) with $k = 2m+2$. We square both sides of (2.8) and integrate the resulting equality in x and y. On the RHS we can perform the integration in y explicitly. Note that

$$\int y_{i_{r_1}}\cdots y_{i_{r_t}}y_{i_{q_1}}\cdots y_{i_{q_t}}\eta^2 dy = \frac{1}{p^3}\sum_{\sigma'}\delta_{i_{\alpha_1}i_{\alpha_2}}\cdots\delta_{i_{\alpha_{2t-1}}i_{\alpha_{2t}}}\ ,$$

where $\sigma' = (\alpha_1,\dots,\alpha_{2t})$ is a permutation of $(r_1,\dots,r_t,q_1,\dots,q_t)$, and $\sum_{\sigma'}$ is the sum over the set of all such permutations. Note that the number of such permutations depends only on t, but not on n.

Thus, using (2.9), we see that $||u||^2$ is estimated from above by a sum of terms, the number of terms being independent of n, each of which has the following form

$$\frac{c(\sigma)}{p^3}\int\limits_{\mathbb{T}^3_p}\sum_{i_1,...,i_{4m+4}=1}^{2n}|v_{i_{4m+4}}\nabla(...\nabla(v_{i_{2m+4}}\nabla v^1_{i_{2m+3}})...)$$

$$v_{i_{2m+2}}\nabla(...\nabla(v_{i_2}\nabla v^1_{i_1})...)\delta_{i_{l_1}i_{k_1}}...\delta_{i_{l_{2m+2}}i_{k_{2m+2}}}|dx \leq \tag{2.15}$$

$$c(\sigma)\sup_x\sum_{i_1,...,i_{4m+4}=1}^{2n}|v_{i_{4m+4}}\nabla(...\nabla(v_{i_{2m+4}}\nabla v^1_{i_{2m+3}})...)$$

$$v_{i_{2m+2}}\nabla(...\nabla(v_{i_2}\nabla v^1_{i_1})...)\delta_{i_{l_1}i_{k_1}}...\delta_{i_{l_{2m+2}}i_{k_{2m+2}}}| \ ,$$

where $\sigma = (l_1, ..., l_{2m+2}, k_1, ..., k_{2m+2})$ is a permutation of $(1, ..., 4m+4)$.

Distributing the derivatives in the RHS of (2.15) we see that it can be estimated from above by a sum of the terms, the number of terms being independent of n, each of which has the following form

$$c(\sigma)\sup_x\sum_{i_1,...,i_{4m+4}=1}^{2n}|D^{\alpha_{4m+4}}_{4m+4}v_{i_{4m+4}}\otimes...\otimes D^{\alpha_1}_1 v_{i_1}\delta_{i_{l_1}i_{k_1}}...\delta_{i_{l_{2m+2}}i_{k_{2m+2}}}| \ . \tag{2.16}$$

Note that $\max_i \alpha_i \leq 2m+1$. For an arbitrary tensor T let $|T|$ be the supremum of its component's absolute values. Then

$$\sup_x |D^\alpha \cos(k_i x)| = \sup_x |D^\alpha \sin(k_i x)| \leq c(D)|k_i|^\alpha \ .$$

Thus due to the particular form (2.5) of the vector fields v_i the expression (2.16) is estimated from above by a constant independent of n times the product of $2m+2$ factors, each of which has the form $\sum_{i=1}^{2n} \lambda_i |k_i|^q$ with $q \leq 4m+2$, where

$$\{\lambda_i, i = 1, ..., 2n\} = \{\lambda_i^1, \lambda_i^2, i = 1, ..., n\}$$

are the eigenvalues of M_i^n. Each of the factors $\sum_{i=1}^{2n} \lambda_i |k_i|^q$ is bounded uniformly in n by assumption C_m. Therefore the absolute value of the expression in (2.16) is bounded uniformly in n. Thus the RHS of (2.15) is bounded uniformly in n. This completes the proof of Lemma 2.1.6.

Proof of Theorem 2.1.2 The right side of (2.2) belongs to $\mathcal{R}^1$. From Lemmas 2.1.3, 2.1.4 and 2.1.5 it follows that (2.2), (2.3) can be used as the inductive definition of the sequence $\{u_k\}$. Moreover, $u_k \in \mathcal{R}^{k+1}$. Thus u_{2k} is an odd function and u_{2k+1} is an even function for all k.

Since $u_k \in \mathcal{R}^{k+1}$ the function Au_k belongs to $\mathcal{R}^{k+2}$. Thus, by Lemma 2.1.6 the norm of the RHS of (2.4) is estimated as follows

$$|| - \varepsilon^{\frac{k+1}{2}} Au_k|| \leq C_1(k)\varepsilon^{\frac{k+1}{2}} .$$

By lemmas 2.1.3 and 2.1.4 the RHS of (2.4) satisfies (1.77). Therefore we can employ Theorem 1.5.8 to conclude that $||w_k|| \leq C_2(k)\varepsilon^{\frac{k+1}{2}}$. Thus (2.1) is proved. This completes the proof of Theorem 2.1.2.

We next employ Theorem 2.1.2 and Corollary 1.4.4 to obtain the asymptotic expansion for $D_\varepsilon^{n,ab}$. While the asymptotic expansion for the solution of equation (1.46) is in the powers of $\varepsilon^{\frac{1}{2}}$, only the terms with integer powers of ε contribute to the asymptotic expansion of $D_\varepsilon^{n,ab}$. The fractional powers in ε vanish because the integrals in y of the odd terms vanish.

Theorem 2.1.7. *Suppose Assumptions A, B, and C_m hold. Then the effective diffusivity $D_\varepsilon^{n,ab}$ has the asymptotic expansion*

$$D_\varepsilon^{n,ab} = d_0^{n,ab} + d_1^{n,ab}\varepsilon + \ldots + d_m^{n,ab}\varepsilon^m + \varepsilon^m \phi^n(\varepsilon, m) \text{ , where} \tag{2.17}$$

$$d_k^{n,ab} = \frac{1}{2}\sum_{i=1}^{2n} \int\int (u_{2k}^{n,a} v_i^b + u_{2k}^{n,b} v_i^a) y_i \eta dxdy \text{ ,}$$

and $\lim_{\varepsilon\to 0} \phi^n(\varepsilon, m) = 0$ *uniformly in* n.

Proof As in the proof of Theorem 2.1.2 we consider the case $a = b = 1$. By Corollary 1.4.4

$$D_\varepsilon^{n,11} = \sum_{k=0}^{2m} \varepsilon^{\frac{k}{2}} \sum_{i=1}^{2n} \int\int u_k^n v_i^1 y_i \eta dxdy +$$

$$\int\int [u^n - (u_0^n + \varepsilon^{\frac{1}{2}} u_1^n + \ldots + \varepsilon^m u_{2m}^n)] v_i^1 y_i \eta dxdy \text{ .} \tag{2.18}$$

Note that $|| \sum_{i=1}^{2n} v_i^1 y_i \eta || \leq C_1$ by (1.96). By Theorem 2.1.2

$$||u^n - (u_0^n + \varepsilon^{\frac{1}{2}} u_1^n + \ldots + \varepsilon^m u_{2m}^n)|| \leq C_2(m)\varepsilon^{m+\frac{1}{2}} \text{ ,}$$

therefore the last term on the RHS of (2.18) does not exceed $C_3(m)\varepsilon^{m+\frac{1}{2}}$.

It remains to show that the terms with non integer powers of ε vanish. By Theorem 2.1.2 the integrand in the first term on the RHS of (2.18) is a product of η times an odd function if k is odd, and therefore the integral is equal to zero. This completes the proof of Theorem 2.1.7.

Proof of Theorem 1.2.2 We introduce the induction hypothesis:

(I_k) The limit $d_k^{ab} = \lim_{n\to\infty} d_k^{n,ab}$ exists.

From the proof of Theorem 1.2.1 it follows that $\lim_{n\to\infty} D_\varepsilon^{n,ab} = D_\varepsilon^{ab}$. By Theorem 2.1.7 $\lim_{\varepsilon\to 0} D_\varepsilon^{n,ab} = d_0^{n,ab}$ uniformly in n. Therefore by the theorem on uniform convergence the following limits exist

$$d_0^{ab} = \lim_{n\to\infty} d_0^{n,ab} = \lim_{\varepsilon\to 0} D_\varepsilon^{ab} \ .$$

Thus I_0 holds. Assume a value of k, $k < m$, for which I_k holds. We shall verify that I_{k+1} holds. Consider

$$\psi_k^{ab}(n,\varepsilon) = \frac{D_\varepsilon^{n,ab} - d_0^{n,ab} - \varepsilon d_1^{n,ab} - \ldots - \varepsilon^k d_k^{n,ab}}{\varepsilon^{k+1}} \ .$$

By I_k

$$\lim_{n\to\infty} \psi_k^{ab}(n,\varepsilon) = \frac{D_\varepsilon^{ab} - d_0^{ab} - \varepsilon d_1^{ab} - \ldots - \varepsilon^k d_k^{ab}}{\varepsilon^{k+1}} \ .$$

By Theorem 2.1.7 $\lim_{\varepsilon\to 0} \psi_k^{ab}(n,\varepsilon) = d_{k+1}^{n,ab}$ uniformly in n. Therefore by the theorem on uniform convergence the following limits exist

$$d_{k+1}^{ab} = \lim_{n\to\infty} d_{k+1}^{n,ab} = \lim_{\varepsilon\to 0} \frac{D_\varepsilon^{ab} - d_0^{ab} - \varepsilon d_1^{ab} - \ldots - \varepsilon^k d_k^{ab}}{\varepsilon^{k+1}} \ . \tag{2.19}$$

Therefore I_{k+1} holds for all $k < m$. From (2.19) with $k = m - 1$ we obtain (1.13). This completes the proof of Theorem 1.2.2.

2.2 Explicit Calculations

Now we calculate explicitly the first two terms of the expansion (1.13) in terms of the correlation matrix G of the velocity field.

Theorem 2.2.1. *Suppose Assumptions A, B, and C_1 hold. Then the effective diffusivity D_ε^{ab} has the asymptotic expansion*

$$D_\varepsilon^{ab} = \int_0^\infty G^{ab}(0,t)dt + \tag{2.20}$$

$$\varepsilon \sum_{l,m=1}^{3} \int_0^\infty \int_{t_3}^\infty \int_{t_2}^\infty \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m} G^{ab}(x,t_1)|_{x=0} G^{lm}(0,t_3)dt_1dt_2dt_3 + o(\varepsilon) \ .$$

Proof Solving (2.2) we obtain

$$u_0^{n,a} = \sum_{i=1}^{2n} y_i \frac{1}{\Omega_i} \eta v_i^a(x) \ .$$

From (2.3) with $k=0$ and $k=1$ we obtain consecutively

$$u_1^{n,a} = \sum_{i,j=1}^{2n} \frac{1}{(\Omega_i+\Omega_j)\Omega_j} y_i y_j \eta v_i \nabla v_j^a \ .$$

$$\begin{aligned} u_2^{n,a} \ = \ & \sum_{i,j,k=1}^{2n} \frac{1}{(\Omega_i+\Omega_j+\Omega_k)(\Omega_j+\Omega_k)\Omega_k} y_i y_j y_k \eta v_i \nabla(v_j \nabla v_k^a) + \qquad (2.21) \\ & \sum_{i,j,k=1}^{2n} \delta_{ij} \frac{2}{\Omega_k} \frac{\Omega_i}{(\Omega_i+\Omega_j+\Omega_k)(\Omega_j+\Omega_k)\Omega_k} y_k \eta v_i \nabla(v_j \nabla v_k^a) + \\ & \sum_{i,k,j=1}^{2n} \delta_{ik} \frac{2}{\Omega_j} \frac{\Omega_i}{(\Omega_i+\Omega_j+\Omega_k)(\Omega_j+\Omega_k)\Omega_k} y_j \eta v_i \nabla(v_j \nabla v_k^a) + \\ & \sum_{i,j,k=1}^{2n} \delta_{jk} \frac{2}{\Omega_i} \frac{\Omega_k}{(\Omega_i+\Omega_j+\Omega_k)(\Omega_j+\Omega_k)\Omega_k} y_i \eta v_i \nabla(v_j \nabla v_k^a) \ . \end{aligned}$$

Using Theorem 2.1.7 together with the expression for $u_0^{n,a}$ we see that the first coefficient (with ε^0) in the expansion (2.17) is equal to

$$d_0^{n,ab} = \frac{1}{2}\sum_{i=1}^{2n} \int\int y_i \frac{p^{-3}}{\Omega_i}(v_i^a v_i^b + v_i^b v_i^a)\sigma_i y_i \eta^2 dxdy = \sum_{i=1}^{2n} \frac{1}{\Omega_i}\int v_i^a v_i^b dx \; . \tag{2.22}$$

The coefficient at the first power of ε is equal to

$$d_1^{n,ab} = \frac{1}{2}\sum_{i=1}^{2n} \int\int (u_2^a v_i^b + v_i^a u_2^b) y_i \eta dxdy \; .$$

Using (2.21) and (2.12) this is seen to be equal to

$$-\sum_{i,j=1}^{2n} \frac{p^{-3}}{\Omega_i^2(\Omega_i + \Omega_j)} \int (v_j \nabla v_i^a)(v_j \nabla v_i^b) dx \; . \tag{2.23}$$

Using (2.5) and (2.6) we evaluate the integrals in (2.22) and (2.23). Thus we obtain

$$d_0^{n,ab} = \sum_{i=1}^{n} \frac{M_i^{n,ab}}{\Omega_i} \; ,$$

$$d_1^{n,ab} = -\sum_{i,j=1}^{n} \frac{M_i^{n,ab} k_i^l k_i^m M_j^{n,lm}}{\Omega_i^2(\Omega_i + \Omega_j)} \; .$$

The expression (1.9) for the Fourier transform of the correlation matrix $G(x,t)$ implies that

$$d_0^{ab} = \lim_{n\to\infty} d_0^{n,ab} = \int_0^\infty G^{ab}(0,t) dt \; ,$$

$$d_1^{ab} = \lim_{n\to\infty} d_1^{n,ab} =$$

$$\sum_{l,m=1}^{3} \int_{0}^{\infty} \int_{t_3}^{\infty} \int_{t_2}^{\infty} \frac{\partial}{\partial x_l} \frac{\partial}{\partial x_m} G^{ab}(x, t_1)|_{x=0} G^{lm}(0, t_3) dt_1 dt_2 dt_3 \ .$$

Therefore (2.20) is valid. This completes the proof of Theorem 2.2.1.

Chapter 3

Vector Fields with Kolmogorov Spectrum

3.1 Effective Diffusivity in Vector Fields with Kolmogorov Spectrum

In this section we study the dependence of the effective diffusivity on the cutoff in the spectrum of the Kolmogorov velocity field.

We start with a vector field whose spectral measure is of order $|k|^\alpha$ at infinity. The case of pure Kolmogorov spectrum corresponds to $\alpha = -\frac{5}{3}$. In this case a typical realization of the field V is not continuous, and V is understood as a generalized random field.

In order to make sense of the equation of motion (1.1) we introduce cutoffs at infinity for the spectral measure, thus regularizing V by ordinary random fields V^m. The spectral measure of V^m is defined to be equal to the spectral measure of V on the cube of size m centered at the origin in wave number space, and equal to zero elsewhere.

We provide a bound on the effective diffusivity $D^{m,ab}$ of the approximating fields V^m in terms of the size of the cutoff m. We obtain

$$D^{m,ab} \leq cm^{2/3} \tag{3.1}$$

for the vector fields with Kolmogorov spectrum.

Let us discuss the physical interpretation of formula (1.9) in the case of the three dimensional turbulence with Kolmogorov spectrum. In the Kolmogorov picture turbulence is looked upon as a system of eddies corresponding to different velocity frequencies. The matrix $M(k)$ is of order of the density of the kinetic energy corresponding to modes with frequency k. From the dimensional considerations it follows that

$$M(k) \sim |k|^{-5/3} \tag{3.2}$$

for large $|k|$ ([15]).

With the characteristic length scale, and the characteristic velocity fluctuation fixed, the size of the domain in which (3.2) is valid is determined by the fluid viscosity. Namely, outside of the cube of size m, the matrix $M(k)$ is assumed to be rapidly decreasing. The size of the cube, is inversely proportional to the ratio of the viscosity ν and the characteristic velocity U, and proportional to the ratio of the Reynolds number R and the macroscopic length scale l of the flow.

Since we are interested in the upper estimates on the effective diffusivity, we assume that $|M^{ab}(k)| \leq c(1+|k|)^{-5/3}$ for some c and all a, b, and k. We shall study the dependence of the effective diffusivity on the Reynolds number R, or more precisely, on the cutoff m in the velocity spectrum.

As seen from (1.9) the function $\Omega(k)$ is the decay rate, or inverse life time of an eddy with frequency k. As follows from [15], classical Kolmogorov turbulence corresponds to the case $\Omega(k) \sim |k|^{2/3}$ for large $|k|$.

We now state the assumptions on the stream function, in somewhat more generality than needed for the case of pure Kolomogorov spectrum of the velocity field.

Assumption $\mathbf{A}_1$ *$V(x,t), x \in \mathbb{R}^3$ is a zero mean Gaussian field, stationary in x and t, isotropic in x, and Markov in time.*

Assumption $\mathbf{B}_1$ *The spectral matrix of the field V is given by (1.9), where $\Omega(k)$ is scalar, and the matrix $M(k)$ is symmetric. There exists a constant $c > 0$, and a compact set $K \subset \mathbb{R}^3$, such that*

$$\int_K dM(k) > cI \ ,$$

where I is the identity matrix. There exist constants $c_1, c_2 > 0$, and α, β, such that

$$|M^{ab}|(k) \leq c_1(1+|k|)^{\alpha} \ , \tag{3.3}$$

$$\Omega(k) \geq c_2(1+|k|)^\beta \ . \tag{3.4}$$

Moreover, $\Omega(k)$ is Lipschitz continuous uniformly on any compact. The case of pure Kolmogorov spectrum for 3 dimensional turbulence corresponds to $\alpha = -5/3$ and $\beta = 2/3$. The vector field $V^m(x,t)$ is defined to be the real valued Gaussian random field whose spectral matrix $\widehat{G^m}$ is given by (1.9) on the set $\{|k^a| < m, a = 1, ..., 3\}$, and is equal to zero outside this set. Notice that if a field V satisfies assumptions A_1 and B_1, then the field V^m satisfies the assumptions A and B of Section 1.2 for $m \geq m_0$ for some m_0. Therefore the effective diffusivity of the field V^m exists and is finite for $m \geq m_0$. We shall denote it by $D^{m,ab}$. We now formulate the main theorem of this section.

Theorem 3.1.1. *Suppose Assumptions A_1 and B_1 hold. Then there exists a constant c such that $D^{m,ab} < cm^{\alpha-\beta+3}$, $a, b = 1, ..., 3$ for all $m \geq m_0$.*

Remark *With $\alpha = -5/3$ and $\beta = 2/3$ we recover (3.1).*

We use the discretization of the spectrum of the field V^m, as described in Section 1.3, to obtain the approximating vector fields V^{mn}. The effective diffusivity of the field V^{mn} will be denoted by $D^{mn,ab}$. From the proof of Theorem 1.2.1 it follows that $D^{mn,ab} \to D^{m,ab}$ as $n \to \infty$ for $m \geq m_0$. Therefore Theorem 3.1.1 is a consequence of the following

Theorem 3.1.2. *Suppose Assumptions A_1 and B_1 hold. Then there exists a constant c such that $D^{mn,ab} < cm^{\alpha-\beta+3}$, $a, b = 1, ..., 3$ for $m \geq m_0, n \geq n_0(m)$.*

Proof We are using the notations introduced in Chapter 1. By Corollary 1.4.4 the effective diffusivity $D^{mn,ab}$ is expressed in terms of the solution u^a of equation

$$\sum_{i=1}^{2n} \Omega_i \left(\partial_{y_i}^2 - \frac{y_i^2}{4} + \frac{1}{2}\right) u^a + \sum_{i=1}^{2n} y_i v_i \nabla_x u^a = -\sum_{i=1}^{2n} y_i v_i^a \eta \; . \tag{3.5}$$

by the formula

$$D^{mn,ab} = \frac{1}{2} \sum_{i=1}^{2n} \int\int_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (u^a v_i^b + u^b v_i^a) y_i \eta dx dy \; . \tag{3.6}$$

Recall that the coefficients $v_i(x)$ of (3.5) are defined by the spectral matrix of the field V^m, and thus depend on m.

We represent the solution u^a of (3.5) uniquely as a sum of two functions which are orthogonal in $\mathcal{L}_2(\mathbb{R}^{2n})$ for all x, that is

$$u^a(x,y) = w(x,y) + u_0(x)\eta(y) \; , \tag{3.7}$$

where $\int u_0(x)dx = 0$, and $\int w(x,y)\eta(y)dy = 0$. Since $\int \eta^2 y_i dy = 0$, the term $u_0(x)\eta(y)$ does not contribute to the integral on the RHS of (3.6). Therefore it is sufficient to estimate the contribution of w to the integral.

As in the proof of Theorem 1.5.8 we obtain

$$\int\int wLwdxdy = \int\int fwdxdy \; , \tag{3.8}$$

where

$$f = -\sum_{i=1}^{2n} y_i v_i^a \eta \tag{3.9}$$

is the RHS of (3.5). We now use (3.8) to bound the RHS of (3.6). Note that $\{p^{3/2}y_i\eta, i = 1, ..., 2n\}$ is an orthonormal system in $\mathcal{L}_2(\mathbb{R}^{2n})$. Let $a_i(x) = p^{3/2}\int wy_i\eta dy$, and let $w_i(x,y) = p^{3/2}a_i(x)y_i\eta$. Thus for each x the sum $\sum_{i=1}^{2n} w_i$ is the projection in $\mathcal{L}_2(\mathbb{R}^{2n})$ of w on the subspace spanned by the functions $\{y_i\eta\}$. In particular by (3.9) the function $\widetilde{w} = w - \sum_{i=1}^{2n} w_i$ is orthogonal to f in $\mathcal{L}_2(\mathbb{R}^{2n})$ for each x. Since $L(y_i\eta) = -\Omega_i y_i\eta$, substituting

$$w = \widetilde{w} + \sum_{i=1}^{2n} w_i \tag{3.10}$$

into (3.8) we obtain

$$\int\int \widetilde{w}L\widetilde{w}dxdy - \sum_{i=1}^{2n}\Omega_i\int a_i^2(x)dx = p^{-3/2}\sum_{i=1}^{2n}\int a_i(x)v_i^a(x)dx\ . \tag{3.11}$$

Since $\int\int \widetilde{w}L\widetilde{w}dxdy \le 0$ we conclude from (3.11) that

$$\sum_{i=1}^{2n}\Omega_i\int a_i^2(x)dx \le p^{-3/2}\sum_{i=1}^{2n}\int |a_i(x)v_i^a(x)|dx\ . \tag{3.12}$$

By Schwartz inequality, the RHS of (3.12) is estimated as follows

$$p^{-3/2}\sum_{i=1}^{2n}\int |a_i(x)v_i^a(x)|dx \le$$

$$\frac{c_2}{2}\sum_{i=1}^{2n}|k_i|^{\beta}\int a_i^2(x)dx + \frac{2}{p^3c_2}\sum_{i=1}^{2n}|k_i|^{-\beta}\int (v_i^a)^2(x)dx\ , \tag{3.13}$$

where c_2 and β are the same as in (3.4). Note that by (1.21) the second term on the RHS of (3.13) is bounded above by $\frac{2}{c_2}\sum_{i=1}^{n}|k_i|^{-\beta}(\lambda_i^1+\lambda_i^2)$, where

λ_i^1 and λ_i^2 are the eigenvalues of M_i^{mn}. This sum in turn is bounded above by $c(m+1)^{\alpha+3-\beta}$ due to (3.3). Therefore by (3.12), (3.13), and (3.4) there exists a constant C such that

$$\sum_{i=1}^{2n} |k_i|^\beta \int a_i^2(x)dx \leq C(m+1)^{\alpha-\beta+3} . \tag{3.14}$$

In order to bound the RHS of (3.6) is is sufficient to bound $\sum_{i=1}^{2n} \int\int u^a v_i^b y_i \eta dxdy$. By (3.7) and (3.10)

$$\sum_{i=1}^{2n} \int\int u^a v_i^b y_i \eta dxdy = \sum_{i=1}^{2n} \int\int w_i v_i^b y_i \eta dxdy =$$

$$p^{-3/2} \sum_{i=1}^{2n} \int a_i(x) v_i^b dx . \tag{3.15}$$

By Schwartz inequality, the RHS of (3.15) is bounded above by

$$\sum_{i=1}^{2n} |k_i|^\beta \int a_i^2(x)dx + p^{-3/2} \sum_{i=1}^{2n} |k_i|^{-\beta} \int (v_i^b)^2(x)dx .$$

The first term is bounded above by $C(m+1)^{\alpha-\beta+3}$ due to (3.14), and the second term is bounded by the same expression, with maybe different C, due to (1.21) and (3.3). Therefore the RHS of (3.6) is bounded above by $C(m+1)^{\alpha-\beta+3}$. This completes the proof of Theorem 3.1.2.

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