Einstein-Thorpe Manifolds

A Dissertation Presented

by

Jaeman Kim

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1998

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation
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One of the most fascinating facts in mathematics is that the local geometry of a manifold provides us with information about its global topology. For instance, the Euler-Poincaré characteristic χ of a compact oriented Riemannian manifold M^{4k} of even dimension can be written as an integral

$$\chi = \frac{2}{V}\frac{[(2k)!]^2}{4k}\int_M trace * \mathbb{R}_{2k} * \mathbb{R}_{2k}\ dV$$

where $\mathbb{R}_{2k}$ is the $2k^{\text{th}}-$curvature operator and $*$ is the Hodge star operator and V is the volume of the Euclidean unit $4k$ sphere and dV is the volume element of M.

If $\mathbb{R}_{2k}$ commutes with $*$, we say that the metric is a Thorpe metric. In the 4−dimensional case, a metric is Thorpe metric if and only if it is Einstein. On the other hand, as we shall see in Section 2.4, Thorpe metrics need not be Einstein in higher dimensions.

We shall say that a Riemannian $4k$ manifold is Einstein-Thorpe if it is both Einstein and Thorpe.

In this dissertation, we shall see that

1. There is an infinite dimensional moduli space of Thorpe metrics on T^{4k} $(k > 1)$. Most of these are not Einstein metrics. The same construction also yields Thorpe metrics on $M^{4k-p} \times T^{4k+p}$ $(k, p \geq 1, 4k - p \geq 2)$ where M^{4k-p} is any compact oriented manifold.

2. However, every Einstein-Thorpe metric on T^8 must be flat. On compact oriented hyperbolic manifolds of dimension 8, every Einstein-Thorpe metric is a hyperbolc metric up to rescalings and diffeomorphisms.

3. There are some manifolds of dimension 8 which have $\chi = 0$ and $P_2 = 0$ but which never carry an Einstein-Thorpe metric. In particular, a compact orientable Einstein-Thorpe manifold (M^8, g) that satisfies

$$\chi = \frac{2!2!}{4!}|P_2|$$

must be $(T^8/\Gamma, \text{flat})$ where Γ is of finite order.

Dedicated to my parents.

Contents

Acknowledgements

I would like to thank, first and foremost, my thesis advisor, Prof. Claude LeBrun for his inspiring guidance and encouragements, without which this dissertation could have never been completed.

My sincere thanks also go to the faculty, staff and all the other members of the Department of Mathematics, in particular Prof. Michael T. Anderson and Prof. Myonghi Kim.

I would like to give special thanks to Prof. Kyung Tae Chung, who first introduced me to the wonders of Differential Geometry.

I am also obliged to thank all my friends at Stony Brook who brought joy to my life and helped me all the way. In particular, I would like to thank Ramón, Susana, Harish, Seung Mok, Wonkuk, Chanyoung, Choong Seok, and Jaseong.

Finally, I want to thank my parents for all the things which they have done for me. I cannot say enough for their help and encouragement.

My sincere gratitude goes to the Korean Government and the Korean Consulate in New York for offering me the Government fellowship for completing this dissertation.

Chapter 1

Introduction

1.1 Motivation

One of the most beautiful aspects of mathematics is that the local geometry of a manifold provides us with information about its global topology. For instance, it is the generalized Gauss-Bonnet theorem [Che44, Tho69] that the Euler-Poincaré characteristic χ of a compact oriented Riemannian manifold M^{4k} of even dimension can be written as an integral

$$\chi = \frac{2}{V}\frac{[(2k)!]^2}{4k}\int_M trace * \mathbb{R}_{2k} * \mathbb{R}_{2k}\, dV$$

where V is the volume of the Euclidean unit $4k-$sphere and dV is the volume element of M and $*$ is the Hodge star operator and $\mathbb{R}_{2k}$ is $2k^{th}$ curvature operator. If $\mathbb{R}_{2k}$ commutes with $*$, i.e. $\mathbb{R}_{2k} * = * \mathbb{R}_{2k}$, we call this condition a Thorpe condition and this metric a Thorpe metric.

In the 4−dimensionl case, the Thorpe condition is equivalent to the Einstein condition. It is an interesting fact that in the 4−dimensional case the Einstein condition, which is the extremal condition of the total scalar curvature of suitable normalized metrics on compact Riemannian manifolds, can be read off from a purely algebraic condition. For dimensions higher than 4, nothing is known about topological conditions for the existence of an Einstein metric on a manifold. This is reflected the fact that the Thorpe condition does not imply the Einstein condition in higher dimensions. As a matter of fact, in section 2.4 we shall see some examples of manifolds whose given metrics are Thorpe metrics, but not Einstein metrics.

The following questions still remain open:

- Are there any non-flat Einstein metrics on the T^{4k} torus, for $k > 1$?
- Does every compact Riemannian manifold M^{4k} admit at least one Einstein metric, for $k > 1$?

On the other hand, the purpose of this dissertation is to prove the following statements:

1. There is an infinite dimensional moduli space of Thorpe metrics on T^{4k}

$(k > 1)$. Most of these are not Einstein metrics. The same construction also yields Thorpe metrics on $M^{4k-p} \times T^{4k+p} \quad (k, p \geq 1, 4k - p \geq 2)$ where M^{4k-p} is any compact oriented manifold.

2. However, every Einstein-Thorpe metric on T^8 must be flat. On compact oriented hyperbolic manifolds of dimension 8, every Einstein-Thorpe metric is a hyperbolc metric up to diffeomorphisms and rescalings.

3. There are some manifolds of dimension 8 which have $\chi = 0$ and $P_2 = 0$ but which never carry an Einstein-Thorpe metric. In particular, a compact orientable Einstein-Thorpe manifold (M^8, g) that satisfies

$$\chi = \frac{2!2!}{4!}|P_2|$$

must be $(T^8/\Gamma, \text{flat})$ where Γ is of finite order.

1.2 Basic properties of the $2k^{th}$ curvature operator

Let M be a Riemannian manifold of even dimension n and let $\bigwedge^p(M)$ denote the bundle of $p-$vectors of M. $\bigwedge^p(M)$ is a Riemannian vector bundle, with inner product on the fiber $\bigwedge^p(x)$ over the point x related to the inner product on the tangent space T_xM of M at x by $< u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p >= det\,[< u_i, v_j >]$ with $u_i, v_j \in T_xM$.

Let R denote the covariant curvature tensor of M. For each even $p > 0$, we define the $p-$th curvature tensor R_p of M to be the covariant tensor field of order $2p$ given by

$$R_p(u_1, \ldots, u_p, v_1, \ldots, v_p) =$$
$$\frac{1}{2^{\frac{p}{2}}\, p!} \sum_{\alpha,\beta \in S_p} \varepsilon(\alpha)\, \varepsilon(\beta)\, R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \cdots R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)})$$

where $u_i, v_j \in T_xM$, and S_p denotes the group of permutations of $(1, \ldots, p)$ and, for $\alpha \in S_p$, $\varepsilon(\alpha)$ is the sign of the permutation α.

The tensor R_p has the following properties: it is alternating in the first p variables, alternating in the last p variables and is invariant under the operation of interchanging the first p variables with the last p variables. Hence, at each point $x \in M$, R_p can be regarded as a symmetric bilinear form on $\bigwedge^p(x)$. By use of the inner product on $\bigwedge^p(x)$, R_p at x may then be identified with a self-adjoint linear operator $\mathbb{R}_p$ on $\bigwedge^p(x)$. Explicitly, this identification is given by

$$< \mathbb{R}_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p > \equiv R_p(u_1, \ldots, u_p, v_1, \ldots, v_p)$$

with $u_i, v_j \in T_xM$.

Taking $p = n$, the space $\bigwedge^n(x)$ is one dimensional and hence the self-adjoint linear operator $\mathbb{R}_n : \bigwedge^n(x) \longrightarrow \bigwedge^n(x)$ is a scalar multiple of the identity. More explicitly, when expressed globally, the line bundle homomorphism

$\mathbb{R}_n : \bigwedge^n(M) \longrightarrow \bigwedge^n(M)$ is

$$\mathbb{R}_n = K\, I$$

where I is the identity automorphism of $\bigwedge^n(M)$ and K is the Lipschitz-Killing curvature of M. Furthermore, for $x \in M$, $K(x) = R_n(e_1, \ldots, e_n, e_1, \ldots, e_n)$ where $\{e_1, \ldots, e_n\}$ is any orthonormal basis for T_xM.

The generalized Gauss-Bonnet theorem [Che44] expresses the Euler-Poincaré characteristic χ of a compact oriented Riemannian manifold of even dimension n as an integral

$$\chi = \frac{2}{c_n} \int_M K\, dV$$

where K is the Lipschitz-Killing curvature of M, c_n is the volume of Euclidean unit $n-$sphere and dV is the volume element of M.

The tensor R_p satisfies the Bianchi identity which can be expressed in the following way [Tho69]:

$$Alt\ R_p = 0$$

where Alt is the skew symmetrization operator given by

$$Alt\ R_p(v_1, \ldots, v_{2p}) = \frac{1}{(2p)!} \sum_{r \in S_{2p}} \varepsilon(r)\, R_p(v_{r(1)}, \ldots, v_{r(2p)})$$

with $v_i \in T_xM$.

When n is a multiple of 4, $p = \frac{n}{2}$ and M is oriented, the Bianchi identity for R_p admits another interpretation in terms of the Hodge star operator on

$\bigwedge^p(M)$:

$$\begin{aligned} Alt\ R_p(e_1,\ldots,e_n) &= \frac{1}{n!}\sum_{r\in S_n}\varepsilon(r) < \mathbb{R}_p(e_{r(1)}\wedge\cdots\wedge e_{r(p)}), e_{r(p+1)}\wedge\cdots\wedge e_{r(n)} > \\ &= \frac{1}{n!}\sum_{r\in S_n} < \mathbb{R}_p(e_{r(1)}\wedge\cdots\wedge e_{r(p)}), *(e_{r(1)}\wedge\cdots\wedge e_{r(p)}) > \\ &= \frac{1}{n!}\sum_{r\in S_n} < *\ \mathbb{R}_p(e_{r(1)}\wedge\cdots\wedge e_{r(p)}), e_{r(1)}\wedge\cdots\wedge e_{r(p)} > \\ &= \frac{p!\,p!}{n!}\,tr\ *\ \mathbb{R}_p \end{aligned}$$

and hence for the case $p = \frac{n}{2}$, the Bianchi identity for $\mathbb{R}_p$ reduces to

$$trace\ *\ \mathbb{R}_p = 0.$$

Let $\mathcal{P} \in G_p(M)$, where the Grassmann bundle $G_p(M)$ of oriented tangent $p-$planes of M shall be viewed as a subbundle of the unit sphere bundle of $\bigwedge^p(M)$ by identifying $\mathcal{P} \in G_p(M)$ with $e_1 \wedge \cdots \wedge e_p \in \bigwedge^p(M)$, where $\{e_1, \cdots, e_p\}$ is any oriented orthonormal basis for $\mathcal{P}$. Let $\{e_1, \ldots, e_p\}$ be an oriented orthonormal basis for $\mathcal{P}$. Then

$$\mathbb{R}_p(\mathcal{P}) = \frac{1}{p!}\sum_{\alpha\in S_p}\varepsilon(\alpha)\mathbb{R}(e_{\alpha(1)}\wedge e_{\alpha(2)})\wedge\cdots\wedge\mathbb{R}(e_{\alpha(p-1)}\wedge e_{\alpha(p)})$$

and suppose $p \geq 0$ and $q \geq 0$ are even integers with $p + q \leq n$. For $\mathcal{P} \in G_{p+q}(M)$, let $\{e_1, \ldots, e_{p+q}\}$ be an orthonormal basis for $\mathcal{P}$ and let us consider $\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq p + q\}$, then $\mathcal{B} \subset G_p(M)$ and

$$\mathbb{R}_{p+q}(\mathcal{P}) = \frac{p!\ q!}{(p+q)!}\sum_{Q\in\mathcal{B}}\mathbb{R}_p(Q)\wedge\mathbb{R}_q(Q^{\perp})$$

where $Q^{\perp}$ is the oriented orthogonal complement of Q in $\mathcal{P}$ [Tho69].

Now we shall see that the Lipschitz-Killing curvature K of M can be expressed by $\mathbb{R}_p$ and the Hodge star operator $*$.

Let M be an oriented Riemannian manifold of even dimension n, then according to [Tho69], the Lipschitz-Killing curvature K of M is the function whose value at $x \in M$ is

$$\frac{p!\ (n-p)!}{n!}\ trace\,(\,*\,\mathbb{R}_{n-p} * \mathbb{R}_p)\,.$$

For an oriented Riemannian manifold of dimension $n = 4k$, we can consider the middle curvature operator $\mathbb{R}_{2k}$, and if this operator satisfies the condition

$$\mathbb{R}_{2k} * = * \,\mathbb{R}_{2k}\,,$$

then, since $*^2 = Identity$, the trace formula for K reduces to

$$K = \frac{[(2k)!]^2}{(4k)!}\ trace\ \mathbb{R}_{2k}^2 \geq 0$$

Given a Riemannian manifold of dimension n and an even integer p, with $2 \leq p \leq n$, the p^{th}–sectional curvature of M is the function $\sigma_p : G_p(M) \longrightarrow \mathbb{R}$ defined by

$$\sigma_p(\mathcal{P}) =< \mathbb{R}_p\mathcal{P}, \mathcal{P} >\,.$$

If $\mathbb{R}_p$ and $\mathbb{R}'_p$ are two self-adjoint operators on $\bigwedge^p(x)$, each satisfying the Bianchi identity, such that

$$< \mathbb{R}'_p\mathcal{P}, \mathcal{P} >=< \mathbb{R}_p\mathcal{P}, \mathcal{P} >$$

for all $\mathcal{P}$ in the fiber of $G_p(M)$ over x. Then $\mathbb{R}'_p = \mathbb{R}_p$ [Tho64, Tho69]. This allows another interpretation of the Thorpe condition.

Theorem 1.2.1 *Let M be an oriented Riemannian manifold of dimension $4k$, then we have $\mathbb{R}_{2k} * = * \mathbb{R}_{2k}$ if and only if $\sigma_{2k}(\mathcal{P}^{\perp}) = \sigma_{2k}(\mathcal{P})$, for all $\mathcal{P} \in G_{2k}(M)$.*

Proof. We have

$$\begin{aligned} \sigma_{2k}(\mathcal{P}^{\perp}) &= < \mathbb{R}_{2k} * \mathcal{P}, * \mathcal{P} > \\ &= < * \mathbb{R}_{2k} * \mathcal{P}, \mathcal{P} > \end{aligned}$$

And hence $\mathbb{R}_{2k} * = * \mathbb{R}_{2k}$ implies $\sigma_{2k}(\mathcal{P}^{\perp}) = \sigma_{2k}(\mathcal{P})$ for all $\mathcal{P} \in G_{2k}(M)$.

Conversely, if $\sigma_{2k}(\mathcal{P}^{\perp}) = \sigma_{2k}(\mathcal{P})$ for all $\mathcal{P} \in G_{2k}(M)$, then

$$\begin{aligned} < \mathbb{R}_{2k} * \mathcal{P}, * \mathcal{P} > &= < * \mathbb{R}_{2k} * \mathcal{P}, \mathcal{P} > \\ &= < \mathbb{R}_{2k} \mathcal{P}, \mathcal{P} > \end{aligned}$$

for all $\mathcal{P} \in G_{2k}(M)$. Furthermore, both the self-adjoint operators, $\mathbb{R}_{2k}$ and $* \mathbb{R}_{2k} *$ satisfy the Bianchi identity:

$$tr * (* \mathbb{R}_{2k} *) = tr\ \mathbb{R}_{2k} * = tr * \mathbb{R}_{2k} = 0$$

Hence, by the above statements, $* \mathbb{R}_{2k} * = \mathbb{R}_{2k}$; i.e. $\mathbb{R}_{2k} * = * \mathbb{R}_{2k}$. ☐

Now we can consider the necessary condition for the existence of a Thorpe metric [Tho69]:

Theorem 1.2.2 *Let M be a compact orientable $4k-$dimensional Riemannian manifold such that $\mathbb{R}_{2k} \, * = * \, \mathbb{R}_{2k}$, then*

$$\chi \geq \frac{k! \; k!}{(2k)!} \, |P_k|$$

where χ is the Euler characteristic of M and P_k is the k^{th} Pontrjagin number of M. And in particular $\chi \geq 0$.
Furthermore $\chi = 0$ if and only if M is $2k-$flat ($\sigma_{2k} \equiv 0$).

Proof. The de Rham representation for the k^{th} Pontrjagin [Tho64] class of M is the differential $4k-$form

$$\frac{[(2k)!]^3}{(2^k \, k!)^2 (2\pi)^{2k}} \; trace\,(\mathbb{R}_{2k} \; * \; \mathbb{R}_{2k}) \; dV$$

Since $\mathbb{R}_{2k}$ commutes with $*$, it also commutes with $I \pm *$, where I denotes the identity operator on $\bigwedge^{2k}$. Hence $\mathbb{R}_{2k}(I \pm *)$ is self adjoint and

$$0 \leq tr \; [\mathbb{R}_{2k}(I \pm *)]^2 = 2 \left[tr \, (\mathbb{R}_{2k})^2 \pm tr \; (\mathbb{R}_{2k} \; * \; \mathbb{R}_{2k}) \right]$$

and so

$$tr \, (\mathbb{R}_{2k})^2 \; \geq |tr \; (\mathbb{R}_{2k} \; * \; \mathbb{R}_{2k})|$$

that means

$$\chi \geq \frac{k! \; k!}{(2k)!} \, |P_k| \, ,$$

and since $K \geq 0$, we have $\chi = 0$ if and only if K is identically zero. $K \equiv 0$ is equivalent to $\mathbb{R}_{2k} = 0$, which in turn is equivalent to $\sigma_{2k} = 0$. ☐

Remark.

1. Let M be any compact orientable hyperbolic manifold of dimension $4k$. Then the Euler characteristic of M is positive.

2. Let N be any compact complex hyperbolic manifold of complex dimension $2k$. Then the Euler characteristic of N is positive.

1.3 Examples

We can now describe some examples of Riemannian manifolds which allow a Thorpe metric.

- (H^{4k}, g_{-1}), (S^{4k}, g_1), $(\mathbb{R}^{4k}, g_o)$ where g_{-1} is the $(-1)-$constant sectional curvature metric, g_1 is the $(1)-$constant sectional curvature metric and g_o is the $(0)-$constant sectional curvature metric.

  We can verify that constant sectional curvature metrics are Thorpe metrics.

- $\mathbb{C}H^{2k}$, $\mathbb{C}P^{2k}$ **and** $\mathbb{C}^{2k}$. The curvature tensor R of the Kähler manifold of c-constant holomorphic sectional curvature satisfies

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c}{4}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W) \\ &\quad +2g(IX,Y)g(IZ,W) + g(IX,Z)g(IY,W) \\ &\quad -g(IY,Z)g(IX,W)\} \end{aligned}$$

where I is the almost complex structure. For example the curvature tensor R of a Kähler manifold of (-1)-constant holomorphic sectional curvature satisfies

$$\begin{aligned} R(e_i, e_j, e_k, e_l) &= \frac{(-1)}{4}\{g(e_i, e_k)g(e_j, e_l) - g(e_j, e_k)g(e_i, e_l) \\ &\quad +2g(Ie_i, e_j)g(Ie_k, e_l) + g(Ie_i, e_k)g(Ie_j, e_l) \\ &\quad -g(Ie_j, e_k)g(Ie_i, e_l)\} \end{aligned}$$

where I is the almost complex structure and $\{e_1, Ie_1, e_2, Ie_2, \ldots, e_{2k}, Ie_{2k}\}$ is an oriented orthonormal frame. So by the fact

$$\begin{aligned} R(e_i, Ie_i, e_i, Ie_i) &= -1 \\ R(e_i, Ie_i, e_j, Ie_j) &= -\frac{1}{2} \quad (i \neq j) \\ R(e_i, e_j, e_i, e_j) &= -\frac{1}{4} \quad (i \neq j) \\ R(e_i, Ie_j, e_i, Ie_j) &= -\frac{1}{4} \quad (i \neq j) \\ R(Ie_i, Ie_j, Ie_i, Ie_j) &= -\frac{1}{4} \quad (i \neq j) \end{aligned}$$

and all the other cases are 0, we know that

$$R_{2k}\left(e_{i_1}, \ldots, e_{i_{2k}}, e_{j_1}, \ldots, e_{j_{2k}}\right) = R_{2k}\left(e_{i_{2k+1}}, \ldots, e_{i_{4k}}, e_{j_{2k+1}}, \ldots, e_{j_{4k}}\right)$$

where $\{e_{i_1}, \ldots, e_{i_{4k}}\} = \{e_{j_1}, \ldots, e_{j_{4k}}\} = \{e_i, Ie_i\}_{i=1}^{2k}$ is an oriented orthonormal frame. Therefore we can verify the Thorpe condition in this case.

- **Quaternion projective spaces $\mathbb{H}P^1$, $\mathbb{H}P^2$.**

In the case of $\mathbb{H}P^1$, the standard metric is an Einstein metric and the

Einstein condition is equivalent to the Thorpe condition in case of dimension 4.

On the other hand , in the case of $\mathbb{H}P^2$ the curvature tensor R of the canonical quaternion projective space satisfies

$$\begin{aligned} 4R(X,Y,Z,W) &= g(X,Z)g(Y,W) - g(Y,Z)g(X,W) \\ &+2g(IX,Y)g(IZ,W) + g(IX,Z)g(IY,W) \\ &-g(IY,Z)g(IX,W) \\ &+2g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) \\ &-g(JY,Z)g(JX,W) \\ &+2g(KX,Y)g(KZ,W) + g(KX,Z)g(KY,W) \\ &-g(KY,Z)g(KX,W). \end{aligned}$$

where I, J and K are the almost complex structures and so we know that

$$R_4\left(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}, e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4}\right) = R_4\left(e_{i_5}, e_{i_6}, e_{i_7}, e_{i_8}, e_{j_5}, e_{j_6}, e_{j_7}, e_{j_8}\right)$$

where $\{e_{i_1}, \ldots, e_{i_8}\} = \{e_{j_1}, \ldots, e_{j_8}\} = \{e_i, Ie_i, Je_i, Ke_i\}_{i=1}^{2}$ is an oriented orthonormal frame, therefore we can verify the Thorpe condition in this case.

Remark. The curvature tensor R of the canonical quaternion projective space $\mathbb{H}P^n$ $(n \geq 3)$ does not satisfy the Thorpe condition. For example, on $\mathbb{H}P^3$

$$R_6\left(e_1, e_2, e_3, Ie_3, Je_3, Ke_3, e_1, e_2, e_3, Ie_3, Je_3, Ke_3\right) \neq$$

$$R_6\left(Ie_1, Je_1, Ke_1, Ie_2, Je_2, Ke_2, Ie_1, Je_1, Ke_1, Ie_2, Je_2, Ke_2\right)$$

where R_6 is the 6^{th} curvature tensor.

Question: Is $\mathbb{H}P^n$ $(n \geq 3)$ a Thorpe manifold or not?

Chapter 2

Thorpe metrics on T^{4k} and $M^{4k-p} \times T^{4k+p}$

2.1 An irreducible decomposition for the curvature tensor

The curvature tensor of a Riemannian manifold M^n splits naturally into three components involving respectively its scalar curvature, the traceless part of its Ricci tensor and its Weyl curvature tensor (if $n \geq 4$).

The bundle in which the curvature tensor naturally lives is not irreducible under the action of the orthogonal group, and consequently has a natural decomposition into irreducible components.

Let E be an $n-$dimensional real vector space then each tensor space $T^{(k,l)}E = \left(\otimes^k E^*\right)\otimes\left(\otimes^l E\right)$ is a representation space for the linear group $GL(E)$.

For any $r \in GL(E)$, $\xi_1, \ldots, \xi_k \in E^*$ and $x_1, \ldots, x_l \in E$, the natural action of $GL(E)$ satisfies.

$$r\left(\xi_1 \otimes \cdots \otimes \xi_k \otimes x_1 \otimes \cdots \otimes x_l\right) = \left({}^t r^{-1}\xi_1\right) \otimes \cdots \otimes \left({}^t r^{-1}\xi_k\right) \otimes \left(rx_1\right) \otimes \cdots \otimes \left(rx_l\right)$$

Let g be a non-degenerate quadratic form on E, then g induces a canonical identification between E and E^*. Moreover, if r belongs to the orthogonal group $O(g)$ of g, we have ${}^t r^{-1} = r$, so E and E^* are isomorphic as $O(g)-$modules, and we may consider tensor products of E only.

Of course, the $O(g)-$module E is irreducible. It is well known that $\otimes^2 E$ is not irreducible. We denote by $S^2_o E$ the space of traceless symmetric $2-$tensors and we may consider a linear map $tr_g : S^2 E \longrightarrow \mathbb{R}$.

The irreducible orthogonal decomposition of the $O(g)-$module $\otimes^2 E$ is the following

$$\otimes^2 E = \bigwedge^2 E \oplus S^2_o E \oplus \mathbb{R}g.$$

We define the Bianchi map b to be the following idempotent of $\otimes^4 E$

$$b(R)(x,y,z,t) = \frac{1}{3}\left(R(x,y,z,t) + R(y,z,x,t) + R(z,x,y,t)\right)$$

for any R in $\otimes^4 E$ and $x, y, z, t \in E^*$, b is $GL(E)-$equivariant, $b^2 = b$

and b maps $S^2 \bigwedge^2 E$ into itself. So we have the $GL(E)-$equivariant decomposition

$$S^2 \bigwedge^2 E = Ker\ b \oplus\ Im\ b.$$

We let $\mathcal{R}E = Ker\ b$ in $S^2 \bigwedge^2 E$ and we call it the vector space of "algebraic curvature tensors". The Ricci contraction is the $O(g)-$equivariant map $C : S^2 \bigwedge^2 E \longrightarrow S^2 E$ defined for any $R \in S^2 \bigwedge^2 E$ and any $x, y \in E^*$ by $C(R)(x,y) = tr\, R(x, \cdot, y, \cdot)$. The Kulkarni-Nomizu product of two symmetric $2-$tensors h and k is the $4-$tensor $h \owedge k$ given by

$$\begin{aligned} (h \owedge k)(x,y,z,t) &= h(x,z)k(y,t) + h(y,t)k(x,z) \\ &\quad -h(x,t)k(y,z) - h(y,z)k(x,t) \end{aligned}$$

for any $x, y, z, t \in E^*$. Now we come to the following fundamental result [Bes86]:

If $n \geq 4$, the $O(g)-$ module $\mathcal{R}E$ has the following orthogonal decomposition into (unique) irreducible subspaces

$$\mathcal{R}E = \mathcal{U}E \oplus \mathcal{Z}E \oplus \mathcal{W}E$$

where

$$\begin{aligned} \mathcal{U}E &= \mathbb{R}(g \owedge g) \\ \mathcal{Z}E &= g \owedge \left(S^2_o E\right) \\ \mathcal{W}E &= Ker\ C \cap Ker\ b. \end{aligned}$$

If we consider now g as a Riemannian metric and E as $T_x^*(M)$ and then write $r = C(R)$ and $s = tr\ r$, we get the formula

$$R = \frac{s}{2n(n-1)} g \owedge g + \frac{1}{(n-2)} ric_o \owedge g + W$$

where s is the scalar curvature, ric_o is the traceless Ricci curvature and W is the Weyl curvature.

If we consider subgroups of $O(g)$, then more refined decompositions may appear. We consider the case of the special orthogonal group $SO(g)$, which corresponds geometrically to the case where M is oriented.

It is well known that the irreducible decomposition of the $O(n)-$module $\mathcal{R}E$ is also $SO(n)-$irreducible if $n \neq 4$.

But new phenomena occur when $n = 4$. The irreducible decomposition of $\mathcal{R}E$ as an $SO(4)-$module is the following

$$\mathcal{R}E = \mathcal{U}E \oplus \mathcal{Z}E \oplus \mathcal{W}^+E \oplus \mathcal{W}^-E$$

where the Weyl tensor splits in two parts, which we have denoted by $\mathcal{W}^+$ and $\mathcal{W}^-$ [Bes86].

If we consider g as a Riemannian metric and E as $T_x^*(M)$ and then write $r = C(R)$ and $s = tr\ r$, we get the formula

$$R = \frac{s}{2n(n-1)} g \owedge g + \frac{1}{(n-2)} ric_o \owedge g + W^+ + W^-$$

where s is the scalar curvature, ric_o is the traceless Ricci curvature, W^+ is the self-dual Weyl curvature, and W^- is the anti-self-dual Weyl curvature.

2.2 About Thorpe metrics on 4-dimensional manifolds

In this section we shall see that in the 4−dimensional case, the Thorpe condition is equivalent to that of Einstein.

If we consider $R_{2k} \equiv R$ in $S^2 \bigwedge^2 T^*M$ as a linear map of $\bigwedge^2 T^*M$ and if we decompose

$$\bigwedge^2 T^*M = \bigwedge^+ T^*M \oplus \bigwedge^- T^*M$$

where $\bigwedge^+ T^*M$ is the $(+1)$−eigenspace (self-dual space) and $\bigwedge^- T^*M$ is the (-1)−eigenspace (anti-self-dual space) of the Hodge $*$ operator, respectively, we get the following expression for R [Bes86],

$$\begin{array}{cccc} & \textit{self-dual} & \textit{anti-self-dual} & \\ R = & \left(\begin{array}{c|c} W^+ + \frac{s}{12} Id & ric_o \\ \hline {}^t ric_o & W^- + \frac{s}{12} Id \end{array}\right) & & \begin{array}{c} \textit{self-dual} \\ \\ \textit{anti-self-dual} \end{array} \end{array}$$

where s is the scalar curvature, ric_o is the traceless Ricci curvature, W^+ is the self-dual Weyl curvature, and W^- is the anti-self-dual Weyl curvature.

We show now that it is possible to interpret $\mathbb{R}* = *\mathbb{R}$ as $\mathbb{R}(A_1) = A_2$ and $\mathbb{R}(B_1) = B_3$ where A_1, $A_2 \in \bigwedge^+ T^*M$ and $B_1, B_3 \in \bigwedge^- T^*M$ in the following way:

$$\mathbb{R} * (A_1) = \mathbb{R}(A_1) = A_2 + B_2$$

on the other hand

$$*\mathbb{R}(A_1) = *(A_2 + B_2) = A_2 - B_2$$

and hence, $\mathbb{R}* = *\mathbb{R}$ implies $B_2 \equiv 0$, i.e. $\mathbb{R}(A_1) = A_2$.

In the same way

$$\mathbb{R} * (B_1) = \mathbb{R}(-B_1) = -A_3 - B_3$$

on the other hand

$$*\mathbb{R}(B_1) = *(A_3 + B_3) = A_3 - B_3$$

and hence $\mathbb{R}* = *\mathbb{R}$ implies $A_3 \equiv 0$, i.e. $\mathbb{R}(B_1) = B_3$.

Therefore we can see that $*\mathbb{R} = \mathbb{R}*$ is equivalent to the vanishing of the traceless Ricci curvature, which implies the Einstein condition.

Furthermore the Euler characteristic of oriented compact 4−dimensional manifolds M can be expressed in the following way [Bes81]:

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(|W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|ric_o|^2}{2} \right) dV$$

The Thorpe condition implies $\chi \geq 0$ and $\chi = 0$ if and only if the Thorpe metric is flat, in particular, every Thorpe metric on T^4 is flat and $S^1 \times S^3$ never carries a Thorpe metric by the Cartan-Hadamard theorem.

2.3 Thorpe metrics on T^{4k} and $M^{4k-p} \times T^{4k+p}$

The purpose of this section is to prove the following statement.

Theorem 2.3.1 *There are non flat Thorpe metrics on T^{4k} for $k > 1$.*

Proof. In the case of $k = 1$, we have already seen in section 2.2 that every Thorpe metric on T^4 is flat. On the other hand, we can construct infinitely many Thorpe metrics on T^{4k} for $k > 1$. First we can consider T^{4k} $(k > 1)$ as $T^p \times T^{4k-p}$ $(2 \leq p \leq 2k-1)$ and take the product metric $(g_{T^p} + g_{T^{4k-p}})$ such that $g_{T^{4k-p}}$ is a standard flat metric on the torus T^{4k-p} and g_{T^p} is a non-flat Riemannian metric on T^p. This is always possible since we can construct non-flat metrics on small open sets and use a partition of unity function to construct a global non-flat metric on T^P. On the other hand, the curvature R of a product metric $g = g_{T^p} + g_{T^{4k-p}}$ is

$$R = R_{T^p} + R_{T^{4k-p}}$$

and this implies that the product metric g is $2k-$flat and so we can see that R satisfies the Thorpe condition $\mathbb{R}_{2k} * = * \mathbb{R}_{2k}$. Hence our constructed metric, which is not-flat, is a Thorpe metric on T^{4k}. Furthermore we can see that there are infinitely many choices of non-flat metrics on small open sets and a partition of unity function and so there are infinitely many non-flat Thorpe metrics on T^{4k}. And this completes the proof. □

We can apply the same argument in the following case:

Theorem 2.3.2 *There are infinitely many Thorpe metrics on $M^{4k-p} \times T^{4k+p}$ for $p, k \geq 1$, where M^{4k-p} is any compact oriented manifold of dimension $4k - p \geq 2$.*

Proof. We can consider a product metric $g = g_{M^{4k-p}} + g_{T^{4k+p}}$ on $M^{4k-p} \times T^{4k+p}$ for $p, k \geq 1$, such that $g_{T^{4k+p}}$ is a standard flat metric on the Torus T^{4k+p} and $g_{M^{4k-p}}$ is a non-flat Riemannian metric on M^{4k-p} $(4k - p \geq 2)$. This is always possible since we can construct non-flat metrics on small open sets and use a partition of unity function to construct a global non-flat metric on M^{4k-p}. On the other hand, the curvature R of a product metric $g = g_{M^{4k-p}} + g_{T^{4k+p}}$ is

$$R = R_{M^{4k-p}} + R_{T^{4k+p}}$$

and this implies that the product metric g is $4k-$flat and so we can see that R satisfies the Thorpe condition

$$\mathbb{R}_{4k} * = * \mathbb{R}_{4k}.$$

Hence our constructed metric is a Thorpe metric on $M^{4k-p} \times T^{4k+p}$. Furthermore we can see that there are infinitely many choices of non-flat metrics on small open sets and a partition of unity function and so there are infinitely many non-flat Thorpe metrics on $M^{4k-p} \times T^{4k+p}$ and this completes the proof.

□

Example 2.3.1 *The following manifolds are Thorpe manifolds*

- $RP^{4k} \times T^{4p} \quad (p > k > 0)$
- $\mathbb{C}P^{2q} \times T^{4s} \quad (s > q > 0)$
- $\mathbb{H}P^{k} \times T^{4s} \quad (s > k > 0)$
- $H^{4k} \times T^{4p} \quad (p > k > 0)$
- $\mathbb{C}H^{2k} \times T^{4p} \quad (p > k > 0)$

Proof. By theorem2.3.2 the above described manifolds are Thorpe manifolds.

□

2.4 Other examples

In Section 2.3 we have seen that the product manifolds of T^{4k+p} with any compact oriented manifold of dimension $4k - p \geq 2$ for $p, k \geq 1$ admit Thorpe product metrics. In this Section we can see that the product manifolds of S^{4k} with hyperbolic manifold of dimension $4k$ admit a Thorpe product metric which is not an Einstein metric and also that the product manifold of $\mathbb{C}P^2$ with complex hyperbolicc manifold of complex dimension 2 admits a Thorpe product metric which is not an Einstein metric and hence in the case of dimensions greater than 4, a Thorpe metric, in general, does not imply an Einstein metric.

Theorem 2.4.1 *If M^{4k} carries a metric with positive constant sectional curvature and if N^{4k} carries a metric with negative constant sectional curvature, then $M^{4k} \times N^{4k}$ is a Thorpe manifold.*

Proof. After rescaling, the curvature R of the Riemannian product manifold is

$$\begin{aligned} R &= R_M + R_N \\ &= \frac{1}{2} g_M \owedge g_M - \frac{1}{2} g_N \owedge g_N \end{aligned}$$

and hence, the only non-zero terms in the $4k^{th}$ curvature tensor are those which are products of sectional curvatures. All the other terms are zero. And so we can verify, case by case, that the product metric is a Thorpe metric. □

The above theorem provides us with a nice example of a Thorpe product manifold, for instance, if we take

$$(M^{4k}, g_M) = (S^{4k}, g_1)$$

$$(N^{4k}, g_N) = (H^{4k}, g_{-1})$$

then

$$(S^{4k} \times H^{4k}, g_1 + g_{-1})$$

is a Thorpe manifold.

On the other hand, this product metric is not an Einstein metric and hence in the case of dimensions greater than 4, a Thorpe metric, in general, does not imply an Einstein metric.

Remark. We can construct more Thorpe product manifolds in the following way:

- $S^4 \times \cdots \times S^4$, with the product metric of standard ones.
- $H^4 \times \cdots \times H^4$, with the product metric of standard ones.
- $H^4 \times \cdots \times H^4 \times S^4 \times \cdots \times S^4$, with the product metric of standard ones.

- $\prod_{p\geq 1}^{2p}(S^2 \times H^2)$, with the product metric of standard ones.

We have seen $H^{4k} \times S^{4k}$ is a Thorpe manifold and so we can raise the natural question of whether $\mathbb{C}H^{2k} \times \mathbb{C}P^{2k}$ is a Thorpe manifold or not?

More explicitly, is the product of two Riemannian manifolds, whose holomorphic sectional curvatures are constant, a Thorpe manifold?

The following examples are Thorpe manifolds:

$$\begin{array}{ccc} \mathbb{C}P^2 & \times & \mathbb{C}P^2 \\ \mathbb{C}H^2 & \times & \mathbb{C}H^2 \\ \mathbb{C}P^2 & \times & \mathbb{C}H^2 \end{array}$$

It is not difficult to prove, by using a Kähler property and duality, that the product metric of standard metrics gives us a Thorpe metric.

On the other hand, the product metric of standard metrics on the manifold $\mathbb{C}P^{2k} \times \mathbb{C}H^{2k}$, with $k \geq 2$, is not a Thorpe metric.

Question: Is $\mathbb{C}P^{2k} \times \mathbb{C}H^{2k}$, with $k \geq 2$, a Thorpe manifold or not?

Remark 2.4.1 *We can construct more Thorpe product manifolds in the following way:*

- $\mathbb{C}P^2 \times \cdots \times \mathbb{C}P^2$, with the product metric of standard ones.
- $\mathbb{C}H^2 \times \cdots \times \mathbb{C}H^2$, with the product metric of standard ones.
- $\mathbb{C}P^2 \times \cdots \times \mathbb{C}P^2 \times \mathbb{C}H^2 \times \cdots \times \mathbb{C}H^2$, with the product metric of standard ones.
- $\mathbb{H}P^1 \times \mathbb{H}P^1$, with the product metric of standard ones.

Remark 2.4.2 *If a compact orientable manifold M^{4k} has non-zero Euler characteristic, then the product manifolds $M^{4k} \times T^{4k}$; which have Euler characteristic and $2k^{th}$ Pontrjagin class equal to zero, have the following property: The product metric of any metrics on M^{4k} with any metrics on T^{4k} is not a Thorpe metric. The following compact orientable Riemannian manifolds are not Thorpe manifolds.*

- $\left(\mathbb{C}P^{2k} \times T^{4k}, g_{\mathbb{C}P^{2k}} + g_{T^{4k}}\right)$
- $\left(\mathbb{H}P^{k} \times T^{4k}, g_{\mathbb{H}P^{k}} + g_{T^{4k}}\right)$
- $\left(H^{4k}/\Gamma \times T^{4k}, g_{H^{4k}/\Gamma} + g_{T^{4k}}\right)$

- $\left(\mathbb{C}H^{2k}/\Gamma \times T^{4k}, g_{\mathbb{C}H^{2k}/\Gamma} + g_{T^{4k}}\right)$

Remark 2.4.3 *The following simply connected manifolds do not carry a Thorpe metric because of their negative Euler charateristic,* $\chi < 0$.

- $\left(S^{4p+3} \times S^{4q+1}\right) \# \cdots \# \left(S^{4p+3} \times S^{4q+1}\right)$, *with* $p, q \geq 1$, *and where* $\#$ *denotes connected sum.*

- $\left(S^{4p+3} \times S^{4q+1} \times S^{4r}\right) \# \cdots \# \left(S^{4p+3} \times S^{4q+1} \times S^{4r}\right)$, *with* $p, q, r \geq 1$, *and where* $\#$ *denotes connected sum.*

- $\left(S^{4p+3} \times S^{4q+1} \times S^{4r} \times S^{4s}\right) \# \cdots \# \left(S^{4p+3} \times S^{4q+1} \times S^{4r} \times S^{4s}\right)$, *with* $p, q, r, s \geq 1$, *and where* $\#$ *denotes connected sum.*

- $\left(S^{4p+1} \times S^{4q+1} \times S^{4r+1} \times S^{4s+1}\right) \# \cdots \# \left(S^{4p+1} \times S^{4q+1} \times S^{4r+1} \times S^{4s+1}\right)$, *with* $p, q, r, s \geq 1$, *and where* $\#$ *denotes connected sum.*

- $\left(S^{4p+3} \times S^{4q+3} \times S^{4r+3} \times S^{4s+3}\right) \# \cdots \# \left(S^{4p+3} \times S^{4q+3} \times S^{4r+3} \times S^{4s+3}\right)$, *with* $p, q, r, s \geq 1$, *and where* $\#$ *denotes connected sum.*

Chapter 3

Einstein-Thorpe metrics

3.1 Uniqueness for Einstein-Thorpe metrics

In this section, we shall show the main theorems in this dissertation, namely that every Einstein-Thorpe metric on T^8 is flat and that every Einstein-Thorpe metric on a compact oriented hyperbolic 8-dimensional manifold is a hyperbolic metric.

Lemma 3.1.1 *Let (M, g) be a Riemannian manifold of dimension 8, then*

$$\text{trace } \mathbb{R}_4 = \frac{1}{2^2}\left(\frac{1}{6}\right)\left\{\frac{1}{2}S^2 - 4\left|ric_o\right|^2 + 4\left|R\right|^2\right\}$$

where S is the scalar curvature, ric_o is the traceless Ricci curvature and R is *the curvature.*

Proof. For 4−forms $\{e_a \wedge e_b \wedge e_c \wedge e_d\}$ and with the Einstein summation,

$$\begin{aligned} \text{trace}\ \mathbb{R}_4 &= \frac{1}{2^2} R^{[ab}_{[ab}\, R^{cd]}_{cd]} \\ &= \frac{1}{2^2} R^{ab}_{[ab}\, R^{cd}_{cd]} \\ &= \frac{1}{2^2}\left(\frac{1}{6}\right)\{R^{ab}_{ab}\, R^{cd}_{cd} + R^{ab}_{ac}\, R^{cd}_{db} + R^{ab}_{ad}\, R^{cd}_{bc} \\ &\qquad + R^{ab}_{bc}\, R^{cd}_{ad} + R^{ab}_{bd}\, R^{cd}_{ca} + R^{ab}_{cd}\, R^{cd}_{ab}\} \end{aligned}$$

where a, b, c and d run from 1 to 8, [ ] is a skew symmetrization, and $\{e_k\}_{k=1}^{8}$. is an orthonormal frame. We analyze the terms of this sum individually.

(i) $R^{ab}_{ab}\, R^{cd}_{cd} = \frac{1}{2}S^2 - 4\, ric^c_{o\,c}\, ric^c_{o\,c} + 2R^{cd}_{cd}\, R^{cd}_{cd}$.

(ii) $R^{ab}_{ac}\, R^{cd}_{db} = -\, ric^b_{o\,c}\, ric^c_{o\,b} + R^{db}_{dc}\, R^{dc}_{db}$.

(iii) $R^{ab}_{cd}\, R^{cd}_{ab}$

and so we obtain

$$\begin{aligned} \text{trace}\ \mathbb{R}_4 &= \frac{1}{2^2}\left(\frac{1}{6}\right)\left\{\frac{1}{2}S^2 - 4\, ric^c_{o\,c}\, ric^c_{o\,c} - 4\, ric^b_{o\,c}\, ric^c_{o\,b} \right. \\ &\qquad \left. + 2R^{cd}_{cd}\, R^{cd}_{cd} + 4R^{db}_{dc}\, R^{dc}_{db} + R^{ab}_{cd}\, R^{cd}_{ab}\right\} \end{aligned}$$

and this completes the proof. □

Remark. On Einstein manifolds of dimension $8k$ ($k \geq 2$), trace $\mathbb{R}_{4k}$ cannot be expressed as an addition of S^{2k} with $|R|^{2k}$.

Now we obtain one of the main theorems in this dissertation.

Theorem 3.1.1 *Every Einstein-Thorpe metric on T^8 must be flat.*

Proof. A Thorpe metric on T^8 which has Euler characteristic equal to zero must be 4-th flat, hence trace $\mathbb{R}_4$ must be zero. From Lemma 3.1.1 and the Einstein condition, we can read that the curvature tensor must be zero. Hence every Einstein-Thorpe metric on T^8 must be flat. □

Now we obtain the other main theorem in this dissertation.

Theorem 3.1.2 *On compact oriented hyperbolic manifolds of dimension* 8, *every Einstein-Thorpe metric is a hyperbolic metric up to rescalings and diffeomorphisms.*

Proof. First of all, we know that every Einstein metric on compact hyperbolic manifolds has the negative Einstein constant: If this were not true, the Cheeger-Gromoll's splitting theorem tells us that compact hyperbolic manifolds allow a flat metric. But this is impossible because compact hyperbolic manifolds have fundamental groups different from those of flat manifolds. Therefore we can fix the scale that makes the given Einstein-Thorpe metric g have the same Einstein constant as that of the hyperbolic metric h_o. So from now, we consider the Einstein-Thorpe metric g which has the same Einstein constant as that of hyperbolic metric h_o.

Since the metric g is an Einstein metric, the pieces of trace $\mathbb{R}_4^g$ can be calculated as follows: For an orthonormal frame,

$$\begin{aligned} R_{ab}^{ab}\, R_{ab}^{ab} &= \frac{S^2}{8\cdot 7} + 2\frac{S}{8\cdot 7} W_{ab}^{ab} + W_{ab}^{ab}\, W_{ab}^{ab} \\ &= \frac{S^2}{8\cdot 7} + W_{ab}^{ab}\, W_{ab}^{ab} \end{aligned}$$

by the property of traceless W and

$$\begin{aligned} R_{ac}^{ab}\, R_{ab}^{ac} &= W_{ac}^{ab}\, W_{ab}^{ac} \\ R_{cd}^{ab}\, R_{ab}^{cd} &= W_{cd}^{ab}\, W_{ab}^{cd} \end{aligned}$$

where S is the scalar curvature, W is the Weyl curvature and $\mathbb{R}_4^g$ is the 4^{th} curvature operator with respect to the metric g. So from lemma 3.1.1, we can conclude that

$$\text{trace } \mathbb{R}_4^g \geq \text{trace } \mathbb{R}_4^{h_o}$$

by $|W|^2$ terms. Therefore we can obtain the following inequality

$$\text{trace } \mathbb{R}_4^g \mathbb{R}_4^g \;\geq\; \binom{8}{4} \left(\frac{\text{trace } \mathbb{R}_4^g}{\binom{8}{4}} \right)^2$$

$$\begin{aligned} &\geq \binom{8}{4}\left(\frac{\text{trace } \mathbb{R}_4^{h_o}}{\binom{8}{4}}\right)^2 \\ &= \text{trace } \mathbb{R}_4^{h_o}\mathbb{R}_4^{h_o} \end{aligned}$$

where $\binom{8}{4} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}$ and that equality holds if and only if the curvature is constant.

Furthermore, from the Thorpe condition, we get

$$\text{trace } \mathbb{R}_4^g\ \mathbb{R}_4^g = \text{trace } \ * \ \mathbb{R}_4^g\ * \ \mathbb{R}_4^g$$

and by the generalized Gauss-Bonnet theorem, we get

$$\int_M \text{trace } \ \mathbb{R}_4^g\ \mathbb{R}_4^g\ dV_g = \int_M \text{trace } \ \mathbb{R}_4^{h_o}\ \mathbb{R}_4^{h_o}\ dV_{h_o}$$

and so we obtain

$$\text{vol}_g(M) \leq \text{vol}_{h_o}(M).$$

Hence the Bishop-Gunther theorem tells us that

$$h^8(M,g)\text{vol}_g(M) \leq h^8(M,h_o)\text{vol}_{h_o}(M),$$

where $h(M,g)$ and $h(M,h_o)$ are the volume-entropies of g and h_o respectively.

On the other hand, we know that there is a universal inequality [BG95]

$$h^8(M,g)\text{vol}_g(M) \geq h^8(M,h_o)\text{vol}_{h_o}(M),$$

and so we can conclude that

$$h^8(M,g)\text{vol}_g(M) = h^8(M,h_o)\text{vol}_{h_o}(M),$$

and that g is isometric to h_o up to rescalings and diffeomorphisms, [BG95]. □

Remark. (i) On compact oriented hyperbolic manifolds of dimension 4, every Einstein metric is a hyperbolic metric up to rescalings and diffeomorphisms, [BG95].

(ii) On compact complex-hyperbolic manifolds of complex dimension 2, every Einstein metric is a standard complex- hyperbolic metric up to rescalings and diffeomorphisms, [LeB95]

3.2 Non-existence of Einstein-Thorpe metrics

In this section we can see that Einstein-Thorpe metrics do not exist on some manifolds of dimension 8 which satisfy $\chi = 0$ and $P_2 = 0$.

Theorem 3.2.1

(i) The product manifold of T^4 with any compact orientable hyperbolic manifold of dimension 4 does not admit an Einstein-Thorpe metric.

(ii) The product manifold of T^4 with any compact complex hyperbolic manifold of complex dimension 2 does not admit an Einstein-Thorpe metric.

The manifolds described in (i) and (ii) satisfy $\chi = 0$ and $P_2 = 0$.

Proof. Since the manifolds described in part (i) and (ii) of the statement of the theorem admit a nowhere vanishing vector field, the Hopf index theorem implies that the Euler characteristic of those manifolds must be zero.

The tangent bundle of $(H^4/\Gamma) \times T^4$ splits as a Whitney sum

$$\pi_1^*T\left(H^4/\Gamma\right) \oplus \pi_2^*T\left(T^4\right)$$

and the total Pontrjagin class satisfies the following relation:

$$P\left(\pi_1^*T\left(H^4/\Gamma\right) \oplus \pi_2^*T\left(T^4\right)\right) = P\left(\pi_1^*T\left(H^4/\Gamma\right)\right)$$

and so the 2^{th} Pontrjagin class,

$$P_2\left(\pi_1^*T\left(H^4/\Gamma\right) \oplus \pi_2^*T\left(T^4\right)\right) = P_2\left(\pi_1^*T\left(H^4/\Gamma\right)\right)$$

and furthermore

$$\begin{aligned} P_2\left(\pi_1^*T\left(H^4/\Gamma\right)\right) &= e\left(\pi_1^*T\left(H^4/\Gamma\right)\right) \cup e\left(\pi_1^*T\left(H^4/\Gamma\right)\right) \\ &= \pi_1^*\left[e\left(T\left(H^4/\Gamma\right)\right) \cup e\left(T\left(H^4/\Gamma\right)\right)\right] \\ &= 0 \end{aligned}$$

Where e denotes the Euler class and $\cup$ is the cup product. Therefore

$$P_2\left(\pi_1^*T\left(H^4/\Gamma\right) \oplus \pi_2^*T\left(T^4\right)\right) = 0$$

The same procedure can be applied to the manifold described in (ii) and so the manifolds described in (i) and (ii) have $\chi = 0$ and $P_2 = 0$.

So any Thorpe metric on the manifold described in (i) and (ii) must be 4^{th} flat, therefore an Einstein-Thorpe metric must be flat by Lemma 3.1.1. This implies that compact (complex) hyperbolic manifolds must allow flat metrics but this is impossible because compact (complex) hyperbolic manifolds have fundamental groups different from those of flat manifolds. Therefore the manifolds described in the statement of the theorem do not admit any Einstein-Thorpe metric. □

Example 3.2.1 *The following manifolds do not carry an Einstein-Thorpe metric*

(i)	$S^7 \times S^1$	(ii)	$S^6 \times S^1 \times S^1$
(iii)	$S^5 \times S^3$	(iv)	$S^5 \times S^2 \times S^1$
(v)	$S^4 \times S^3 \times S^1$	(vi)	$S^3 \times S^1 \times S^3 \times S^1$
(vii)	$S^3 \times S^2 \times S^2 \times S^1$	(viii)	$S^2 \times S^2 \times S^2 \times S^1 \times S^1$

Proof. Suppose that the manifolds described in the above statements allow an Einstein-Thorpe metric. Since these manifolds have $\chi = 0$, every Thorpe metric on them must be 4^{th} flat. And so trace $\mathbb{R}_4$ must be zero and this implies that the curvature must be zero by the Lemma 3.1.1 and the Einstein condition. This contradicts the Cartan-Hadamard theorem and completes the proof. □

3.3 In the case of $\chi = (\frac{2!\,2!}{4!})|P_2|$

In this section we can see that a compact orientable Einstein-Thorpe manifold of dimension 8 that satisfies the above topological equality must be flat.

Theorem 3.3.1 *Suppose that (M^8, g) is a compact orientable Einstein-Thorpe manifold and that*

$$\chi = \left(\frac{2!\,2!}{4!}\right)|P_2|$$

then (M^8, g) must be a flat manifold, i.e. $(T^8/\Gamma,\ flat)$ where Γ is of finite order.

Proof. The above topological condition together with the Thorpe condition can be expressed by

$$\text{trace}\,\mathbb{R}_4\ \mathbb{R}_4 = |\text{trace}\,\mathbb{R}_4\ *\ \mathbb{R}_4|$$

Now consider any orthonormal basis $A_i \in \bigwedge^+(M)$, and any orthonormal basis $B_i \in \bigwedge^-(M)$ where $\bigwedge^+(M)$ and $\bigwedge^-(M)$ denote the self dual space and the anti selfdual space with respect to the Hodge $*$ operator repectively. Then we get

$$< \mathbb{R}_4(A_i), \mathbb{R}_4(A_i) > \ = |R_4(A_i, A_j)|^2 + |R_4(A_i, B_j)|^2$$

$$\begin{aligned} <\mathbb{R}_4(B_i), \mathbb{R}_4(B_i)> &= |R_4(B_i, B_j)|^2 + |R_4(B_i, A_j)|^2 \\ <*\mathbb{R}_4(A_i), \mathbb{R}_4(A_i)> &= |R_4(A_i, A_j)|^2 - |R_4(A_i, B_j)|^2 \\ <*\mathbb{R}_4(B_i), \mathbb{R}_4(B_i)> &= |R_4(B_i, A_j)|^2 - |R_4(B_i, B_j)|^2 \end{aligned}$$

So we get

$$\text{trace } \mathbb{R}_4 \ \mathbb{R}_4 = |R_4(A_i, A_j)|^2 + |R_4(A_i, B_j)|^2 + |R_4(B_i, B_j)|^2 + |R_4(B_i, A_j)|^2$$

and

$$\text{trace}\, \mathbb{R}_4 \ * \ \mathbb{R}_4 = |R_4(A_i, A_j)|^2 - |R_4(A_i, B_j)|^2 + |R_4(B_i, A_j)|^2 - |R_4(B_i, B_j)|^2$$

If we assume $\text{trace}\, \mathbb{R}_4 \ * \ \mathbb{R}_4 \geq 0$, then by the given condition

$$\text{trace } \mathbb{R}_4 \ \mathbb{R}_4 = \text{trace}\, \mathbb{R}_4 \ * \ \mathbb{R}_4$$

and this implies

$$R_4(A_i, B_j) = R_4(B_i, B_j) = 0$$

therefore

$$\mathbb{R}_4^- = \frac{\mathbb{R}_4 - * \mathbb{R}_4}{2} \equiv 0.$$

Furthermore by the Bianchi identity,

$$\text{trace } * \ \mathbb{R}_4 \equiv 0$$

and so we obtain

$$\text{trace } \mathbb{R}_4 \equiv 0.$$

Hence by Lemma 3.1.1, we can see that the Einstein-Thorpe metric g is a flat metric. On the other hand, if we assume

$$\text{trace}\, \mathbb{R}_4 \ * \ \mathbb{R}_4 \leq 0,$$

then by the given condition

$$\operatorname{trace} \mathbb{R}_4 \ \mathbb{R}_4 = -\operatorname{trace} \mathbb{R}_4 \ * \ \mathbb{R}_4$$

and this implies

$$R_4(A_i, A_j) = R_4(B_i, A_j) = 0$$

and so we get

$$\mathbb{R}_4^+ = \frac{\mathbb{R}_4 + * \mathbb{R}_4}{2} \equiv 0.$$

Furthermore by the Bianchi identity,

$$\operatorname{trace} \ * \ \mathbb{R}_4 \equiv 0$$

and so we obtain

$$\operatorname{trace} \mathbb{R}_4 \equiv 0.$$

Hence by Lemma 3.1.1, we can see that the Einstein-Thorpe metric g is a flat metric. Therefore under the given topological condition, the Einstein-Thorpe metric g must be flat. Hence we can conclude that (M, g) must be $(T^8/\Gamma, \ flat)$, where Γ is of finite order and this completes the proof. □

Remark. In the case of a compact oriented Einstein manifold M of dimension 4 with $\chi = \frac{1}{2}|P_1|$, N. Hitchin in [Hit74], has classified these manifolds as follows:

1. M is flat.

2. M is a $K3$ surface $(\pi_1(M) = \{1\})$.

3. M is an Enriques surface ($\pi_1(M) = \mathbb{Z}_2$)

4. M is the quotient of an Enriques surface by a free antiholomorphic involution ($\pi_1(M) = \mathbb{Z}_2 \times \mathbb{Z}_2$).

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