Characterizations of various classes of Einstein metrics

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Abstract of the Dissertation

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A rigidity theorem of the complete n-dimensional spin Ricci flat manifolds admitting a certain S^1 action is proved, provided that the action has smooth fixed points and the metric is asymptotically flat. Such manifolds are isometric to the n-dimensional Riemannian Schwarzschild metric.

Also the critical point of the scalar curvature functional restricted to the Yamabe space of constant scalar curvature metrics with volume one is studied. It is proved that there are strong topological and geometrical restrictions on the critical point metrics. For some cases, it is proved that the metric is Einstein.

To my parents, my wife, and my family of God.

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Chapter 0

Introduction

In Riemannian geometry, it has been an interesting question to study Einstein metrics on manifolds. A metric g on a manifold M is called an Einstein metric if the Ricci curvature is proportional to the metric g, in other words,

$$Ric(g) = \lambda g.$$

for some constant λ. Einstein metrics serve as a good candidate for best or nicest Riemannian structures on the manifold M. In dimension 2 and 3, having such structures means that the metrics are of constant curvatures. We know that not every 3−dimensional and 4−dimensional manifold has an Einstein metric, compared with the fact that that any 2−dimensional manifold admits a complete metric with constant curvature[Bes87]. In dimension greater than 4, it may be that any manifold admits a (negative) Einstein metric.

The existence of such metrics does pose restrictions on the manifold itself in some cases. For example, when the constant λ is positive, the manifold itself is compact. It is not known whether the nonpositive λ might pose some restrictions on the manifold, except that if $\lambda < 0$, the isometry group is finite. It is our concern to classify such structures. It is usually hard to classify such metrics in general setting, so it is often necessary to introduce some additional conditions of symmetries or topologies on the manifolds.

In the chapter 1, we restrict our attention to a special class of Einstein metrics, namely Ricci flat metrics on a manifold with a compact $1-$parameter group of isometries. In other words, we study complete, noncompact Ricci flat $n-$dimensional manifolds with smooth S^1 action. Then the projection map π from the manifold M to the space N of orbits is a Riemannian submersion. Our study in chapter 1 is focused on the case when the space N is a submanifold of M. Such a metric is called *static*. In dimension 4, it is known that, under natural conditions, static metric is unique if the action is smooth. But as shown in Theorem 2, in dimension greater than 4, an analogous result fails. In fact, there are the same number of such $n-$dimensional metrics as $(n-2)-$dimensional positive Einstein manifolds. But in our Uniqueness Theorem, it is shown that under "pre-determined" geometric behavior of its end, such a structure is unique. We refer the study of nonzero λ in more general setting to [Hw].

If the Ricci flat metric has "large" symmetries, we conjecture that the metric can be completely determined by the behavior of the norms of Killing vector fields. Not only we confirm this in Section 1.3, but also we show that the norms are harmonic functions.

In the chapter 2, we study the scalar curvature functional and Einstein metrics on compact manifolds. Let $\mathcal{M}$ denote the set of all smooth Riemannian structures on a closed n-manifold M, and $\mathcal{M}_1$ those of volume 1. The scalar curvature functional on $\mathcal{M}_1$ is defined as a function $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$, which is given by

$$g \to \mathcal{S}(g) = \int_M s_g dv_g,$$

where s_g denote the scalar curvature of the metric g, and dv_g is the volume form. It is well known that a compact Riemannian manifold (M, g) of volume 1 is Einstein if the metric g is a critical point of $\mathcal{S}$ restricted to $\mathcal{M}_1$.

Since the solution to the Yamabe Problem shows that any compact manifold M carries a lot of metrics of constant scalar curvature, we will introduce the set

$$\mathcal{C} = \{g \in \mathcal{M}_1, \quad s_g \quad \text{constant}\}.$$

It should be pointed out that a metric g in $\mathcal{C}$ is a critical point of $\mathcal{S}$ restricted to the set $Conf_0(g)$ of metrics pointwise conformal to g and having the same total volume[Bes87].

When we restrict the domain of the scalar curvature functional $\mathcal{S}$ to $\mathcal{C}$, the equation for a critical point(metric) is given by

$$z_g = D_g df - (\Delta_g f)g - f r_g \tag{0.1}$$

where z_g is the traceless Ricci tensor, r_g is the Ricci tensor, and f is a function on M with vanishing mean value. A solution (g, f) to the equation 0.1 will be the focus of our study in the chapter 2.

First we proved that if the function f is greater than or equal to -1, the metric is Einstein. It will also be proved that in dimension 3, there are quite strong topological(Theorem 6) and geometrical(Theorem 11) restrictions on the structure of the solutions. Especially the regions where $f < -1$ and $f = -1$ are both characterized. A result(Theorem 7) analogous to the static metric case will be also proved in the chapter 2.

Chapter 1

Ricci flat manifolds with S^1 symmetry

1.1 Motivation

Let M be a complete, noncompact Riemannian $n-$dimensional manifold such that the Ricci curvature vanishes. As mentioned in chapter 0, it is hard to know the structure of such manifolds without any simplification of the metric in higher dimensions. As a natural simplification, we assume that there is an isometric S^1 action on the manifold M. Let $(N, \check{g})$ be the $(n-1)-$dimensional space of non-trivial orbits, where the action is free. Let $M' \subset M$ be the set of points where the action is free and let $\pi : M' \to N$ be the projection onto the space of orbits. Since π is a Riemannian submersion,

$$g_{M'} = \pi^* g_N + \theta^2$$

where θ is a connection 1-form on S^1-bundle. Using the formula of O'Neill for Riemannian submersions[Bes87], it is easy to see, c.f. [Hw], that

$$\check{s} \cdot \pi = 3|A|^2,$$

where $\check{s}$ is the scalar curvature of the metric $\check{g}$ on N and A is the $(2,1)$ tensor field on M whose values on vector fields E_1, E_2 are given by

$$A_{E_1}E_2 = \mathcal{H}D_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}D_{\mathcal{H}E_1}\mathcal{H}E_2.$$

Here $\mathcal{H}$ the horizontal distribution of the Riemannian submersion π and $\mathcal{V}$ the vertical distribution of π. The tensor field A is related to the obstruction to integrability of the horizontal distribution of the submersion π.

A metric with a smooth isometric S^1 action is called *stationary.* Moreover if $A \equiv 0$ on M, such a metric is called *static.* These metrics have been of great interest in general relativity of dimension 4, c.f. [KSHM80]. Our goal of this chapter is to study those metrics in any dimension. For a more discussion of S^1 action on Riemannian manifolds, we refer to [Bes87], [An], and [Hw].

In this chapter, we study static metrics. A static metric on M can be written as the warped product metric given by

$$g = \check{g}(x) + h^2(x)d\theta^2_{S^1}, \tag{1.1}$$

where $\check{g}$ is the $(n-1)$ dimensional metric on N, and h is the norm of the Killing vector field of the action. Then the equations that (M, g) is Ricci-flat take the form

$$h\check{r} = D_{\check{g}}dh \tag{1.2}$$

$$\Delta_{\check{g}}h = 0, \tag{1.3}$$

on N. In particular, the scalar curvature $\check{s}$ of $\check{g}$ vanishes. Even though it is the simplest possible metric, it still hard to classify these metrics. Locally there are infinitely many solutions to the equation 1.2 and 1.3. Thus in order to obtain some rigidity results, we need to impose conditions on the behavior of the infinity of such manifolds. Among conditions at infinity, asymptotic flatness is perhaps the simplest and most natural, but also quite strong condition.

It is known that if the isometric S^1 action on the static manifold does not have any fixed point and the function h is bounded, then the manifold M is isometric to $N^{n-1} \times S^1$ due to Cheng and Yau's result, c.f., [An]. Thus from now on, we study the case when the S^1 action has a fixed point.

Example of a static metric The n-dimensional Riemannian Schwarzschild metric on $\mathbb{R}^2 \times S^{n-2}$[Bes87] with the metric g can be written as

$$g = \frac{dr^2}{1 - r^{3-n}} + r^2 \hat{g}_0 + 4(1 - r^{3-n})d\theta^2,$$

where $r \geq 1$, $\hat{g}_0$ is the canonical metric, of curvature 1, on S^{n-2}, and θ is the angle variable on S^1 with period 4π. This is an example of a complete static metric. The fixed point set, where $h = 0$, is isometric to S^{n-2} of curvature 1.

Our direct motivation for the study of static metrics stems from the so-called *Black Hole Uniqueness Theorem* due to Israel[Isr67], Robinson[Rob77], and Bunting-Masood-ul-Alam[BM87].

Theorem 1 [Black Hole Uniqueness Theorem] *Let M^4 be a complete 4-dimensional static manifold. Assume that the metric on N^3 is asymptotically flat. Then M^4 is isometric to the 4-dimensional Riemannian Schwarzschild metric.*

First it should be noted that in these works, the space M^4 is usually given the metric of Lorentzian signature $3 + 1$. However the Ricci equations 1.2 and

1.3 on N^3, of the Riemannian version, have the same form for static metrics as in the Lorentzian case. Fixed point sets of the action correspond to black holes in Lorentzian case. The idea of the proof of the above theorem involves the use of the positive mass theorem in dimension three. M. T. Anderson has proved that this theorem still holds without the asymptotic decay condition of the metric, only if the fixed point set of the smooth S^1 action is compact, [An].

We consider the generalizations of this theorem to dimensions greater than or equal to 5. Without an asymptotic decay condition on the metric, the uniqueness is not true:

Theorem 2 [Bes87] *Given any Einstein manifold* $(V^{n-2}, \hat{g})$*,* $n \geq 5$*, with* $Ric_V = (n-3)\hat{g}$*, the manifold* $(\mathbb{R}^2 \times V^{n-2}, g)$*, where the metric* g *is given by*

$$g = dt^2 + f^2(t)\hat{g} + \frac{4(f')^2}{(n-2)^2} d\theta^2,$$

and f *is the unique function on* $[0, \infty)$ *with* $f(0) = 1$*,* $f' \geq 0$*,* $f'^2 = 1 - f^{1-n}$*, is complete and static.*

If we let N be the space of orbits, given by $\mathbb{R}_+ \times V^{n-2}$, with the metric $\check{g} = dt^2 + f^2(t)\hat{g}$, then N is asymptotic to the Ricci flat metric cone on V^{n-2}, which is smooth except at the origin. The metric has curvature decay of order 2, i.e., as the distance function r from some fixed point in N goes to ∞,

$$|K| \leq \frac{C}{r^2},$$

where K is the sectional curvature of the metric given by

$$\breve{g} = dt^2 + f^2(t)g$$

on N. This metric $\breve{g}$ is asymptotically flat only when $(V^{n-2}, \hat{g}) = S^{n-2}(1)$, which corresponds to the n-dimensional Riemannian Schwarzschild metric. Asymptotically flatness is a faster decay condition of the curvature of $\breve{g}$. Theorem 2 tells us that there are at least as many as $n-$dimensional complete Ricci flat metrics as $(n-2)-$dimensional positive Einstein manifolds. Then we can pose the following question:

Question A: Does every complete static Ricci flat metric with quadratic curvature decay arise in this way, i.e., any such metric is of the form given in Theorem 2?

Even though the answer to the question A is not known yet, in the case where the metric is asymptotically flat, the answer is yes, and is shown in our Uniqueness Theorem. More specifically, we will describe it in the following. With abuse of notation, we denote $\breve{g}$ by g. We assume that the manifold (N^{n-1}, g) is *asymptotically flat* in the following sense:

Assumption 1. There exists a compact set $K \in N$ such that $N \setminus K$ has a *structure of infinity*, i.e., there is a C^∞ diffeomorphism $\Phi : N \setminus K \to \mathbb{R}^{n-1} \setminus B_1(0)$ which satisfies

$$\Phi_* g = (1 + \frac{2m}{(n-3)^2}\frac{1}{r^{n-3}})\delta_{\alpha\beta} + \eta_{\alpha\beta},$$

and

$$h = 1 - \frac{1}{(n-3)}\frac{m}{r^{n-3}} + \frac{cm^2}{r^{n-2}} + \frac{c_\alpha y^\alpha}{r^{n-1}} + v$$

where h is the norm of the Killing vector field of the action, $\eta_{\alpha\beta} = O(1/r^{n-2})$, $\partial\eta \in O(1/r^{n-1})$ with $\partial^2\eta \in L^q_{-(\tau+n-2)}(E_2)$ for $q > n-1$ and $v = O(1/r^{n-1})$, $Dv = O(1/r^n)$, where m and c are constant, as $r = |x|$ goes to infinity, and $D^2 v \in L^q_{-\tau-(n-1)}(E_2)$, with $n-1 < q < \infty$ for some $\tau \in (1,2)$ where $E_2 = R^{n-1} \setminus B_2(0)$.

Here the asymptotic behavior of the harmonic function h on N is chosen in order to correspond to the behavior of the Green function on N. Also the metric $g_{\alpha\beta}$ is chosen so that the *mass* m is given by

$$m = \lim_{r\to\infty} \frac{1}{volS^{n-2}(1)} \int_{S^{n-2}(r)} |dh|_g.$$

L^q_δ, $W^{k,q}$ are defined as usual, c.f. [Bar86],[GT83]. The main result of this chapter is the following Uniqueness Theorem.

Theorem 3 [Uniqueness Theorem] *Let M^n be a complete $n-$dimensional static, spin manifold, with $n \geq 5$. Also let the metric be asymptotically flat in the sense of Assumption 1. Then M^n is isometric to the $n-$dimensional Riemannian Schwarzschild metric.*

The fixed point set of the S^1 action of the asymptotically flat metric should be compact by definition.

We recall that a spin manifold is an oriented Riemannian manifold (X^m, g) together with a lift of the structural group $SO(m)$ of its principal bundle

$SO(X^m, g)$ of oriented orthonormal frames to its simply connected double cover $Spin(m)$. It is well known that X has a spin structure if and only if its second Stiefel-Whitney class $w_2(X)$ vanishes. Note that every oriented manifold of dimension ≤ 3 is spin. This theorem is based on the following special case of the positive mass theorem due to Witten[Wit81], c.f., [Bar86, Theorem 6.3].

Theorem 4 *Suppose that (M, g) is a complete spin $n-$dimensional manifold and that there is an asymptotic structure Φ such that*

$$(\Phi_* g - \delta) \in W^{2,q}_{-\beta}(E_{R_0}) \tag{1.4}$$

for some $R_0 > 1$, $q > n$, and $\beta \geq \frac{1}{2}(n-2)$. If the scalar curvature and the mass are zero, then M is flat.

Note that this version of the positive mass theorem only requires the regularity of the metric to be $C^{1,1}(M) \cap W^{2,q}_{loc}(M)$ for $q > n$.

1.2 The proof of Uniqueness Theorem

In this section, we generalize the idea of Bunting and Masood-ul-Alam[BM87] to higher dimensions. We turn (N, g) into an asymptotically flat complete Riemannian manifold with zero scalar curvature and zero mass by making suitable conformal transformations and by gluing suitably along the boundary. Then the positive mass theorem for a spin manifold stated above applies, which implies that (N, g) is conformally flat. Since there is no structure theorem in dimension greater than or equal to 5, as in dimension 4, we need to devise an analogous result for any dimensions. Lemma 4(Structure Lemma) serves this purpose. Having this, Theorem 3 follows.

Without loss of generality, we can assume that $0 < h < 1$ on N, and $h = 0$ on the boundary ∂N, which is the fixed point set of the isometric action; g and h are assumed to be smooth in $\bar{N} = N \cup \partial N$. It follows that $|dh|$ is constant on each component of the boundary ∂N, which is totally geodesic in $\bar{N}$. The end of the manifold N, which is the region when h tends to 1, is simply connected by Assumption 1.

In order to prove theorem 3, we need the following lemmas.

Lemma 1 *The metric $\gamma_\pm = \Omega_\pm^2 g_{\alpha\beta}$ is of scalar curvature zero, where $\Omega_\pm = 2^{-\frac{2}{n-3}}(1 \pm h)^{\frac{2}{n-3}}$. Also the metric γ_+ is asymptotically flat with mass zero.*

Proof For $g^1 = \Omega^2 g$, $s_g = 0$, and $\Omega = \Omega(h)$, the scalar curvature s^1 of g^1 is given by

$$\begin{aligned} s^1 &= \Omega^{-2}[s_g - \Omega^{-2}(n-2)(2\Omega\Delta\Omega + (n-5)|d\Omega|^2)] \\ &= -\frac{(n-2)|dh|^2}{\Omega^4}(2\Omega\frac{d^2\Omega}{dh^2} + (n-5)(\frac{d\Omega}{dh})^2), \end{aligned}$$

since $\Delta h = 0$. Taking $\Omega = b(1 \pm h)^{\frac{2}{n-3}}$ with a positive constant b, we have $s^1 = 0$. We shall take $b = 2^{-\frac{2}{n-3}}$ from now on.

For the other part, for sufficiently large r,

$$\begin{aligned} \gamma_{+\alpha\beta} &= (\frac{1+h}{2})^{\frac{4}{n-3}}[(1 + \frac{2m}{(n-3)^2}\frac{1}{r^{n-3}})\delta_{\alpha\beta} + \eta_{\alpha\beta}] \\ &= (1 - \frac{m}{2(n-3)r^{n-3}} + \frac{cm^2}{r^{n-2}} + \rho)^{\frac{4}{n-3}}[(1 + \frac{2m}{(n-3)^2}\frac{1}{r^{n-3}})\delta_{\alpha\beta} + \eta_{\alpha\beta}] \end{aligned}$$

$$
\begin{aligned}
&= (1-\frac{2m}{(n-3)^2}\frac{1}{r^{n-3}}+\frac{cm^2}{r^{n-2}}+\tilde{\rho})[(1+\frac{2m}{(n-3)^2}\frac{1}{r^{n-3}})\delta_{\alpha\beta}+\eta_{\alpha\beta}] \\
&= \delta_{\alpha\beta}+\tilde{\eta}_{\alpha\beta},
\end{aligned}
$$

where $\tilde{\rho}=O(1/r^{d-1})$, and $\tilde{\eta}_{\alpha\beta}=O(1/r^{d-1})$, for some $d\geq n-1$, which implies that the metric is asymptotically flat satisfying 1.4 and has zero mass.

□

Lemma 2 γ_- *compactifies the infinity: If p is the point at infinity, then there is a* $W^{2,q}$ *extension of* γ_- *to* $N\cup\{p\}$.

Proof From Assumption 1, we get, for sufficiently large r,

$$
\begin{aligned}
\gamma_{-\alpha\beta} &= \left(\frac{1-h}{2}\right)^{\frac{4}{n-3}} g_{\alpha\beta} \\
&= \left(\frac{1}{2(n-3)}\frac{m}{r^{n-3}}+\frac{cm^2}{r^{n-2}}+\frac{c_\alpha y^\alpha}{r^{n-1}}+v\right)^{\frac{4}{n-3}} g_{\alpha\beta} \\
&= \left(\frac{m}{2(n-3)}\right)^{\frac{4}{n-3}}\frac{1}{r^4}[1+\frac{\tilde{c}m}{r}+\frac{c_\alpha y^\alpha}{r^2}+r^{n-3}v]^{\frac{4}{n-3}} g_{\alpha\beta}
\end{aligned}
$$

Thus the metric $\gamma_{-\alpha\beta}$ becomes

$$
\begin{aligned}
&\left(\frac{m}{2(n-3)}\right)^{\frac{4}{n-3}}\frac{1}{r^4}[1+\frac{\tilde{c}m}{r}+\frac{c_\alpha y^\alpha}{r^2}+r^{n-3}v]^{\frac{4}{n-3}}[(1+\frac{2m}{(n-3)^2}\frac{1}{r^{n-3}})\delta_{\alpha\beta}+\eta_{\alpha\beta}] \\
&= \left(\frac{m}{2(n-3)}\right)^{\frac{4}{n-3}}\frac{1}{r^4}[1+\frac{c'm}{r}+\frac{c'_\alpha y^\alpha}{r^2}+\tilde{v}][(1+\frac{2m}{(n-3)^2}\frac{1}{r^{n-3}})\delta_{\alpha\beta}+\eta_{\alpha\beta}] \\
&= \left(\frac{m}{2(n-3)}\right)^{\frac{4}{n-3}}\frac{1}{r^4}[(1+\phi)\delta_{\alpha\beta}+\Phi_{\alpha\beta}],
\end{aligned}
$$

where $\tilde{v}=O(\frac{1}{r^2})$, $|D\tilde{v}|=O(\frac{1}{r^3})$, $\phi=\sum_\lambda O(1/r^\lambda)$ with $1\leq\lambda\leq n-2$, and $\Phi_{\alpha\beta}=O(1/r^{n-1})$, $|D\Phi_{\alpha\beta}|=O(1/r^n)$,

$$D^2_{(y)}\Phi\in L^q_{-\tau-(n-1)}(E_2),$$

where $\tau \in (1,2)$, and $\tilde{c}$, c' are some constant. Making coordinate transformation $z^\alpha = y^\alpha/r^2$, so that

$$\frac{\partial y^\alpha}{\partial z^\beta} = (\delta_{\alpha\beta} - 2z^\alpha z^\beta/|z|^2)|z|^{-2}.$$

Hence

$$\gamma_- = \left(\frac{m}{2(n-3)}\right)^{\frac{4}{n-3}} [(1+\tilde{\phi})\delta_{\alpha\beta} + \tilde{\Phi}_{\alpha\beta}]dz^\alpha dz^\beta$$

where $\tilde{\Phi}_{\alpha\beta} = O(|z|^{n-1})$, $|D\tilde{\Phi}_{\alpha\beta}| = O(|z|^{n-2})$, and

$$\begin{aligned}
&\int_{|z|<\epsilon} |D^2_{(z)}\tilde{\Phi}_{\alpha\beta}|^q dz \\
\leq\ & C\int_{|y|>\frac{1}{\epsilon}} |D^2_{(y)}\Phi_{\alpha\beta}|^q |y|^{4q-2(n-1)} dy \\
=\ & C\int_{|y|>\frac{1}{\epsilon}} |D^2_{(y)}\Phi_{\alpha\beta}|^q |y|^{(\tau+n-1)q-(n-1)} |y|^{-(\tau+n-1-4)q-(n-1)} dy \\
=\ & O(\epsilon^{(\tau+n-5)q+(n-1)}),
\end{aligned}$$

where C is a constant independent of ϵ. Note that $\tau + n - 5 \geq \tau - 1 > 0$ for $n \geq 4$. Thus the compactified metric

$$\begin{aligned}
\gamma_-^* &= \gamma_-(z) \quad on \quad \{z : 0 < |z| < \epsilon\} \\
&= \left(\frac{m}{2(n-3)}\right)^{\frac{4}{n-3}} \delta_{\alpha\beta} dz^\alpha dz^\beta \quad at \quad p
\end{aligned}$$

is $W^{2,q}_{loc}$.

□

Lemma 3 *The second fundamental form of the boundary in the $\gamma_\pm$ is given by*

$$II(\gamma_\pm)_{\alpha\beta} = \pm\frac{2^{\frac{n-5}{n-3}}}{(n-3)}|dh|\hat{\gamma}_{\pm\alpha\beta} \quad \alpha,\beta = 1,..n-2 \tag{1.5}$$

where $\hat{\gamma}_\pm$ is the metric on the boundary induced from $\gamma_\pm$, and $|dh|$ is positive constant on the each component of the boundary.

Proof For $g^1 = \Omega^2 g$, $\Omega = \Omega(h)$, the second fundamental forms on the boundary relative to g^1 and g are related by

$$II(g^1)_{\alpha\beta} = \Omega II(g)_{\alpha\beta} + \Omega^{-2}(\frac{d\Omega}{dh})|dh|\hat{g}_{\alpha\beta} \tag{1.6}$$

where $\hat{g}$ is the metric induced on the boundary from g^1. Note that the boundaries with respect to the metric g are totally geodesic. Then the lemma follows by taking $\Omega = 2^{-\frac{2}{n-3}}(1 \pm h)^{\frac{2}{n-3}}$.

□

Then we construct a metric γ on the $\tilde{N} \cup \{p\}$, where $\tilde{N}$ is the double of N glued along with the appropriate boundaries; having two copies Σ_+ and Σ_-, paste these along their respective boundaries to form the double $\tilde{N}$, with the metric

$$\begin{aligned} \gamma^*(x) &= \gamma_+(x) \quad x \in N_+ \\ &= \gamma_-(x) \quad x \in N_-. \end{aligned}$$

By the above lemma we can compactify one end of $\tilde{N}$ by adding a point p representing the infinity of N_-, and extend γ^* to a metric γ on $\tilde{N} \cup \{p\}$ such that γ is $W^{2,q}$ in a neighborhood of P in $\tilde{N} \cup \{p\}$. Also we have γ is locally $W^{2,q}$ on $\tilde{N} \cup \{p\}$, since γ is $C^{1,1}$ in a neighborhood of the boundaries of N_+ and N_-.

By this construction, $(\tilde{N} \cup \{p\}, \gamma)$ is complete, of scalar curvature zero, γ is $W^{2,q}$, and asymptotically flat with zero mass. Thus, by the positive mass theorem mentioned in the introduction, $\tilde{N} \cup \{p\}$ is isometric to $\mathbb{R}^{n-1}$. In particular, $(\tilde{N} \cup \{p\}, \gamma)$ is conformally flat.

Lemma 4 [Structure Lemma] *The function $W = |dh|^2$ on N depends only on h. Also the second fundamental forms of level sets $h^{-1}(c)$ for each c, $0 < c < 1$, depend only on h. In fact, $II_{h^{-1}(c)} = \frac{2h|dh|}{(n-3)(1+h)} g$.*

Proof For $g^1 = \Omega^2 g$,

$$r^1 = r - (n-3)(\frac{Dd\Omega}{\Omega} - 2\frac{d\Omega \cdot d\Omega}{\Omega^2}) - (\frac{\Delta\Omega}{\Omega} + (n-4)|dlog\Omega|^2)g. \qquad (1.7)$$

Take an orthonormal frame $\{E_i\}_{i=1..n-2}$ on $h^{-1}(c)$ for a regular value c of $h : \Omega \to \mathbb{R}_+$, and let $\nu = dh/W^{1/2}$ be the outward normal vector field on $h^{-1}(c)$. By taking $\Omega = 2^{-\frac{2}{n-3}}(1+h)^{\frac{2}{n-3}}$ in the above formula,

$$\begin{aligned} r^1(X,\nu) &= r(X,\nu) - (n-3)\Omega^{-2}(\Omega\langle D_X d\Omega, \nu\rangle - 2d\Omega(X)d\Omega(\nu)) \\ &= r(X,\nu) - 2r(X,\nu)\frac{h}{1+h} = r(X,\nu)\frac{1-h}{1+h} \end{aligned}$$

where X is a vector field tangent to $h^{-1}(c)$. Since $r^1 \equiv 0$, we have $r(X,\nu) = 0$ or $h = 1$. By the assumption on h(otherwise $h \equiv 1$ on N, and N is Ricci-flat), we have

$$\begin{aligned} E_i(W) &= 2\langle D_{E_i} dh, dh\rangle = 2hr(E_i, dh) \\ &= 2hW^{1/2} r(E_i, \nu) = 0. \end{aligned}$$

Now we claim that there is no critical point of h in N. We will prove this in the following lemma. Having this, it implies that the boundary has one connected component. Since the Weyl tensor $\mathcal{W}$ vanishes, the Riemann curvature tensor is determined by the Ricci tensor only. For $\gamma_+ = \Omega^2 g$, $\Omega = \Omega(h)$, and for $X \in Th^{-1}(c)$ for some c,

$$\begin{aligned} r^1(X,X) &= r(X,X) - (n-3)\Omega^{-1}\frac{d\Omega}{dh}hr(X,X) - \Omega^{-2}|dh|^2[\Omega\frac{d^2\Omega}{dh^2} + (n-4)(\frac{d\Omega}{dh})^2] \\ &= r(X,X)[1-(n-3)\Omega^{-1}\frac{d\Omega}{dh}h] - \Omega^{-2}|dh|^2[\Omega\frac{d^2\Omega}{dh^2} + (n-4)(\frac{d\Omega}{dh})^2]. \end{aligned}$$

Let $\Omega = 2^{-\frac{2}{n-3}}(1+h)^{\frac{2}{n-3}}$, then we have $r^1(X,X) = 0$, and

$$\begin{aligned} r(X,X) &= \Omega^{-2}|dh|^2[\Omega\frac{d^2\Omega}{dh^2} + (n-4)(\frac{d\Omega}{dh})^2]/[1-(n-3)\Omega^{-1}\frac{d\Omega}{dh}h] \\ &= \frac{2|dh|^2}{(n-3)(1+h)}. \end{aligned}$$

Hence

$$\begin{aligned} II(X,X) &= \langle D_X\nu, X\rangle = \frac{1}{|dh|}\langle D_X dh, X\rangle = \frac{h}{|dh|}r(X,X) \\ &= \frac{2h|dh|}{(n-3)(1+h)}, \end{aligned}$$

and

$$II(X,Y) = \langle D_X\nu, X\rangle = \frac{1}{|dh|}\langle D_X dh, Y\rangle = \frac{h}{|dh|}r(X,Y) = 0,$$

where the last equality follows from a similar computation for $r(X,Y)$, with orthonormal vector fields X and Y tangent to $h^{-1}(c)$.

□

Lemma 5 *There is no critical point of h in N.*

Proof By Assumption 1, we have $|dh| = O(1/r^{n-2})$. From the Bochner formula

$$0 = |Ddh|^2 - \frac{1}{2}\Delta|dh|^2 + r(dh,dh),$$

$$\frac{1}{2}\Delta W - \frac{\langle dW, dh\rangle}{2h} = |Ddh|^2 \geq 0,$$

which implies that the function $W = |dh|^2$ cannot have its maximum in the interior of N. Suppose there is a critical point q of h with $h(q) = c_q$. Consider the domain $S = \{x \in N | c_q \leq h(x) < 1\}$. Note that W is a function of h only. Since $W = 0$ on $h^{-1}(c_q)$ and W tends to 0 as h tends to 1, it should have its maximum in the interior of S, which is a contradiction.

□

By the elementary Morse theory and Lemma 5, the topology of N is homeomorphic to $h^{-1}(0) \times \mathbb{R}_+$, and since we assume that the topology at the infinity is simple, the boundary $h^{-1}(0)$ is simply connected and diffeomorphic to S^{n-2}. Also each $h^{-1}(c)$ is diffeomorphic to S^{n-2}.

It is also easy to see that $h^{-1}(c)$ are constant curvature hypersurfaces of codimension 2 in M from Gauss equation

$$K_{\alpha\beta} = \bar{K}_{\alpha\beta} + II^2,$$

and from

$$R(x,y,x,y) = -\frac{s}{(n-2)(n-3)} + \frac{1}{n-3}(r(x,x) + r(y,y)) + \mathcal{W}(x,y,x,y),$$

for any orthonormal x, y in $T\Sigma$. Note that $\mathcal{W}$ vanishes since the metric $\breve{g}$ is conformally flat. It follows that each level set of h is isometric to spheres of appropriate radii. In fact, the sectional curvature of each level set is given by

$$K_{h^{-1}(c)} = \frac{4|dh|^2}{(n-3)^2}\frac{1+c+c^2}{(1+c)^2}. \tag{1.8}$$

Thus the metric g is spherically symmetric and g is isometric to the hypersurface of $n-$dimensional Riemannian Schwarzschild metric.

Remark. (1) A 4−dimensional manifold is *spherically symmetric* if there is an isometric action of the special orthogonal group $SO(3)$ on M each of whose orbits is either a 2−surface of constant positive curvature or a single point($n-$dimensional spherically symmetric manifold is similarly defined). It should be noted that even having the spherical symmetry, an Einstein manifold of nonzero constant λ does not have to be unique. It is easy to show that a spherically symmetric Einstein manifold M^4 of constant λ(not zero) can be written, locally at points where $df \neq 0$, as

$$\bar{g} = dt^2 + f'(t)^2 du^2 + f(t)^2(d\theta^2 + sin^2\theta d\phi^2), \tag{1.9}$$

where $f'^2 = 1 - \frac{\lambda}{3}f^2 + \frac{c}{f}$, for some constant c [Bes87]. Especially, for any positive a, take $b = (\frac{3}{2}a + \frac{1}{2a})^{-1}$, $u = b\varphi$, and f the unique function on $[0, \infty)$ such that $f(0) = a, f'(0) = 0, f' \geq 0$ and $2ff'' + f'^2 - 3f^2 = 1$, $f'^2 = 1 + f^2 - a(1+a^2)f^{-1}$. Then the above metric on $\mathbb{R}^2 \times_{f^2} S^2$ is complete and Einstein with $\lambda = -3$. Thus we have a one-parameter family of such metrics, depending on a. It should also be noted that a spherically symmetric Einstein manifold M^n is homeomorphic to $\mathbb{R}^2 \times S^{n-2}$ or $S^1 \times \mathbb{R} \times S^{n-2}$.

(2) Assumption 1 is a strong condition on the asymptotic geometry of the manifold. It may be the case that the assumption can be weakened to the bound

$$|R| = \frac{C}{r^{2+\epsilon}}, \tag{1.10}$$

where R is the full Riemannian curvature tensor and ϵ is a positive constant. In order to obtain a similar result with the weaker condition 1.10, one should apply the arguments of [BKN]. It will appear later.

□

1.3 A metric with large symmetries

By an ingenious argument, Weyl[Syn66] proved that a 4−dimensional axially symmetric static metric is determined by one axially symmetric harmonic function on a domain in $\mathbb{R}^3$. In other words, such a metric corresponds to a harmonic function of S^1 symmetry locally defined on $\mathbb{R}^3$. In this section we generalize this fact to higher dimensions. Assume that a Riemannian manifold (M^n, ds^2) admits $(n-2) - S^1$ actions such that the metric can be written as:

$$\bar{g} = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2 + \sum_{i=3}^{n} g_{ii}dx_i^2, \tag{1.11}$$

where the g's are functions of x_1, and x_2. The topology of the manifold is $\mathbb{R}^2 \times (S^1)^{n-2}$. Let $\Sigma = \mathbb{R}_+ \times S^1 = \mathbb{R}^2$. Using isothermal coordinate on Σ, equation 1.11 becomes

$$\bar{g} = \alpha^2(dx_1^2 + dx_2^2) + \sum_{i=3}^{n} f_i^2 dx_i^2, \tag{1.12}$$

where $f_i^2 = g_{ii}$. By straightforward calculation, the surviving components of the Ricci tensor are the following:

$$-r_{11} = \partial_1(\frac{\partial_1\alpha}{\alpha}) + \partial_2(\frac{\partial_2\alpha}{\alpha}) + \sum_{i=3}^{n}\frac{\partial_{11}f_i}{f_i} - \frac{\partial_1\alpha}{\alpha}(\sum_{i=3}^{n}\frac{\partial_1 f_i}{f_i}) + \frac{\partial_2\alpha}{\alpha}(\sum_{i=3}^{n}\frac{\partial_2 f_i}{f_i}),$$

$$
\begin{aligned}
-r_{12} &= \sum_{i=3}^{n}\left(\frac{\partial_2\alpha\partial_1 f_i + \partial_1\alpha\partial_2 f_i}{\alpha f_i} - \frac{\partial_{12} f_i}{f_i}\right),\\
-r_{22} &= \partial_1(\frac{\partial_1\alpha}{\alpha}) + \partial_2(\frac{\partial_2\alpha}{\alpha}) + \sum_{i=3}^{n}\frac{\partial_{22} f_i}{f_i} - \frac{\partial_2\alpha}{\alpha}(\sum_{i=3}^{n}\frac{\partial_2 f_i}{f_i}) + \frac{\partial_1\alpha}{\alpha}(\sum_{i=3}^{n}\frac{\partial_1 f_i}{f_i}),\\
-r_{ii} &= \frac{f_i}{\alpha^2}\left(\Delta f_i + \sum_{j\neq i, j\geq 3}^{n}\frac{\partial_1 f_i\partial_1 f_j + \partial_2 f_i\partial_2 f_j}{f_j}\right),
\end{aligned}
$$

where $i \geq 3$, and

$$\Delta f_i = \partial_{11} f_i + \partial_{22} f_i \tag{1.13}$$

We note that

$$\sum_{i\geq 3} r_i^i = \frac{\Delta(\Pi_{j\geq 3} f_j)}{\alpha^2 \Pi_{j\geq 3} f_j}. \tag{1.14}$$

Now consider the case when the metric is Ricci flat(in fact, it suffices to assume that $\sum_{i\geq 3} r_i^i = 0$). Then by the equation 1.14, $r = \Pi_{j\geq 3} f_j$ is a harmonic function of (x_1, x_2). Then there exists globally a conjugate harmonic function $z(x_1, x_2)$ using the simply connectedness of $\mathbb{R}^2$, such that

$$r + iz = l(x_1 + ix_2),$$

where l is an analytic function. Making a conformal change of the metric we have

$$\alpha^2(dx_1^2 + dx_2^2) = A^2(dr^2 + dz^2). \tag{1.15}$$

Thus we can get the metric in the form of

$$\bar{g} = A^2(dr^2 + dz^2) + \left(\frac{r}{\Pi_{j\geq 4} f_j}\right)^2 dx_3^2 + \sum_{k\geq 4}^{n} f_k^2 dx_k^2 \tag{1.16}$$

where (A, f_i), $i \geq 4$ are functions of (r, z).

Let $A = e^{\nu - \sum_{k=4}^{n}\lambda_k}$, $f_k = e^{\lambda_k}$, $k \geq 4$. Then

$$ds^2 = e^{2(\nu - \sum_{k=4}^{n}\lambda_k)}(dr^2 + dz^2) + r^2 e^{-2\sum_{k=4}^{n}\lambda_k} dx_3^2 + \sum_{k\geq 4} e^{2\lambda_k} dx_k^2. \tag{1.17}$$

Thus (in the following computations, 1 means r-coordinate and 2 means z-coordinate) we have

$$\begin{aligned}
-r_{11} &= \Delta\nu - \sum_{k\geq 4}\Delta\lambda_k - \frac{1}{r}\sum_{k\geq 4}\partial_1\lambda_k + (\sum_{k\geq 4}\partial_1\lambda_i)^2 + \sum_{k\geq 4}(\partial_1\lambda_k)^2 - \frac{\partial_1\nu}{r}, \\
-r_{12} &= \frac{\partial_2\nu}{r} - (\sum_{k\geq 4}\partial_1\lambda_k)(\sum_{j\geq 4}\partial_2\lambda_j) - \sum_{k\geq 4}\partial_1\lambda_k\partial_2\lambda_k, \\
-r_{22} &= \Delta\nu - \sum_{k\geq 4}\Delta\lambda_k - \frac{1}{r}\sum_{k\geq 4}\partial_1\lambda_k + (\sum_{k\geq 4}\partial_2\lambda_i)^2 + \sum_{k\geq 4}(\partial_2\lambda_k)^2 + \frac{\partial_1\nu}{r}, \\
-r_{33} &= -r^2e^{-2\nu}(\sum_{k\geq 4}\Delta\lambda_k + \frac{1}{r}\sum_{k\geq 4}\partial_1\lambda_k), \\
-r_{ii} &= \frac{e^{2\lambda_i}}{e^{2(\nu-\sum_{k\geq 4}\lambda_k)}}\left(\Delta\lambda_i + \frac{\partial_1\lambda_i}{r}\right), \quad i\geq 4.
\end{aligned}$$

Moreover, we find that

$$\begin{aligned}
-\frac{1}{2}(r_{11}+r_{22}) &= \Delta\nu - \sum_{k\geq 4}\Delta\lambda_k - \frac{1}{r}\sum_{k\geq 4}\partial_1\lambda_k + \frac{1}{2}[(\sum_{k\geq 4}\partial_1\lambda_i)^2 \\
&\quad +(\sum_{k\geq 4}\partial_2\lambda_i)^2 + \sum_{k\geq 4}(\partial_1\lambda_k)^2 + \sum_{k\geq 4}(\partial_2\lambda_k)^2], \\
-r_{11}+r_{22} &= (\sum_{k\geq 4}\partial_1\lambda_i)^2 - (\sum_{k\geq 4}\partial_2\lambda_i)^2 + \sum_{k\geq 4}(\partial_1\lambda_k)^2 - \sum_{k\geq 4}(\partial_2\lambda_k)^2 - 2\frac{\partial_1\nu}{r}.
\end{aligned}$$

For a Ricci flat metric, we have

$$\Delta\lambda_i + \frac{\partial_1\lambda_i}{r} = 0, \quad i\geq 4, \tag{1.18}$$

$$\partial_1\nu = \frac{r}{2}[(\sum_{k\geq 4}\partial_1\lambda_i)^2 - (\sum_{k\geq 4}\partial_2\lambda_i)^2 + \sum_{k\geq 4}(\partial_1\lambda_k)^2 - \sum_{k\geq 4}(\partial_2\lambda_k)^2], \tag{1.19}$$

$$\partial_2\nu = r[(\sum_{k\geq 4}\partial_1\lambda_k)(\sum_{j\geq 4}\partial_2\lambda_j) + \sum_{k\geq 4}\partial_1\lambda_k\partial_2\lambda_k], \tag{1.20}$$

$$2\Delta\nu + (\sum_{k\geq 4}\partial_1\lambda_i)^2 + (\sum_{k\geq 4}\partial_2\lambda_i)^2 + \sum_{k\geq 4}(\partial_1\lambda_k)^2 + \sum_{k\geq 4}(\partial_2\lambda_k)^2 = 0. \tag{1.21}$$

If the equation 1.18 is satisfied, then 1.19 and 1.20 are integrable, and 1.21 is implied by the other equations. Thus in any domain Ω in which the metric is Ricci flat, we obtain the metric with $\lambda_k, k \geq 4$ satisfying 1.18, and ν defined by

$$\begin{aligned} \nu &= \int \frac{r}{2}[(\sum_{k\geq 4}\partial_1\lambda_i)^2 - (\sum_{k\geq 4}\partial_2\lambda_i)^2 + \sum_{k\geq 4}(\partial_1\lambda_k)^2 - \sum_{k\geq 4}(\partial_2\lambda_k)^2]dr \\ &\quad + r[(\sum_{k\geq 4}\partial_1\lambda_k)(\sum_{j\geq 4}\partial_2\lambda_j) + \sum_{k\geq 4}\partial_1\lambda_k\partial_2\lambda_k]dz \end{aligned}$$

The equation 1.18 is recognized as Laplace's equation in cylindrical coordinates (r, ϕ, z) in Euclidean space $\mathbb{R}^3$ for a function which is independent of ϕ. After appropriate relabeling, we have the following

Theorem 5 *A metric of form 1.11 is Ricci flat if and only if the metric $\bar{g}$ is given by*

$$\bar{g} = e^{2(\nu - \sum_{i=1}^{n-3}\lambda_i)}(dr^2 + dz^2) + r^2 e^{-2\sum_{i=1}^{n-3}\lambda_i}dx_0^2 + \sum_{i=1}^{n-3} e^{2\lambda_i}dx_i^2 \tag{1.22}$$

where $\Delta\lambda_i + \frac{\partial_r\lambda_i}{r} = 0$, for $i = 1, ..n-3$, and ν is given by

$$\begin{aligned} \nu &= \int \frac{r}{2}[(\sum_i \partial_r\lambda_i)^2 - (\sum_i \partial_z\lambda_i)^2 + \sum_i(\partial_r\lambda_i)^2 - \sum_i(\partial_z\lambda_i)^2]dr \\ &\quad + r[(\sum_i \partial_r\lambda_i)(\sum_j \partial_z\lambda_j) + \sum_i \partial_r\lambda_i\partial_z\lambda_i]dz, \end{aligned}$$

where each summation takes from 1 to $n-3$.

We can say that the metric $\bar{g}$ is generated by $n-3$ functions. We call a metric in Theorem 5 a G_{n-3}-*metric*, and a manifold with such metric a G_{n-3}-*manifold*.

Example. (1) For a 5-dimensional G_2-manifold with $\lambda_1 = a \; ln(r)$ and $\lambda_2 = bz$, $\nu = a^2 \; ln(r) - \frac{1}{2}b^2r^2 + abz$, and the metric becomes

$$\bar{g} = r^{2a(a-1)}e^{-b^2r^2+2b(a-1)z}(dr^2 + dz^2) + r^2e^{-2(a \; ln(r)+bz)}dx_0^2 + r^{2a}dx_1^2 + e^{2bz}dx_2^2, \tag{1.23}$$

which is Ricci-flat. Then we have the curvature of order

$$r^{-2a^2+2a-2}e^{b^2r^2-2abz+2bz},$$

as r goes to ∞.

(2) For $(k+3)$-dimensional G_k-manifold with all $\lambda_i = ln(r)$, $i = 1, ..., k$, $\nu = \frac{1}{2}k(k+1)ln(r)$, and the metric becomes

$$\bar{g} = r^{k(k-1)}(dr^2 + dz^2) + r^{2-2k}dx_0^2 + r^2 \sum_{j=1}^{k} dx_j^2, \tag{1.24}$$

which has curvatures of order $\frac{1}{r^4}$ when $k = 2$, and of order $\frac{1}{r^8}$ when $k = 3$. In fact, it is of order $r^{-k(k-1)-2}$ for general k.

If we take $\lambda_i = z$, $i = 1, ..., k$, then $\nu = -\frac{1}{4}r^2k(k+1)$, and the metric is given by

$$\bar{g} = e^{-k(\frac{1}{2}r^2(k+1)+2z)}(dr^2 + dz^2) + r^2e^{-2kz}dx_0^2 + \sum_{i=1}^{k} e^{2z}dx_i^2,$$

which has curvatures of order $e^{\frac{1}{2}r^2k(k+1)+2kz}$ as r goes to ∞.

Chapter 2

Metrics of Constant Scalar Curvature

2.1 Motivation

Let $\mathcal{M}$ denote the set of all smooth Riemannian structures on a closed n-manifold M, and $\mathcal{M}_1$ those of volume 1. Einstein and Hilbert showed that critical points of the scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$, which is given by $g \to \mathcal{S}(g) = \int_M s_g dv_g$ are exactly Einstein metrics, where s_g denote the scalar curvature of the metric g, and dv_g is the volume form. As mentioned in the chapter 0, we introduce the space of Yamabe metrics,

$$\mathcal{C} = \{g \in \mathcal{M}_1, \quad s_g \quad \text{constant}\}.$$

When we restrict the domain of the scalar functional $\mathcal{S}$ to $\mathcal{C}$, the equation for a critical point(metric) becomes

$$z_g = D_g df - (\Delta_g f)g - fr_g \tag{2.1}$$

where z_g is the traceless Ricci tensor, r_g is the Ricci tensor, and f is a function on M with vanishing mean value.

J. Lafontaine showed that if a solution to the equation is conformally flat, such a metric is Einstein[Laf83,p71]. It has been conjectured that the only solutions to these equations are Einstein metrics, with either $f \equiv 0$ or f a first eigenfunction of the Laplacian on $S^n(1)$[Bes87, 4.48]. If a solution is Einstein with nonzero f, it is isometric to the standard sphere due to [Oba62](see Section 2.2). It is not known yet whether it is true or not in general, but we have obtained several convincing partial results in dimension 3. An immediate consequence of the equation is that $s_g/(n-1)$ is in the spectrum of the Laplacian, if f is not identically zero. Using a simple calculation we have proved that if $f \geq -1$, then g is Einstein. In general, we have developed numerous so-called *Robinson-type identities* for these equations in dimension 3. These impose quite strong restrictions on the structure of solutions. Especially we prove the following result:

Theorem 6 *Let (g, f) be a $3-$dimensional solution to* 2.1. *Then the region where $f < -1$ is either diffeomorphic to a 3-ball with convex boundary homeomorphic to S^2, or there is compact minimal 2-sphere(or projective plane) contained in it which is locally of least area.*

The boundary B of the set $f < -1$ has special properties. It is a convex submanifold of M. Also the functions $W = |df|^2$ and $|dW|$ are constant on B(Theorem 9). Especially, on B, we have

$$|z|^2 = \frac{3}{2} z(N,N)^2,$$

where N is the normal vector field on B. This is equivalent to saying that the traceless Ricci tensor z has eigenvalues $z_1 = z_2 = -2z_3$, where z_3 means the eigenvalue corresponding to the normal vector field N, and z_1 and z_2 to the tangent vector fields to the submanifold B. Note that if we set $s_g = 0$, the equation 2.1 reduces to the static equations 1.2 and 1.3 in the chapter 1. Thus we can think of a solution to 2.1 as a compact manifold version of the static equations. In the static metric case, the solutions with eigenvalues of $z_1 = z_2 = -2z_3$ everywhere are called "degenerate", and classified by Levi-Civita[KSHM80]. If a solution to 2.1 has the same eigenvalues everywhere, it is shown that the solution is Einstein. More generally, we prove the following result as an analogous result holds for static equations[Kün71].

Theorem 7 *Let (g, f) be a 3−dimensional solution to 2.1. If $g(df, df)$ is a function of f only, then the metric g is Einstein.*

2.2 Properties of Critical metrics

We start with the following observation, see 4.47 in [Bes87].

Proposition 1 *Let g be a metric in $\mathcal{C}$ with scalar curvature s_g such that $s_g/(n-1)$ is not in the spectrum $Sp^+\Delta_g$ of the Laplacian. If the metric g is a critical point of the scalar curvature functional restricted to $\mathcal{C}$, then (M, g) is Einstein.*

From now on we investigate the case when $s_g/(n-1) \in Sp^+\Delta_g$. The restriction of the scalar curvature functional $\mathcal{S}$ to $\mathcal{C}$ will have a critical point metric g if and only if there is a function f with vanishing mean value such that

$$z_g = D_g df - (\Delta_g f)g - fr_g.$$

This is equivalent to the one of following equations

$$(1+f)r_g = D_g df + \frac{n-1+nf}{n(n-1)} s_g g, \tag{2.2}$$

$$(1+f)z_g = D_g df + \frac{s_g f}{n(n-1)} g, \tag{2.3}$$

where r_g, z_g is the Ricci tensor, and the traceless Ricci curvature tensor, respectively, and $\Delta_g f = \frac{s_g}{1-n} f$.

Theorem 8 [Oba62] *Let* (N, γ) *be a compact* $n-$*dimensional manifold of constant scalar curvature. Assume that it admits a nontrivial solution* ϕ *of* $D_\gamma d\phi = \frac{\Delta_\gamma \phi}{n} g$. *Then* (N, γ) *is isometric to the standard sphere of appropriate radius.*

Thus, if g is both a solution to 2.1 and Einstein, (M, g) is isometric to the standard sphere.

We shall denote by B_c the set $\{x \in M | (1+f)(x) = c\}$, and M_c the set $\{x \in M | (1+f)(x) \leq c\}$. We identify B by B_0. Also we denote by $|z|$ the norm of the traceless Ricci tensor of the metric g.

Proposition 2 *If either* $f \geq -1$ *or* $|z| \equiv 0$ *on* M_0, *then* (M, g) *is Einstein.*

Proof From 2.3,

$$\begin{aligned} \int_M (1+f)|z|^2 &= \int_M (1+f) z_{ij} z^{ij} \\ &= \int_M (Ddf + \frac{sf}{n(n-1)} g)_{ij} z^{ij} \\ &= \int_M (Ddf)_{ij} z^{ij} = \int_M (df)_i (\delta z)_k^{ik} = 0, \end{aligned}$$

where the last equality comes from the following equation:

$$\delta z = \delta(r - \frac{s}{n}g) = \delta r + d(\frac{s}{n}) = \delta r = -\frac{1}{2}d(s) = 0.$$

□

Remark. We show later that $z = 0$ on B suffices to prove that the metric is Einstein(see (3) in the Remark after Theorem 9).

□

Proposition 2 suggests that the study of the sets $M_0 = \{x \in M | (1+f)(x) \leq 0\}$ or $B = \{x \in M | (1+f)(x) = 0\}$ may be useful. The rest of this section will be devoted to such a study.

Suppose that the function f is critical at a point $p \in B$. On the set B, we have

$$D_g df(\xi, \xi) = \frac{s}{n(n-1)} g(\xi, \xi) > 0,$$

for any tangent vector ξ in the tangent space T_pM at p. Thus the point p is a non-degenerate critical point of f. Since the index of f at p is 0, p is a local minimum(case I in figure 2.2). Since such points are isolated, the set B' of critical points in the compact set B should be finite. Suppose that $B = B'$, i.e., every point in the set $B = f^{-1}(-1)$ is a critical point of the function f. Then the value -1 should be the global minimum of f, which implies that the metric g is Einstein, by Proposition 2. From now on we can assume that $B \neq B'$. Then components of $B \backslash B'$ are co-dimension 1 submanifolds of M(case II, III, IV in figure 2.2). The following is the summary of behavior of tensors on the set(submanifold) B.

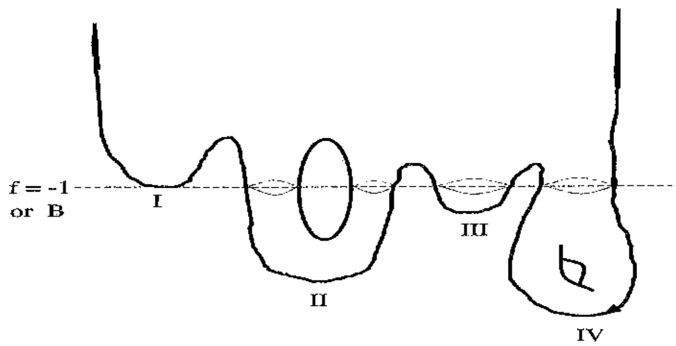

Figure 2.1: Topology of the manifold near the submanifold B

Theorem 9 *We have the following results:*

(i) *the function* $W = |df|^2$ *is constant* W_B *on each component of* B,

(ii) *each component of* $B\backslash B'$ *is a convex submanifold of* M. *The second fundamental form* II_B *at the set* B *is given by* $\frac{s}{n(n-1)W_B^{1/2}}g$, *the mean curvature* m_B *at* B *is* $\frac{s}{nW_B^{1/2}}$, *and the norm of the second fundamental form at* B *is given by* $|II|_B^2 = \frac{s^2}{n^2(n-1)W_B}$,

(iii) *the function* $\Omega = |dW|^2$ *is constant on* B,

(iv) $N(z(df, N)) = -\frac{s}{n}z(N, N)$ *on* B.

Proof (i) By definition, on the set B', $W = |df|^2 = 0$. On a component of $B\backslash B'$, for the tangent vector field $X \in TB$,

$$X(W) = 2\langle D_X df, df\rangle = 2D_g df(X, df) = 2(1+f)z(X, df) = 0,$$

from the equation 2.3.

(ii) For a normal vector field $N = df/W^{1/2}$ to the hypersurface B in M,

$$II_B = D_g N = W^{-1/2} D_g df = W^{-1/2}\frac{s}{n(n-1)}g,$$

where the last equality comes from 2.3,

(iii) We have

$$\begin{aligned} Ddf(dW, df) &= \frac{s}{n(n-1)}\langle dW, df\rangle = \frac{2s^2}{n^2(n-1)^2}W \\ &= \langle D_{dW}df, df\rangle = \frac{1}{2}\langle dW, dW\rangle, \end{aligned}$$

where $W = |df|^2$. Hence $\Omega = |dW|^2$ is constant on $f = -1$, or B, and equals to $\frac{4s^2}{n^2(n-1)^2}W$. In fact,

$$dW = \frac{2s}{n(n-1)}df$$

at B.

(iv) From the Bochner formula and the equation 2.2,

$$\begin{aligned} \frac{s}{n-1}|df|^2 &= -g(d\Delta_g f, df) = |Ddf|^2 - \frac{1}{2}\Delta_g|df|^2 + r(df, df) \\ &= |Ddf|^2 - \frac{1}{2}\Delta_g W + \frac{g(df, dW)}{2(1+f)} + \frac{n-1+nf}{n(n-1)(1+f)}sW, \end{aligned}$$

where $W = g(df, df) = |df|^2$. Thus we have

$$|D_g df|^2 = \frac{1}{2}\Delta_g W - \frac{g(df, dW)}{2(1+f)} + \frac{sW}{n(n-1)(1+f)}. \tag{2.4}$$

Note that we also have

$$\begin{aligned} 2V|z|^2 &= 2V^{-1}\{|Ddf|^2 - \frac{s^2f^2}{n(n-1)^2}\} = V^{-1}\{2|Ddf|^2 + \frac{2sf\Delta f}{n(n-1)}\} \\ &= V^{-1}\Delta W - V^{-2}\langle df, dW\rangle + \frac{2s}{n(n-1)}(\frac{W}{V^2} + \frac{f\Delta f}{V}) \\ &= div(\frac{dW}{V}) + \frac{2s}{n(n-1)}div(\frac{fdf}{V}), \end{aligned}$$

since

$$\begin{aligned} div(\frac{fdf}{V}) &= \frac{W}{V} + \frac{f\Delta f}{V} - \frac{fW}{V^2} \\ &= \frac{f\Delta f}{V} + \frac{VW}{V^2} - \frac{fW}{V^2} = \frac{f\Delta f}{V} + \frac{W}{V^2}, \end{aligned}$$

where $V = 1 + f$. Hence we have the following equality;

$$V|z|^2 = \frac{1}{2} div(\frac{dW}{V} + \frac{2s}{n(n-1)}\frac{fdf}{V}). \tag{2.5}$$

From the equation 2.5 we have

$$\begin{aligned} 2|z|^2 &= V^{-1} div(\frac{dW}{V}) + \frac{2s}{n(n-1)} V^{-1} div(\frac{fdf}{V}) \\ &= div(\frac{dW}{V^2} + \frac{2s}{n(n-1)}\frac{fdf}{V^2}) + \frac{1}{V^3}(\langle dW, dV\rangle + \frac{2s}{n(n-1)}\frac{fW}{V^3}). \end{aligned}$$

Take an ϵ so that ϵ is not a critical value of V. Then

$$\begin{aligned} \int_{M_{-\epsilon}} 2|z|^2 &= \int_{M_{-\epsilon}} div(\frac{dW}{V^2} + \frac{2s}{n(n-1)}\frac{fdf}{V^2}) + \int_{M_{-\epsilon}} \frac{1}{V^3}(\langle dW, dV\rangle + \frac{2s}{n(n-1)} W) \\ &= \int_{B_{-\epsilon}} V^{-2}\{\langle dW, N\rangle + \frac{2s}{n(n-1)} f W^{1/2}\} + \int_{M_{-\epsilon}} \frac{2}{V^2} z(\nabla f, \nabla f) \\ &= 2\int_{B_{-\epsilon}} V^{-1} z(\nabla f, N) + 2\int_{M_{-\epsilon}} V^{-2} z(\nabla f, \nabla f). \end{aligned}$$

Using the co-area formula, we take the derivative with respect to ϵ at $\epsilon = 0$ of the following equality:

$$\int_{M_{-\epsilon}} |z|^2 = \int_{B_{-\epsilon}} V^{-1} z(df, N) + \int_{M_{-\epsilon}} V^{-2} z(\nabla f, \nabla f).$$

Then we have

$$\begin{aligned} \int_B |z|^2 &= \int_B -V^{-2} z(df, df) + V^{-1}(N(z(df, N)) + z(df, N)m) + V^{-2} z(df, df) \\ &= \int_B V^{-1}(N(z(df, N)) + z(df, N)m). \end{aligned}$$

In other words, on B, with (ii) above,

$$N(z(df, N)) = -z(df, N)m = -z(df, N)(\frac{s}{nW^{1/2}}) = -\frac{s}{n} z(N, N). \tag{2.6}$$

□

Remark. (1) The equation 2.4 implies that the function W cannot have its maximum on the set $\{x \in M; f(x) < -1\}$. Hence if the function f is nontrivial, the maximum of W should be taken on the set $\{x \in M; f(x) \geq -1\}$.

(2) Let us consider the pointwise conformal class of an Einstein metric g_0, denoted by $Conf_0(g)$, and take the intersection of this class with the constant scalar curvatures space $\mathcal{C}$, denoted by $\mathcal{C}_{g_0}$. It is easy to prove that every metric in the class $\mathcal{C}_{g_0}$ is Einstein. The proof follows from the following arguments, c.f. [LP87]: we can take any metric g in $\mathcal{C}_{g_0}$. By definition, there is a positive function ϕ such that $g = \phi^2 g_0$, or $g_0 = \phi^{-2} g$. Then we have

$$0 = z_{g_0} = z_g + \frac{(n-2)}{\phi}(D_g d\phi - \frac{1}{n}\Delta_g \phi g). \tag{2.7}$$

Let δ denote the divergence. Then

$$\begin{aligned}
\int_M \phi |z|^2 &= \int_M \phi z_{ij} z^{ij} \\
&= -(n-2)\int_M (Dd\phi - \frac{1}{n}\Delta\phi g)_{ij} z^{ij} = -(n-2)\int_M (Dd\phi)_{ij} z^{ij} \\
&= -(n-2)\int_M d\phi_i (\delta z)^{ik}_k = 0,
\end{aligned}$$

where the last equality comes from the equation in the proof of Proposition 2.

(3) From the equation 2.5, we obtain the following equality;

$$\int_{M_0} V|z|^2 = \int_{M_0} div\left(\frac{dW}{2V} + \frac{s f df}{n(n-1)V}\right) = \int_B z(df, N). \tag{2.8}$$

Thus if $z = 0$ at B, $|z|$ vanishes in M_0. It implies that the metric g is Einstein by Proposition 2. In other words, $z = 0$ at B suffices to prove that the metric g is Einstein.

□

2.3 The 3 dimensional Case

In this section we consider the case when the dimension is 3. First we prove Theorem 6. We shall need the following results. Let M_0' be a component of M_0. and $B = \cup_\alpha B_\alpha$ be the boundary of M_0', c.f. case II in Figure 2.1. Then we have

Theorem 10 *At least one of B_α is homeomorphic to S^2.*

Proof Let $\{e_1, e_2, e_3 = N\}$ be an orthonormal basis nearby B, with the normal vector field N on B. Also let K_{e_1,e_2} be the sectional curvature of the subspace generated by X and Y, and K_B the intrinsic Gauss curvature of B. Then we have

$$\begin{aligned} K_{e_1,e_2} &= \frac{s}{2} - r(N,N), \\ K_B &= K_{e_1,e_2} + II_B^2. \end{aligned}$$

Since $II_B = \frac{s}{6W_B^{1/2}} g$ as in (ii) of Theorem 9,

$$\begin{aligned} K_B &= \frac{s}{2} - r(N,N) + \frac{s^2}{18W_B} = \frac{s}{2} - (z(N,N) + \frac{s}{3}) + \frac{s^2}{18W_B} \\ &= \frac{s}{6} + \frac{s^2}{18W_B} - z(N,N), \end{aligned}$$

i.e., for each α,

$$W_{B_\alpha}^{1/2} K_B = \frac{s}{6} W_{B_\alpha}^{1/2} (1 + \frac{s}{3W_{B_\alpha}}) - z(N, \nabla f).$$

By taking integration, $\sum_\alpha \int_{B_\alpha}$,

$$2\pi \sum_\alpha W_{B_\alpha}^{1/2} \chi(B_\alpha) = \sum_\alpha W_{B_\alpha}^{1/2} \int_{B_\alpha} K_{B_\alpha}$$

$$
\begin{aligned}
&= \sum_{\alpha} \frac{s}{6} W_{B_\alpha}^{1/2}(1 + \frac{s}{3W_{B_\alpha}}) Area B_\alpha - \int_B z(N, \nabla f) \\
&= \sum_{\alpha} \frac{s}{6} W_{B_\alpha}^{1/2}(1 + \frac{s}{3W_{B_\alpha}}) Area B_\alpha - \int_{M_0'} (1+f)|z|^2 > 0,
\end{aligned}
$$

where the last equality comes from the equation 2.5, i.e.,

$$
\int_{M_0'} (1+f)|z|^2 = \int_{M_0'} div\left(\frac{dW}{2V} + \frac{sf\,df}{6V}\right) = \int_B z(df, N).
$$

And so we can conclude that at least one of the sign of $\chi(B_\alpha)$ is positive.

□

The following lemma is the topological Lemma in [Gal93]. In our setting, the "black holes" are empty.

Lemma 6 *Let M be a 3 dimensional orientable compact Riemannian manifold with boundary ∂M which is mean convex, and has at least one component diffeomorphic to S^2. Then either there is a compact minimal 2-sphere(or projective plane) σ contained in $M \setminus \partial M$ which is locally of least area, or else ∂M is a 2-sphere and M is diffeomorphic to a closed 3-ball.*

Combining Theorem 10 and Lemma 6 it follows that either the region M_0 is diffeomorphic to a closed 3-ball with a convex 2-sphere boundary, or there is a compact minimal 2-sphere(or projective plane) in $M_0 \setminus B$. This proves Theorem 6.

Now we prove Theorem 7. We first need the following technical theorem(Theorem 11).

Definition 1 *Let $\check{\mathcal{W}}$ be the Weyl-Schouten tensor field defined by*

$$\check{\mathcal{W}}_{abc} \equiv R_{abc} = r_{ab;c} - r_{ac;b} + \frac{1}{4}(g_{ac}s_{;b} - g_{ab}s_{;c}),$$

where we denote by ";" the covariant differentiation.

It is well known that the tensor $\check{\mathcal{W}}$ vanishes if and only if (M^3, g) is conformally flat. The equation (i) in the following theorem has the same flavor as a Robinson equality, c.f. [Rob77]. The equation (ii) follows from (i).

Theorem 11 *We have the following equalities:*

(i) *On M, we have*

$$\mathcal{Y} \equiv V^4|\check{\mathcal{W}}|^2 + 3|dW + \frac{sf}{3}df|^2 = 8V^2W|z|^2, \tag{2.9}$$

or

$$\frac{1}{4}V^{-1}W^{-1}\mathcal{Y} = div\left(\frac{dW}{V} + \frac{s}{3}\frac{fdf}{V}\right) = 2V|z|^2.$$

(ii) *At B, we have*

$$|z|^2 = \frac{3}{2}z(N,N)^2, \tag{2.10}$$

where $V = 1 + f$, $W = |df|^2$, and $N = W^{-1/2}df$.

Proof (i) For the time being, we write tensors in a given coordinate system. From the equation 2.2 we have

$$r_{ab} = \frac{f_{;ab}}{1+f} + \frac{2+3f}{6(1+f)}sg_{ab}, \tag{2.11}$$

and from the definition 1

$$
\begin{aligned}
R_{abc} &= r_{ab;c} - r_{ac;b} + \frac{1}{4}(g_{ac}s_{;b} - g_{ab}s_{;c}) \\
&= \frac{f_{;abc}}{1+f} - \frac{f_{;ab}f_{;c}}{(1+f)^2} + \frac{3f_{;c}(1+f) - (2+3f)f_{;c}}{6(1+f)^2} s g_{ab} \\
&\quad - \frac{f_{;acb}}{1+f} + \frac{f_{;ac}f_{;b}}{(1+f)^2} - \frac{3f_{;b}(1+f) - (2+3f)f_{;b}}{6(1+f)^2} s g_{ac} \\
&= \frac{f_{;abc} - f_{;acb}}{1+f} - \frac{(f_{;ab}f_{;c} - f_{;ac}f_{;b})}{(1+f)^2} + \frac{s}{6(1+f)^2}(f_{;c}g_{ab} - f_{;b}g_{ac}).
\end{aligned}
$$

By Ricci identity, we have

$$
f_{;abc} - f_{;acb} = R_{bcla}f^{;l}. \tag{2.12}
$$

In dimension 3, we have

$$
R_{ijkl} = -\frac{s}{2}(g_{ik}g_{jl} - g_{il}g_{jk}) + (r_{ik}g_{jl} + r_{jl}g_{ik} - r_{il}g_{jk} - r_{jk}g_{il}). \tag{2.13}
$$

Thus

$$
\begin{aligned}
R_{bcla}f^{;l} &= -\frac{s}{2}(g_{bl}g_{ca} - g_{ba}g_{cl})f^{;l} + (r_{bl}g_{ca} + r_{ca}g_{bl} - r_{ba}g_{cl} - r_{cl}g_{ba})f^{;l} \\
&= \frac{s}{2}(g_{ba}f_{;c} - g_{ca}f_{;b}) + (r_{bl}g_{ca}f^{;l} + r_{ca}f_{;b} - r_{ba}f_{;b} - r_{cl}g_{ba}f^{;l}) \\
&= \frac{s}{2}(g_{ba}f_{;c} - g_{ca}f_{;b}) + (r_{bl}g_{ca}f^{;l} - r_{cl}g_{ba}f^{;l}) + (r_{ca}f_{;b} - r_{ba}f_{;b}).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
r_{bl}g_{ca}f^{;l} &= \frac{f_{;bl}f^{;l}}{1+f}g_{ca} + \frac{2+3f}{6(1+f)}sf_{;b}g_{ca} \\
&= \frac{W_{;b}}{2(1+f)}g_{ca} + \frac{2+3f}{6(1+f)}sf_{;b}g_{ca},
\end{aligned}
$$

and

$$
r_{ca}f_{;b} = \frac{f_{;ca}f_{;b}}{1+f} + \frac{2+3f}{6(1+f)}sg_{ca}f_{;b}.
$$

Therefore we obtain

$$
\begin{aligned}
R_{abc} &= \frac{s}{2(1+f)}(g_{ba}f_{;c}-g_{ca}f_{;b}) \\
&\quad+\frac{1}{2(1+f)^2}(W_{;b}g_{ca}+\frac{2+3f}{3}sf_{;b}g_{ca}-W_{;c}g_{ba}-\frac{2+3f}{3}sf_{;c}g_{ba}) \\
&\quad+\frac{1}{(1+f)^2}(f_{;ca}f_{;b}+\frac{2+3f}{6}sg_{ca}f_{;b}-f_{;ba}f_{;c}-\frac{2+3f}{6}sg_{ba}f_{;c}) \\
&\quad-\frac{(f_{;ab}f_{;c}-f_{;ac}f_{;b})}{(1+f)^2}+\frac{s}{6(1+f)^2}(f_{;c}g_{ab}-f_{;b}g_{ac}) \\
&= \frac{s}{6(1+f)^2}(g_{ba}f_{;c}-g_{ca}f_{;b})(3(1+f)-2(2+3f)+1) \\
&\quad+\frac{2(f_{;ac}f_{;b}-f_{;ab}f_{;c})}{(1+f)^2}+\frac{1}{2(1+f)^2}(W_{;b}g_{ca}-W_{;c}g_{ba}) \\
&= \frac{sf}{2(1+f)^2}(g_{ca}f_{;b}-g_{ba}f_{;c}) \\
&\quad+\frac{2(f_{;ac}f_{;b}-f_{;ab}f_{;c})}{(1+f)^2}+\frac{1}{2(1+f)^2}(W_{;b}g_{ca}-W_{;c}g_{ba}).
\end{aligned}
$$

Then, since $|\check{\mathcal{W}}|^2 = R_{abc}R^{abc}$, we have

$$
\begin{aligned}
|\check{\mathcal{W}}|^2 &= \frac{s^2f^2}{4(1+f)^4}(4W)+\frac{4(2|Ddf|^2W-\frac{1}{2}|dW|^2)}{(1+f)^4} \\
&\quad+\frac{1}{4(1+f)^4}(4|dW|^2)+\frac{2sf}{(1+f)^4}(2\Delta fW-\langle df,dW\rangle) \\
&\quad+\frac{2}{(1+f)^4}(2\Delta f\langle df,dW\rangle-|dW|^2)+\frac{sf}{2(1+f)^4}(4\langle df,dW\rangle) \\
&= \frac{s^2f^2W}{(1+f)^4}+\frac{(8|Ddf|^2W-2|dW|^2)}{(1+f)^4}+\frac{|dW|^2}{(1+f)^4} \\
&\quad+\frac{-2sf}{(1+f)^4}(sfW+\langle df,dW\rangle)-\frac{2}{(1+f)^4}(sf\langle df,dW\rangle+|dW|^2) \\
&\quad+\frac{2sf\langle df,dW\rangle}{(1+f)^4} \\
&= -\frac{s^2f^2W}{(1+f)^4}-\frac{2sf\langle df,dW\rangle}{(1+f)^4}-\frac{3|dW|^2}{(1+f)^4}+\frac{8|Ddf|^2W}{(1+f)^4}.
\end{aligned}
$$

From the equation 2.3,

$$V^2|z|^2 = |Ddf|^2-\frac{s^2f^2}{12}, \tag{2.14}$$

where $V = 1 + f$, we have

$$\begin{aligned} V^4|\check{\mathcal{W}}|^2 &= -s^2f^2W - 2sf\langle df, dW\rangle - 3|dW|^2 + 8|Ddf|^2W \\ &= -s^2f^2W - 2sf\langle df, dW\rangle - 3|dW|^2 + 8(V^2|z|^2 + \frac{s^2f^2}{12})W \\ &= -\frac{1}{3}s^2f^2W - 2sf\langle df, dW\rangle - 3|dW|^2 + 8WV^2|z|^2. \end{aligned}$$

Also from Bochner identity 2.4,

$$\begin{aligned} V^4|\check{\mathcal{W}}|^2 &= -s^2f^2W - 2sf\langle df, dW\rangle - 3|dW|^2 \\ &\quad +4W\Delta W - 4W\frac{\langle df, dW\rangle}{V} + \frac{4sW^2}{3V}. \end{aligned}$$

Note that, if g is Einstein, then

$$\mathcal{L} \equiv dW + \frac{sf}{3}df \tag{2.15}$$

will vanish. Since we have

$$3|\mathcal{L}|^2 = 3|dW|^2 + 2sf\langle df, dW\rangle + \frac{s^2f^2}{3}W, \tag{2.16}$$

we have

$$\begin{aligned} \mathcal{Y} &\equiv V^4|\check{\mathcal{W}}|^2 + 3|\mathcal{L}|^2 = 8V^2W|z|^2 \\ &= 4W\Delta W - 4WV^{-1}\langle df, dW\rangle + \frac{4}{3}sW^2V^{-1} - \frac{2}{3}s^2f^2W \\ &= 4W\Delta W - 4WV^{-1}\langle df, dW\rangle + \frac{4}{3}sW^2V^{-1} + \frac{4}{3}sfW\Delta f. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{4}V^{-1}W^{-1}\mathcal{Y} &= V^{-1}\Delta W - V^{-2}\langle df, dW\rangle + \frac{s}{3}(\frac{W}{V^2} + \frac{f\Delta f}{V}) \\ &= div(\frac{dW}{V}) + div(\frac{s}{3}\frac{fdf}{V}) = 2V|z|^2, \end{aligned}$$

where the last equation is 2.5. □

(ii) From (i) we have

$$\begin{aligned} |z|^2 &= \frac{V^4|\check{W}|^2 + 3|dW + \frac{sf}{3}df|^2}{8V^2W} \\ &= \frac{V^2}{8W}|\check{W}|^2 + \frac{3}{8WV^2}|dW + \frac{sf}{3}df|^2. \end{aligned}$$

We observe that nearby B, in fact, where df is not zero,

$$dW = 2D_{df}df. \tag{2.17}$$

Thus we obtain that

$$Vz(df,\cdot) = D_{df}df + \frac{sf}{6}g(df,\cdot) = \frac{1}{2}dW + \frac{sf}{6}df. \tag{2.18}$$

Therefore we have

$$|\mathcal{L}|^2 = |dW + \frac{sf}{3}df|^2 = 4V^2\sum_i z(df,e_i)^2, \tag{2.19}$$

for an orthonormal basis $\{e_i\}_{i=1,2,3}$ with $e_3 = N$. Thus we obtain the following equality near B:

$$|z|^2 = \frac{V^2}{8W}|\check{W}|^2 + \frac{3}{2W}\sum_i z(df,e_i)^2. \tag{2.20}$$

Now at B, or $V = 0$, we have

$$|z|^2 = \frac{3}{2}\sum_i z(N,e_i)^2. \tag{2.21}$$

For a tangent vector field X to $h^{-1}(c)$ nearby B, we have

$$z(df,X) = \frac{\langle D_{df}df, X\rangle}{V} = \frac{\langle dW, X\rangle}{2V},$$

and at B we have

$$z(df,X)=\frac{\langle D_N dW,X\rangle+\langle dW,D_N X\rangle}{2W^{1/2}}. \qquad (2.22)$$

On B we have

$$\langle D_N dW,X\rangle=\langle D_X dW,N\rangle=\langle D_X dW,\frac{dW}{\Omega^{1/2}}\rangle=\frac{1}{2\Omega^{1/2}}X|dW|^2=0,$$

since $\Omega=|dW|^2$ is constant on B by (iii) in Theorem 9. Also we note that on B

$$dW=\frac{s}{3}df,$$

and so

$$\langle dW,D_N X\rangle=\frac{s}{3}\langle df,D_X N\rangle=\frac{s}{3W^{1/2}}\langle df,D_X df\rangle=\frac{s}{6W^{1/2}}X(W)=0. \qquad (2.23)$$

It follows that, on B,

$$z(df,X)=0.$$

In other words, $z(N,e_i)=0$ for $i=1,2$. From the equation 2.21, we can conclude that

$$|z|^2=\frac{3}{2}z(N,N)^2.$$

□

Now we are in a position to prove the theorem.

Proof of Theorem 7 Let $\{X=e_1,Y=e_2,N=e_3\}$ be a coordinate frame on each B_c, being N a normal vector field of B_c. From

$$\begin{aligned} II(X,X) &= \langle D_X N,X\rangle=W^{-1/2}\langle D_X df,X\rangle \\ &= W^{-1/2}[Vz(X,X)-\frac{sf}{6}], \\ II(X,Y) &= W^{-1/2}\langle D_X df,Y\rangle=W^{-1/2}Vz(X,Y), \end{aligned}$$

and $K_{XY} = \frac{s}{2} - r(N,N)$, we have

$$\begin{aligned} K_{B_c} &= K_{XY} + II(X,X)II(Y,Y) - II(X,Y)^2 \\ K_{B_c} &= \frac{s}{2} - r(N,N) + II(X,X)II(Y,Y) - II(X,Y)^2 \\ &= \frac{s}{6} - z(N,N) + \frac{1}{W}(Vz(X,X) - \frac{sf}{6})(Vz(Y,Y) - \frac{sf}{6}) - \frac{V^2}{W}z(X,Y)^2 \\ &= \frac{s}{6}(1 + \frac{sf^2}{6W}) + z(N,N)(\frac{sfV}{6W} - 1) + \frac{V^2}{W}(z(X,X)z(Y,Y) - z(X,Y)^2). \end{aligned}$$

Thus if $W = W(f)$, then K_{B_c} is constant on each B_c if and only if $z(N,N)$ and $z(X,X)z(Y,Y) - z(X,Y)^2$ are constant on B_c. Note that

$$z(N,N) = \frac{1}{2V}(\langle dW, df\rangle + \frac{sf}{3})$$

is a function of f only. Also, from the equation below 2.16, the function $|z|^2$ is a function of f only. Thus $G_c \equiv |z|^2 - \frac{3}{s}z(N,N)^2$ is also a function of f only. Since

$$G_c = \frac{1}{2}(z(X,X) - z(Y,Y))^2 + 2z(X,Y)^2,$$

we have

$$\begin{aligned} z(X,Y)^2 &= \frac{1}{2}G_c - \frac{1}{4}(z(X,X) - z(Y,Y))^2 \\ z(X,X)z(Y,Y) - z(X,Y)^2 &= z(X,X)z(Y,Y) - \frac{1}{2}G_c + \frac{1}{4}(z(X,X) - z(Y,Y))^2 \\ &= \frac{1}{4}z(X,X)^2 + \frac{1}{4}z(Y,Y)^2 + \frac{1}{2}z(X,X)z(Y,Y) - \frac{1}{2}G_c \\ &= \frac{1}{4}(z(X,X) + z(Y,Y))^2 - \frac{1}{2}G_c \\ &= \frac{1}{4}z(N,N)^2 - \frac{1}{2}G_c, \end{aligned}$$

implying that $z(X,X)z(Y,Y) - z(X,Y)^2$ is a function of f only. This implies that B_c is of constant curvature.

Note that there is no critical point of f in M_0 due to (1) of the second Remark in section 2.2. That each hypersurface is of constant curvature means that the manifold M^3 is foliated by constant curvature surfaces. Especially M_0 is foliated by constant curvature spheres since there is no critical point of f in M_0 and ∂M_0 is isometric to S^2 up to some constant factor. Thus in M_0 we can write a metric by

$$g = c^2 dt^2 + H^2(t)(d\theta^2 + sin^2\theta d\phi^2), \tag{2.24}$$

with $t < 0$, and constant c. It comes from the metric form

$$g = \frac{1}{W} df^2 + \tilde{H}^2(f)(d\theta^2 + sin^2 d\phi^2),$$

by taking

$$cdt = \sqrt{\frac{1}{W}} df. \tag{2.25}$$

Note that $W \neq 0$ in M_0. Since this metric 2.24 is conformally flat, by applying the result of Lafontaine, we can conclude that the metric is standard. But we will prove this by straight computation.

Claim. The metric 2.24 is standard on M_0, i.e., $H(t) = c\ \ cos(t)$.

Proof Given the metric form of 2.24, we have

$$\begin{aligned}
K(\partial_t, \partial_\theta) &= -\frac{H''}{c^2 H} \equiv \beta, \\
K(\partial_t, \partial_\phi) &= \beta, \\
K(\partial_\theta, \partial_\phi) &= \frac{1}{H^2}(1 - \frac{H'^2}{c^2}) \equiv \alpha, \\
r(\partial_t, \partial_t) &= 2\beta c^2, \\
r(\partial_\theta, \partial_\theta) &= (\alpha + \beta) H^2,
\end{aligned}$$

$$
\begin{aligned}
r(\partial_\phi, \partial_\phi) &= (\alpha+\beta)H^2 sin^2\theta, \\
r(\partial_t, \partial_\theta) &= r(\partial_t, \partial_\phi) = r(\partial_\theta, \partial_\phi) = 0, \\
s_g &= 2(\alpha+2\beta) = \frac{2}{H^2}(1 - \frac{H'^2}{c^2} - \frac{2H''H}{c^2}).
\end{aligned}
$$

Without loss of generality, we can take c be $(\frac{6}{s})^{1/2}$. Then we have

$$\frac{c^2}{H^2} = 3 + \frac{(H')^2}{H^2} + \frac{2H''}{H}.$$

Thus

$$c^2 = 3H^2 + (H')^2 + 2H''H. \tag{2.26}$$

Using 2.25, it is obvious that f is depends only on t, since $|df|^2 = W \neq 0$. From 2.3, we have

$$
\begin{aligned}
Vz(\partial_t, \partial_t) &= \langle D_{\partial_t} df, \partial_t \rangle + \frac{sf}{6}c^2, \\
Vz(\partial_\theta, \partial_\theta) &= \langle D_{\partial_\theta} df, \partial_\theta \rangle + \frac{sf}{6}H^2, \\
Vz(\partial_\phi, \partial_\phi) &= \langle D_{\partial_\phi} df, \partial_\phi \rangle + \frac{sf}{6}H^2 sin^2\theta, \\
-\frac{s}{2}f &= \Delta f,
\end{aligned}
$$

which is equivalent to

$$Vc^2(2\beta - \frac{s}{3}) = f_{tt} + \frac{sf}{6}c^2, \tag{2.27}$$

$$VH^2(\alpha + \beta - \frac{s}{3}) = \frac{HH'}{c^2}f_t + \frac{sf}{6}H^2, \tag{2.28}$$

$$VH^2(\alpha + \beta - \frac{s}{3})sin^2\theta = \frac{HH'}{c^2}f_t sin^2\theta + \frac{sf}{6}H^2 sin^2\theta. \tag{2.29}$$

$$\frac{1}{c^2}f_{tt} + f_t(\frac{2H'}{c^2H}) + \frac{sf}{2} = 0. \tag{2.30}$$

From the definition of α and β and 2.27, we have

$$\begin{aligned} Vc^2(2(-\frac{H''}{c^2H}) - \frac{s}{3}) &= f_{tt} + f, \\ V(-\frac{2H''}{H} - 2) &= f_{tt} + f, \end{aligned}$$

$$f_{tt} + f = -2V(\frac{H''}{H} + 1). \qquad (2.31)$$

In the metric 2.24, let $t \in [t_0, t_2]$ with $V(t_2) = 0$. We observe from 2.24 that $H(t)$ is not zero in M_0 except the point where $t = t_0$. It can be proved that $H'(t)$ is not zero in M_0. Suppose that $H'(t_1) = 0$ for $V(t_1) \leq 0$. From 2.30,

$$f_{tt} + f_t\frac{2H'}{H} + 3f = 0,$$

or

$$2f_tH' + (f_{tt} + 3f)H = 0, \qquad (2.32)$$

Since $H'(t_1) = 0$, we obtain an equality

$$f_{tt}(t_1) + 3f(t_1) = 0. \qquad (2.33)$$

Let $\epsilon = -V(t_1)$. Then by definition of t_1, $\epsilon > 0$ (Note that on B_0, $H'(t) \neq 0$ and $f_t(t) \neq 0$. In fact, $f_t(t)H'(t) = H(t)$ on B_0). By 2.31 and 2.33, at t_1,

$$\begin{aligned} f_{tt} + f &= -2f = 2(1+\epsilon) \\ &= -2V\left(\frac{H''}{H} + 1\right) = 2\epsilon\left(\frac{H''}{H} + 1\right) \\ \frac{1+\epsilon}{\epsilon} &= \frac{H''}{H} + 1, \end{aligned}$$

which gives the equality $H'' = \frac{H}{\epsilon} > 0$. In other words, if there is a critical point of H in M_0, then it is always a local minimum. But this is impossible, since $H(t_0) = 0$ for $t_0 < t_1$. Hence $H'(t) \neq 0$ in M_0.

By writing $P = H'$, the equation 2.26

$$c^2 = 3H^2 + (H')^2 + 2H''H$$

reduces to a first-order differential system of the form

$$\frac{dH}{dt} = P, \ \frac{dP}{dt} = Q(H,P) = -(\frac{P}{2H})P - G(H), \tag{2.34}$$

where $G(H) = \frac{1}{2}(3H - \frac{c^2}{H})$. Then, by Cauchy's existence theorem, if the function $F = F(H,P)$ satisfying

$$\frac{dP}{dH} = F(H,P) = \frac{Q(H,P)}{P} = -(\frac{P}{2H}) - \frac{G(H)}{P}, \tag{2.35}$$

is Lipschitz continuous in P in a neighborhood U of a point (H_0, P_0), then there is a unique and continuous integral $P = P(H)$ with $P(H_0) = P_0$. Note that the function F is Lipschitz if the domain U is away from $H(t) = 0$ and $P = H' = 0$. But from above, $P \neq 0$ on M_0 and $H(t) = 0$ only at $t = t_0$. In other words, the nonlinear ordinary differential equation 2.26 has a unique solution on $(t_0, t_2]$. Since H is a smooth function on M_0, we can extend it to $[t_0, t_2]$, which is M_0. Since $H_1(t) = c.cos(t)$ is a solution to 2.26 with $H_1(t_0) = 0$ and $H_1'(t_0) = c$, by the uniqueness of the solutions to 2.26 by the preceding arguments, this $H_1(t)$ is the unique solution to 2.26, and so the metric g is standard.

□

This claim implies that on region M_0, the metric is Einstein. Then Theorem 7 follows from Proposition 2.

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