Indefinite Kähler-Einstein metrics on Compact Complex Surfaces

A Dissertation Presented

by

Jimmy Petean

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1997

State University of New York
at Stony Brook

The Graduate School

Jimmy Petean

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Claude LeBrun
Professor of Mathematics
Dissertation Director

Dusa McDuff
Professor of Mathematics
Chairman of Defense

Michael Anderson
Professor of Mathematics

Santiago Simanca
Research Scientist, Department of Applied Mathematics and Statistics
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation

Indefinite Kähler-Einstein metrics on Compact Complex Surfaces

by

Jimmy Petean

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1997

We completely classify those compact complex surfaces which admit indefinite Ricci-flat Kähler metrics. Slightly weaker results are also obtained for indefinite Kähler-Einstein metrics with non-zero scalar curvature.

Contents

1 Introduction 1

2 Preliminaries 6

3 Indefinite Kähler Metrics 17

4 Indefinite Kähler-Einstein Metrics 25

Acknowledgements

I would like to thank my advisor, Claude LeBrun, for all his support during these years. He has always been very generous sharing his time and thoughts. Every time that I knocked at his door, and that was often, I was welcome and left with a handful of ideas and new, interesting problems to think.

There are many people in the Department of Mathematics at Stony Brook that have helped me in one way or another during these four years. I would like to thank them all for this. I want to thank especially to Dusa McDuff, Michael Anderson and Lowell Jones for their courses, support and sharing of ideas.

Chapter 1

Introduction

A pseudo-Riemannian metric on a smooth manifold is called Einstein if the Ricci tensor of the Levi-Civita connection equals a scalar multiple of the metric. These equations first appeared as the vacuum case of the Einstein field equations 'with cosmological constant' and were introduced by Einstein as a system of hyperbolic partial differential equations for an unknown *Lorentzian*-signature metric on a 4-manifold. Since then, mathematicians have been highly interested in these equations, but the attention have been focused in the Riemannian case. While we still have only limited knowledge about general Riemannian solutions, strong results have been obtained about the existence of (positive definite) *Kähler*-Einstein metrics (cf. [1, 23]). In [3] the reader can find a detailed discussion of these topics and an extensive list of references.

In the last years, especially since the work of Ooguri and Vafa [20] on $N = 2$ string theory, indefinite Ricci-flat metrics of Kähler type on complex surfaces have attracted considerable attention from physicists. We will study indefinite Kähler-Einstein metrics on compact complex surfaces, focusing in the existence problem. We will completely classify surfaces admitting indefinite Ricci-flat metrics of Kähler type, and almost completely classify those admitting indefinite Kähler-Einstein metrics with non-zero Einstein constant. We will also display some non-locally-homogeneous examples. These examples will show that the moduli spaces of these metrics can be highly non-trivial and surprisingly different from those encountered in the positive definite case. In particular, we will see that indefinite Ricci-flat metrics on tori need not be flat.

Let us begin by considering a compact complex manifold (M^{2n}, J). Here M is a $2n$-dimensional smooth compact manifold and J is an integrable almost complex structure on M. If $n = 2$, M is called a (compact) complex surface.

A pseudo-Riemannian metric g on M^{2n} is said to be Hermitian (or J-compatible) if $g(x, y) = g(Jx, Jy)$ for all x, y. At any point of the manifold one can choose an orthogonal basis of the tangent space of the form $\{x_1, Jx_1, ..., x_n, Jx_n\}$; so, if g is Hermitian, its signature is of the form $(2k, 2l)$. In particular if M is a complex surface, any indefinite Hermitian metric on M has signature $(2, 2)$.

If g is a Hermitian pseudo-Riemannian metric then $\omega(x, y) = g(Jx, y)$ is a 2-form, called the Kähler form of g.

Definition 1 *: A Hermitian pseudo-Riemannian metric g is called* Kähler *if its Kähler form is closed. In particular, if g is not positive or negative definite, it is called an* indefinite Kähler metric.

Consider now the Levi-Civita connection ∇ of g on M. Assume that g

is Kähler; then J is parallel with respect to ∇. This is usually stated only in the Riemannian case, but it is not difficult to check that it is also valid in the indefinite case (by exactly the same proof). Let Ric be the Ricci tensor of ∇. Then Ric is J-invariant and hence $\rho(x,y) = Ric(Jx,y)$ is a 2-form. It is called the Ricci form of g. It is also true in the indefinite case that $-i\rho$ is the curvature of the canonical line bundle of M (the bundle of holomorphic 2-forms); the proof is the same as in the Riemannian case. In particular ρ is closed and the de Rham class $[\rho/2\pi]$ is equal to the first Chern class of M in cohomology with real coefficients.

Definition 2 *: An indefinite Kähler metric g is called indefinite* Kähler-Einstein *if there exists $\lambda \in \mathbf{R}$ such that $Ric = \lambda g$ (or $\rho = \lambda\omega$). In this case λ is called the* Einstein constant.

If g is an indefinite Kähler-Einstein metric on M and $k \in \mathbf{R}$, then $\hat{g} = kg$ is also an indefinite Kähler-Einstein metric (even if $k < 0$). The Kähler form of $\hat{g}$ is $\hat{\omega} = k\omega$ while the Ricci form is $\hat{\rho} = \rho$. If $\rho = \lambda\omega$, then $\hat{\rho} = (\lambda/k)\hat{\omega}$. Without loss of generality, we may therefore assume that λ is either 0 or 1.

Indefinite Kähler-Einstein metrics on compact complex surfaces is the object of study of this paper. The following are the simplest examples.

Complex Tori: Let $M = \mathbf{C}^2/\Lambda$ be a complex 2-dimensional torus. Let z_1, z_2 be the standard coordinates on $\mathbf{C}^2$. The 1-forms $dz_1, dz_2, d\bar{z}_1, d\bar{z}_2$ then descend to M. If $A = (a_{jk})$ is a 2×2 (constant) Hermitian non-degenerate matrix, then $\omega = \Sigma a_{jk} dz_j \wedge d\bar{z}_k$ defines a closed, real, (1,1)-form on M. So ω is the Kähler form of a Kähler metric g. Moreover, this pseudo-metric is flat.

If we choose A to be indefinite, then g is an indefinite Kähler-Einstein metric on M with Einstein constant 0.

Minimal Ruled Surfaces: Let S be a Riemann surface of genus $\mathbf{g} \geq 2$. There is a unique Riemannian metric h_1 compatible with the complex structure of S with constant scalar curvature -2; h_1 is a Kähler-Einstein metric on S with Einstein constant -1. In the same way we have a Kähler-Einstein metric h_2 on $\mathbf{CP}^1$ with Einstein constant 1. Then $h_2 - h_1$ is a well defined indefinite Kähler-Einstein metric on $M = \mathbf{CP}^1 \times S$ with Einstein constant 1.

The general ruled surface is of the form $\mathbf{P}(E)$, where E is a 2-dimensional complex vector bundle over a Riemann surface S. We will later construct indefinite Kähler-Einstein metrics on 'most' of these twisted products (assuming always that the genus of S is greater than 1).

More examples (including non locally-homogeneous ones) will be presented in the last section. The following theorem determines on which surfaces solutions could be found.

Theorem 1 : *Let M be a compact complex surface. If M admits an indefinite Kähler-Einstein metric, then M is one of the following:*

a) a Complex Torus ;

b) a Hyperelliptic surface ;

c) a Primary Kodaira surface ;

d) a minimal ruled surface over a curve of genus $\mathbf{g} \geq 2$ *; or*

e) a minimal surface of class VII_0 with no global spherical shell, and with second Betti number even and positive.

Remark 1: No surface of type (e) is known, and it has been conjectured that they simply do not exist (cf. [17, section 5]). Moreover, if such a surface existed and admitted an indefinite Kähler metric, providing it with the opposite orientation, would then yield a symplectic manifold with $b^+ > 1$ violating the Bogomolov inequality $2\chi \geq 3\tau$; no such symplectic manifold is known at present.

Remark 2: We will display indefinite Kähler-Einstein metrics with Einstein constant 0 on the surfaces (a) , (b) and (c) and with Einstein constant 1 on 'most' surfaces of type (d).

Chapter 2

Preliminaries

In this chapter we will summarize some basic facts that we will use in this work; we will mainly describe some well-known results about the classification of compact complex surfaces and the Seiberg-Witten invariants.

Let us first recall some standard notation. For a compact connected smooth 4-manifold X, $b_k(X)$ will denote its k-th Betti number. If X is an oriented manifold, $\overline{X}$ is the oriented manifold obtained by reversing the given orientation. Let also $b^+(X)$ ($b^-(X)$) be the dimension of a maximal subspace of $H^2(X, \mathbf{R})$ where the intersecion form is positive (negative) definite. So $b_2 = b^+ + b^-$ and the signature of X is $\tau(X) = b^+ - b^-$.

Throughout this thesis M will always denote a compact complex surface. And as usual $c_k(M)$ will denote the k-th Chern class of M and $c_1^2(M)$ and $c_2(M)$ will denote the integers obtained by evaluating the corresponding 4-form in the fundamental homology class of M. So $c_2(M) = \chi(M)$ is the Euler characteristic of M and $c_1^2 = 2\chi + 3\tau$.

Compact Complex Surfaces

We will now give a brief description of the Enriques-Kodaira classification of compact complex surfaces. See [2] for a complete analysis of this topic. Let M be such a surface, and consider the *canonical* line bundle K_M of M; i.e. the line bundle of holomorphic 2-forms on M. For any positive integer m consider its m-th tensor power K_M^m and let p_m denote the dimension of the linear space of holomorphic sections of this line bundle (these number are called the *plurigenera* of M).

Definition 3 *: The* Kodaira number *of M is*

$$\limsup_{m\to\infty} \frac{log(p_m)}{log(m)}$$

We will usually denote the Kodaira number of a surface M by $Kod(M)$.

There is a more geometric description of this important invariant of a complex surface M (in general, of any compact complex manifold); any holomorphic line bundle L over M induces a holomorphic map

$$\sigma_L : M - - \to \mathbf{CP}^{k-1}$$

where k is the dimension of the space of holomorphic sections of the line bundle L. The map is defined at those points $x \in M$ where at least one holomorphic section of L does not vanish; in this case $\sigma_L(x) = [\phi_1(x) : ... : \phi_k(x)]$, where $\phi_1, ..., \phi_k$ is any basis of the space of holomorphic sections of L. $Kod(M)$ then gives the maximal dimension of the images of M through the maps induced by the line bundles K_M^m. Accordingly, $Kod(M)$ can only be $-\infty, 0, 1$ or 2.

We will now describe the *blowing up* process. Given any point $x \in M$, replace a neighborhood of x by a neighborhood of the zero-section of the line bundle $L \mapsto \mathbf{CP}^1$, $L = \{(x, z) \in \mathbf{CP}^1 \times \mathbf{C}^2 \;\; | \;\; z \in x\}$. The identification can be made holomorphically and so we get a new compact complex surface, the *blow up* of M at x, which is diffeomorphic to $M \# \overline{\mathbf{CP}}^2$ (where $\overline{\mathbf{CP}}^2$ refers to the oriented manifold obtained by considering the non-standard orientation of $\mathbf{CP}^2$).

We will denote the blow up of M at one point by $\widehat{M}$. The zero-section of the line bundle L is, of course, embedded into $\widehat{M}$; so the blow-up of any complex surface has a holomorphically embedded copy of $\mathbf{CP}^1$ with self-intersection -1. Conversely, if a complex surface N has such a curve, then there is a complex surface M whose blow-up at one point is N. M is then said to be obtained by *blowing down* N.

Definition 4 *: A compact complex surface is called* minimal *if it has no holomorphically embedded copy of* $\mathbf{CP}^1$ *with self-intersection -1.*

Every compact complex surface is obtained from a minimal complex surface by blowing up a finite number of times. To classify compact complex surfaces one can therefore consider only those which are minimal.

The Enriques-Kodaira classification of compact complex surfaces gives a description of minimal compact complex surfaces in terms of its Kodaira number. Roughly it says that if M is a minimal compact complex surface, then:

i) Suppose that $Kod(M) = -\infty$. If $b_1(M)$ is even then M is $\mathbf{CP}^2$ or a ruled surface; recall that M is called a ruled surface if there is a Riemann surface S and a holomorphic 2-dimensional bundle E over S such that $M = \mathbf{P}(E)$ (this is the $\mathbf{CP}^1$ bundle over S whose fiber over $x \in S$ is the projective space of the fiber of E over x). If $b_1(M)$ is odd then it is actually 1 and the intersection form of M is negative definite (i.e. $b^+(M) = 0$); but the structure of these surfaces is not fully understood yet (see [17] for an account of what is known at present).

ii) Suppose that $Kod(M) = 0$. One can give good descriptions of all these surfaces. They are Complex Tori, Hyperelliptic surfaces, K3 surfaces, Enriques surfaces, or (Primary or Secondary) Kodaira surfaces. See [2] for details about them. It is important to note that for all these surfaces $c_1^2 = 0$.

iii) Suppose that $Kod(M) = 1$. Then M is an elliptic surface; i.e. there is a Riemann surface S and a holomorphic map $\pi : M \to S$ such that the general fiber of the map is an elliptic curve. A very complete description of these surfaces can be found in [5]. For every minimal elliptic surface $c_1^2 = 0$ and the Euler characteristic is non-negative.

iv) Suppose that $Kod(M) = 2$. Then M is called a surface of *general type*, and not much is known about its structure. We do know that M is an algebraic surface (i.e. is the smooth zero locus of a set or homogeneous polynomials in $\mathbf{CP}^n$ for some n) and that $0 < c_1^2(M) \leq 3c_2$.

Seiberg-Witten invariants

The other important tool we will need in this work is the theory of Seiberg-Witten invariants. We will give here a very brief description of these invariants. We refer to [16],[21] and [22] for more details.

Let (V, q) be a 4-dimensional real vector space V with a positive definite inner product q. Consider the tensor algebra of V,

$$T(V) = \bigoplus_{n \in \mathbf{N}_0} \underbrace{V \otimes ... \otimes V}_{n}$$

and the ideal $\mathcal{I}$ generated by elements of the form $v + q(v, v)$, for $v \in V$. The quotient of $T(V)$ by $\mathcal{I}$ is called the *Clifford algebra* of (V, q), and it is denoted by $Cl(V, q)$. The subgroup of $Cl(V, q)$ generated by elements of the form $v.w$, for $v, w \in V$ of norm 1, is called $Spin(V, q)$ (here $v.w$ means the class of $v \otimes w$). It is easy to see that the elements of $Spin(V, q)$ acting by conjugation on $Cl(V, q)$ leave V invariant; moreover, if we consider the restriction of this action to V we get an element of $SO(V, q)$. One can check that this map $p : Spin(V, q) \to SO(V, q)$ actually gives the universal (double) covering of $SO(V, q)$.

Now consider the complexified Clifford algebra $Cl(V, q) \otimes \mathbf{C}$ of (V, q), and let $Spin^c(V, q)$ be the subgroup generated by $Spin(V, q)$ and the complex numbers of norm 1. It is easy to see that there is an isomorphism

$$Spin^c(V, q) = Spin(V, q) \times_{\{1,-1\}} S^1$$

The complexified Clifford algebra is isomorphic to the algebra of 4×4 complex matrices (see [13] or [16]). Hence there is a unique irreducible complex representation of this algebra; the canonical action on $\mathbf{C}^4$. Restricting the action one gets a complex representation of $Spin^c(V,q)$. We will denote it by $S_{\mathbf{C}}$ and call it the *spin representation.* It turns out that this representation of $Spin^c(V,q)$ is not irreducible but factors as the sum of two irreducible non-equivalent complex representations of dimension 2. Let us denote these spaces by ${S_{\mathbf{C}}}^+$ and ${S_{\mathbf{C}}}^-$ and call them the *plus* and *minus spin representations.*

We will denote by $Cl(4)$ the Clifford algebra of $(\mathbf{R}^4, q)$, where q is the Euclidean metric. Similarly $Spin(4) = Spin(\mathbf{R}^4, q)$ and $Spin^c(4) = Spin^c(\mathbf{R}^4, q)$. Note that for any 4-dimensional inner product space $(V, \hat{q})$, $Cl(V, \hat{q})$ is isomorphic to $Cl(4)$ (and the same happens for the spin groups). Of course, this isomorphism is not natural but depends on the choice of an orthonormal basis.

The previous constructions provide an important tool in 4-dimensional geometry. Let (X, g) be a 4-dimensional compact Riemannian manifold. Let P be the $SO(4)$ bundle of oriented orthonormal frames of (X, g). A *Spin* structure on (X, g) is a $Spin(4)$ principal bundle over X such that the quotient by the action of $\{1, -1\}$ is isomorphic to P. Similarly, a $Spin^c$ structure on (X, g) is a $Spin^c(4)$ principal bundle whose quotient by the action of $\{1, -1\} \times_{\{1,-1\}} S^1$ is isomorphic to P.

An important reason to introduce the $Spin^c$ groups is that while admitting a *Spin* structure is a very restrictive condition for a compact 4-manifold (it is equivalent to ask that the second Stiefel-Whitney class of the tangent bundle vanishes), every compact 4-manifold does admit a $Spin^c$ structure. Actually, given a complex line bundle over the manifold whose first Chern class agrees mod 2 with the second Stiefel-Whitney class of the tangent bundle, we can associate to it a $Spin^c$ structure. So, for instance, suppose that X admits an almost complex structure J. J provides the tangent bundle of X with the structure of a 2-dimensional complex vector bundle. The first Chern class of this bundle agrees mod 2 with the second Stiefel-Whitney class and so an almost complex structure on a manifold induces a canonical $Spin^c$ structure.

There is a natural map $Spin^c \to S^1$; given a $Spin^c$ principal bundle this map produces a complex line bundle. The line bundle induced by a $Spin^c$ structure is called the *determinant* line bundle of the $Spin^c$ structure. Note also that the Levi-Civita connection of the Riemannian manifold and any connection A on the determinant line bundle induce a connection on the $Spin^c$ bundle.

Given a $Spin^c$ structure $\tilde{P}$ on X, the spin representation induces a 4-dimensional complex vector bundle $\mathbf{S_C} = \tilde{P} \times_{Spin^c} S_{\mathbf{C}}$. $\mathbf{S_C}$ is called the *spin bundle* of $\tilde{P}$. The spin bundle splits as $\mathbf{S_C} = \mathbf{S_C^+} \oplus \mathbf{S_C^-}$, according to the splitting of the spin representation. Recall that any connection on $\tilde{P}$ induces a connection on $\mathbf{S_C}$.

We also have the bundle of Clifford algebras $Cl(X,g) = P \times_{SO(4)} Cl(4)$. And it is not difficult to check that the usual Clifford multiplication extends to these bundles; i.e. there is a bundle map $Cl(X,g) \otimes \mathbf{S_C} \to \mathbf{S_C}$ which restricted to any fiber gives the Clifford multiplication.

Given a connection A on the determinant line bundle of a $Spin^c$ structure we can now define the *twisted* Dirac operator:

$$\begin{aligned} D_A: \ C^\infty(\mathbf{S_C}) &\rightarrow C^\infty(\mathbf{S_C}) \\ \sigma &\mapsto \Sigma e_i \cdot \nabla^A_{e_i}\sigma \end{aligned}$$

where the point is Clifford multiplication, and ∇^A is the connection induced on $\mathbf{S_C}$ by the connection on the $Spin^c$ bundle induced by the Levi-Civita connection of the manifold and A. It can be checked that D_A is a self-adjoint operator and that under the splitting of the spin bundles, $D_A : \mathbf{S}^{\pm}_{\mathbf{C}} \rightarrow \mathbf{S}^{\mp}_{\mathbf{C}}$.

We can now write the Seiberg-Witten equations.

$$\begin{cases} D_A(\phi) = 0 \\ F_A^+ = \phi \otimes \phi^* - |\phi|^2 Id \end{cases}$$

The unknowns are a section ϕ of $\mathbf{S}^{+}_{\mathbf{C}}$ and and a unitary connection A on the determinant line bundle of the $Spin^c$ structure. The second equation needs to be explained. In dimension 4, the Hodge-star operator gives an endomorphism of the bundle of 2-forms, of square 1. So we have a decomposition of the space of 2-forms into the eigenspaces of eigenvalue 1 and -1; the self-dual and anti-self-dual 2-forms. The left-hand side of the second equation is the self-dual part of the curvature of the connection A. On the other side there is a natural isomorphism of vector spaces between the exterior algebra of a vector space and its Clifford algebra (this of course depends on an inner product). Using the inner product we identify the tangent and cotangent bundles and so the exterior algebra of the cotangent bundle with the Clifford algebra. But the Clifford algebra is isomorphic to $End(S_{\mathbf{C}})$. Under these identifications forms of even degree correspond to endomorphisms which preserve $\overset{+}{S_{\mathbf{C}}^{-}}$. Moreover, self-dual 2-forms correspond to endomorphisms which act trivially on $S_{\mathbf{C}}^{-}$ and as endomorphisms of $S_{\mathbf{C}}^{+}$ have trace 0. The left hand side of the second equation is the traceless element of $End(S_{\mathbf{C}}^{+})$ given by $\chi \mapsto \langle \chi, \phi \rangle \phi - |\phi|^2 \chi$; we identify it with a self-dual 2-form by the previous considerations.

Given the equations, there is a long process to produce invariants out of them. There are many technical problems that one has to treat to do this. We refer to [16] for that discussion. We will now only state the main results that we will use in this work. For simplicity we will only consider the simplest case: we will assume that $b^+ > 1$ and that the $Spin^c$ structure is induced by an almost complex structure. Fortunately, this is all what we will need.

In order to obtain smooth solution spaces it is necessary to consider perturbations of the equations. This is done by adding an imaginary self-dual 2-form $i\eta$ to one side of the second equation. The space of smooth funcions of X to S^1, $C^\infty(X, S^1)$, acts on the space of solutions of the *perturbed* Seiberg Witten equations by $f.(A, \phi) \mapsto (A + 2d\log(f), f\phi)$. The quotient of the space of solutions by this action is called the *moduli* space. For generic perturbations it is a smooth compact orientable manifold. In the case of a $Spin^c$ structure induced by an almost complex structure and for a generic perturbation the moduli space is a finite set of points. When $b^+ > 1$ the number of these points (counted with the respective orientation) depends neither on the metric nor on the generic perturbation. This number is the **Seiberg-Witten invariant** of the $Spin^c$ structure of the manifold.

Let us now state the main results that we will need. First, there are two important vanishing theorems.

Theorem: *If a compact connected smooth oriented 4-manifold X can be expressed as the connected sum of two oriented 4-manifolds and for each of these 2 manifolds the intersection form is not negative definite (i.e. $b^+ > 0$), then the Seiberg-Witten invariant of any $Spin^c$ structure on X is 0.*

Theorem: *If $b^+(X) > 1$ and X admits a Riemannian metric of positive scalar curvature, then all the Seiberg-Witten invariants of X vanish.*

The invariants were first computed [22] for the canonical $Spin^c$ structure of a Kähler manifold. An important generalization of this result was obtained by Taubes [21].

Recall that a *symplectic* form on a 4-manifold is a closed 2-form ω which is non-degenerate everywhere in the manifold (i.e. $\omega \wedge \omega \neq 0$ everywhere).

Any symplectic form ω induces almost complex structures J on the 4-manifold which are compatible with ω in the sense that for any vector fields $v, \hat{v}$ on the manifold $\omega(Jv, J\hat{v}) = \omega(v, \hat{v})$ and moreover the bilinear form $g(v, \hat{v}) = \omega(v, J\hat{v})$ is positive definite (i.e. a Riemannian metric). The space of such almost complex structures is path-connected and hence all of them induce the same $Spin^c$ structure on the manifold. This is called the $Spin^c$ structure induced by the symplectic structure. If one of these almost complex structures J is integrable (i.e. a complex structure) then the metric g is Kähler and ω is its Kähler form. Taubes computed the invariants for general symplectic manifolds:

Theorem: *Let (X, ω) be a symplectic manifold, $b^+(X) > 1$. The $Spin^c$ structure induced by ω has Seiberg-Witten invariant 1 or -1.*

Chapter 3

Indefinite Kähler Metrics

A natural question to consider, independently of the Einstein equations, is the existence of indefinite Kähler metrics on complex manifolds. Our main tool to study this problem in the case of compact complex surfaces will be the Seiberg-Witten invariants, which we discussed in the previous chapter.

Let M be a compact complex surface. As usual, the complex structure induces a standard orientation on M and we denote by $\overline{M}$ the manifold M provided with the opposite orientation. If M admits an indefinite Kähler metric, its Kähler form ω is a symplectic form compatible with the orientation of $\overline{M}$ (since $\omega \wedge \omega < 0$). In particular $b^-(M) > 0$. But much more can be said: the work of Taubes (see [21] or the previous chapter) shows that, assuming that $b^-(M) > 1$, the Seiberg-Witten invariant of the canonical $Spin^c$ structure induced by ω on $\overline{M}$ is different from 0. This turns out to be a very strong obstruction, as we can see in the following lemma.

Lemma 1 *: If a compact complex surface admits an indefinite Kähler metric, then it is minimal or a one-point blow-up of* $\mathbf{CP}^2$*.*

Proof: Let N be the blow-up of the compact complex surface M and suppose that it admits an indefinite Kähler metric. From the discussion above we know that there is at least one $Spin^c$ structure on $\overline{N}$ with non-trivial Seiberg-Witten invariant. All the Seiberg-Witten invariants of a connected sum of 4-manifolds vanish unless one of them has a negative-definite intersection form (see [22, 21] or the previous chapter). Since $\overline{N} = \overline{M}\#\mathbf{CP}^2$ we must have $b^-(M) = 0$ (in particular M is minimal) and hence $b_1(N)$ must be even; the symplectic form on $\overline{N}$ provides it with an almost complex structure and for almost complex compact 4-manifolds $b_1 + b^+$ is odd (note that $b^+(\overline{N}) = 1$). The first Betti number is invariant under blowing-ups and so $b_1(M)$ is also even.

We have proved that if $N = \overline{M}$ admits an indefinite Kähler metric, then $b^-(M) = 0$ and $b_1(M)$ is even. Let us now invoke the classification of compact complex surfaces to see what are the possibilities for M.

The only surface with $Kod(M) = -\infty$, $b^-(M) = 0$ and even first Betti number is $\mathbf{CP}^2$.

If $Kod(M)$ is 0 or 1, then $c_1^2(M) = 0$. When $b_1(M)$ is even we have $c_1^2 + 8q + b^- = 10p_g + 9$ (see [8, p.755]; p_g is the geometric genus of M and q the irregularity. We only need to know that they are integers). This clearly implies that $b^-(M) > 0$.

Now assume that $Kod(M) = 2$. Then $0 < c_1^2 = 2c_2 + 3\tau \leq 3c_2$. Another way to write this inequality is $2b^+ \leq 2 - 2b_1 + 4b^-$; assuming that $b^- = 0$ we get that $b_1 = 0$ and $b^+ = 1$ (note that $b^+ > 0$ because M is Kähler). But then it is known that M is a quotient of the unit ball in $\mathbf{C}^2$ (cf. [2, p.136]). By a theorem of Mal'tsev (cf. [24, p.151]) the fundamental group of M is then 'residually finite' and hence M admits non-trivial finite coverings. Consider a covering of order $k > 1$; then N is covered by a surface which is the blow-up of another surface at k points. Such a surface can not admit an indefinite Kähler metric and so neither can N. □

Remark: The blow-up of $\mathbf{CP}^2$ at one point is a ruled surface. We will see now that every ruled surface M admits an indefinite Kähler metric. Let $\pi : E \to S$ be a 2-dimensional holomorphic vector bundle over a Riemann surface S so that $M = \mathbf{P}(E)$. Let $\hat{\pi} : M \to S$ and $p : E - E_0 \to M$ be the natural projections. There are local holomorphic sections of p. Moreover, if σ and $\tilde{\sigma}$ are two local holomorphic sections of p, then there is a holomorphic function f sucht that $\tilde{\sigma} = f\sigma$. Choose a Hermitian metric on E and a Kähler form ω_0 on S. Given a local holomorphic section σ of p, define $\omega = \hat{\pi}^*(\omega_0) - is\partial\bar{\partial}\log(\|\sigma\|)$. It can be checked that ω is well defined globally and that, for small s, it is the Kähler form of an indefinite Kähler metric on M.

Now we have to study which minimal compact complex surfaces do admit indefinite Kähler metrics. The Kodaira classification will be again of much help. The previous remark deals with the surfaces of Kodaira number $-\infty$ of Kähler type. Those which are not of Kähler type are called surfaces of class

VII_0. Of course, to study the existence of indefinite Kähler metrics we need to consider only those surfaces for which $b^- > 0$. When the first Betti number is odd, the existence of an indefinite Kähler metric would actually imply that $b^- > 1$. The following lemma shows that none of the known examples of VII_0 surfaces admits such a metric. Nevertheless the classification here is not complete and we cannot decide if new examples could hold indefinite Kähler metrics; but, as we said in the introduction, it seems very unlikely.

Lemma 2 *: If M is a surface of class VII_0 with a global spherical shell (cf. [17]) and $b_2(M) = b^-(M) > 0$, then M does not admit an indefinite Kähler metric.*

Proof: Such a surface is diffeomorphic to the connected sum of $S^1 \times S^3$ with $b_2(M)$ copies of $\overline{\mathbf{CP}}^2$ (cf. [17]). Then M admits Riemannian metrics of positive scalar curvature; hence all the Seiberg-Witten invariants of $\overline{M}$ vanish and the lemma follows from previous remarks.

□

Now we turn our attention to elliptic surfaces. Let us recall a few standard notations. All the results we will use can be found in [5],[8] or [9]. Let M be a minimal elliptic surface with projection $\pi : M \to S$ onto the Riemann surface S. All the fibers of π are elliptic curves except for a finite number of them. The possibilities for these finite *singular* fibers is very limited. They are classified in [9, p.564]. For us it is important to recall that a singular fiber of type mI_0 is an elliptic fiber of multiplicity m, and a singular fiber of type I_0^* is a union of holomorphic spheres of self-intersection -2. Recall that the *base orbifold* of the elliptic surface is $(S, \{p_i\}, \{m_i\})$, where $\pi^{-1}(p_i)$ are the multiple fibers and m_i is the corresponding multiplicity.

Proposition 1 : *A minimal elliptic surface of Kähler type admits an indefinite Kähler metric if and only if its Euler characteristic is 0.*

Proof: Let M be an elliptic surface of positive Euler characteristic $12d$. The smooth structure of M is determined by its base orbifold and d (cf. [5, p.122]). For any orbifold and any $d > 0$, it is easy to construct (as in [9, p.578]) an elliptic surface with constant $\mathcal{J}$ invariant (this means that the complex structure of the regular fibers is constant), only singular fibers of types mI_0 and I_0^* and the given base orbifold and Euler characteristic. The number of singular fibers of type I_0^* of this surface will then be $2d$ (see [10, p.14]). It follows then that $\overline{M}$ has embedded 2-spheres of self-intersection 2. Note also that $b^+(\overline{M}) > 1$. This implies that M does not admit indefinite Kähler metrics because all the Seiberg-Witten invariants of a 4-manifold X that admits embedded 2-spheres with positive self-intersection vanish (cf. [4, 11]). Indeed, if X has a $Spin^c$ structure with non-trivial Seiberg-Witten invariant so does $X\#\overline{\mathbf{CP}^2}$ [4]. We can then assume that X has an embedded sphere S whose cohomology class is non-trivial and has self-intersection 0. Let E be the exceptional curve in $X\#\overline{\mathbf{CP}^2}$ (i.e. $E = \mathbf{CP}^1 \subset \mathbf{CP}^2$). For any positive integer k, the cohomology class of $kS + E$ can be represented by an embedded sphere of self-intersection -1 and hence, there is a diffeomorphism of $X\#\overline{\mathbf{CP}^2}$ realizing the reflection on the orthogonal complement of this class (with respect to the intersection form). It is easy to check that these diffeomorphisms produce infinite different $Spin^c$ structures with non-trivial Seiberg-Witten invariants. But this is a contradiction, since Witten has observed [22] that the number of $Spin^c$ structures with non-trivial invariants is always finite.

Now assume that the Euler characteristic of M is 0. We will construct an indefinite Kähler metric on M. First note that M has only singular fibers of type mI_0. Let $\pi : M \to S$ be the projection onto the base curve and ω be a Kähler form on M. At any $p \in M$ the fiber of π through p is smooth and hence it has a tangent plane; this is of course contained in the kernel of $\pi_* : T_pM \to T_{\pi(p)}S$. It is not equal to the kernel exactly when p is in a multiple fiber. Assume that this is the case; there exists a neighborhood U of $\pi(p)$ such that $\pi^{-1}(U)$ is isomorphic to the quotient of the product $V \times T$ of the unit disc V and a torus T by the action of a finite cyclic group G generated by an automorphism χ of the form $\chi(z,t) = (e^{2i\pi/m}z, t + h(z))$ (cf. [8, p.767]). Let z be a holomorphic coordinate in V and f be a smooth positive function of $\|z\|$ with compact support in V which is equal to 1 in a neighborhood of 0. Consider $\psi_p = f dz \wedge d\bar{z}$; ψ_p is invariant through G and so descends to $\pi^{-1}(U)$. Since this form has compact support in $\pi^{-1}(U)$, it can be extended to the whole M. Construct such a form for each multiple fiber. Summing up these forms and the pull-back of a Kähler form on S, we get a (1,1)-form $\hat{\psi}$ on M which is closed, vanishes on the tangent plane to the fibers and is strictly positive in the orthogonal plane. For a big positive constant λ, $\bar{\omega} = \omega - \lambda\hat{\psi}$ is then the Kähler form of an indefinite Kähler metric on M. □

Remark: Every $K3$ surface is diffeomorphic to an elliptic surface. Since the Euler characteristic of a $K3$ surface is 24, the previous proposition implies that $K3$ surfaces do not admit indefinite Kähler metrics. Every Enriques surface is covered by a $K3$ surface and hence Enriques surfaces do no admit indefinite Kähler metrics either. Of course, to prove this last assertion we could also argue that Enriques surfaces are also elliptic surfaces of positive Euler characteristic. Note also that Secondary Kodaira surfaces do not admit indefinite Kähler metrics since their second Betti number is 0.

Finally, we turn our attention to a surface M of general type. In this case, it has been proved (cf. [11], [15]) that the existence of non-trivial Seiberg-Witten invariants for $\overline{M}$ implies that $\tau(M) \geq 0$. The proof goes as follows; we can assume that $b_+(\overline{M}) > 1$, because if $b^+(\overline{M}) = 1$ then it is clear that $\tau(M) \geq 0$. Hence, for any Riemannian metric g on M we have (cf. [14]),

$$\int_M s_g^2 dvol_g \geq 32\pi^2 c_1^2(\overline{M})$$

When M is a minimal surface of general type with no (-2)-sphere (what is guaranteed by the presence of a non-trivial Seiberg-Witten invariant for $\overline{M}$), $c_1(M)$ is negative-definite and hence [23], M admits a (Riemannian) Kähler-Einstein metric g_0. For this metric

$$\int_M s_{g_0}^2 dvol_{g_0} = 32\pi^2 c_1^2(M)$$

This shows that $c_1^2(\overline{M}) \leq c_1^2(M)$ and hence $\tau(M) \geq 0$.

Since surfaces of general type have even first Betti number, it follows that b^+ is odd. If $\overline{M}$ admits an almost complex structure, then b^- is also odd and therefore $\tau(M)$ is even.

Let us now summarize the results of this chapter:

Theorem 2 *Suppose M admits an indefinite Kähler metric. Then*

i) If $Kod(M) = -\infty$, then M is a ruled surface or is as in Theorem 1 (e).

ii) If $Kod(M) = 0$, then M is a torus, an Hyperelliptic surface or a Primary Kodaira surface.

iii) If $Kod(M) = 1$, then M is minimal and $\tau(M) = 0$

iv) If $Kod(M) = 2$, then M is minimal and $\tau(M)$ is non-negative and even.

We have seen that ruled surfaces and the surfaces in (iii) of Kähler type do admit indefinite Kähler metrics. We will see in the next section that all the surfaces in (ii) actually admit indefinite Kähler-Einstein metrics. The existence of indefinite Kähler metrics in the other cases remains unknown.

Chapter 4

Indefinite Kähler-Einstein Metrics

We will first see that the existence of an indefinite Kähler-Einstein metric completely determines the Kodaira number of a compact complex surface. We will show that the Kodaira number must be $-\infty$ or 0. Together with the results of the last section, this will finish the proof of Theorem 1.

Let us consider first the case when the Einstein constant is not 0.

Proposition 2 : *If M admits an indefinite Kähler-Einstein metric with Einstein constant $\neq 0$, then $Kod(M) = -\infty$ and $c_1^2(M) < 0$.*

Proof: Suppose that M admits such a metric g. Let $\omega(x, y) = g(Jx, y)$ be its Kähler form and $\rho(x, y) = Ric(Jx, y)$ be its Ricci form. Then $\rho = k\omega$, with $k \neq 0$, and so it is everywhere non-degenerate and indefinite.

Suppose now that for some positive integer m the m-th tensor power K_M^m of the canonical line bundle admits a non-trivial holomorphic section γ. Since M is compact $|\gamma|$, must attain its positive maximum at some point $x \in M$. In a neighborhood of x, γ can be written as $\gamma = \sigma^m$, where σ is a local section of K_M.

Then, we see that

$$\rho = \frac{1}{im}\partial\bar{\partial}\log|\gamma|^2$$

See [3, p.82]; the proof is given in the Riemannian case, but the same proof works in the indefinite case. The last equation implies that ρ is semi-negative definite at x. This is of course a contradiction. Hence, for all $m > 0$, K_M^m has no non-trivial global holomorphic section; and $Kod(M) = -\infty$.

The second assertion follows from the facts that $[\rho] = 2\pi c_1$ and $\omega \wedge \omega < 0$.

□

Corollary 1 : *If M admits an indefinite Kähler-Einstein metric with Einstein constant* $\neq 0$, *then M is as in (d) or (e) of Theorem 1.*

Proof: The corollary follows almost immediately from the previous result and Theorem 2 in the previous chapter. The only thing to remark is that a ruled surface has $c_1^2 < 0$ if and only if the base Riemann surface has genus g> 1.

□

Now let us consider the case when the Einstein constant is 0.

Proposition 3 : *If M admits an indefinite Kähler-Einstein metric with Einstein constant 0, then* $Kod(M) = 0$ *and* $c_1(M, \mathbf{R}) = 0$.

Proof: Suppose that M admits such a metric g. Then the Ricci form of g vanishes and so $c_1(M, \mathbf{R}) = 0$. Note that this implies that M is a minimal surface.

The only surfaces with Kodaira number $-\infty$ and vanishing real first Chern class are the minimal surfaces of class VII with 0 second Betti number, which do not admit indefinite Kähler metrics. So we can assume that there exists $m > 0$ and a non-trivial holomorphic section γ of K_M^m.

Let $\widetilde{M}$ be the universal covering of M. The pull-back of g gives an indefinite Kähler-Einstein metric on $\widetilde{M}$ (with Einstein constant 0). Since this metric is Ricci flat, there are holomorphic 2-forms of constant length in a neighborhood of any point of $\widetilde{M}$ (this fact is usually stated only in the Riemannian case, but it is not difficult to check that the proof also works in the indefinite case. See [3, p.82]). Of course, two local holomorphic 2-forms of constant length can only differ by multiplication by a constant. Since $\widetilde{M}$ is simply connected, we can glue these local holomorphic 2-forms together and get a global non trivial holomorphic 2-form φ of constant length.

The pull back $\hat{\gamma}$ of γ to $\widetilde{M}$ is a holomorphic section of $K_{\widetilde{M}}^m$. Hence there must exist a holomorphic function f on $\widetilde{M}$ such that $\hat{\gamma} = f\varphi^m$.

Hence f is constant, because $|f|$ achieves its maximum where $\|\hat{\gamma}\|$ does. It follows that $\|\gamma\|$ is constant. Then γ is never zero and K_M^m is trivial. This implies that $Kod(M) = 0$.

□

Corollary 2 *: If M admits an indefinite Kähler-Einstein metric with Einstein constant 0, then M is as is (a), (b) or (c) of Theorem 1.*

By now we have already proved Theorem 1. The only thing remaining is to display the promised examples.

Non flat solutions on Tori : On the torus $M = \mathbf{C}/\Lambda_1 \times \mathbf{C}/\Lambda_2$ consider

$$\gamma = f(z)dz \wedge d\bar{z} + dz \wedge d\bar{w} + dw \wedge d\bar{z}$$

where z and w are holomorphic coordinates on each complex plane and f is a smooth positive function on $M = \mathbf{C}/\Lambda_1$. It is clear that γ is i times the Kähler form of an indefinite Kähler metric g on M. We will now compute the curvature of g. If $z = x_1 + ix_2$ and $w = x_3 + ix_4$, then $\{\frac{\partial}{\partial x_i}\}$ is the standard basis of the tangent space of $\mathbf{R}^4$. This basis, of course, descends to M and one can apply the Gram-Schmidt process to it to get a basis $\{v_i\}$ orthonormal with respect to g. Let $\{e_i\}$ be the basis of T^*M dual to $\{v_i\}$. The canonical orientation of M and g induce a splitting of the space of 2-forms $\Lambda^2(T^*M) = \Lambda^+ \bigoplus \Lambda^-$ into self-dual and anti-self-dual 2-forms. And

$$\phi_1 = e_1 \wedge e_2 + e_3 \wedge e_4,$$

$$\phi_2 = e_1 \wedge e_3 + e_2 \wedge e_4,$$

$$\phi_3 = e_1 \wedge e_4 - e_2 \wedge e_3,$$

form an orthogonal basis of the space of self-dual 2-forms. Considering the curvature tensor $\mathcal{R}$ as a section of $End(\Lambda^2(T^*M))$ and taking into account the splitting we have

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

Direct computations show that $B = D = 0$ and in the basis of Λ^+ considered above,

$$A = (2f^{-2}\Delta f) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

so the metric is Einstein, and is locally homogeneous (and flat) iff f is contant.

It is interesting to compare this computation with [12]. Our metric is Einstein on a manifold with vanishing Euler characteristic; nevertheless the metric is not flat. This is possible because, as shown in this example, a symmetric operator A with respect to a metric of signature (2,2) (instead of a Riemannian metric) can verify $trace(A^2) = 0$ while $A \neq 0$.

Primary Kodaira surfaces : As described in [8, p.786], such a surface M is of the form $M = \mathbf{C}^2/G$ where $G = \langle \psi_1, \psi_2, \psi_3, \psi_4 \rangle$, each ψ_i is an affine automorphism of $\mathbf{C}^2$ and G is fixed point free. More precisely ψ_i is of the form

$$\psi_i(w_1, w_2) = (w_1 + \alpha_i, w_2 + \bar{\alpha}_i w_1 + \beta_i)$$

where $\alpha_i, \beta_i \in \mathbf{C}$ and $\alpha_1 = \alpha_2 = 0$.

Let S be the torus given by the lattice $\langle \alpha_3, \alpha_4 \rangle$ and f be a smooth function on S. On $\mathbf{C}^2$ consider

$$\gamma = (f(w_1) - 2Re(w_1)) dw_1 \wedge d\bar{w}_1 + dw_1 \wedge d\bar{w}_2 + dw_2 \wedge d\bar{w}_1$$

The same computations as in the torus show that γ defines an indefinite Kähler-Einstein metric on $\mathbf{C}^2$, which is homogeneous (and flat) only when f is constant. Moreover, γ is invariant through the ψ_i's and hence the metric descends to M.

Hyperelliptic surfaces : It is shown in [6, p.585] that any Hyperelliptic surface M is of the form $M = F \times C/G$, where F and C are elliptic curves and G is finite group of fixed-point-free automorphisms of $F \times C$. Moreover, let $F = \mathbf{C}/\Lambda$ with $\Lambda = \langle 1, \tau \rangle$; then $G = \langle \phi, \varphi \rangle$, where ϕ is of the form $\phi(z, w) = (z + \tau/m, e^{2k\pi i/m} w)$ and φ is a translation of order m.

If z, w are holomorphic coordinates in $\mathbf{C}^2$ then $dz \wedge d\bar{z} - dw \wedge d\bar{w}$ is the Kähler form of an indefinite Kähler flat metric (on $\mathbf{C}^2$). This form is invariant through translations and so projects to a $(1,1)$-form on $F \times C$. A direct computation shows that this form is invariant through ϕ and φ and hence defines a $(1,1)$-form on M; this is the Kähler form of an indefinite Kähler-Einstein metric on M with Einstein constant 0.

Minimal irrational ruled surfaces : Now let $M = \mathbf{P}(E)$, where E is a 2-dimensional holomorphic vector bundle over a curve S of genus $\mathrm{g} \geq 2$. We will construct indefinite Kähler-Einstein metrics on M when the bundle E is stable or the direct sum of two line bundles of the same degree (see [7], [19]).

Note that given vector bundles E and $\widehat{E}$, $\mathbf{P}(E)$ and $\mathbf{P}(\widehat{E})$ are isomorphic if and only if $\widehat{E} = E \otimes L$ for a line bundle L; and that $\widehat{E}$ verifies any of the conditions above if and only if E does. So both conditions are really properties of M.

Consider M as a $\mathbf{CP}^1$-bundle over S. Let $\{U_i\}_{i=1}^N$ be an open cover of S and $g_{ij} : U_i \bigcap U_j \rightarrow Gl(2, \mathbf{C})$ be a set of transition functions for E. Then $[g_{ij}] : U_i \bigcap U_j \rightarrow \mathbf{P}\ (Gl(2, \mathbf{C}))$ are transition functions for M. Under the conditions stated above, Narasimhan and Seshadri [19] proved that M admits constant transition functions in $\mathbf{P}(U(2))$. Let g_1 be the Fubini-Study metric on $\mathbf{CP^1}$; then g_1 is a Kähler-Einstein metric on $\mathbf{CP^1}$, invariant through the action of $\mathbf{P}(U(2))$. Renormalize g_1 so that the Einstein constant is 1 and let g_2 be a Kähler-Einstein metric on S with Einstein constant -1. Then $g_1 - g_2$ is invariant through the transition functions and so, it defines an indefinite Kähler-Einstein metric on M with Einstein constant 1.

Remark: In [18, p.395] M.S. Narasimhan and S. Ramanan proved that every vector bundle (over a curve of genus greater than 1) can be 'approximated' by stable vector bundles. A little more precisely, every vector bundle is contained in an analytic family of vector bundles for which the set of stable bundles is open and dense.

The cases considered above therefore contain 'most' of the minimal ruled surfaces (over curves of genus greater than 1).

Bibliography

[1] T. Aubin, *Equations du type de Monge-Ampere sur les varietes kähleriennes compactes*, C.R. Acad. Sci. Paris 283 (1976) 119-121.

[2] W. Barth, C. Peters, and A. Van de Ven, Compact Complex Surfaces, Springer-Verlag, 1984.

[3] A. Besse, Einstein Manifolds, Springer-Verlag, 1987.

[4] R. Fintushel, R.J. Stern, *Immersed spheres in 4-dimensions and the immersed Thom conjecture*, Proc. of the Gokova Geometry-Topology Conference 1994, Turkish J. of Math. 19 (2) (1995) 27-39.

[5] R. Friedman, J.W. Morgan, Smooth 4-manifolds and Complex Surfaces, Springer-Verlag, 1994.

[6] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978.

[7] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Iwanami Shoten and Princeton Univ. Press, 1987.

[8] K. Kodaira, *On the structure of compact complex analytic surfaces I*, Am. J. Math. 86 (1964) 751-798.

[9] K. Kodaira, *On Compact Analytic Surfaces II*, Ann. of Math. 77 (1963), 563-626.

[10] K. Kodaira, *On Compact Analytic Surfaces III*, Ann. of Math. 78 (1963), 1-40.

[11] D. Kotschick, *Orientations and Geometrizations of Compact Complex Surfaces*, Bull. London Math. Soc. 29 (1997), no2, 145-149.

[12] P.R. Law, *Neutral metrics in four dimensions*, J. of Math. Phys. 32 (11), 1991, 3039-3042.

[13] H.B. Lawson, M.L. Michelsohn, Spin Geometry, Princeton Univ. Press, 1989.

[14] C. LeBrun, *Polarized 4-manifolds, extremal Kähler metrics, and Seiberg-Witten theory*, Math. Res. Lett. 2, 653-662 (1995).

[15] N.C. Leung, *Seiberg-Witten invariants and uniformizations*, Math. Ann. 306, 31-46 (1996).

[16] J.W. Morgan, The Seiberg-Witten Equations and applications to the Topology of Smooth Four-Manifolds, Princeton Univ. Press, 1996.

[17] I. Nakamura, *Towards classification of Non-Kählerian complex surfaces*, Sugaku Expositions 2, 209-229 (1989).

[18] M.S. Narasimhan, S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. of Math. 101 (1975) 391-417.

[19] M.S. Narasimhan, C.S. Seshadri, *Stable and unitary bundles on a compact Riemann surface*, Ann. of Math. 82 (1965) 540-567.

[20] H. Ooguri, C. Vafa, *Self-Duality and N=2 String magic*, Mod. Phys. Lett. A Vol. 5, no 18 (1990), 1389-1398.

[21] C.H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1, 809-822 (1994).

[22] E. Witten, *Monopoles and Four-manifolds*, Math. Res. Lett. 1, 769-796 (1994).

[23] S.T. Yau, *On Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA 74, 1798-1799 (1977).

[24] A.E. Zalesskij, Encyclopedia of Mathematical Sciencies, Vol. 37, Part II, Springer-Verlag, 1993.