Langlands Parameters of Subquotients of Derived Functor Modules

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Abstract of the Dissertation
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Let G be a linear, noncompact, simple Lie group with finite center, and K a maximal compact subgroup of G. Suppose that rank G = rank K; Harish-Chandra showed that this equal rank condition is necessary and sufficient for G to have discrete series representations. In the ongoing effort to classify the unitary dual of G, one technique used to search for unusual unitary representations of G is "to continue the discrete series analytically" by allowing a parameter to vary outside the range that produces discrete series.

Let $\mathfrak{g}$ be the complexified Lie algebra of G such that the underlying algebraic object for a representation of G is a $(\mathfrak{g}, K)$ module. Fix a Cartan

subgroup T of K and its complexified Lie algebra $\mathfrak{t}$, and let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. In 1978, Zuckerman gave an algebraic construction of a series $A_{\mathfrak{b}}(\lambda)$ of $(\mathfrak{g}, K)$ modules such that $A_{\mathfrak{b}}(\lambda)$ underlies a discrete series representation when a certain translate of λ is dominant (the "good" zone). Moreover, as $\mathfrak{b}$ varies, the representations obtained this way exhaust the discrete series.

It is known that when $A_{\mathfrak{b}}(\lambda)$ is a discrete series module, then it contains a special K type whose highest weight we denote Λ. When $A_{\mathfrak{b}}(\lambda)$ is no longer in the "good" zone, but Λ is still K-dominant, the theme of continuation beyond the discrete series leads one to consider the unique irreducible subquotient, V, of $A_{\mathfrak{b}}(\lambda)$ containing the K type τ_{Λ}.

In 1997 Knapp conjectured that a certain recursive procedure would produce the Langlands parameters of V. These are parameters that locate V in the classification of all irreducible $(\mathfrak{g}, K)$ modules. He proved, via a combinatorial argument, that in particular cases his algorithm succeeds in identifying the Langlands parameters. Roughly, the proof extracts the infinitesimal character and the minimal K type and then shows that the Langlands parameters produced from the process are the only possible ones that can have these invariants.

This thesis provides a different approach to the problem studied by Knapp. Via techniques of cohomological induction, we produce a simple set of criteria on roots of $\mathfrak{g}$ that, when satisfied, allows one to construct an explicit mapping from which one can often read off the Langlands parameters of V. We show that this approach handles the cases considered by Knapp in 1997, as well as some other cases. The approach taken in this paper gives deeper insight into why Knapp's process works, and suggests some lines of reasoning for how to proceed more generally.

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Chapter 1: Introduction

Section 1.1: Overview

Let G be a linear, noncompact, simple Lie Group with finite center, K be a maximal compact subgroup of G corresponding to a global Cartan involution Θ. Suppose that rank $G =$ rank K, so that there is a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{k}_0$ that is also a Cartan subalgebra of $\mathfrak{g}_0$; Harish-Chandra showed that this equal-rank condition is a necessary and sufficient condition for G to have discrete series representations. In analyzing a representation (π, V) of G, one often examines the restriction of π to K. The theorem of the highest weight parametrizes irreducible representations of K, and we call an equivalence class of irreducible representations of K with highest weight Λ a **K-type**, denoted τ_Λ. A representation π of G is **admissible** if each K-type occurs with only finite multiplicity in $\pi|_K$. Work by Langlands [L], and subsequent work by Knapp and Zuckerman, parametrized irreducible admissible representations of G. The **Langlands parameters** of such a representation consist of a cuspidal parabolic subgroup MAN, a discrete series or a limit of discrete series on M, and a complex-valued linear functional on the Lie algebra of A satisfying a positivity condition.

When attempting to handle a representation (π, V) algebraically, one often studies its underlying "$(\mathfrak{g}, K)$ module." This is a vector space naturally associated with V that carries compatibly both a $U(\mathfrak{g})$ module structure, where $U(\mathfrak{g})$ is the universal enveloping algebra of the complexified Lie algebra of G, and a representation of K in which every vector lies in a finite-dimensional invariant subspace. General $(\mathfrak{g}, K)$ modules may also be defined, and theorems of Harish-Chandra, Lepowsky, and Rader show that every irreducible $(\mathfrak{g}, K)$ module is the underlying $(\mathfrak{g}, K)$ module of an irreducible admissible representation of G. Generally, the terminology of G-representations is transferred to $(\mathfrak{g}, K)$ modules. In particular, by the **Langlands parameters** of an irreducible $(\mathfrak{g}, K)$ module V, we mean those of an associated irreducible admissible representation of G.

Cohomological induction, introduced by Zuckerman in the late 1970s, is an algebraic technique used to construct admissible $(\mathfrak{g}, K)$ modules. Let us describe a functor of cohomological induction: Let $(\mathfrak{g}, K)$ be a reductive pair, let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ stable parabolic subalgebra with Levi factor $\mathfrak{l}$ and nilpotent radical $\mathfrak{u}$, and let $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ be the opposite parabolic of $\mathfrak{q}$. Let Z be an $(\mathfrak{l}, L \cap K)$ module, and define $Z^{\#}$ to be the $(\mathfrak{l}, L \cap K)$ module $Z \otimes \bigwedge^{\text{top}} \mathfrak{u}$. Extend $Z^{\#}$ to a $(\bar{\mathfrak{q}}, L \cap K)$ module by having $\bar{\mathfrak{u}}$ act as zero. Form the $(\mathfrak{g}, K)$ module

$$\mathcal{L}_j(Z) = \Pi_j(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^{\#})$$

where Π_j is the j^{th} derived functor of the so-called "Bernstein functor" Π, whose precise definition will not concern us at this time. Let $S = \dim(\mathfrak{u} \cap \mathfrak{k})$; this is the middle dimension among all degrees for which Π_j can be nonzero. Zuckerman sketched an argument that $\mathcal{L}_S(Z)$ is in the discrete series of G if L is compact, Z is one-dimensional, and a certain translate $\lambda + \delta$ of the unique weight λ of Z has positive inner product with the roots of $\mathfrak{u}$ (in which case, we say that λ or Z is in the **good zone**).

One technique used to search for unusual unitary representations of G

is "to continue the discrete series analytically" by allowing a parameter to vary outside the range that produces discrete series. Wallach [W1] in effect was one of the first to apply this approach, treating the case that G/K is Hermitian symmetric, Z is one-dimensional, and $\mathfrak{u}$ is built from all the noncompact positive roots (L still being compact). In this case, $S = 0$. Wallach determined the exact range of the parameters in which $\mathcal{L}_S(Z)$ is irreducible and infinitesimally unitary. Outside this range $\mathcal{L}_S(Z)$ can be reducible, and Wallach determined exactly when the unique irreducible quotient of $\mathcal{L}_S(Z)$ is infinitesimally unitary. Enright, Howe, and Wallach [EHW] and Jakobsen [J] independently extended Wallach's results to Z finite-dimensional. It is known that these unitary representations obtained via analytic continuation of discrete series play an important role in the classification of the unitary dual for certain groups G. Furthermore some of these representations, which include certain "ladder representations," are of interest in mathematical physics.

Enright, Parthasarathy, Wallach, and Wolf [EPWW] considered a generalization in which G/K is no longer Hermitian symmetric but L is still compact. Again they considered "analytic continuations." Their standing hypothesis was that a certain K-type parameter Λ remained dominant for K; this condition had automatically been satisfied in the Hermitian case. Now, the parameter S was no longer 0. The paper [EPWW] was chiefly concerned with unitarizability, and the work was a predecessor of Vogan's Unitarizability Theorem [V2], which tidily extends the results of [EPWW] by allowing L noncompact and Z infinite-dimensional.

Knapp undertook the task of determining for the setting of [EPWW] the Langlands parameters of the unique irreducible subquotient of $\mathcal{L}_S(Z)$ containing the K-type τ_Λ. In [K3], extrapolating from work of Wallach in the Hermitian case, he proposed a recursive process for doing so, and in certain cases Knapp proved via a combinatorial argument that his procedure worked. Roughly, the proof extracts the infinitesimal character and the minimal K-type and then shows that the Langlands parameters produced from the process are the only possible ones that can have these invariants.

We mention some features of the Knapp process. For a discrete series, the cuspidal parabolic subgroup is G itself, and $A = 1$. As the parameter $\lambda + \delta$ moves outside the initial range (the good zone), the process increases the dimension of A by 1 at each step, essentially projecting data to get the new M and A parameters.

Knapp's combinatorial argument has a limited scope. It becomes more complicated for more complicated groups, and there appear to be cases not settled by Knapp where the infinitesimal character and the minimal K-type that it uses do not uniquely determine a set of Langlands parameters.

This thesis provides a different, more representation-theoretic approach to the question of Langlands parameters and analytic continuations of discrete series. We start by exploiting some basic properties of representations of an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra naturally embedded in $\mathfrak{g}$. Then we apply techniques of cohomological induction to produce a set of criteria on roots of $\mathfrak{g}$ that, when satisfied, allows us to construct a mapping, Φ, which can be use to read off Langlands parameters. The criteria given provide a true reduction of the problem, because they are simple and can be checked in a finite number of steps in any particular example. We then show that the cases handled by Knapp, as well as some other cases, are handled by the approach of this thesis. Moreover, the approach taken here gives deeper insight into why Knapp's process works, and suggests some lines of reasoning for how to proceed more generally.

Section 1.2: Basic Notational Conventions

Let G be a connected semisimple Lie group with finite center and let K be a maximal compact subgroup. We denote corresponding Lie algebras by the corresponding Gothic letters with subscripts 0, and we denote complexifications by dropping the subscripts. We let bar denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Let θ be the Cartan involution of $\mathfrak{g}_0$ corresponding to K and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the associated Cartan decomposition. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. We introduce in the usual way an inner product $\langle \cdot, \cdot \rangle$ and a norm squared $|\cdot|^2$ on the real linear span of the roots. We use a hat to denote a coroot; that is, if $\alpha \in \Delta$ then $\widehat{\alpha} = 2\alpha/|\alpha|^2$. If the Cartan subalgebra $\mathfrak{h}_0$ lies in $\mathfrak{k}_0$ then each root vector lies in $\mathfrak{k}$ or in $\mathfrak{p}$, and the roots are called compact or noncompact accordingly. We denote the subset of compact roots by Δ_K and the subset of noncompact roots by $\Delta(\mathfrak{p})$.

Chapter 2: $SL(2,\mathbb{R})$ and Some of its Representations

In this thesis, we use some basic relationships among the representations of $SL(2,\mathbb{R})$. In this chapter, we recall basic information about some representations of $SL(2,\mathbb{R})$. All of the material in this section may be found in [K1] or [D] .

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \middle| ad - bc = 1 \right\}$. Let $K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| 0 \le \theta < 2\pi \right\}$. Let the $n^{th} K$ type be

$$\tau_n \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{in\theta}.$$

(1) Principal Series Representations: $\mathcal{P}^{\pm,w}$

Let $w \in \mathbb{C}$ and let V be the space of complex-valued functions in $L^2(\mathbb{R}, (1+x^2)^{Re\ w}\, dx)$. The **principal series** of $SL(2,\mathbb{R})$ is a family of representations of G on V given by

$$\mathcal{P}^{\pm,w} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \begin{cases} |-bx+d|^{-1-w}\, f\left(\frac{ax-c}{-bx+d}\right) & \text{if } + \\ sgn(-bx+d)\, |-bx+d|^{-1-w}\, f\left(\frac{ax-c}{-bx+d}\right) & \text{if } - . \end{cases}$$

$\mathcal{P}^{\pm,w}$ is not unitary unless w is a purely imaginary parameter. (However, if $w \in \mathbb{R}$ and $0 < w < 1$ then $\mathcal{P}^{\pm,w}$ can be renormed so as to become unitary. When this is done, we have the **complementary series.**) The K types are given by all even n in the $+$ case, and by all odd n in the $-$ case. The K types all occur with multiplicity one.

The principal series representations can also be realized as induced representations. Let $S = MAN$ the upper triangular subgroup, and let $\nu \in \mathfrak{a}^*$. Define σ on $M = \{\pm I\}$ by

$$\sigma \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} = \begin{cases} 1 & \text{if } + \\ \epsilon & \text{if } - . \end{cases}$$

Then we may realize the principal series representations as

$$U(S, \sigma, \nu)(\cdot) = \mathrm{ind}_{MAN}^{G}(\sigma \otimes \exp \nu \otimes 1).$$

(2) Discrete Series Representations: $\mathcal{D}_n^+$ and $\mathcal{D}_n^-$

Let $n \geq 2$ be an integer. The Hilbert space for $\mathcal{D}_n^+$ is

$$\{f \text{ analytic for } Im\ z > 0 \mid \|f\|^2 = \iint_{Im\ z>0} |f(z)|^2\, y^n\, \frac{dx\,dy}{y^2} < \infty\}$$

and the group action is

$$\mathcal{D}_n^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (-bz+d)^{-n}\, f\left(\frac{az-c}{-bz+d}\right).$$

The K types of $\mathcal{D}_n^+$ are given by $n + 2m$, where m is a nonnegative integer, all occurring with multiplicity one.

The Hilbert space for $\mathcal{D}_n^-$ is the complex conjugate of the Hilbert space for $\mathcal{D}_n^+$ and the group action is

$$\mathcal{D}_n^- \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = \overline{(-bz+d)}^{\,-n}\, f\left(\frac{az-c}{-bz+d}\right).$$

The K types of $\mathcal{D}_n^-$ are given by $-(n+2m)$, where m is a nonnegative integer, all occurring with multiplicity one.

(3) Finite Dimensional Representations: Φ_n

Let n be a nonnegative integer and V_n be the space of polynomials of degree $\leq n$. Then G acts on this $(n+1)$-dimensional space by

$$\Phi_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} P(x) = (-bx+d)^n\, P\left(\frac{ax-c}{-bx+d}\right).$$

The K types of Φ_n are given by $n, n-2, \ldots, -(n-2), -n$, all occurring with multiplicity one.

(4) Reducibility

By comparing the actions in (1) and (3), we see that

$$\Phi_n \subseteq \begin{cases} \mathcal{P}^{+,-(n+1)} & \text{if } n \text{ even} \\ \mathcal{P}^{-,-(n+1)} & \text{if } n \text{ odd.} \end{cases}$$

Similarly, by restricting the functions in the space for the discrete series representations from the upper half-plane to $\mathbb{R}$, we see that

$$\mathcal{D}_n^+ \oplus \mathcal{D}_n^- \subseteq \begin{cases} \mathcal{P}^{+,n-1} & \text{if } n \text{ even} \\ \mathcal{P}^{-,n-1} & \text{if } n \text{ odd.} \end{cases}$$

Chapter 3: Introduction to $(\mathfrak{g}, K)$ Modules and Cohomological Induction

In this chapter, we summarize some of the basic definitions and properties of cohomological induction. All of what follows may be found in [KV] or [K2].

Section 3.1: $(\mathfrak{g}, K)$ Modules

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra and let K be a compact Lie group. We call $(\mathfrak{g}, K)$ a **pair** if the following compatibility conditions are satisfied:

(i) the complexified Lie algebra $\mathfrak{k}$ of K is a subalgebra of $\mathfrak{g}$;

(ii) K acts on $\mathfrak{g}$ by automorphisms $\mathrm{Ad}(k)$ extending the adjoint action on $\mathfrak{k}$; and

(iii) the differential of $\mathrm{Ad}(k)$ is $\mathrm{ad}(\mathfrak{k}) \subseteq \mathrm{ad}(\mathfrak{g})$.

Throughout this paper, we shall always assume that G is unimodular. Additionally, we operate under the assumption that whenever we are working with a pair, $(\mathfrak{g}, K)$, that there is a real Lie group G with complexified Lie algebra $\mathfrak{g}$ such that K is a compact subgroup of G for which $\mathfrak{k}$ and Ad are compatible with the definitions imposed by G.

For any pair $(\mathfrak{g}, K)$, a $(\mathfrak{g}, K)$ **module** is a complex vector space V carrying representations of $\mathfrak{g}$ and K such that

(i) the K representation is locally K finite;

(ii) the differentiated version of the K action is the restriction to $\mathfrak{k}$ of the $\mathfrak{g}$ action; and

(iii) $(\mathrm{Ad}(k)u)x = k(u(k^{-1}x)) \quad$ for $k \in K, u \in U(\mathfrak{g})$, and $x \in V$.

Naturally associated to a representation (π, V) of G is its **underlying $(\mathfrak{g}, K)$ module** (or **Harish-Chandra module**), that is, the subspace of K finite C^∞ vectors of V, denoted $C^\infty(V)_K$, with its $U(\mathfrak{g})$ and K structures. Due to the following theorem of Harish-Chandra, Lepowsky, and Rader, the study of admissible representations of G is essentially equivalent to the study of $(\mathfrak{g}, K)$ modules.

Theorem. Every irreducible $(\mathfrak{g}, K)$ module is the underlying $(\mathfrak{g}, K)$ module of an irreducible admissible representation of G.

Section 3.2: The Hecke Algebra $R(\mathfrak{g}, K)$

The study of $\mathfrak{g}$ modules amounts to the same thing as the study of unital left $U(\mathfrak{g})$ modules. Since we are interested in $(\mathfrak{g}, K)$ modules, we would like to have something analogous to $U(\mathfrak{g})$ and unital left $U(\mathfrak{g})$ modules to assist us. The "Hecke algebra," $R(\mathfrak{g}, K)$, which we shall define presently, will play the role of $U(\mathfrak{g})$. Further, since $R(\mathfrak{g}, K)$ usually does not have an identity, only an approximate identity, approximately unital left $R(\mathfrak{g}, K)$ modules will play the role of unital left $U(\mathfrak{g})$ modules.

The **Hecke algebra of** $K, R(K)$, is the algebra of matrix coefficients of all finite dimensional (unitary) representations of K with convolution as multiplication. Unless K is a finite group, $R(K)$ does not have an identity, only an approximate identity. The **Hecke algebra of** $(\mathfrak{g}, K), R(\mathfrak{g}, K)$, is the vector space

$$R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g})$$

equipped with the following multiplication: choose an $\mathrm{Ad}(K)$–invariant inner product on $U^n(\mathfrak{g})$ for sufficiently large n, and let $\{u_i\}$ be an orthonormal basis of $U^n(\mathfrak{g})$. Then define

$$(c \otimes u)(c' \otimes u) = \sum_i c * c'(\cdot)(\mathrm{Ad}(\cdot)u, u_i) \otimes u_i u'.$$

This multiplication descends to $R(\mathfrak{g}, K)$ and makes $R(\mathfrak{g}, K)$ into a complex associatice algebra. The relevance of $R(\mathfrak{g}, K)$ in the study of $(\mathfrak{g}, K)$ modules can be seen in the following theorem.

Theorem. A $(\mathfrak{g}, K)$ module in a natural way is an approximately unital left $R(\mathfrak{g}, K)$ module. Conversely, any approximately unital left $R(\mathfrak{g}, K)$ module comes from a $(\mathfrak{g}, K)$ module by an inverse construction. For any two such modules V and W,

$$\mathrm{Hom}_{\mathfrak{g},K}(V, W) = \mathrm{Hom}_{R(\mathfrak{g},K)}(V, W).$$

Let $\mathcal{C}(\mathfrak{g}, K)$ be the category of all $(\mathfrak{g}, K)$ modules and $(\mathfrak{g}, K)$ maps. Because of the above theorem, we may regard $\mathcal{C}(\mathfrak{g}, K)$ as the category of all left $R(\mathfrak{g}, K)$ modules that are approximately unital. This is a good category with enough injectives and enough projectives.

Section 3.3: The Master Functors: P and I

Definition: A map of pairs $i : (\mathfrak{h}, L) \to (\mathfrak{g}, K)$ consists of two maps

$i_{\mathrm{alg}} : \mathfrak{h} \to \mathfrak{g}$	a Lie algebra homomorphism
$i_{\mathrm{gp}} : L \to K$	a Lie group homomorphism

satisfying the compatibility conditions

(i) $i_{\text{alg}} \circ \iota_L = \iota_K \circ di_{\text{gp}}$, where di_{gp} is the differential of i_{gp};

(ii) $i_{\text{alg}} \circ \text{Ad}_L(l) = \text{Ad}_K(i_{\text{gp}}(l)) \circ i_{\text{alg}}$ for $l \in L$,

where $\iota_L : \mathfrak{l} \to \mathfrak{h}$ and $\iota_K : \mathfrak{k} \to \mathfrak{g}$ are the Lie algebra inclusions arising from the definition of the pairs $(\mathfrak{h}, L)$ and $(\mathfrak{g}, K)$.

Next we introduce the two master functors of cohomological induction: P and I.

Definition: The functor $P(\cdot) : \mathcal{C}(\mathfrak{h}, L) \to \mathcal{C}(\mathfrak{g}, K)$ is defined by

$$P(V) = P_{\mathfrak{h},L}^{\mathfrak{g},K}(V) = R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h},L)} V.$$

The functor $I(\cdot) : \mathcal{C}(\mathfrak{h}, L) \to \mathcal{C}(\mathfrak{g}, K)$ is defined by

$$I(V) = I_{\mathfrak{h},L}^{\mathfrak{g},K}(V) = \text{Hom}_{R(\mathfrak{h},L)}(R(\mathfrak{g}, K), V)_K$$

where the subscript K denotes "the K-finite part."

The functor P is right-exact, covariant and sends projectives to projectives. The functor I is left-exact, covariant and sends injectives to injectives. In this thesis, we will be concerned chiefly with two particular specializations of these functors:

(1) When $K = L$, so that the pairs are $(\mathfrak{h}, L)$ and $(\mathfrak{g}, L)$, P and I reduce to functors ind and pro, defined by

$$P_{\mathfrak{h},L}^{\mathfrak{g},L}(V) \cong \text{ind}_{\mathfrak{h},L}^{\mathfrak{g},L}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V,$$
$$I_{\mathfrak{h},L}^{\mathfrak{g},L}(V) \cong \text{pro}_{\mathfrak{h},L}^{\mathfrak{g},L}(V) = \text{Hom}_{U(h)}(U(\mathfrak{g}), V)_L$$

These, in fact, are exact functors.

(2) When $\mathfrak{g} = \mathfrak{h}$, so that the pairs are $(\mathfrak{g}, L)$ and $(\mathfrak{g}, K)$, the functors are given special names, Π and Γ. Π is called the **Bernstein functor** and Γ is called the **Zuckerman functor.** We have the following natural isomorphisms:

$$P_{\mathfrak{g},L}^{\mathfrak{g},K}(V) = \Pi_{\mathfrak{g},L}^{\mathfrak{g},K}(V) = \Pi(V) \cong R(K) \otimes_{R(\mathfrak{k},L)} V$$
$$I_{\mathfrak{g},L}^{\mathfrak{g},K}(V) = \Gamma_{\mathfrak{g},L}^{\mathfrak{g},K}(V) = \Gamma(V) \cong \mathrm{Hom}_{R(\mathfrak{k},L)}(R(K), V)_K$$

Of primary interest in this thesis will be the derived functors of the right exact Π,

$$\Pi_j : \mathcal{C}(\mathfrak{h}, L) \to \mathcal{C}(\mathfrak{g}, K)$$

obtained on modules by applying P to projective resolutions in $\mathcal{C}(\mathfrak{h}, L)$ and taking homology, and the derived functors of the left exact Γ,

$$\Gamma^j : \mathcal{C}(\mathfrak{h}, L) \to \mathcal{C}(\mathfrak{g}, K)$$

obtained on modules by applying I to injective resolutions in $\mathcal{C}(\mathfrak{h}, L)$ and taking cohomology.

Section 3.4: Parabolic Subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$

Before presenting the above functors in the form that we shall use them, we need to introduce some more notation and terminology. Again, the main reference for this material is [KV].

Definition. A **reductive pair** $(\mathfrak{g}, K)$ is a tuple $((\mathfrak{g}, K), \mathfrak{g}_0, \theta, \langle\cdot,\cdot\rangle)$ consisting of a pair $(\mathfrak{g}, K)$, a real form $\mathfrak{g}_0$ of $\mathfrak{g}$, a Lie algebra involution θ of $\mathfrak{g}_0$, and a nondegenerate form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_0$ that is $\mathrm{Ad}(K)$ invariant and is skew symmetric under ad $\mathfrak{g}_0$. Moreover, it is assumed that

(i) $\mathfrak{g}_0$ is a reductive Lie algebra

(ii) the decomposition of $\mathfrak{g}_0$ into $+1$ and -1 eigenspaces under θ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where $\mathfrak{k}_0$ is the Lie algebra of K

(iii) $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under $\langle \cdot, \cdot \rangle$, and $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$.

As an example, if G is a connected semisimple Lie group with finite center, K maximal compact, θ a Cartan involution and $\langle \cdot, \cdot \rangle$ equal to the Killing form, then $(\mathfrak{g}, K)$ is a reductive pair.

Fix a θ stable Cartan subalgebra, $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ and choose a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{b}$ denote the corresponding Borel subalgebra and define

$$H = N_G(\mathfrak{b}) \cap N_G(\theta\mathfrak{b})$$

$$T = N_K(\mathfrak{b}) = N_K(\theta\mathfrak{b})$$

We then call $(\mathfrak{h}, T)$ a **Cartan subpair.**

Starting from a reductive pair $(\mathfrak{g}, K)$, we define a **parabolic subalgebra** of $\mathfrak{g}$ to be any Lie subalgebra of $\mathfrak{g}$ containing a Borel subalgebra $\mathfrak{b}$ built from a Cartan subpair. The parabolic subalgebras $\mathfrak{q}$ containing $\mathfrak{b}$ are parametrized by the set of subsets of simple roots; the one corresponding to a subset Π is of the form

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha,$$

where $\Gamma = \Delta^+(\mathfrak{g}, \mathfrak{h}) \cup \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) | \alpha \in \text{span}(\Pi)\}$. Now define

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{u} = \bigoplus_{\substack{\alpha \in \Gamma \\ \alpha \notin -\Gamma}} \mathfrak{g}_\alpha,$$

so that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. In this decomposition, $\mathfrak{l}$ is called the **Levi factor** and $\mathfrak{u}$ is called the **nilpotent radical**. We call a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ **germane** if it satisfies the equivalent conditions below.

(a) $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$

(b) $\mathfrak{l}$ is closed under conjugation (of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$)

(c) $\Gamma \cap -\Gamma$ is closed under bar

(d) $\Gamma \cap -\Gamma$ is closed under θ

(e) $\mathfrak{l}$ is θ stable.

The two types of germane parabolics that are of particular significance to us are **real** parabolic subalgebras, i.e., $\mathfrak{q}$ is closed under conjugation, and θ **stable** parabolic subalgebras, i.e., $\mathfrak{q} = \theta\mathfrak{q}$. If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a real or θ stable parabolic subalgebra, then there exists $h \in i\mathfrak{t}_0 \cup \mathfrak{a}_0$ such that $\mathfrak{l} = Z_{\mathfrak{g}}(h)$ and $\mathfrak{u}$ is the sum of eigenspaces of ad h for positive eigenvalues. The element h may be taken in $\mathfrak{a}_0$ if $\mathfrak{q}$ is real, and it may be taken in $i\mathfrak{t}_0$ if $\mathfrak{q}$ is θ stable. Conversely, we can construct real and θ stable parabolic subalgebras by selecting a linearly independent set $\{h_i\} \subset i\mathfrak{t}_0 \cup \mathfrak{a}_0$, and then defining $\mathfrak{l} = Z_{\mathfrak{g}}(\mathbb{C}h_1 + \cdots \mathbb{C}h_m)$ and $\mathfrak{u}$ = sum of simultaneous eigenspaces of $\{\text{ad } h_i\}$ for positive simultaneous eigenvalues.

Section 3.5: Functors of Cohomological Induction

We need one final definition before defining functors of cohomological induction. Let $i : (\mathfrak{h}, L) \to (\mathfrak{g}, K)$ be a map of pairs. Define the **forgetful functor** $\mathcal{F} : \mathcal{C}(\mathfrak{g}, K) \to \mathcal{C}(\mathfrak{h}, L)$ by setting $\mathcal{F}^{\mathfrak{h},L}_{\mathfrak{g},K}(X)$ to be the $(\mathfrak{h}, L)$ module with the same underlying vector space as X and with actions

$$hx = i_{\mathrm{alg}}(h)x \qquad \text{and} \qquad lx = i_{\mathrm{gp}}(l)x.$$

In particular, if $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic subalgebra and X is an $(\mathfrak{l}, L \cap K)$ module, then $\mathcal{F}_{\mathfrak{l},L\cap K}^{\mathfrak{q},L\cap K}(X)$ becomes a $(\mathfrak{q}, L \cap K)$ module by letting $\mathfrak{u}$ act as 0. The functor $\mathcal{F}$ is covariant and exact. Moreover, the functor I is right adjoint to $\mathcal{F}$. That is, there is a natural isomorphism

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(X, I(V)) \cong \mathrm{Hom}_{R(\mathfrak{h},L)}(\mathcal{F}(X), V)$$

for $V \in \mathcal{C}(\mathfrak{h}, L)$ and $X \in \mathcal{C}(\mathfrak{g}, K)$. This relationship is commonly referred to as **Frobenius reciprocity** [KV, pg.110].

Let $(\mathfrak{g}, K)$ be a reductive pair, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ a θ stable parabolic subalgebra containing a θ stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Let Z be in $\mathcal{C}(\mathfrak{l}, L \cap K)$, and define

$$Z^{\#} = Z \otimes_{\mathbb{C}} \bigwedge\nolimits^{\mathrm{top}} \mathfrak{u}.$$

Since $\mathfrak{u}$ is an $(\mathfrak{l}, L \cap K)$ module, $\bigwedge^{\mathrm{top}}\mathfrak{u}$ is a one-dimensional $(\mathfrak{l}, L \cap K)$ module with unique weight $2\delta(\mathfrak{u})$ relative to $\mathfrak{h}$. Then $Z^{\#}$ is in $\mathcal{C}(\mathfrak{l}, L \cap K)$. Let $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ be the members of $\mathcal{C}(\mathfrak{g}, K)$ given by

$$\mathcal{L}_j(Z) = (\Pi_{\mathfrak{g},L\cap K}^{\mathfrak{g},K})_j(\mathrm{ind}_{\bar{\mathfrak{q}},L\cap K}^{\mathfrak{g},L\cap K}(\mathcal{F}_{\mathfrak{l},L\cap K}^{\bar{\mathfrak{q}},L\cap K}(Z^{\#})))$$
$$\mathcal{R}^j(Z) = (\Gamma_{\mathfrak{g},L\cap K}^{\mathfrak{g},K})^j(\mathrm{pro}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(\mathcal{F}_{\mathfrak{l},L\cap K}^{\mathfrak{q},L\cap K}(Z^{\#}))).$$

The functors $\mathcal{L}_j$ and $\mathcal{R}^j$ are the functors of **cohomological induction.**

Let $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. It is easy to see [KV, Cor 2.125b] that Π_j and Γ^j (and consequently $\mathcal{L}_j$ and $\mathcal{R}^j$) are 0 for $j > 2S$. In fact, $\mathcal{L}_j$ and $\mathcal{R}^j$ are 0 for

$j > S$ [KV, Th 5.35]. This dimension S, called the **middle dimension**, is the one of primary interest.

Let λ be an analytically integral linear functional on $\mathfrak{h}$ that is orthogonal to all members of $\Delta(\mathfrak{l})$, and let $\mathbb{C}_\lambda$ be the one-dimensional $(\mathfrak{l}, L\cap K)$ module with highest weight λ. We define $(\mathfrak{g}, K)$ modules by

$$A_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda) \qquad \text{and} \qquad A^{\mathfrak{q}}(\lambda) = \mathcal{R}^S(\mathbb{C}_\lambda).$$

We can extend the definition of functors of cohomological induction from one using a θ stable parabolic subalgebra and a twist by $2\delta(\mathfrak{u})$ to one applicable to any germane parabolic subalgebra . If $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ is a germane parabolic subalgebra we now allow L to be a subgroup of G satisfying $L_0 \subseteq L \subseteq N_G(\mathfrak{q}) \cap N_G(\theta\mathfrak{q})$ and define the "unnormalized" functors ${}^{\mathrm{u}}\mathcal{L}$ and ${}^{\mathrm{u}}\mathcal{R}$ from $\mathcal{C}(\mathfrak{l}, L\cap K)$ to $\mathcal{C}(\mathfrak{g}, K)$ by

$${}^{\mathrm{u}}\mathcal{L}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K} = \Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}(\mathrm{ind}^{\mathfrak{g},L\cap K}_{\mathfrak{q},L\cap K}(\mathcal{F}^{\mathfrak{q},L\cap K}_{\mathfrak{l},L\cap K}(Z)))$$
$${}^{\mathrm{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K} = \Gamma^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}(\mathrm{pro}^{\mathfrak{g},L\cap K}_{\mathfrak{q},L\cap K}(\mathcal{F}^{\mathfrak{q},L\cap K}_{\mathfrak{l},L\cap K}(Z)))$$

The derived functors of interest are

$$({}^{\mathrm{u}}\mathcal{L}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})_j(Z) = (\Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})_j(\mathrm{ind}^{\mathfrak{g},L\cap K}_{\mathfrak{q},L\cap K}(\mathcal{F}^{\mathfrak{q},L\cap K}_{\mathfrak{l},L\cap K}(Z)))$$
$$({}^{\mathrm{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^j(Z) = (\Gamma^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})^j(\mathrm{pro}^{\mathfrak{g},L\cap K}_{\mathfrak{q},L\cap K}(\mathcal{F}^{\mathfrak{q},L\cap K}_{\mathfrak{l},L\cap K}(Z))).$$

The unnormalized functors also arise naturally. According to Propositions 11.47 and 11.65 of [KV], if $I^G_{MAN}(\xi,\nu)$ is a continuous-series representation and $V^\xi_{K\cap M}$ is the underlying $(\mathfrak{m}, M\cap K)$ module of ξ, then the underlying $(\mathfrak{g}, K)$ module of $I^G_{MAN}(\xi,\nu)$ is

(3.5.1) $\quad X_K(\xi,\nu) \cong {}^{\mathrm{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},K\cap M}(V^\xi_{K\cap M}\otimes\mathbb{C}_{\nu+\rho}) \cong {}^{\mathrm{u}}\mathcal{L}^{\mathfrak{g},K}_{\mathfrak{q},K\cap M}(V^\xi_{K\cap M}\otimes\mathbb{C}_{\nu-\rho}),$

where $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{m}$ acts in $V^{\xi}_{K\cap M}$, and $\mathfrak{a}$ acts in $\mathbb{C}_{\nu\pm\rho}$. [We will sometimes use $X_K(\cdot,\nu)$ to denote the obvious functor from $\mathcal{C}(\mathfrak{m}, M\cap K)$ to $\mathcal{C}(\mathfrak{g}, K)$.] Notice that this functor has an adjustment by ρ. Modulo a technical matter involving double covers, we may define the "normalized" functors of cohomological induction, ${}^{\mathrm{n}}\mathcal{R}$ and ${}^{\mathrm{n}}\mathcal{L}$, to incorporate this shift (cf. [KV §XI.7]).

Section 3.6: The Bottom-Layer Map

All of the material in this section may be found in [KV §V.6].

A principle in representation theory is that when analyzing a representation of the connected semisimple G, it is often helpful to study either the restriction of the representation to K or another type of K analog. The study of $(\mathfrak{g}, K)$ modules has a similiar principle, so we shall presently introduce the K analogs, $\mathcal{L}^K_j(Z)$ and $\mathcal{R}^j_K(Z)$, of $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$. The bottom-layer map then links these two modules. To obtain the nicest results, we assume that $(\mathfrak{g}, K)$ is a reductive pair, so that it arises from a reductive group,G, [cf. KV Prop 4.31], and we assume that L meets every component of G.

To start, we continue to write $Z^{\#}_{\bar{\mathfrak{q}}}$ and $Z^{\#}_{\mathfrak{q}}$ for the K analogs

$$\mathcal{F}^{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}_{\bar{\mathfrak{q}}, L\cap K}(Z^{\#}_{\bar{\mathfrak{q}}}) = \mathcal{F}^{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}_{\mathfrak{l}, L\cap K}(Z^{\#}).$$
$$\mathcal{F}^{\mathfrak{q}\cap\mathfrak{k}, L\cap K}_{\mathfrak{q}, L\cap K}(Z^{\#}_{\mathfrak{q}}) = \mathcal{F}^{\mathfrak{q}\cap\mathfrak{k}, L\cap K}_{\mathfrak{l}, L\cap K}(Z^{\#}),$$

where the superscript $(\cdot)^{\#}$ continues to refer to the tensor product $\bigwedge^{\mathrm{top}}\mathfrak{u}$. Define

$$\Pi^K_j = (P^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})_j \qquad \text{and} \qquad \Gamma^j_K = (I^{\mathfrak{k},K}_{\mathfrak{k},L\cap K})^j.$$

The K analogs of $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ are the functors from $\mathcal{C}(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ to $\mathcal{C}(\mathfrak{k}, K)K$

$$\mathcal{L}_j^K(Z) = \Pi_j^K(P_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z_{\bar{\mathfrak{q}}}^{\#}))$$
$$\mathcal{R}_K^j(Z) = \Gamma_K^j(I_{\mathfrak{q}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z_{\mathfrak{q}}^{\#})).$$

The advantage of studying $\mathcal{L}_j^K(Z)$ and $\mathcal{R}_K^j(Z)$ is that we usually know exactly what they are. When G is connected and Z is an irreducible $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ module, these modules split so that study of them is reduced to the case in which they are irreducible. This case is then handled by an algebraic Bott-Borel-Weil theorem. Therefore, if μ_L is the highest weight of Z, then $\mathcal{L}_j^K(Z) = 0$ unless the weight $\mu_G = \mu_L + 2\delta(\mathfrak{u}\cap\mathfrak{p})$ is dominant for K. When it is dominant for K, $\mathcal{L}_j^K(Z)$ is an irreducible representation of K with highest weight μ_G.

To take advantage of this concrete knowledge, we need to link $\mathcal{L}_j^K(Z)$ and $\mathcal{L}_j(Z)$. To do so, one sets $\mathfrak{m} = \mathfrak{k}, \mathfrak{m}' = \bar{\mathfrak{q}}$, and $Z = Z_{\bar{\mathfrak{q}}}^{\#}$ in [KV Lemma 5.26], to obtain a one-one $(\mathfrak{k}, L\cap K)$ map

$$\text{(3.6.1)} \qquad \beta_Z : P_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z_{\bar{\mathfrak{q}}}^{\#}) \longrightarrow P_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z_{\bar{\mathfrak{q}}}^{\#}).$$

Because of the isomorphism from [KV Prop. 2.115],

$$\Pi_j^K \circ \mathcal{F}_{\mathfrak{g}, L\cap K}^{\mathfrak{k}, L\cap K} \cong \mathcal{F}_{\mathfrak{g}, K}^{\mathfrak{k}, K} \circ \Pi_j,$$

it is meaningful to form $\mathcal{B}_Z = \Pi_j^K(\beta_Z)$ from (3.6.1); the result is the map

$$\text{(3.6.2)} \qquad \mathcal{B}_Z : \mathcal{L}_j^K(Z) \longrightarrow \mathcal{L}_j(Z).$$

This map is called the **bottom-layer map.**

Theorem 5.80(a) [KV] If τ is a K type in $\mathcal{L}_S^K(Z)$, then the bottom-layer map $\mathcal{B}_Z$ maps the τ subspace one-one onto the τ subspace of $\mathcal{L}_S(Z)$.

The K types of $\mathcal{L}_S(Z)$ that appear in $\mathcal{L}_S^K(Z)$ are called the **bottom-layer** K types of $\mathcal{L}_S(Z)$.

Section 3.7: Some Properties of $A_{\mathfrak{q}}(\lambda)$

In this subsection, we highlight some basic facts about the $A_{\mathfrak{q}}(\lambda)$ modules. The notation is as in §1.2.

• If Z is an $(\mathfrak{l}, L\cap K)$ module with infinitesimal character λ then $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ have infinitesimal character $\lambda+\delta(\mathfrak{u})$ [KV Cor. 5.25]. In particular, $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda+\delta$ since in this case, the $(\mathfrak{l}, L\cap K)$ module $\mathbb{C}_\lambda$ has infinitesimal character $\lambda+\delta(\mathfrak{l})$ and $(\lambda+\delta(\mathfrak{l}))+\delta(\mathfrak{u}) = \lambda+\delta$.

• If $\Lambda := \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})$ is $\Delta^+(\mathfrak{k},\mathfrak{t})$ dominant, then $A_{\mathfrak{q}}(\lambda)$ contains the K type Λ with multiplicity one [KV §V.6].

• If L is compact, then $A_{\mathfrak{q}}(\lambda)$ is isomorphic to $A_{\mathfrak{b}}(\lambda)$, where $\mathfrak{b}$ is a Borel subalgebra and $\mathfrak{q}\supseteq\mathfrak{b}$. (To see this, one can combine an algebraic Borel-Weil theorem with an induction-in-stages result.)

• If λ is in the **good zone**, that is, the infinitesimal character $\lambda+\delta$ is strictly $\Delta(\mathfrak{u})$ dominant, then

(a) Λ is $\Delta^+(\mathfrak{k},\mathfrak{t})$ dominant

(b) $A_{\mathfrak{q}}(\lambda)$ is irreducible

(c) $A_{\mathfrak{q}}(\lambda)$ is unitarizable

(d) if also rank G = rank K and $\mathfrak{t}_0\subset\mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, then $A_{\mathfrak{b}}(\lambda)$ is a discrete series module. In fact, the good $A_{\mathfrak{b}}(\lambda)$'s, as $\mathfrak{b}$ varies, exhaust the discrete series of G [KV Thm. 11.178].

The main result of this work concerns a natural subquotient V of the $(\mathfrak{g}, K)$ module $A_{\mathfrak{b}}(\lambda)$ when λ is no longer in the good zone. In particular, we are interested in determining the Langlands parameters of V.

Chapter 4: Setting the Stage

Section 4.1: Langlands Parameters

As previously mentioned, theorems of Harish-Chandra, Lepowsky and Rader show that every irreducible $(\mathfrak{g}, K)$ module globalizes to an irreducible admissible representation of G. Therefore it is reasonable to transfer the language of G representations to that of $(\mathfrak{g}, K)$ modules. The Langlands classification of irreducible admissible representations of G is well known (see, for example, [K1, Th. 14.91]) and by the **Langlands parameters** of an irreducible $(\mathfrak{g}, K)$ module V we mean a triple (MAN, σ, ν) such that

(i) MAN is a cuspidal parabolic subgroup of G

(ii) σ is a discrete series or limit of discrete series on M with infinitesimal character λ_σ

(iii) ν is a complex-valued linear functional on the Lie algebra $\mathfrak{a}_0$ of A with Re ν in the closed positive Weyl chamber

(iv) the induced representation $\text{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$ has a unique irreducible quotient, called the **Langlands quotient** and denoted $J(MAN, \sigma, \nu)$

(v) V is equivalent with the underlying $(\mathfrak{g}, K)$ module of $J(MAN, \sigma, \nu)$.

Section 4.2: The Conjectural Method

The setting for this section is as follows: G is a linear, simple non-compact Lie group with finite center, K a maximal compact subgroup, and rank G = rank K. Let $T \subseteq K$ be a Cartan subroup, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the set of roots. Fix a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ and *assume* that there is exactly one noncompact simple root and this root has multiplicity two in the highest root. Define $\Delta_K^+, \delta, \delta_K$ as usual. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the θ stable parabolic subalgebra with $\mathfrak{l}$ formed from the compact simple roots, and $\mathfrak{u}$ formed from the remaining positive roots. Let λ be an analytically integral form on $\mathfrak{t}$ that is orthogonal to the roots of Δ_L and let $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}) = \lambda + 2\delta(\mathfrak{p}) = (\lambda + \delta) + (\delta - 2\delta_K)$. Assume that Λ is K dominant, and consider $A_{\mathfrak{q}}(\lambda)$.

In [K3], Knapp outlined a recursive procedure which he conjectured would produce the Langlands parameters of the irreducible subquotient V of $A_{\mathfrak{q}}(\lambda)$ containing the K type Λ. Using combinatorial arguments, he proved that this method gives the correct parameters in some cases. In this thesis, I will approach this problem from another point of view which will allow us to reduce the question about the success of Knapp's method to a simple question about dominance properties of a finite set of roots. As a result, we will be able to prove that the procedure works for a wider class of $A_{\mathfrak{b}}(\lambda)$-modules, not just those which are isomorphic to the $A_{\mathfrak{q}}(\lambda)$-types above. First we describe, with slight modifications, the Conjectural Method of [K3].

Assume that

(1) $\langle \lambda + \delta, \beta \rangle > 0$ for all compact simple roots, β, of $\Delta^+(\mathfrak{g}, \mathfrak{t})$, and

(2) Λ is Δ_K^+ dominant

Roughly, if the infinitesimal character $\lambda + \delta$ of $A_{\mathfrak{q}}(\lambda)$ is nondominant versus a noncompact root α then split, by the Cayley transform relative to α, the Cartan subalgebra $\mathfrak{t}$ into $\mathfrak{t}' \oplus \mathfrak{a}'$. Project the infinitesimal character onto the dual of each of these pieces, but negate the projection onto the $\mathfrak{a}'$

piece. Label these projections $\lambda_{\sigma'}$ and ν. Form $M' = Z_G(\mathfrak{a}')$ and the roots $\Delta^+(\mathfrak{m}', \mathfrak{t}')$, which may be identified with the roots of $\Delta^+(\mathfrak{g})$ orthogonal to α. The functional $\lambda_{\sigma'}$ will be dominant versus the compact simple roots of $\Delta^+(\mathfrak{m}')$ and the corresponding weight Λ' will be $M' \cap K$ dominant. Continue this process on M' and the corresponding $A_{\mathfrak{q}_{\mathfrak{m}'}}(\lambda')$ with infinitesimal character $\lambda_{\sigma'}$, increasing the dimension of $\mathfrak{a}$ at each step until you produce a discrete series module on a subsequent M.

More precisely Set $M_0 = G, A_0 = \{I\}, \mathfrak{t}_0^0 = \mathfrak{t}_0, \mathfrak{a}_0^0 = 0, \mathfrak{h}_0^0 = \mathfrak{t}_0^0 \oplus \mathfrak{a}_0^0, \lambda_0 = \lambda, \delta_0 = \delta, \lambda_{\sigma_0} = \lambda_0 + \delta_0, \nu_0 = 0, \Lambda_0 = \lambda_{\sigma_0} + (\delta_0 - 2\delta_{0,K})$. Suppose $M_j, A_j, \mathfrak{t}_0^j, \mathfrak{a}_0^j, \mathfrak{h}_0^j, \lambda_j, \delta_j, \lambda_{\sigma_j}, \Lambda_j$ and ν_j are given with $\dim A_j = j$ and with λ_{σ_j} dominant nonsingular with respect to all simple roots of M_j that are M_j compact. There are now two cases:

(a) If $\langle \lambda_{\sigma_j}, \alpha \rangle \geq 0$ for all simple roots α of M_j that are M_j noncompact, the recursive construction ends. Define $M = M_j, A = A_j, \lambda_\sigma = \lambda_{\sigma_j}$, and $\nu = \nu_0 + \cdots + \nu_j$. Define N so that ν is dominant relative to N. Then MAN, λ_σ, and ν are the cuspidal parabolic subgroup, the infinitesimal character of the M representation, and the parameter on $\mathfrak{a}_0$ of a set of Langlands parameters for the irreducible subquotient of $A_{\mathfrak{q}}(\lambda)$ containing the K type Λ.

(b) Otherwise, of the M_j noncompact simple roots α with $\langle \lambda_{\sigma_j}, \alpha \rangle < 0$, set α_{j+1} to be the one for which $-\frac{\langle \lambda_{\sigma_j}, \alpha \rangle}{|\alpha|^2}$ is greatest. Further, set

$$c_{j+1} = -\frac{\langle \lambda_{\sigma_j}, \alpha_{j+1} \rangle}{|\alpha_{j+1}|^2} = \frac{\langle s_{\alpha_{j+1}}(\lambda_{\sigma_j}), \alpha_{j+1} \rangle}{|\alpha_{j+1}|^2} \tag{4.2.1}$$

where $s_{\alpha_{j+1}}$ is the Weyl group reflection corresponding to α_{j+1}. Applying the Cayley transform relative to α_{j+1} [K4 §VI.7], we write $\mathfrak{h}_0^{j+1} = \mathfrak{t}_0^{j+1} \oplus \mathfrak{a}_0^{j+1}$

for the transformed version of $\mathfrak{h}_0^j$ and let $A_{j+1} = \exp(\mathfrak{a}_0^{j+1})$ with $\dim A_{j+1} = j+1$. Identifying α_{j+1} with its Cayley transform, set

$$\nu_{j+1} = c_{j+1}\alpha_{j+1} \qquad \text{and} \qquad \nu^{j+1} = \nu^j + \nu_{j+1}. \tag{4.2.2}$$

Define N_{j+1} so that ν^{j+1} is dominant relative to N_{j+1}. Let $M_{j+1}A_{j+1} = Z_G(A_{j+1})$, and we identify $\Delta(\mathfrak{m}_{j+1}, \mathfrak{t}^{j+1})$ with the subset of $\Delta(\mathfrak{m}_j, \mathfrak{t}^j)$ orthogonal to α_{j+1}. Set $\Delta^+_{M_{j+1}} = \Delta_{M_{j+1}} \cap \Delta^+_{M_j}$, let δ_{j+1} be half the sum of the positive roots and $\delta_{j+1,K}$ be half the sum of the positive M_{j+1} compact roots. Define $\lambda_{\sigma_{j+1}}$ to be the projection of λ_{σ_j} orthogonal to α_{j+1}, so that

(4.2.3)

$$\begin{aligned}\lambda_{\sigma_{j+1}} &= \lambda_{\sigma_j} - \frac{\langle \lambda_{\sigma_j}, \alpha_{j+1}\rangle}{|\alpha_{j+1}|^2}\alpha_{j+1} \\ &= \lambda_{\sigma_j} + \frac{\langle s_{\alpha_{j+1}}(\lambda_{\sigma_j}), \alpha_{j+1}\rangle}{|\alpha_{j+1}|^2}\alpha_{j+1} = \lambda_{\sigma_j} + c_{j+1}\alpha_{j+1} = \lambda_{\sigma_j} + \nu_{j+1}.\end{aligned}$$

We also define λ_{j+1} so that $\lambda_{\sigma_{j+1}} = \lambda_{j+1} + \delta_{j+1}$ and set $\Lambda_{j+1} = \lambda_{\sigma_{j+1}} + (\delta_{j+1} - 2\delta_{j+1,K})$. Then $\lambda_{\sigma_{j+1}}$ is dominant nonsingular relative to the M_{j+1} compact simple roots, and the recursive construction continues.

From these definitions, we also note that repeated iterations of (4.2.3) yield

$$\begin{aligned}\lambda_{\sigma_{j+1}} &= \lambda_{\sigma_0} + \nu_1 + \cdots + \nu_{j+1} \\ &= (\lambda + \delta) + \nu^{j+1}.\end{aligned} \tag{4.2.4}$$

Moreover, (4.2.3) shows that

- $\lambda_{\sigma_{j+1}}$ is also the projection of $s_{\alpha_{j+1}}(\lambda_{\sigma_j})$ orthogonal to α_{j+1}, and
- $\lambda_{\sigma_{j+1}} + \nu_{j+1} = s_{\alpha_{j+1}}(\lambda_{\sigma_j})$.

Proposition 10 of [K3] shows that the Conjectural Method runs into no obstruction in finding parameters MAN, λ_σ, and ν. In fact, the hypotheses used in Proposition 10 are relaxed from those initially described in the setting of this section. In particular, we still take $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ to be a parabolic subalgebra formed from the compact simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{t})$ and $\Delta(\mathfrak{u}) \subseteq \Delta^+$. But, we no longer assume that λ is orthogonal to the roots of $\mathfrak{l}$, only that λ is dominant for $\Delta_L^+ = \Delta^+ \cap \Delta_L$. Further, we do not make any assumption on the number of noncompact simple roots. Proposition 10 then shows that

(1) λ_j is analytically integral

(2) Λ_j is $\Delta^+_{M_{j,K}}$ dominant

(3) λ_j is dominant for the compact simple roots of $\Delta^+_{M_j}$.

Later, we shall restrict ourselves to the situation in which $\Delta^+(\mathfrak{g})$ contains only one noncompact simple root. This characteristic, however, is not necessarily retained by the subsequent $\Delta^+_{M_j}$. We will handle this possibility later.

Section 4.3: The Approach

Knapp proved via combinatorial arguments that his Method does produce the Langlands parameters in some cases. We now approach the problem from another, more representation-theoretic, point of view. This new approach will ultimately reduce the problem of verifying the Method to checking a finite number of computations. First, we sketch the approach.

We start by forming a θ-stable parabolic subalgebra $\mathfrak{q}_j = \mathfrak{l}_j \oplus \mathfrak{u}_j$ of $\mathfrak{m}_{j-1}$ with $(\mathfrak{l}_j)_0 \cong \mathfrak{t}_0^j \oplus \mathfrak{sl}(2, \mathbb{R})$ where the $\mathfrak{sl}(2, \mathbb{R})$ is built from α_j, and $\mathfrak{u}_j$ is built from the remaining positive root spaces of $\mathfrak{m}_{j-1}$. Then, at the level of $(\mathfrak{l}_j, L_j \cap K)$ modules - think $\mathfrak{sl}(2, \mathbb{R})$ - we have a short exact sequence, roughly described as

$$(4.3.1) \qquad 0 \to \begin{pmatrix} \text{discrete} \\ \text{series} \\ \text{module} \end{pmatrix} \to \begin{pmatrix} \text{principal} \\ \text{series} \\ \text{module} \end{pmatrix} \to \mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}_j}, T^{j-1}}^{\mathfrak{l}_j, L_j \cap K}(\mathbb{C}_\lambda) \to 0$$

where the principal series has ν_j as its $\mathfrak{a}_{\mathfrak{l}_j}^*$ parameter. We then form a long exact sequence of $(\mathfrak{m}_{j-1}, M_{j-1} \cap K)$ derived functor modules from (4.3.1) and the covariant, right exact $\mathcal{L}$ functor of cohomological induction. Focusing on the resulting

$$(4.3.2) \qquad \longrightarrow (\mathcal{L})_{S_j} \begin{pmatrix} \text{principal} \\ \text{series} \\ \text{module} \end{pmatrix} \longrightarrow (\mathcal{L})_{S_j} \mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}_j}, T^{j-1}}^{\mathfrak{l}_j, L_j \cap K}(\mathbb{C}_\lambda) \longrightarrow$$

section of the long exact sequence, where S_j is the so-called middle dimension, $\dim(\mathfrak{u}_j \cap \mathfrak{k})$, we

(1) show, via a "bottom layer"-type argument, that the $M_{j-1} \cap K$ type with highest weight Λ_{j-1}, which is nonzero in the right hand side, is in the image of the map,

(2) use induction-in-stages to write the right hand side of (4.3.2) as a single functor of cohomological induction,

(3) show that the left hand side of (4.3.2) may be regarded as a principal series $(\mathfrak{m}_{j-1}, M_{j-1} \cap K)$ module, and then

(4) induce this principal series $(\mathfrak{m}_{j-1}, M_{j-1} \cap K)$ module up to a $(\mathfrak{g}, K)$ module by applying a covariant, exact functor modelling ordinary parabolic induction with A_{j-1} parameter $\nu_1 + \cdots + \nu_{j-1}$. Then, using double induction and a Frobenius reciprocity argument, we show that the K type with highest weight Λ behaves as desired.

Chapter 5: Reduction of the Problem

Section 5.1: Steps (1) and (2) of the Approach

In this section, we carry out steps (1) and (2) of the Approach (see §4.3). To do so, we will prove the following theorem.

Theorem 5.1: Let G be a linear, noncompact Lie group, K a maximal compact subgroup, and rank $G =$ rank K. Let $T \subseteq K$ be a Cartan subgroup, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the set of roots. Fix a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ and let $\mathfrak{b}$ be the Borel subalgebra formed by the positive roots. Define $\Delta_K^+, \delta, \delta_K$ and $\delta(\mathfrak{p})$ in the usual way. Let λ be an analytically integral form on $\mathfrak{t}$. Assume

(1) $\langle \lambda + \delta, \beta \rangle > 0$ for all compact simple roots, β, of $\Delta^+(\mathfrak{g}, \mathfrak{t})$, and

(2) $\Lambda = \lambda + 2\delta(\mathfrak{p})$ is Δ_K^+ dominant.

Suppose α_1 is a noncompact simple root such that $\langle \lambda + \delta, \widehat{\alpha}_1 \rangle < 0$. Then form the θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ by building $\mathfrak{l}$ from α_1 and $\mathfrak{u}$ from the remaining positive root spaces. Let $\mathfrak{b}_\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_1}$ be a Borel subalgebra of $\mathfrak{l}$, $L = N_G(\mathfrak{q}) \cap N_G(\theta\mathfrak{q})$ and $S = \dim(\mathfrak{u} \cap \mathfrak{k})$.

Apply the Cayley transform [K4 §VI.7] relative to α_1 and write $\mathfrak{t}_0' \oplus \mathfrak{a}_0'$ for the transformed version of $\mathfrak{t}_0$. Identify α_1 with its image under the Cayley transform, and define λ_{σ_1} to be the projection of $\lambda + \delta$ orthogonal to α_1. Build $A = \exp(\mathfrak{a}_0')$ and form the minimal parabolic subgroup of L with Langlands decomposition $M_L A N_L$ in the usual way, taking N_L to be formed from the transformed $\mathfrak{g}_{\alpha_1}$. Since L is split modulo center, we have that $M_L = T'$.

Consider the parabolically induced $(\mathfrak{l}, L \cap K)$ module $X_{L\cap K}(\xi_L, \nu)$ (see

(3.5.1)) where

(i) $\xi_L = \widetilde{Z}_L$ is the irreducible $(\mathfrak{t}', \widetilde{T}_L)$ module of infinitesimal character $\lambda_{\sigma_1} - \delta(\mathfrak{u})$ (which we can regard as a $(\mathfrak{t}', T_L)$ module), that matches the action of T' in $\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda)$, and

(ii) $\nu = -\frac{\langle \lambda+\delta, \alpha_1 \rangle}{|\alpha_1|^2}\alpha_1 \in (\mathfrak{a}')^*$.

Then there exists a $(\mathfrak{g}, K)$ module map,

$$\mathcal{L}_S(\varphi) : \mathcal{L}_S(X_{L\cap K}(\xi_L, \nu)) \longrightarrow A_{\mathfrak{b}}(\lambda) \tag{5.1.1}$$

whose image contains the multiplicity-one K type with highest weight Λ.

Note: For the rest this paper, any numerical label beginning with 11 refers to Chapter XI of [KV].

Proof: Let α_1 be the noncompact simple root with $\langle \lambda + \delta, \widehat{\alpha}_1 \rangle < 0$. Build $\mathfrak{l}$ from α_1, and $\mathfrak{u}$ from the remaining positive root spaces so that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ stable parabolic subalgebra . We have $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \cong \mathfrak{t}' \oplus \mathfrak{sl}(2, \mathbb{C})$.

First, we will form a short exact sequence of $(\mathfrak{l}, L \cap K)$ modules. Form $\mathfrak{b}_{\mathfrak{l}} = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_1}$, a Borel subalgebra of $\mathfrak{l}$. Since $\mathfrak{t}$ is θ stable and α_1 is imaginary, we have $\bar{\mathfrak{b}}_{\mathfrak{l}} = \mathfrak{t} \oplus \mathfrak{g}_{-\alpha_1}$. Then $\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda)$ is an upside down Verma module for $\mathfrak{t}' \oplus \mathfrak{sl}(2, \mathbb{C})$ with infinitesimal character $\lambda + \delta(\mathfrak{l})$ and lowest weight $\lambda + 2\delta(\mathfrak{l})$ relative to the Cartan subalgebra $\mathfrak{t}_0$.

Next we consider $X_{L\cap K}(\xi_L, \nu)$ (see (3.5.1)) as the Harish-Chandra module of a principal series representation of L. This has a positive $(\mathfrak{a}')^*$

parameter, and according to [KV Prop. 11.43] its infinitesimal character is

$$\begin{aligned}\lambda_{\sigma_1} - \delta(\mathfrak{u}) + \nu = \lambda_{\sigma_1} + \nu - \delta(\mathfrak{u}) \\ &= s_{\alpha_1}(\lambda + \delta) - \delta(\mathfrak{u}) \\ &= s_{\alpha_1}(\lambda + \delta - \delta(\mathfrak{u})) \qquad \text{since } \alpha_1 \perp \delta(\mathfrak{u}) \text{ [KV 4.69]}.\end{aligned}$$

Accordingly, as L is locally isomorphic to $T' \times SL(2,\mathbb{R})$ we know from Chapter 2 that $X_{L\cap K}(\xi_L, \nu)$ contains the underlying $(\mathfrak{g}, K)$ module of a representation we can call a

> (matching character on T') $\times$ ($SL(2,\mathbb{R})$ discrete series), with infinitesimal character $(\lambda+\delta) - \delta(\mathfrak{u})$

as a submodule. By matching infinitesimal characters, $L\cap K$ types and $L\cap K$ actions, we see that the resulting quotient of $X_{L\cap K}(\xi_L, \nu)$ by this discrete series module is $\mathcal{L}^{\mathfrak{l}, L\cap K}_{\bar{\mathfrak{b}}_{\mathfrak{l}}, T}(\mathbb{C}_\lambda)$. Therefore, we have the short exact sequence

$$0 \to \begin{pmatrix}\text{discrete}\\ \text{series}\\ \text{module}\end{pmatrix} \to X_{L\cap K}(\xi_L, \nu) \to \mathcal{L}^{\mathfrak{l}, L\cap K}_{\bar{\mathfrak{b}}_{\mathfrak{l}}, T}(\mathbb{C}_\lambda) \to 0$$

We let

$$\varphi : X_{L\cap K}(\xi_L, \nu) \longrightarrow \mathcal{L}^{\mathfrak{l}, L\cap K}_{\bar{\mathfrak{b}}_{\mathfrak{l}}, T}(\mathbb{C}_\lambda)$$

be the quotient map.

To continue the proof, we use an argument not unlike that on page 765 of [KV], with φ above replacing the φ in [KV]. Accordingly, we form the diagram

(5.1.2)

$$\begin{array}{ccc} U(\mathfrak{k}) \otimes_{U(\bar{\mathfrak{q}}\cap\mathfrak{k})} X_{L\cap K}(\xi_L, \nu)^{\#}_{\bar{\mathfrak{q}}} & \xrightarrow{\operatorname{ind}^{\mathfrak{k}, L\cap K}_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}(\varphi^{\#}_{\bar{\mathfrak{q}}})} & U(\mathfrak{k}) \otimes_{U(\bar{\mathfrak{q}}\cap\mathfrak{k})} [\mathcal{L}^{\mathfrak{l}, L\cap K}_{\bar{\mathfrak{b}}_{\mathfrak{l}}, T}(\mathbb{C}_\lambda)]^{\#}_{\bar{\mathfrak{q}}} \\ \beta_X \downarrow & & \downarrow \beta_A \\ U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} X_{L\cap K}(\xi_L, \nu)^{\#}_{\bar{\mathfrak{q}}} & \xrightarrow{\operatorname{ind}^{\mathfrak{g}, L\cap K}_{\bar{\mathfrak{q}}, L\cap K}(\varphi^{\#}_{\bar{\mathfrak{q}}})} & U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} [\mathcal{L}^{\mathfrak{l}, L\cap K}_{\bar{\mathfrak{b}}_{\mathfrak{l}}, T}(\mathbb{C}_\lambda)]^{\#}_{\bar{\mathfrak{q}}} \end{array}$$

where the maps β_X and β_A are the one-one $(\mathfrak{k}, L\cap K)$ maps given by (3.6.1). This diagram commutes since effectively β_X and β_A act in the respective first factors while $\mathrm{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k},L\cap K}^{\mathfrak{k},L\cap K}(\varphi_{\bar{\mathfrak{q}}}^{\#})$ and $\mathrm{ind}_{\bar{\mathfrak{q}},L\cap K}^{\mathfrak{g},L\cap K}(\varphi_{\bar{\mathfrak{q}}}^{\#})$ act in the respective second factors. Applying Π_S^K to this diagram, where $S = \dim(\mathfrak{u}\cap\mathfrak{k})$ is the appropriate middle dimension, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_S^K(X_{L\cap K}(\xi_L,\nu)) & \xrightarrow{\mathcal{L}_S^K(\varphi)} & \mathcal{L}_S^K(\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda)) \\ {\scriptstyle \mathcal{B}_X}\downarrow & & \downarrow{\scriptstyle \mathcal{B}_A} \\ \mathcal{L}_S(X_{L\cap K}(\xi_L,\nu)) & \xrightarrow{\mathcal{L}_S(\varphi)} & \mathcal{L}_S(\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda)) \end{array} \tag{5.1.3}$$

in which $\mathcal{B}_X$ and $\mathcal{B}_A$ are bottom-layer maps as in (3.6.2).

The $L\cap K$ type with highest weight $\lambda + 2\delta(\mathfrak{l})$ occurs with multiplicity-one in the the $\mathfrak{sl}(2,\mathbb{R})$ principal series type module, $X_{L\cap K}(\xi_L,\nu)$, and φ is one-one on that $L\cap K$ type. Since $\Lambda = (\lambda + 2\delta(\mathfrak{l})) + 2\delta(\mathfrak{u}\cap\mathfrak{p}) = \lambda + 2\delta(\mathfrak{p})$ is assumed to be K dominant, the K type τ_Λ with highest weight Λ occurs with multiplicity one in $\mathcal{L}_S(X_{L\cap K}(\xi_L,\nu))$ and $\mathcal{L}_S^K(\varphi)$ is one-one on τ_Λ. By [KV Th. 5.80a, see §3.6], $\mathcal{B}_A$ is one-one onto for the K type with highest weight Λ. Consequently, $\mathcal{B}_A \circ \mathcal{L}_S^K(\varphi)$ maps onto the multiplicity-one K type of $\mathcal{L}_S(\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda))$ with highest weight Λ. Thus, by the commutativity of the diagram, the same thing must be true of $\mathcal{L}_S(\varphi)\circ\mathcal{B}_X$, and we conclude that the image of $\mathcal{L}_S(\varphi)$ contains the multiplicity-one K type with highest weight Λ.

To complete the proof, we must identify the range representation of the map $\mathcal{L}_S(\varphi)$ as $A_{\mathfrak{b}}(\lambda)$. Since α_1 is a noncompact root, we have that the appropriate middle dimension of $(\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K})_i(\mathbb{C}_\lambda)$ is 0. Because the functors of cohomological induction vanish above the middle dimension, we have that

$(\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K})_i(\mathbb{C}_\lambda)$ is nonvanishing in only one degree. Moreover, the 0^{th} derived functor is nothing more than $\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda)$. Therefore the double induction result in [KV, Cor. 11.86a] is applicable. When combined with a supplementary argument to take $\bigwedge^{\text{top}}(\mathfrak{u})$ into account (cf. [KV, §XI.7]), it gives $\mathcal{L}_S(\mathcal{L}_{\bar{\mathfrak{b}}_{\mathfrak{l}},T}^{\mathfrak{l},L\cap K}(\mathbb{C}_\lambda)) \cong (\mathcal{L}_{\bar{\mathfrak{b}},T}^{\mathfrak{g},K})_S(\mathbb{C}_\lambda) \cong A_{\mathfrak{b}}(\lambda)$ where $\mathfrak{b}$ is the natural Borel subalgebra formed from $\Delta^+(\mathfrak{g},\mathfrak{t})$. $\square$

Let us now specialize the above theorem to produce a corollary which will be used in proving Knapp's Conjectural Method for $A_{\mathfrak{b}}(\lambda)$. Set $G = M_{j-1}, \mathfrak{q}_j = \mathfrak{l}_j \oplus \mathfrak{u}_j$ the parabolic subalgebra of $\mathfrak{m}_{j-1}$ with $\mathfrak{l}_j$ formed from α_j and $\mathfrak{u}_j$ from the remaining positive root spaces, ξ_{L_j} to be the appropriate character on T^j, and abbreviate $(\mathcal{L}_{\bar{\mathfrak{q}}_j,L_j\cap K}^{\mathfrak{m}_{j-1},M_{j-1}\cap K})_{S_j}$ by $\mathcal{L}_{S_j}$, so that we obtain the corollary below.

Corollary 5.2: In the setting of §4.2 there exists an $(\mathfrak{m}_{j-1}, M_{j-1}\cap K)$ module map

$$(5.1.4) \qquad \mathcal{L}_{S_j}(\varphi_j) : \mathcal{L}_{S_j}(X_{L_j\cap K}(\xi_{L_j}, \nu_j)) \longrightarrow A_{\mathfrak{b}_{\mathfrak{m}_{j-1}}}(\lambda_{j-1})$$

whose image contains the multiplicity one $(M_{j-1} \cap K)$ type with highest weight Λ_{j-1}.

Section 5.2: Reduction of Step (3) to Calculable Conditions

At this point, we would like to rewrite the domain space of $\mathcal{L}_{S_j}(\varphi_j)$ in (5.1.4) as an $(\mathfrak{m}_{j-1}, M_{j-1} \cap K)$ principal series module as follows:

$$(5.2.1) \qquad \mathcal{L}_{S_j}(X_{L_j\cap K}(\xi_{L_j}, \nu_j)) \cong X_{M_{j-1}\cap K}(\xi_j^h, \nu_j),$$

where ξ_j^h is an M_j representation with underlying $(\mathfrak{m}_j, M_j \cap K)$ module $A_{\mathfrak{b}_{\mathfrak{m}_j}}(\lambda_j)$. This amounts to a change of polarization and is the subject of Theorem 11.225 of [KV]. Unfortunately, the dominance condition (11.220) required to apply Theorem 11.225 will not usually be satisfied. However, in our situation, since A^j is one dimensional, we are able to run through the proof of Theorem 11.225 to extract weaker conditions under which the isomorphism (5.2.1) holds.

Theorem 5.3: Assume the same set-up as in Theorem 5.1. Let $MA = Z_G(\mathfrak{a}_0')$, $\Delta^+(\mathfrak{m}) = \Delta^+(\mathfrak{g}) \cap \Delta(\mathfrak{m}, \mathfrak{t}')$, $\delta(\mathfrak{m})$ be half the sum of the positive roots of $\mathfrak{m}$, and $\mathfrak{b}_{\mathfrak{m}}$ the Borel subalgebra formed from $\Delta^+(\mathfrak{m})$. Let $\lambda_1 = \lambda_{\sigma_1} - \delta(\mathfrak{m})$. Let

(5.2.2) $$C = \{\gamma \in \Delta^+(\mathfrak{g}) - \{\alpha_1\} \mid \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\}\}.$$

If

(5.2.3) $$\langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma} \rangle > 0 \qquad \text{for all } \gamma \in C$$

then the domain space of $\mathcal{L}_S(\varphi)$ in (5.1.1) is a $(\mathfrak{g}, K)$ principal series module. Specifically

(5.2.4) $$\mathcal{L}_S(X_{L \cap K}(\xi_L, \nu)) \cong X_K(\xi^h, \nu)$$

where ξ^h has $A_{\mathfrak{b}_{\mathfrak{m}}}(\lambda_1)$ as its underlying $(\mathfrak{m}, M \cap K)$ module.

Remark: The proof of this theorem follows the lines of the proof of Theorem 11.225 [KV], except that we replace condition (11.220) of that theorem, with condition (5.2.3) above.

Proof: We restate the setup of [KV Th 11.225] in our notation as it applies to our situation, dropping the hypothesis (11.220) on the functional λ_{σ_1} in (iv) below. We start with

(i) the θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$,

(ii) the Levi subgroup $L = N_G(\mathfrak{q}) \cap N_G(\theta\mathfrak{q})$ for $\mathfrak{l}$

(iii) the Cartan pair $(\mathfrak{h}', T')$ for both $(\mathfrak{l}, L \cap K)$ and $(\mathfrak{g}, K)$ with $\mathfrak{h}'_0 = \mathfrak{t}'_0 \oplus \mathfrak{a}'_0$ and $T' = Z_{L\cap K}(\mathfrak{h}') = Z_K(\mathfrak{h}')$.

(iv) the functional $\lambda_{\sigma_1} \in i\mathfrak{t}'^*_0$

Let $A_L = A = \exp(\mathfrak{a}'_0)$ and consider two continuous-series representations

$$I^G_{MAN}(\xi, \nu) \qquad \text{and} \qquad I^L_{M_L A N_L}(\xi_L, \nu) \tag{5.2.5}$$

and their underlying modules subject to the following conditions:

(5.2.6)

$$MA = Z_G(\mathfrak{a}'_0) \qquad \text{and} \quad M_L A = Z_L(\mathfrak{a}'_0),$$

$$N \supseteq N_L,$$

$$\langle \text{Re } \nu, \beta\rangle \geq 0 \qquad \text{for every positive } \mathfrak{a}\text{-root } \beta \text{ of } \mathfrak{g},$$

$$\mathfrak{b}_{\mathfrak{m}_L} = \text{ Borel subalgebra of } \mathfrak{m}_L,$$

$$\mathfrak{b}_{\mathfrak{m}} = \text{ Borel subalgebra of } \mathfrak{m},$$

$$\mathfrak{b}_{\mathfrak{m}} \supseteq \mathfrak{b}_{\mathfrak{m}_L} \qquad \text{and} \qquad \mathfrak{b}_{\mathfrak{m}} \supseteq \mathfrak{m}_\beta \text{ if } \mathfrak{m}_\beta \subseteq \mathfrak{u},$$

$$\widetilde{Z}_L = \text{irreducible } (\mathfrak{t}', \widetilde{T}_L) \text{ module of infinitesimal character } \lambda_{\sigma_1} - \delta(\mathfrak{u}),$$

$$\widetilde{Z} = \text{irreducible } (\mathfrak{t}', \widetilde{T}') \text{ module of infinitesimal character } \lambda_{\sigma_1},$$

$$\widetilde{Z} = \widetilde{Z}_L \otimes \mathbb{C}_{\delta(\mathfrak{u})},$$

$$\xi_L = \xi(\widetilde{Z}_L, \mathfrak{b}_{\mathfrak{m}_L}) \text{ as a representation of } M_L, \text{ and}$$

$$\xi = \xi(\widetilde{Z}, \mathfrak{b}_{\mathfrak{m}}) \text{ as a representation of } M.$$

Accordingly, the underlying $(\mathfrak{m}, M\cap K)$ module of ξ is $({}^{\mathfrak{n}}\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{b}_\mathfrak{m},T'})^{S_\mathfrak{m}}(\widetilde{Z})$. By ξ^h we shall mean the representation of M with underlying $(\mathfrak{m}, M\cap K)$ module $({}^{\mathfrak{n}}\mathcal{L}^{\mathfrak{m},M\cap K}_{\mathfrak{b}_\mathfrak{m},T'})_{S_\mathfrak{m}}(\widetilde{Z})$ - which after tracing through the definitions is $A_{\mathfrak{b}_\mathfrak{m}}(\lambda)$, apart from technicalities involving double covers. Also, $\xi_L^h \cong \xi_L$ since $M_L = T'$. Further, we remark that $\mathfrak{b}_{\mathfrak{m}_L}$ is nothing more than $\mathfrak{t}'$ because L is split modulo center. Also, we identify the positive roots as follows:

$$(5.2.7)\qquad \begin{aligned} \Delta^+_{\text{imag}}(\mathfrak{g},\mathfrak{h}') &= \text{ members of } \Delta(\mathfrak{m},\mathfrak{t}') \text{ contributing to } \mathfrak{b}_\mathfrak{m} \\ \Delta^+_{\text{imag}}(\mathfrak{l},\mathfrak{h}') &= \text{ members of } \Delta(\mathfrak{m}_L,\mathfrak{t}') \text{ contributing to } \mathfrak{b}_{\mathfrak{m}_L} = \emptyset \\ \Delta^+_{\text{real}}(\mathfrak{g},\mathfrak{h}') &= \text{ real roots contributing to } \mathfrak{n} \\ \Delta^+_{\text{real}}(\mathfrak{l},\mathfrak{h}') &= \text{ real roots contributing to } \mathfrak{n}_L = \alpha_1. \end{aligned}$$

In this set-up, we argue as in the proof of Theorem 11.225 by letting $X'_K(\xi,\nu)$ and $X'_{L\cap K}(\xi_L,\nu)$ be the underlying Harish-Chandra modules for $I^G_{MAN^-}(\xi,\nu)$ and $I^L_{M_LAN_L^-}(\xi_L,\nu)$, respectively. We shall prove, in a moment, that

$$(5.2.8)\qquad \mathcal{R}^S(X'_{L\cap K}(\xi_L,\nu)) \cong X'_K(\xi,\nu).$$

Assuming (5.2.8), we have, just as in the proof of Theorem 11.225,

$$\begin{aligned} \mathcal{L}_j(X_{L\cap K}(\xi_L,\nu))^h &\cong \mathcal{L}_j(X_{L\cap K}(\xi_L^h,\nu))^h && \text{since } \xi_L \text{ is unitary} \\ &\cong \mathcal{R}^j(X_{L\cap K}(\xi_L^h,\nu)^h) && \text{by [KV (6.24), (6.21a)]} \\ &\cong \mathcal{R}^j(X_{L\cap K}((\xi_L^h)^h,(\nu)^h)) && \text{by [KV Cor. 11.59]} \\ &\cong \mathcal{R}^j(X_{L\cap K}(\xi_L,-\bar\nu)) && \text{by admissibility of } \xi_L \\ &\cong X_K(\xi,-\bar\nu) && \text{by (5.2.8)} \\ &\cong X_K((\xi^h)^h,\nu^h) && \text{by admissibility of } \xi \\ &\cong (X_K(\xi^h,\nu))^h && \text{by [KV Cor. 11.59].} \end{aligned}$$

Setting $j = S$, taking $(\cdot)^h$ of both sides and again using admissibility, we see that (5.2.8) implies the result of the theorem.

To begin the proof of (5.2.8) we start by writing, via (11.210),

$$X'_{L\cap K}(\xi_L, \nu) = ({}^{\mathfrak{n}}\mathcal{R}^{\mathfrak{l}, L\cap K}_{\mathfrak{h}'+\mathfrak{n}_L^-})^p(\widetilde{Z}_L \otimes \mathbb{C}_\nu).$$

Since $\Delta^+(\mathfrak{l}, \mathfrak{h}')$ does not contain any imaginary roots, the index $p = 0$. Therefore, a Mackey isomorphism and an induction-in-stages result give

$$
\begin{aligned}
&\mathcal{R}^S(X'_{L\cap K}(\xi_L, \nu)) \\
&= ({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S(({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{l},L\cap K}_{\mathfrak{h}'+\mathfrak{n}_L^-,T'})(\widetilde{Z}_L \otimes \mathbb{C}_{\delta(\mathfrak{n}_L^-)'} \otimes \mathbb{C}_\nu) \otimes \mathbb{C}_{2\delta(\mathfrak{u})}) \\
&= ({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^S({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{l},L\cap K}_{\mathfrak{h}'+\mathfrak{n}_L^-,T'})(\widetilde{Z}_L \otimes \mathbb{C}_{\delta(\mathfrak{n}_L^-)'} \otimes \mathbb{C}_\nu \otimes \mathbb{C}_{2\delta(\mathfrak{u})}) \\
&= ({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{h}'+\mathfrak{u}+\mathfrak{n}_L^-,T'})^S(\widetilde{Z}_L \otimes \mathbb{C}_{\delta(\mathfrak{n}_L^-)'} \otimes \mathbb{C}_\nu \otimes \mathbb{C}_{2\delta(\mathfrak{u})}) \\
&= ({}^{\mathfrak{n}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{h}'+\mathfrak{u}+\mathfrak{n}_L^-,T'})^S(\widetilde{Z}_L \otimes \mathbb{C}_{\delta(\mathfrak{n}_L^-)'} \otimes \mathbb{C}_\nu \otimes \mathbb{C}_{2\delta(\mathfrak{u})} \otimes \mathbb{C}_{-\delta(\mathfrak{u}+\mathfrak{n}_L^-)'}).
\end{aligned}
\tag{5.2.9}
$$

At this point in the proof, we would like to change the Borel subalgebra $\mathfrak{h}'+\mathfrak{u}+\mathfrak{n}_L^-$ to the Borel subalgebra $\mathfrak{h}'+\mathfrak{n}^-+\mathfrak{n}_\mathfrak{m}$. In order to make this change, we shall apply Lemma 11.128 of [KV], which we restate here:

Lemma 11.128 [KV] Let $\{(\mathfrak{h}, T), \lambda, \Delta^+_{\text{imag}}, \Delta^+_{\text{real}}, \{Z(\mathfrak{b})\}$ be a set of data for standard $(\mathfrak{g}, K)$ modules satisfying (i) and either (ii) or (ii') in (11.110), and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$ be two Borel subalgebras satisfying the conditions

(i) $\Delta^+_{\text{imag}} \subseteq \Delta(\mathfrak{n}) \cap \Delta(\mathfrak{n}')$

(ii) $\Delta^+_{\text{real}} \subseteq \Delta(\mathfrak{n}) \cap \Delta(\mathfrak{n}')$

(iii) whenever α is a complex root with $\alpha \in \Delta(\mathfrak{n}')$ but not in $\Delta(\mathfrak{n})$ and is such that $\langle \lambda, \widehat{\alpha} \rangle$ is a nonzero integer, then the integer is positive, and $\theta\alpha$ is in $\Delta(\mathfrak{n}) \cap \Delta(\mathfrak{n}')$.

If $p = \dim(\mathfrak{n} \cap \mathfrak{k})$ and $p' = \dim(\mathfrak{n}' \cap \mathfrak{k})$, then for all q

$$({}^{\mathfrak{u}}\mathcal{L}^{\mathfrak{g},K}_{\mathfrak{b}^-,T})_{p+q}(Z(\mathfrak{b})) \cong ({}^{\mathfrak{u}}\mathcal{L}^{\mathfrak{g},K}_{\mathfrak{b}'^-,T})_{p'+q}(Z(\mathfrak{b}'))$$
$$({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{b}^-,T})^{p+q}(Z(\mathfrak{b})) \cong ({}^{\mathfrak{u}}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{b}'^-,T})^{p'+q}(Z(\mathfrak{b}')).$$

In our application of this lemma, $\mathfrak{h}' + \mathfrak{u} + \mathfrak{n}_L^-$ will play the role of $\mathfrak{b}'$ and $\mathfrak{h}' + \mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}}$ will play the role of $\mathfrak{b}$. Condition (11.110) is used only in the definition of "standard module" [cf. KV §XI.6] and has no role in the proof.

In this case, conditions (i) and (ii) in the lemma are immediate from (5.2.7). Therefore, we only need to show that condition (iii) is satisfied. First, let us be a bit more explicit about what needs to be shown.

We set

$$C = \{\gamma \in \Delta_{\mathrm{cplx}}(\mathfrak{u} + \mathfrak{n}_L^-) \mid \gamma \notin \Delta(\mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}}) \text{ and } \langle \lambda_{\sigma_1} + \nu, \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\}\}.$$

We are to show that if $\gamma \in C$ then

(a) $\theta\gamma \in \Delta(\mathfrak{u} + \mathfrak{n}_L^-) \cap \Delta(\mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}})$ and

(b) $\langle \lambda_{\sigma_1} + \nu, \widehat{\gamma} \rangle > 0$.

Let $\gamma \in C$. If $\gamma \in \Delta(\mathfrak{n}_L^-)$, then $\gamma \in \Delta(\mathfrak{n}^-) \subseteq \Delta(\mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}})$. So when $\gamma \in C$ we must have $\gamma \in \Delta(\mathfrak{u})$. Since $\Delta(\mathfrak{u})$ is θ stable $\theta\gamma \in \mathfrak{u}$ and hence also in $\Delta(\mathfrak{u} + \mathfrak{n}_L^-)$. Moreover, $-\gamma \in \Delta(\mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}})$ and the fact that γ is a complex root gives that $-\gamma \in \Delta(\mathfrak{n}^-)$ which is closed under conjugation. Therefore $-\bar{\gamma} = \theta\gamma$ is in $\Delta(\mathfrak{n}^-)$ and also in $\Delta(\mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}})$. Hence we have (a), i.e., $\theta\gamma \in \Delta(\mathfrak{u} + \mathfrak{n}_L^-) \cap \Delta(\mathfrak{n}^- + \mathfrak{n}_{\mathrm{m}})$.

For (b), using the information in the above paragraph we can reexpress C as

$$C = \{\gamma \in \Delta_{\mathrm{cplx}}(\mathfrak{u}) \mid -\gamma \in \Delta(\mathfrak{n}^-) \text{ and } \langle \lambda_{\sigma_1} + \nu, \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\}\}.$$

Further, because A is one dimensional, we have the following equivalences

$$\begin{aligned} -\gamma \in \Delta(\mathfrak{n}^-) &\Longleftrightarrow \langle -\gamma, \nu \rangle < 0 \\ &\Longleftrightarrow \langle -\gamma, \alpha_1 \rangle < 0 \qquad \text{since } \nu = c\alpha_1 \text{ with } c > 0 \\ &\Longleftrightarrow \langle \gamma, \alpha_1 \rangle > 0 \end{aligned}$$

which allow us to write

$$C = \left\{ \gamma \in \Delta_{\text{cplx}}(\mathfrak{u}) \mid \langle \gamma, \alpha_1 \rangle > 0 \text{ and } \langle \lambda_{\sigma_1} + \nu, \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\} \right\}.$$

If we apply the inverse Cayley transform with respect to α_1 to $\mathfrak{h}'_0$ so that we once again have a compact Cartan subalgebra , and identify the roots with their image under this transformation we then may write

$$\begin{aligned} C &= \left\{ \gamma \in \Delta(\mathfrak{u}) \mid \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\} \right\} \\ &= \left\{ \gamma \in \Delta^+(\mathfrak{g}) - \{\alpha_1\} \mid \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\} \right\}. \end{aligned}$$

Hence, by hypothesis (5.2.3), we have that (b) holds as well.

Therefore, Lemma 11.128 is applicable. So, in the last line of (5.2.9) we change the Borel subalgebra $\mathfrak{h}' + \mathfrak{u} + \mathfrak{n}_L^-$ to the Borel subalgebra $\mathfrak{h}' + \mathfrak{n}^- + \mathfrak{n}_{\mathfrak{m}}$. At this point, the arguments given with [KV Th. 11.216] complete the proof of (5.2.8) and therefore the proof of Theorem 5.3. □

We again specialize to the setup of the Conjectural Method and apply Theorem 5.3 to the various steps of Knapp's Conjectural Method to obtain the following corollaries.

Corollary 5.4: Using the notation of the Conjectural Method, let

$$C_j = \left\{ \gamma \in \Delta^+(\mathfrak{m}_{j-1}) - \{\alpha_j\} \mid \langle \gamma, \alpha_j \rangle > 0, \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\} \right\}$$

If

$$(5.2.10) \qquad \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma}\rangle > 0 \qquad \text{for all } \gamma \in C_j$$

then (5.2.1) holds; that is, the domain space of $\mathcal{L}_{S_j}(\varphi_j)$ in (5.1.4) is a $(\mathfrak{m}_{j-1}, M_{j-1} \cap K)$ principal series module, which has been written in §5.2 as

$$(5.2.1) \qquad \mathcal{L}_{S_j}(X_{L_j \cap K}(\xi_{L_j}, \nu_j)) \cong X_{M_{j-1} \cap K}(\xi_j^h, \nu_j),$$

where ξ_j^h is an M_j representation with underlying $(\mathfrak{m}_j, M_j \cap K)$ module $A_{\mathfrak{b}_{\mathfrak{m}_j}}(\lambda_j)$.

Furthermore, Corollary 5.4 combined with Corollary 5.2 gives

Corollary 5.5: Assuming (5.2.10), there exists an $(\mathfrak{m}_{j-1}, M_{j-1} \cap K)$ module map which has been written in §5.1 as

$$(5.1.4) \qquad \mathcal{L}_{S_j}(\varphi_j) : X_{M_{j-1} \cap K}(\xi_j^h, \nu_j) \longrightarrow A_{\mathfrak{b}_{\mathfrak{m}_{j-1}}}(\lambda_{j-1})$$

whose image contains the $(M_{j-1} \cap K)$ type with highest weight Λ_{j-1}.

Section 5.3: Step (4) of the Approach

We conclude this chapter with the final reduction:

Corollary 5.6: Using the notation of the Conjectural Method, let

$$C_j = \{\gamma \in \Delta^+(\mathfrak{m}_{j-1}) - \{\alpha_j\} \mid \langle \gamma, \alpha_j \rangle > 0, \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma}\rangle \in \mathbb{Z} - \{0\}\}.$$

Suppose the recursive process of the Conjectural Method stops after n steps. If (5.2.10) holds for all $j, 1 \leq j \leq n$, then there exists a $(\mathfrak{g}, K)$ map from a standard continuous series module to $A_{\mathfrak{b}}(\lambda)$,

$$(5.2.11) \qquad \Phi : X_K(\xi_n^h, \nu^n) \longrightarrow A_{\mathfrak{b}}(\lambda),$$

whose image contains the K type with highest weight Λ.

Proof: Apply the functor

$$X_K(\cdot, \nu^{j-1}) : \mathcal{C}(\mathfrak{m}_{j-1}, M_{j-1} \cap K) \longrightarrow \mathcal{C}(\mathfrak{g}, K)$$

to the mapping (5.1.4) of Corollary 5.5. Denote the resulting $(\mathfrak{g}, K)$ map by ϕ_j, and use double induction [pg. 740, KV] to combine the resulting domain space as $X_K(\xi_j^h, \nu_j + \nu^{j-1})$. Since $\nu_j + \nu^{j-1} = \nu^j$ and $A_{\mathfrak{b}_{\mathfrak{m}_{j-1}}}(\lambda_{j-1})$ can be identified with ξ_j^h, we have the $(\mathfrak{g}, K)$ maps

$$\phi_j : X_K(\xi_j^h, \nu^j) \longrightarrow X_K(\xi_{j-1}^h, \nu^{j-1}).$$

By Corollary 5.5 and Frobenius reciprocity §3.5, the K type with highest weight Λ lies in the image of each ϕ_j. Therefore, if the Conjectural Method terminates after n steps, we can compose the ϕ_j maps to create a $(\mathfrak{g}, K)$ map

$$\Phi : X_K(\xi_n^h, \nu^n) \longrightarrow X_K(\xi_0^h, \nu^0)$$

whose image contains the K type with highest weight Λ. We have $M_0 = G$ and $A_0 = I$ so that the range representation of Φ is $\xi_0^h = A_{\mathfrak{b}}(\lambda)$:

$$\Phi : X_K(\xi_n^h, \nu^n) \longrightarrow A_{\mathfrak{b}}(\lambda)$$

Since the Conjectural Method stops when the infinitesimal character of ξ_n^h, namely λ_{σ_n}, is $\Delta^+(\mathfrak{m}_{j-1})$ dominant, ξ_n^h is a discrete series (or limit of discrete series) module. □

Remark: It is the mapping Φ that allows one to read off the Langlands parameters of the submodule V of $A_{\mathfrak{b}}(\lambda)$ generated by the K type Λ, provided that (5.2.10) holds for all j.

Chapter 6: Application of the Results

In order to use Corollary 5.6, we need a situation in which condition (5.2.10) holds at each step, j, of the Conjectural Method. In this chapter, we provide such a situation, culminating with Theorem 6.9 and Corollary 6.10.

Section 6.1: Main Hypotheses

The set-up for this chapter will be similiar to that in Section 4.2. G is a linear, simple, noncompact Lie group with finite center, K a maximal compact subgroup, and rank $G =$ rank K. Let $T \subseteq K$ be a Cartan subroup, and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ be the set of roots. Fix a positive system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$. To obtain the best results we will *assume*

(∗) there is exactly one noncompact simple root of $\Delta^+(\mathfrak{g})$, and this root has multiplicity at most two in the highest root of $\Delta^+(\mathfrak{g})$.

Define $\Delta_K^+, \delta, \delta_K$, and $\delta(\mathfrak{p})$ as usual. Let λ be an analytically integral form on $\mathfrak{t}$, and let

$$\Lambda = \lambda + 2\delta(\mathfrak{p}) = (\lambda + \delta) + (\delta - 2\delta_K).$$

Main Hypotheses: The hypotheses that we will invoke this chapter are

(ia) $\langle \lambda, \beta \rangle \geq 0$ for all simple roots β, except for one noncompact simple root, α_1, for which $\langle \lambda + \delta, \widehat{\alpha}_1 \rangle$ is a negative integer, or

(ib) $\langle \lambda + \delta, \beta \rangle \geq 0$ for all simple roots β, except for one noncompact simple root, α_1, for which $\langle \lambda + \delta, \widehat{\alpha}_1 \rangle$ is a negative integer, and

(ii) $\Lambda := \lambda + 2\delta(\mathfrak{p}) = (\lambda + \delta) + (\delta - 2\delta_K)$ is Δ_K^+ dominant.

Note: Since $\langle \delta, \beta \rangle > 0$ for simple roots β, if condition (ia) holds, then so does condition (ib).

Section 6.2: Satisfying the Reduced Conditions when $j = 1$

Let us recall what needs to be shown. Fix $j \geq 1$. Let

$$C_j = \{\gamma \in \Delta^+(\mathfrak{m}_{j-1}) - \{\alpha_j\} \mid \langle \gamma, \alpha_j \rangle > 0, \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\}\}.$$

We need to show that

$$\text{if } \gamma \in C_j \text{ then } \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma} \rangle > 0. \tag{6.2.1}$$

To start the proof of (6.2.1), we let $j = 1$ so that

$$C_1 = \{\gamma \in \Delta^+(\mathfrak{g}) - \{\alpha_1\} | \langle \gamma, \alpha_1 \rangle > 0, \langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\}\}.$$

First, note that if $\gamma \in C_1$ then the condition $\langle \gamma, \alpha_1 \rangle > 0$ forces the coefficient of α_1 in the $\Delta^+(\mathfrak{g})$-simple expansion of γ to be ≥ 1.

Proposition 6.1: Let $\gamma \in C_1$ and suppose that the coefficient of α_1 in the $\Delta^+(\mathfrak{g})$-simple expansion of γ is one. Assuming (ib), we have $\langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma} \rangle > 0$.

Remark: We are imposing no compactness/noncompactness restrictions on the remaining simple roots of $\Delta^+(\mathfrak{g})$. In particular, we are not assuming $(*)$. Further, we are not assuming (ii).

Proof: We write γ in its $\Delta^+(\mathfrak{g})$-simple expansion as

$$\gamma = \sum a_i \epsilon_i + \sum b_i \beta_i + a\alpha_1,$$

where the first sum is over the simple roots non-adjacent to α_1, the second sum is over the simple roots adjacent to α_1, $a_i \geq 0, b_i \geq 0$ with some $b_i > 0$. Then $0 < \langle \gamma, \widehat{\alpha}_1 \rangle = \sum b_i \langle \beta_i, \widehat{\alpha}_1 \rangle + 2a$. If we set $\kappa = \sum b_i \langle \beta_i, \widehat{\alpha}_1 \rangle$, then κ is a strictly negative integer. Therefore, $0 < -\kappa < 2a$ with each inequality strict. So, if $a = 1$, then $\kappa = -1, \kappa + a = 0$, and $\langle \gamma, \widehat{\alpha}_1 \rangle = \kappa + 2a = 1$. We compute

$$\begin{aligned}
\langle s_{\alpha_1}(\lambda+\delta), \gamma \rangle &= \langle \lambda+\delta, s_{\alpha_1}\gamma \rangle \\
&= \langle \lambda+\delta, \gamma - \langle \gamma, \widehat{\alpha}_1 \rangle \alpha_1 \rangle \\
&= \langle \lambda+\delta, \gamma - \alpha_1 \rangle \\
&= \langle \lambda+\delta, \sum a_i \epsilon_i + \sum b_i \beta_i \rangle \\
&\geq 0, \qquad \text{by (ib).}
\end{aligned}$$

Further, since $\gamma \in C_1$, we in fact have $\langle s_{\alpha_1}(\lambda+\delta), \widehat{\gamma} \rangle > 0$. □

Proposition 6.2: Assume (ia) and (ii). Let $\gamma \in C_1$ and suppose that the coefficient of α_1 in the $\Delta^+(\mathfrak{g})$-simple expansion of γ is two. Assume further that $\gamma \in \Delta^+_K$ and that $|\gamma|^2 \geq |\alpha_1|^2$. Then $\langle s_{\alpha_1}(\lambda+\delta), \widehat{\gamma} \rangle > 0$.

Proof: We have

$$\langle s_{\alpha_1}(\lambda+\delta), \widehat{\gamma} \rangle = \langle \lambda+\delta, \widehat{\gamma} \rangle - \langle \lambda+\delta, \widehat{\alpha}_1 \rangle \langle \alpha_1, \widehat{\gamma} \rangle,$$

and therefore

$$\langle s_{\alpha_1}(\lambda+\delta), \widehat{\gamma} \rangle = \langle \Lambda, \widehat{\gamma} \rangle + \langle 2\delta_K - \delta, \widehat{\gamma} \rangle - \langle \lambda+\delta, \widehat{\alpha}_1 \rangle \langle \alpha_1, \widehat{\gamma} \rangle.$$

Summing, we get

$$\begin{aligned}
&2\langle s_{\alpha_1}(\lambda+\delta), \widehat{\gamma} \rangle \\
&\quad = \langle \Lambda, \widehat{\gamma} \rangle + [\langle 2\delta_K - \delta, \widehat{\gamma} \rangle - 2\langle \lambda+\delta, \widehat{\alpha}_1 \rangle \langle \alpha_1, \widehat{\gamma} \rangle + \langle \lambda+\delta, \widehat{\gamma} \rangle] \\
&\quad = \langle \Lambda, \widehat{\gamma} \rangle + [\langle 2\delta_K, \widehat{\gamma} \rangle - 2\langle \lambda+\delta, \widehat{\alpha}_1 \rangle + \langle \lambda, \widehat{\gamma} \rangle] \qquad \text{since } \langle \alpha_1, \widehat{\gamma} \rangle = 1 \\
&\quad \geq \langle \Lambda, \widehat{\gamma} \rangle + [2 - 2\langle \lambda, \widehat{\alpha}_1 \rangle - 2 + \langle \lambda, \widehat{\gamma} \rangle] \qquad \text{since } \langle 2\delta_K, \widehat{\gamma} \rangle \geq 2 \text{ and } \langle \delta, \widehat{\alpha}_1 \rangle = 1 \\
&\quad = \langle \Lambda, \widehat{\gamma} \rangle + \langle \lambda, \widehat{\gamma} - 2\widehat{\alpha}_1 \rangle \\
&\quad \geq \langle \lambda, \widehat{\gamma} - 2\widehat{\alpha}_1 \rangle, \qquad \text{by (ii).}
\end{aligned}$$

Now, $\widehat{\gamma} - 2\widehat{\alpha}_1 = \frac{2}{|\gamma|^2}(\gamma) - \frac{2}{|\alpha_1|^2}(2\alpha_1) = \frac{2}{|\gamma|^2}(\gamma - \frac{|\gamma|^2}{|\alpha_1|^2}(2\alpha_1))$, where $\frac{|\gamma|^2}{|\alpha_1|^2} \geq 1$. We let $c = \frac{|\gamma|^2}{|\alpha_1|^2} \in \{1, 2, 3\}$. Writing $\gamma = \sum b_i\beta_i + a\alpha_1$ as a sum of simple roots of $\Delta^+(\mathfrak{g})$ so that each $b_i \geq 0$ and $a = 2$, we get that $\widehat{\gamma} - 2\widehat{\alpha}_1 = \frac{2}{|\gamma|^2}(\gamma - 2c\alpha_1) = \frac{2}{|\gamma|^2}[\sum b_i\beta_i + (a - 2c)\alpha_1]$. So,

$$\begin{aligned} 2\langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma}\rangle &\geq \langle \lambda, \widehat{\gamma} - 2\widehat{\alpha}_1\rangle \\ &= \frac{2}{|\gamma|^2}[\sum b_i\langle \lambda, \beta_i\rangle + (a - 2c)\langle \lambda, \alpha_1\rangle]. \end{aligned} \tag{6.2.2}$$

By (ia), each $\langle \lambda, \beta_i\rangle \geq 0$, and

$$\begin{aligned} \langle \lambda, \alpha_1\rangle &= \langle \lambda, \widehat{\alpha}_1\rangle \cdot \frac{|\alpha_1|^2}{2} \\ &= \langle \lambda + \delta, \widehat{\alpha}_1\rangle \cdot \frac{|\alpha_1|^2}{2} - \frac{|\alpha_1|^2}{2} \qquad \text{since } \langle \delta, \widehat{\alpha}_1\rangle = 1 \\ &\leq -|\alpha_1|^2 \qquad \text{by (ia)} \\ &< 0. \end{aligned}$$

Hence, since $a = 2, c \in \{1, 2, 3\}$ gives $(a - 2c) \leq 0$, we have from (6.2.2) that $2\langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma}\rangle \geq 0$. Finally, since $\gamma \in C_1$, we get that this is a strict inequality. □

Proposition 6.3: Suppose the Dynkin diagram of $\Delta^+(\mathfrak{g})$ is of type B_n. Assume (ia), (ii), and that α_1 is a long simple root. If $\gamma \in C_1 \cap \Delta^+_K$, then $\langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma}\rangle > 0$.

Proof: First suppose γ is a long root, so that $|\gamma|^2 = |\alpha_1|^2$. Moreover, the coefficient of α_1 in γ is ≤ 2. Therefore, by the above proposition, we have $\langle s_{\alpha_1}(\lambda + \delta), \widehat{\gamma}\rangle > 0$. Next, suppose γ is a short root. In this case, the coefficient of α_1 in γ is one. Therefore, Proposition 6.1 gives the result. □

Putting these three propositions together, along with a further assumption that α_1 is the lone noncompact simple root of $\Delta^+(\mathfrak{g})$ yields the following conclusion:

Theorem 6.4: Assume conditions (*), (ia) and (ii). Then for all $\gamma \in C_1$, we have $\langle s_{\alpha_1}(\lambda+\delta), \widehat{\gamma}\rangle > 0$.

Remark. The lone-noncompact simple root hypothesis of (*) placed on α_1 allows us to characterize Δ_K^+ as the set of roots which contain α_1 an even number of times in its $\Delta^+(\mathfrak{g})$-simple expansion.

Proof: Let $\gamma \in C_1$. If the coefficient of α_1 in γ is one, then we get the result from Proposition 6.1 - without using condition (ii). So assume that the coefficient of α_1 in γ is two. Since α_1 is the lone noncompact simple root of $\Delta^+(\mathfrak{g})$, γ is compact. Now we just run through the allowable cases.

If the Dynkin diagram is a single line diagram, then Proposition 6.2 gives the result.

If the Dynkin diagram of $\Delta^+(\mathfrak{g})$ is of type B_n and α_1 is a short root, then Proposition 6.2 gives the result. If α_1 is a long root, then Proposition 6.3 gives the result.

If the Dynkin diagram of $\Delta^+(\mathfrak{g})$ is of type C_n, then α_1 must be a short root by the coefficient two assumption on γ. Therefore, Proposition 6.2 gives the result.

If the Dynkin diagram of $\Delta^+(\mathfrak{g})$ is of type F_4, then α_1 can be either node. If α_1 is the short root node, then Proposition 6.2 gives the result. If α_1 is the long root node, then the coefficient two assumption on γ forces γ to be the highest root of F_4, which is a long root. So then $|\gamma|^2 = |\alpha_1|^2$ and Proposition 6.2 applies.

If the Dynkin diagram of $\Delta^+(\mathfrak{g})$ is of type G_2, then α_1 is the long simple root. Again, the coefficient two assumption on γ forces γ to be the highest root of G_2, which is a long root. So then $|\gamma|^2 = |\alpha_1|^2$ and Proposition 6.2 applies. □

Section 6.3: Satisfying the Reduced Conditions when $j > 1$

As previously noted, from [K3] we know that

(1) λ_j is analytically integral

(2) Λ_j is $\Delta^+_{M_{j,K}}$ dominant

(3) λ_j is dominant for the compact simple roots of $\Delta^+_{M_j}$.

so that the Conjectural Method runs into no obstacles. However, our Approach does hit a slight snag; part of the Main Hypotheses on λ and $\Delta^+(\mathfrak{g})$ may not be inherited by λ_j and $\Delta^+(\mathfrak{m}_j)$. In particular, although λ is nondominant versus only **one** simple root of $\Delta^+(\mathfrak{g})$, when we apply the Conjectural Method, this may no longer hold for λ_{j-1} and $\Delta^+(\mathfrak{m}_{j-1})$ when $j > 1$. As a result, we need to supplement some the arguments of Section 6.2 to handle this possibility.

Recall what the goal is: Let $j > 1$ and

$$C_j = \{\gamma \in \Delta^+(\mathfrak{m}_{j-1}) - \{\alpha_j\} \mid \langle \gamma, \alpha_j \rangle > 0, \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma} \rangle \in \mathbb{Z} - \{0\}\}.$$

We are to prove, as in §6.2 that

$$\text{(6.2.1)} \qquad \text{if } \gamma \in C_j, \text{ then } \langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma} \rangle > 0.$$

We start by investigating the noncompact roots of $\Delta^+(\mathfrak{m}_{j-1})$. To do so, we shall assume (*) throughout this section:

($*$) there is exactly one noncompact simple root of $\Delta^+(\mathfrak{g})$, and this root has multiplicity at most two in the highest root of $\Delta^+(\mathfrak{g})$.

Proposition 6.5: Assuming ($*$), every $\Delta^+(\mathfrak{m}_i)$ noncompact root is $\Delta^+(\mathfrak{g})$ noncompact.

Note. First we recall why this proposition is not entirely obvious. The property of an imaginary root being compact or noncompact is not always preserved by the Cayley transform [cf. KV Prop. 11.249]. If β is a compact (resp. noncompact) root in $\Delta^+(\mathfrak{g}, \mathfrak{t}^0)$ that is strongly orthogonal to α_1, then it remains compact (resp. noncompact) as a root in $\Delta^+(\mathfrak{m}_1, \mathfrak{t}')$. But if β is orthogonal to α_1 but not strongly orthogonal, then β is noncompact (resp. compact) as a root in $\Delta^+(\mathfrak{m}_1, \mathfrak{t}')$.

Proof. The proof proceeds by induction on i. Let $i = 1$. Let $\beta \in \Delta^+(\mathfrak{m}_1, \mathfrak{t}')$ be noncompact. Then either

(a) β is noncompact in $\Delta^+(\mathfrak{g})$ and β and α_1 are strongly orthogonal, or

(b) β is compact in $\Delta^+(\mathfrak{g})$, but β and α_1 are not strongly orthogonal.

We show that case (b) cannot exist.

Suppose (b) holds. Since β and α_1 are not strongly orthogonal, and α_1 is simple, both $\beta \pm \alpha_1$ are positive roots. Moreover, since β is $\Delta^+(\mathfrak{g})$ compact, ($*$) implies that β contains α_1 with coefficent 0 or 2 in its $\Delta^+(\mathfrak{g})$ simple expansion. If the coefficient of α_1 in β is 0, then $\beta - \alpha_1$ has a negative α_1 coefficient while the coefficient of another simple $\Delta^+(\mathfrak{g})$ root is positive. This is not possible [K1, Prop. 4.6], so we must have that the coefficient of α_1 in β is 2. But in this case the root $\beta + \alpha_1$ has α_1 coefficient 3, which contradicts assumption $(*)$. Hence any root with α_1 coefficient 0 or 2 which is also orthogonal to α_1 must be strongly orthogonal to α_1. Therefore, case (b) never occurs, and we have the base step of the induction.

Next suppose the proposition holds for $i = j - 1$, and let β be a noncompact root of $\Delta^+(\mathfrak{m}_j)$. Suppose β is compact as a root of $\Delta^+(\mathfrak{m}_{j-1})$. Then $\beta \pm \alpha_j$ are also roots of $\Delta^+(\mathfrak{m}_{j-1})$. In fact, since α_j is $\Delta^+(\mathfrak{m}_{j-1})$ noncompact, both $\beta \pm \alpha_j$ are $\Delta^+(\mathfrak{m}_{j-1})$ noncompact. Hence, by the inductive hypothesis, both $\beta \pm \alpha_j$ are $\Delta^+(\mathfrak{g})$ noncompact and therefore each contains α_1 with coefficient one in its $\Delta^+(\mathfrak{g})$ simple expansion. This forces the α_1 coefficient of α_j in its $\Delta^+(\mathfrak{g})$ simple expansion to be 0, which is a contradiction. Therefore β cannot be a compact root of $\Delta^+(\mathfrak{m}_{j-1})$.

On the other hand, if β is noncompact as a root of $\Delta^+(\mathfrak{m}_{j-1})$ then the inductive hypothesis gives the result. □

This proposition and the assumption that α_1 is the lone noncompact simple root of $\Delta^+(\mathfrak{g})$ combine to give the following:

Corollary 6.6: Assuming $(*)$, any noncompact root of $\Delta^+(\mathfrak{m}_i)$ has α_1 coefficient one in its $\Delta^+(\mathfrak{g})$ simple expansion.

Recall that we are trying to handle the situation in which $\lambda_{\sigma_{j-1}}$ is nondominant with respect to some noncompact simple roots of $\Delta^+(\mathfrak{m}_{j-1})$, and $j > 1$. If $\gamma \in C_j$ then $\langle \gamma, \alpha_j \rangle \neq 0$ so that γ is a root in the same connected component of the Dynkin diagram of $\Delta^+(\mathfrak{m}_{j-1})$ as α_j. Therefore, if there is at most one noncompact simple root per connected component of the Dynkin diagram of $\Delta^+(\mathfrak{m}_{j-1})$, then our Main Hypotheses do indeed pass to this next stage, and we have no difficulties applying the propositions of the previous sections.

Proposition 6.7: Assume (ib), and that the coefficient of α_1 in the highest root of $\Delta^+(\mathfrak{g})$ is one. Then, for all j, there is at most one noncompact simple root of $\Delta^+(\mathfrak{m}_{j-1})$ against which $\lambda_{\sigma_{j-1}}$ is nondominant. Therefore, for all j, (6.2.1) holds.

Remark: We are making no compactness/noncompactness assumptions on the remaining simple roots of $\Delta^+(\mathfrak{g})$. Further, we are not assuming (ii).

Proof: By (ib), $\lambda_{\sigma_{j-1}}$ can be nondominant versus a root only if that root contains α_1 in its $\Delta^+(\mathfrak{g})$ simple expansion. Further, since the sum of all simple roots of $\Delta^+(\mathfrak{m}_{j-1})$ is also a root, the coefficient one restriction implies that there can be at most one simple root per component of $\Delta^+(\mathfrak{m}_{j-1})$ containing α_1. By changing appropriate indices, Proposition 6.1 then implies (6.2.1). □

Let us now consider the case in which there is more than one noncompact simple root in some connected component of the Dynkin diagram for $\Delta^+(\mathfrak{m}_{j-1})$ against which $\lambda_{\sigma_{j-1}}$ is nondominant. First, under our Main Hypotheses, $\lambda_{\sigma_{j-1}}$ can be nondominant versus a root only if that root contains α_1 in its $\Delta^+(\mathfrak{g})$ simple expansion. Second, since the sum of the simple roots of a Dynkin diagram is also a root, and since, assuming $(*)$ so that α_1 occurs with coefficient ≤ 2 in the $\Delta^+(\mathfrak{g})$ highest root, there can be at most two simple roots in any component of $\Delta^+(\mathfrak{m}_{j-1})$ which contain α_1 in their $\Delta^+(\mathfrak{g})$ simple expansion. Third, if there are two simple roots in a component of $\Delta^+(\mathfrak{m}_{j-1})$ which contain α_1, then each of these roots has coefficient one in the $\Delta^+(\mathfrak{m}_{j-1})$–component highest root, for otherwise, the highest root in $\Delta^+(\mathfrak{m}_{j-1})$ contains α_1 with multiplicity greater than two. Fourth, we note that if a Dynkin diagram contains two simple roots with coefficient one in the highest root, then that diagram is a single line diagram. Finally, with the exception of the A_n-type diagrams, no Dynkin diagram has adjacent simple roots contained with coefficient one in its highest root.

According to the recursive procedure, of the two $\lambda_{\sigma_{j-1}}$-nondominant, simple roots, we let α_j be the root for which $-\frac{\langle \lambda_{\sigma_{j-1}}, \cdot \rangle}{|\cdot|^2}$ is greatest. Even though the other simple root may not turn out to be α_{j+1} in the process, for ease of notation in the next few arguments, we call this root α_{j+1}.

Theorem 6.8: Assuming $(*)$, (ia) and (ii) of the Main Hypotheses, if $\gamma \in C_j$ then $\langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma} \rangle > 0$ for all j.

Proof. By the above paragraph, we only need to consider the case in which $\lambda_{\sigma_{j-1}}$ is nondominant against two simple roots of $\Delta^+(\mathfrak{m}_{j-1})$ which lie in the same single-lined Dynkin component, and which are contained in the $\Delta^+(\mathfrak{m}_{j-1})$ highest root with coefficient one.

We know that in order to have $\langle \gamma, \alpha_j \rangle > 0$, when expanded into $\Delta^+(\mathfrak{m}_{j-1})$ simple roots, γ must contain α_j. In fact, since α_j is contained in the $\Delta^+(\mathfrak{m}_{j-1})$ highest root with coefficient one, γ contains α_j with coefficient one. Moreover, since we are in a single-line diagram, we have $\langle \gamma, \widehat{\alpha}_j \rangle = 1$.

Expand γ into its $\Delta^+(\mathfrak{m}_{j-1})$ simple expansion as

$$\gamma = \sum k_i \kappa_i + x\alpha_{j+1} + \alpha_j$$

so that

$$
\begin{aligned}
\langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \gamma\rangle &= \langle \lambda_{\sigma_{j-1}}, s_{\alpha_j}(\gamma)\rangle \\
&= \langle \lambda_{\sigma_{j-1}}, \gamma - \alpha_j\rangle && \text{since } \langle \gamma, \widehat{\alpha}_j\rangle = 1 \\
&> \langle \lambda_{\sigma_{j-1}}, \sum k_i\kappa_i + x\alpha_{j+1}\rangle && \text{since } \langle \lambda_{\sigma_{j-1}}, -\alpha_j\rangle > 0 \\
&\geq \langle \lambda_{\sigma_{j-1}}, x\alpha_{j+1}\rangle && \text{since only } \alpha_j \text{ and } \alpha_{j+1} \text{ contain } \alpha_1.
\end{aligned}
$$

Therefore, we have the result when $x = 0$.

So suppose $x \neq 0$. By our assumptions, x must be one. So, as a root of $\Delta^+(\mathfrak{g})$, γ contains α_1 with coefficient two and is therefore compact. Hence, by Proposition 6.5, γ is $\Delta^+(\mathfrak{m}_{j-1})$ compact. Writing $\lambda_{\sigma_{j-1}} = \lambda_{j-1} + \delta(\mathfrak{m}_{j-1})$, and using the fact [K3, Prop. 10] that Λ_{j-1} is $\Delta^+(\mathfrak{m}_{j-1} \cap \mathfrak{k})$ dominant, we imitate the proof of Proposition 6.3 to write

$$
\begin{aligned}
2\langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \gamma\rangle &\geq \langle \lambda_{j-1}, \widehat{\gamma} - 2\alpha_j\rangle \\
&= \langle \lambda_{j-1}, \sum k_i\kappa_i + \alpha_{j+1} - \alpha_j\rangle
\end{aligned}
$$

Now, $\langle \lambda_{j-1}, \sum k_i\kappa_i\rangle \geq 0$, and

$$
\begin{aligned}
\langle \lambda_{j-1}, \alpha_{j+1} - \alpha_j\rangle &= \langle \lambda_{\sigma_{j-1}}, \alpha_{j+1} - \alpha_j\rangle && \text{since } \alpha_{j+1} \text{ and } \alpha_j \text{ are } \Delta^+(\mathfrak{m}_{j-1}) \text{ simple} \\
&= -|\alpha_{j+1}|^2 c_{j+1} + |\alpha_j|^2 c_j \\
&= |\alpha_j|^2 (c_j - c_{j+1}) && \text{since} |\alpha_j|^2 = |\alpha_{j+1}|^2 \\
&\geq 0 && \text{by our choice of } c_j.
\end{aligned}
$$

Therefore, $\langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \gamma\rangle > 0$. □

Remark: If one wishes to verify (6.2.1) in a particular example, it is often easier to use the following simplification:

$$
\begin{aligned}
\langle s_{\alpha_j}(\lambda_{\sigma_{j-1}}), \widehat{\gamma}\rangle &= \langle s_{\alpha_j}((\lambda + \delta) + \nu^{j-1}), \widehat{\gamma}\rangle && \text{by (4.2.4)} \\
&= \langle s_{\alpha_j}(\lambda + \delta) + \nu^{j-1}, \widehat{\gamma}\rangle && \text{since } \alpha_j \perp \alpha_i \text{ for } 1 \leq i \leq j-1 \\
&= \langle s_{\alpha_j}(\lambda + \delta), \widehat{\gamma}\rangle && \text{since } \gamma \perp \alpha_i \text{ for } 1 \leq i \leq j-1.
\end{aligned}
$$

Section 6.4: Langlands Parameters of Subquotients of $A_{\mathfrak{b}}(\lambda)$

Theorem 6.8 allows us to conclude

Theorem 6.9: Let G be a linear, noncompact simple Lie group with finite center, let K be a maximal compact subgroup, and suppose rank $G =$ rank K. Let $T^0 \subset K$ be a Cartan subgroup, and let $\Delta^+(\mathfrak{g}, \mathfrak{t}^0)$ be a positive system of roots such that

(i) there is exactly one noncompact simple root; call it α_1, and

(ii) the coefficient of α_1 in the highest root is ≤ 2.

Let λ be an analytically integral form on $\mathfrak{t}^0$, and set $\Lambda = \lambda + 2\delta(\mathfrak{p}) = (\lambda + \delta) + (\delta - 2\delta_K)$. Suppose

(iii) $\langle \lambda, \beta \rangle \geq 0$ for all compact simple roots, β, of $\Delta^+(\mathfrak{g})$

(iv) Λ is Δ^+_K dominant.

Then there exists a $(\mathfrak{g}, K)$ map from a standard continuous series module to $A_{\mathfrak{b}}(\lambda)$,

$$(*) \qquad \Phi : X_K(\xi^h_n, \nu^n) \longrightarrow A_{\mathfrak{b}}(\lambda),$$

whose image contains the (nonzero) K type with highest weight Λ.

Corollary 6.10: In the setting of the above Theorem, let V be the irreducible subquotient of $A_{\mathfrak{b}}(\lambda)$ containing the K type with highest weight Λ.

(1) If $A_{\mathfrak{b}}(\lambda)$ is irreducible then the Conjectural Method produces the Langlands parameters of $A_{\mathfrak{b}}(\lambda)$.

(2) If $A_{\mathfrak{b}}(\lambda)$ is infinitesimally unitary, then the Conjectural Method produces the Langlands parameters of V.

(3) If Λ is the minimal K type of $X_K(\sigma, \nu)$, then the Conjectural Method produces the Langlands parameters of V.

(4) If Λ is the minimal K type of $A_{\mathfrak{b}}(\lambda)$, then the Conjectural Method produces the Langlands parameters of V.

(5) If λ is orthogonal to the compact simple roots of $\Delta^+(\mathfrak{g})$, then the Conjectural Method produces the Langlands parameters of V.

Proof: (1) is clear. For (2), since $A_{\mathfrak{b}}(\lambda)$ is infinitesimally unitary, we can compose the map Φ of the theorem with a projection onto V to yield the result. (3) is also clear.

For (4), if Λ is not also a minimal K type in $X_K(\sigma, \nu)$, then every minimal K type in $X_K(\sigma, \nu)$ maps to 0 under Φ. This is a contradiction. Therefore (3) applies.

For (5), setting $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is formed from the compact simple roots and $\mathfrak{u}$ from the remaining positive root spaces, we have $A_{\mathfrak{b}}(\lambda) \cong A_{\mathfrak{q}}(\lambda)$. By Corollary 8 of [K3], Λ is the minimal K type of $A_{\mathfrak{b}}(\lambda)$ and therefore (4) applies.

Final Note: Recently, I have shown that the restriction in (ii) on the multipicity of the lone noncompact root in the highest root can be removed if G is E_6, F_4, or G_2. If $G = E_7$ (resp. $G = E_8$), then label the simple roots $\{\beta\}$ in the standard fashion, and let $E_{7,i}$ (resp. $G = E_{8,i}$) be a real form such that β_i is the lone noncompact simple root of $\Delta^+(\mathfrak{g})$. Then we can remove (ii) for $E_{7,3}, E_{7,4}, E_{8,3}, E_{8,4}$, and $E_{8,7}$. For the remaining cases, $E_{7,5}, E_{8,2}, E_{8,5}$ and $E_{8,6}$, we need to impose a minor nonsingularity condition on Λ in order to remove (ii).

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