Distinguished Kähler Metrics and Equivariant Cohomological Invariants

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Abstract of the Dissertation

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This work is concerned with cohomological aspects of Kähler geometry. We approach these via invariants of Futaki type. The Futaki invariants are characters on the Lie algebra of holomorphic vector fields of a compact Kähler manifold, one attached to each Kähler class. The non-vanishing of one of them provides an obstruction to the existence of a representative of constant scalar curvature in the corresponding Kähler class.

In chapter Two we give an equivariant cohomological description of the Futaki invariant. This is used to derive localization formulas, expressing its value on a given holomorphic vector field

in terms of local data at the latter's zero locus. The cohomological approach naturally leads to a Kählerian version of a well-known formula of Duistermaat-Heckman. We also use a holomorphic version of equivariant cohomology to give a topological explanation of the Kähler class invariance of the Futaki character.

In chapter Three we give two constructions relating Futaki type invariants to distinguished Kähler metrics. In the first we use a similar invariant to determine a special potential function, conditions on which give rise to a new notion of a distinguished Kähler metric. We call such metrics central. The whole process mimics the relation between the Futaki invariant and Calabi's notion of extremal Kähler metrics. Both types of metrics have distinguished holomorphic vector fields and we describe relations between them. We also give existence results for central metrics for which the central potential is constant.

In the second construction we define a new invariant that depends on two distinct Kähler classes. Its non-vanishing gives an obstruction to the existence of Calabi-Yau metric pairs having identical harmonic components of their shared Ricci form. Calling such pairs harmonic, we describe relations between harmonic and extremal pairs. Finally we show how to construct out of known examples of extremal metrics harmonic Calabi-Yau pairs.

In chapter Four we apply methods of Kähler geometry to give a weak topological uniqueness result for Hermitian Einstein metrics

in dimension four. LeBrun's refinement of early results of Derdzinski shows that if such a metric is non-Kähler, it can live on at most three distinct compact complex surfaces. On these it is conformal to an extremal Kähler metric which gives a Kähler cone critical point of the L^2 norm of the scalar curvature. We give a computer-assisted proof that on one of these spaces, the general position two point blow-up of the complex projective plane, there exists a unique such critical point. This shows that all possible Hermitian Einstein metrics on this space must be conformal to *cohomologous* extremal Kähler metrics.

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Chapter 0

Introduction

Within the general study of spaces of Riemannian metrics on a given compact manifold, Kähler metrics are immediately distinguishable by their cohomological character. A simple picture describes their parameterization by pairs, each of which consists of a point of a cone in the $(1,1)$-Dolbeault cohomology group together with a smooth real valued function. A second illustration is provided by a cohomological reflection of the close relationship between a Kähler metric and the complex structure: the Ricci form of any such metric represents (a multiple of) the first Chern class of the manifold. Thus it is natural that the history of the subject is rich with methods and results of a topological type. This work is based on a more recent development of the last fifteen years, in which various cohomological invariants are tied up with questions of existence of *distinguished* Kähler metrics.

These invariants are of practical significance for complex manifolds admitting continuous groups of biholomorphisms. Typically they are characters on the Lie algebra of holomorphic vector fields, attached to each Kähler class.

The primary example is the invariant defined by Futaki [Ft1], originally for the first Chern class and then for an arbitrary Kähler class. This character is defined using a Kähler metric, but can be shown to depend only on the Kähler class. Its most basic utility is in giving obstructions to the existence of Kähler-Einstein metrics, or more generally of Kähler metrics of constant scalar curvature.

While readily defined, the Futaki invariant is not readily computable. However, when the Kähler class is the first Chern class, a localization formula is known [FMo1], expressing the value of the invariant on a given holomorphic vector field in terms of topological information and data given at the zeros of the vector field. Such formulas have a long history, but have reappeared more recently in symplectic geometry [DH1], for vector fields generating circle actions. From this point of view the above formula can be understood as essentially resulting only from the symplectic portion of the information contained in the Kähler form.

In Chapter Two we derive new localization formulas that are valid for arbitrary Kähler classes. As in one of the derivations of the Duistermaat-Heckman Theorem [AB, BV1], our derivation uses equivariant cohomology, where we form the relevant equivariantly closed forms from both the Kähler form and the Ricci form of an invariant Kähler metric in the Kähler class. In dimension four we actually have two formulas, one of which we use to re-derive and slightly generalize known formulas [LSn, LSm, KLP]. We then proceed to the higher dimensional case. Finally we use a recent holomorphic version of equivariant cohomology [Lu], to extend the validity of the formula to more general holomorphic vector fields, and arbitrary Kähler metrics. We have:

Theorem 0.1 *Let (M_n, J) be a Kähler manifold, Ξ a non-degenerate gradient holomorphic vector field and ω a Kähler form of an arbitrary Kähler metric on M. Denote by ρ the Ricci form of the metric, and f the holomorphy potential of Ξ with respect to the metric. If M_0, the zero set of Ξ, forms a complex submanifold, then*

$$\mathcal{F}_{[\omega]}(\Xi) = (-2\pi)^{rk(N)} \left\{ \tfrac{1}{2(n+1)!} s_0 \int_{M_0} \tfrac{(\omega+f)^{\wedge n+1}}{\det(L_\Xi+\Omega)} - \tfrac{1}{n!} \int_{M_0} \tfrac{(\rho+\Delta f)\wedge(\omega+f)^{\wedge n}}{\det(L_\Xi+\Omega)} \right\}.$$

See Sections 1.4, 2.2.2 and 2.2.3 for exact definitions.

The realization that a Kähler metric defines two (holomorphically) equivariantly closed forms leads to a Kählerian generalization of the Duistermaat-Heckman formula.

Theorem 0.2 *Let (M_n, J) be a Kähler manifold, ω, ρ as above, and Ξ a non-degenerate gradient holomorphic vector field having isolated fixed points. Then*

$$\int_M e^{t(f+\Delta f)} \frac{(\omega+\rho)^n}{n!} = (-2\pi)^n \sum_p \frac{e^{t(f(p)+\Delta f(p))}}{t^n \det B_p}, \tag{0.1}$$

where the sum is taken over the fixed point set.

Regarding the above exponential as a power series, this formula embodies the Futaki invariant as well as many other new invariants. The use of *holomorphic* equivariant cohomology allows for a direct cohomological proof of their invariance under change of representative in the Kähler class.

In Chapter Three we turn to applications of invariants of Futaki type to the search for distinguished Kähler metrics in a given Kähler class. The first such notion, potentially usable for an arbitrary Kähler class, was given by Calabi [C1] in his paper on extremal Kähler metrics. An extremal Kähler metric can be regarded as defined via a holomorphicity condition on the Laplacian of a certain potential function. The latter is essentially the Green function of the scalar curvature, and the holomorphicity condition gives rise to a distinguished (gradient) holomorphic vector field. The special case of constant scalar curvature occurs when the potential function is constant, and is detected by the vanishing of the Futaki invariant. The value of the Futaki invariant on the distinguished vector field gives a uniform lower bound for the L^2 norm of the scalar curvature of metrics in the class [H2], achieved exactly by extremal Kähler metrics. This follows from a duality of the Futaki invariant with the distinguished vector field. The duality in question is with respect to a bilinear form on gradient holomorphic vector fields. This bilinear form is another Kähler class invariant [FMa], determined solely by the symplectic portion of the Kählerian information.

Starting with another invariant Lie algebra character defined in [FMo1] and related to classical invariants [Bt2], we reverse the above process and arrive via duality at another potential function, giving a new candidate for a notion of a distinguished Kähler metric. Unlike extremal metrics, these metrics behave uniformly throughout the Kähler cone, in the sense that on a given space the potential functions of all of them are either all constant, or all non-constant. If a manifold is Fano, this constancy (or lack of) can essentially be detected by the existence (or lack of) of a Kähler-Einstein metric. In other words, the behavior throughout the Kähler cone can be understood via the behavior at a central point in the cone, namely the first Chern class. This motivates us to name such metrics *central.*

This contrast with extremal metrics results from the fact that the invariant character we use to define central metrics is not a Kähler class invariant, but only a complex manifold invariant. Despite the distinction, an analogous uniform bound holds also in this case, with similar corollaries. We quote:

Theorem 0.3 *Let M be a compact Kähler manifold with a Kähler metric g having Kähler form ω and central potential G. Then:*

$$\int_M (\Delta C)^2 \frac{\omega^{\wedge n}}{n!} \geq -\mathcal{B}(\Xi_{\pi_g(\Delta C)}). \tag{0.2}$$

The right hand side of the inequality is a real non-negative Kähler class invariant, and equality occurs exactly when g is central.

Here $\mathcal{B}$ denotes the character and C the potential function, with $\Xi_{\pi_g(\Delta C)}$ (a generalization of) its corresponding holomorphic vector field. For details see Section 3.1.

The bilinear form is further used to derive relations between the gradient vector fields related to the central and the extremal potentials of an arbitrary Kähler metric.

We infer existence results for central Kähler metrics of constant central potential from Yau's solution to the Calabi conjecture [Yu]. Yau's result turns out to have other interesting ties with the invariants we study. A known result [FMo1] uses it to show that on a Fano manifold $\mathcal{B} \equiv \mathcal{F}_{c_1}$, where the right hand side denotes the Futaki invariant at the first Chern class. We use it to define a new invariant, called the *reflection character*, which depends, however, on a *pair* of Kähler classes. For classes of fixed volume, it simply equals the difference of the Futaki invariants at the two classes. Its non-vanishing obstructs the existence of Calabi-Yau pairs of metrics having equal harmonic parts of their shared Ricci form. We call such a pair *harmonic*, and proceed to relate its existence to the existence of Calabi-Yau pairs of *extremal* metrics. This works very simply for the case of constant scalar curvature, while for the non-constant case an extra condition involving the Laplacians of the various scalar curvatures is required to insure that an extremal pair is harmonic. We then proceed using results in [H3] to find non-trivial examples of Calabi-Yau pairs which are both extremal and harmonic. The reflection character of the corresponding classes consequently vanishes, but as these examples include pairs of extremal metrics having non-constant extremal potential, the individual Futaki invariants remain non-zero. We summarize:

Theorem 0.4 *Let M be a compact Kähler manifold, and $\Omega, \tilde{\Omega}$ a pair of Kähler classes. If the reflection character $\mathcal{R}_{\Omega}^{\tilde{\Omega}}$ does not vanish identically, then there does not exist a harmonic pair of Kähler metrics with Kähler forms $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$. When this obstruction vanishes, there are examples of harmonic pairs which are also extremal.*

For details see Sections 3.3 and 3.4.

In Chapter Four we apply the invariants, and the theory of extremal Kähler metrics as a whole, to the question of existence of Hermitian Einstein metrics on compact complex surfaces. The possibility of application lies in a result of Derdzinski [Dr], which implies that such a metric is conformal to an extremal Kähler metric. The classification of the complex surfaces admitting such metrics was refined by LeBrun [Le], who showed that only the general position blow-ups of $\mathbb{C}P^2$ at one, two or three points might admit such Einstein metrics which are also non-Kähler. He also showed that the isometry group (of the Einstein metric and of its extremal conformal partner) contains a 2-torus.

Of these three cases the two point blow-up is the least understood, and we are able to give the following weak topological uniqueness result:

Theorem 0.5 *If there exists Hermitian Einstein metrics on the general position two point blow-up of the complex projective plane, then any two such metrics are conformal to* **cohomologous** *extremal Kähler metrics. The latter metrics evaluate equal volumes for the two exceptional divisors.*

The proof begins with the observation made in [Le], that the corresponding extremal metric is a critical point, over extremal Kähler metrics in neighboring Kähler classes, of the L^2 norm of the scalar curvature. This in turn is computable from the above mentioned lower bound result using the Futaki invariant evaluated on the extremal vector field. We are able to systematize and improve on the calculations in [Le] by transferring the computation to one of elementary integrals on the image polygon of the T^2- moment map. The Futaki invariant formulas of Chapter Two, and the bilinear form of Chapter Three all appear in the calculation. A rather complicated cohomological expression results for the L^2 norm of the scalar curvature.

The second part of the proof is a computer-assisted computation of the critical points of this cohomological function. A numerical calculation easily gives a unique critical point, but much care is taken to symbolically insure that no other critical points in the Kähler cone were left out. Details appear in an appendix to Chapter Four.

Of the two remaining cases, results for the one point blow-up are well known. For the three point blow-up, the first part of the proof works just as well, but gives a much more complicated answer. At the second stage the computer program is unable to determine all critical points. An improvement here will be of much value, as this space is known to admit Kähler-Einstein metrics, for which uniqueness results are known [BM], and so a proof of mere topological uniqueness will translate into a uniqueness of a stronger type.

The determination of the critical point above indicates that it is a global minimum. This agrees with evidence from all other known cases, and so appears to suggest that a stronger condition than criticality distinguishes the Einstein metric and its extremal conformal partner.

Chapter 1

Holomorphy Potentials and Equivariant Forms in Kähler Geometry

This chapter contains consequences and constructions related to the existence of gradient holomorphic vector fields on compact Kähler manifolds. Such a vector field can be constructed from a complex valued function, which we call a holomorphy potential. The construction involves the metric, or alternatively, the closed Kähler 2-form. In Section 1.2 we compute holomorphy potentials with respect to other distinguished closed 2-forms, namely the Ricci form and its harmonic part. The results are then reinterpreted in the next two sections in terms of moment maps, and ultimately in terms of equivariant cohomology. This cohomology theory can be regarded as associated to the vector field, with the differential built out of both the standard exterior one and interior multiplication by the vector field. A distinguishing feature of the theory is that the contribution to cohomological non-triviality is completely determined by the zero set of the vector field. This results in various localization formulas for integrals of equivariantly closed forms. Section 1.3 develops the theory in the special case where the imaginary part of the vector field generates an isometric circle action, and Section 1.4 treats the general case. The upshot of the chapter is that the Kähler geometry determines a number of distinguished equivariantly closed forms.

1.1 Some Conventions, General Background

We give here a summary of repeatedly occuring notations, and review some of the underlying material. This is intended merely to facilitate a reading, and some of the concepts will be re-introduced later.

A smooth manifold will be denoted with or without its dimension, e.g. M, M^{2n}. If it admits a complex structure J, its complex dimension will be given as M_n. If a complex manifold (M, J) admits a Riemannian metric g with respect to which the complex structure, thought of as an endomorphism of the real tangent bundle, is an isometry, it is called *Hermitian*. If for such g the 2-form $\omega(\_,\_) = g(J\_,\_)$ is closed, g, ω and M are called *Kähler* (M is sometimes called *Kählerian*). We will at times not distinguish between g and ω. For example, we might refer to g as belonging to a DeRham cohomology class (the *Kähler class*). A Kähler manifold (M, ω) is an example of a *symplectic* manifold, which is one admitting a closed non-degenerate 2-form. We will sometimes call a closed 2-form pre-symplectic.

When working in local complex coordinates z^i on a complex manifold, we will make moderate use of complex index notation. The complex structure induces decompositions of the complexified tensor bundles. A bar/no bar over an index gives indication of where the object belongs. For example $\phi = \phi_{\bar{\alpha}} dz^{\bar{\alpha}}$ is a $(0,1)$-form, $\Xi = \Xi^{\beta}\frac{\partial}{\partial z^{\beta}}$ a $(1,0)$ vector field. Given a Hermitian metric, commas will denote covariant differentiation with respect to the (complexified) Levi-Civita connection. For example $\Xi^{\alpha}_{,\alpha} = \sum_{\alpha} \nabla_{\alpha}\Xi^{\alpha}$. We will generally put one comma for repeated covariant differentiations. Also note the use of the summation convention in the previous equation. We will make use of the inverse matrix $g^{\alpha\bar{\beta}}$ of the coefficient matrix of g to raise indices: $\phi^{\alpha} = g^{\alpha\bar{\beta}}\phi_{\bar{\beta}}$. The corresponding invariant notation will be $\phi^{\#}$. Recall that a for the Levi-Civita connection $\nabla g = 0$, so one can (consistently) raise a covariant derivative index $(\zeta^{,\gamma})$.

A Hermitian metric h on (M, J) also induces inner products on all bundles in the above decompositions. The sesqui-linear ones will be denoted by $(\cdot,\cdot)_h$, or, more usually, without the subscript. We will also denote it via indices. Recall that this implies a bar over the second factor. If, for example, M is compact, one uses the integration with respect to the metric volume form to get associated inner products on global sections: $< \phi, \xi >_h = \int_M (\phi, \xi) d\mu_h$. Here if ϕ, ξ are, say, functions, then the integrand is simply the product $\phi\bar{\xi}$. Note that when g is Kähler with Kähler form ω, $d\mu_g = \omega^{\wedge n}/n!$.

The exterior derivative on a complex manifold decomposes as $d = \partial + \bar{\partial}$, where ∂ raises by 1 the p sub-degree of a (p,q)-form, and $\bar{\partial}$ raises its q sub-degree similarly. One has $\partial^2 = \bar{\partial}^2 = 0, \partial\bar{\partial} = -\bar{\partial}\partial$. From these operators three Laplacians can be formed: $\Delta = \Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$, $\Delta_{\partial} = \Delta_{\partial} = \partial^*\partial + \partial\partial^*$ and $\Delta_d = \Delta_d = d^*d + dd^*$. Here the superscript * denotes the formal adjoint with respect to the inner product $< \cdot, \cdot >$. For a Kähler metric on a compact manifold these satisfy the basic relation: $\Delta = \frac{1}{2}\Delta_d = \Delta_{\partial}$, and, as follows from Hodge theory, Δ commutes with $\bar{\partial}$. Note that for smooth functions only the first of the summands in each of these acts non-trivially, and with these conventions, it turns out that at a maximum of a real valued function they are all positive. Recall the Divergence Theorem, that states that on a compact manifold (without boundary), the integral of a divergence is zero. This works for complex divergences as well, see Chapter 2. Here we merely state a special case, namely that for a Kähler metric and any smooth complex valued function f, $\int_M \Delta f \, \omega^{\wedge n}/n! = 0$.

If r denotes the Ricci tensor of a Kähler metric g, a fundamental result is that the 2-form ρ, called the Ricci form, is closed, and the DeRham cohomology class it represents is 2π times the *first Chern class* of the manifold, denoted c_1, or $c_1(M)$. Using this, one can understand ρ as a curvature of a Chern connection on a holomorphic line bundle, the anticanonical line bundle, and obtain the simple expression $\rho = -i\partial\bar{\partial}\log\det(g)$. The right hand side defines the first Chern form of g even if g is only Hermitian.

The scalar curvature s of a Kähler metric g can be computed by the formula $s\omega^n = 2n\rho \wedge \omega^{n-1}$. This follows from Hodge theory, as the Kähler form ω is *harmonic* (i.e., belongs to the kernel of Δ), and s is (up to factor) the ω-trace of ρ. A basic result of Hodge theory is that on a compact (Kähler) manifold any (Dolbeault) cohomology class admits a unique harmonic representative. The Hard Lefschetz Theorem gives a cohomological decomposition resulting from the operator given by wedging with the Kähler form (and its adjoint). It holds since these operators commute with Laplacian. A particular consequence we will make use of is that the inner product of ω with a harmonic 2-form is harmonic, and therefore constant (constants are the only harmonic functions on compact manifolds).

The relations between g, J and ω in Kähler geometry (e.g. $\nabla J = 0$) induce strong ties between vector fields preserving each of these three structures, to be detailed in the next section. A vector field X is called *symplectic* if $\mathcal{L}_X\omega = 0$, *Killing* if $\mathcal{L}_X g = 0$ and *holomorphic* if $\mathcal{L}_X J = 0$. Here $\mathcal{L}$ denotes the Lie derivative. A vector field X is called *Hamiltonian* if it satisfies $\imath_X\omega := \omega(X, \_) = df$, where f is a smooth real valued function on the manifold. f is uniquely determined up to an additive constant. Such vector fields are always symplectic. With the other two categories we will also be mainly interested in vector fields that are related to functions as above. However, in these latter cases the functions will automatically satisfy an extra differential equation. This accounts for the finite-dimensionality of the spaces of such vector fields.

Another notational choice is to regard a holomorphic vector field as living in the complexified category, i.e., as a $(1,0)$ vector field. So $\Xi = JX + iX$ will be called holomorphic if X is holomorphic. Remaining in the context of a compact Kähler manifold, if X is Killing and generates a circle action, Ξ is holomorphic and JX is a gradient vector field [Fr]. For more details and references, see the next section.

If a compact connected commutative Lie group (a torus) T acts on a symplectic manifold M, preserves the symplectic form, and induces an infinitesimal action of the Lie algebra t such that associated to every vector in t corresponds a Hamiltonian vector field, it is called a *Hamiltonian T-action*, and M is called a Hamiltonian T-space. For such a space, the dimension of T can be at most half the dimension of M. When M is compact and the action is effective (i.e. no non-trivial element acts trivially), the manifold admits a complex structure with respect to which T acts holomorphically, and the symplectic form is Kähler. In this case M is called *toric*. The theory of these will be of interest in Chapter 4, whereas in Chapter 2 we will be interested in the other extreme case, where T is the circle group S^1. For these, recall that an action is called *semi-free* if the only stabilizer subgroups of points in M are the identity and the entire group.

More specialized background will be described later. We mention that in Chapter 3 some Lie algebraic information will be needed on one occasion, and the background for Chapter 4 includes a limited amount of 4-dimensional Riemannian geometry.

1.2 Laplacians and Holomorphy Potentials

Let (M, J) be a complex manifold, and Ξ a holomorphic vector field with a non-empty zero set. Then for a Kähler metric g on M, there exists a complex valued smooth function f on M such that Ξ is the type $(1,0)$ part of the gradient of f with respect to g (cf. [LSm]). We write $\Xi := \Xi_f = \partial_g^\# f := (\bar{\partial} f)^\#$, or, in local complex coordinates, $\Xi = \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} \frac{\partial f}{\partial \overline{z^\beta}} \frac{\partial}{\partial z^\alpha}$. The function f then satisfies $f_{,\bar{\beta}}^{,\alpha} = 0$.

If ω is the Kähler form of a Kähler metric g, one can also write

$$\imath_\Xi \omega := \omega(\Xi, \_) = \bar{\partial} f.$$

f is called the **holomorphy potential** of Ξ, and is determined up to an additive constant. Ξ will sometimes be called a **gradient (holomorphic) vector field.**

Now writing $\Xi = JX - iJ(JX) = JX + iX$, the real vector field X is Killing, i.e. a generator of isometries, if and only if the imaginary part of f is a constant, which we will choose to be zero (cf. [LSm]. For the opposite convention, of having X be the real part of Ξ, at the price of making f purely imaginary, see [Ft2, Lemma 2.3.8]). In this case, X is also the Hamiltonian vector field of the Hamiltonian f, with respect to the symplectic form ω, and we have

$$\imath_X\omega := \omega(X, \_) = df. \tag{1.1}$$

Since g is Kähler, its Ricci form ρ is a closed 2-form. As with ω one can ask whether the interior product $\imath_\Xi\rho$ is $\bar{\partial}$-exact, and if so, with respect to what function.

Proposition 1.1 *Let (M, J) be a complex manifold and Ξ a gradient holomorphic vector field on it. Suppose g is a Kähler metric on M with Kähler form ω and Ricci form ρ. Then if f is a smooth complex valued function on M satisfying*

$$\imath_\Xi\omega = \bar{\partial}f, \tag{1.2}$$

we have

$$\imath_\Xi\rho = \bar{\partial}(\Delta f). \tag{1.3}$$

Moreover, if M is compact, the second equation implies the first. Here $\Delta := \Delta_{\bar{\partial}}$ is the $\bar{\partial}$-Laplacian: $\Delta := \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$.

Remark 1.2 *Equivalently, using the $\bar{\partial}$-Laplacian on 1-forms, one can write:*

$$\imath_\Xi \rho = \Delta(\imath_\Xi \omega).$$

Proof of Proposition 1.1 This proposition is well known (cf. [Kb, Theorem 4.2]), and goes back to Bochner [Bc2] and Yano [Yn], but we focus specifically on our main concern, the gradient vector fields. Let ϕ be the $(0,1)$-form associated to Ξ using the metric. Then

$$\bar{\partial}^* \phi = -\sum_\alpha \phi^{\alpha}_{,\alpha}$$

$$\bar{\partial}^* \bar{\partial} \phi = -\sum_{\alpha,\bar{\beta}} (\phi^{;\alpha}_{\bar{\beta}} - \phi^{;\alpha}_{,\bar{\beta}})_{,\alpha} dz^{\bar{\beta}}$$

together give

$$\Delta\phi = \sum_{\alpha,\bar{\beta}} (-\phi_{\bar{\beta}}{}^{,\alpha}{}_{,\alpha} + \phi^{;\alpha}_{,\bar{\beta}\alpha} - \phi^{;\alpha}_{,\alpha\bar{\beta}}) dz^{\bar{\beta}} = \sum_{\alpha,\bar{\beta}} (-\phi_{\bar{\beta}}{}^{,\alpha}{}_{,\alpha} + \rho_{\alpha\bar{\beta}} \phi^{,\alpha}) dz^{\bar{\beta}}, \qquad (1.4)$$

by the Ricci identities. But $\phi_{\bar{\beta}} = f_{,\bar{\beta}}$, so

$$\phi_{\bar{\beta}}{}^{,\alpha} = f_{,\bar{\beta}}{}^{,\alpha} = \sum_{\bar{\delta},\gamma} g^{\alpha\bar{\delta}} g_{\gamma\bar{\beta}} f^{,\gamma}{}_{,\bar{\delta}} = 0,$$

since f is a holomorphy potential. So the first term on the right hand side of equation (1.4) drops out, while the second is exactly $\imath_\Xi \rho$, and we get

$$\bar{\partial}(\Delta f) = \Delta(\bar{\partial} f) = \imath_\Xi \rho$$

as required. In the other direction, Since Ξ has zeros, $\imath_\Xi \omega = \bar{\partial} h$ for some holomorphy potential h, so by the first part,

$$\bar{\partial}(\Delta h) = \imath_\Xi \rho = \bar{\partial}(\Delta f).$$

We see that $\Delta(f-h)$ is a constant, which by harmonic theory must be zero on a compact manifold, and moreover f and h themselves must also differ by a constant, so f is also a holomorphy potential for Ξ. $\square$

As before, if the imaginary part X of Ξ is Killing, then in the above setting we have

$$\imath_X \rho = d(\Delta f) = \frac{1}{2} d(\Delta_d f). \tag{1.5}$$

Here $\Delta_d := d^* d + dd^*$ is the d-Laplacian, and we have used the fact that for Kähler metrics $\Delta_{\bar\partial} = \frac{1}{2}\Delta_d$. Now even though the 2-form ρ might be degenerate, it will be useful in the following to regard it as a pre-symplectic form, and thus consider the real function Δf as the ρ-Hamiltonian of X.

Aside from ω and ρ, another closed 2-form determined by a Kähler metric g is ρ_H, the harmonic part of ρ. As with ρ, one can determine the ρ_H-holomorphy potential of a holomorphic vector field Ξ, and the ρ_H-Hamiltonian of a Killing vector field X.

Since ρ and ρ_H belong to the same cohomology class, there exists (cf. [GH, Chapter 1, Section 2]) a smooth real valued function F, called the **Ricci potential**, such that

$$\rho - \rho_H = i\partial\bar\partial F. \tag{1.6}$$

Unless otherwise stated, we normalize F to be perpendicular to the constants.

Definition. The **modified Laplacian** of a Kähler metric g is the operator $\Delta_F : C^\infty_{\mathbb{C}} \to C^\infty_{\mathbb{C}}$ given by $\Delta_F f = \Delta f - (\partial F, \partial f)$. Here $(.,.)$ is the pointwise inner product on forms of type $(1,0)$ induced by g.

This operator was first used in [FMo1] and [Ft2, Section 2.4]. See also [FMS].

Proposition 1.3 *With the assumptions of Proposition 1.1, equation (1.2) implies*

$$\imath_{\Xi}\rho_H = \bar{\partial}(\Delta_F f), \tag{1.7}$$

and is implied by it if M is compact.

Proof. Since $\imath_{\Xi}\rho_H = \imath_{\Xi}\rho - \imath_{\Xi}(i\partial\bar{\partial}F)$, by Proposition 1.1 we need only verify $\imath_{\Xi}(i\partial\bar{\partial}F) = (\partial F, \partial f)$. Verifying this tensor equality locally, and using the summation convention, we have

$$\begin{aligned}
(i\partial\bar{\partial}F) &= F_{,\alpha\bar{\beta}}dz^{\alpha} \wedge \overline{dz^{\beta}} \\
\Xi &= g^{\gamma\bar{\beta}}f_{,\bar{\beta}}\frac{\partial}{\partial z^{\gamma}} \\
&= f^{,\gamma}\frac{\partial}{\partial z^{\gamma}},
\end{aligned}$$

So,

$$\begin{aligned}
\imath_{\Xi}(i\partial\bar{\partial}F) &= F_{,\alpha\bar{\beta}}f^{,\gamma}dz^{\alpha}(\frac{\partial}{\partial z^{\gamma}})\overline{dz^{\beta}} \\
&= F_{,\alpha\bar{\beta}}f^{,\alpha}\overline{dz^{\beta}} \\
&= (F_{,\alpha}f^{,\alpha})_{\bar{\beta}}\overline{dz^{\beta}} \\
&= \bar{\partial}(\partial F, \partial f),
\end{aligned}$$

where in the penultimate equality we have used $f^{,\alpha}_{\ ,\bar{\beta}} = 0$, which holds because f is a holomorphy potential. □

And again, when $X = \Im m(\Xi)$ is Killing, remarks similar to those regarding ρ apply to ρ_H, since Killing fields preserve harmonic forms (see [Bc1], or [BY, Chapter 2, Section 7]).

1.3 Moment Maps and Equivariant Localization

We describe certain notions from symplectic geometry which will be required later.

Let (M^{2n}, ω) be a symplectic manifold with a Hamiltonian T-action, where T is a torus with Lie algebra t whose dual is t^*. Denote by X_ξ the vector field corresponding to a Lie algebra element $\xi \in t$.

Definition. In this setting, a **moment map** is a smooth map $\Phi : M \to t^*$ satisfying :

i) $\imath_{X_\xi}\omega = d < \Phi, \xi >$ for all $\xi \in t$,

ii) $\Phi \circ g = g \circ \Phi \circ g^{-1}$ for all $g \in T$.

For some time we will only be interested in the case $T = S^1$. Then Φ reduces to a single smooth function $f : M \to \mathbb{R} \cong t^*$, condition i) becomes equation (1.1) and the flow ϕ_t of $X_\xi := X$, is periodic: $\phi_1 = id$.

Ignoring for the moment the possible degeneracies of a closed 2-form, we have seen in the previous section that a vector field with zeros preserving a Kähler metric gives rise, at least if it is periodic, to three possibly distinct moment maps relative to ω, ρ and ρ_H. We also saw there the holomorphic counterpart of this situation.

The next important notion is that of localization. For this we give a rather concise introduction to equivariant cohomology.

Denote by $\Omega^m_{S^1}$ the space of smooth S^1-invariant m-forms on M, i.e. those m-forms α for which $\mathcal{L}_X \alpha = 0$, where $\mathcal{L}$ denotes the Lie derivative. Also let $\Omega^*_{S^1}[q]$ be the ring of polynomials in the degree 2 indeterminate q with coefficients in $\Omega^*_{S^1}$. Consider the degree 1 operator

$$d_X = d - q\imath_X.$$

Assuming both d and $\imath_X$ act trivially on q, we have $d_X^2 = -q\mathcal{L}_X$, and so $(\Omega^*_{S^1}[q], d_X)$ constitutes a differential complex. The resulting cohomology $H^*_{S^1}(M)$ is called (the DeRham model for) equivariant cohomology.

We define integration of an equivariant differential form over a submanifold via term by term integration of each component, where of course the latter is non-zero only for forms of degree equal to the dimension of the submanifold. Integration over M induces a map

$$\int_M : H^*_{S^1}(M) \to H^*_{S^1}(point),$$

which is well-defined since an equivariantly exact form has an exact component of the top degree.

Remark 1.4 *Although we regarded q as an indeterminate, we can also view it as a non-zero complex parameter (see [AB, Section 5]). Setting $q = 1$, one can view equivariant differential forms as polynomials on the Lie algebra of S^1 taking values in ordinary differential forms. Although we will not make much use of this interpretation, we will at times set $q = 1$. Then we regard the complex as $\mathbb{Z}_2$-graded by the parity of the ordinary degrees in the components of an equivariant form. This has the advantage that the integration map above* *gives a map to the real numbers, or, in the holomorphic version of the next section, to the complex numbers.*

A first result leading to localization, is the following:

Lemma 1.5 *Let*

$$\alpha = \alpha_{2n} + q\alpha_{2n-2} + \ldots + q^n\alpha_0$$

be an equivariantly closed $2n$-form such that α_0 vanishes on the fixed points of the S^1-action. Then α is d_X-exact. In particular,

$$\int_M \alpha_{2n} = 0.$$

A proof can be found in [BGV, Proposition 7.10] or [MS, Lemma 5.54].

To state the Localization formula, we examine the circle action in more detail. We here assume that M is compact.

Let M_0 denote the (disconnected) submanifold of fixed points of the action, and $N \to M_0$ its normal bundle, with locally constant rank denoted by $\text{rk}(N)$. Let $L_X \in \Gamma(End(N))$ be the induced action of X on N. Then ω pulls back to a symplectic form on M_0, and N becomes a symplectic vector bundle with symplectic structure (infinitesimally) preserved by L_X. L_X can be made into an (invertible) complex automorphism of N by choosing a complex structure on N commuting with L_X and compatible with the symplectic structure on M_0. As such it has weights (eigenvalues as a complex operator) which are purely imaginary, and, since L_X stems from an S^1-action, actually integer-valued. We have:

Theorem 1.6 (Localization [BV2, AB]) *Let M, M_0 and N be as above. Given an equivariantly closed form α,*

$$\int_M \alpha = (-2\pi)^{\mathrm{rk}(N)/2} \int_{M_0} \frac{\alpha}{\det(qL_X + \Omega)}, \tag{1.8}$$

where Ω is the (endomorphism valued) curvature 2-form of an L_X-invariant connection on N, and the determinant is complex, taken with respect to the complex structure on N.

Note that the statement implicitly includes both the formula's independence of the choice of Ω, and the invertibility of the denominator, which follows from that of L_X. More explicitly, if the normal bundle to a connected component of M_0 splits as $L_1 \oplus L_2 \oplus ... \oplus L_m$, with L_X acting with weight k_j on L_j, and Ω_j is the L_j-component of the curvature, then

$$\frac{1}{\det(qL_X + \Omega)} = \frac{1}{\det(\oplus_{j=1}^m (qL_X|_{E_j} + \Omega_j))} = \frac{1}{\prod_{j=1}^m \det(qL_X|_{E_j} + \Omega_j)} =$$
$$\prod_{j=1}^m \frac{1}{\det(qk_jI + \Omega_j)} = \prod_{j=1}^m \frac{1}{(qk_j + \Omega_j)} = \prod_{j=1}^m \frac{1}{qk_j} \sum_{i=0}^n (-\frac{\Omega_j}{qk_j})^i,$$

where in the penultimate step the determinants can be dropped since they are (complex) determinants of endomorphisms on (complex, one dimensional) line bundles.

The basic relation between moment maps and equivariant cohomology is that $\omega + qf$, for f as above, is a very simple example of an equivariantly *closed* form. It is closed exactly because of condition i) above (or equation (1.1)). Note that for this to hold, the non-degeneracy of the symplectic form is immaterial. This enables later applications involving the Ricci form.

For this particular closed equivariant form the Localization formula gives:

Corollary 1.7 *For any positive integer m,*

$$\int_M (\omega + qf)^m = (-2\pi)^{\mathrm{rk}(N)/2} \int_{M_0} \frac{(\omega + qf)^m}{\det(qL_x + \Omega)}. \tag{1.9}$$

We will mostly be interested in the cases $k = n+1$ and, less directly, $k = n+2$.

An important special case arises when X has only isolated zeros:

Corollary 1.8 *for $m \geq n$, if X has only isolated zeros, then*

$$\binom{m}{n} \int_M f^{m-n} \omega^{\wedge n} = (-2\pi)^n \sum_p \frac{(f(p))^m}{e(p)}, \tag{1.10}$$

where the sum is over the critical points of f, and $e(p) \in \mathbb{Z}$ is the product of the weights at p (treated as real integers).

(Note that the factor q^{m-n} that should have appeared on the left hand side canceled with a q^m from the numerator, and a q^n in the denominator, on the right hand side.)

Multiplying (1.9) by $1/m!$ and summing the series gives

Theorem 1.9 (Duistermaat-Heckman) *For (M^{2n}, ω) and f as in the above corollary, and for every non-zero $q \in \mathbb{C}$,*

$$\int_M e^{qf} \frac{\omega^{\wedge n}}{n!} = \int_M e^{\omega + qf} = (-2\pi)^n \sum_p \frac{e^{qf(p)}}{q^n e(p)}. \tag{1.11}$$

Note that for q indeterminate, the above series summation involves a power series completion of the polynomial ring $\Omega^*_{S^1}[q]$.

The last result will serve as a paradigm in what follows. The original proof of the Duistermaat-Heckman Theorem involved a computation of the volumes of symplectically reduced spaces (see [DH1, DH2] and [G, Chapter 2]). Similar notions were used in the proof of the localization formula obtained in [LSm] for the invariant that will be introduced in the next chapter. The equivariant cohomological approach adopted here resembles later proofs of Theorem 1.9 (see [AB, BV1, BV2], [Au, Chapter 5] and [MS, Chapter 5]).

1.4 Holomorphic Equivariant Cohomology

Following an approach beginning with [W], a holomorphic version of the previous theory was systematically developed in [Lu], where many other references to related earlier works are included. We summarize the elements essential to our applications.

For a compact complex manifold (M_n, J), let

$$\Omega^{(r)}(M) = \bigoplus_{q-p=r} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M)$ denotes the smooth (p,q)-forms. Given a holomorphic vector field Ξ on M, define the differential operator

$$\bar{\partial}_{\Xi} = \bar{\partial} - t\imath_{\Xi}.$$

As before, t can be regarded either as a non-zero complex parameter, or as a formal variable of bidegree $(2,0)$. Since now

$$\bar{\partial}_{\Xi}^2 = -t(\bar{\partial}\imath_{\Xi} + \imath_{\Xi}\bar{\partial}) = 0,$$

we see that $(\Omega^{(*)}(M), \bar{\partial}_{\Xi})$ constitutes a differential complex, where $^{(*)}$ denotes the range of $r = -n, -n+1, ...n-1, n$. We denote the resulting cohomology by $H_\Xi^{(r)}(M)$, if r is non-zero, and $H_\Xi(M)$ otherwise. The relevant localization formulas are valid for the latter. To state these, we first carefully describe the degeneracy behavior of Ξ at its zero locus.

Assume M_0 is a (disconnected) complex submanifold of zeros of Ξ (here this is not automatic, and weaker assumptions are possible), and $N = T^{(1,0)}M/T^{(1,0)}M_0$ its holomorphic normal bundle, with locally constant *complex* rank denoted by $\mathrm{rk}(N)$.

Let $L_\Xi \in \Gamma(End(N))$ be the induced (complex Lie derivative) action of Ξ on N. Another major difference from the case of a circle action is that L_Ξ is now explicitly *assumed* to be an invertible complex automorphism of N. In this case we call Ξ a non-degenerate vector field.

As before, integration over M is well-defined as a map

$$\int_M : H_\Xi(M) \to H_\Xi(point),$$

which as we have already mentioned, can also be thought of as a map to $\mathbb{C}$ once t is set to 1. We have:

Theorem 1.10 (Holomorphic Localization [Lu]) *Let M, M_0 and N be as above, with Ξ non-degenerate. Given α with $[\alpha] \in H_\Xi(M)$,*

$$\int_M \alpha = (-2\pi)^{\mathrm{rk}(N)} \int_{M_0} \frac{\alpha}{\det(tL_\Xi + \Omega)}. \tag{1.12}$$

where Ω is the (complex endomorphism valued) curvature 2-form of an L_Ξ-invariant connection on N, induced from any Hermitian metric on M, and the determinant is complex, taken with respect to the complex structure on N.

Assume the fixed point locus of Ξ consists of isolated fixed points, and near such a point p write

$$\Xi = \sum_i v_i^p(z) \frac{\partial}{\partial z_i}.$$

Denote by B_p the (Hessian) matrix $(\partial v_i^p / \partial z_j)_{n\times n}$.

Corollary 1.11 *With notations as in Theorem 1.10, if M_0 consists of isolated fixed points at which Ξ is non-degenerate , then*

$$\int_M \alpha = (-2\pi)^{\mathrm{rk}(N)} \sum_p \frac{\alpha_0(p)}{t^n \det B_p}, \qquad (1.13)$$

where the sum is taken over the fixed point set.

Generalizations abound [Lu]. For example, when the zero set of the vector field does not form a complex manifold, or for degenerate holomorphic vector fields with the formula involving the Grothendieck residue. Even more strikingly, a corresponding formula holds for meromorphic vector fields. This extends the utility of such results to a much larger class of complex manifolds.

Chapter 2

The Futaki Invariant and Localization Formulas

This chapter introduces the Futaki invariant $\mathcal{F}_{[\omega]}$, a character on the Lie algebra of holomorphic vector fields. Its definition depends on a choice of Kähler (metric or) form. However, it is in fact invariant under changes of representative within the Kähler class. We begin in Section 2.1 with another character $\mathcal{B}$ [FMo2], which is in fact a complex manifold invariant. After deriving some of its basic forms for gradient holomorphic vector fields, we define the Futaki invariant and recall that when the Kähler class coincides with the first Chern class, $\mathcal{F}_{c_1} \equiv \mathcal{B}$. Once this invariant's form is given for a gradient holomorphic vector field, a basic difference emerges between the two invariants: using the localization techniques of the previous Chapter, $\mathcal{B}$ can be localized using a single equivariantly closed form, whereas $\mathcal{F}_{[\omega]}$ cannot. Section 2.2 nevertheless describes its localization via two of the equivariantly closed forms given in the previous chapter. First we give a formula valid only in complex dimension two, and derive from it a slight generalization of a more explicit known form [LSn, LSm, KLP]. Then we derive a formula valid in any dimension. This leads to a Kählerian generalization of the Duistermaat-Heckman formula. These later formulas are computed first for Killing vector fields generating isometric circle actions, and then for more general gradient holomorphic vector fields. Finally we show how to understand the above mentioned Kählerian invariance of the Futaki character via holomorphic equivariant cohomology.

2.1 Preliminaries, Definition

For a Hermitian metric h and a holomorphic vector field Ξ, we denote by $\mathrm{div}_h\Xi$ the *(complex) divergence* of Ξ: $\mathrm{div}_h\Xi = \sum_\alpha \nabla_\alpha \Xi^\alpha$. We retain the notation ρ even in this case, but now for the first Chern form of h.

Theorem 2.1 (Futaki-Morita [FMo2]) *Let (M_n, J) be a compact complex manifold and Ξ a holomorphic vector field. Then the number*

$$\mathcal{B}(\Xi) := \int_M \mathrm{div}_h \Xi \frac{\rho^{\wedge n}}{n!} \tag{2.1}$$

is independent of the choice of Hermitian metric h used for its computation. In particular it is invariant under biholomorphisms of M.

Now when h is Kähler, if Ξ has zeros and f is its holomorphy potential, then $\mathrm{div}_h\Xi = -\Delta f$. While varying h, ρ and f also vary (in general), but we still have:

Corollary 2.2 *In the above setting,*

$$\int_M \Delta f \frac{\rho^{\wedge n}}{n!}$$

does not depend on the choice of Kähler metric.

We will also denote the invariant in the previous theorem $\mathcal{B}_{c_1}(\Xi)$, since for certain purposes we will be restricitng its evaluation to Kähler metrics whose Kähler class is $c_1(M)$. We now give it another form, which is useful for a later comparison with the Futaki invariant.

Lemma 2.3 *Let (M_n, g) be a compact Kähler manifold with Ricci potential F. Given a gradient holomorphic vector field Ξ with corresponding holomorphy potential f, we have:*

$$\int_M \Delta f \frac{\rho^{\wedge n}}{n!} = \int_M \Delta_F f \frac{\rho_H^{\wedge n}}{n!}.$$

Since ρ and ρ_H belong to the same cohomology class, we will see in the next section that this result really follows from an invariance that can be established via the techniques of Section 1.4. We give, however, a Kähler-geometric argument, for which we will need the following important theorem.

Theorem 2.4 (Calabi-Yau [Yu]) *Let M be a compact Kähler manifold. If $\bar{\rho}$ a real closed $(1,1)$-form representing $c_1(M)$, then in every Kähler class there exists a unique Kähler form ω, whose Ricci form equals $\bar{\rho}$.*

Note also that two Kähler metrics having identical Ricci forms have proportional volume forms.

Proof of Lemma 2.3. By the Calabi-Yau Theorem there exists a unique Kähler metric $\tilde{g}$ in the Kähler class of g such that its Ricci form $\tilde{\rho}$ satisfies $\tilde{\rho} = \rho_H$. Then $\imath_\Xi \tilde{\rho} = \imath_\Xi \rho_H$, and so by Proposition 1.1

$$\bar{\partial}(\tilde{\Delta} \tilde{f}) = \bar{\partial}(\Delta_F f),$$

where $\tilde{f}$ is the $\tilde{g}$-holomorphy potential of Ξ, and $\tilde{\Delta}$ is the $\tilde{g}$-Laplacian. So

$$\tilde{\Delta}\tilde{f} = \Delta_F f - k, \tag{2.2}$$

where k is a constant. Hence all the above gives

$$\begin{aligned} \int_M \Delta_F f \rho_H^{\wedge n} &= \int_M \tilde{\Delta}\tilde{f}\tilde{\rho}^{\wedge n} + k\int_M \tilde{\rho}^{\wedge n} \\ &= \int_M \Delta f \rho^{\wedge n} + k \int_M \tilde{\rho}^{\wedge n}, \end{aligned}$$

where in the last equality we have used Corollary 2.2. We will thus be done if we show $k = 0$. But $\int_M \tilde{\Delta}\tilde{f}\tilde{\omega}^{\wedge n} = 0$ by the Divergence Theorem, so equation (2.2) implies that k is the average value of $\Delta_F f$ with respect to $\tilde{\omega}^{\wedge n}$, i.e.

$$k = \frac{\int_M \Delta_F f \tilde{\omega}^{\wedge n}}{\int_M \tilde{\omega}^{\wedge n}}.$$

To see the relation between the volume forms of $\tilde{\omega}$ and ω we note that $\tilde{\rho} = \rho_H = \rho - i\partial\bar{\partial}F$, or

$$-i\partial\bar{\partial}\log\det(\tilde{g}) = -i\partial\bar{\partial}\log\det(g) - i\partial\bar{\partial}F.$$

Choosing F to make this equation hold for the functions in it, we see upon taking exponents, that $\det(\tilde{g})/\det(g) = e^F$, or

$$\tilde{\omega}^{\wedge n} = e^F\omega^{\wedge n}.$$

We now compute:

$$\begin{aligned} \int_M \Delta_F f \tilde{\omega}^{\wedge n} &= \int_M \Delta f \tilde{\omega}^{\wedge n} - \int_M (\partial F, \partial f)_\omega \tilde{\omega}^{\wedge n} \\ &= \int_M \Delta f e^F \omega^{\wedge n} - \int_M (\partial F, \partial f)_\omega \tilde{\omega}^{\wedge n} \end{aligned}$$

$$
\begin{aligned}
&= \; < \bar{\partial}^* \bar{\partial} f, e^F >_\omega - \int_M (\partial F, \partial f)_\omega \tilde{\omega}^{\wedge n} \\
&= \; < \bar{\partial} f, \bar{\partial}(e^F) >_\omega - \int_M (\partial F, \partial f)_\omega \tilde{\omega}^{\wedge n} \\
&= \; < \bar{\partial} f, e^F \bar{\partial}(F) >_\omega - \int_M (\partial F, \partial f)_\omega \tilde{\omega}^{\wedge n} \\
&= \; \int_M (\bar{\partial} f, \bar{\partial} F)_\omega e^F \omega^{\wedge n} - \int_M (\partial F, \partial f)_\omega \tilde{\omega}^{\wedge n} \\
&= \; 0,
\end{aligned}
$$

since $(\bar{\partial} f, \bar{\partial} F)_\omega = f_{,\bar{\alpha}} F^{,\bar{\alpha}} = F_{,\beta} f^{,\beta} = (\partial F, \partial f)_\omega$, and so $k = 0$. □

Remark 2.5 *Requiring in the proof that $\tilde{\omega}$ belong to the same class as ω was a choice made strictly for the sake of specificity, and is not necessary.*

We now introduce the Futaki character.

Definition. Let (M_n, J, ω) be a Kähler manifold with Ricci potential F. The **Futaki character** is the map $\mathcal{F}_{[\omega]} : h(M) \to \mathbb{C}$, where $h(M)$ denotes the Lie algebra of holomorphic vector fields on M, given by

$$
\mathcal{F}_{[\omega]}(\Xi) = \int_M \Xi(F) \frac{\omega^{\wedge n}}{n!}. \tag{2.3}
$$

The values of this character do not depend on the choice of metric in the Kähler class ω (see [B, C2, Ft3]), i.e., it is a Kähler class invariant. This will essentially be shown in the end of the next section, using the techniques of Section 1.4.

When (M, J) is a *Fano manifold*, i.e. when the class c_1 contains positive definite $(1,1)$-forms, we have the following relation between $\mathcal{B}$ and $\mathcal{F}_{c_1}$:

Proposition 2.6 (Futaki-Morita [FMo2]) *Let (M, J) be a compact Fano manifold. Let g be a Kähler metric with Kähler form ω having Kähler class c_1. If F denotes the Ricci potential of g, then for any holomorphic vector field Ξ,*

$$\mathcal{F}_{c_1}(\Xi) = \int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} = \int_M \mathrm{div}_g \Xi \frac{\rho^{\wedge n}}{n!} = \mathcal{B}_{c_1}(\Xi).$$

Proof. We reproduce the original proof for the convenience of the reader, although the result also follows from the previous and next propositions. By Theorem 2.4 there exists a unique Kähler metric h in the class c_1 satisfying $\rho_h = \omega$, where ρ_h is the Ricci form of h. Then $\rho - \omega = i\partial\bar{\partial}F$ can be rewritten as

$$\rho - \rho_h = i\partial\bar{\partial}F,$$

so we can take

$$F = -\log\frac{\det(g)}{\det(h)},$$

and then:

$$\begin{aligned}
\mathcal{F}_{c_1}(\Xi) &= \int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} \\
&= -\int_M \Xi(\log\frac{\det(g)}{\det(h)})\frac{\omega^{\wedge n}}{n!} \\
&= -\int_M \frac{\det(h)}{\det(g)}\Xi(\frac{\det(g)}{\det(h)})\frac{\omega^{\wedge n}}{n!} \\
&= -\int_M \Xi(\frac{\det(g)}{\det(h)})\frac{\omega_h^{\wedge n}}{n!} &\quad (2.4)\\
&= \int_M \mathrm{div}_h \Xi \frac{\omega^{\wedge n}}{n!} &\quad (2.5)\\
&= \int_M \mathrm{div}_h \Xi \frac{\rho_h^{\wedge n}}{n!} \\
&= \int_M \mathrm{div}_g \Xi \frac{\rho_g^{\wedge n}}{n!}, &\quad (2.6)
\end{aligned}$$

where to get (2.4) we used $\frac{1}{\det(g)}\frac{\omega^{\wedge n}}{n!} = \frac{1}{\det(h)}\frac{\omega_h^{\wedge n}}{n!}$, to get (2.5) the same relation combined with integration by parts, and for equality (2.6), Theorem 2.1. □

Combining this with Lemma 2.3, we get

$$\mathcal{F}_{c_1}(\Xi) = -\int_M \Delta_F f \frac{\rho_H^{\wedge n}}{n!} \left(= -\int_M \Delta f \frac{\rho^{\wedge n}}{n!}\right), \tag{2.7}$$

when M is of Fano type. On the other hand we have:

Proposition 2.7 *For any Kähler metric g on a compact complex manifold M,*

$$\mathcal{F}_{[\omega]}(\Xi) = -\int_M \Delta_F f \frac{\omega^{\wedge n}}{n!}, \tag{2.8}$$

for any gradient holomorphic vector field Ξ having holomorphy potential f.

Proof. Using the Divergence Theorem,

$$\begin{aligned}
&\mathcal{F}_{[\omega]}(\Xi) = \\
&\int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} = \int_M \Xi^\alpha F_\alpha \frac{\omega^{\wedge n}}{n!} = \int_M f^{,\alpha} F_{,\alpha}\frac{\omega^{\wedge n}}{n!} = \\
&\int_M (\partial F, \partial f)\frac{\omega^{\wedge n}}{n!} = \int_M (\partial F, \partial f)\frac{\omega^{\wedge n}}{n!} - \int_M \Delta f \frac{\omega^{\wedge n}}{n!} = \\
&\int_M ((\partial F, \partial f) - \Delta f)\frac{\omega^{\wedge n}}{n!} = -\int_M \Delta_F f \frac{\omega^{\wedge n}}{n!}.
\end{aligned}$$

□

Comparing the expressions for $\mathcal{B}_{c_1}$ and for $\mathcal{F}$ at a general Kähler class $[\omega]$, and recalling Proposition 1.3, we see that since $\Delta_F f$ is a ρ_H-holomorphy potential, $\mathcal{B}_{c_1}$ admits a localization formula via the techniques of Section 1.4. Also, if M is Fano, and ω is a Kähler form in the class c_1, then, being the unique harmonic form in its class, $\omega = \rho_H$; and so, via Proposition 1.3, we see that f

and $\Delta_F f$ *differ by a constant.* This means that, in *this* case, two of the three moment maps induced by a Kähler metric, or two of the three equivariantly closed forms, coincide (up to translation). This has been understood in [FMo1] and [FMo2], and here we have mainly reorganized the information to fit the equivariant setting. In the next section we will study the Futaki invariant at a *general* Kähler class, for the purpose of obtaining localization formulas. As for the latter, by Proposition 2.7 $\mathcal{F}_{[\omega]}$ does not, in general, have the form appearing in Corollary 1.8, which is easier to localize. Nevertheless, we will show that it admits a localization formula.

To end this section, we record some other forms of the Futaki invariant $\mathcal{F}_{[\omega]}(\Xi)$ of a gradient holomorphic vector field, that will be needed later (M is assumed compact throughout):

$$\int_M (\partial F, \partial f)\frac{\omega^{\wedge n}}{n!} = \int_M (\Delta f) F \frac{\omega^{\wedge n}}{n!} = \int_M f \Delta F \frac{\omega^{\wedge n}}{n!} = -\frac{1}{2}\int_M f(s - s_0)\frac{\omega^{\wedge n}}{n!}. \tag{2.9}$$

Here s is the scalar curvature of the given Kähler metric g,

$$s_0 = \frac{\int_M s \frac{\omega^{\wedge n}}{n!}}{\int_M \frac{\omega^{\wedge n}}{n!}}$$

its average value and the last equality in equation (2.9) follows from relation (1.6) by taking traces (notice that in our conventions $(\omega, i\partial\bar{\partial}F) = -\Delta F = -\Delta_{\bar{\partial}} F$, and also that $2(\omega, \rho_H)$ is the constant s_0, as follows from the Hard Lefschetz Theorem). Recall that the same theorem (or Hodge theory) also implies that

$$s\omega^n = 2n\rho \wedge \omega^{n-1}, \tag{2.10}$$

we get the additional useful form

$$-\frac{1}{(n-1)!}\int_M f\rho\wedge\omega^{\wedge n-1}+\frac{s_0}{2n!}\int_M f\omega^n. \tag{2.11}$$

2.2 Localization Formulas

We have already mentioned that when $[\omega]=c_1$, the Futaki invariant can be localized. Since in that case we really have an invariant of the complex manifold, a localization formula can be found without resorting to symplectic geometry. This was the approach in [FMo2], where localization results were obtained following methods of Bott [Bt1, Bt2], while demonstrating that the Futaki invariant is the 'barycenter' of the moment map (see equation (2.7) and the remarks after Proposition 2.7) was derived separately (see [FMo1]).

In complex dimension 2, localization results were obtained for any Kähler class, assuming the holomorphic vector field generates a $\mathbb{C}^*$ action, in [LSn, LSm, KLP]. We first present another approach for that case, and give a slightly more general formula. Both methods compute the Futaki invariant with respect to an invariant Kähler metric obtained by averaging a given Kähler metric in the class with respect to the S^1-component of the $\mathbb{C}^*$-action. The method given here, as well as its later generalization to higher dimensions, requires minimum knowledge of the explicit form of this metric.

2.2.1 A Formula for Complex Surfaces

The following expression for the invariant can also be thought of as relating $\mathcal{F}_{[\omega]}$ with $\mathcal{F}_{c_1}$. In all our formulas, the periodic flow of a circle action generating vector field is of period 1.

Proposition 2.8 *Let (M_2, J, g) be a Kähler surface. If X a Killing vector field generating an S^1-action, and Ξ the corresponding holomorphic vector field, then*

$$\begin{aligned}
\mathcal{F}_{[\omega]}(\Xi) &= \frac{1}{4}s_0 \int_M f\omega^{\wedge 2} - \frac{1}{4}\int_M (f+\Delta f)(\omega+\rho)^{\wedge 2} \\
&\quad + \frac{1}{4}\int_M (f-\Delta f)(\omega-\rho)^{\wedge 2} + \frac{1}{2}\int_M \Delta f \rho^{\wedge 2} \\
&= \frac{1}{4}s_0 \int_M f\omega^{\wedge 2} - \frac{1}{4}\int_M (f+\Delta f)(\omega+\rho)^{\wedge 2} \\
&\quad + \frac{1}{4}\int_M (f-\Delta f)(\omega-\rho)^{\wedge 2} - \frac{1}{2}\mathcal{F}_{c_1}(\Xi). \qquad (2.12)
\end{aligned}$$

Here ω is the Kähler form of g and f any Hamiltonian of X.

Proof. By equations (2.9) and (2.11)

$$\mathcal{F}_{[\omega]}(\Xi) = \frac{1}{2}\int_M (s_0 - s) f \frac{\omega^{\wedge 2}}{2} = \frac{1}{4}s_0 \int_M f\omega^{\wedge 2} - \int_M f\rho \wedge \omega.$$

Further algebra gives

$$\begin{aligned}
\int_M (f+\Delta f)(\omega+\rho)^{\wedge 2} - \int_M (f-\Delta f)(\omega-\rho)^{\wedge 2} &= \qquad (2.13) \\
4\int_M f\rho\wedge\omega + 2\int_M \Delta f\omega^{\wedge 2} + 2\int_M \Delta f \rho^{\wedge 2} &= \\
4\int_M f\rho\wedge\omega + 2\int_M \Delta f\rho^{\wedge 2},
\end{aligned}$$

where last step follows from the Divergence Theorem. Rearranging terms, one arrives at the first equation, while the last one results from equation (2.7). □

Theorem 2.9 *Let (M_2, J, g) be a Kähler surface, X a Killing vector field generating an S^1 action, and Ξ the corresponding holomorphic vector field.*

Let M_0 be the fixed point manifold of the action, and N its normal bundle of locally constant (real) rank $\mathrm{rk}(N)$. *Then*

$$\mathcal{F}_{[\omega]}(\Xi) = \frac{(-2\pi)^{\mathrm{rk}(N)/2}}{q}\left\{\frac{1}{12}s_0\int_{M_0}\frac{(\omega+qf)^3}{\det(qL_X+\Omega)} - \frac{1}{12}\int_{M_0}\frac{((\omega+\rho)+q(f+\Delta f))^3}{\det(qL_X+\Omega)} + \frac{1}{12}\int_{M_0}\frac{((\omega-\rho)+q(f-\Delta f))^3}{\det(qL_X+\Omega)} + \frac{1}{6}\int_{M_0}\frac{(\rho+q\Delta f)^3}{\det(qL_X+\Omega)}\right\}, \tag{2.14}$$

where q, det*, L_X and Ω are as in Theorem 1.6.*

Proof. Using both the previous proposition together with the expansion of the equivariantly closed forms $\omega + qf, (\omega+\rho)+q(f+\Delta f), (\omega-\rho)+q(f-\Delta f)$ and $\rho + q\Delta f$, as in the passage from the left hand sides of equation (1.9) and equation (1.10), with $m = n+1 = 3$,

$$\begin{aligned} q\mathcal{F}_{[\omega]}(\Xi) &= \frac{1}{12}s_0\int_M(\omega+qf)^3 - \frac{1}{12}\int_M((\omega+\rho)+q(f+\Delta f))^3 \\ &\quad + \frac{1}{12}\int_M((\omega-\rho)+q(f-\Delta f))^3 + \frac{1}{6}\int_M(\rho+q\Delta f)^3, \end{aligned}$$

and the result follows directly from the Localization Theorem 1.6. □

We would like to derive from the above formula the more manifestly topological ones appearing in [LSn, LSm, KLP]. While the terms in formula (2.14) appear to require extra information, namely the values of Δf and f at the fixed points of the S^1-action, these can actually be obtained from topological information alone. The differences of values of the ρ-Hamiltonian Δf at the fixed points depend only on the cohomology class $[\rho] = 2\pi c_1$, and that of the ω-Hamiltonian f on the cohomology class $[\omega]$ (this is related to the independence of the measure, induced on the image of the moment map, on the particular (pre-)symplectic representative, see [Mb1]. See [Ka, Lemma 2.7] for an explicit calculation in dimension four).

Our derivation will use the following known properties of Hamiltonians, for which good references are [Au] and [MS]. The relevant original papers are [At, Bt3] and [GuS].

a. Every Hamiltonian f is a (perfect) Bott-Morse function, i.e., with respect to a compatible metric (such as the invariant one we have been using), the manifold M_0 of critical points of f satisfies $T_xM_0 = ker\nabla^2 f(x)$, where the linear operator $\nabla^2 : T_xM \to T_xM$ is obtained from the Hessian via the metric.

b. The connected components of M_0 are even dimensional and of even index.

c. A local minimum (maximum) of f is also a global minimum (maximum), and is obtained on a connected set.

d. The Hamiltonian f has a canonical local expression near a fixed point p given, in Darboux coordinates for the Kähler form, by:

$$f(z^+, z^-) = f(p) + \frac{k^+}{2}|z^+|^2 + \frac{k^-}{2}|z^-|^2,$$

where k^+, k^- are the weights, and in particular, integers.

e. For the four-manifolds we consider, a concrete description exists of the submanifolds connecting two flow-adjacent fixed points, and of the relation between their Kählerian area and the moment map values at the fixed points [Ka].

It follows from c. that for a four manifold M, there are at most two fixed surfaces, C_- and C_+, on which the global minimum and maximum of f

are respectively obtained. We will consider the most generic case, where M contains exactly two fixed surfaces. M is then ruled and $C_\pm$ are sections, so in particular have equal Euler number χ. We make a convenient choice for f by requiring

$$f(C_+) = -f(C_-). \tag{2.15}$$

The volume of the fiber F is then given by

$$\omega(F) = 2\pi(2f(C_+)) = 4\pi f(C_+). \tag{2.16}$$

Let us first consider the contribution of C_- and C_+ to the first term in equation (2.14). Recall that in general

$$\frac{1}{\det(qL_X + \Omega)} = \prod_{j=1}^{m} \frac{1}{qk_j} \sum_{i=0}^{n} (-\frac{\Omega_j}{qk_j})^i,$$

where k_j are the weights, and Ω_j components of the curvature of the normal bundle over a connected component of the fixed point set.

Let $\Omega_\pm$ be the curvature 2-forms of the normal bundles $N_{C_\pm}$ of $C_\pm$, respectively, and $k_\pm = \pm 1$ their respective non-zero weights. Here the sign is determined by our convention in formula (1.1) for the sign relation between the moment map and the action. The contribution from $C_\pm$ to the first integral in (2.14) is therefore

$$\begin{aligned}
&\tfrac{-2\pi}{q}\tfrac{s_0}{12}\left\{\textstyle\int_{C_-}(\omega+qf)^3/(-q+\Omega_-) + \int_{C_+}(\omega+qf)^3/(q+\Omega_+)\right\} = \\
&\tfrac{-2\pi}{q}\tfrac{s_0}{12}\left\{\textstyle\int_{C_-}(\omega+qf)^3 \wedge (-(\frac{1}{q})(1+\frac{\Omega_-}{q})) + \int_{C_+}(\omega+qf)^3 \wedge (\frac{1}{q})(1-\frac{\Omega_+}{q})\right\} = \\
&\tfrac{-2\pi}{q}\tfrac{s_0}{12}\left\{\textstyle\int_{C_-}(-3qf^2\omega - qf^3\Omega_-) + \int_{C_+}(3qf^2\omega - qf^3\Omega_+)\right\} = \\
&-2\pi\tfrac{s_0}{12}3f^2(C_+)\omega(C_+ - C_-) - 2\pi\tfrac{s_0}{12}f^3(C_+)(\textstyle\int_{C_-}\Omega_- - \int_{C_+}\Omega_+) = \\
&-\tfrac{s_0(\omega(F))^2}{32\pi}\omega(C_+ - C_-) - \tfrac{s_0(\omega(F))^3}{192\pi}(C_-^2 - C_+^2),
\end{aligned}$$

where we have used (2.15), (2.16) and $\int_{C_\pm} \Omega_\pm = 2\pi C_\pm^2$.

Next, the contribution from the isolated fixed points p_j, having weights $e(p_j) = k_j^+ k_j^- = \det(L_X|_{p_j})$ is

$$\frac{(-2\pi)^2}{q}\frac{s_0}{12}\sum_j \frac{(\omega+qf)^3|_{p_j}}{\det(qL_X|_{p_j})} = \frac{(2\pi)^2}{q}\frac{s_0}{12}\sum_j \frac{(\omega+qf)^3|_{p_j}}{q^2 e(p_j)} =$$
$$\frac{(2\pi)^2}{q}\frac{s_0}{12}\sum_j \frac{q^3 f(p_j)^3}{q^2 e(p_j)} = (2\pi)^2\frac{s_0}{12}\sum_j \frac{f(p_j)^3}{e(p_j)}.$$

Proceeding, $f(p_j)$ is replaced with a more cohomological expression:

$$\begin{aligned}
&= (2\pi)^2\frac{s_0}{12}\sum_j \frac{(-(f(C_+)-f(p_j))+f(C_+))^3}{e(p_j)} \\
&= (2\pi)^2\frac{s_0}{12}\sum_j \frac{(-\sum_{l>j} k_l^+\frac{\omega(E_l)}{2\pi}+\frac{\omega(F)}{4\pi})^3}{e(p_j)} \\
&= \frac{s_0}{192\pi}\sum_j \frac{(-\sum_{l>j} 2k_l^+\omega(E_l)+\omega(F))^3}{e(p_j)},
\end{aligned}$$

where E_l denotes the rational curves which are closures of non-trivial $\mathbb{C}^*$-orbits (*gradient spheres* in the terminology of [Ka]), belonging to the singular fiber connecting C_- and C_+ on which the given fixed point p_j lies (to which a particular *branch* with a vertex labeled $f(p_j)$ corresponds in the extended graph of [Ka]), and the inequality under the last summation means that one sums only over those gradient spheres connecting p_j with C_+. k_l^+ denotes the order of the isotropy subgroup of E_l, which is also a k^+ weight for a fixed point, namely E_l's *south pole*, i.e. the $-\infty$ limit of its gradient flow. We have also used relation (2.16).

Remark 2.10 *Denoting by $\hat{E}_j'$ ($\hat{E}_j$) the chain of rational curves obtained by following the flow from p_j to C_+ (C_-), we this that because $-2\omega(\hat{E}_j')+\omega(F) =$*

$-\omega(\hat{E}_j') + (\omega(F) - \omega(\hat{E}_j')) = -\omega(\hat{E}_j') + \omega(\hat{E}_j)$, *the expression reduces, for a* semi-free *action (i.e. all weights equal to one), to*

$$(2\pi)^2 \frac{s_0}{12} \sum_j \frac{(\omega(\hat{E}_j - \hat{E}_j'))^3}{(4\pi)^3} = \frac{s_0}{192\pi}(\omega(\hat{E}_j - \hat{E}_j'))^3.$$

This agrees (up to a conventional factor) with the result in [LSm].

We now consider the last three terms in (2.14). With little algebra, we rewrite them as follows:

$\frac{(-2\pi)^{\mathrm{rk}(N)/2}}{q} \Big\{ -\frac{1}{12} \int_{M_0} [((\omega+\rho) + q(f+\Delta f))^3 / \det(qL_X + \Omega)]$
$+\frac{1}{12} \int_{M_0} [((\omega-\rho) + q(f-\Delta f))^3 / \det(qL_X + \Omega)] +$
$\frac{1}{6} \int_{M_0} [(\rho + q\Delta f)^3 / \det(qL_X + \Omega)] \Big\}$
$= \frac{(-2\pi)^{\mathrm{rk}(N)/2}}{q} \int_{M_0} \Big[\Big(-\frac{1}{2}(\omega^{\wedge 2} \wedge \rho + q\Delta f \omega^{\wedge 2} + q^3 f^2 \Delta f + q^2 f^2 \rho)$
$-qf\rho \wedge \omega - q^2 f \Delta f \omega) \,/\, \det(qL_X + \Omega) \Big].$

Hence the contribution to this term from $C\pm$ is

$\frac{-2\pi}{q} \Big\{ \int_{C\_} (-\frac{1}{2}(\omega^{\wedge 2} \wedge \rho + q\Delta f \omega^{\wedge 2} + q^3 f^2 \Delta f + q^2 f^2 \rho) - qf\rho \wedge \omega - q^2 f \Delta f \omega)$
$/ \det(-q + \Omega\_)+$
$\int_{C_+} (-\frac{1}{2}(\omega^{\wedge 2} \wedge \rho + q\Delta f \omega^{\wedge 2} + q^3 f^2 \Delta f + q^2 f^2 \rho) - qf\rho \wedge \omega - q^2 f \Delta f \omega)$
$/ \det(q + \Omega_+) \} =$
$\frac{-2\pi}{q} \Big\{ \int_{C\_} (-\frac{1}{2}(\omega^{\wedge 2} \wedge \rho + q\Delta f \omega^{\wedge 2} + q^3 f^2 \Delta f + q^2 f^2 \rho) - qf\rho \wedge \omega - q^2 f \Delta f \omega)$
$\wedge(-(\frac{1}{q})(1 + \frac{\Omega\_}{q}))+$
$\int_{C_+} (-\frac{1}{2}(\omega^{\wedge 2} \wedge \rho + q\Delta f \omega^{\wedge 2} + q^3 f^2 \Delta f + q^2 f^2 \rho) - qf\rho \wedge \omega - q^2 f \Delta f \omega)$
$\wedge(\frac{1}{q})(1 - \frac{\Omega_+}{q}) \Big\} =$
$\frac{-2\pi}{q} \Big\{ \int_{C\_} \frac{1}{2} q f^2 \rho + qf\Delta f \omega + \frac{1}{2} q f^2 \Delta f \Omega\_ +$
$\int_{C_+} (-\frac{1}{2} q f^2 \rho - qf\Delta f \omega + \frac{1}{2} q f^2 \Delta f \Omega_+) \Big\} =$

$-2\pi\left\{\frac{1}{2}f^2(C_-)\int_{C_-}\rho + f(C_-)(\Delta f)(C_-)\int_{C_-}\omega + \frac{1}{2}f^2(C_-)(\Delta f)(C_-)\int_{C_-}\Omega_- -\right.$
$\left.\frac{1}{2}f^2(C_+)\int_{C_+}\rho - f(C_+)(\Delta f)(C_+)\int_{C_+}\omega + \frac{1}{2}f^2(C_+)(\Delta f)(C_+)\int_{C_+}\Omega_+\right\} =$
$-2\pi\left\{-\frac{1}{2}f^2(C_-)2\pi K\cdot C_- + f(C_-)(\Delta f)(C_-)\int_{C_-}\omega + \frac{1}{2}f^2(C_-)(\Delta f)(C_-)2\pi C_-^2 -\right.$
$\left.(-\frac{1}{2}f^2(C_+)2\pi K\cdot C_+) - f(C_+)(\Delta f)(C_+)\int_{C_+}\omega + \frac{1}{2}f^2(C_+)(\Delta f)(C_+)2\pi C_+^2\right\},$

where K is the canonical bundle. Now as the d-Laplacian is the trace of the Hessian, which is degenerate at the minimum and maximum of f, we have

$$\Delta f(C_\pm) = \frac{1}{2}\Delta_d f(C_\pm) = \frac{\pm 2}{2} = \pm 1.$$

(Recall that our Laplacians are minus the naive ones). Using this and the adjunction formula, the calculation continues:

$-2\pi\left\{-\frac{1}{2}f^2(C_-)2\pi\chi - f(C_-)\omega(C_-) + \frac{1}{2}f^2(C_+)2\pi\chi - f(C_+)\omega(C_+)\right\} =$
$2\pi f(C_+)\omega(C_+ - C_-) = -\frac{1}{2}\omega(F)\omega(C_- - C_+),$

by (2.15) and (2.16). This term already appeared in [LSn].

And lastly, the contribution to the second part of the expression for the Futaki invariant from the isolated fixed points p_j is

$\frac{(-2\pi)^2}{q}\sum_j(-\frac{1}{2}(\omega^{\wedge 2}\wedge\rho + q\Delta f\omega^{\wedge 2} + q^3f^2\Delta f + q^2f^2\rho) - qf\rho\wedge\omega - q^2f\Delta f\omega)|_{p_j}$
$/\det(qL_X|_{p_j}) = \frac{(2\pi)^2}{q}\sum_j(-\frac{1}{2}q^3f^2(p_j)(\Delta f)(p_j))/(q^2e(p_j)) =$
$-2\pi^2\sum_j f^2(p_j)(\Delta f)(p_j)/e(p_j).$

Now if the S^1-action is semi-free then $e(p_j) = 1$, and since the index is even and the points p_j are neither minima nor maxima, the index has to be 2, and the normal form for f as a moment map and as a Bott-Morse function coincide. Therefore, the weights are ± 1, so $\Delta f(p_j) = 0$, and this last term does not contribute, in agreement with [LSm]. For a general action, if the weights at p_j are k_j^+, k_j^-, we get:

$$-2\pi^2 \sum_j f^2(p_j) \frac{k_j^+ + k_j^-}{k_j^+ k_j^-} = -2\pi^2 \sum_j f^2(p_j)(\frac{1}{k_j^-} + \frac{1}{k_j^+}).$$

Now as we have already seen, one can substitute $-\sum_{l>j} 2k_l^+ \omega(E_l) + \omega(F)$ for $f(p_j)$. However, to reproduce the form of [KLP] for the above term, we rewrite it as a sum over gradient spheres. For this we use the following information. A fixed point p_j belongs to two of the gradient spheres E_l, having isotropy subgroups $\mathbb{Z}_{k_j^-}$ and $\mathbb{Z}_{k_j^+}$, respectively. Also, if p_l is the south pole of E_l, and p_{l+1} its north pole (∞ gradient limit of its $\mathbb{C}^*$ orbit), then $k_{l+1}^- = -k_l^+$. Note that this relation is also used when the poles lie on the fixed surfaces $C_\pm$. However, since gradient spheres with poles lying on $C_\pm$ have isotropy subgroups of equal (and trivial) order [Ka], the terms involving $f^2(C_\pm)$ cancel by relation (2.15). Given all this, the new form for the sum is given by:

$$\begin{aligned}
&= 2\pi^2 \sum_l (f^2(p_{l+1}) - f^2(p_l)) \frac{1}{k_l^+} \\
&= 2\pi^2 \sum_l (f(p_{l+1}) + f(p_l))(f(p_{l+1}) - f(p_l)) \frac{1}{k_l^+} \\
&= 2\pi^2 \sum_l (f(p_{l+1}) + f(p_l))(\frac{1}{2\pi}\omega(E_l)) \\
&= -2\pi^2 \sum_l [(f(C_+) - f(p_{l+1})) - (f(p_l) - (-f(C_+)))](\frac{1}{2\pi}\omega(E_l)) \\
&= -2\pi^2 \sum_l [\sum_{m>l} \frac{k_m^+}{2\pi}\omega(E_m) - \sum_{m<l} \frac{k_m^+}{2\pi}\omega(E_m)](\frac{1}{2\pi}\omega(E_l)) \\
&= \frac{1}{2} \sum_{m<l} (k_m^+ - k_l^+)[\omega(E_m)][\omega(E_l)],
\end{aligned}$$

where in the penultimate equality we have used relation (2.15) again, and the inequality under the various summations refers to the partial ordering on these curves, where one precedes the other with respect to the flow (of the real part of Ξ) in some singular fiber of the ruling.

So our final expression for the Futaki Invariant is:

$$\begin{aligned}\mathcal{F}_{[\omega]}(\Xi) &= -\frac{s_0(\omega(F))^2}{32\pi}\omega(C_+ - C_-) - \frac{s_0(\omega(F))^3}{192\pi}(C_-^2 - C_+^2) \\ &+ \frac{s_0}{192\pi}\sum_j \frac{(-(\sum_{l>j} 2k_l^+\omega(E_l)) + \omega(F))^3}{e(p_j)} - \frac{1}{2}\omega(F)\omega(C_- - C_+) \\ &+ \frac{1}{2}\sum_{m<l}(k_m^+ - k_l^+)[\omega(E_m)][\omega(E_l)]. \qquad (2.17)\end{aligned}$$

2.2.2 The Higher Dimensional Case

The Futaki invariant formula of the previous section does not hold in higher dimensions. However, using wedge products of equivariantly closed forms one can obtain a localization formula in any dimension. For this calculation we set the indeterminate q to equal 1.

Proposition 2.11 *Let (M_n, J) be a Kähler manifold, and X a Killing vector field with respect to a Kähler metric g. Assume X generates an S^1-action, let Ξ denote the corresponding holomorphic vector field, and f a Hamiltonian of X.*

Then:

$$\mathcal{F}_{[\omega]}(\Xi) = \frac{1}{2(n+1)!}s_0\int_M (\omega + f)^{\wedge n+1} - \frac{1}{n!}\int_M (\rho + \Delta f) \wedge (\omega + f)^{\wedge n}. \quad (2.18)$$

Proof. By the expressions (2.9) and (2.11), we have

$$\mathcal{F}_{[\omega]}(\Xi) = \frac{1}{2n!}s_0\int_M f\omega^{\wedge n} - \frac{1}{(n-1)!}\int_M f\rho \wedge \omega^{\wedge n-1}.$$

The right hand side of expression (2.18), on the other hand, evaluates to

$\frac{1}{2(n+1)!}s_0(n+1)\int_M f\omega^n - \frac{1}{n!}\int_M \rho \wedge (\sum_{k=0}^{n}\binom{n}{k} f^{n-k}\omega^k) -$

$\frac{1}{n!}\int_M \Delta f(\sum_{k=0}^{n}\binom{n}{k} f^{n-k}\omega^k) =$

$\frac{1}{2n!}s_0\int_M f\omega^n - \frac{1}{n!}n\int_M f\rho\wedge\omega^{n-1} - \frac{1}{n!}\int_M \Delta f\omega^{\wedge n} =$

$\frac{1}{2n!}s_0\int_M f\omega^n - \frac{1}{(n-1)!}\int_M f\rho\wedge\omega^{n-1},$

where we have used the Divergence Theorem in the last step. □

Theorem 2.12 *Under the assumptions of the previous proposition, we have the following localization formula for the Futaki invariant:*

$$\mathcal{F}_{[\omega]}(\Xi) = (-2\pi)^{\mathrm{rk}(N)/2}\left\{\frac{1}{2(n+1)!}s_0\int_{M_0}\frac{(\omega+f)^{\wedge n+1}}{\det(L_X+\Omega)} - \frac{1}{n!}\int_{M_0}\frac{(\rho+\Delta f)\wedge(\omega+f)^{\wedge n}}{\det(L_X+\Omega)}\right\},$$

where rk, det, L_Ξ, Ω *are as in Theorem 1.6.*

Proof. This follows immediately from the previous proposition and from the Localization Theorem 1.6. □

Note that although this formula differs in form from the one given in the previous section, it is completely equivalent to it.

Corollary 2.13 *Under the above assumptions, if the circle action admits only isolated fixed points, we have*

$$\mathcal{F}_{[\omega]}(\Xi) = (-2\pi)^n\{\frac{1}{2(n+1)!}s_0\sum_p\frac{(f(p))^{n+1}}{e(p)} - \frac{1}{n!}\sum_p\frac{(\Delta f)(p)(f(p))^n}{e(p)}\},$$

where, as usual, the sums are over the isolated fixed points.

Note that other combined powers of $\omega + qf$ and $\rho + q\Delta f$ also admit localization formulas. As we will see they are in fact Kähler class invariants. The following result includes them all in a power series, and can be regarded as a generalization, valid in the context of Kähler geometry, of the Duistermaat-Heckman formula (Theorem 1.9).

Theorem 2.14 *Let (M^{2n}, J) be a Kähler manifold, ω, ρ as above, and X a Killing vector field generating a circle action with isolated fixed points. Then*

$$\int_M e^{q(f+\Delta f)} \frac{(\omega + \rho)^n}{n!} = \int_M e^{(\omega+qf)+(\rho+q\Delta f)} = (-2\pi)^n \sum_p \frac{e^{q(f(p)+\Delta f(p))}}{q^n e(p)},$$

where the sum is taken over the fixed point set.

2.2.3 Generic Vector Fields, Invariance

The holomorphic version of equivariant cohomology described in Section 1.4 provides a more natural setting for understanding the Futaki invariant. First, the above results (and proofs) extend directly to give localization formulas for much more general classes of holomorphic vector fields. For example, we have, setting the indeterminate t to 1:

Theorem 2.15 *Let (M_n, J) be a Kähler manifold, Ξ a non-degenerate gradient holomorphic vector field and ω a Kähler form of an arbitrary Kähler metric on M. Denote by ρ the Ricci form of the metric, and f the holomorphy potential of Ξ with respect to the metric. If M_0, the zero set of Ξ, forms a complex submanifold, then*

$$\mathcal{F}_{[\omega]}(\Xi) = (-2\pi)^{\mathrm{rk}(N)} \{ \tfrac{1}{2(n+1)!} s_0 \int_{M_0} \tfrac{(\omega+f)^{\wedge n+1}}{\det(L_\Xi+\Omega)} - \tfrac{1}{n!} \int_{M_0} \tfrac{(\rho+\Delta f)\wedge(\omega+f)^{\wedge n}}{\det(L_\Xi+\Omega)} \},$$

where rk, det, L_Ξ, Ω *are as in Theorem 1.10.*

The proof is identical to that of Theorem 2.12, except that it uses the *Holomorphic* Localization Theorem 1.10. Also,

Corollary 2.16 *Under the above assumptions, if* Ξ *has only (non-degenerate) isolated fixed points with Hessian matrix* B_p *at each such point, then*

$$\mathcal{F}_{[\omega]}(\Xi) = (-2\pi)^n \{ \frac{1}{2(n+1)!} s_0 \sum_p \frac{(f(p))^{n+1}}{\det B_p} - \frac{1}{n!} \sum_p \frac{(\Delta f)(p)(f(p)^n}{\det B_p} \},$$

with the sums taken over the isolated fixed points.

Likewise, we have a formula extending Theorem 2.14 for non-degenerate gradient holomorphic vector fields having only isolated zeros:

$$\int_M e^{t(f+\Delta f)} \frac{(\omega+\rho)^n}{n!} = \int_M e^{(\omega+tf)+(\rho+t\Delta f)} = (-2\pi)^n \sum_p \frac{e^{t(f(p)+\Delta f(p))}}{t^n \det B_p}. \quad (2.19)$$

Remark 2.17 *In the case of isolated zero points, the requirement that the holomorphic vector field be non-degenerate may be dropped, and one could still obtain a Localization formula involving the Grothendieck residue (see [Lu, Theorem 8.1]).*

Remark 2.18 *It is an intriguing remaining question whether it is useful to attempt to define the Futaki invariant for meromorphic vector fields, as the equivariant cohomology works in this setting as well.*

Next, we examine the question of invariance. We have stated in the previous section that the Futaki invariant does not depend on the choice of Kähler metric used in its definition, within a fixed Kähler class. The holomorphic version of equivariant cohomology allows for a natural proof of this invariance, at least for gradient holomorphic vector fields (For vector fields having an empty zero set, the Futaki invariant vanishes, (see for example the end of the proof of Theorem 3.7), but the proof of this presupposes its invariance). We will use the (more topologically proven) invariance of $\mathcal{B}$, and add an assumption on the complex structure.

Theorem 2.19 *Let (M_n, J) be a Kähler manifold with $c_1^n \neq 0$, and a gradient holomorphic vector field Ξ. If ω, $\tilde{\omega}$ are two Kähler forms in the same Kähler class, with Ricci potentials F and $\tilde{F}$, respectively, then*

$$\int_M \Xi F \frac{\omega^{\wedge n}}{n!} = \int_M \Xi \tilde{F} \frac{\tilde{\omega}^{\wedge n}}{n!}.$$

Proof. We continue to work setting $t = 1$. By the holomorphic equivalent of Proposition 2.11, we can write the Futaki invariant as a difference of integrals of holomorphically equivariantly closed forms:

$$\int_M \Xi F \frac{\omega^{\wedge n}}{n!} = \frac{1}{2(n+1)!} s_0 \int_M (\omega + f)^{\wedge n+1} - \frac{1}{n!} \int_M (\rho + \Delta f) \wedge (\omega + f)^{\wedge n},$$

where f is an ω-holomorphy potential for Ξ. Now if $\tilde{\omega} = \omega + i\partial\bar{\partial}\phi$ for a smooth real valued function ϕ, then

$$\tilde{f} = f + \imath_{\Xi}(i\partial\phi) + K \tag{2.20}$$

is the general form of an $\tilde{\omega}$-holomorphy potential, with K a constant. Now as it follows from expression (2.9) that the Futaki invariant does not depend on the choice of normalization constant K, we take $K = 0$. But then, as

$$\bar{\partial}_\Xi(-i\partial\phi) = i\partial\bar{\partial}\phi + \imath_\Xi(i\partial\phi),$$

we see that $\omega + f$ and $\tilde{\omega} + \tilde{f}$ differ by a $\bar{\partial}_\Xi$-exact form. The result will follow if we can show the same for $\rho + \Delta f$ and $\tilde{\rho} + \tilde{\Delta}\tilde{f}$. Let β be a $(1,0)$-form such that $\tilde{\rho} = \rho + \bar{\partial}\beta$. Then again,

$$\tilde{\rho} + \tilde{\Delta}\tilde{f} = \rho + \Delta f + \bar{\partial}\beta - \imath_\Xi\beta + D = \rho + \Delta f + \bar{\partial}_\Xi\beta + D,$$

with D constant. Therefore,

$$\int_M (\tilde{\rho} + \tilde{\Delta}\tilde{f})^{n+1} = \int_M (\rho + \Delta f + \bar{\partial}_\Xi\beta + D)^{n+1}. \tag{2.21}$$

But by Corollary 2.2, the left hand side also equals $\int_M(\rho + \Delta f)^{n+1}$ which can be also written as $\int_M(\rho + \Delta f + \bar{\partial}_\Xi\beta)^{n+1}$, as what is added is holomorphically equivariantly exact. This differs from the right hand side of (2.21) by a non-zero multiple of $D\int_M \rho^n$. So, if $c_1^n \neq 0$, then $\int_M \rho^n \neq 0$, $D = 0$, and we are done. □

In the next two chapters we will be using another invariant, a bilinear form on gradient holomorphic vector fields given by,

$$\mathcal{K}_{[\omega]}(f_{\Xi_1}, f_{\Xi_2}) = \int_M f_{\Xi_1} f_{\Xi_2} \frac{\omega^{\wedge n}}{n!},$$

where the holomorphy potentials f_{Ξ_i} are normalized to be orthogonal to the constants. Note that this invariant can also be easily understood via equivariant cohomology, as it can be written as an integral of equivariantly closed forms:

$$\left\{\int_M (\omega + f_{\Xi_1}) \wedge (\omega + f_{\Xi_2})^{\wedge n+1} - \tbinom{n+1}{2}/\tbinom{n+2}{1} \int_M (\omega + f_{\Xi_2})^{\wedge n+2}\right\} / \tbinom{n+1}{1}.$$

Although this is certainly a useful observation, we will nevertheless approach this invariant somewhat differently.

Chapter 3

Cohomological Kählerian Constructions

The invariants of the previous Chapter are applied here to the study of distinguished Kähler metrics. Section 3.1 begins with a brief review of the notion of an *extremal* Kähler metric [C1] and its relation to the Futaki invariant. Such a metric has a distinguished gradient holomorphic vector field, called *extremal*, whose holomorphy potential is given by the scalar curvature minus its average value, i.e., roughly by the Laplacian of the Ricci potential. Now for *any* Kähler metric one can construct a gradient $(1,0)$-vector field from the same function. The value of the Futaki invariant on the L^2-projection of this vector field onto the space of gradient *holomorphic* vector fields is an invariant of the Kähler class. (Minus) this real number gives a lower bound on the L^2 norm of the (normalized) scalar curvature, which is achieved exactly when the metric is extremal [H2]. The role of the Futaki invariant in distinguishing classes admitting extremal Kähler metrics of non-constant versus constant scalar curvature can be deduced from this bound.

We proceed to give an analogous result for the invariant $\mathcal{B}$ defined in the beginning of the previous chapter (equation (2.1)). The primary tool used above is another Kähler class invariant $\mathcal{K}_{[\omega]}$, a bilinear form on the space of gradient holomorphic vector fields. Under this pairing the Futaki invariant is dual to an (essentially) unique vector field, which for an extremal metric coincides with its extremal vector field. We determine that $\mathcal{B}$ is likewise dual to another gradient vector field, whose holomorphy potential is the Laplacian of the ratio of the n-form determined by the Ricci form and the Kählerian volume form. (Minus) the L^2 norm of this function is bounded by the value of $\mathcal{B}$ on the vector field, again a Kähler class invariant.

We call metrics for which the above lower bound is achieved (and also their above-mentioned distinguished potential and vector field), *central*. The latter invariant number distinguishes classes admitting central metrics of constant versus non-constant central potential. In Section 3.2 we give existence results for the case of constant central potential. Unlike extremal metrics, central metrics have potentials which are either all constant, or all non-constant, throughout the Kähler cone. We describe circumstances under which the (not necessarily holomorphic) $(1,0)$-vector field associated to the scalar curvature of a central metric projects onto the (holomorphic) *central* vector field, and vice versa. This works well when the Kähler class is c_1, but can sometimes be extended to other Kähler classes.

The question of the relation of different Kähler classes is taken up in Section 3.3, where we construct a Lie algebra character associated to a *pair* of Kähler classes, termed the *reflection character*. For classes of fixed volume it turns out to equal the difference of the Futaki invariants at the two classes.

It vanishes if the classes admit two metrics having identical Ricci forms *and* identical harmonic Ricci forms. We then give a partial determination of when this can occur for extremal metrics. The case of *non-constant* extremal potential is the most difficult, and in Section 3.4 we construct examples out of known families of extremal metrics [H3].

3.1 Extremal and Central Kähler Metrics

In [C1] Calabi defined a new notion of a distinguished Kähler metric.

Definition. A Kähler metric g with Kähler form ω on a complex manifold M will be called an **extremal** Kähler metric if it is critical point of the functional

$$g \longrightarrow \int_M s_g^2 \frac{\omega^{\wedge n}}{n!} \tag{3.1}$$

among Kähler metrics in the class $[\omega]$. Here s_g denotes the scalar curvature of g.

We now specialize to the case where M is compact.

Proposition 3.1 (Calabi [C1]) *For M compact a Kähler metric is extremal if and only if its scalar curvature is a holomorphy potential, i.e.* $s{'}^{\alpha}_{,\bar\beta} = 0$.

In particular, if g is a Kähler-Einstein metric, or, more generally, any Kähler metric of constant scalar curvature, it is extremal.

The Futaki invariant has been found to contain the following information regarding Kähler metrics:

Theorem 3.2 (Futaki-Mabuchi [FMa]) *Let M be a compact Kähler manifold. For a Kähler metric g with Kähler form ω, denote by $C_0^\infty(M, \mathbb{C}, g)$ the space of all smooth functions on M g-perpendicular to the constants, and $\Gamma_0 := \Gamma_{0,g}$ the subspace of all g-holomorphy potentials also perpendicular to the constants. Then, for any Kähler metric g in a Kähler class $[\omega]$, the number $\mathcal{F}_{[\omega]}(\Xi_{\pi_g(s-s_0)})$ depends only on the Kähler class. Here s denotes the scalar curvature of g, s_0 its average value, and $\pi_g : C_0^\infty(M, \mathbb{C}, g) \to \Gamma_0$ the orthogonal projection with respect to the inner product*

$$< f, h >= \int_M f\bar{h}\frac{\omega^{\wedge n}}{n!}.$$

$\Xi_{\pi_g(s-s_0)}$ is called the **extremal vector field** of g.

Theorem 3.3 (Hwang [H1, H2], see also [Sm]) *Let M be a compact Kähler manifold, g a Kähler metric on it with Kähler form ω. Then:*

$$\int_M (s - s_0)^2 \frac{\omega^{\wedge n}}{n!} \geq -\mathcal{F}_{[\omega]}(\Xi_{\pi_g(s-s_0)}), \tag{3.2}$$

where the notations are as in Theorem 3.2. The right hand side of inequality (3.2) is real and non-negative, and equality occurs exactly when g is extremal.

Remark 3.4 *Since by Theorem 3.2 the real number on the right hand side of inequality (3.2) depends only on the Kähler class $[\omega]$, it gives, by the above, a lower bound for the left hand side over all Kähler metrics in the class.*

One can obtain as corollaries the following two results, which have actually been known earlier:

Corollary 3.5 (Calabi [C2]) *Under the conditions of Theorem 3.3, if the right hand side of inequality (3.2) vanishes (in particular if $\mathcal{F}_{[\omega]}$ is identically zero), an extremal Kähler metric in the class $[\omega]$ has constant Ricci potential (equivalently, constant scalar curvature).*

Corollary 3.6 (Futaki [Ft2]) *Under the conditions of Theorem 3.3, if there exists a Kähler metric of constant Ricci potential (or constant scalar curvature) in the class $[\omega]$, then $\mathcal{F}_{[\omega]}(\cdot) \equiv 0$.*

We introduce now a new definition of a distinguished Kähler metric, which broadens the notion of a Kähler-Einstein metric in a manner similar to, but somewhat more uniform than that of an extremal Kähler metric. All of the above results will have parallels for this new notion.

Definition. For M a compact complex manifold, a Kähler metric is called **central** if and only if the function

$$\Delta C := \Delta(\frac{\det \rho}{\det \omega}) \tag{3.3}$$

is a holomorphy potential, i.e. $(\Delta C)^{,\alpha}{}_{,\bar{\beta}} = 0$.

Note that det refers to the complex determinant, and that C is well defined because ω is non-degenerate.

Regarding c_1 as a central element in the second cohomology group of the complex manifold M (or, more precisely, in $H^{1,1}(M)$), the terminology in the definition is meant to suggest that such a metric is determined to a large degree by corresponding Kähler metrics in c_1 (– for a Fano manifold. Or, more generally, by the behaviour of the Ricci forms in c_1).

To obtain more symmetric expressions in what follows, it is convenient to define the **extremal potential** E to be twice the Ricci potential F. The function C, which we will term the **central potential**, is then the analog of the extremal potential E, and so ΔC is the analog of $s - s_0$. We have:

Theorem 3.7 *Let M be a compact Kähler manifold. Maintaining the notations of Theorem 3.2, for any Kähler metric g in a Kähler class $[\omega]$, the number $\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)})$ depends only on the Kähler class.*

We will call $\Xi_{\pi_g(\Delta C)}$ the **central vector field** of g.

Theorem 3.8 *Let M be a compact Kähler manifold with a Kähler metric g having Kähler form ω and central potential C. Then*

$$\int_M (\Delta C)^2 \frac{\omega^{\wedge n}}{n!} \geq -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}), \tag{3.4}$$

where $\pi_g : C_0^\infty(M, \mathbb{C}, g) \to \Gamma_0$ is as in Theorem 3.3. The right hand side of inequality (3.4) is real non-negative, and equality occurs exactly when g is central.

The following corollaries of Theorem 3.8 are the analogs of Corollaries 3.5 and 3.6.

Corollary 3.9 *Under the conditions of Theorem 3.8, if the right hand side of inequality (3.4) vanishes (in particular if $\mathcal{B}_{c_1}$ is identically zero), a central Kähler metric in M has constant central potential.*

Corollary 3.10 *Under the conditions of Theorem 3.8, if there exists on M a Kähler metric with constant central potential, then $\mathcal{B}_{c_1}(\cdot) \equiv 0$.*

The proofs of the above results are very similar to those of the first set of statements. We review the most important steps. The pertinent Lie theoretic background here is that the Lie algebra of gradient vector fields on a compact Kähler manifold is the Lie algebra of an algebraic group, and as such splits into a sum of a reductive subalgebra and a nilpotent radical [Fj]. Both the former's embedding and, consequently, the splitting, are non-canonical. See the relevant papers quoted below for further information.

Proof of Theorem 3.7. Let $f_{\Xi_1}, f_{\Xi_2} \in \Gamma_{0,g}$. Define a symmetric $\mathbb{C}$-bilinear form on $\Gamma_{0,g}$ by

$$\hat{\mathcal{K}}_g(\Xi_1, \Xi_2) = \int_M f_{\Xi_1} f_{\Xi_2} \frac{\omega^{\wedge n}}{n!}.$$

Futaki and Mabuchi showed in [FMa] that within a given Kähler class, $\hat{\mathcal{K}}_g$ is independent of the choice of Kähler metric, and so defines a bilinear form $\hat{\mathcal{K}}_{[\omega]}$ on gradient vector fields. Following Hwang [H2] we extend this form to all holomorphic vector fields by fixing an arbitrary (non-degenerate) bilinear form on nowhere vanishing vector fields, and declaring the latter orthogonal to the gradient vector fields. We denote the resulting bilinear form $\mathcal{K}_{[\omega]}$. Let ρ be the Ricci form of g, note that C is real valued and satisfies $\rho^{\wedge n} = C\omega^{\wedge n}$. We now have, for any gradient vector field Ξ_f:

$$\begin{aligned} -\mathcal{B}_{c_1}(\Xi_f) &= \int_M (\Delta f_\Xi) \frac{\rho^{\wedge n}}{n!} = \\ \int_M (\Delta f_\Xi) C \frac{\omega^{\wedge n}}{n!} &= \int_M f_\Xi \Delta C \frac{\omega^{\wedge n}}{n!} = \\ \int_M f_\Xi \, \pi_g(\Delta C) \frac{\omega^{\wedge n}}{n!} &= \mathcal{K}_{[\omega]}(\Xi_f, \Xi_{\pi_g(\Delta C)}), \end{aligned}$$

where we have used integration by parts, and then in the penultimate step, the orthogonality of the projection π_g. (By the construction of $\mathcal{K}_{[\omega]}$, and because (as we will see) $\mathcal{B}_{c_1}$ vanishes on nowhere vanishing vector fields, both sides of the above equation are zero for such vector fields, and so these can be safely neglected). Therefore, because for any g belonging to the Kähler class, $\Xi_{\pi_g(\Delta C)}$ is $\mathcal{K}_{[\omega]}$-dual to the fixed functional $-\mathcal{B}_{c_1}$, as g varies in $[\omega]$ this vector field can only vary in the subspace of $h(M)$ on which $\mathcal{K}_{[\omega]}$ degenerates, i.e. it is fixed up to an additive element of the nilpotent radical of $h(M)$: in [FMa] it is shown that $\mathcal{K}_{[\omega]}$ has this degeneracy subspace for metrics having a maximal compact group of isometries, and since this subspace, as well as $\mathcal{K}_{[\omega]}$, are metric independent, this follows for any metric in the class. But $-\mathcal{B}_{c_1}$ vanishes on such elements – as in [Mb1], or, alternatively, since the invariance of this character implies that it is invariant under the adjoint action of the identity component of the group of biholomorphisms (as in, e.g., [C2]). Its codimension one kernel is therefore also invariant under the adjoint action, and so induces the zero functional after dividing by the kernel of this action, i.e. the center of the reductive part of the Lie algebra. The quotient includes the nilpotent radical (and also the nowhere vanishing vector fields). Thus although $\Xi_{\pi_g(\Delta C)}$ may vary with the metric, $-\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)})$ does not, and we are done. □

In the rest of this chapter we will ignore the slight ambiguity in the definitions of the central and extremal vector fields, as we have just seen that these do not cause an ambiguity in the value of the above lower bounds.

Proof of Theorem 3.8. We write the L^2_g-orthogonal decomposition of ΔC as

$$\Delta C = \pi_g(\Delta C) + T. \tag{3.5}$$

Then,

$$\begin{aligned} 0 \leq < \pi_g(\Delta C), \pi_g(\Delta C) > &= < \pi_g(\Delta C), \Delta C - T >= \\ < \pi_g(\Delta C), \Delta C > &= < \Delta(\pi_g(\Delta C)), C >= \\ -\int_M \mathrm{div}_g \Xi_{\pi_g(\Delta C)} C \frac{\omega^{\wedge n}}{n!} = -\int_M \mathrm{div}_g \Xi_{\pi_g(\Delta C)} \frac{\rho^{\wedge n}}{n!} &= -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}), \end{aligned}$$

where first we have used the orthogonality of the relation (3.5), then the fact that ΔC was real-valued meant that conjugation was unnecessary in the integral, and finally integration by parts. Combining this again with the orthogonality of (3.5), we have

$$\begin{aligned} ||\Delta C||_g^2 &= ||\pi_g(\Delta C)||_g^2 + ||T||_g^2 = \\ -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}) + ||T||_g^2 &\geq -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}), \end{aligned}$$

as required. Equality here exactly means $T = 0$, or $\Delta C = \pi_g(\Delta C)$, so g is central. □

Remark 3.11 *In the next section we will show that there are indeed cases where the bound on $||\Delta C||_g^2$ given by Theorem 3.8 is non-trivial.*

3.2 Some Existence Results

Consideration of extremal Kähler metrics can be divided into two main cases: constant and non-constant scalar curvature. Existence questions in both are challenging, and not completely solved. In contrast, for central Kähler metrics having constant central potential, the question of existence has been *implicitly* understood for some time.

Theorem 3.12 *Let M be a complex manifold admitting a central Kähler metric of constant central potential. Then M admits such metrics in every Kähler class.*

Proof. Let g be a central metric with Kähler form ω and Ricci form ρ. Given a fixed Kähler class, let $\tilde{g}$ be the unique Calabi-Yau metric in it having Ricci form $\tilde{\rho}$ equal to ρ. Then in particular $\det\tilde{\rho} = \det\rho$, and, again by the Calabi-Yau Theorem, $\det\tilde{\omega} = A\det\omega$ for some positive constant A. Therefore, the central potential $\tilde{C}$ of $\tilde{g}$ satisfies $\tilde{C} = \frac{1}{A}C$, and so is constant. □

Note also that the sign of $\tilde{C}$ (if non-zero) is the same as that of C.

Kähler-Einstein metrics certainly have constant central potential, but even if a manifold does not admit any Kähler-Einstein metrics, it could admit metrics of constant central potential. For example, generalized Kähler-Einstein metrics in the sense of [Mt] (i.e. metrics with eigenvalues of the Ricci tensor constant with respect to the metric) have constant central potential. We also mention that every central Kähler metric of constant non-zero central potential has a symplectic Ricci form. We will show that more is true later on.

Thus, for example, the Kähler cone of a manifold admitting a Kähler-Einstein metric (or having $\mathcal{B} \equiv 0$ and admitting one central metric) is completely filled with central Kähler representatives, but none have non-constant central potential, by Corollary 3.9. $\mathcal{B}$ not identically zero, on the other hand, implies that every central metric must have non-constant central potential, by Corollary 3.10. This should be contrasted with the behavior of extremal Kähler metrics.

Although the case of non-constant central metrics seems more difficult, we will see shortly that known results about extremal metrics can be used to show at least that non-zero central vector fields do exist.

Lemma 3.13 *The following relation holds between the extremal and the central vector fields, of any Kähler metric:*

$$\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) = \mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta E)}). \tag{3.6}$$

Proof. We use orthogonality, reality, and integration by parts, as before.

$$\begin{aligned} \mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) &= \\ -\int_M \pi_g(\Delta C)\Delta E\frac{\omega^{\wedge n}}{n!} &= -\int_M \pi_g(\Delta C)\pi_g(\Delta E)\frac{\omega^{\wedge n}}{n!} = \\ -\int_M \pi_g(\Delta E)\Delta C\frac{\omega^{\wedge n}}{n!} &= -\int_M \Delta(\pi_g(\Delta E))C\frac{\omega^{\wedge n}}{n!} = \\ -\int_M \Delta(\pi_g(\Delta E))\frac{\rho^{\wedge n}}{n!} &= \mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta E)}). \end{aligned}$$

□

Combining this with Proposition 2.6, we get:

Corollary 3.14 *For any Kähler metric in a class $[\omega]$ on a Fano manifold M,*

$$\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) = \mathcal{F}_{c_1}(\Xi_{\pi_g(\Delta E)}). \tag{3.7}$$

Proposition 3.15 *There exist manifolds with Kähler metrics satisfying*

$$-\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}) > 0.$$

Proof. Let M be a Fano manifold having an extremal Kähler metric of non-constant scalar curvature in the class c_1. An example of such a manifold would be the one point blow-up of CP^2 (cf. [C1]). Taking $[\omega] = c_1$, Corollary 3.14 gives

$$-\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}) = -\mathcal{F}_{c_1}(\Xi_{\pi_g(\Delta C)}) = -\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) = -\mathcal{F}_{c_1}(\Xi_{\Delta E}) > 0,$$

by Corollary 3.5. □

Corollary 3.14 also implies:

Theorem 3.16 *Let g be a Kähler metric with Kähler form in c_1. Then, if g is central and extremal, its central and extremal vector fields coincide.*

Proof. Let E and C be the extremal potential and central potential of g, respectively. By (3.7),

$$\mathcal{F}_{c_1}(\Xi_{\Delta C} - \Xi_{\Delta E}) = \mathcal{F}_{c_1}(\Xi_{\Delta C - \Delta E}) = 0. \tag{3.8}$$

Suppose $\Xi_{\Delta E}$ and $\Xi_{\Delta C}$ span a real 2-dimensional subspace in the space of gradient (holomorphic) Killing vector fields. The restriction of the bilinear form $\mathcal{K}_{c_1}(\cdot,\cdot)$ to this subspace is a positive definite inner product. Now $\Xi_{\Delta C - \Delta E}$ is also a nonzero vector in this subspace, and (minus) the relation (3.8) can be rewritten in two ways:

$$\mathcal{K}_{c_1}(\Xi_{\Delta C - \Delta E}, \Xi_{\Delta C}) = \mathcal{K}_{c_1}(\Xi_{\Delta C - \Delta E}, \Xi_{\Delta E}) = 0. \tag{3.9}$$

Since a nonzero vector cannot be orthogonal to all vectors of a basis, $\Xi_{\Delta C}$ and $\Xi_{\Delta E}$ coincide up to constant multiple. But then relations (3.9) force them to coincide exactly (even if one of them was zero). □

It is also worth noting that some information can be drawn with even weaker assumptions. For example:

Proposition 3.17 *Let g be a Kähler metric with Kähler form in c_1. Using previous notations, if g is central, then $\Xi_{\pi_g(\Delta E)} = \Xi_{\Delta C}$. A similar relation holds when g is extremal.*

Proof. Suppose g is central, and let E, C and π_g be as above. We will see that since ΔC is a real holomorphy potential, an argument essentially analogous to the previous one works. We write the information contained in equation (3.7) in the following asymmetric form:

$$\mathcal{K}_{c_1}(\Xi_{\Delta C - \pi_g(\Delta E)}, \Xi_{\Delta C}) = 0,$$

$$\int_M (\Delta C - \pi_g(\Delta E))\pi_g(\Delta E)\frac{\omega^{\wedge n}}{n!} = 0.$$

Using only the first equation, and taking $\pi_g(\Delta E) + T$ to be the orthogonal splitting of ΔE with respect to the hermitian inner product on $C_0^\infty(M, \mathbb{C}, g)$, we have:

$$\int_M (\Delta E - T - \Delta C)\Delta C\frac{\omega^{\wedge n}}{n!} = 0.$$

Now even though T might be complex valued, it is in any case orthogonal to the entire space of (normalized) holomorphy potentials. Also ΔC is *real*, so the above equation can be regarded as expressing a *hermitian* inner product. So we see that the term $T\Delta C$ drops out, and we are left with

$$\int_M \Delta E \Delta C \frac{\omega^{\wedge n}}{n!} = \int_M (\Delta C)^2 \frac{\omega^{\wedge n}}{n!}.$$

Since this relation holds between two *real* functions, it implies that ΔE projects onto ΔC in the real subspace generated by both. To see that this projection is the same as π_g, we adjoin the second equation, and get:

$$\int_M (\pi_g \Delta E)^2 \frac{\omega^{\wedge n}}{n!} = \int_M \Delta E \Delta C \frac{\omega^{\wedge n}}{n!} = \int_M (\Delta C)^2 \frac{\omega^{\wedge n}}{n!},$$

and we are done. The proof in the case where g is extremal is similar. □

One might suppose, e.g., that centrality implies extremality, at least if $[\omega] = c_1$. The exact situation is possibly more delicate, even in the constant central case.

Proposition 3.18 *If an extremal Kähler metric on a Fano manifold M, having Kähler class c_1, has constant scalar curvature, then it is central. If a central Kähler metric, having Kähler class c_1, has constant non-zero central potential, then it is extremal if and only if it is Kähler-Einstein.*

Proof. The first part follows simply because a constant scalar curvature Kähler metric in c_1 is Kähler-Einstein. This is well known, see [Ft2, Lemma 2.2.3], or [Ft4]. On the other hand, if g is (constant central and) Kähler-Einstein, it is clearly extremal. Suppose now that g is central, with constant non-zero central potential C, and is not Kähler-Einstein. The (constant) centrality implies that $\mathcal{F}_{c_1}(\cdot) \equiv 0$, so if g were extremal, it would have to have constant scalar curvature. But this cannot happen, by the first part. □

We can characterize such central metrics somewhat more precisely.

Theorem 3.19 *A central Kähler metric g having constant non-zero Ricci potential C has Einstein-symplectic Ricci form (for this terminology see [Mb1], or the proof).*

Proof. For such g, its Ricci form ρ is symplectic, so $\log C$ is well defined. We have:

$$0 = i\partial\bar{\partial}\log C = i\partial\bar{\partial}\log\det(\rho) - i\partial\bar{\partial}\log\det(\omega) = i\partial\bar{\partial}\log\det(\rho) + \rho.$$

The above equation precisely says that ρ is Einstein-symplectic. □

3.3 Reflection, Transference

We can extend some of the previous conclusions to other Kähler classes.

Theorem 3.20 *On a Kähler manifold M, suppose that for some Kähler class $[\omega]$, $\mathcal{F}_{[\omega]}(\cdot) \equiv \mathcal{B}(\cdot)$. Then the central and extremal vector fields of any Kähler metric that is both extremal and central coincide.*

Proof. Lemma 3.13 and the premise together yield:

$$\mathcal{F}_{[\omega]}(\Xi_{\Delta C}) = \mathcal{B}_{c_1}(\Xi_{\Delta E}) = \mathcal{F}_{[\omega]}(\Xi_{\Delta E}).$$

From here the proof continues just as in Theorem 3.16, with $\mathcal{F}_{[\omega]}$ replacing $\mathcal{F}_{c_1}$. □

To proceed further in relating distinguished Kähler metrics of different Kähler classes (but this time, mostly extremal), we define the following invariant.

Definition. Let M_n be a Kähler manifold, Ω and $\tilde{\Omega}$ two Kähler classes. Let ω be a Kähler form in Ω with Calabi-Yau representative $\tilde{\omega}$ in $\tilde{\Omega}$. Define the **reflection potential** Φ of the pair $(\omega, \tilde{\Omega})$ to be the smooth real valued function given up to an additive constant by

$$\rho_H - \tilde{\rho_H} = i\partial\bar{\partial}\Phi,$$

where ρ_H, $\tilde{\rho_H}$ are the ω-harmonic and $\tilde{\omega}$-harmonic representatives, respectively, in the class c_1. The $\Omega \to \tilde{\Omega}$ **reflection character** is defined to be the Lie algebra character $\mathcal{R}_{\Omega}^{\tilde{\Omega}} : h(M) \to \mathbb{C}$, where $h(M)$ is the Lie algebra of holomorphic vector fields on M, given by

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi) = \int_M \Xi(\Phi)\frac{\omega^{\wedge n}}{n!}. \tag{3.10}$$

We call $(\omega, \tilde{\omega})$ a **Calabi-Yau (metric) pair**, and say they form a **harmonic Calabi-Yau pair** if $\rho_H = \tilde{\rho_H}$.

We proceed immediately to show that this invariant is well defined.

Proposition 3.21 *Keeping notations as in the definition, $\mathcal{R}_{\Omega}^{\tilde{\Omega}}$ does not depend on the choice of ω in Ω. Furthermore, we have $\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv -A\mathcal{R}_{\tilde{\Omega}}^{\Omega}$, where $A := A_{\tilde{\Omega}}^{\Omega} = \frac{\Omega^{\wedge n}}{\tilde{\Omega}^{\wedge n}}$ (the volume ratio of the classes).*

Proof. Since ω, $\tilde{\omega}$ are Calabi-Yau related, one has $A\frac{\tilde{\omega}^{\wedge n}}{n!} = \frac{\omega^{\wedge n}}{n!}$ $(A > 0)$. Also, if F and $\tilde{F}$ are the Ricci potentials of ω and $\tilde{\omega}$, respectively, we have, $i\partial\bar{\partial}\Phi = \rho_H - \tilde{\rho_H} = \rho_H - \rho + \rho - \tilde{\rho_H} = \rho_H - \rho + \tilde{\rho} - \tilde{\rho_H} = -i\partial\bar{\partial}F + i\partial\bar{\partial}\tilde{F}$. Choosing $\Phi = -F + \tilde{F}$, and using the volume form proportionality of the pair, we get

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi) = \int_M \Xi(\Phi)\frac{\omega^{\wedge n}}{n!} = -\int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} + \int_M \Xi(\tilde{F})\frac{\omega^{\wedge n}}{n!} =$$

$$-\int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} + A\int_M \Xi(\tilde{F})\frac{\tilde{\omega}^{\wedge n}}{n!} = -\mathcal{F}_{[\omega]}(\Xi) + A\mathcal{F}_{[\tilde{\omega}]}(\Xi),$$

which is an expression depending only on Kähler classes. The last statement follows from the definition of A together with the relation $\Phi_{\omega,\tilde{\Omega}} = -\Phi_{\tilde{\omega},\Omega}$. □

Remark 3.22 *Note that on Kähler classes of metrics of fixed volume, the reflection character satisfies cocycle condition:*

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv -\mathcal{R}_{\tilde{\Omega}}^{\Omega}$$
$$\mathcal{R}_{\Omega}^{\tilde{\Omega}} + \mathcal{R}_{\tilde{\Omega}}^{\tilde{\tilde{\Omega}}} \equiv \mathcal{R}_{\Omega}^{\tilde{\tilde{\Omega}}}.$$

For simplicity we will work only with such Kähler classes in what follows.

Proposition 3.23 *Let M be a compact Kähler manifold and Ω, $\tilde{\Omega}$ two Kähler classes (having the same volume). Suppose $\omega \in \Omega$ is the Kähler form of a Kähler metric g, $\tilde{\omega} \in \tilde{\Omega}$ the Kähler form of its Calabi-Yau related $\tilde{g}$, ρ their shared Ricci form, F, $\tilde{F}$ the corresponding Ricci potentials and Φ their reflection potential. We will assume these potentials are normalized to be orthogonal to the constants. Then:*

A *If $(\omega, \tilde{\omega})$ form a harmonic pair then $\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv 0$.*

B *If $\Delta\Phi$ is a holomorphy potential and $\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi_{\Phi}) = 0$ then $(\omega, \tilde{\omega})$ form a harmonic pair.*

C *If $(\omega, \tilde{\omega})$ form a harmonic pair then g has constant scalar curvature if and only if $\tilde{g}$ has constant scalar curvature.*

D *If* $(\omega, \tilde{\omega})$ *form a harmonic pair, and* g *is extremal, then* $\tilde{g}$ *is extremal with respect to the same holomorphic vector field, if and only if* $\Delta^2 F = \tilde{\Delta}^2 F + D$*, where* D *is a constant vanishing at least when* $c_1^{\wedge n} \neq 0$.

E *If both* g *and* $\tilde{g}$ *are extremal with respect to the same holomorphic vector field, then* $(\omega, \tilde{\omega})$ *form a harmonic pair if and only if* $\Delta^2 F = \Delta^2 \tilde{F}$.

Proof. **A** and **B** are proved in the same way as for the Futaki character. **C** follows since the assumptions imply $\tilde{\rho_H} = \rho_H = \rho = \tilde{\rho}$. For **D**, $\Phi = 0$ implies $F = \tilde{F}$, and by Proposition 1.1,

$$\bar{\partial}\Delta^2 F = \imath_{\Xi_{\Delta F}}\rho = \imath_{\Xi_{\Delta F}}\tilde{\rho}. \tag{3.11}$$

Now if $\tilde{g}$ is extremal (with respect to the same vector field), then $\imath_{\Xi_{\Delta F}}\tilde{\rho} = \bar{\partial}\tilde{\Delta}^2\tilde{F} = \bar{\partial}\tilde{\Delta}^2 F$, so combining with (3.11), $\Delta^2 F$ and $\tilde{\Delta}^2 F$ differ by a constant. If this is assumed, on the other hand, and again combined with equation (3.11), it results in $\imath_{\Xi_{\Delta F}}\tilde{\rho} = \bar{\partial}\tilde{\Delta}^2\tilde{F}$, which, via a second use of Proposition 1.1, implies that $\tilde{g}$ is extremal with respect to $\Xi_{\Delta F}$. Given this equivalence, since $\Delta^2 F$ and $\Delta^2 \tilde{F}$ have integrals with respect to (the shared) ρ^n both equal to $-\mathcal{B}(\Xi_{\Delta F})$, they can only differ by a constant if $c_1^n \neq 0$. Finally, to show **E**, we note in one direction again that $\Phi = 0$ implies $F = \tilde{F}$. The assumption in the other direction means that $\Delta(\Delta\tilde{F} - \Delta F) = 0$, so the difference inside the brackets must be a constant. Integrating with respect to $\omega^{\wedge n}$ and using the Divergence Theorem eliminates the constant. Repeating this argument eliminates the remaining Laplacian, and so we see that $F = \tilde{F}$, $\Phi = 0$, and the pair is harmonic. □

In the next section we give examples of harmonic Calabi-Yau pairs of extremal metrics.

3.4 Extremal Pairs

In [H3], families of extremal Kähler metrics are constructed in a manner analogous to the one given in [Ki, KS1, KS2] for the Kähler-Einstein case. We demonstrate the existence of extremal Calabi-Yau pairs using only special cases of these constructions.

We begin with a summarized description of the construction. We refer the reader to these papers, as well as to [HS], for further details. Let $p : (L,h) \to N = M \times M$ be a holomorphic Hermitian line bundle over a product of two copies of a Kähler-Einstein manifold M of positive Ricci curvature, second Betti number equal to one and dimension l. Let ω be an indivisible integral Kähler-Einstein form on M with Ricci form $c_1(M,\omega) = k\omega$, $k > 0$. Write $c_1(L,h) = n\omega_1 + n\omega_2$, with $n \neq 0$, where $\omega_i = \pi_i^*\omega$ and π_i the corresponding projections on the factors of N, and take B to be the symmetric two-tensor associated with $2\pi c_1(L,h)$ via the complex structure on N. Fix positive real numbers a_1, a_2 and b with $a_i \pm bn > 0, a_1 \neq a_2$. Let g_N be the Kähler metric on N with Kähler form $a_1\omega_1 + a_2\omega_2$. The Ricci tensor r_N of g_N has constant eigenvalues $\frac{k}{a_1}, \frac{k}{a_2}$, each of multiplicity l with respect to g_N, and B has eigenvalues $\frac{n}{a_1}, \frac{n}{a_2}$, also of multiplicity l.

Define two functions

$$Q(x) := \det(I - xg_n^{-1}B) = (1 - \frac{n}{a_1}x)^l(1 - \frac{n}{a_2}x)^l,$$

$$T(x) := tr_{g_N - xB}r_N = \frac{kl}{a_1 - nx} + \frac{kl}{a_2 - nx}.$$

Q and $T \cdot Q$ are everywhere defined and positive on $(-b, b)$. To emphasize the dependence on the Kähler class, we will sometimes write Q_{a_1,a_2}, T_{a_1,a_2}.

Now use Q and T to define $\phi : [-b, b] \to \mathbb{R}$ by

$$(\phi Q)(x) = 2(x + b)Q(-b) - 2\int_{-b}^{x}(\sigma_0 + \lambda y - T(y))(x - y)Q(y)\,dy,$$

where the constants σ_0 and λ can be written in terms of b, a_i, and n, by solving the equations

$$\sigma_0\alpha_0 + \lambda\alpha_1 = Q(b) + Q(-b) + \int_{-b}^{b} T(x)Q(x)\,dx,$$

$$\sigma_0\alpha_1 + \lambda\alpha_2 = b(Q(b) + Q(-b)) + \int_{-b}^{b} xT(x)Q(x)\,dx.$$

Here $\alpha_i = \int_{-b}^{b} x^i Q(x)\,dx, i = 0, 1, 2$. ϕ is smooth on [-b,b], non-negative, zero exactly at the endpoints, and satisfies $\phi'(\pm b) = \mp 2$.

The above data determine a metric

$$g = dt^2 + (dt \cdot J)^2 + p^*g_N - up^*B := dt^2 + (dt \cdot J)^2 + g_t$$

on the complement L_0 of the zero section in L, which extends to an extremal Kähler metric on the compactification $\mathbb{P}(\mathbb{L} \oplus \mathbb{C})$ of L_0. Here J is the complex structure on L_0, and the two functions $u : L_0 \to [-b, b]$ and $t : L_0 \to (0, \int_{-b}^{b} \frac{dx}{\sqrt{\phi(x)}})$ are recovered from ϕ by precomposing the hermitian norm on L_0 with

$$\log r = \int_{-b}^{u(r)} \frac{dx}{\phi(x)}, \quad t(r) = \int_{-b}^{u(r)} \frac{dx}{\sqrt{\phi(x)}},$$

respectively. t measures the distance from a section of $\mathbb{P}(\mathbb{L} \oplus \mathbb{C})$ (the one that corresponds to zero hermitian norm in L), u is the moment map for the S^1-part of the natural $\mathbb{C}^*$-action on L_0, and if H is the real gradient vector field generating the $\mathbb{R}^+$-part of this action, $\phi(u) = g(H, H)$. The map $(L_0, g) \to (L_0/S^1 \approx N \times (0, \infty), dt^2 + g_t)$ is a Riemannian submersion.

The explicitness of the description of g allows one to give local coordinate expressions for the various quantities of interest. If z_0 is a fiber coordinate such that $\frac{\partial}{\partial z_0}$ is the generator of the C^*-action, and $z_1, ..., z_{2l}$ are coordinates on N, then on a *fiber where* $\frac{\partial u}{\partial z_i} = 0, \;\; i = 1, ..2l$, we get

$$g_{0\bar{0}} = 2\phi(u), \quad g_{0\bar{\beta}} = 0, \quad g_{\alpha\bar{\beta}} = (g_N)_{\alpha\bar{\beta}} - uB_{\alpha\bar{\beta}}.$$

For the Ricci tensor one has

$$r_{0\bar{0}} = -\phi(\phi' + \frac{Q'}{Q}\phi)'(u), \quad r_{0\bar{\beta}} = 0, \quad r_{\alpha\bar{\beta}} = (r_N)_{\alpha\bar{\beta}} + \frac{1}{2}(\phi(\log(\phi Q))')(u)B_{\alpha\bar{\beta}},$$

where the prime denotes differentiation with respect to u. The scalar curvature is now

$$s(u) = T(u) - \frac{1}{2Q(u)}(\phi Q)''(u). \tag{3.12}$$

Finally we record the following expressions that hold for any smooth function $f : [-b, b] \to \mathbb{R}$:

$$\int_{\mathbb{P}(\mathbb{L}\oplus\mathbb{C})} f(u)\, \mathrm{dvol}(g) = 2\pi \mathrm{Vol}(N, g_N) \int_{-b}^{b} f(x)Q(x)\, dx,$$

$$\text{grad } f(u) = f'(u)H, \quad \Delta f(u) = \frac{(\phi Q f')'}{2Q}(u).$$

The condition for such a function to be a holomorphy potential is simply that it be affine in u, a condition that can be verified directly for s from the expression (3.12). Thus one obtains extremal Kähler metrics in every Kähler class of the Kähler cone of $\mathbb{P}(\mathbb{L} \oplus \mathbb{C})$. These will not in general be (products and not be) of constant scalar curvature, since the required extra condition $\lambda = 0$ is only obtained in (at most) a real-algebraic hypersurface of the Kähler cone.

To obtain extremal harmonic Calabi-Yau pairs from this family, we make the simple observation that

$$Q_{a_1,a_2} = Q_{a_2,a_1}, \quad T_{a_1,a_2} = T_{a_2,a_1}.$$

Going through the above expressions in succession we see that σ_0, λ, then ϕ, u, t and finally the Ricci tensor, s and Δs all remain invariant under this permutation of the a_i's. So g_{a_1,a_2} and g_{a_2,a_1}, for every allowable value of the a_i's form an extremal harmonic Calabi-Yau pair, usually of non-constant scalar curvature. The expression for the Laplacian of a function of u together with the coincidence of the scalar curvatures of the two metrics show that condition **E** in Proposition 3.23 is verified, so the reflection potential of the two metrics is zero, and then by condition **A** of the same proposition,

$$\mathcal{R}_{g_{a_1,a_2}}^{g_{a_2,a_1}} \equiv 0.$$

Chapter 4

Einstein Metrics Conformal to Extremal Kähler Metrics

The lower bound on the L^2 norm of the (normalized) scalar curvature, which is achieved, as described in the beginning of the previous chapter, by extremal Kähler metrics, is applied in what follows to a uniqueness question regarding non-Kähler Einstein metrics conformal to extremal Kähler metrics. Section 4.1 describes background results in which such metrics are shown to exist on at most three compact complex surfaces [Dr, Le]. We then recall that varying over Kähler classes, such metrics are conformal to extremal Kähler metrics that are critical points of the Kähler class infimum of the L^2 norm of the scalar curvature. Now since for the first of these surfaces, the one point blow-up of $\mathbb{C}P^2$, a unique such metric is known to exists up to scaling and isometries [P], we turn to the next case, the two point general position blow-up of $\mathbb{C}P^2$. We give a moment theoretic method of computing the above L^2 norm. After a possible biholomorphism, an extremal Kähler metric gives rise to a standardized moment map, with respect to which we view our space as a Hamiltonian T^2-space. Choosing an appropriate basis for the torus, the image of the moment map can be described explicitly as a pentagon, for which (a multiple of) Lebesgue measure equals the push-forward of the Kählerian Liouville measure. It follows that the bilinear form $\mathcal{K}_{[\omega]}$ on basis elements can be computed as elementary integrals on the pentagon. Combining this with the Futaki invariant formulas of Chapter 2 and some linear algebra, one gets a cohomological formula for the Futaki invariant on the extremal vector field, and consequently for the required L^2 norm. We carry this out explicitly in Section 4.2, with the factorization of the final expression provided by a symbolic computational program. This part of the method works also for the last possible space, the three point general position blow-up of $\mathbb{C}P^2$, whose moment map image is a hexagon. We summarize and then include in an appendix the computer-assisted proof of the existence of a unique critical point, which implies a type of topological uniqueness for such Einstein metrics.

4.1 Background

An interesting relation between extremal Kähler metrics and Hermitian Einstein metrics was described in [Dr].

Theorem 4.1 (Derdzinski [Dr]) *Let (M, h) be a compact oriented Einstein four manifold for which W^+ has less than three eigenvalues, and is not parallel. Then, up to a double covering, M is topologically $S^2 \times S^2$ or $\mathbb{C}P^2\#(-k\mathbb{C}P^2)$, $0 \leq k \leq 8$, and h is Hermitian non-Kähler, and conformal to an extremal Kähler metric of non-constant and positive scalar curvature.*

Here W^+ is the self-dual Weyl curvature of h, regarded as a symmetric trace-free endomorphism of the rank 3 bundle Λ^+ of self-dual 2-forms on M.

This result was refined in [Le] in a number of ways. First, it was pointed out that a result of Goldberg-Sachs [GoS], adopted to the Riemannian signature, allows one to drop the first assumption on W^+, if one assumes the Einstein metric is Hermitian. Second, as the Hermitian property of h is in any case implied in the theorem, the original result in [Dr] also contained a biholomorphic classification, which was improved by a direct differential geometric argument as follows:

Theorem 4.2 (LeBrun [Le]) *Let (M^4, J) be a compact complex surface. If M admits a J-Hermitian non-Kähler Einstein metric h, then (M, J) is obtained from $\mathbb{C}P^2$ by blowing up one, two or three points in general position (i.e., no two points coincide and no three are collinear). Moreover, the isometry group of h contains a 2-torus.*

Of these three cases, the one point blow-up is known to admit such a metric [P]. Regarding the two and three point blow-ups, all that is known is information about the existence of Kähler-Einstein metrics on these spaces. Namely, the two point blow-up cannot admit such a metric (by a theorem of Lichnerowicz [Lc]), and the three point blow-up admits a Kähler-Einstein metric [Su, TY].

With regards to uniqueness, it is known that the Page metric [P] is the unique one up to isometry and rescaling satisfying the hypothesis of Theorem 4.2. We further develop here a method initiated in [Le] to help settle the issue in the remaining two cases, especially the two point blow-up. It is worth mentioning that a final resolution of the problem would also yield an answer to the following question asked in [Le]: If a compact complex surface admits a Kähler-Einstein metric, is every Hermitian Einstein metric on it also Kähler?

We first review the basic idea. Let h be as above, g the extremal Kähler metric conformal to it, ω the Kähler form of g and s its scalar curvature. Now h, being Einstein, is a critical point over the space of all Riemannian metrics on M, of the functional $\int_M |W_h^+|^2 d\mu_h$. As this functional is conformally invariant, g is also such a critical point. But g is also Kähler, and therefore satisfies $24\int_M |W_g^+|^2 \frac{\omega^{\wedge 2}}{2} = \int_M s_g^2 \frac{\omega^{\wedge 2}}{2} := \mathcal{A}$, and so can be regarded as critical with respect to the latter functional. See [Be, Chapter 1, Sections G and H] and [Dr] for the facts used here.

Now g is also extremal, so $[\omega]$ has a neighborhood in $H^{1,1}(M)$ of classes which admit extremal Kähler representatives [LSm]. Restricting $\mathcal{A}$ to these metrics, g is still critical, and so is $[\omega]$, if $\mathcal{A}$ is regarded as a functional on the Kähler cone in $H^{1,1}(M)$. One now proceeds to compute this functional and attempt to determine its critical points, with M being one of the spaces in Theorem 4.2.

To this end, one uses inequality (3.2), which is an equality for the extremal metric g, and is rewritten:

$$\mathcal{A} := \int_M s^2 \frac{\omega^{\wedge 2}}{2} = \int_M s_0^2 \frac{\omega^{\wedge 2}}{2} - \mathcal{F}_{[\omega]}(\Xi_s), \tag{4.1}$$

where Ξ_s is the extremal vector field of g. As the term involving s_0 is well known (from equation (2.10)) to be $32\pi^2 \frac{(c_1\cdot[\omega])^2}{[\omega]^2}$, we are left with the task of computing the Futaki invariant at the extremal vector field. By Theorem 3.2 this is a Kähler class invariant. It can be computed via the methods of Chapter 2. We will however, proceed differently. First note, that for the spaces we are considering, the Lie algebra of holomorphic vector fields has as center of its reductive part an abelian (real) two dimensional subalgebra, to which Ξ_s belongs [FMa]. Suppose Ξ_1, Ξ_2 form a basis for this subalgebra, and $\Xi_s = \alpha\Xi_1 + \beta\Xi_2$, for some real constants α, β. Recall that associated to each Kähler class there is a symmetric bilinear form $\mathcal{K}_{[\omega]}$ on holomorphic vector fields with respect to which Ξ_s is (essentially) dual to $\mathcal{F}_{[\omega]}$, (See Theorem 3.2 and the beginning of the proof of Theorem 3.7). This duality is completely well defined on the reductive quotient of the Lie algebra of gradient holomorphic vector fields, and so in particular when $\mathcal{K}_{[\omega]}$ is restricted to this subalgebra.

We therefore have the equations:

$$\mathcal{F}_{[\omega]}(\Xi_1) = \mathcal{K}_{[\omega]}(\Xi_1, \Xi_s) = \alpha\mathcal{K}_{[\omega]}(\Xi_1, \Xi_1) + \beta\mathcal{K}_{[\omega]}(\Xi_1, \Xi_2),$$

$$\mathcal{F}_{[\omega]}(\Xi_2) = \mathcal{K}_{[\omega]}(\Xi_2, \Xi_s) = \alpha\mathcal{K}_{[\omega]}(\Xi_2, \Xi_1) + \beta\mathcal{K}_{[\omega]}(\Xi_2, \Xi_2).$$

We can solve for α, β in terms of the $\mathcal{K}_{[\omega]}(\Xi_i, \Xi_j), i, j = 1, 2$. By Cramer's Rule,

$$\alpha = \frac{\det \mathcal{K}^{\alpha}_{[\omega]}}{\det \mathcal{K}_{[\omega]}}$$

$$\beta = \frac{\det \mathcal{K}^{\beta}_{[\omega]}}{\det \mathcal{K}_{[\omega]}},$$

where

$$\det \mathcal{K}_{[\omega]} = \mathcal{K}_{[\omega]}(\Xi_1, \Xi_1)\mathcal{K}_{[\omega]}(\Xi_2, \Xi_2) - \mathcal{K}_{[\omega]}(\Xi_1, \Xi_2)^2,$$

$$\det \mathcal{K}^{\alpha}_{[\omega]} = \mathcal{F}_{[\omega]}(\Xi_1)\mathcal{K}_{[\omega]}(\Xi_2, \Xi_2) - \mathcal{F}_{[\omega]}(\Xi_2)\mathcal{K}_{[\omega]}(\Xi_1, \Xi_2),$$

$$\det \mathcal{K}^{\beta}_{[\omega]} = \mathcal{K}_{[\omega]}(\Xi_1, \Xi_1)\mathcal{F}_{[\omega]}(\Xi_2) - \mathcal{K}_{[\omega]}(\Xi_1, \Xi_2)\mathcal{F}_{[\omega]}(\Xi_1).$$

Then, denoting for brevity $\mathcal{K}_{[\omega]}(\Xi_i, \Xi_j) = \mathcal{K}^{ij}_{[\omega]}$, we have

$$\begin{aligned}
&\mathcal{F}_{[\omega]}(\Xi_s) = \\
&\alpha\mathcal{F}_{[\omega]}(\Xi_1) + \beta\mathcal{F}_{[\omega]}(\Xi_2) = \\
&\tfrac{1}{\det \mathcal{K}_{[\omega]}} \left\{ \det \mathcal{K}^{\alpha}_{[\omega]}\mathcal{F}_{[\omega]}(\Xi_1) + \det \mathcal{K}^{\beta}_{[\omega]}\mathcal{F}_{[\omega]}(\Xi_2) \right\} = \\
&\tfrac{1}{\det \mathcal{K}_{[\omega]}} \Big\{ (\mathcal{F}_{[\omega]}(\Xi_1)\mathcal{K}^{22}_{[\omega]} - \mathcal{F}_{[\omega]}(\Xi_2)\mathcal{K}^{12}_{[\omega]})\mathcal{F}_{[\omega]}(\Xi_1) + \\
&(\mathcal{K}^{11}_{[\omega]}\mathcal{F}_{[\omega]}(\Xi_2) - \mathcal{K}^{12}_{[\omega]}\mathcal{F}_{[\omega]}(\Xi_1))\mathcal{F}_{[\omega]}(\Xi_2) \Big\} = \\
&\frac{\mathcal{K}^{22}_{[\omega]}(\mathcal{F}_{[\omega]}(\Xi_1))^2 - 2\mathcal{K}^{12}_{[\omega]}\mathcal{F}_{[\omega]}(\Xi_1)\mathcal{F}_{[\omega]}(\Xi_2) + \mathcal{K}^{11}_{[\omega]}(\mathcal{F}_{[\omega]}(\Xi_2))^2}{\mathcal{K}^{11}_{[\omega]}\mathcal{K}^{22}_{[\omega]} - (\mathcal{K}^{12}_{[\omega]})^2},
\end{aligned} \tag{4.2}$$

and so we arrive at the following expression for $\mathcal{A}$:

$$\begin{aligned}
\mathcal{A} = 32\pi^2 \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} - \\
\frac{\mathcal{K}^{22}_{[\omega]}(\mathcal{F}_{[\omega]}(\Xi_1))^2 - 2\mathcal{K}^{12}_{[\omega]}\mathcal{F}_{[\omega]}(\Xi_1)\mathcal{F}_{[\omega]}(\Xi_2) + \mathcal{K}^{11}_{[\omega]}(\mathcal{F}_{[\omega]}(\Xi_2))^2}{\mathcal{K}^{11}_{[\omega]}\mathcal{K}^{22}_{[\omega]} - (\mathcal{K}^{12}_{[\omega]})^2}.
\end{aligned} \tag{4.3}$$

Now if Ξ_1, Ξ_2 are chosen as generators of $\mathbb{C}^*$-actions, by Theorem 2.9 we have cohomological expressions for $\mathcal{F}_{[\omega]}(\Xi_j)$, $j = 1, 2$. It remains to find similar expressions for $\mathcal{K}_{[\omega]}(\Xi_i, \Xi_j)$, $i, j = 1, 2$.

For this we need to understand the moment map for the T^2-action generated by the imaginary parts of Ξ_1 and Ξ_2. Our spaces M are toric varieties, (i.e. admit an effective $\mathbb{C}^{*n}$-action, where n is the complex dimension). As will be reviewed below, this torus subgroup can be made into a group of isometries for g (from which the statement about the isometry group of h in Theorem 4.2 follows). Therefore, with respect to ω, our spaces become Hamiltonian T^2-spaces. The induced measure on the image of the moment map $\Phi : M \to t^*$ is given as follows:

Proposition 4.3 *For a Kähler toric manifold with moment map Φ, the push-forward measure $\Phi_*\frac{\omega^{\wedge 2}}{2}$ (called the* **Duistermaat-Heckman measure***) of the Liouville measure on M (i.e. the measure determined by $\frac{\omega^{\wedge 2}}{2}$) is equal to* $(2\pi)^2 \times$ (Lebesgue measure on the moment map polygon).

For a proof see for example [G, exercises 2.20 – 2.24].

Proceeding as in [Mb3, FMa], let f_1, f_2 be the real valued holomorphy potentials *orthogonal to the constants* of Ξ_1, Ξ_2, respectively. If x_1, x_2 are coordinates on t^* such that $\Phi^* x_i = f_i$, $i = 1, 2$, then, using the definition of $\mathcal{K}_{[\omega]}$ and the above proposition, we have:

$$\mathcal{K}_{[\omega]}(\Xi_i, \Xi_j) = (2\pi)^2 \int_{Im(\Phi)} x_i x_j \, dx_1 \, dx_2,$$

for $i, j = 1, 2$, and Φ the moment map given by (f_1, f_2).

Note that although the above condition on the holomorphy potentials prohibits translation of the moment map, we can always allow translation if we evaluate the right hand side via the simple substitution

$(2\pi)^2 \int_{Im(\bar{\Phi})}(x_i - \bar{x}_i)(x_j - \bar{x}_i) \, dx_1 \, dx_2 = (2\pi)^2 [\int_{Im(\bar{\Phi})} x_i x_j \, dx_1 \, dx_2 - \bar{x}_i \bar{x}_j A] =$
$(2\pi)^2 [\int_{Im(\bar{\Phi})} x_i x_j \, dx_1 \, dx_2 - \int_{Im(\bar{\Phi})} \bar{x}_i \, dx_1 \, dx_2 \int_{Im(\bar{\Phi})} \bar{x}_j \, dx_1 \, dx_2 / A]$,

where $\bar{\Phi} = \Phi + (\bar{x}_i, \bar{x}_j)$ and A is the area of the image. This is useful in choosing a convenient location for the image.

We now wish to understand the shape of the image. By the convexity Theorem [At, GuS], the image of the moment map $\Phi : M \to t^*$ is convex. Delzant [Dl] showed that Hamiltonian T^n-spaces (of dimension $2n$) are determined (up to isomorphism) by such convex polyhedra, and classified which polyhedra can occur. We note in passing that an understanding of the Futaki invariant at the first Chern class of a toric Fano manifold has been achieved, from this point of view, in [Mb2].

To get concrete formulas, we now describe the image polygons for our manifolds. For a general toric surface, these have edges with rational slope (including ∞), such that any two consecutive edges with slopes m/n and m'/n', (in reduced form) satisfy $nm' - n'm = \pm 1$ [Dl, Ka]. The pre-image of a vertex is a fixed point for the torus action, that of an interior point is a free torus orbit, and that of an edge of slope m/n is a 2-sphere fixed by the action of a subgroup $\{e^{im\theta}, e^{in\theta}\}$ of the torus.

For the space $\mathbb{C}P^2$, a moment polygon is a triangle, with vertices $(0,0)$, $(0,r)$ and $(r,0)$. Here $2\pi r = \omega(F)$, where F is a complex projective line in $\mathbb{C}P^2$. Operating on vertices by an element of $SL(2,\mathbb{Z})$ yields another admissible polygon. Although this won't change the cohomological calculation we are about to perform, we can eliminate this freedom by fixing a basis in t^* as follows. Given homogeneous coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{C}P^2$, an (effective) holomorphic T^2-action is given by $[z_0 : z_1 : z_2] \to [e^{i\theta}z_0 : e^{i\phi}z_1 : z_2]$. As the isometry group of an extremal Kähler metric is maximal compact in the identity component of the automorphism group [C2], and since the latter is unique up to conjugation, by changing homogeneous coordinates we can make g invariant under this action. Then we choose the Lie algebra basis to be the pair of Killing vector fields which generate the two circle subgroups $(e^{i\theta}, 1, 1)$ and $(1, e^{i\phi}, 1)$, and Ξ_1, Ξ_2 the corresponding vector fields.

We now proceed to describe the polygons for blow-ups of $\mathbb{C}P^2$. The simple prescription for this (see [Au, Chapter 6] or [Ka, Section 6.4]) is as follows: cut the polygon near a vertex, where "near" means that the resulting new edge intersects the two edges that formerly met at the vertex. To determine the angle that this new edge makes with the other two, choose its outward normal to be $u_i + u_{i+1}$, where u_i, u_{i+1} are primitive inward-pointing vectors (of the $\mathbb{Z}^2$ lattice in t^*), normal to the neighboring edges. For $\mathbb{C}P^2$, the general position condition means that we cut off only near the one, two or three vertices of the original triangle, getting a trapezoid, pentagon or hexagon for the one, two or three point blow-ups, respectively. The condition on the normals results in each of the new edges being parallel to an original edge of the triangle (see Figure 4.1). Their lengths are $a, b, \sqrt{2}c$, where a, b, c are (up to a 2π factor) the symplectic areas of each of the exceptional divisors (see subsequent figures). The various integrations will actually be performed by integrating over the triangle and then subtracting from the result integrals over each "chopped-off" triangle.

Note that although our spaces are originally defined in terms of blow ups in the *complex* category, as our spaces *are* T^2-Hamiltonian, we can get from them to $\mathbb{C}P^2$ by a sequence of *equivariant symplectic* blow- downs (of well chosen exceptional divisors), and therefore return to them by reversing the procedure, i.e. by equivariantly blowing up $\mathbb{C}P^2$. This justifies the above procedure for determining the corresponding polygons (of course, in any case, our calculations are purely cohomological).

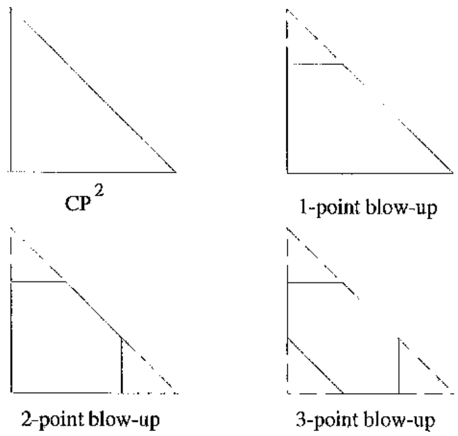

Figure 4.1: Moment polygons for the various spaces of interest.

In terms of the concrete action above, the points $[0:0:1], [0:1:0], [1:0:0]$ are fixed by the T^2-action, and so the action extends to the blow-ups at each of these points. Keeping the same basis for the Lie algebra still eliminates the $SL(2,\mathbb{Z})$ degrees of freedom in the choice of the polygon.

4.2 Uniqueness for the Two Point Blow-up

We outlined in the previous section a method for computing $\mathcal{A}$. Proceeding with the calculation for the 2-point blow-up, we saw how to reduce terms related to the bilinear form $\mathcal{K}_{[\omega]}$ to elementary integrals over a pentagon. As functions of the lengths of sides of the polygon, we have:

$$\mathcal{K}_{[\omega]}(\Xi_1,\Xi_1) = 4\pi^2 \Big\{ [\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] x^2 \, dy \, dx$$
$$-([\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] x \, dy \, dx)^2$$
$$/[\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] 1 \, dy \, dx \Big\}$$

$$= 4\pi^2(\frac{r^4}{12} - \frac{a^4}{12} + \frac{2rb^3}{3} - \frac{b^4}{4} - \frac{r^2b^2}{2} - \frac{((r^3 - a^3 - 3rb^2 - 2b^3)/6)^2}{(r^2 - a^2 - b^2)/2}),$$

$$\mathcal{K}_{[\omega]}(\Xi_1, \Xi_2) = 4\pi^2 \Big\{ [\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] xy\, dy\, dx$$

$$-([\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] x\, dy\, dx)$$

$$([\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] y\, dy\, dx)$$

$$/[\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] 1\, dy\, dx \Big\}$$

$$= 4\pi^2(\frac{r^4}{24} + \frac{a^4}{8} - \frac{rb^3}{6} + \frac{b^4}{8} - \frac{ra^3}{6} -$$

$$\frac{((r^3 - a^3 - 3rb^2 + 2b^3)/6)((r^3 - b^3 + 2a^3 - 3ra^2)/6)}{(r^2 - a^2 - b^2)/2}),$$

$$\mathcal{K}_{[\omega]}(\Xi_2, \Xi_2) = 4\pi^2 \Big\{ [\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] y^2 dy\, dx$$

$$-([\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] y\, dy\, dx)^2$$

$$/[\int_0^r \int_0^{r-x} - \int_0^a \int_{r-a}^{r-x} - \int_{r-b}^r \int_0^{r-x}] 1\, dy\, dx \Big\}$$

$$= 4\pi^2(\frac{r^4}{12} - \frac{a^4}{4} + \frac{2ra^3}{3} - \frac{b^4}{12} - \frac{r^2a^2}{2} - \frac{((r^3 - b^3 - 3ra^2 + 2a^3)/6)^2}{(r^2 - a^2 - b^2)/2}).$$

We are left with determining the Futaki invariant at our basis vector fields, for which we also need the average scalar curvature.

To compute s_0, we take the homology basis h, e_1, e_2, where h is the proper transform of a projective line in $\mathbb{C}P^2$ with area $2\pi(r-b)$, e_2 the exceptional divisor intersecting h of area $2\pi b$ and e_1 the other exceptional divisor of area $2\pi a$. The intersection matrix is given in this basis by:

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

The first Chern class c_1 is Poincarè dual to $3h - e_1 + 2e_2$ (for the general scheme for computing it for blow-ups, see [GH, Chapter 1, Section 4]). If $[\omega]$ is Poincarè dual to $Ah + Be_1 + Ce_2$, we can use Poincarè duality to find the coefficients. Intersecting with h, e_1 and e_2 in succession we get:

$$2\pi(r-b) = \int_h \omega = C$$

$$2\pi a = \int_{e_1} \omega = -B$$

$$2\pi b = \int_{e_2} \omega = A - C,$$

so the Poincarè dual to $[\omega]$ is $2\pi(rh - ae_1 + (r-b)e_2)$. We therefore have,

$$\begin{aligned} s_0 &= 8\pi\frac{c_1 \cdot [\omega]}{[\omega]\cdot[\omega]} \\ &= 8\pi\frac{(3h - e_1 + 2e_2)\cdot 2\pi(rh - ae_1 + (r-b)e_2)}{2\pi(rh - ae_1 + (r-b)e_2)\cdot 2\pi(rh - ae_1 + (r-b)e_2)} \\ &= 8\pi\frac{2\pi(3(r-b) - a + 2r - 2(r-b))}{4\pi^2(r(r-b) - a^2 + r(r-b) - (r-b)^2)} \\ &= 4\frac{3r-a-b}{r^2-a^2-b^2}. \end{aligned} \tag{4.4}$$

This also gives the first term in $\mathcal{A}$, namely,:

$$\int_M s_0^2\frac{\omega^{\wedge 2}}{2} = \frac{s_0^2}{2}[\omega]\cdot[\omega] = 8\frac{(3r-a-b)^2}{(r^2-a^2-b^2)^2}4\pi^2(r^2-a^2-b^2) = 32\pi^2\frac{(3r-a-b)^2}{r^2-a^2-b^2}$$

Although the T^2-action is not semi-free, the action of the circle subgroups generated by each basis element is. This means that the formulas for $\mathcal{F}_{[\omega]}(\Xi_j), j = 1, 2$, simplify to those of [LSm]. Note that the other requirement for the use of those formulas, namely that the action of (each of) the circle group(s) has two fixed surfaces, holds for both the 2-point and 3-point blow-ups. One can read off the data for these S^1 actions directly from the polygon data (see [Ka, Section 2.5]). We detail this in Figure 4.2.

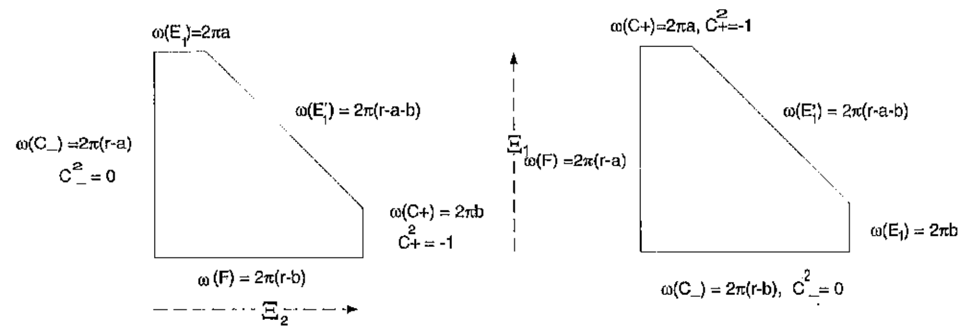

Figure 4.2: Futaki invariant data for the two point blow-up.

We thus have:

$$
\begin{aligned}
\mathcal{F}_{[\omega]}(\Xi_1) &= 1/2(2\pi(r-b) - 2\pi a)2\pi(r-a) \\
&+ \frac{4(3r-a-b)[2\pi(r-a)]^3}{192\pi(r^2-a^2-b^2)}(0-(-1) \\
&+ 6\frac{2\pi a - 2\pi(r-b)}{2\pi(r-a)} + (\frac{2\pi(r-a-b)-2\pi b}{2\pi(r-a)})^3) \\
&= \frac{4\pi^2}{3}\frac{(r-a-b)(2a^3-4a^2r+2ar^2+arb-r^2b+2rb^2-b^3)}{r^2-a^2-b^2}, \\
\mathcal{F}_{[\omega]}(\Xi_2) &= 1/2(2\pi(r-a) - 2\pi b)2\pi(r-b) \\
&+ \frac{4(3r-a-b)[2\pi(r-b)]^3}{192\pi(r^2-a^2-b^2)}(0-(-1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 6\frac{2\pi b - 2\pi(r-a)}{2\pi(r-b)} + (\frac{2\pi(r-a-b) - 2\pi a}{2\pi(r-b)})^3) \\
&= \quad \frac{4\pi^2}{3}\frac{(r-a-b)(2b^3 - 4b^2 r + 2br^2 + bra - r^2 a + 2ra^2 - a^3)}{r^2 - a^2 - b^2}.
\end{aligned}
$$

We thus completed the computation of all terms in formula (4.3). Due to their complexity, we use a symbolic computational program (Maple) to determine $\mathcal{A}$. This turns out to be the following rational function in r,a and b, whose numerator and denominator are homogeneous polynomials with total degree equal to 14.

$$
\begin{aligned}
\mathcal{A} := 32\pi^2 (&2536\, r^6\, b^5\, a^3 + 2746\, r^4\, b^4\, a^6 + 618\, r^6\, b^4\, a^4 - 9\, r^{14} + 6\, r^{13}\, b \\
&- 151\, r^4\, a^{10} - 613\, r^8\, a^6 - 1003\, r^6\, a^8 + 992\, r^7\, a^7 + 131\, r^{10}\, a^4 \\
&+ 190\, r^9\, a^5 + 578\, r^5\, a^9 + 131\, r^{10}\, b^4 + a^{14} + b^{14} - 824\, r^{10}\, a^2\, b^2 \\
&+ 1792\, r^9\, a^2\, b^3 + 1792\, r^9\, a^3\, b^2 - 3264\, r^5\, b^5\, a^4 - 1532\, r^3\, b^5\, a^6 \\
&- 925\, r^8\, b^4\, a^2 - 1960\, r^5\, b^7\, a^2 + 2184\, r^6\, b^6\, a^2 + 1189\, r^4\, b^8\, a^2 \\
&- 454\, r^3\, b^9\, a^2 - 1124\, r^7\, b^5\, a^2 + 104\, r^2\, b^2\, a^{10} + 104\, r^2\, b^{10}\, a^2 \\
&+ 1476\, r^7\, b^4\, a^3 - 38\, r^2\, b^9\, a^3 + 472\, r^2\, b^6\, a^6 + 2184\, r^6\, b^2\, a^6 \\
&+ 1189\, r^4\, b^2\, a^8 - 545\, r^2\, b^4\, a^8 - 1960\, r^5\, b^2\, a^7 + 8\, r^2\, b^5\, a^7 \\
&- 545\, r^2\, b^8\, a^4 - 925\, r^8\, b^2\, a^4 - 420\, r^3\, b^8\, a^3 + 2746\, r^4\, b^6\, a^4 \\
&- 3532\, r^5\, b^6\, a^3 - 3630\, r^8\, b^3\, a^3 - 3532\, r^5\, b^3\, a^6 - 3264\, r^5\, b^4\, a^5 \\
&+ 3068\, r^4\, b^5\, a^5 - 420\, r^3\, b^3\, a^8 - 1532\, r^3\, b^6\, a^5 + 1880\, r^4\, b^3\, a^7 \\
&+ 1476\, r^7\, b^3\, a^4 + 1880\, r^4\, b^7\, a^3 - 1124\, r^7\, b^2\, a^5 - 454\, r^3\, b^2\, a^9
\end{aligned}
$$

$$
\begin{aligned}
&- 38\, r^2\, b^3\, a^9 + 8\, r^2\, b^7\, a^5 - 404\, a^7\, r^3\, b^4 - 24\, a^{11}\, r\, b^2 + 36\, a^3\, r\, b^{10} \\
&+ 264\, a^4\, r\, b^9 + 12\, a^7\, r\, b^6 + 264\, a^9\, r\, b^4 + 258\, a^8\, r\, b^5 + 258\, a^5\, r\, b^8 \\
&+ 2536\, a^5\, r^6\, b^3 - 404\, a^4\, r^3\, b^7 + 36\, a^{10}\, r\, b^3 + 12\, a^6\, r\, b^7 \\
&- 24\, r\, b^{11}\, a^2 + 118\, r^{10}\, a^3\, b + 292\, r^7\, a^6\, b + 550\, r^5\, a^8\, b - 524\, r^6\, a^7\, b \\
&- 64\, r^9\, a^4\, b - 296\, r^4\, a^9\, b - 86\, r^8\, b^5\, a - 64\, r^9\, b^4\, a - 296\, r^4\, b^9\, a \\
&+ 550\, r^5\, b^8\, a + 292\, r^7\, b^6\, a + 118\, r^{10}\, b^3\, a - 524\, r^6\, b^7\, a - 86\, a^5\, r^8\, b \\
&+ 50\, a^{10}\, r^3\, b + 22\, a^{11}\, r^2\, b + 50\, r^3\, b^{10}\, a + 992\, r^7\, b^7 - 613\, r^8\, b^6 \\
&- 1003\, r^6\, b^8 + 578\, r^5\, b^9 + 190\, r^9\, b^5 - 151\, r^4\, b^{10} + 17\, a^{12}\, r^2 \\
&- 8\, a^{11}\, r^3 + 5\, a^{12}\, b^2 - 33\, a^4\, b^{10} - 54\, a^5\, b^9 - 29\, a^8\, b^6 - 33\, a^{10}\, b^4 \\
&- 54\, a^9\, b^5 - 29\, a^6\, b^8 - 2\, a^{11}\, b^3 - 4\, a^7\, b^7 - 8\, r^3\, b^{11} + 17\, b^{12}\, r^2 \\
&+ 5\, b^{12}\, a^2 - 2\, b^{11}\, a^3 + 91\, r^{12}\, b^2 - 216\, r^{11}\, b^3 + 91\, r^{12}\, a^2 - 216\, r^{11}\, a^3 \\
&- 6\, a^{13}\, r + 2\, a^{13}\, b - 6\, b^{13}\, r + 2\, b^{13}\, a + 22\, b^{11}\, r^2\, a - 54\, r^{11}\, a\, b^2 \\
&- 4\, r^{12}\, a\, b - 54\, r^{11}\, a^2\, b - 12\, a^{12}\, r\, b - 12\, b^{12}\, a\, r + 6\, r^{13}\, a) \Big/ \Big((\\
&24\, a^6\, r^2\, b^2 - 66\, a^2\, r^4\, b^4 + 112\, a^4\, r^3\, b^3 - 200\, a^3\, r^4\, b^3 - 66\, a^4\, r^4\, b^2 \\
&+ 112\, a^3\, r^3\, b^4 - 24\, a^3\, r\, b^6 + 120\, a^3\, r^5\, b^2 - 24\, a^7\, r\, b^2 - 24\, a^2\, r\, b^7 \\
&- 66\, a^4\, b^4\, r^2 + 10\, r^6\, b^4 - 9\, r^2\, b^8 + 9\, r^8\, b^2 - 10\, r^4\, b^6 - 16\, r^7\, b^3 \\
&+ 16\, r^3\, b^7 - r^{10} + 120\, a^2\, r^5\, b^3 + 24\, a^2\, r^2\, b^6 - 72\, a^2\, r^6\, b^2 + 10\, a^4\, b^6 \\
&+ 10\, a^4\, r^6 - 16\, a^3\, r^7 + 8\, a^3\, b^7 + 9\, a^2\, b^8 + 9\, a^2\, r^8 + 10\, a^6\, b^4 \\
&- 10\, a^6\, r^4 - 24\, a^6\, r\, b^3 - 9\, a^8\, r^2 + 9\, a^8\, b^2 + 16\, a^7\, r^3 + 8\, a^7\, b^3 + b^{10} \\
&+ a^{10})(-r^2 + a^2 + b^2)^2 \Big) .
\end{aligned}
$$

We now wish to determine the critical points of $\mathcal{A}$. As we are any-way only interested in metrics up to homothety (and $\mathcal{A}$ is scale invariant in dimension four), we can eliminate one variable. To keep in conformity with the conventions in [Le], we rescale by defining the two new variables as $s = a/(r-a-b), t = b/(r-a-b)$. A part of the graph of the resulting function on the (relevant part of the) Kähler cone, which corresponds to $s, t > 0$, is given in Figure 4.3.

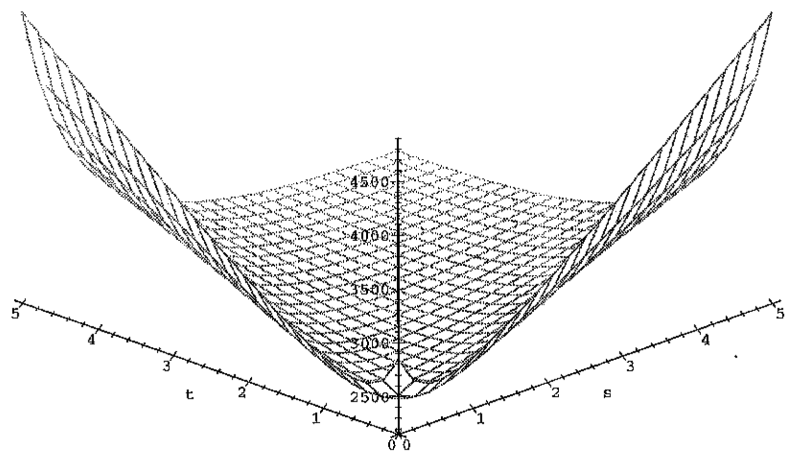

Figure 4.3: $\mathcal{A}$ for the two point blow-up.

The graph suggests the following property, which was nearly confirmed computationally: not only does $\mathcal{A}$ have a unique critical point, lying on the line $s = t$, but also, a unique critical point when restricted to any line orthogonal to that one. As it is difficult for Maple to find critical points directly, this observation suggests a more natural coordinate system for evaluation, namely (p, q) with $s = p + q, t = p - q$. Note that $p = (s+t)/2$ is positive.

With this choice of coordinates, Maple finds numerically a critical point with $q = 0$ $(s = t)$, and $p = .9913521825$. One has to verify more carefully that the numerical process did not discard other critical points. To this end we use algebraic routines to solve for each of the partial derivatives separately, and then proceed to find all the intersection points of the resulting two curves in the p, q plane. One can then go through the list and see that other than the above mentioned critical point, all other solutions are either complex valued, or else have $p < 0$, and so our critical point is indeed unique. The relevant portion of the Maple session will be given in the appendix.

We thus have a computer-assisted proof for the following weak uniqueness result.

Theorem 4.4 *If there exists Hermitian Einstein metrics on the general position two point blow-up of the complex projective plane, then any two such metrics are conformal to* **cohomologous** *extremal Kähler metrics. The latter metrics evaluate equal volumes for the two exceptional divisors.*

The proof of course gives a more accurate determination of the Kähler class. Note that even the existence of extremal Kähler metrics alone is not known for this space.

The first part of the proof, namely the evaluation of $\mathcal{A}$, works also for the 3-point blow-up of $\mathbb{C}P^2$. We give in Figure 4.4 only the relevant polygon data. The computer-assisted evaluation of the critical set has not been successful thus far.

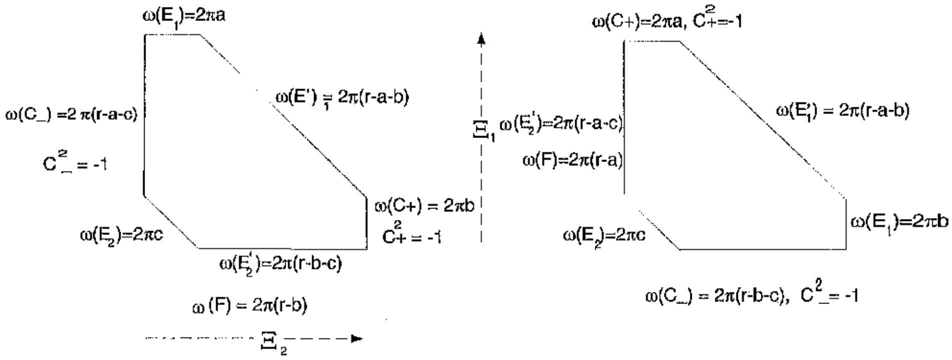

Figure 4.4: Futaki invariant data for the three point blow-up.

4.3 Appendix: Critical Set Computer Check

We continue the evaluation of the critical points of

$$\int_{\mathbb{C}P^2 \# 2\bar{\mathbb{C}P}^2} s^2 d\mu.$$

Preparatory substitution:

SS1:=factor(subs(r=a+b+delta,A));

$$\begin{aligned} SS1 := 32\pi^2 \Big(& 22496\, a^6\, b\, \delta^7 + 7896\, a^7\, b\, \delta^6 + 35312\, a^5\, b\, \delta^8 \\ & + 1152\, a^8\, b\, \delta^5 + 20276\, a^3\, b\, \delta^{10} + 192\, b^8\, \delta^6 + 1152\, b^7\, \delta^7 \\ & + 2908\, b^6\, \delta^8 + 1932\, a^3\, \delta^{11} + 3532\, a^4\, \delta^{10} + 4088\, a^5\, \delta^9 \\ & + 2908\, a^6\, \delta^8 + 1152\, a^7\, \delta^7 + 192\, a^8\, \delta^6 + 9\, \delta^{14} + 1932\, b^3\, \delta^{11} \\ & + 120\, a\, \delta^{13} + 650\, b^2\, \delta^{12} + 120\, b\, \delta^{13} + 4088\, b^5\, \delta^9 + 3532\, b^4\, \delta^{10} \\ & + 7434\, a^2\, b\, \delta^{11} + 1486\, a\, b\, \delta^{12} + 7434\, a\, b^2\, \delta^{11} + 650\, a^2\, \delta^{12} \\ & + 33764\, a^4\, b\, \delta^9 + 192\, a^8\, b^6 + 192\, a^6\, b^8 + 384\, a^7\, b^7 \\ & + 71008\, a^2\, b^6\, \delta^6 + 123968\, a^2\, b^5\, \delta^7 + 130678\, a^2\, b^4\, \delta^8 \end{aligned}$$

$$+ 2844\,a^8\,b^2\,\delta^4 + 22176\,a^7\,b^2\,\delta^5 + 71008\,a^6\,b^2\,\delta^6 + 3768\,a^8\,b^3\,\delta^3$$
$$+ 33648\,a^7\,b^3\,\delta^4 + 121344\,a^6\,b^3\,\delta^5 + 1152\,a^5\,b^8\,\delta$$
$$+ 15360\,a^5\,b^7\,\delta^2 + 74976\,a^5\,b^6\,\delta^3 + 186420\,a^5\,b^5\,\delta^4$$
$$+ 268596\,a^5\,b^4\,\delta^5 + 85898\,a^2\,b^3\,\delta^9 + 34286\,a^2\,b^2\,\delta^{10}$$
$$+ 235688\,a^5\,b^3\,\delta^6 + 2844\,a^4\,b^8\,\delta^2 + 29928\,a^4\,b^7\,\delta^3$$
$$+ 123348\,a^4\,b^6\,\delta^4 + 268596\,a^4\,b^5\,\delta^5 + 345812\,a^4\,b^4\,\delta^6$$
$$+ 273932\,a^4\,b^3\,\delta^7 + 3768\,a^3\,b^8\,\delta^3 + 33648\,a^3\,b^7\,\delta^4$$
$$+ 121344\,a^3\,b^6\,\delta^5 + 235688\,a^3\,b^5\,\delta^6 + 273932\,a^3\,b^4\,\delta^7$$
$$+ 197274\,a^3\,b^3\,\delta^8 + 123968\,a^5\,b^2\,\delta^7 + 130678\,a^4\,b^2\,\delta^8$$
$$+ 85898\,a^3\,b^2\,\delta^9 + 2844\,a^2\,b^8\,\delta^4 + 22176\,a^2\,b^7\,\delta^5 + 2844\,a^8\,b^4\,\delta^2$$
$$+ 29928\,a^7\,b^4\,\delta^3 + 4032\,a^6\,b^7\,\delta + 25392\,a^6\,b^6\,\delta^2 + 74976\,a^6\,b^5\,\delta^3$$
$$+ 123348\,a^6\,b^4\,\delta^4 + 1152\,a^8\,b^5\,\delta + 4032\,a^7\,b^6\,\delta + 15360\,a^7\,b^5\,\delta^2$$
$$+ 35312\,a\,b^5\,\delta^8 + 33764\,a\,b^4\,\delta^9 + 1152\,a\,b^8\,\delta^5 + 7896\,a\,b^7\,\delta^6$$
$$+ 22496\,a\,b^6\,\delta^7 + 20276\,a\,b^3\,\delta^{10}\Big) \Big/ \Big(\Big(\delta^{10} + 448\,a^3\,b\,\delta^6$$
$$+ 240\,a^2\,b^5\,\delta^3 + 816\,a^2\,b^4\,\delta^4 + 1152\,a^2\,b^3\,\delta^5 + 828\,a^2\,b^2\,\delta^6$$
$$+ 288\,a^2\,b\,\delta^7 + 120\,a\,b^5\,\delta^4 + 360\,a\,b^4\,\delta^5 + 448\,a\,b^3\,\delta^6$$
$$+ 288\,a\,b^2\,\delta^7 + 90\,a\,b\,\delta^8 + 64\,a^3\,\delta^7 + 36\,a^2\,\delta^8 + 10\,a\,\delta^9$$
$$+ 24\,b^5\,\delta^5 + 60\,b^4\,\delta^6 + 64\,b^3\,\delta^7 + 36\,b^2\,\delta^8 + 10\,b\,\delta^9 + 24\,a^5\,b^5$$
$$+ 24\,a^5\,\delta^5 + 60\,a^4\,\delta^6 + 120\,a^5\,b^4\,\delta + 240\,a^5\,b^3\,\delta^2 + 240\,a^5\,b^2\,\delta^3$$
$$+ 120\,a^5\,b\,\delta^4 + 120\,a^4\,b^5\,\delta + 516\,a^4\,b^4\,\delta^2 + 912\,a^4\,b^3\,\delta^3$$
$$+ 816\,a^4\,b^2\,\delta^4 + 360\,a^4\,b\,\delta^5 + 240\,a^3\,b^5\,\delta^2 + 912\,a^3\,b^4\,\delta^3$$

$$+ 1440\,a^3\,b^3\,\delta^4 + 1152\,a^3\,b^2\,\delta^5\Big)\left(2\,a\,b + 2\,a\,\delta + 2\,b\,\delta + \delta^2\right)^2\Big)$$

Restriction to constant volume metrics:

AA:=factor(subs(a=s*delta,b=t*delta,SS1));

$$AA := 32\pi^2(9 + 120\,t + 120\,s + 192\,t^8 + 1152\,t^7 + 1932\,t^3 + 3532\,s^4$$
$$+ 4088\,s^5 + 650\,t^2 + 650\,s^2 + 74976\,s^5\,t^6 + 186420\,s^5\,t^5$$
$$+ 268596\,s^5\,t^4 + 85898\,s^2\,t^3 + 34286\,s^2\,t^2 + 235688\,s^5\,t^3$$
$$+ 29928\,s^4\,t^7 + 123348\,s^4\,t^6 + 268596\,s^4\,t^5 + 22496\,s^6\,t$$
$$+ 7896\,s^7\,t + 1152\,s^8\,t + 20276\,s^3\,t + 192\,s^8\,t^6 + 192\,s^6\,t^8$$
$$+ 384\,s^7\,t^7 + 7434\,s^2\,t + 15360\,s^7\,t^5 + 25392\,s^6\,t^6 + 74976\,s^6\,t^5$$
$$+ 123348\,s^6\,t^4 + 1152\,s^8\,t^5 + 4032\,s^7\,t^6 + 35312\,s\,t^5 + 33764\,s\,t^4$$
$$+ 1152\,s\,t^8 + 7896\,s\,t^7 + 22496\,s\,t^6 + 20276\,s\,t^3 + 2844\,s^4\,t^8$$
$$+ 35312\,s^5\,t + 1486\,s\,t + 7434\,s\,t^2 + 33764\,s^4\,t + 71008\,s^2\,t^6$$
$$+ 123968\,s^2\,t^5 + 130678\,s^2\,t^4 + 2844\,s^8\,t^2 + 22176\,s^7\,t^2$$
$$+ 71008\,s^6\,t^2 + 3768\,s^8\,t^3 + 33648\,s^7\,t^3 + 121344\,s^6\,t^3 + 1152\,s^5\,t^8$$
$$+ 15360\,s^5\,t^7 + 1932\,s^3 + 4088\,t^5 + 2908\,s^6 + 2908\,t^6 + 3532\,t^4$$
$$+ 192\,s^8 + 1152\,s^7 + 345812\,s^4\,t^4 + 273932\,s^4\,t^3 + 3768\,s^3\,t^8$$
$$+ 33648\,s^3\,t^7 + 121344\,s^3\,t^6 + 235688\,s^3\,t^5 + 273932\,s^3\,t^4$$
$$+ 197274\,s^3\,t^3 + 123968\,s^5\,t^2 + 130678\,s^4\,t^2 + 85898\,s^3\,t^2$$
$$+ 2844\,s^2\,t^8 + 22176\,s^2\,t^7 + 2844\,s^8\,t^4 + 29928\,s^7\,t^4 + 4032\,s^6\,t^7)$$
$$\Big/\Big((1 + 10\,t + 10\,s + 64\,t^3 + 60\,s^4 + 24\,s^5 + 36\,t^2 + 36\,s^2 + 24\,s^5\,t^5$$
$$+ 120\,s^5\,t^4 + 1152\,s^2\,t^3 + 828\,s^2\,t^2 + 240\,s^5\,t^3 + 120\,s^4\,t^5 + 448\,s^3\,t$$

$$+ 288\, s^2\, t + 120\, s\, t^5 + 360\, s\, t^4 + 448\, s\, t^3 + 120\, s^5\, t + 90\, s\, t$$

$$+ 288\, s\, t^2 + 360\, s^4\, t + 240\, s^2\, t^5 + 816\, s^2\, t^4 + 64\, s^3 + 24\, t^5 + 60\, t^4$$

$$+ 516\, s^4\, t^4 + 912\, s^4\, t^3 + 240\, s^3\, t^5 + 912\, s^3\, t^4 + 1440\, s^3\, t^3$$

$$+ 240\, s^5\, t^2 + 816\, s^4\, t^2 + 1152\, s^3\, t^2)(\, 2\, s\, t + 2\, s + 2\, t + 1\,)^2\Big)$$

Change of coordinates:

BB:=subs(s=p+q,t=p-q,AA);

$$BB := 32\pi^2\, \Big(9 + 240\, p + 235688\,(\, p+q\,)^5\,(\, p-q\,)^3$$

$$+ 29928\,(\, p+q\,)^4\,(\, p-q\,)^7 + 74976\,(\, p+q\,)^5\,(\, p-q\,)^6$$

$$+ 186420\,(\, p+q\,)^5\,(\, p-q\,)^5 + 268596\,(\, p+q\,)^5\,(\, p-q\,)^4$$

$$+ 85898\,(\, p+q\,)^2\,(\, p-q\,)^3 + 34286\,(\, p+q\,)^2\,(\, p-q\,)^2$$

$$+ 3532\,(\, p+q\,)^4 + 4088\,(\, p+q\,)^5 + 650\,(\, p+q\,)^2 + 1932\,(\, p+q\,)^3$$

$$+ 2908\,(\, p+q\,)^6 + 192\,(\, p+q\,)^8 + 1152\,(\, p+q\,)^7 + 192\,(\, p-q\,)^8$$

$$+ 1152\,(\, p-q\,)^7 + 1932\,(\, p-q\,)^3 + 650\,(\, p-q\,)^2 + 4088\,(\, p-q\,)^5$$

$$+ 2908\,(\, p-q\,)^6 + 3532\,(\, p-q\,)^4 + 123348\,(\, p+q\,)^4\,(\, p-q\,)^6$$

$$+ 268596\,(\, p+q\,)^4\,(\, p-q\,)^5 + 22496\,(\, p+q\,)^6\,(\, p-q\,)$$

$$+ 7896\,(\, p+q\,)^7\,(\, p-q\,) + 1152\,(\, p+q\,)^8\,(\, p-q\,)$$

$$+ 20276\,(\, p+q\,)^3\,(\, p-q\,) + 192\,(\, p+q\,)^8\,(\, p-q\,)^6$$

$$+ 192\,(\, p+q\,)^6\,(\, p-q\,)^8 + 384\,(\, p+q\,)^7\,(\, p-q\,)^7$$

$$+ 7434\,(\, p+q\,)^2\,(\, p-q\,) + 15360\,(\, p+q\,)^7\,(\, p-q\,)^5$$

$$+ 25392\,(\, p+q\,)^6\,(\, p-q\,)^6 + 74976\,(\, p+q\,)^6\,(\, p-q\,)^5$$

$$+ 123348\,(\, p+q\,)^6\,(\, p-q\,)^4 + 1152\,(\, p+q\,)^8\,(\, p-q\,)^5$$

$$+ 4032\,(p+q)^7\,(p-q)^6 + 35312\,(p+q)\,(p-q)^5$$

$$+ 33764\,(p+q)\,(p-q)^4 + 1152\,(p+q)\,(p-q)^8$$

$$+ 7896\,(p+q)\,(p-q)^7 + 22496\,(p+q)\,(p-q)^6$$

$$+ 20276\,(p+q)\,(p-q)^3 + 2844\,(p+q)^4\,(p-q)^8$$

$$+ 35312\,(p+q)^5\,(p-q) + 1486\,(p+q)\,(p-q)$$

$$+ 7434\,(p+q)\,(p-q)^2 + 33764\,(p+q)^4\,(p-q)$$

$$+ 71008\,(p+q)^2\,(p-q)^6 + 123968\,(p+q)^2\,(p-q)^5$$

$$+ 130678\,(p+q)^2\,(p-q)^4 + 2844\,(p+q)^8\,(p-q)^2$$

$$+ 22176\,(p+q)^7\,(p-q)^2 + 71008\,(p+q)^6\,(p-q)^2$$

$$+ 3768\,(p+q)^8\,(p-q)^3 + 33648\,(p+q)^7\,(p-q)^3$$

$$+ 121344\,(p+q)^6\,(p-q)^3 + 1152\,(p+q)^5\,(p-q)^8$$

$$+ 15360\,(p+q)^5\,(p-q)^7 + 345812\,(p+q)^4\,(p-q)^4$$

$$+ 273932\,(p+q)^4\,(p-q)^3 + 3768\,(p+q)^3\,(p-q)^8$$

$$+ 33648\,(p+q)^3\,(p-q)^7 + 121344\,(p+q)^3\,(p-q)^6$$

$$+ 235688\,(p+q)^3\,(p-q)^5 + 273932\,(p+q)^3\,(p-q)^4$$

$$+ 197274\,(p+q)^3\,(p-q)^3 + 123968\,(p+q)^5\,(p-q)^2$$

$$+ 130678\,(p+q)^4\,(p-q)^2 + 85898\,(p+q)^3\,(p-q)^2$$

$$+ 2844\,(p+q)^2\,(p-q)^8 + 22176\,(p+q)^2\,(p-q)^7$$

$$+ 2844\,(p+q)^8\,(p-q)^4 + 29928\,(p+q)^7\,(p-q)^4$$

$$+ 4032\,(p+q)^6\,(p-q)^7\Big) \Big/ \Big(\Big(1 + 20\,p + 240\,(p+q)^5\,(p-q)^3$$

$$+ 24\,(p+q)^5\,(p-q)^5 + 120\,(p+q)^5\,(p-q)^4$$

$$+ 1152 (p+q)^2 (p-q)^3 + 828 (p+q)^2 (p-q)^2 + 60 (p+q)^4$$

$$+ 24 (p+q)^5 + 36 (p+q)^2 + 64 (p+q)^3 + 64 (p-q)^3$$

$$+ 36 (p-q)^2 + 24 (p-q)^5 + 60 (p-q)^4 + 120 (p+q)^4 (p-q)^5$$

$$+ 448 (p+q)^3 (p-q) + 288 (p+q)^2 (p-q)$$

$$+ 120 (p+q)(p-q)^5 + 360 (p+q)(p-q)^4$$

$$+ 448 (p+q)(p-q)^3 + 120 (p+q)^5 (p-q) + 90 (p+q)(p-q)$$

$$+ 288 (p+q)(p-q)^2 + 360 (p+q)^4 (p-q)$$

$$+ 240 (p+q)^2 (p-q)^5 + 816 (p+q)^2 (p-q)^4$$

$$+ 516 (p+q)^4 (p-q)^4 + 912 (p+q)^4 (p-q)^3$$

$$+ 240 (p+q)^3 (p-q)^5 + 912 (p+q)^3 (p-q)^4$$

$$+ 1440 (p+q)^3 (p-q)^3 + 240 (p+q)^5 (p-q)^2$$

$$+ 816 (p+q)^4 (p-q)^2 + 1152 (p+q)^3 (p-q)^2\Big)$$

$$\left(2 (p+q)(p-q) + 4p + 1\right)^2\Big)$$

Evaluation of partial derivatives:

BBp:=factor(diff(BB,p));BBq:=factor(diff(BB,q));

$$BBp := 128\pi^2(-3 + 19439196576\, p^{11}\, q^8 - 16193640384\, p^{13}\, q^6$$

$$+ 101952\, p^{23} + 7465955520\, p^{15}\, q^4 - 10730224512\, p^{10}\, q^{10}$$

$$- 5416128\, p^{21}\, q^2 + 44540352\, p^{19}\, q^4 - 173959488\, p^{17}\, q^6$$

$$+ 404514432\, p^{15}\, q^8 + 12249292608\, p^{12}\, q^8 + 2010911616\, q^{12}\, p^{10}$$

$$- 73584\, q^{18} + 371718720\, q^{16}\, p^6 - 68795712\, q^{18}\, p^4$$

$$+ 5204160\, q^{20}\, p^2 - 1100504448\, q^{14}\, p^8 - 134\, p$$

$$- 8446618368\, p^{14}\, q^6 + 3312435744\, p^{16}\, q^4 + 5663843136\, p^{13}\, q^8$$

$$+ 1108576800\, p^{17}\, q^4 - 178765632\, p^{19}\, q^2 - 3288432384\, p^{15}\, q^6$$

$$+ 269258688\, p^{18}\, q^4 + 1854029952\, p^{14}\, q^8 - 920306880\, p^{16}\, q^6$$

$$- 6465733920\, p^{14}\, q^2 + 7945920\, p^3\, q^{20} - 13548257952\, p^9\, q^{10}$$

$$- 54378432\, p^5\, q^{18} - 429812352\, p^9\, q^{14} + 623645568\, p^{11}\, q^{12}$$

$$+ 195041088\, p^7\, q^{16} - 2384774784\, p^{12}\, q^{10} - 611839872\, p^{13}\, q^{10}$$

$$+ 5674423104\, p^8\, q^{12} - 13370688\, p^2\, q^{18} + 2272248\, p\, q^{16}$$

$$- 43788096\, p^3\, q^{18} + 404672544\, p^5\, q^{16} - 1721675520\, p^7\, q^{14}$$

$$+ 4127360832\, p^9\, q^{12} + 19322598336\, p^{12}\, q^4$$

$$+ 23166590688\, p^{10}\, q^8 - 6079425408\, p^{11}\, q^{10} - 37707840\, p^{20}\, q^2$$

$$- 23738270400\, p^{12}\, q^6 - 3236538096\, p^{16}\, q^2 - 615829824\, p^{18}\, q^2$$

$$- 5138724864\, p^{15}\, q^2 + 14866296624\, p^8\, q^8 + 3556895040\, p^6\, q^{12}$$

$$- 27040838016\, p^{11}\, q^6 + 592051200\, p^4\, q^{12} - 1098828480\, p^5\, q^{14}$$

$$- 1712358144\, p^6\, q^{14} - 24211394592\, p^{10}\, q^6 - 12501252384\, p^8\, q^{10}$$

$$+ 13009843776\, p^{14}\, q^4 + 17805113568\, p^{13}\, q^4 - 4476432480\, p^6\, q^{10}$$

$$+ 5367729600\, p^7\, q^{12} + 21083863056\, p^9\, q^8 - 25996896\, p^2\, q^{14}$$

$$+ 92186544\, p^3\, q^{16} + 255548448\, p^4\, q^{16} - 459767232\, p^4\, q^{14}$$

$$- 63936\, q^{22} + 8233139592\, p^7\, q^8 - 6446363400\, p^{13}\, q^2 + 68256\, q^{20}$$

$$+ 171000\, q^{16} - 1650096\, p\, q^{18} - 1603706160\, p^{17}\, q^2$$

$$+ 16708365432\, p^{11}\, q^4 - 383040\, p\, q^{22} - 17183678184\, p^9\, q^6$$

$$- 8583881472\, p^7\, q^{10} - 1812669336\, p^5\, q^{10} + 110702592\, q^{12}\, p^{12}$$

$$
\begin{aligned}
&- 93063168\, q^{14}\, p^{10} + 158634408\, p^{3}\, q^{12} + 53982720\, q^{16}\, p^{8} \\
&- 20782080\, q^{18}\, p^{6} + 202752\, p^{21}\, q^{4} + 1293984\, p\, q^{20} \\
&+ 398440728\, p^{16} - 1013760\, p^{19}\, q^{6} + 3041280\, p^{17}\, q^{8} \\
&- 6082560\, p^{15}\, q^{10} + 1686984288\, p^{5}\, q^{12} - 18432\, p^{23}\, q^{2} \\
&- 19768320\, p^{18}\, q^{6} - 91846656\, p^{14}\, q^{10} + 52462080\, p^{16}\, q^{8} \\
&+ 4460544\, p^{20}\, q^{4} + 4866048\, q^{20}\, p^{4} - 562176\, q^{22}\, p^{2} \\
&- 470016\, p^{22}\, q^{2} - 8515584\, p^{11}\, q^{14} + 6082560\, p^{9}\, q^{16} \\
&+ 8515584\, p^{13}\, q^{12} - 3041280\, p^{7}\, q^{18} + 1013760\, p^{5}\, q^{20} \\
&- 202752\, p^{3}\, q^{22} + 18652944\, p^{2}\, q^{16} + 13824\, q^{24} - 3665016\, p\, q^{14} \\
&+ 4608\, p^{24} + 18432\, p\, q^{24} + 53940440\, p^{8}\, q^{2} + 194401008\, p^{6}\, q^{4} \\
&- 89921744\, p^{4}\, q^{6} + 10273336\, p^{2}\, q^{8} + 90363656\, p^{7}\, q^{2} \\
&+ 21714476\, p^{5}\, q^{4} - 12615320\, p^{3}\, q^{6} + 915162\, p\, q^{8} - 8116052\, p^{6} \\
&+ 3732\, q^{6} - 28833490\, p^{7} - 373586868\, p^{11} - 292717608\, p^{10} \\
&- 79907210\, p^{8} - 2833684\, p^{4}\, q^{4} - 964576\, p^{2}\, q^{6} + 3805468\, p^{4}\, q^{2} \\
&- 355804\, p^{2}\, q^{4} + 16172078\, p^{5}\, q^{2} - 1763046\, p^{3}\, q^{4} - 4214\, p\, q^{6} \\
&- 298394108\, p^{9}\, q^{2} + 867915384\, p^{7}\, q^{4} - 436101496\, p^{5}\, q^{6} \\
&+ 71135740\, p^{3}\, q^{8} - 2902348\, p\, q^{10} - 33922\, p^{3} - 286863\, p^{4} \\
&- 1551\, q^{4} - 1758997\, p^{5} + 36462\, q^{8} - 2735\, p^{2} + 129\, q^{2} + 4418\, p\, q^{2} \\
&+ 67982\, p^{2}\, q^{2} + 623746\, p^{3}\, q^{2} - 36333\, p\, q^{4} + 47869968\, p^{6}\, q^{2} \\
&- 331159820\, p^{12} + 264276\, q^{12} - 134951508\, p^{13} \\
&- 129302400\, p^{3}\, q^{14} - 173442422\, p^{9} - 166152\, q^{10}
\end{aligned}
$$

$$+ 1265245780\, p^5 q^8 - 141457384\, p^3 q^{10} + 4152652\, p\, q^{12}$$

$$- 3017588584\, p^{11} q^2 + 6294641716\, p^9 q^4 - 4390366768\, p^7 q^6$$

$$+ 346236504\, p^{15} - 1286889368\, p^{10} q^2 + 2688898092\, p^8 q^4$$

$$- 1570928848\, p^6 q^6 + 345880844\, p^4 q^8 - 25175064\, p^2 q^{10}$$

$$+ 314513592\, p^{17} + 183919248\, p^{18} + 31695936\, p^2 q^{12}$$

$$+ 3620019456\, p^6 q^8 - 576682080\, p^4 q^{10} + 137761152\, p^{14}$$

$$- 252576\, q^{14} + 6339744\, p^{21} + 81014832\, p^{19} + 26633376\, p^{20}$$

$$- 5044328352\, p^{12} q^2 + 11516520768\, p^{10} q^4 - 9722483040\, p^8 q^6$$

$$+ 1030464\, p^{22}) \Big/ \Big((1 + 20\, p - 120\, p^8 q^2 + 240\, p^6 q^4 - 240\, p^4 q^6$$

$$+ 120\, p^2 q^8 - 960\, p^7 q^2 + 1440\, p^5 q^4 - 960\, p^3 q^6 + 240\, p\, q^8$$

$$+ 3312\, p^6 - 48\, q^6 + 2304\, p^7 + 24\, p^{10} + 996\, p^8 + 3096\, p^4 q^4$$

$$- 1104\, p^2 q^6 - 4752\, p^4 q^2 + 1488\, p^2 q^4 - 4992\, p^5 q^2 + 3072\, p^3 q^4$$

$$- 384\, p\, q^6 + 704\, p^3 + 1844\, p^4 + 52\, q^4 + 3072\, p^5 + 36\, q^8 + 162\, p^2$$

$$- 18\, q^2 - 192\, p\, q^2 - 936\, p^2 q^2 - 2688\, p^3 q^2 + 384\, p\, q^4 - 3024\, p^6 q^2$$

$$+ 240\, p^9 - 24\, q^{10})^2 (\, 2\, p^2 - 2\, q^2 + 4\, p + 1\,)^3\Big)$$

$$BBq := 128\pi^2\, q(6 + 14648610816\, p^{11} q^8 - 15376112640\, p^{13} q^6$$

$$+ 433152\, p^{23} + 9785318400\, p^{15} q^4 - 5472061056\, p^{10} q^{10}$$

$$- 4359168\, p^{21} q^2 + 19768320\, p^{19} q^4 - 53222400\, p^{17} q^6$$

$$+ 94279680\, p^{15} q^8 + 7504104384\, p^{12} q^8 + 488218752\, q^{12} p^{10}$$

$$- 92880\, q^{18} + 69066432\, q^{16} p^6 - 11492928\, q^{18} p^4 + 795456\, q^{20} p^2$$

$$- 231389568\, q^{14} p^8 + 318\, p - 6578668800\, p^{14} q^6$$

$$+ 3589568352\, p^{16}\, q^{4} + 2716519680\, p^{13}\, q^{8} + 961248384\, p^{17}\, q^{4}$$

$$- 263672064\, p^{19}\, q^{2} - 2026073088\, p^{15}\, q^{6} + 176482368\, p^{18}\, q^{4}$$

$$+ 655648128\, p^{14}\, q^{8} - 422615232\, p^{16}\, q^{6} - 23740777728\, p^{14}\, q^{2}$$

$$+ 709632\, p^{3}\, q^{20} - 8603642880\, p^{9}\, q^{10} - 5575680\, p^{5}\, q^{18}$$

$$- 57784320\, p^{9}\, q^{14} + 97929216\, p^{11}\, q^{12} + 22809600\, p^{7}\, q^{16}$$

$$- 686206080\, p^{12}\, q^{10} - 114960384\, p^{13}\, q^{10} + 2514939840\, p^{8}\, q^{12}$$

$$- 5598144\, p^{2}\, q^{18} + 1837152\, p\, q^{16} - 11301120\, p^{3}\, q^{18}$$

$$+ 109496448\, p^{5}\, q^{16} - 511847424\, p^{7}\, q^{14} + 1391572224\, p^{9}\, q^{12}$$

$$+ 41464557216\, p^{12}\, q^{4} + 20902420320\, p^{10}\, q^{8}$$

$$- 2398201344\, p^{11}\, q^{10} - 43221312\, p^{20}\, q^{2} - 26834519616\, p^{12}\, q^{6}$$

$$- 8309887824\, p^{16}\, q^{2} - 1113600960\, p^{18}\, q^{2} - 15679820544\, p^{15}\, q^{2}$$

$$+ 17956164528\, p^{8}\, q^{8} + 2460984384\, p^{6}\, q^{12} - 35809310976\, p^{11}\, q^{6}$$

$$+ 512486688\, p^{4}\, q^{12} - 580654080\, p^{5}\, q^{14} - 687543552\, p^{6}\, q^{14}$$

$$- 37139236224\, p^{10}\, q^{6} - 9596577888\, p^{8}\, q^{10} + 20235624768\, p^{14}\, q^{4}$$

$$+ 32556544128\, p^{13}\, q^{4} - 4507840512\, p^{6}\, q^{10} + 3020645376\, p^{7}\, q^{12}$$

$$+ 22266325056\, p^{9}\, q^{8} - 18794880\, p^{2}\, q^{14} + 50664960\, p^{3}\, q^{16}$$

$$+ 98413920\, p^{4}\, q^{16} - 309027648\, p^{4}\, q^{14} - 8640\, q^{22}$$

$$+ 11071360608\, p^{7}\, q^{8} - 29198529888\, p^{13}\, q^{2} + 59616\, q^{20}$$

$$+ 70056\, q^{16} - 1299456\, p\, q^{18} - 3471446016\, p^{17}\, q^{2}$$

$$+ 42319363488\, p^{11}\, q^{4} - 27648\, p\, q^{22} - 30291101664\, p^{9}\, q^{6}$$

$$- 7715314944\, p^{7}\, q^{10} - 1920448032\, p^{5}\, q^{10} + 8515584\, q^{12}\, p^{12}$$

$$- 6082560\, q^{14}\, p^{10} + 126983136\, p^3\, q^{12} + 3041280\, q^{16}\, p^8$$

$$- 1013760\, q^{18}\, p^6 + 355968\, p\, q^{20} + 5473370856\, p^{16}$$

$$+ 1366193280\, p^5\, q^{12} - 3041280\, p^{18}\, q^6 - 8515584\, p^{14}\, q^{10}$$

$$+ 6082560\, p^{16}\, q^8 + 1013760\, p^{20}\, q^4 + 202752\, q^{20}\, p^4 - 18432\, q^{22}\, p^2$$

$$- 202752\, p^{22}\, q^2 + 14191056\, p^2\, q^{16} - 1713312\, p\, q^{14} + 18432\, p^{24}$$

$$- 4319625748\, p^8\, q^2 + 2035938088\, p^6\, q^4 - 353510568\, p^4\, q^6$$

$$+ 18274004\, p^2\, q^8 - 1619635136\, p^7\, q^2 + 581526432\, p^5\, q^4$$

$$- 69409984\, p^3\, q^6 + 1881264\, p\, q^8 + 38999658\, p^6 - 39234\, q^6$$

$$+ 160506632\, p^7 + 5248215616\, p^{11} + 2968887364\, p^{10}$$

$$+ 524415294\, p^8 + 129079300\, p^4\, q^4 - 9814664\, p^2\, q^6$$

$$- 20821270\, p^4\, q^2 + 2475502\, p^2\, q^4 - 115223608\, p^5\, q^2$$

$$+ 21391704\, p^3\, q^4 - 892968\, p\, q^6 - 9366006656\, p^9\, q^2$$

$$+ 5675965056\, p^7\, q^4 - 1371522304\, p^5\, q^6 + 117200192\, p^3\, q^8$$

$$- 2058624\, p\, q^{10} + 108376\, p^3 + 1050269\, p^4 + 5781\, q^4$$

$$+ 7365848\, p^5 + 97902\, q^8 + 7574\, p^2 - 342\, q^2 - 13704\, p\, q^2$$

$$- 249394\, p^2\, q^2 - 2754256\, p^3\, q^2 + 176760\, p\, q^4 - 487146792\, p^6\, q^2$$

$$+ 7676559412\, p^{12} + 58644\, q^{12} + 9323754624\, p^{13}$$

$$- 100845312\, p^3\, q^{14} + 1381560368\, p^9 - 99204\, q^{10}$$

$$+ 1930052736\, p^5\, q^8 - 133255296\, p^3\, q^{10} + 1744128\, p\, q^{12}$$

$$- 24370298496\, p^{11}\, q^2 + 23403059712\, p^9\, q^4 - 10117308672\, p^7\, q^6$$

$$+ 7894085664\, p^{15} - 16648949080\, p^{10}\, q^2 + 12780306028\, p^8\, q^4$$

$$
\begin{aligned}
&- 4175364688\, p^6\, q^6 + 546235084\, p^4\, q^8 - 20772568\, p^2\, q^{10} \\
&+ 3114392160\, p^{17} + 1436885328\, p^{18} + 19838136\, p^2\, q^{12} \\
&+ 5257200984\, p^6\, q^8 - 595816968\, p^4\, q^{10} + 9414937416\, p^{14} \\
&- 51624\, q^{14} + 31902336\, p^{21} + 527915520\, p^{19} + 150386400\, p^{20} \\
&- 29407276248\, p^{12}\, q^2 + 34909311144\, p^{10}\, q^4 - 19598142840\, p^8\, q^6 \\
&+ 4722624\, p^{22}) \Big/ \Big((1 + 20\, p - 120\, p^8\, q^2 + 240\, p^6\, q^4 - 240\, p^4\, q^6 \\
&+ 120\, p^2\, q^8 - 960\, p^7\, q^2 + 1440\, p^5\, q^4 - 960\, p^3\, q^6 + 240\, p\, q^8 \\
&+ 3312\, p^6 - 48\, q^6 + 2304\, p^7 + 24\, p^{10} + 996\, p^8 + 3096\, p^4\, q^4 \\
&- 1104\, p^2\, q^6 - 4752\, p^4\, q^2 + 1488\, p^2\, q^4 - 4992\, p^5\, q^2 + 3072\, p^3\, q^4 \\
&- 384\, p\, q^6 + 704\, p^3 + 1844\, p^4 + 52\, q^4 + 3072\, p^5 + 36\, q^8 + 162\, p^2 \\
&- 18\, q^2 - 192\, p\, q^2 - 936\, p^2\, q^2 - 2688\, p^3\, q^2 + 384\, p\, q^4 - 3024\, p^6\, q^2 \\
&+ 240\, p^9 - 24\, q^{10})^2 (\, 2\, p^2 - 2\, q^2 + 4\, p + 1\,)^3\Big)
\end{aligned}
$$

Symbolic solution of the first critical point equation:

M:=[solve(BBq=0,q)];

$$
\begin{aligned}
M := \Big[&0, \mathrm{RootOf}(-6 - 433152\, p^{23} - 318\, p + (353510568\, p^4 \\
&+ 53222400\, p^{17} + 26834519616\, p^{12} + 37139236224\, p^{10} \\
&+ 19598142840\, p^8 + 35809310976\, p^{11} + 4175364688\, p^6 \\
&+ 1371522304\, p^5 + 10117308672\, p^7 + 422615232\, p^{16} \\
&+ 30291101664\, p^9 + 892968\, p + 39234 + 15376112640\, p^{13} \\
&+ 2026073088\, p^{15} + 6578668800\, p^{14} + 9814664\, p^2 + 3041280\, p^{18} \\
&+ 69409984\, p^3)\_Z^3 + (11492928\, p^4 + 1013760\, p^6 + 1299456\, p
\end{aligned}
$$

$$+ 5575680\,p^5 + 11301120\,p^3 + 92880 + 5598144\,p^2)\_Z^9$$

$$+ (-202752\,p^4 - 355968\,p - 59616 - 709632\,p^3 - 795456\,p^2)\,\_Z^{10}$$

$$+ (-546235084\,p^4 - 2716519680\,p^{13} - 20902420320\,p^{10}$$

$$- 1930052736\,p^5 - 1881264\,p - 18274004\,p^2 - 14648610816\,p^{11}$$

$$- 17956164528\,p^8 - 117200192\,p^3 - 655648128\,p^{14}$$

$$- 22266325056\,p^9 - 5257200984\,p^6 - 97902 - 6082560\,p^{16}$$

$$- 94279680\,p^{15} - 7504104384\,p^{12} - 11071360608\,p^7)\_Z^4$$

$$- 5473370856\,p^{16} + (1619635136\,p^7 + 487146792\,p^6 + 13704\,p$$

$$+ 249394\,p^2 + 16648949080\,p^{10} + 342 + 202752\,p^{22}$$

$$+ 4359168\,p^{21} + 9366006656\,p^9 + 4319625748\,p^8$$

$$+ 29198529888\,p^{13} + 1113600960\,p^{18} + 263672064\,p^{19}$$

$$+ 29407276248\,p^{12} + 8309887824\,p^{16} + 3471446016\,p^{17}$$

$$+ 15679820544\,p^{15} + 43221312\,p^{20} + 2754256\,p^3 + 20821270\,p^4$$

$$+ 24370298496\,p^{11} + 115223608\,p^5 + 23740777728\,p^{14})\_Z + ($$

$$18794880\,p^2 + 511847424\,p^7 + 687543552\,p^6 + 231389568\,p^8$$

$$+ 580654080\,p^5 + 309027648\,p^4 + 6082560\,p^{10} + 57784320\,p^9$$

$$+ 100845312\,p^3 + 51624 + 1713312\,p)\_Z^7$$

$$+ (8640 + 27648\,p + 18432\,p^2)\,\_Z^{11} + (-8515584\,p^{12}$$

$$- 2514939840\,p^8 - 3020645376\,p^7 - 19838136\,p^2 - 58644$$

$$- 1366193280\,p^5 - 1744128\,p - 126983136\,p^3 - 2460984384\,p^6$$

$$- 512486688\,p^4 - 97929216\,p^{11} - 1391572224\,p^9 - 488218752\,p^{10}$$

$$)\_Z^6 + (8603642880\,p^9 + 99204 + 4507840512\,p^6$$
$$+ 9596577888\,p^8 + 133255296\,p^3 + 1920448032\,p^5$$
$$+ 20772568\,p^2 + 5472061056\,p^{10} + 8515584\,p^{14}$$
$$+ 2398201344\,p^{11} + 2058624\,p + 114960384\,p^{13} + 595816968\,p^4$$
$$+ 7715314944\,p^7 + 686206080\,p^{12})\_Z^5 + (-22809600\,p^7$$
$$- 109496448\,p^5 - 69066432\,p^6 - 14191056\,p^2 - 1837152\,p$$
$$- 3041280\,p^8 - 50664960\,p^3 - 98413920\,p^4 - 70056)\_Z^8 + ($$
$$-41464557216\,p^{12} - 961248384\,p^{17} - 581526432\,p^5 - 1013760\,p^{20}$$
$$- 129079300\,p^4 - 20235624768\,p^{14} - 23403059712\,p^9$$
$$- 3589568352\,p^{16} - 176482368\,p^{18} - 42319363488\,p^{11}$$
$$- 2035938088\,p^6 - 19768320\,p^{19} - 5781 - 32556544128\,p^{13}$$
$$- 5675965056\,p^7 - 176760\,p - 34909311144\,p^{10} - 9785318400\,p^{15}$$
$$- 21391704\,p^3 - 12780306028\,p^8 - 2475502\,p^2)\_Z^2 - 18432\,p^{24}$$
$$- 38999658\,p^6 - 160506632\,p^7 - 5248215616\,p^{11}$$
$$- 2968887364\,p^{10} - 524415294\,p^8 - 108376\,p^3 - 1050269\,p^4$$
$$- 7365848\,p^5 - 7574\,p^2 - 7676559412\,p^{12} - 9323754624\,p^{13}$$
$$- 1381560368\,p^9 - 7894085664\,p^{15} - 3114392160\,p^{17}$$
$$- 1436885328\,p^{18} - 9414937416\,p^{14} - 31902336\,p^{21}$$
$$- 527915520\,p^{19} - 150386400\,p^{20} - 4722624\,p^{22})^{1/2}, -\mathrm{RootOf}(-6$$
$$- 433152\,p^{23} - 318\,p + (353510568\,p^4 + 53222400\,p^{17}$$
$$+ 26834519616\,p^{12} + 37139236224\,p^{10} + 19598142840\,p^8$$

$$+ 35809310976\,p^{11} + 4175364688\,p^{6} + 1371522304\,p^{5}$$

$$+ 10117308672\,p^{7} + 422615232\,p^{16} + 30291101664\,p^{9}$$

$$+ 892968\,p + 39234 + 15376112640\,p^{13} + 2026073088\,p^{15}$$

$$+ 6578668800\,p^{14} + 9814664\,p^{2} + 3041280\,p^{18} + 69409984\,p^{3})$$

$$\_Z^{3} + (11492928\,p^{4} + 1013760\,p^{6} + 1299456\,p + 5575680\,p^{5}$$

$$+ 11301120\,p^{3} + 92880 + 5598144\,p^{2})\_Z^{9}$$

$$+ (-202752\,p^{4} - 355968\,p - 59616 - 709632\,p^{3} - 795456\,p^{2})\,\_Z^{10}$$

$$+ (-546235084\,p^{4} - 2716519680\,p^{13} - 20902420320\,p^{10}$$

$$- 1930052736\,p^{5} - 1881264\,p - 18274004\,p^{2} - 14648610816\,p^{11}$$

$$- 17956164528\,p^{8} - 117200192\,p^{3} - 655648128\,p^{14}$$

$$- 22266325056\,p^{9} - 5257200984\,p^{6} - 97902 - 6082560\,p^{16}$$

$$- 94279680\,p^{15} - 7504104384\,p^{12} - 11071360608\,p^{7})\_Z^{4}$$

$$- 5473370856\,p^{16} + (1619635136\,p^{7} + 487146792\,p^{6} + 13704\,p$$

$$+ 249394\,p^{2} + 16648949080\,p^{10} + 342 + 202752\,p^{22}$$

$$+ 4359168\,p^{21} + 9366006656\,p^{9} + 4319625748\,p^{8}$$

$$+ 29198529888\,p^{13} + 1113600960\,p^{18} + 263672064\,p^{19}$$

$$+ 29407276248\,p^{12} + 8309887824\,p^{16} + 3471446016\,p^{17}$$

$$+ 15679820544\,p^{15} + 43221312\,p^{20} + 2754256\,p^{3} + 20821270\,p^{4}$$

$$+ 24370298496\,p^{11} + 115223608\,p^{5} + 23740777728\,p^{14})\_Z + ($$

$$18794880\,p^{2} + 511847424\,p^{7} + 687543552\,p^{6} + 231389568\,p^{8}$$

$$+ 580654080\,p^{5} + 309027648\,p^{4} + 6082560\,p^{10} + 57784320\,p^{9}$$

$$+\,100845312\,p^3+51624+1713312\,p)\_Z^7$$

$$+\,(\,8640+27648\,p+18432\,p^2\,)\,\_Z^{11}+(-8515584\,p^{12}$$

$$-\,2514939840\,p^8-3020645376\,p^7-19838136\,p^2-58644$$

$$-\,1366193280\,p^5-1744128\,p-126983136\,p^3-2460984384\,p^6$$

$$-\,512486688\,p^4-97929216\,p^{11}-1391572224\,p^9-488218752\,p^{10}$$

$$)\_Z^6+(8603642880\,p^9+99204+4507840512\,p^6$$

$$+\,9596577888\,p^8+133255296\,p^3+1920448032\,p^5$$

$$+\,20772568\,p^2+5472061056\,p^{10}+8515584\,p^{14}$$

$$+\,2398201344\,p^{11}+2058624\,p+114960384\,p^{13}+595816968\,p^4$$

$$+\,7715314944\,p^7+686206080\,p^{12})\_Z^5+(-22809600\,p^7$$

$$-\,109496448\,p^5-69066432\,p^6-14191056\,p^2-1837152\,p$$

$$-\,3041280\,p^8-50664960\,p^3-98413920\,p^4-70056)\_Z^8+($$

$$-41464557216\,p^{12}-961248384\,p^{17}-581526432\,p^5-1013760\,p^{20}$$

$$-\,129079300\,p^4-20235624768\,p^{14}-23403059712\,p^9$$

$$-\,3589568352\,p^{16}-176482368\,p^{18}-42319363488\,p^{11}$$

$$-\,2035938088\,p^6-19768320\,p^{19}-5781-32556544128\,p^{13}$$

$$-\,5675965056\,p^7-176760\,p-34909311144\,p^{10}-9785318400\,p^{15}$$

$$-\,21391704\,p^3-12780306028\,p^8-2475502\,p^2)\_Z^2-18432\,p^{24}$$

$$-\,38999658\,p^6-160506632\,p^7-5248215616\,p^{11}$$

$$-\,2968887364\,p^{10}-524415294\,p^8-108376\,p^3-1050269\,p^4$$

$$-\,7365848\,p^5-7574\,p^2-7676559412\,p^{12}-9323754624\,p^{13}$$

$$- 1381560368\,p^9 - 7894085664\,p^{15} - 3114392160\,p^{17}$$
$$- 1436885328\,p^{18} - 9414937416\,p^{14} - 31902336\,p^{21}$$
$$- 527915520\,p^{19} - 150386400\,p^{20} - 4722624\,p^{22})^{1/2}\Big]$$

Symbolic solution of the second critical point equation:

N:=[solve(BBp=0,p)];

$$N := [\mathrm{RootOf}(-3 + 4608\,\_Z^{24} - 73584\,q^{18} - 63936\,q^{22} + 68256\,q^{20}$$
$$+ 171000\,q^{16} + (-4390366768\,q^6 - 1721675520\,q^{14}$$
$$+ 867915384\,q^4 + 195041088\,q^{16} + 8233139592\,q^8 - 3041280\,q^{18}$$
$$+ 90363656\,q^2 + 5367729600\,q^{12} - 8583881472\,q^{10} - 28833490)$$
$$\_Z^7 + (-20782080\,q^{18} + 3620019456\,q^8 - 1712358144\,q^{14}$$
$$+ 194401008\,q^4 - 4476432480\,q^{10} + 47869968\,q^2$$
$$+ 371718720\,q^{16} + 3556895040\,q^{12} - 1570928848\,q^6 - 8116052)$$
$$\_Z^6 + (1013760\,q^{20} + 16172078\,q^2 - 1098828480\,q^{14}$$
$$- 54378432\,q^{18} - 1812669336\,q^{10} - 1758997 + 404672544\,q^{16}$$
$$+ 1686984288\,q^{12} - 436101496\,q^6 + 21714476\,q^4$$
$$+ 1265245780\,q^8)\_Z^5 + (3805468\,q^2 - 89921744\,q^6$$
$$- 2833684\,q^4 + 592051200\,q^{12} + 4866048\,q^{20} - 576682080\,q^{10}$$
$$+ 345880844\,q^8 - 68795712\,q^{18} - 286863 + 255548448\,q^{16}$$
$$- 459767232\,q^{14})\_Z^4 + (-129302400\,q^{14} + 158634408\,q^{12}$$
$$- 43788096\,q^{18} - 33922 - 12615320\,q^6 - 1763046\,q^4 + 623746\,q^2$$
$$+ 71135740\,q^8 - 141457384\,q^{10} + 7945920\,q^{20} - 202752\,q^{22}$$

$$+ 92186544\,q^{16})\_Z^3 + (18652944\,q^{16} - 25175064\,q^{10}$$

$$+ 5204160\,q^{20} - 964576\,q^6 - 562176\,q^{22} + 31695936\,q^{12}$$

$$- 25996896\,q^{14} - 355804\,q^4 + 67982\,q^2 - 13370688\,q^{18}$$

$$+ 10273336\,q^8 - 2735)\_Z^2 + (-134 + 4418\,q^2 - 36333\,q^4$$

$$- 2902348\,q^{10} + 2272248\,q^{16} - 3665016\,q^{14} + 4152652\,q^{12}$$

$$+ 1293984\,q^{20} - 1650096\,q^{18} + 915162\,q^8 - 4214\,q^6 - 383040\,q^{22}$$

$$+ 18432\,q^{24})\_Z + (\,101952 - 18432\,q^2\,)\,\_Z^{23}$$

$$+ (\,1030464 - 470016\,q^2\,)\,\_Z^{22}$$

$$+ (\,202752\,q^4 + 6339744 - 5416128\,q^2\,)\,\_Z^{21}$$

$$+ (\,26633376 + 4460544\,q^4 - 37707840\,q^2\,)\,\_Z^{20}$$

$$+ (\,-178765632\,q^2 - 1013760\,q^6 + 81014832 + 44540352\,q^4\,)\,\_Z^{19}$$

$$+ (\,-615829824\,q^2 + 183919248 + 269258688\,q^4 - 19768320\,q^6\,)$$

$$\_Z^{18} + (-1603706160\,q^2 + 1108576800\,q^4 + 314513592$$

$$- 173959488\,q^6 + 3041280\,q^8)\_Z^{17} + (-3236538096\,q^2$$

$$+ 3312435744\,q^4 - 920306880\,q^6 + 398440728 + 52462080\,q^8)$$

$$\_Z^{16} + (-3288432384\,q^6 - 5138724864\,q^2 + 346236504$$

$$+ 7465955520\,q^4 + 404514432\,q^8 - 6082560\,q^{10})\_Z^{15} + ($$

$$1854029952\,q^8 + 13009843776\,q^4 - 8446618368\,q^6$$

$$- 91846656\,q^{10} + 137761152 - 6465733920\,q^2)\_Z^{14} + ($$

$$-6446363400\,q^2 + 5663843136\,q^8 - 16193640384\,q^6 - 134951508$$

$$- 611839872\,q^{10} + 17805113568\,q^4 + 8515584\,q^{12})\_Z^{13} + ($$

$$-5044328352\,q^2 - 23738270400\,q^6 - 331159820 - 2384774784\,q^{10}$$
$$+ 110702592\,q^{12} + 12249292608\,q^8 + 19322598336\,q^4)\_Z^{12} + ($$
$$-3017588584\,q^2 + 623645568\,q^{12} + 19439196576\,q^8$$
$$+ 16708365432\,q^4 - 6079425408\,q^{10} - 8515584\,q^{14} - 373586868$$
$$- 27040838016\,q^6)\_Z^{11} + (-1286889368\,q^2 + 11516520768\,q^4$$
$$- 24211394592\,q^6 + 2010911616\,q^{12} + 23166590688\,q^8$$
$$- 292717608 - 10730224512\,q^{10} - 93063168\,q^{14})\_Z^{10} + ($$
$$-17183678184\,q^6 + 4127360832\,q^{12} + 6082560\,q^{16} - 173442422$$
$$+ 21083863056\,q^8 + 6294641716\,q^4 - 13548257952\,q^{10}$$
$$- 298394108\,q^2 - 429812352\,q^{14})\_Z^9 + (-12501252384\,q^{10}$$
$$+ 2688898092\,q^4 - 79907210 - 1100504448\,q^{14} - 9722483040\,q^6$$
$$+ 53940440\,q^2 + 14866296624\,q^8 + 5674423104\,q^{12}$$
$$+ 53982720\,q^{16})\_Z^8 + 13824\,q^{24} + 3732\,q^6 - 1551\,q^4 + 36462\,q^8$$
$$+ 129\,q^2 + 264276\,q^{12} - 166152\,q^{10} - 252576\,q^{14})]$$

Evaluation of the first set of critical point candidates:

VV:=[intercept(M[2]=0,N[1]=0,p,q)];

$$VV := \Big[\Big\{q = \mathrm{RootOf}(-1 + 4608\,\_Z^{12} + 22752\,\_Z^{10} - 21312\,\_Z^{11}$$
$$+ 12154\,\_Z^4 - 24528\,\_Z^9 + 57000\,\_Z^8 - 84192\,\_Z^7 + 88092\,\_Z^6$$
$$+ 1244\,\_Z^3 - 55384\,\_Z^5 - 517\,\_Z^2 + 43\,\_Z)^{1/2}, p = \mathrm{RootOf}(6$$
$$+ 18432\,\_Z^{24} + 527915520\,\_Z^{19} + 1436885328\,\_Z^{18}$$
$$+ 9414937416\,\_Z^{14} + 433152\,\_Z^{23} + 7676559412\,\_Z^{12} + 318\,\_Z$$

$$+ 1050269\_Z^4 + 108376\_Z^3 + 7574\_Z^2 + 1381560368\_Z^9$$

$$+ 524415294\_Z^8 + 2968887364\_Z^{10} + 5248215616\_Z^{11}$$

$$+ 7365848\_Z^5 + 160506632\_Z^7 + 38999658\_Z^6 + 4722624\_Z^{22}$$

$$+ 150386400\_Z^{20} + 5473370856\_Z^{16} + 31902336\_Z^{21}$$

$$+ 7894085664\_Z^{15} + 9323754624\_Z^{13} + 3114392160\_Z^{17})\Big\}, \Big\{$$

$$p = \text{RootOf}(6 + 18432\_Z^{24} + 527915520\_Z^{19} + 1436885328\_Z^{18}$$

$$+ 9414937416\_Z^{14} + 433152\_Z^{23} + 7676559412\_Z^{12} + 318\_Z$$

$$+ 1050269\_Z^4 + 108376\_Z^3 + 7574\_Z^2 + 1381560368\_Z^9$$

$$+ 524415294\_Z^8 + 2968887364\_Z^{10} + 5248215616\_Z^{11}$$

$$+ 7365848\_Z^5 + 160506632\_Z^7 + 38999658\_Z^6 + 4722624\_Z^{22}$$

$$+ 150386400\_Z^{20} + 5473370856\_Z^{16} + 31902336\_Z^{21}$$

$$+ 7894085664\_Z^{15} + 9323754624\_Z^{13} + 3114392160\_Z^{17}), q =$$

$$-\text{RootOf}(-1 + 4608\_Z^{12} + 22752\_Z^{10} - 21312\_Z^{11} + 12154\_Z^4$$

$$- 24528\_Z^9 + 57000\_Z^8 - 84192\_Z^7 + 88092\_Z^6 + 1244\_Z^3$$

$$- 55384\_Z^5 - 517\_Z^2 + 43\_Z)^{1/2}\Big\}\Big]$$

Numerical values of first part of the above symbolically found critical points. One inspects that the solutions are either complex or have $p < 0$, so are not in the Kähler cone.

allvalues(VV[1]);

$$\{\%18, p = -3.802974653\}, \{\%18, p = -3.170767370\},$$

$$\{p = -2.183469616, \%18\}, \{\%18, p = -1.840896415\},$$

$\{\%18, p = -1.707106781\}, \{\%18, p = -1.331931115\},$

$\{\%18, \%10\}, \{\%18, \%9\}, \{\%18, \%8\}, \{\%18, \%7\},$

$\{\%18, p = -.6717383310\}, \{\%18, \%6\}, \{\%18, \%5\},$

$\{\%18, \%4\}, \{\%18, \%3\}, \{\%18, \%2\}, \{\%18, \%1\},$

$\{\%18, p = -.4209714630\}, \{\%18, p = -.3673473418\},$

$\{\%18, p = -.2928932188\}, \{\%18, p = -.1708834708\},$

$\{\%18, p = -.1607167876\}, \{\%18, p = -.1591035847\},$

$\{\%18, p = -.08568766797\}, \{p = -3.802974653, \%17\},$

$\{p = -3.170767370, \%17\}, \{p = -2.183469616, \%17\},$

$\{p = -1.840896415, \%17\}, \{p = -1.707106781, \%17\},$

$\{p = -1.331931115, \%17\}, \{\%10, \%17\}, \{\%9, \%17\},$

$\{\%8, \%17\}, \{\%7, \%17\}, \{p = -.6717383310, \%17\},$

$\{\%6, \%17\}, \{\%5, \%17\}, \{\%4, \%17\}, \{\%3, \%17\},$

$\{\%2, \%17\}, \{\%1, \%17\}, \{p = -.4209714630, \%17\},$

$\{p = -.3673473418, \%17\}, \{p = -.2928932188, \%17\},$

$\{p = -.1708834708, \%17\}, \{p = -.1607167876, \%17\},$

$\{p = -.1591035847, \%17\}, \{p = -.08568766797, \%17\},$

$\{p = -3.802974653, q = .4399400557\, I\},$

$\{p = -3.170767370, q = .4399400557\, I\},$

$\{p = -2.183469616, q = .4399400557\, I\},$

$\{p = -1.840896415, q = .4399400557\, I\},$

$\{p = -1.707106781, q = .4399400557\,I\}$,

$\{p = -1.331931115, q = .4399400557\,I\}$,

$\{\%10, q = .4399400557\,I\}, \{\%9, q = .4399400557\,I\}$,

$\{\%8, q = .4399400557\,I\}, \{\%7, q = .4399400557\,I\}$,

$\{p = -.6717383310, q = .4399400557\,I\}$,

$\{\%6, q = .4399400557\,I\}, \{\%5, q = .4399400557\,I\}$,

$\{\%4, q = .4399400557\,I\}, \{\%3, q = .4399400557\,I\}$,

$\{\%2, q = .4399400557\,I\}, \{\%1, q = .4399400557\,I\}$,

$\{p = -.4209714630, q = .4399400557\,I\}$,

$\{p = -.3673473418, q = .4399400557\,I\}$,

$\{p = -.2928932188, q = .4399400557\,I\}$,

$\{p = -.1708834708, q = .4399400557\,I\}$,

$\{p = -.1607167876, q = .4399400557\,I\}$,

$\{p = -.1591035847, q = .4399400557\,I\}$,

$\{p = -.08568766797, q = .4399400557\,I\}$,

$\{p = -3.802974653, q = .2004472566\}$,

$\{p = -3.170767370, q = .2004472566\}$,

$\{p = -2.183469616, q = .2004472566\}$,

$\{p = -1.840896415, q = .2004472566\}$,

$\{p = -1.707106781, q = .2004472566\}$,

$\{p = -1.331931115, q = .2004472566\}$,

$\{\%10, q = .2004472566\}, \{\%9, q = .2004472566\},$

$\{\%8, q = .2004472566\}, \{\%7, q = .2004472566\},$

$\{p = -.6717383310, q = .2004472566\}, \{\%6, q = .2004472566\},$

$\{\%5, q = .2004472566\}, \{\%4, q = .2004472566\},$

$\{\%3, q = .2004472566\}, \{\%2, q = .2004472566\},$

$\{\%1, q = .2004472566\}, \{p = -.4209714630, q = .2004472566\},$

$\{p = -.3673473418, q = .2004472566\},$

$\{p = -.2928932188, q = .2004472566\},$

$\{p = -.1708834708, q = .2004472566\},$

$\{p = -.1607167876, q = .2004472566\},$

$\{p = -.1591035847, q = .2004472566\},$

$\{p = -.08568766797, q = .2004472566\},$

$\{p = -3.802974653, \%16\}, \{p = -3.170767370, \%16\},$

$\{p = -2.183469616, \%16\}, \{p = -1.840896415, \%16\},$

$\{p = -1.707106781, \%16\}, \{p = -1.331931115, \%16\},$

$\{\%10, \%16\}, \{\%9, \%16\}, \{\%8, \%16\}, \{\%7, \%16\},$

$\{p = -.6717383310, \%16\}, \{\%6, \%16\}, \{\%5, \%16\},$

$\{\%4, \%16\}, \{\%3, \%16\}, \{\%2, \%16\}, \{\%1, \%16\},$

$\{p = -.4209714630, \%16\}, \{p = -.3673473418, \%16\},$

$\{p = -.2928932188, \%16\}, \{p = -.1708834708, \%16\},$

$\{p = -.1607167876, \%16\}, \{p = -.1591035847, \%16\},$

$\{ p = -.08568766797, \%16 \}, \{ p = -3.802974653, \%15 \},$

$\{ p = -3.170767370, \%15 \}, \{ p = -2.183469616, \%15 \},$

$\{ p = -1.840896415, \%15 \}, \{ p = -1.707106781, \%15 \},$

$\{ p = -1.331931115, \%15 \}, \{ \%10, \%15 \}, \{ \%9, \%15 \},$

$\{ \%8, \%15 \}, \{ \%7, \%15 \}, \{ p = -.6717383310, \%15 \},$

$\{ \%6, \%15 \}, \{ \%5, \%15 \}, \{ \%4, \%15 \}, \{ \%3, \%15 \},$

$\{ \%2, \%15 \}, \{ \%1, \%15 \}, \{ p = -.4209714630, \%15 \},$

$\{ p = -.3673473418, \%15 \}, \{ p = -.2928932188, \%15 \},$

$\{ p = -.1708834708, \%15 \}, \{ p = -.1607167876, \%15 \},$

$\{ p = -.1591035847, \%15 \}, \{ p = -.08568766797, \%15 \},$

$\{ p = -3.802974653, \%14 \}, \{ p = -3.170767370, \%14 \},$

$\{ p = -2.183469616, \%14 \}, \{ p = -1.840896415, \%14 \},$

$\{ p = -1.707106781, \%14 \}, \{ p = -1.331931115, \%14 \},$

$\{ \%10, \%14 \}, \{ \%9, \%14 \}, \{ \%8, \%14 \}, \{ \%7, \%14 \},$

$\{ p = -.6717383310, \%14 \}, \{ \%6, \%14 \}, \{ \%5, \%14 \},$

$\{ \%4, \%14 \}, \{ \%3, \%14 \}, \{ \%2, \%14 \}, \{ \%1, \%14 \},$

$\{ p = -.4209714630, \%14 \}, \{ p = -.3673473418, \%14 \},$

$\{ p = -.2928932188, \%14 \}, \{ p = -.1708834708, \%14 \},$

$\{ p = -.1607167876, \%14 \}, \{ p = -.1591035847, \%14 \},$

$\{ p = -.08568766797, \%14 \}, \{ p = -3.802974653, \%13 \},$

$\{ p = -3.170767370, \%13 \}, \{ p = -2.183469616, \%13 \},$

$\{p = -1.840896415, \%13\}, \{p = -1.707106781, \%13\},$

$\{p = -1.331931115, \%13\}, \{\%10, \%13\}, \{\%9, \%13\},$

$\{\%8, \%13\}, \{\%7, \%13\}, \{p = -.6717383310, \%13\},$

$\{\%6, \%13\}, \{\%5, \%13\}, \{\%4, \%13\}, \{\%3, \%13\},$

$\{\%2, \%13\}, \{\%1, \%13\}, \{p = -.4209714630, \%13\},$

$\{p = -.3673473418, \%13\}, \{p = -.2928932188, \%13\},$

$\{p = -.1708834708, \%13\}, \{p = -.1607167876, \%13\},$

$\{p = -.1591035847, \%13\}, \{p = -.08568766797, \%13\},$

$\{p = -3.802974653, q = .6764719278\},$

$\{p = -3.170767370, q = .6764719278\},$

$\{p = -2.183469616, q = .6764719278\},$

$\{p = -1.840896415, q = .6764719278\},$

$\{p = -1.707106781, q = .6764719278\},$

$\{p = -1.331931115, q = .6764719278\},$

$\{\%10, q = .6764719278\}, \{\%9, q = .6764719278\},$

$\{\%8, q = .6764719278\}, \{\%7, q = .6764719278\},$

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$\{\%5, q = .6764719278\}, \{\%4, q = .6764719278\},$

$\{\%3, q = .6764719278\}, \{\%2, q = .6764719278\},$

$\{\%1, q = .6764719278\}, \{p = -.4209714630, q = .6764719278\},$

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$\{ p = -.2928932188, q = .6764719278 \}$,

$\{ p = -.1708834708, q = .6764719278 \}$,

$\{ p = -.1607167876, q = .6764719278 \}$,

$\{ p = -.1591035847, q = .6764719278 \}$,

$\{ p = -.08568766797, q = .6764719278 \}$,

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$\{ p = -1.707106781, \%12 \}, \{ p = -1.331931115, \%12 \}$,

$\{ \%10, \%12 \}, \{ \%9, \%12 \}, \{ \%8, \%12 \}, \{ \%7, \%12 \}$,

$\{ p = -.6717383310, \%12 \}, \{ \%6, \%12 \}, \{ \%5, \%12 \}$,

$\{ \%4, \%12 \}, \{ \%3, \%12 \}, \{ \%2, \%12 \}, \{ \%1, \%12 \}$,

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$\{ p = -.2928932188, \%12 \}, \{ p = -.1708834708, \%12 \}$,

$\{ p = -.1607167876, \%12 \}, \{ p = -.1591035847, \%12 \}$,

$\{ p = -.08568766797, \%12 \}, \{ p = -3.802974653, \%11 \}$,

$\{ p = -3.170767370, \%11 \}, \{ p = -2.183469616, \%11 \}$,

$\{ p = -1.840896415, \%11 \}, \{ p = -1.707106781, \%11 \}$,

$\{ p = -1.331931115, \%11 \}, \{ \%10, \%11 \}, \{ \%9, \%11 \}$,

$\{ \%8, \%11 \}, \{ \%7, \%11 \}, \{ p = -.6717383310, \%11 \}$,

$\{ \%6, \%11 \}, \{ \%5, \%11 \}, \{ \%4, \%11 \}, \{ \%3, \%11 \}$,

$\{ \%2, \%11 \}, \{ \%1, \%11 \}, \{ p = -.4209714630, \%11 \}$,

$\{p = -.3673473418, \%11\}, \{p = -.2928932188, \%11\},$

$\{p = -.1708834708, \%11\}, \{p = -.1607167876, \%11\},$

$\{p = -.1591035847, \%11\}, \{p = -.08568766797, \%11\},$

$\{p = -3.802974653, q = 1.852038703\},$

$\{p = -3.170767370, q = 1.852038703\},$

$\{p = -2.183469616, q = 1.852038703\},$

$\{p = -1.840896415, q = 1.852038703\},$

$\{p = -1.707106781, q = 1.852038703\},$

$\{p = -1.331931115, q = 1.852038703\},$

$\{\%10, q = 1.852038703\}, \{\%9, q = 1.852038703\},$

$\{\%8, q = 1.852038703\}, \{\%7, q = 1.852038703\},$

$\{p = -.6717383310, q = 1.852038703\}, \{\%6, q = 1.852038703\},$

$\{\%5, q = 1.852038703\}, \{\%4, q = 1.852038703\},$

$\{\%3, q = 1.852038703\}, \{\%2, q = 1.852038703\},$

$\{\%1, q = 1.852038703\}, \{p = -.4209714630, q = 1.852038703\},$

$\{p = -.3673473418, q = 1.852038703\},$

$\{p = -.2928932188, q = 1.852038703\},$

$\{p = -.1708834708, q = 1.852038703\},$

$\{p = -.1607167876, q = 1.852038703\},$

$\{p = -.1591035847, q = 1.852038703\},$

$\{p = -.08568766797, q = 1.852038703\}$

$$\%1 := p = -.4460338892 + .2673861622\, I$$

$$\%2 := p = -.4460338892 - .2673861622\, I$$

$$\%3 := p = -.4697369777 + .2934682106\, I$$

$$\%4 := p = -.4697369777 - .2934682106\, I$$

$$\%5 := p = -.6034986533 + 1.065746913\, I$$

$$\%6 := p = -.6034986533 - 1.065746913\, I$$

$$\%7 := p = -1.000000000 + .8408964153\, I$$

$$\%8 := p = -1.000000000 - .8408964153\, I$$

$$\%9 := p = -1.047486572 + .3510224844\, I$$

$$\%10 := p = -1.047486572 - .3510224844\, I$$

$$\%11 := q = .9833861885 + .002804374974\, I$$

$$\%12 := q = .9833861885 - .002804374974\, I$$

$$\%13 := q = .7826336983 + .5736370024\, I$$

$$\%14 := q = .7826336983 - .5736370024\, I$$

$$\%15 := q = .3147997311 + .04558909220\, I$$

$$\%16 := q = .3147997311 - .04558909220\, I$$

$$\%17 := q = .5220842039 + 1.083852290\, I$$

$$\%18 := q = .5220842039 - 1.083852290\, I$$

Numerical values of second part of the above symbolically found critical points. Again one inspects that the solutions are either complex or have $p < 0$, so are not in the Kähler cone.

allvalues(VV[2]);

$$\{\,\%8, p = -3.802974653\,\}, \{\,\%7, p = -3.802974653\,\},$$
$$\{\,q = -.4399400557\,I, p = -3.802974653\,\},$$
$$\{\,q = -.2004472566, p = -3.802974653\,\},$$
$$\{\,\%6, p = -3.802974653\,\}, \{\,\%5, p = -3.802974653\,\},$$
$$\{\,\%4, p = -3.802974653\,\}, \{\,\%3, p = -3.802974653\,\},$$
$$\{\,q = -.6764719278, p = -3.802974653\,\},$$
$$\{\,\%2, p = -3.802974653\,\}, \{\,\%1, p = -3.802974653\,\},$$
$$\{\,p = -3.802974653, q = -1.852038703\,\},$$
$$\{\,\%8, p = -3.170767370\,\}, \{\,\%7, p = -3.170767370\,\},$$
$$\{\,q = -.4399400557\,I, p = -3.170767370\,\},$$
$$\{\,q = -.2004472566, p = -3.170767370\,\},$$
$$\{\,\%6, p = -3.170767370\,\}, \{\,\%5, p = -3.170767370\,\},$$
$$\{\,\%4, p = -3.170767370\,\}, \{\,\%3, p = -3.170767370\,\},$$
$$\{\,q = -.6764719278, p = -3.170767370\,\},$$
$$\{\,\%2, p = -3.170767370\,\}, \{\,\%1, p = -3.170767370\,\},$$
$$\{\,p = -3.170767370, q = -1.852038703\,\},$$
$$\{\,\%8, p = -2.183469616\,\}, \{\,\%7, p = -2.183469616\,\},$$

$\{ q = -.4399400557\, I, p = -2.183469616 \}$,

$\{ q = -.2004472566, p = -2.183469616 \}$,

$\{ \%6, p = -2.183469616 \}, \{ \%5, p = -2.183469616 \}$,

$\{ \%4, p = -2.183469616 \}, \{ \%3, p = -2.183469616 \}$,

$\{ q = -.6764719278, p = -2.183469616 \}$,

$\{ \%2, p = -2.183469616 \}, \{ p = -2.183469616, \%1 \}$,

$\{ p = -2.183469616, q = -1.852038703 \}$,

$\{ \%8, p = -1.840896415 \}, \{ \%7, p = -1.840896415 \}$,

$\{ q = -.4399400557\, I, p = -1.840896415 \}$,

$\{ q = -.2004472566, p = -1.840896415 \}$,

$\{ \%6, p = -1.840896415 \}, \{ \%5, p = -1.840896415 \}$,

$\{ \%4, p = -1.840896415 \}, \{ \%3, p = -1.840896415 \}$,

$\{ q = -.6764719278, p = -1.840896415 \}$,

$\{ \%2, p = -1.840896415 \}, \{ \%1, p = -1.840896415 \}$,

$\{ p = -1.840896415, q = -1.852038703 \}$,

$\{ \%8, p = -1.707106781 \}, \{ \%7, p = -1.707106781 \}$,

$\{ q = -.4399400557\, I, p = -1.707106781 \}$,

$\{ q = -.2004472566, p = -1.707106781 \}$,

$\{ \%6, p = -1.707106781 \}, \{ \%5, p = -1.707106781 \}$,

$\{ \%4, p = -1.707106781 \}, \{ \%3, p = -1.707106781 \}$,

$\{ q = -.6764719278, p = -1.707106781 \}$,

$$\{ \%2, p = -1.707106781 \}, \{ \%1, p = -1.707106781 \},$$

$$\{ p = -1.707106781, q = -1.852038703 \},$$

$$\{ \%8, p = -1.331931115 \}, \{ \%7, p = -1.331931115 \},$$

$$\{ q = -.4399400557\, I, p = -1.331931115 \},$$

$$\{ q = -.2004472566, p = -1.331931115 \},$$

$$\{ \%6, p = -1.331931115 \}, \{ \%5, p = -1.331931115 \},$$

$$\{ \%4, p = -1.331931115 \}, \{ \%3, p = -1.331931115 \},$$

$$\{ q = -.6764719278, p = -1.331931115 \},$$

$$\{ \%2, p = -1.331931115 \}, \{ \%1, p = -1.331931115 \},$$

$$\{ p = -1.331931115, q = -1.852038703 \}, \{ \%8, \%18 \},$$

$$\{ \%7, \%18 \}, \{ q = -.4399400557\, I, \%18 \},$$

$$\{ q = -.2004472566, \%18 \}, \{ \%6, \%18 \}, \{ \%5, \%18 \},$$

$$\{ \%4, \%18 \}, \{ \%3, \%18 \}, \{ q = -.6764719278, \%18 \},$$

$$\{ \%2, \%18 \}, \{ \%1, \%18 \}, \{ \%18, q = -1.852038703 \},$$

$$\{ \%8, \%17 \}, \{ \%7, \%17 \}, \{ q = -.4399400557\, I, \%17 \},$$

$$\{ q = -.2004472566, \%17 \}, \{ \%6, \%17 \}, \{ \%5, \%17 \},$$

$$\{ \%4, \%17 \}, \{ \%3, \%17 \}, \{ q = -.6764719278, \%17 \},$$

$$\{ \%2, \%17 \}, \{ \%1, \%17 \}, \{ \%17, q = -1.852038703 \},$$

$$\{ \%8, \%16 \}, \{ \%7, \%16 \}, \{ q = -.4399400557\, I, \%16 \},$$

$$\{ q = -.2004472566, \%16 \}, \{ \%6, \%16 \}, \{ \%5, \%16 \},$$

$$\{ \%4, \%16 \}, \{ \%3, \%16 \}, \{ q = -.6764719278, \%16 \},$$

$\{\%2, \%16\}, \{\%1, \%16\}, \{\%16, q = -1.852038703\},$

$\{\%8, \%15\}, \{\%7, \%15\}, \{q = -.4399400557\,I, \%15\},$

$\{q = -.2004472566, \%15\}, \{\%6, \%15\}, \{\%5, \%15\},$

$\{\%4, \%15\}, \{\%3, \%15\}, \{q = -.6764719278, \%15\},$

$\{\%2, \%15\}, \{\%1, \%15\}, \{\%15, q = -1.852038703\},$

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$\{q = -.4399400557\,I, p = -.6717383310\},$

$\{q = -.2004472566, p = -.6717383310\},$

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$\{\%4, p = -.6717383310\}, \{\%3, p = -.6717383310\},$

$\{q = -.6764719278, p = -.6717383310\},$

$\{\%2, p = -.6717383310\}, \{\%1, p = -.6717383310\},$

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$\{\%7, \%14\}, \{q = -.4399400557\,I, \%14\},$

$\{q = -.2004472566, \%14\}, \{\%6, \%14\}, \{\%5, \%14\},$

$\{\%4, \%14\}, \{\%3, \%14\}, \{q = -.6764719278, \%14\},$

$\{\%2, \%14\}, \{\%1, \%14\}, \{\%14, q = -1.852038703\},$

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$\{q = -.2004472566, \%13\}, \{\%6, \%13\}, \{\%5, \%13\},$

$\{\%4, \%13\}, \{\%3, \%13\}, \{q = -.6764719278, \%13\},$

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$\{\%8, \%12\}, \{\%7, \%12\}, \{q = -.4399400557\, I, \%12\},$

$\{q = -.2004472566, \%12\}, \{\%6, \%12\}, \{\%5, \%12\},$

$\{\%4, \%12\}, \{\%3, \%12\}, \{q = -.6764719278, \%12\},$

$\{\%2, \%12\}, \{\%1, \%12\}, \{\%12, q = -1.852038703\},$

$\{\%8, \%11\}, \{\%7, \%11\}, \{q = -.4399400557\, I, \%11\},$

$\{q = -.2004472566, \%11\}, \{\%6, \%11\}, \{\%5, \%11\},$

$\{\%4, \%11\}, \{\%3, \%11\}, \{q = -.6764719278, \%11\},$

$\{\%2, \%11\}, \{\%1, \%11\}, \{\%11, q = -1.852038703\},$

$\{\%8, \%10\}, \{\%7, \%10\}, \{q = -.4399400557\, I, \%10\},$

$\{q = -.2004472566, \%10\}, \{\%6, \%10\}, \{\%5, \%10\},$

$\{\%4, \%10\}, \{\%3, \%10\}, \{q = -.6764719278, \%10\},$

$\{\%2, \%10\}, \{\%1, \%10\}, \{\%10, q = -1.852038703\},$

$\{\%8, \%9\}, \{\%7, \%9\}, \{q = -.4399400557\, I, \%9\},$

$\{q = -.2004472566, \%9\}, \{\%6, \%9\}, \{\%5, \%9\}, \{\%4, \%9\},$

$\{\%3, \%9\}, \{q = -.6764719278, \%9\}, \{\%2, \%9\}, \{\%1, \%9\},$

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$\{\%7, p = -.4209714630\},$

$\{q = -.4399400557\, I, p = -.4209714630\},$

$\{q = -.2004472566, p = -.4209714630\},$

$\{\%6, p = -.4209714630\}, \{\%5, p = -.4209714630\},$

$\{\%4, p = -.4209714630\}, \{\%3, p = -.4209714630\},$

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$\{ \%2, p = -.4209714630 \}, \{ \%1, p = -.4209714630 \}$,

$\{ p = -.4209714630, q = -1.852038703 \}$,

$\{ \%8, p = -.3673473418 \}, \{ \%7, p = -.3673473418 \}$,

$\{ q = -.4399400557\, I, p = -.3673473418 \}$,

$\{ q = -.2004472566, p = -.3673473418 \}$,

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$\{ \%4, p = -.3673473418 \}, \{ \%3, p = -.3673473418 \}$,

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$\{ \%2, p = -.3673473418 \}, \{ \%1, p = -.3673473418 \}$,

$\{ p = -.3673473418, q = -1.852038703 \}$,

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$\{ \%6, p = -.2928932188 \}, \{ \%5, p = -.2928932188 \}$,

$\{ \%4, p = -.2928932188 \}, \{ \%3, p = -.2928932188 \}$,

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$\{ \%2, p = -.2928932188 \}, \{ \%1, p = -.2928932188 \}$,

$\{ p = -.2928932188, q = -1.852038703 \}$,

$\{ \%8, p = -.1708834708 \}, \{ \%7, p = -.1708834708 \}$,

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$\{\%4, p = -.1708834708\}, \{\%3, p = -.1708834708\},$

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$\{p = -.1708834708, q = -1.852038703\},$

$\{\%8, p = -.1607167876\}, \{\%7, p = -.1607167876\},$

$\{q = -.4399400557\, I, p = -.1607167876\},$

$\{q = -.2004472566, p = -.1607167876\},$

$\{\%6, p = -.1607167876\}, \{\%5, p = -.1607167876\},$

$\{\%4, p = -.1607167876\}, \{\%3, p = -.1607167876\},$

$\{q = -.6764719278, p = -.1607167876\},$

$\{\%2, p = -.1607167876\}, \{\%1, p = -.1607167876\},$

$\{p = -.1607167876, q = -1.852038703\},$

$\{\%8, p = -.1591035847\}, \{\%7, p = -.1591035847\},$

$\{q = -.4399400557\, I, p = -.1591035847\},$

$\{q = -.2004472566, p = -.1591035847\},$

$\{\%6, p = -.1591035847\}, \{\%5, p = -.1591035847\},$

$\{\%4, p = -.1591035847\}, \{\%3, p = -.1591035847\},$

$\{q = -.6764719278, p = -.1591035847\},$

$\{\%2, p = -.1591035847\}, \{\%1, p = -.1591035847\},$

$\{p = -.1591035847, q = -1.852038703\}$,

$\{\%8, p = -.08568766797\}, \{\%7, p = -.08568766797\}$,

$\{q = -.4399400557\,I, p = -.08568766797\}$,

$\{q = -.2004472566, p = -.08568766797\}$,

$\{\%6, p = -.08568766797\}, \{\%5, p = -.08568766797\}$,

$\{\%4, p = -.08568766797\}, \{\%3, p = -.08568766797\}$,

$\{q = -.6764719278, p = -.08568766797\}$,

$\{\%2, p = -.08568766797\}, \{\%1, p = -.08568766797\}$,

$\{p = -.08568766797, q = -1.852038703\}$

$\%1 := q = -.9833861885 - .002804374974\,I$

$\%2 := q = -.9833861885 + .002804374974\,I$

$\%3 := q = -.7826336983 - .5736370024\,I$

$\%4 := q = -.7826336983 + .5736370024\,I$

$\%5 := q = -.3147997311 - .04558909220\,I$

$\%6 := q = -.3147997311 + .04558909220\,I$

$\%7 := q = -.5220842039 - 1.083852290\,I$

$\%8 := q = -.5220842039 + 1.083852290\,I$

$\%9 := p = -.4460338892 + .2673861622\,I$

$\%10 := p = -.4460338892 - .2673861622\,I$

$\%11 := p = -.4697369777 + .2934682106\,I$

$\%12 := p = -.4697369777 - .2934682106\,I$

$$\%13 := p = -.6034986533 + 1.065746913\, I$$

$$\%14 := p = -.6034986533 - 1.065746913\, I$$

$$\%15 := p = -1.000000000 + .8408964153\, I$$

$$\%16 := p = -1.000000000 - .8408964153\, I$$

$$\%17 := p = -1.047486572 + .3510224844\, I$$

$$\%18 := p = -1.047486572 - .3510224844\, I$$

Evaluation of the second set of critical point candidates:

subs(q=0,N[1]);

$$\begin{aligned}&\mathrm{RootOf}(-3 + 4608\,\_Z^{24} + 81014832\,\_Z^{19} + 183919248\,\_Z^{18}\\ &\quad + 137761152\,\_Z^{14} + 101952\,\_Z^{23} - 331159820\,\_Z^{12} - 134\,\_Z\\ &\quad - 286863\,\_Z^{4} - 33922\,\_Z^{3} - 2735\,\_Z^{2} - 173442422\,\_Z^{9}\\ &\quad - 79907210\,\_Z^{8} - 292717608\,\_Z^{10} - 373586868\,\_Z^{11}\\ &\quad - 1758997\,\_Z^{5} - 28833490\,\_Z^{7} - 8116052\,\_Z^{6} + 1030464\,\_Z^{22}\\ &\quad + 26633376\,\_Z^{20} + 398440728\,\_Z^{16} + 6339744\,\_Z^{21}\\ &\quad + 346236504\,\_Z^{15} - 134951508\,\_Z^{13} + 314513592\,\_Z^{17})\end{aligned}$$

Numerical values of the final set of symbolically found critical point candidates. For these $q = 0$, and so a true critical point from this family will evaluate equal volumes on the two exceptional divisors.

All solutions **but one** are either complex or have $p < 0$, and so are not in the Kähler cone. The unique solution is the first on the fourth row from the bottom. This completes the proof.

allvalues(");

$-4.199614645, -2.931616733, -1.707106781, -1.314201991,$

$-.8506115576 - .06269240613\,I,$

$-.8506115576 + .06269240613\,I, -.4559444654,$

$-.4442866226 - .2945305897\,I, -.4442866226 + .2945305897\,I,$

$-.4389928395 - .2603594925\,I, -.4389928395 + .2603594925\,I,$

$-.3619247455, -.2928932188, -.2163970164, -.1688705464,$

$\mathbf{.9913521825}, -1.840896415, -1.000000000 - .8408964153\,I,$

$-1.000000000 + .8408964153\,I, -.1591035847, -1.840896415,$

$-1.000000000 - .8408964153\,I, -1.000000000 + .8408964153\,I,$

$-.1591035847$

For confirmation, the unique critical point in the Kähler cone is found directly by a numerical method. Though more efficient than the above procedure, the method does not guarantee that no critical point has been dropped in the process.

fsolve(BBp=0,BBq=0,p,q,p=0..infinity);

$$\{\,\mathbf{q = 0, p = .9913521825}\,\}$$

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