Kähler-Einstein Cone Metrics

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Abstract of the Dissertation
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On a compact complex manifold with an irreducible curve D we define cone metrics. The main theorem is that a given cone metric may be deformed, within a certain function space, to a Kähler-Einstein cone metric. This existence theorem is proved by working directly with the complex Monge-Ampère equation, and we follow the continuity method. The continuity method involves two parts, an openness part and a closedness part. We draw upon the work of several other authors, especially Mazzeo [14] and Melrose and Mendoza [15] in the openness part, and Yau [22] and Aubin [1], [2] in the closedness part. The main contributions of the paper are application of the theory of Mazzeo, Melrose, and Mendoza (as

above) to obtain closed range in the first part, and to understand how to obtain a priori bounds for functions which may achieve nonsmooth maxima in the second part.

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Chapter 1

Introduction

On a compact complex manifold with an irreducible curve D we define cone metrics. The main theorem is that a given cone metric may be deformed, within the same Kähler class, to a Kähler-Einstein cone metric. This existence theorem is proved by working directly with the complex Monge-Ampère equation, and we follow the continuity method. What it means to solve this equation is to solve it away from the divisor, that is, on the noncompact set which is the complement of the divisor. This noncompactness, and the singularities that arise as one approaches the divisor, create analytic difficulties.

We use the continuity method, which involves two parts, an openness part and a closedness part. We draw upon the work of several other authors, especially Mazzeo [14] and Melrose and Mendoza [15] in the openness part, and Yau [22] and Aubin [1] and [2] in the closedness part, but as indicated above, nontrivial modifications are needed. The main contributions of this paper therefore are:

- Application of the theory Melrose to obtain closed range in the first part.

and

- To understand how to obtain *a priori* bounds for functions which may achieve nonsmooth maxima in the second part.

Chapter 2

Kähler-Einstein Cone Metrics

2.1 Cone Metrics

Let M be a complex manifold with $\dim_C M = n$, and D a divisor with only one irreducible component, that is, a complex curve. We restrict ourselves to this situation, but use the letter D because it is not difficult to pass to the more general situation of an effective divisor with simple normal crossings. One way to state this condition is the following. Writing $D = \sum_{i=1}^{N} D_i$, where $\{D_i\}$ are the irreducible components, the D_i are smooth and they meet transversely. If U is a neighborhood of p through which exactly k divisors pass, there exist local holomorphic coordinates $(z_1, \cdots, z_n)$ such that $D \cap U = \{z_1 \ldots z_k = 0\}$.

We first give the exact definition of cone metrics. In a separate section below we make some computations which identify the leading terms of the metric and verify that those leading terms really do describe a cone, and hence justify the terminology. Let V be a smooth volume form on M, with D as above, and $0 < \alpha < 1$. With K_M denoting the canonical bundle, we assume that $K_M + \alpha D$ is ample. which means that

$$C_1(K_M) + \alpha C_1(D) \in H^2_{DR}(M)$$

contains a positive definite real closed (1.1) form. Let $[D]$ be the line bundle associated to D and s a global defining function. that is. $D = \{s = 0\}$. Choose a Hermitian metric in $[D]$ and denote the associated norm by $\|\cdot\|$. Put

$$\hat{V} = \frac{V}{\|s\|^{2\alpha}(1 - \varepsilon\|s\|^{2(1-\alpha)})^2}$$

Now compute

$$\begin{aligned}\partial\overline{\partial}\log\hat{V} = \partial\overline{\partial}\log V &- \alpha\partial\overline{\partial}\log\|s\|^2 \\ &- 2\partial\overline{\partial}\log(1 - \varepsilon\|s\|^{2(1-\alpha)}).\end{aligned}$$

A smooth volume form on M is the same as a Hermitian metric on the anti-canonical bundle K_M^{-1}. [9]. Therefore.

$$\begin{aligned}\partial\overline{\partial}\log\hat{V} &= -Ric(K_M^{-1}) + \alpha R([D]) \\ &- 2\partial\overline{\partial}\log(1 - \varepsilon\|s\|^{2(1-\alpha)}).\end{aligned}$$

The cohomology class of $-Ric(K_M^{-1}) + \alpha R([D])$ is

$$-C_1(K_M^{-1}) + \alpha C_1(D) = C_1(K_M) + \alpha C_1(D).$$

Since we assumed $C_1(K_M) + \alpha C_1(D) = C_1 > 0$. it is possible to choose V and $\|\cdot\|$ in such a way that the representative $-Ric(K_m^{-1}) + \alpha R([D])$ is positive definite. Then by choosing ε small enough. we can make sure that $\partial\overline{\partial}\log\hat{V} > 0$.

Now denote $\omega_0 = \partial\overline{\partial} \log \hat{V}$. defined on $\Omega := M - D$. By its very construction as $\partial\overline{\partial}$ of a real function . ω_0 is Hermitian of type (1,1) and as noted above it is positive definite. and so it defines a Kähler metric.

Strictly speaking. it is a Kähler metric only on the noncompact set $M - D$. but we also refer to it as a *singular* Kähler metric on all of M. or a Kähler *cone* metric in M. The parameter α is called the *cone angle*. Taking ω_0 as the original Kähler cone metric then. we seek to deform this to a Kähler-Einstein cone metric. In order to formulate the problem more precisely. first note that the singularities of ω_0 are mild enough that ω_0 can be integrated over all of M against smooth forms of the appropriate degree — so it defines a current. The problem can now be stated as that of finding ω with the following properties.

1. ω is Kähler-Einstein. i.e.. $\rho = -\omega$ on $M - D$.

2. $[\omega] = [\omega_0]$ in the sense of currents.

3. ω has the same kind of singularities as ω_0. Namely. there should exist constants c and C so that $c\omega_0 \leq \omega \leq C\omega_0$.

2.2 Background

We briefly review a few facts about Kähler-Einstein manifolds from the smooth. compact case. A complex manifold is Kähler-Einstein if it admits a metric whose Ricci and Kähler forms are related by $\rho = k\omega$ for a real number k. The first Chern class will be positive. negative. or zero according to the sign of k. This provides a necessary condition for the existence of a Kähler-Einstein metric. In fact, if $C_1(M) = 0$ or $C_1(M) < 0$, this condition is also sufficient. This has been shown by Aubin [1], [2] and by Yau [22]. For $C_1(M) > 0$ there are examples demonstrating its insufficiency. Additionally, for $C_1(M) > 0$, obstructions have been found by Matsushima, Futaki, and others. These are discussed in the book by Futaki [8]. Since these automatically vanish when there are no nontrivial holomorphic vector fields, Calabi asked whether a complex manifold with $C_1 > 0$ and no nontrivial holomorphic vector fields must admit a Kähler-Einstein metric. Tian has given an affirmative answer for the surface case [20].

The study of singular spaces and metrics arises in many contexts. From the complex point of view, the metrics considered here may be considered as higher dimensional generalizations of metrics which arise naturally in the theory of Riemann surfaces. There one encounters two types of distinguished points, the cone (or branch) points and the puncture points. Removal of these points leaves a noncompact manifold. In the case of a puncture, the missing point is infinitely far away so the metric is complete, while in the case of a cone point, the missing point is reached in finite distance and so the metric is incomplete.

On higher dimensional objects the analogues of puncture points have been studied more than the analogues of cone points, but the cone metrics are also interesting and have applications. See [19] for a discussion of some of these applications. Metrics similar to the puncture case have been studied by R. Kobayashi [12] and by Cheng and Yau [7]. The second part of Yau's paper [22] addresses the general question of Monge-Ampère equations with degenerate right hand side. The method used there is to approximate the singular metric by smooth metrics, but uniqueness of solutions prevails in such a way that it is not possible to control the geometry of the solution metric. Tsuji [21] has studied cone metrics, but in that work the cone angle α is assumed to be rational. Therefore not only is the result more limited, but the method itself cannot be applied to arbitrary cone angles, because for α rational it is possible to pass to a smooth cover and then appeal to the earlier work of Yau [22] and Aubin [1], [2].

From the Riemannian viewpoint singular spaces also occur naturally. The analysis of such spaces has been studied by many people: see for example the work of Cheeger [5] and [6], Melrose and Mendoza [15] and that of Mazzeo [14].

2.3 Local Properties of the Metric

Recall that the metric potential is given by

$$\tilde{V} = \frac{V}{\|s\|^{2\alpha}(1-\varepsilon\|s\|^{2(1-\alpha)})^{2}}.$$

In a local coordinate neighborhood U, let e_0 be a local holomorphic basis section of $[D]$. Then s may be written as $s = s_0 \cdot e_0$, where s_0 is a holomorphic *function*. For these local computations it will be convenient to isolate the singular direction, so we will let the dimension of M be $n+1$ so that local holomorphic coordinates $(z, w_1, \ldots, w_n)$ may be defined in which $z = s_0$. In these coordinates. $\|s\|^2 = |z|^2\|e_0\|^2$; note that $\|e_0\|^2$ is smooth but generally not holomorphic. Put $a = \|e_0\|^2$. Then

$$\hat{V} = \frac{V}{|z|^{2\alpha}a^{\alpha}(1-\varepsilon|z|^{2(1-\alpha)}a^{1-\alpha})^2}.$$

$$\begin{aligned}\partial\overline{\partial}\log\hat{V} &= \partial\overline{\partial}\log V - \alpha\partial\overline{\partial}\log|z|^2 - \partial\overline{\partial}\log a^{\alpha} \\ &- 2\partial\overline{\partial}\log\Big(1-\varepsilon|z|^{2(1-\alpha)}a^{1-\alpha}\Big).\end{aligned}$$

One computes directly that $\partial\overline{\partial}\log|z|^2 = 0$. Since a is positive and bounded away from zero, $\partial\overline{\partial}\log a^{\alpha}$ is smooth, as is $\partial\overline{\partial}\log V$.

That leaves $2S := -2\partial\overline{\partial}\log(1-\varepsilon|z|^{2(1-\alpha)}a^{1-\alpha})$ as the most singular term, that is, the term needing further study. Write $b = a^{1-\alpha}$.

$$\begin{aligned}-\partial\overline{\partial}\log(1-\varepsilon|z|^{2(1-\alpha)}b) &= \frac{-1}{(1-\varepsilon|z|^{2(1-\alpha)}b)^2}\Big((1-\varepsilon|z|^{2(1-\alpha)}b)\cdot\partial\overline{\partial}(1-\varepsilon|z|^{2(1-\alpha)}b) \\ &- \partial(1-\varepsilon|z|^{2(1-\alpha)}b)\wedge\overline{\partial}(1-\varepsilon|z|^{2(1-\alpha)}b)\Big) \\ &= \frac{1}{(1-\varepsilon|z|^{2(1-\alpha)}b)^2}\Big(\varepsilon(1-\varepsilon|z|^{2(1-\alpha)}b)(\partial\overline{\partial}|z|^{2(1-\alpha)}b \\ &+ \partial|z|^{2(1-\alpha)}\wedge\overline{\partial}b + \partial b\wedge\overline{\partial}|z|^{2(1-\alpha)} + |z|^{2(1-\alpha)}\partial\overline{\partial}b) \\ &+ \varepsilon^2\Big((\partial|z|^{2(1-\alpha)}b + |z|^{2(1-\alpha)}\partial b)\wedge(\overline{\partial}|z|^{2(1-\alpha)}b \\ &+ |z|^{2(1-\alpha)}\overline{\partial}b)\Big)\Big) \\ &= \frac{1}{(1-\varepsilon|z|^{2(1-\alpha)}b)^2}\Big(\varepsilon(1-\varepsilon|z|^{2(1-\alpha)}b)\cdot(b\partial\overline{\partial}|z|^{2(1-\alpha)} \\ &+ \partial|z|^{2(1-\alpha)}\wedge\overline{\partial}b + \partial b\wedge\overline{\partial}|z|^{2(1-\alpha)} + |z|^{2(1-\alpha)}\partial\overline{\partial}b) \\ &+ \varepsilon^2(b^2\partial|z|^{2(1-\alpha)}\wedge\overline{\partial}|z|^{2(1-\alpha)} + b|z|^{2(1-\alpha)}\partial|z|^{2(1-\alpha)}\wedge\overline{\partial}b \\ &+ b|z|^{2(1-\alpha)}\partial b\wedge\overline{\partial}|z|^{2(1-\alpha)} + |z|^{4(1-\alpha)}\partial b\wedge\overline{\partial}b)\Big).\end{aligned}$$

Examination of this last expression shows that the most singular term contains the factor

$$\partial\bar{\partial}|z|^{2(1-\alpha)} = (1-\alpha)^2|z|^{-2\alpha}dz \wedge d\bar{z}.$$

We therefore have:

$$\omega = \partial\bar{\partial}\log\hat{V} = (\omega - 2S) + 2S$$

where $\omega - 2S$ is a smooth (1.1) form and S is the singular part. Furthermore. the form of S is

$$S = |z|^{-2\alpha} \cdot \text{bounded form.}$$

So locally. ω is equivalent to

$$\omega_0 = \frac{1}{|z|^{2\alpha}}dz \wedge d\bar{z} + \sum_2^n dw \wedge d\bar{w}$$

in the sense that there is a constant $c > 0$ so that

$$c\omega_0 \leq \omega \leq \frac{1}{c}\omega_0.$$

The metric ω_0 is a crude approximation to ω in the same way that in the smooth Riemannian case a small enough neighborhood looks like a piece of the plane. The interesting part. the z factor. is just a flat cone. so we may refer to ω_0 as the flat cone metric. One way to see that the z factor describes a flat cone is to view it in two real variables. Then

$$g_{ij} = \begin{bmatrix} \frac{1}{(x^2+y^2)^\alpha} & 0 \\ 0 & \frac{1}{(x^2+y^2)^\alpha} \end{bmatrix} = \frac{1}{r^{2\alpha}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

in other words a conformal change $g_{ij} = o(r)\delta_{ij}$ from the flat metric. Because it is radially symmetric. it may be seen as a surface of revolution. and because

$\Delta \log \phi(r) = 0$, it is flat, so it is a cone. (This comes from a formula in [11].) One may see by direct computation that the cone point is reached in finite time, so this metric is incomplete.

2.4 Derivation of the Monge-Ampère Equation

With ω_0 the original cone metric described above, we consider new metrics of the form $\omega = \omega_0 + \partial\overline{\partial}u$. For now let us just say that u is a real-valued function and postpone a more precise description until the next section. Since $\omega = \omega_0 + \partial\overline{\partial}u$, $[\omega] = [\omega_0]$. If ρ and ρ_0 denote the Ricci forms corresponding to ω and ω_0, respectively, then

$$\rho = -\partial\overline{\partial} \log \det(g)$$

and

$$\rho_0 = -\partial\overline{\partial} \log \det(g_0).$$

Next, we have

$$[\rho_0] = [\rho] = [-\omega] - [-\omega_0].$$

The first equality holds by computing $\rho = \rho_0 + \partial\overline{\partial} f_1$, in which f_1 is bounded because of the third condition above. The second equality holds because g is Kähler-Einstein, and the third because ω and ω_0 are assumed to be in the same class. Then since $[\rho_0] = [-\omega_0]$, there is a function f such that

$$\rho_0 + \omega_0 = \partial\overline{\partial} f.$$

We would like to specify a unique choice by requiring the normalization

$$\int_M (e^f - 1)\omega_0^m = 1,$$

but we need to know that f is bounded in order for this to make sense.

LEMMA. f is continuous on all of M. Proof: We had $[\omega] = [-\rho]$, so there exists a function f so that $\omega + \rho = \partial\overline{\partial} f$, or

$$\partial\overline{\partial}\log \hat{V} - \partial\overline{\partial}\log \omega^n = \partial\overline{\partial} f.$$

This implies

$$\log \hat{V} - \log \omega^n = f + c.$$

for c some constant, or

$$\log \frac{\hat{V}}{\omega^n} = f + c.$$

Both the numerator and denominator are of the form $| z |^{2\alpha}$ times a smooth, nonvanishing form, so log of the quotient makes sense.

Continuing with the derivation of the Monge-Ampère equation, we then have

$$\rho = \rho_0 - \partial\overline{\partial}\log \frac{\det(g)}{\det(g_0)}.$$

and also

$$\rho = -\omega$$

and

$$\rho_0 = -\omega_0 + \partial\overline{\partial} f.$$

Putting these together gives

$$-\omega = -\omega_0 + \partial\overline{\partial} f - \partial\overline{\partial}\log \frac{\det(g)}{\det(g_0)}.$$

Recalling that $\omega - \omega_0 = \partial\overline{\partial} u$, this may be rewritten as

$$\partial\overline{\partial}\log \frac{\det(g)}{\det(g_0)} = \partial\overline{\partial} f + \partial\overline{\partial} u.$$

As in the smooth case, this implies that

$$\log \frac{\det(g)}{\det(g_0)} = u + f + \text{constant}.$$

Replacing u by u plus a constant, we may assume that the constant above is zero. So we get the Monge-Ampère equation

$$\log \frac{\det(g)}{\det(g_0)} = f + u.$$

We also require $\omega + \partial\bar{\partial}u > 0$, so that the solution will give a metric.

General references for this material are [19] and [9].

2.5 Definition of the Function Spaces

We want to describe functions which are uniformly bounded on the non-compact set $\Omega := M - D$ with respect to the singular metric g. Cover M by finite number of unit polydisks U_k in which either D appears as one of the coordinate axes or else D does not intersect U_i at all. Then define $C_g^{2,\delta}(\Omega)$ to consist in functions u which are continuous on M and twice differentiable on Ω and bounded in the norm

$$\begin{aligned} \|u\|_{2,\delta} &= \sup_k \Bigg(\sup_\Omega |u| \\ &+ \sup_\Omega \|\nabla u\|_g + \sup_\Omega \|\nabla\nabla u\|_g \\ &+ \sup_{\substack{x,y \in U_K \\ x \neq y}} \frac{|\,(g^{i\bar{j}}\partial_i\partial_{\bar{j}}u)(x) - (g^{i\bar{j}}\partial_i\partial_{\bar{j}}u)(y)\,|}{(d_g(x,y))^\delta} \Bigg) \end{aligned}$$

Here ∇ is the covariant derivative with respect to g.

REMARK: If g were complete, then for the Hölder part we could have $|\nabla\nabla u(x) - P_{xy}\nabla\nabla u(y)|$ for the numerator, where P_{xy} is the parallel transport operator.

2.6 Statement of the Theorem

THEOREM. Let M be a compact complex manifold. and D a divisor with one irreducible component. that is. a complex curve. Suppose that ω_g is the Kähler form of a cone metric as defined above. Then there exists a Kähler-Einstein cone metric of the form $\omega_g + \partial\overline{\partial}u$. where u is a function in $C_g^{2,\delta}(\Omega)$.

2.7 Continuity Method and Computation of the Linearized Operator

We want to solve the Monge-Ampère equation

$$\left.\begin{array}{rcl} (\omega_g + \partial\overline{\partial}u)^n & = & e^{f+u}\omega_g^n \\ \omega_g + \partial\overline{\partial}u & > & 0 \end{array}\right\} MA.$$

We set up the continuity method by introducing a parameter t:

$$\left.\begin{array}{rcl} (\omega_g + \partial\overline{\partial}u)^n & = & e^{tf+u}\omega_g^n \\ \omega_g + \partial\overline{\partial}u & > & 0 \end{array}\right\} (MA)_t.$$

The problem we therefore want to solve is to show existence of a solution when $t = 1$. Put $E = \{t \in [0.1] | (MA)_t \text{ can be solved}\}$. E contains zero since $u \equiv 0$ solves $(MA)_0$. If E can be shown to be both open and closed. then $E = [0.1]$ and in particular will contain 1.

That E is open can be shown by the Inverse Function Theorem.

Theorem 2.7.1 (INVERSE FUNCTION THEOREM) (*[2]*). *Suppose B_1 and B_2 are Banach spaces and that $f \in C^1(U)$ for some $U \subset B_1$. If at $u_0 \in U$, $f'(u_0)$ is a homeomorphism of B_1 onto B_2, then there exists a U' neighborhood of u_0 such that $f|U'$ is a homeomorphism of U' into $f(U')$.*

Define an operator $\Phi_t : C^{2,\alpha} \to C^{0,\alpha}$ by

$$\Phi_t(u) = \log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u)}{\det(g_{i\bar{j}})} - tf - u;$$

a solution to $(MA)_t$ occurs when $\Phi_t(u) = 0$. Suppose $t_0 \in E$. We want to show $(t_0 - \delta, t_0 + \delta) \subset E$ also for some small $\delta > 0$.

The first step in applying the Inverse Function Theorem is to compute the linearization of Φ_{t_0}. Abbreviate the notation by writing Φ_0 for Φ_{t_0}.

$$\begin{aligned} \Phi_0(u) &= \log \frac{\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u)}{\det(g_{i\bar{j}})} - t_0 f - u \\ &= \log\det\,(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u) - \log\det(g_{i\bar{j}}) - t_0 f - u \end{aligned}$$

$$\begin{aligned} d\Phi_0(u)(v) &= \tfrac{d}{ds}|_{s=0}(\Phi_0(u+sv)) \\ &= \tfrac{d}{ds}\,|_{s=0}\,(\log\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}u + s\partial_i\partial_{\bar{j}}v) - (u+sv)), \end{aligned}$$

since these are the only terms which involve u. So we get

$$d\Phi_0(u)(v) = \Delta_{g'} v - v,$$

where g' is the metric whose Kähler form is $\omega + \partial\bar{\partial}u$.

2.8 Proof That $d\Phi_{t_0}$ is injective

Recall that u_0 is a solution of $(MA)_{t_0}$ and we want to show that $d\Phi(t_0, u_0)$ is a homeomorphism. To simplify the notation just write $d\Phi$ and u. Similarly let g be the metric corresponding to the Kähler form $\omega + \partial\bar{\partial}u$ and write Δ for Δ_g. The proof in the compact. smooth case is described in several sources. including [18]. We follow the same method. except that we are trying to solve the equation on the noncompact set Ω that remains when the divisor is removed. This means that we will be left to investigate convergence of the various integrals and to ask what happens to the boundary terms that arise.

Assume $v \in C^{2,\omega}$ and $d\Phi(v) = 0$. Then $\int_\Omega v(\Delta v - v)\omega_g^n = 0$.

$$\begin{aligned}\int_\Omega (v\Delta v - v^2)\omega_g^n &= \frac{1}{2}\int_\Omega \Delta(v^2)\omega_g^n \\ &- \int_\Omega \langle \nabla_v, \nabla_v\rangle \omega_g^n - \int_\Omega v^2\omega_g^n.\end{aligned}$$

Here, $\Delta = \text{div grad}$ and $\nabla v = \text{grad } v$. We want first to show that $\int_\Omega \Delta(v^2)\omega_g^n = 0$. noting that Ω is noncompact.

Let T_ε be a tubular neighborhood of D. (Since g is incomplete and the distance to D is finite. this makes sense.) Then

$$\begin{aligned}\int_\Omega \Delta(v^2)\omega_g^n &= \int_\Omega (\text{div grad } v^2)\omega_g^n = \int_\Omega d(i_{\nabla v^2}\omega_g^n) \\ &= \lim_{\varepsilon \to 0}\int_{\Omega - T_\varepsilon} d(i_{\nabla v^2}\omega_g^n) = \lim_{\varepsilon\to 0}\int_{\partial(\Omega - T_\varepsilon)} i_{\nabla v^2}\omega_g^n \\ &= -\lim_{\varepsilon \to 0}\int_{\partial T_\varepsilon} i_{\nabla v^2}\omega_g^n.\end{aligned}$$

The first two equalities are pointwise equations on the integrands and the fourth is application of Stokes' theorem. See for example [10] for the identities involving the interior product.

Now notice that $\nabla(v^2) = 2v\nabla v$, so

$$\begin{aligned}\left|\int_{\partial T_\varepsilon} i_{\nabla v^2}\omega_g^n\right| &= \left|\int_{\partial T_\varepsilon} 2v i_{\nabla v}\omega_g^n\right| \\ &\leq 2\cdot\sup_\Omega |v|\cdot\left|\int_{\partial T_\varepsilon} i_{\nabla v}\omega_g^n\right|.\end{aligned}$$

Therefore it is enough to show that

$$\int_{\partial T_\varepsilon} i_{\nabla v}\omega_g^n \to 0 \text{ as } \varepsilon \to 0, \text{ for } v \in C^{2,\alpha}.$$

Since M itself is compact, we could cover it by a finite number of coordinate charts, each of which either does not intersect D at all, or which intersects D in such a way that, locally, the divisor appears as a coordinate axis. Therefore it is enough to show that this integral goes to zero in each such coordinate neighborhood. Furthermore, this is a question about the metric and the volume which does not involve the complex structure. So we may show, slightly more generally: If U is a coordinate neighborhood in a Riemannian manifold (M, g), and S is an embedded hypersurface, and $\|X\|_g$ is bounded on U, then

$$\int_S i_X\omega_g < \infty \text{ and } \int_S i_X\omega_g \to 0 \text{ as } \mathrm{Vol}(S) \to 0.$$

Here ω_g is the volume form corresponding to g. In local coordinates $(x_1, \cdots x_n)$,

$$\omega_g = \sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n.$$

Choose local coordinates so that $S = \{x_n = 0\}$. Recall the definition of the interior product:

$$i_X\omega_g(V_1, \cdots, V_{n-1}) = \omega_g(X, V_1, \cdots, V_{n-1}).$$

If $X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$, then

$$i_X \omega_g = \sqrt{\det g}(x_1 dx_2 \wedge \cdots \wedge dx_n + \cdots + x_n dx_1 \wedge \cdots \wedge dx_{n-1}).$$

and so

$$\int_S i_x \omega_g \sqrt{\det g} x_n dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Lemma 2.8.1 *Given $p \in S$, it is possible to choose local coordinates so that in U, $S = \{x_n = 0\}$, <u>and</u> at p the metric has the form*

$$g_{ij}(p) = \begin{bmatrix} * & 0 \\ & \\ 0 & g_{nn} \end{bmatrix}$$

PROOF. Define $Y_i = \frac{\partial}{\partial x_i} - \frac{g_{in}}{g_{nn}} \frac{\partial}{\partial x_n}$ for $i = 1, \cdots, n-1$, where g_{ij} is evaluated at p. Then $\left\langle Y_i, \frac{\partial}{\partial x_n} \right\rangle_p = g_{in} - \frac{g_{in}}{g_{nn}} g_{nn} = 0$ as desired.

Define a corresponding choice of coordinates $y_i = x_i - \frac{g_{in}}{g_{nn}} x_n$ for $i = 1, \cdots, n-1$, and $y_n = x_n$. ($g_{ij} = g_{ij}(p)$, i.e., constants.)

Now verify that this defines a change of coordinates:

$$\frac{\partial y_i}{\partial x_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{g_{1n}}{g_{nn}} \\ & & & & \\ 0 & 1 & \cdots & 0 & -\frac{g_{2n}}{g_{nn}} \\ & & & & \\ & & & 1 & -\frac{g_{n-1,n}}{g_{nn}} \\ & & & & \\ 0 & & & 0 & 1 \end{bmatrix}$$

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) = 1$$

Continuing with the proof of the main result.

$$\left|\int_s i_x\omega_g\right| \le \int_s |i_x\omega_g|.$$

At a point we can choose coordinates as in the lemma. At this point.

$$\begin{aligned} \sqrt{\det g}|x_n|dx_1 \wedge \cdots \wedge dx_{n-1} &= \sqrt{\det(g_{ij})_{i,j=1}^{n-1}}\sqrt{g_{nn}}|x_n|dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= \sqrt{g_{nn}}|x_n| \cdot \sqrt{\det(g_{ij})_1^{n-1}}dx_1 \wedge \cdots \wedge dx_{n-1} \\ &\le \sup\|X\| \cdot \omega_s. \end{aligned}$$

Since these are pointwise quantities, this inequality must hold everywhere. So $\left|\int_s i_x\omega_g\right| \le \sup\|X\| \cdot \int_s \omega_S = \sup\|X\| \cdot \mathrm{Vol}(S)$. Applying this to the case $\int_{\partial T_\varepsilon} i_{\nabla v}\omega_g^n$, note that

1. $\|\nabla v\|$ bounded, by definition of the function space $C^{2,\alpha}$, and

2. $\mathrm{Vol}(\partial T_\varepsilon) \to 0$ as $\varepsilon \to 0$. (Proved below.)

So this proves that $\int_\Omega \Delta(v^2)\omega_g^n = 0$.

Finally, we have

$$\begin{aligned} 0 &= \int_\Omega V(\Delta v - v)\omega_g^n = \frac{1}{2}\int_\Omega \Delta(v^2)\omega_g^n \\ &- \int_\Omega (\nabla v, \nabla v)\omega_g^n - \int_\Omega v^2\omega_g^n = -\int_\Omega \|\nabla v\|_g^2\omega_g^n - \int_\Omega v^2\omega_g^n. \end{aligned}$$

(These are finite if $v \in C^{2,\alpha}$, given that $\mathrm{Vol}(\Omega_g) < \infty$.) Therefore, v itself must be zero.

Lemma 2.8.2 *$Vol(\partial T_\varepsilon) \to 0$ as $\varepsilon \to 0$.*

PROOF. ω_g is equivalent to the flat cone metric

$$\frac{1}{|z|^{2\alpha}} dz \wedge d\bar{z} + \sum_{i=2}^{n} dw_i \wedge d\bar{w}_i.$$

so it suffices to prove it for the flat cone metric. And, since this is Euclidean in the w-directions, it suffices to prove it in 1 complex variable, namely for the metric $\frac{1}{|z|^{2\alpha}} dz \wedge d\bar{z}$.

But for this metric, $\mathrm{Vol}(\partial T_\varepsilon) = 2\pi\varepsilon(1-\alpha)$ (see Fig. 1).

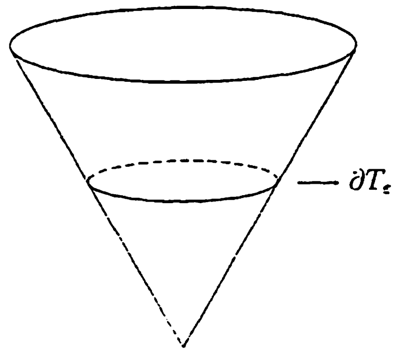

Fig. 1

2.9 $d\Phi$ is surjective

The potential obstacles to surjectivity are that the operator is not uniformly elliptic and the domain Ω is not compact. On the other hand the uniformity was built into the function spaces that we defined. We apply the theory of elliptic boundary and edge operators developed by Melrose.

We first include a simple example to motivate and illustrate some of the ideas of this theory and to introduce some of the terminology. Suppose one were to study the Laplacian in $\mathbf{R}^2$. It is simply

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Now suppose one chose to study it in polar coordinates. It becomes

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}.$$

One may regard this as being defined on a cylinder. a manifold with compact boundary $\mathbf{S}^1$. This operator is singular on this manifold. but we can remove these singularities by considering instead

$$r^2\Delta = r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

In a suitable sense. the operator $P = r^2\Delta$ has the same structure as the original operator Δ: P is not elliptic any longer. but rewriting it as

$$P = (r\frac{\partial}{\partial r})^2 + \frac{\partial^2}{\partial \theta^2}.$$

we observe that it is elliptic if it is regarded as being constructed from the operators or vector fields $r\partial/\partial r$ and $\partial/\partial\theta$. tangent to the boundary $\mathbf{S}^1$. One says that it is a totally characteristic elliptic operator. These are a special case of the edge operators.

To define these notions more generally. suppose M is a manifold with compact boundary. and locally t is a defining function for the boundary. We have the Lie algebra of edge vector fields. $\mathcal{V}_1$ which is generated by

$$\{t\frac{\partial}{\partial t}, t\frac{\partial}{\partial x_1}, \ldots, t\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_1}, \ldots \frac{\partial}{\partial y_l}\}.$$

An operator P which is an elliptic combination (in the usual sense) of elements of $\mathcal{V}_1$ is called an elliptic edge operator. This only summarizes some of the basic definitions. One may consult the works mentioned above to see. for example. that these notions all make sense invariantly.

In the interior, we can study P in the usual way. At the boundary, we can deduce some things indirectly. For any complex number s the conjugated operator $t^{-s}Pt^{s}$ should have the same behavior as P, but if P is elliptic $t^{-s}Pt^{s}$ will have special restriction properties at the boundary. This restriction, $(t^{-s}Pt^{s})_e$, actually makes sense as an elliptic operator on the compact boundary. This ought to mean progress simply because that situation is more understood, but most importantly for these questions, the spectrum, called the boundary spectrum, of such an operator is discrete. We actually have a family of elliptic operators on the boundary, parametrized by s. These are called the indicial operators and denoted by $I_P(s)$.

Let us develop a different point of view which also leads to the indicial operator and will clarify its appearance in the theory. In the same way that application of the Fourier transform converts expressions in $\partial/\partial x_i$ into algebraic equations, application of the Mellin transform converts expressions in $t\partial/\partial t$ into algebraic expressions. When wishing to solve $Pu = f$ for P of elliptic edge type and f a graded conormal distribution, application of the Mellin transform produces

$$I_P(s)u_M = f_M,$$

which suggests trying to solve for u_M :

$$u_M = \frac{f_M}{I_P(s)}$$

and then applying the inverse Mellin transform. However, there are two places where this could fail. Firstly, f_M is only meromorphic, so it may possess poles, and secondly, $I_P(s)$ vanishes for some values of s, namely at the boundary spectrum, also called the indicial roots. Corresponding to each value $\delta = \Re(s)$ there is defined a weighted edge Sobolev space $t^\delta H^k_e$, and $P : t^\delta H^k_e \longrightarrow t^\delta H^{k-m}_e$. The results of Melrose and Mendoza and of Mazzeo show that if δ is not $\Re(s)$ for any indicial root, then the map will have closed range.

We now apply this theory to the operator at hand. We do it first for the model metric

$$\frac{1}{|z|^{2\alpha}} dz \wedge d\bar{z} + \Sigma_1^n dw_i \wedge d\bar{w}_i,$$

the flat cone metric, and then adjust it for the actual metric. At the end, an additional regularity step is required to get back to the Hölder spaces in which we are interested.

It requires some changes of variables to bring this to the description of the edge theory. First introduce polar coordinates by putting $r = e^{i\theta}$ and also since r and θ are real variables, put $w_i = x_i + \sqrt{-1}y_i$. Then put $\rho = r^\beta$, where $\beta = 1 - \alpha$. Finally, putting $t = 1/\beta \cdot \rho$ brings the metric to the form

$$dt^2 + \beta^2 t^2 d\theta^2 + \Sigma_1^n dx_i^2 + \Sigma_1^n dy_i^2.$$

The Laplacian is

$$\Delta u = \frac{\partial^2 u}{\partial t^2} + \frac{1}{\beta^2 t^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{t}\frac{\partial u}{\partial t} + \Sigma_1^n \frac{\partial^2 u}{\partial x_i^2} + \Sigma_1^n \frac{\partial^2 u}{\partial y_i^2},$$

so the operator we will study is

$$(\beta t)^2(\Delta - I) = (\beta t\frac{\partial}{\partial t})^2 + (\frac{\partial}{\partial \theta})^2 + \beta^2\Sigma_1^n(t\frac{\partial}{\partial x_i})^2 + \beta^2\Sigma_1^n(t\frac{\partial}{\partial y_i})^2 - \beta^2 t^2.$$

It is clear that this is elliptic as a combination of the edge vector fields

$$V_e = \{t\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}, t\frac{\partial}{\partial x_i}, t\frac{\partial}{\partial y_i}\}.$$

Put $L = (\beta t)^2(\Delta - I)$. The next step is to compute the indicial roots.

$$t^{-s}Lt^{s} = \beta^2 s^2 + \frac{\partial^2}{\partial\theta^2} + \beta^2\Sigma_1^n(t\frac{\partial}{\partial x_i})^2 + \beta^2\Sigma_1^n(t\frac{\partial}{\partial y_i})^2 - \beta^2 t^2.$$

and at the boundary $\{t = 0\}$ we have

$$(t^{-s}Lt^{s})_e = \beta^2 s^2 + \frac{\partial^2}{\partial\theta^2}.$$

This vanishes for eigenvalues of the one- dimensional Laplacian $\frac{\partial^2}{\partial\theta^2}$ on S^1, that is, when $s = j/\beta$ for any integer j. Although we computed this for the model, or flat cone metric, for the actual metric the indicial roots are the same. This is because the operators differ from each other by a compact perturbation.

We now show that $d\Phi$ has trivial cokernel. We have the dense inclusions

$$C_g^{2,\delta}(\Omega) \subset H_g^1(\Omega)$$

and

$$C_g^{0,\delta}(\Omega) \subset H_g^{-1}(\Omega)$$

where these are the usual definitions of Sobolev spaces on Riemannian manifolds. The proof of injectivity given earlier for $d\Phi$ can be made sense of on the level of the Sobolev spaces, where now $d\Phi$ is defined in the distribution sense. We note in passing that in fact the injectivity can even be improved to about $H^{1/2}$ by investigating more carefully the terms which may appear in the asymptotic expansion of elements of the kernel; this expansion is described in [14] and in [15]. A good illustration of its use for improvement of regularity is [16]. $d\Phi$ is essentially self-adjoint, so this extension is self-adjoint, and so the injectivity also shows that the cokernel vanishes, so on the level of Sobolev spaces the map is a topological isomorphism. ie. $d\Phi$ and its inverse are continuous. We now dualize the above. To begin with. we have inclusions - also dense - of the dual spaces:

$$H^{-1} \subset (C_g^{2,\delta})^*$$

and

$$H_g^1 \subset (C_g^{0,\delta})^*.$$

The boundedness of the operators. together with the triviality of the kernel on a dense subspace implies the triviality of the kernel on the larger space. that is.

$$(d\Phi)^* : (C_g^{0,\delta})^* \longrightarrow (C_g^{2,\delta})^*$$

has trivial kernel. so $d\Phi$ has trivial cokernel. To get the continuity of $d\Phi$ and its inverse on the $C_g^{k,\delta}(\Omega)$ spaces themselves. this is not enough: we need to show that the range is closed.

We have the inclusions in the weighted Sobolev edge spaces:

$$C_g^{2,\delta}(\Omega) \subset t^{-(n+\varepsilon)} H_e^2$$

and

$$C_g^{0,\delta}(\Omega) \subset t^{-(n+\varepsilon)} H_e^0.$$

and we may extend the map L to a map

$$L : t^{-(n+\varepsilon)} H_e^2 \longrightarrow t^{-(n+\varepsilon)} H_e^0.$$

If only we are careful to choose ε so that $-(n+\varepsilon)$ is not one of the indicial roots j/β. this map will have closed range and will be continuous. See Mazzeo's paper [14] Theorem 6.1. So for $f \in C_g^{0,\delta}(\Omega)$ we can find a function $u \in t^{-(n+\epsilon)}H_e^2$ for which $Lu = f$. The remaining piece that we need is to show that actually $u \in C_g^{2,\delta}(\Omega)$. This is given by Proposition 4.21 of the paper by Lee and Melrose [13].

2.10 Outline of the Closedness Step

Recall that we had the parameterized family of equations $(MA)_t$ and defined

$$E = \{t \in [0, 1] | (MA)_t \text{ can be solved}\}.$$

Now show that E is closed. So suppose that $t_0 \in [0, 1]$ and $t_i \to t_0$ with $(MA)_{t_i}$ solvable for each i. We must show that $(MA)_{t_0}$ is solvable as well.

If $t_i \to t_0$, then any subsequence $t_{i_j} \to t_0$ also. So if $\{u_i\}$ has even a convergent *subsequence*, i.e., $u_{i_j} \to u_0$ for some u_0, it will be enough. For if there is such a convergent subsequence $u_{i_j} \to u_0$ then u_0 is a solution of $(MA)_{t_0}$.

To prove the existence of a convergent subsequence, we must show $\{u_i\}$ is bounded in $C_g^3(\Omega)$. Then, since the embedding $C_g^3(\Omega) \in C_g^{2,\delta}(\Omega)$ is compact for $0 < \delta < 1$, $\{u_i\}$ will have a subsequence $\{u_{i_j}\}$ which converges in $C^{2,\delta}$. The goal, therefore, is to bound $\{u_i\}$ in $C_g^3(\Omega)$. Here we follow the method of Yau [22] in which he accomplished these estimates for the case $C_1(M) = 0$ and M smooth.

The main idea in proving these estimates is to obtain an elliptic equality or elliptic inequality on some relevant quantity and then apply the maximum principle. The difficulty encountered in the singular case is that if a maximum occurs over the divisor D, it may not be smooth, so it is not possible to apply the maximum principle directly. The third order quantity one is able to bound is in terms of the solution metric $\omega = \omega_0 + \partial\bar{\partial}u$. In order for this to be a uniform bound we need the solution metric to be uniformly bounded in terms of the original metric. For this purpose, one would like to bound the second order quantity $m + \Delta u$. Since it is not possible to do so directly, one estimates instead $e^{-Cu}(m + \Delta u)$, and this requires in turn an estimate on u itself.

Rephrasing this in the forward direction, the main steps in the closedness argument are:

1. C^0 estimate on u.

2. Bound on $m + \Delta u$.

3. Third order bounds.

A good explanation of the method may be found in [17] or in [4].

2.11 C^0 Estimate

We attempt to use the maximum principle, as in [1], [2], but it does not apply directly; the function space in which we are working allows u to achieve nonsmooth maxima. So we add on a function F which we hope to control uniformly. Suppose u is a solution of the Monge-Ampère equation. Then locally,

$$\frac{\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}u)}{\det(g_{i\bar{j}})} = e^{f+u}.$$

Put $v = u + F$ for an unknown function F to be determined. Then $u = v - F$ so the Monge Ampère equation becomes

$$\frac{\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}v - \partial_i\partial_{\bar{j}}F)}{\det(g_{i\bar{j}})} = e^{f+u}.$$

Suppose v achieves a maximum on Ω. Then at that point, $(\partial_i\partial_{\bar{j}}v)$ is negative semidefinite, so that

$$\frac{\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}v - \partial_i\partial_{\bar{j}}F)}{\det(g_{i\bar{j}})} \leq \frac{\det(g_{i\bar{j}} - \partial_i\partial_{\bar{j}}F)}{\det(g_{i\bar{j}})}$$

or

$$e^{f+u} \leq \frac{\det(g_{i\bar{j}} - \partial_i\partial_{\bar{j}}F)}{\det(g_{i\bar{j}})}.$$

that is,

$$e^{v} \leq e^{-f+F} \cdot \frac{\det(g_{i\bar{j}} - \partial_i\partial_{\bar{j}}F)}{\det(g_{i\bar{j}})}.$$

At that point we have

$$e^{v} \leq e^{-f+F}\frac{\det(g_{i\bar{j}} - \partial_i\partial_{\bar{j}}F)}{\det(g_{i\bar{j}})}$$

Recalling that f is bounded, the choice of F must therefore satisfy:

1. Max v occurs on Ω.

2. $\max u \leq \max v$.

3. F is uniformly bounded.

4. For some $Cg_{i\bar{j}} \leq \partial_i \partial_{\bar{j}} F$, uniformly.

Put $F = \|s\|^{2\beta}$ for a positive power β, $\|s\| \leq 1$. Then $v = u + F$ will agree with u along D, and $\|s\|^{2\beta}$ will be an initially increasing function in directions perpendicular to the divisor. If F increases more rapidly than *any* $u \in C_g^{2,\delta}(\Omega)$, then v will achieve a maximum on Ω.

We compare the gradient of F to that of functions in $C_g^{2,\delta}(\Omega)$: if the gradient is unbounded with respect to the flat cone metric ω_0 then it will also be unbounded with respect to ω_g.

Write $\|s\|^{2\beta} = |z|^{2\beta}\|e\|^{2\beta}$ locally, with e a basis section for $[D]$, or $\|s\|^{2\beta} = |z|^{2\beta} b$. Because $\|e\|$ is bounded away from zero, b is smooth.

In diagonal coordinates at one point

$$\langle \text{grad } f, \text{grad } f \rangle_g = \sum \left| \frac{\partial f}{\partial z_i} \right|^2 g^{i\bar{i}}$$

We compute the first term $i = 1$, since it corresponds to the singular $(z, \bar{z})$ direction

$$\frac{\partial}{\partial z}(|z|^{2\beta} b) \frac{\partial}{\partial \bar{z}}(|z|^{2\beta} b) g_0^{11} = \beta^2 b^2 |z|^{2+4(\beta-1)+2\alpha} + \cdots.$$

This is unbounded if $2 + 4(\beta - 1) + 2\alpha < 0$ or $2\beta < 1 - \alpha$. Because F rises more steeply than u can fall, $\max u + F$ occurs over Ω. Now show $\partial\bar{\partial} F \geq C\omega_g$

for some C. Note the formula $\partial\bar{\partial}e^{f} = e^{f}(\partial\bar{\partial}f + \partial f \wedge \bar{\partial}f)$. Then

$$\begin{aligned}
\partial\bar{\partial}\|s\|^{2\beta} &= \partial\bar{\partial}e^{\beta \log \|s\|} \\
&= \|s\|^{2\beta}(\beta\partial\bar{\partial}\log\|s\|^2 + \beta^2\partial\log\|s\|^2 \wedge \bar{\partial}\log\|s\|^2) \\
&\geq \|s\|^{2\beta}\beta\partial\bar{\partial}\log\|s\|^2.
\end{aligned}$$

since for a real-valued form h. $\partial h \wedge \bar{\partial}h \geq 0$.

But $\partial\bar{\partial}\log\|s\|^2 = -R(\|\cdot\|)$. so we need to show that there exists C' such that

$$-\beta\|s\|^{2\beta}R(\|\cdot\|) \geq C'\omega_g$$

or

$$\beta\|s\|^{2\beta}R(\|\cdot\|) \leq -C'\omega_g.$$

Since $\|s\| \leq 1$ and $R(\|\cdot\|)$ is bounded. there is such a constant C'.

Note that C' depends on the original metric and on the curvature of the line bundle.

2.12 Bound on $m + \Delta u$

We follow the computations in Yau's work [22] although another approach may be found in [17]. Since Yau's paper addressed the case when $C_1(M) = 0$. the first step is to modify the computations to the case $C_1(M) = -1$. In the formulas below. C is a constant to be determined later. and is different than the constant in the last section. One then obtains. as an analogue of Yau's inequality 2.18:

$$\Delta'\left(e^{-Cu}(m+\Delta u)\right) \geq e^{-Cu}(\Delta f - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - m) \qquad (\S 12)$$
$$+ \ e^{-Cu}(1-Cm)(m+\Delta u)$$
$$+ \ (C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}})e^{-Cu}(m+\Delta u)\sum \frac{1}{1+u_{i\bar{i}}}.$$

Here Δ is the Laplacian corresponding to the original metric, and Δ' is the Laplacian corresponding to the solution metric $\omega + \partial\bar{\partial}u$. $\inf_{i\neq l} R_{i\bar{i}l\bar{l}}$ is finite by direct computation. If the maximum of $e^{-Cu}(m+\Delta u)$ occurs over Ω, then the argument continues similarly to Yau's paper. We therefore address the case in which the maximum occurs over D.

Then choose $F = \|s\|^{2\beta}$, for s a defining section of $[D]$ such that $\|s\| \leq 1$. Consider $e^{-Cu}(m+\Delta u + F)$. Note that $e^{-Cu}(m+\Delta u+F) \geq e^{-Cu}(m+\Delta u)$ everywhere, so $\max(e^{-Cu}(m+\Delta u+F)) \geq \max e^{-Cu}(m+\Delta u)$. F is initially increasing since $F \geq 0$ and the properties of defining form of divisors are such that they vanish *only* at the divisor. Similarly to the procedure for the C^0 estimate we try to choose β so that $e^{-Cu}F$ is increasing faster than $e^{-Cu}(m+\Delta u)$ can fall, for *any* solution u. Then we will be certain that the maximum of $e^{-Cu}(m+\Delta u+F)$ occurs over Ω.

$e^{-Cu}(m+\Delta u+F) = e^{-Cu}m + e^{-Cu}\Delta u + e^{-Cu}F$. First compare the rates of increase or decrease of $e^{-Cu}m$ and $e^{-Cu}F$, comparing these gradients

$$\nabla e^{-Cu}m = -Cme^{-Cu}\nabla u$$

and

$$\nabla(e^{-Cu}F) = -CF\nabla u + e^{-Cu}\nabla F.$$

The terms $-Cme^{-Cu}\nabla u$ and $-CF\nabla u$ are bounded in the $C_g^{2,\delta}(\Omega)$ norm. but $e^{-Cu}\nabla F$ will be unbounded if $2\beta < 1-\alpha$. Now compare the terms $e^{-Cu}\nabla u$ and $e^{-Cu}F$. This time. $e^{-Cu}\Delta u$ is only in $C^{0,\delta}$. so we must use the Hölder part of the norm to compare the steepness of ascent and descent. Since $e^{-cu} > 0$ and u has more regularity. then if $e^{-cu}F \in C_g^{0,\delta}(\Omega)$. then $e^{cu}(e^{-cu}F) = F \in C_g^{0,\delta}(\Omega)$ also. So as long as F rises more steeply than anything in $C_g^{0,\delta}(\Omega)$. $e^{-cu}F$ will do so also. So choose $2\beta < \delta$. Putting these together. if $2\beta < \min\{1-\alpha. \delta\}$. then $e^{-cu}(m + \Delta u + F)$ achieves a maximum in Ω.

Continuing with the main argument. we compute

$$\Delta'\left(e^{-Cu}(m + \Delta u + F)\right) = \Delta'\left(e^{-Cu}(m + \Delta u)\right) + \Delta'(e^{-Cu}F)$$

$$\begin{aligned}
\Delta'(e^{-Cu}F) &= \Delta'(e^{-Cu+\log F}) \\
&= \frac{1}{2}e^{-Cu+\log F}\|\nabla(-Cu + \log F)\|_{g'}^2 + e^{-Cu}F\Delta'(-Cu + \log F) \\
&\geq e^{-Cu}F\Delta'(-Cu + \log F) \\
&= -Ce^{-Cu}F\Delta'(u) + e^{-Cu}F\Delta'\log F.
\end{aligned}$$

$$\Delta'\log F = \sum \frac{(\log F)_{i\bar{i}}}{1 + u_{i\bar{i}}}.$$

and $\partial\bar{\partial}\log F = \partial\bar{\partial}\log\|s\|^{2\beta} = \beta\partial\bar{\partial}\log\|s\|^2 = \beta R(\|\cdot\|)$. Since this is smooth and defined on all of the compact manifold M. this is bounded above and below: let a be the lower bound. Then

$$\Delta'\log F = \sum \frac{(\log F)_{i\bar{i}}}{1 + u_{i\bar{i}}} \geq a\sum\frac{1}{1 + u_{i\bar{i}}}. \qquad (\S 12)$$

We also have. as in 2.11 of Yau's paper.

$$\Delta' u = \sum \frac{u_{i\bar{i}}}{1 + u_{i\bar{i}}} = m - \sum \frac{1}{1 + u_{i\bar{i}}}. \qquad (\S 12)$$

Continuing (1), (2), and (3), we get

$$\begin{aligned}
\Delta'\left(e^{-Cu}(m+\Delta u+F)\right) &\geq e^{-Cu}(\Delta f - m^2 \inf R i\bar{i} l\bar{l} - m) \qquad (\S 12)\\
&- e^{-Cu}(1-C'm)(m+\Delta u)\\
&+ e^{-Cu}(m+\Delta u)(C+\inf R i\bar{i} l\bar{l})\sum \frac{1}{1+u_{i\bar{i}}}\\
&- Ce^{-Cu}F\left(m-\sum\frac{1}{1+u_{i\bar{i}}}\right)+e^{-Cu}Fa\sum\frac{1}{1+u_{i\bar{i}}}\\
&= e^{-Cu}(\Delta f - m^2\inf R i\bar{i} l\bar{l} - m - CFm)\\
&- e^{-Cu}(1-C'm)(m+\Delta u)\\
&+ e^{-Cu}\underbrace{[(m+\Delta u)(C+\inf R i\bar{i} l\bar{l})+C'F+Fa]}_{\star}\\
&\cdot \sum\frac{1}{1+u_{i\bar{i}}}.
\end{aligned}$$

Because $(m+\Delta u)>0$, $F\geq 0$, and Fa does not involve C, it is possible to choose C so that $\star$ is larger than 1.

By comparing to Yau's work at this point we can draw an interesting observation. In the smooth case, one chooses C so that $(m+\Delta u)(C+\inf R_{i\bar{i}l\bar{l}}>1$, so that C compensates for the curvature of M. Here, in the singular case, C compensates for the curvature of M *and* of the line bundle $[D]$.

We now observe that

$$\sum\frac{1}{1+u_{i\bar{i}}}\geq\left(\frac{\sum(1+u_{i\bar{i}})}{\Pi(1+u_{i\bar{i}})}\right)^{\frac{1}{m-1}}. \qquad (\S 12)$$

At the computing point, the original Monge-Ampère equation

$$\frac{\det(g_{i\bar{j}}+\partial_i\partial_{\bar{j}}u)}{\det(g_{i\bar{j}})}=e^{f+u}$$

becomes $\Pi(1+u_{i\bar{i}}) = e^{f+u}$, so that the inequality (5) may be written

$$\sum \frac{1}{1+u_{i\bar{i}}} \geq \left(\frac{\sum(1+u_{i\bar{i}})}{e^{f+u}}\right)^{\frac{1}{m-1}}$$

or

$$\sum \frac{1}{1+u_{i\bar{i}}} \geq \left(\frac{m+\Delta u}{e^{f+u}}\right)^{\frac{1}{m-1}}.$$

Then (4) becomes

$$\begin{aligned} \Delta'\left(e^{-Cu}(m+\Delta u+F)\right) &\geq e^{-Cu}(\Delta f - m^2 \inf R_{i\bar{i}l\bar{l}} - m - CFm) \\ &- e^{-Cu}(1-Cm)(m+\Delta u) \\ &+ e^{-Cu}\left((u+\Delta u)(C+\inf Ri\bar{i}l\bar{l}) + CF + Fa\right) \\ &\cdot e^{-\frac{(f+u)}{m-1}}(m+\Delta u)^{\frac{1}{m-1}}. \end{aligned}$$

At a maximum point of $e^{-Cu}(m+\Delta u+F)$, $0 \geq \Delta'\left(e^{-Cu}(m+\Delta u+F)\right) \geq$ right-hand-side.

Combine like powers of $(m+\Delta u)$. Then at the maximum point, if $y = (m+\Delta u)$, we have obtained above an expression of the form

$$y^{1+\frac{1}{m-1}} \leq cy^{\frac{1}{m-1}} + ay + b.$$

Take the maximum of the three terms on the right. Then we have

$$y^{1+\frac{1}{m-1}} \leq 3cy^{\frac{1}{m-1}}, \text{ so that } y \leq 3c,$$

or

$$y^{1+\frac{1}{m-1}} \leq 3ay, \text{ so that } y \leq (3a)^{m-1},$$

or

$$y^{1+\frac{1}{m-1}} \leq 3b, \text{ so that } y \leq (3b)^{\frac{1}{1+\frac{1}{m-1}}}.$$

In any case. this gives a bound for $(m+\Delta u)$. $\left(y^{1+\frac{1}{m-1}} - cy^{\frac{1}{m-1}} - ay - b\right) \leq 0$ at the maximum. Since $y \geq 0$. $a.b.c$ cannot all be nonpositive. So at least one is non-negative. In particular. the maximum of the three will be nonnegative. so the inequalities above make sense. Finally. note the following.

- $0 \leq F \leq 1$.
- Δf is bounded since the curvature is bounded and $\Delta f = n+$scalar curvature.
- a depends on the curvature of $[D]$.
- C depends on the curvature of M and of $[D]$.
- we already have $\| u \|_{C^0}$.

So this is indeed an a priori bound.

2.13 Third Order Estimates

Again we follow the computations of Yau and ask what happens if the maximum occur over the divisor D.

Put $S = \sum g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} u_{i\bar{j}k} u_{\bar{r}s\bar{t}}$. S is obtained by taking the third covariant derivatives of u with respect to g. and then taking the norm of this object with respect to g'. By the previous step. in which we found that $m + \Delta u$ is uniformly bounded. g' is uniformly bounded in terms of g so now uniform bounds in terms of g' imply uniform bounds in terms of g.

Yau shows that $\Delta'(S + C_7\Delta u) \geq C_8 S - C_9$ for certain constants. As pointed out in Bourguignon [4], the same holds for $C_1 < 0$ where $S + C_7\Delta u$ has a maximum, say P.

$$0 \geq \Delta'(S + C_7\Delta u) \geq C_8 S - C_9.$$

This implies that for all x,

$$\begin{aligned} C_8(S + C_7\Delta u)(x) &\leq C_8(S + C_7\Delta u)(P) \\ &= (C_8 S - C_9)(P) + C_9 + C_8 C_7 \Delta u(P) \\ &\leq 0 + C_9 + C_8 C_7 \Delta u(P) \leq C_9 + c_7 C_8 \max \Delta u. \end{aligned}$$

We again consider the possibility that a maximum may occur over D, and consider $S + C_7\Delta u + F$. This time F must satisfy

1. $\max(S + C_7\Delta u + F) \geq \max(S + C_7\Delta u)$

2. $\max(S + C_7\Delta u + F)$ occurs over Ω

3. $\Delta' F$ bounded uniformly from below

4. F bounded uniformly from above.

As in the bound for $m + \Delta u$, $F = \|s\|^{2\beta}$, $\|s\| \leq 1$ and $2\beta < \min\{1 - \alpha, \delta\}$ works.

Then for all x,

$$\begin{aligned} C_8(S + C_7\Delta u + F)(x) &\leq C_8(S + C_7\Delta u + F)(P) \\ &= (C_8 S - C_9)(P) + C_9 \end{aligned}$$

$$
\begin{aligned}
&+ \quad C_8 C_7 \Delta u(P) + C_8 F(P) \\
&\leq \quad -\min \Delta' F + C_9 + C_8 C_7 \Delta u(P) \\
&+ \quad C_8 \max F(P).
\end{aligned}
$$

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