Generalized Cauchy-Riemann Operators in Symplectic Geometry

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Abstract of the Dissertation

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We study applications of generalized Cauchy-Riemann operators (on Riemann surfaces) to symplectic geometry in dimension four. Such operators (for the definition see Chapter 2) appear naturally in the theory of J-holomorphic curves, and that is where we look for examples. The following questions/problems are being considered:

(1) A topological condition for a generalized Cauchy-Riemann operator to be surjective. Surjectivity of certain operators implies that spaces of appropriate J-holomorphic curves are manifolds. The almost-complex structure J is then called regular.

In Chapter 4 we prove a simple but useful criterion of regularity of almost-complex structures on 4-dimensional symplectic manifolds (Proposition 4.3.2). The criterion is due to Gromov, and it is an extension of a corollary to the Riemann-Roch theorem. Having applications in mind we formulate the criterion for singular as well as immersed pseudo-holomorphic curves.

(2) Next, we look at the simplest non-generic almost-complex structures on a four-dimensional manifold. We study them through the associated moduli spaces of J-holomorphic tori of virtual dimension 0. We show that the actual dimension of the moduli spaces $\mathcal{M}(J, A, 1)$ can be $0, 1$, or 2 but no higher, and stratify the space of almost-complex structures according to that dimension (Proposition 5.0.5). We show that all strata are non-empty, give examples of generic almost complex structures with arbitrarily large number of J-holomorphic tori, as well as examples of generic almost-complex structures which are non-homotopic (through the space of generic structures).

(3) In the last chapter we study the evaluation mapping on a moduli space of J-holomorphic curves. We give an elementary construction of a generic almost-complex structure J, on a four-dimensional manifold, for which an appropriate evaluation mapping is not orientation preserving. This happens for all values of the first Chern class for which the moduli space is generically of positive dimension. If, on the other hand, the complex structure is integrable then all evaluation maps are holomorphic, hence preserve orientation. We study pseudo-holomorphic curves of genus $g \geq 1$ in a symplectic 4-manifold and show, by examples, that their behavior is typically different from that of pseudo-holomorphic spheres.

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Chapter 1

Introduction.

In this paper we give examples of applications of generalized Cauchy-Riemann operators in symplectic geometry, specifically in the theory of J-holomorphic curves.

In Chapter 2 we define generalized Cauchy-Riemann operators, and give several versions of the Similarity Principle, which is one of our main tools. The simplest example is Proposition 2.0.9:

Proposition 1.0.1 *Let $a(z)$, and $b(z)$ be complex valued functions of class L^p. We consider the operator $\overline{\partial}\xi(z)+a(z)\xi(z)+b(z)\overline{\xi(z)}$ defined on complex-valued functions $\xi(z)$ on the unit disc $D \subset \mathbb{C}$.*

(A) *Let $\xi(z)$ be a pseudo-analytic function i.e satisfying:*

$$\overline{\partial}\xi(z) + a(z)\xi(z) + b(z)\overline{\xi(z)} = 0$$

Then there exists an analytic function $\eta(z)$, and a continuous $\mathbb{C}$-valued function $s(z)$,both defined on D, such that

$$\xi(z) = e^{s(z)}\eta(z).$$

Moreover, we can assume that $s(z_0) = 0$ *at any prescribed point.*

(B) *Let* η *be a holomorphic function on the unit disc* $D \subset \mathbb{C}$. *Then there is a continuous* $\mathbb{C}$*-valued function* $t(z)$, *and a pseudo-analytic function* ξ *satisfying:*

$$\overline{\partial}\xi(z) + a(z)\xi(z) + b(z)\overline{\xi(z)} = 0$$

such that $\eta(z) = e^{t(z)}\xi(z)$. *Moreover we can assume that* $t(z_0) = 0$ *at any prescribed point.*

(see "Methods of Mathematical Physics," (vol. 2) by Courant, Hilbert). Next, we consider operators acting on sections of holomorphic vector bundles (for our purposes mostly line bundles) over Riemann surfaces. Here an operator is called a generalized Cauchy-Riemann operator if it is equal to the classical Cauchy-Riemann operator (the Dolbeaut operator) plus a zero-order term. The point being that the zero-order perturbation is not necessarily complex-linear. This has global consequences described in Chapter 3 that at least locally sections in the kernel of a generalized Cauchy-Riemann operator (i.e pseudo-holomorphic sections) behave much like holomorphic sections, e.g every zero contributes positively to the Euler characteristic of the underlying bundle (see Corollary 2.0.10). An application to pseudo-holomorphic curves is given in Corollary 2.0.18.

Chapter 3 deals with global properties of Cauchy-Riemann operators, and the spaces of such operators. We stratify the space of operators on a given complex line bundle according to the dimension of the kernel of an operator. The main stratum consists, of course, of operators whose kernel has maximal dimension allowed by the Riemann-Roch theorem. Our interests lie however in lower dimensional strata, especially in the space of operators one dimensional kernel. Since our operators are not necessarily complex-linear that stratum is, in general non-empty. As we show in Chapter 3 , this holds if the base Riemann surface is not a sphere. Here a sample result is Proposition 3.2.3:

Proposition 1.0.2 *Let* $(a(\lambda_1, \lambda_2, ..., \lambda_k), b(\lambda_1, \lambda_2, ..., \lambda_k)$ *be a generic k-dimensional deformation of an operator* $\overline{\partial}_{a(0)b(0)}$ *with two dimensional kernel. Parameters* $(\lambda_1, \cdots, \lambda_k)$ *corresponding to operators with one dimensional kernel form a codimension one (non-linear) cone with vertex at the origin.*

We give "hands on" examples of operators with one dimensional kernel, the first being the example of a generalized Cauchy-Riemann operator with constant coefficients on the trivial bundle over a torus see Example 3.1.1.

In Chapter 4 we prove a simple but useful criterion of regularity of almost-complex structures on 4-dimensional symplectic manifolds i.e Proposition 4.3.2:

Proposition 1.0.3 *Let* (M, J) *be a 4-dim manifold with an almost complex structure* J*, and let* A *be a 2-dim homology class in* $H^2(M, Z)$*.*

(a) *If* $c_1(A) \geq 1$ *then* J *is regular for all immersed* J*-holomorphic curves in class* A*.*

(b) J *is regular for all singular curves* u *such that* $c_1(A) - \sum(k_i - 1) \geq 1$*, where* $\{k_i\}$ *is the set of multiplicities of all singular points of* u*.*

Here $c_1 = c_1(TM, J)$ *is the first Chern class of* (M, J)*.*

The criterion is due to Gromov, and it is an extension of a corollary to the Riemann-Roch theorem. Regularity of an almost-complex structure J implies that spaces of J-holomorphic curves are in fact smooth manifolds. Having that application in mind we formulate the criterion for singular as well as immersed pseudo-holomorphic curves. The criterion was also proved by Hofer, Lizan, Sikorav [HLJ94] (in the non-singular case), and by Ivashkovich, Sevchishin [IS95].

In Chapter 5 we study the space of ω-tame non-regular almost-complex structures J on a compact symplectic 4-manifold (M, ω) by looking at the associated moduli spaces of J-holomorphic curves. If the almost-complex structure J is sufficiently generic then the space of all J-holomorphic curves of genus g in a homology class $A \in H_2(M, \mathbb{Z})$ is a smooth manifold $\mathcal{M}(J, A, g)$ of dimension equal to the virtual dimension. If $c_1(A) \geq 1$ (where c_1 is the first Chern class of (TM, J)), any j is generic (regular) in this sense, provided that we restrict to immersed curves ([Gro85], [HLJ94]) or curves with controlled singularities (see Chapter 4).

Here we consider the simplest non-generic case of J-holomorphic tori in a class A with $A \cdot A = 0$. Now, in contrast to the generic case, the moduli spaces $\mathcal{M}(J, A, 1)$ depend a great deal on J. As explained in Chapter 5 it is then reasonable to consider only the case of embedded tori, where the virtual dimension of $\mathcal{M}(J, A, 1)$ is zero. However, the actual dimension of the moduli spaces $\mathcal{M}(J, A, 1)$ can be $0, 1$, or 2 but no higher. We then stratify the space of almost-complex structures. We denote by $\mathcal{J}_{i,j}$ the space of all J's such that for all $u \in \mathcal{M}(J, a, 1)$, $\dim \ker(D_u) \leq i$ (with equality for some u), and so that

$\dim \mathcal{M}(J, A, 1) = j$. (Here D_u is the Cauchy-Riemann operator cutting out the moduli space of pseudo-holomorphic curves, see Chapter 2) . Our main results are in Section 4, where we show that all but two strata have codimension two or higher:

Proposition 1.0.4 *The set $\mathcal{J}_{0,0}$ is open, and dense but with infinitely many connected components. The sets $\mathcal{J}_{2,0}$, $\mathcal{J}_{2,1}$, and $\mathcal{J}_{1,1}$ all have codimension 2 (or higher), while $\mathcal{J}_{1,0}$ has codimension one. Therefore two generic (in $\mathcal{J}_{0,0}$) almost complex structures can be connected by a path $\{J_t\} \subset \mathcal{J}_{0,0} \cup \mathcal{J}_{1,0}$.*

Here a set $S \subset \mathcal{J}$ has codimension 2 etc. if a generic one parameter family $\{J_t\}$ avoids S.

We show also that all strata are non-empty, give examples of generic almost complex structures with arbitrarily large number of J-holomorphic tori, as well as examples of generic almost-complex structures which are non-homotopic (through the space of generic structures).

In Chapter 6 almost-complex structure J on a four-dimensional manifold with the property that an evaluation mapping defined on an moduli space of J-holomorphic curves is not orientation preserving. This happens for all values of the first Chern class for which the moduli space is generically of positive dimension. Such behavior is in contrast to the case of integrable complex structures where evaluation maps are holomorphic, hence preserve orientation. We study pseudo-holomorphic curves of genus $g \geq 1$ in a symplectic 4-manifold and show, by examples, that their behavior is typically different from that of pseudo-holomorphic spheres.

While writing this paper I learned that C. Taubes [Tau95] proved the equivalence of the newly introduced Seiberg-Witten invariants and Gromov invariants which are defined on symplectic manifolds by counting appropriately pseudo-holomorphic curves (see [MS94]). One of the corollaries is the existence of pseudo-holomorphic curves in symplectic 4-manifolds. In particular, there are symplectic spheres in $(\mathbb{C}P^2, \omega)$ for an arbitrary symplectic form ω, which implies the uniqueness of symplectic structures on $\mathbb{C}P^2$. Such an existence theorem provides further motivation for studying pseudo-holomorphic curves. So far, spheres have been the most useful and the most utilized of all pseudo-holomorphic curves. In this chapter we explain why this is so, and point out the main difference between pseudo-holomorphic spheres and pseudo-holomorphic curves of higher genus.

We give two constructions (sections 6 and 7) of regular almost-complex structures J for which the evaluation mapping ev is not orientation preserving. To achieve regularity of J we work with homology classes B such that $c_1(B) \geq 1$ (see Chapter 2). However, it is not difficult to construct examples where ev does not preserve orientation for small values of $c_1(B)$.

In section 6, by cutting and pasting, we prove Theorem 6.4.1:

Theorem 1.0.5 *Let (M, ω) be a 4-dimensional, compact, symplectic manifold and $B \in H_2(M, \mathbb{Z})$ a homology class such that $c_1(B) \geq 1$. Assume that there is a positively symplectically immersed surface Σ of genus $g \geq 1$ in the homology class B. Then there exist ω-tame almost-complex structures J such that the evaluation mapping* ev *defined on $\mathcal{M}(J, B, g)$ does not preserve orientation.*

Here an almost-complex structure J is ω-tame if the Riemannian metric $\omega(v, Jv)$ $(v \in TM)$ is positive definite. A curve Σ is called positively symplectically immersed if all self-intersection points are two-fold with positive orientation, and the symplectic form ω restricted to Σ never vanishes ([McD92a]). We need such curves to ensure that the space of pseudo-holomorphic curves in class B is nonempty (since every positively symplectically immersed curve is J-holomorphic for an appropriate almost-complex structure J).

In section 7 we show that a typical small perturbation of an integrable complex structure will have the property that the evaluation mapping is not orientation preserving. Thus, we can find almost-complex structures with the required property which are arbitrarily close to integrable, complex structures. This is the content of Theorem 6.5.1:

Theorem 1.0.6 *Let (M^4, J) be an almost-complex (compact, smooth) manifold, $B \in H_2(M, \mathbb{Z})$ a homology class with $c_1(B) \geq 1$, and $u : \Sigma \to M$ an embedded J-holomorphic curve in class B of genus $g \geq 1$ such that J is integrable on a neighborhood of $u(\Sigma)$. Then there are almost-complex structures J' arbitrarily close to J (in the C^1 topology) for which the evaluation mapping* ev *on $\mathcal{M}(J', B, g)$ does not preserve orientation.*

Chapter 2

Generalized Cauchy-Riemann operators. The Similarity Principle.

Throughout this paper we will use the notion of a generalized Cauchy-Riemann operator. This chapter contains some facts about such operators (including their definition) necessary for applications. The main part of this chapter are different versions of the Similarity Principle (Proposition 2.0.2, 2.0.13, 14, 15).

Definition 2.0.7 A generalized Cauchy-Riemann operator on a holomorphic line bundle L over a Riemann surface Σ is a linear (over $\mathbb{R}$) operator of the form

$$\overline{\partial} + A : \Gamma(L) \to \Gamma(\Lambda^{0,1} \otimes L)$$

where $A \in \Gamma(\Lambda^{0,1} \otimes \mathrm{End}_{\mathbb{R}}(L))$ is a $(0,1)$-form on Σ with values in the bundle of real endomorphisms of L, and $\overline{\partial}$ is the Dolbeault operator on L. In general we will assume that A is either smooth, or in L^{∞}, or at least of Sobolev class $W^{0,p}$. A section $\xi \in \Gamma(L)$ solving the equation $\overline{\partial}(\xi) + A(\xi) = 0$ will be called pseudo-holomorphic or pseudo-analytic.

Remark 2.0.8 Such operators arise as linearisations of the equation $du + J \circ du \circ j = 0$ characterizing pseudo-holomorphic curves $u : (\Sigma, j) \to (E, J)$. The term A is typically anti-holomorphic.

Locally, the study of generalized Cauchy-Riemann operators is the same as that of the generalized functions of Bers and Vekua [Vek62]. The first part of the following proposition is known as the Carleman Similarity Principle (see [Hei57], [FHS94]).

Proposition 2.0.9 *Let $a(z)$, and $b(z)$ be complex valued functions of class L^p. We consider the operator $\overline{\partial}\xi(z) + a(z)\xi(z) + b(z)\overline{\xi(z)}$ defined on complex-valued functions $\xi(z)$ on the unit disc $D \subset \mathbb{C}$.*

(A) *Let $\xi(z)$ be a pseudo-analytic function i.e satisfying:*

$$\overline{\partial}\xi(z) + a(z)\xi(z) + b(z)\overline{\xi(z)} = 0$$

Then there exists an analytic function $\eta(z)$, and a continuous $\mathbb{C}$-valued function $s(z)$,both defined on D, such that

$$\xi(z) = e^{s(z)}\eta(z).$$

Moreover, we can assume that $s(z_0) = 0$ at any prescribed point.

(B) *Let η be a holomorphic function on the unit disc $D \subset \mathbb{C}$. Then there is a continuous $\mathbb{C}$-valued function $t(z)$, and a pseudo-analytic function ξ satisfying:*

$$\overline{\partial}\xi(z) + a(z)\xi(z) + b(z)\overline{\xi(z)} = 0$$

such that $\eta(z) = e^{t(z)}\xi(z)$. *Moreover we can assume that* $t(z_0) = 0$ *at any prescribed point.*

Proof: See, e.g, "Methods of Mathematical Physics" by Courant, Hilbert (vol. 2). □

As a standard corollary we quote:

Corollary 2.0.10 *([CH63])Suppose that the function* $f(z)$ *satisfies the above equation only in the punctured disc* $0 < |z| < 1$. *Then one of three possibilities holds:*

(a) $f(z)$ *has an essential singularity at* $z = 0$, *i.e images by* $f(z)$ *of arbitrarily small neighborhoods of the origin are dense in* $\mathbb{C}$,

(b) $f(z)$ *has a pole at the origin, i.e for some positive integer* n, *and a continuous function* $s(z)$ *we have:*

$$f(z) = e^{s(z)}\frac{1}{z^n}$$

(c) $f(z)$ *is regular at the origin. If* $f(0) = 0$, *then for a positive integer* n, *and a continuous function* $s(z)$ *one has:*

$$f(z) = e^{s(z)}z^n$$

If we assume simply that the functions $a(z)$, *and* $b(z)$ *are* C^∞, *and if* $\xi(z)$ *is a pseudo-analytic function in the whole disc satisfying:*

$$\begin{aligned} \overline{\partial}\xi(z) + a(z)\xi(z) + b(z)\overline{\xi(z)} &= 0 \\ \xi(z_0) &= 0 \end{aligned}$$

then

$$\xi(z) = c(z - z_0) + o(z - z_0), \quad c \in \mathbb{C}$$

i.e the first jet of $\xi(z)$ is holomorphic.

This proposition as well as the similarity principle are true under much weaker assumptions see below or [Hei57], [Ber54]. We will also use a global version of the similarity principle true on the sphere S^2 (see [FHS94]).

Proposition 2.0.11 *Let $L \to S^2$ be a holomorphic line bundle of degree $d \geq 1$, and $\overline{\partial} + A$ a generalized Cauchy-Riemann operator on L, where A is of Sobolev class $W^{1,p}$ with $p > 2$. Then:*

(a) $\dim_{\mathbb{R}} \ker(\overline{\partial} + A) = 2(d+1)$.

(b) *For any holomorphic section η and a point $q \in \Sigma$ there is a continuous function s, $s(q) = \frac{\pi\sqrt{-1}}{2}$ such that $e^s\eta$ is pseudo-holomorphic. Moreover the imaginary part of $e^{s(z)}$ is strictly positive for all z.*

(c) *For any pseudo-holomorphic section ξ and a point q there is a continuous function t, $t(q) = 0$ such that $e^t\xi$ is holomorphic.*

Proof: To prove (a) compute: $\mathrm{index}_{\mathbb{R}}(\overline{\partial} + A) = 2(d+1)$, and use the fact that $\overline{\partial} + A$ is surjective (just as $\overline{\partial}$ is)(see Chapter 3, or [Gro85], [HLJ94]). The second part is an application of Schauder's fixed-point theorem, and is proved e.g in [CH63]. Part (c) is again the similarity principle. If $e^t\xi$ were to be holomorphic then:

$$0 = \overline{\partial}(e^t\xi) = e^t(\overline{\partial}t)\xi + e^t\overline{\partial}\xi$$

which is equivalent to

$$\overline{\partial}(t) + \frac{\overline{\partial}\xi}{\xi} = 0.$$

Since $\frac{\overline{\partial}\xi}{\xi}$ is a $(0,1)$-form of class L^∞ and $H^{0,1}(S^2, \mathbb{C}) = 0$, this last equation can be solved in the Sobolev space $W^{1,p}$ for any p (with a prescribed value of t at any given point). For details see again [CH63]and Floer et al. [FHS94]. The assertion about positivity of the imaginary part of $e^{s(z)}$ follows from Liouville's theorem, see [Rod87]. □

This proposition, specifically part (b), fails on surfaces other than the sphere. One can think of (b) above as a way of putting a linear complex structure on the kernel of the operator $\overline{\partial} + A$ (specifically, multiplying by a function given in (b) corresponds to multiplying by $\sqrt{-1}$). Nothing like that is possible on surfaces of higher genus. It will shown in the next chapter that there are Cauchy-Riemann operators with one-dimensional (over $\mathbb{R}$) kernel. Here we give the following example.

Example 2.0.12 On a given line bundle $L \to T^2$ of positive degree there is an operator $\overline{\partial} + A$, and a section v of class $W^{1,p}, 2 < p$ such that:

(a) $\overline{\partial}v + A(v) = 0$ i.e v is a pseudo-holomorphic section

(b) If s is a bounded function then sv is not in the kernel of $\overline{\partial} + A$, unless s is constant, and real.

To construct such an operator we first pick a $(0,1)$-form ω such that the equation:

$$\overline{\partial}v + \omega(\overline{v} - v) = 0$$

has only one regular (i.e bounded) solution (one can insist that v = const, be the unique solution.) Take e.g ω to be a non-zero form with constant coefficients, and use Fourier series to find the unique section (see Lemma 3.1.2). Let then v be a solution to the equation:

$$\overline{\partial}v + \omega v = 0,$$

which exists by positivity of degree of L. Next, choose a complex anti-linear endomorphism A ($A \in \Gamma(\Lambda^{(0,1)} \otimes \mathrm{End}_{\mathbb{R}}(L))$) of L which preserves v. For example if h is a hermitian metric on L then let : $A(w) = \frac{h(v,w)}{h(v,v)} v \otimes \omega$, where $w \in L_z$, $z \in \Sigma$. Now v is also a solution to: $\overline{\partial}v + A(v) = 0$. However there is no section of the form sv where s is a bounded, non-constant function which would also solve this equation. Otherwise s would satisfy equation: $\overline{\partial}v + \omega(\overline{v} - v) = 0$, which is impossible.

The following technical proposition will be needed later on.

Proposition 2.0.13 *Let $L \to \Sigma$ be a holomorphic line bundle of degree g such that* $\dim_{\mathbb{R}}(\ker\overline{\partial}) = 2$. *There is an $\epsilon > 0$ such that if for a smooth $A \in \Gamma(\Lambda^{0,1} \otimes \mathrm{End}_{\mathbb{R}}(L))$ we have $||A||_{L^\infty} < \epsilon$ then* $\dim_{\mathbb{R}} \ker(\overline{\partial} + A) = 2$.

Proof: We will prove the proposition by contradiction. First note that $\mathrm{index}_{\mathbb{R}}(\overline{\partial} + A) = 2$ therefore $\dim_{\mathbb{R}} \ker(\overline{\partial} + A) \geq 2$. Suppose that there is a sequence $A_n \in \Gamma(\Lambda^{0,1} \otimes \mathrm{End}_{\mathbb{R}}(L))$ such that $||A||_{L^\infty} \to 0$, but $\dim_{\mathbb{R}} \ker(\overline{\partial} + A) \geq 3$. Let $p > 2$. Choose triples of sections $\tau_n^1, \tau_n^2, \tau_n^3 \in \ker(\overline{\partial} + A_n)$ orthogonal with respect to the L^2 norm, and such that $||\tau_n^j||_{W^{0,p}} = 1$, $j = 1, 2, 3$. We will show that there is a convergent (in L^2 norm) subsequence $\{\tau_{n_k}^1, \tau_{n_k}^2, \tau_{n_k}^3\}$, say

$\lim_{n_k \to \infty} \tau^j_{n_k} = \tau^j$, $j = 1, 2, 3$. It follows from the standard theory of elliptic operators (applied to $\overline{\partial}$) that $\tau^1, \tau^2, \tau^3 \in \ker(\overline{\partial})$, and hence $\dim_R(\ker \overline{\partial}) > 2$, contrary to hypothesis.

To prove the existence of a convergent subsequence $\{\tau^1_{n_k}, \tau^2_{n_k}, \tau^3_{n_k}\}$ we simply notice that

$$||\tau^j_n||_{W^{1,p}} \leq c(||\overline{\partial}(\tau^j_n)||_{W^{0,p}} + ||\tau^j_n||_{W^{0,p}}) \leq c'(||\tau^j_n||_{W^{0,p}}) \leq c'$$

for suitable constants c, $c' > 0$ (see e.g [MS94]). Since the inclusion of the space of $W^{1,p}$-sections of L into the space of $W^{0,p}$-sections is a compact operator we can, indeed, choose a convergent subsequence of $\{\tau^1_n, \tau^2_n, \tau^3_n\}$. That is enough to prove the proposition. □

Remark 2.0.14 The assumption that $\dim_{\mathbb{R}} \ker(\overline{\partial}) = 2$ is satisfied if the holomorphic structure on the line bundle L is generic. This is a special case of the "geometric" version of the Riemann-Roch theorem, see e.g [GH78].

The above proposition is also true if we ask that A be generic (i.e in an open, dense set in C^1 toplogy) rather than small. We will not need this however.

We will now go back to the similarity principles. The similarity principle is a useful tool that allows to reduce the study of almost-complex objects to that of holomorphic objects, besides its many applications in complex analysis and geometry. Above (Proposition 2.0.9) we gave a first version of the Similarity Principle as in [CH63]. We will state now a useful generalization (see [Ber54]) which extends to higher dimensions.

Proposition 2.0.15 *If $f(z,\overline{z})$ is a complex valued function on a disc $D \subset D^2(0,R) \subset \mathbb{C}$ such that $|\frac{\overline{\partial} f}{\partial \overline{z}}| \leq M|f|$ then for a continuous function s, and a holomorphic $g(z)$ (in $D^2(0,R)$) we have:*

$$f(z) = e^{s(z)}g(z).$$

Moreover:

$$\begin{aligned} |s(z)| &\leq 4MR \quad for \quad z \in D \\ |s(z_2) - s(z_1)| &\leq c(R,M,\beta)|z_2 - z_1|^{\beta} \end{aligned}$$

for $0 < \beta < 1$. Here $c(R,M,\beta)$ is a constant.

Here is a generalization of the similarity principle to higher dimensions. It is due to Yau.

Proposition 2.0.16 *([JY83]) Let $f \in C^m(B,\mathbb{C})$, be a complex valued function on a bounded domain $B \subset \mathbb{C}^m$. Suppose that for every $\tau = \{\tau_1, \cdots, \tau_l\} \subset \{1, \cdots, m\}$ there is a constant c_τ such that*

$$\left|\frac{\overline{\partial}^{\tau_1 + \cdots + \tau_l}}{\partial \overline{z^{\tau_1}} \cdots \partial \overline{z^{\tau_l}}}\right| \leq c_\tau |f|$$

in the domain B. Then:

$$f(z) = e^{s(z)}g(z)$$

where $s(z)$ is Holder continuous, and $g(z)$ is holomorphic.

We will now give an example of a similarity principle for vector-valued functions, which was used J.C. Sikorav [Sik95] to show that singularities of J-holomorphic curves are C^1-equivalent to singularities of holomorphic curves.

Proposition 2.0.17 *([HZ94], [Sik95]) Let $j(z)$ be a complex structure on $\mathbb{C}$ of class C^α (here $0 < \alpha < 1$), and let $J(z) \in GL_{\mathbb{R}}(\mathbb{C}^n)$ be a Lipschitz (in z) complex structure on $\mathbb{C}^n$. If $f : (\mathbb{C}, 0) \to \mathbb{C}^n$ is a (germ of) continuous map, such that:*

$$|\mathrm{d}f \circ j - J \circ \mathrm{d}f| \leq C|f|$$

then there exists a (germ of) map $\Phi : (\mathbb{C}, 0) \to (GL_{\mathbb{R}}(C^n), \mathrm{Id})$ of class C^{1-}, a holomorphic (germ of) mapping $h : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, and a local diffeomorphism ψ of $\mathbb{C}$ of class $C^{1,\alpha}$ such that:

$$f = \Phi(h \circ \psi).$$

Moreover, if $j(z) = i$ (the standard complex structure on $\mathbb{C}$) one can take $\psi = \mathrm{Id}$. If $J(z) = i$, one can assume $\Phi(z) \in GL_{\mathbb{C}}(\mathbb{C}^n)$. Here the class C^{1-} denotes the intersection $\bigcap_{\beta<1} C^\beta$.

The proof is given in [Sik95]. A standard corollary, analogous to Corollary 2.0.10 follows:

Corollary 2.0.18 *([Sik95]) If $f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ is a non-zero (germ of) J-holomorphic curve then:*

$$f(z) = z^k a + o_{1-}(z^k)$$

for some positive integer k, and a vector $a \in \mathbb{C}^n$. Here $o_{1-}(z^k)$ denotes a function α-Holder continuous for all $0 < \alpha < 1$, whose Holder constant (computed at a point z) behaves like $o(|z|^{k-\alpha})$ (for every α).

Chapter 3

Number of solutions of a Cauchy-Riemann equation. Local structure of the space of Cauchy-Riemann operators.

3.1 Cauchy-Riemann operators on functions.

In this section we will prove elementary lemmas on Cauchy-Riemann operators acting on the space of complex functions on the standard two dimensional torus T^2, and use them to give an example of a 1-dimensional moduli space of pseudo-holomorphic tori. We consider equations of the form: $\overline{\partial} u + u\, a d\overline{z} + \overline{u}\, b d\overline{z} = 0$, where u is a complex valued function $u \in W^{k,2}, k > 2$, and $a d\overline{z}$, $b d\overline{z}$ are smooth forms of type $(0, 1)$.

Lemma 3.1.1 *The equation* $\overline{\partial} u + \overline{u}\, a d\overline{z} = 0$ *has no solution if* $a d\overline{z}$ *is a non-zero* $(0, 1)$*-form with constant coefficients.*

Proof: We will make use of Fourier series. Let $z = (x + iy)$ be coordinates on the torus T^2, and let $u = \Sigma\, e^{imx+iny}\, u_{mn}$ be the Fourier series of the function

u. The necessary and sufficient condition that u lies in $W^{k,2}$ is that the series $\sum |u_{mn}|^2 |(m+in)|^{2k}$ converges. Since $k > 2$ we have:

$$\overline{\partial} u \;=\; \Sigma\, e^{imx+iny}\, u_{mn}\, \frac{i}{2}\,(m+in)\, d\overline{z}.$$

In terms of the Fourier series our equation $\overline{\partial} u + \overline{u} a d\overline{z} \;=\; 0$ becomes

$$\Sigma\, e^{imx+iny}\, [(m+in)\,\frac{i}{2} u_{mn} + a\overline{u}_{-m-n}] \;=\; 0$$

Therefore we get $(m+in)\,\frac{i}{2}\, u_{mn} + a\,\overline{u}_{-m-n} \;=\; 0$ for all (m,n). Replacing (m,n) with $(-m,-n)$ yields

$$-(m+in)\,\frac{i}{2}\, u_{-m-n} + a\overline{u}_{mn} \;=\; 0$$

If we solve these equations for u_{mn} we get

$$\frac{u_{mn}}{(m-in)}\,(\frac{(m^2+n^2)}{2} + 2|a|^2) \;=\; 0.$$

It follows $u_{mn} \;=\; 0$ for $(m,n) \neq 0$. The assumption $a \neq 0$ guarantees that $u \;=\; u_{00}$ is a solution only if $u = 0$. □

Using Fourier series we can also find examples of operators with one-dimensional kernel.

Lemma 3.1.2 *If* $c d\overline{z}$ *is a non-zero* $(0,1)$*-form with constant coefficients then*

$$\overline{\partial} u + (\overline{u} - u) c d\overline{z} = 0$$

has exactly one solution (up to multiplication by real constants).

Proof: As above let $u = \Sigma u_{mn} e^{imx+iny}$, then

$$\overline{\partial} u + (\overline{u} - u) c d\overline{z} = \Sigma e^{imx+iny} (\frac{i}{2}(m+in) u_{mn} + (\overline{u}_{-m-n} - u_{mn}) c) d\overline{z}$$

and we get

$$\overline{u}_{-m-n} = u_{mn} - \frac{i}{2c}(m+in)u_{mn}$$

By substitution $(m,n) = -(m,n)$ we get a second equation

$$\overline{u}_{mn} = u_{-m-n} + \frac{i}{2c}(m+in)u_{-m-n}$$

which combined with the first one yields

$$\overline{u}_{mn} = (1 + \frac{i}{2c}(m+in))(1 + \frac{i}{2\overline{c}}(m-in))\overline{u}_{mn}$$

Thus if $u_{mn} \neq 0$ we must have $1 = (1 + \frac{i}{2c}(m+in))(1 + \frac{i}{2\overline{c}}(m-in))\overline{u}_{mn}$, or, by separating into real and imaginary parts:

$$\frac{1}{4|c|^2}|m+in|^2 = 0,$$
$$\frac{i}{2|c|^2}[c(m-in) + \overline{c}(m+in)] = 0.$$

It follows that $u_{mn} = 0$ for $(m,n) \neq (0,0)$, hence u is constant. That is enough to complete the proof of the lemma. □

Before proceeding to the next section we give a short example of a space of pseudo-holomorphic curves for a non-generic almost-complex structure. We will take $T^2 \times \mathbb{C}$ to be our ambient space, and consider pseudo-holomorphic tori. For an embedded torus the normal bundle N_Σ is trivial, hence the Cauchy-Riemann operator acts on topologically trivial bundle $T^2 \times \mathbb{C}$, and we have:

Proposition 3.1.3 *Let $\overline{\partial} + A$, $A \in \Omega^{0,1}(\mathrm{End}_{\mathbb{R}}(\mathbb{C}))$, be a Cauchy-Riemann operator on $T^2 \times \mathbb{C}$. Then the index of $\overline{\partial} + A$ is 0, while the real dimension of its kernel can be* $0, 1, 2$.

Proof: Suppose that the kernel of $\overline{\partial} + A$ is at least three dimensional. In that case, we could find a pseudo-holomorphic section η vanishing at a point. By the similarity principle (see Corollary 2.0.8 or Lemma 4.3.4) each zero of η would contribute positively to the Euler characteristic of $T^2 \times \mathbb{C}$ making it strictly positive which is not possible. (For an alternative proof see [Rod87].)
□

Therefore we can expect 2, 1, or 0-dimensional spaces of embedded tori. The standard integrable complex structure admits two dimensional family of pseudo-holomorphic tori.

Example 3.1.4 One dimensional space of pseudo-holomorphic curves. To find a one-dimensional space of tori we take J to be given by $\begin{bmatrix} J_0 & 0 \\ B & J_0 \end{bmatrix}$ where B is a real 2×2 matrix, of the form $\begin{bmatrix} b & c \\ c & -b \end{bmatrix}$, J_0 denotes the standard complex structure on $\mathbb{C}$, and $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We will take B to be $\begin{bmatrix} y_2 a & 0 \\ 0 & -y_2 a \end{bmatrix}$ where $z_2 = x_2 + iy_2$ are coordinates on the second factor in $T^2 \times \mathbb{C}$, and a is a real constant. $T^2 \times \mathbb{C}$ is a symplectic manifold with the standard product symplectic structure ω_0, and J is ω_0-tame on a neighbourhood of $T^2 \times 0$. We will check that J-holomorphic tori form a 1-parameter family of the form:

$$\begin{aligned} u : T^2 &\rightarrow T^2 \times \mathbb{C} \\ u(z) &= (z, t) \end{aligned}$$

with t being a real constant.

If $u = (u_1, u_2)$ is a J-holomorphic torus then we have

$$du + J \circ du \circ j = (du_1 + idu_1 \circ j, du_2 + Bdu_1 \circ j + idu_2 \circ j)$$

where j is a complex structure on T^2. It follows that j is equivalent to the standard complex structure, and by composing with the inverse of u_1 we can assume $j = J_0$, and $u_1(z, \overline{z}) = z$. Next we show that u_2 is constant. This follows from Hopf's maximum principle for elliptic equations (see [CH63], page 326). Functions $\mathrm{Re}u_2, \mathrm{Im}u_2$ satisfy

$$\partial_x \mathrm{Re}u_2 - \partial_y \mathrm{Im}u_2 = 0$$

$$\partial_x \mathrm{Im}u_2 + \partial_y \mathrm{Re}u_2 = (\mathrm{Im}u_2)a$$

hence $\partial_x^2(\mathrm{Re}u_2) + \partial_y^2(\mathrm{Re}u_2) = \partial_x(\mathrm{Re}u_2 a)$. Since this equation is satisfied on a closed surface $\mathrm{Re}u_2$ is constant (by the Hopf's maximum principle). It is then immediate $\mathrm{Im}u_2 = 0$. Finally we get $(u_1, u_2) = (z, t)$, where t is a real constant. This is a one parameter family of curves.

In Chapter 4 we construct more examples of non-regular almost-complex structures.

Example 3.1.5 Here is another example of a Cauchy-Riemann equation with one dimensional space of solutions which works on a Riemann surface Σ of arbitrary genus $g > 0$. This example is due to Y. Rodin [Rod87]. For simplicity we take once again $\Sigma = T^2$ (the construction however, is much the same regardless of genus), and the line bundle under consideration will be trivial. We will find a form $ad\overline{z}$ of type $(0,1)$ such that the equation $\overline{\partial}u - uad\overline{z} + \overline{u}ad\overline{z} = 0$, where u is a complex-valued function, has exactly one solution $u = 1$. This is equivalent to constructing the adjoint equation $\overline{\partial}u - u \wedge ad\overline{z} + \overline{u \wedge ad\overline{z}}$ with a singular solution u admitting a simple pole (see [Rod87]). Now u is a differential form of type $(1,0)$. To this end we'll identify the torus T^2 with $\mathbb{C}$ mod a lattice, in fact without any loss of generality it can be assumed $T^2 = \mathbb{C}/\mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}$. In order to find a one starts with a solution u of the form $u = i\partial v$, where v has a function of special form, e.g $v = \ln(x^2+y^2)g + h$, where both g, and h are real-valued functions. Then $u = \frac{1}{2}(iv_x + v_y)dz$ and $\overline{\partial}u - u \wedge ad\overline{z} + \overline{u \wedge ad\overline{z}}$ is equivalent to the following equation for $ad\overline{z}$:

$$\frac{1}{4}(v_{xx} + v_{yy}) = (a_1 v_x + a_2 v_y)$$

where $a_1 + ia_2 = a$. Solving this equation for a_1 one gets: $a_1 = \frac{1}{v_x}(\frac{1}{4}(v_{xx} + v_{yy}) - a_2 v_y)$. To insure that a_1 is non-singular we make a few choices:

(a) Let $g(x,y) = g(x^2+y^2)$, and $g = \text{const}$ for $r^2 = x^2 + y^2 < \delta + \epsilon$.

Both δ, and ϵ are as usual small positive numbers to be determined later. Notice that $v_x = \frac{2x}{x^2+y^2}g + g_x ln(x^2+y^2) + h_x$, and with g as in (a) we have for $r^2 > \delta$:

$$\frac{2x}{x^2+y^2}g + g_x \ln(x^2+y^2) < A$$

for some constant A. To make the partial derivative v_x non-zero:

(b) Let $h(x,y) = h_1(x^2+y^2) + cx + dy$, where $h(x^2+y^2)$ is constant on the disc $r^2 < \delta + \epsilon$, and $c + h_x > A$.

This last condition implies that v_x can vanish only on a disc $r^2 < \delta$, and that's where potential singularities of a_1 are located. However, in that disc both g and h_1 are constant; that implies $v_{xx} + v_{yy} = 0$ and $a_1 = a_2 \frac{v_y}{v_x}$. Finally choose a_2 so that $a_2 \frac{v_y}{v_x}$ is non-singular.

3.2 Local structure of the space of generalized Cauchy-Riemann operators.

In the previous section we found examples of Cauchy-Riemann operators with one-dimensional (over $\mathbb{R}$) kernel. Such examples are the reason why spaces of J-holomorphic curves in almost-complex manifolds do not behave exactly like moduli spaces of holomorphic curves in algebraic varieties. A simple study of the local structure of spaces of generalized Cauchy-Riemann operators shows that operators with one dimensional kernel form a subspace of codimension one, with cone-like singularities. To see this we will first study bifurcations of Cauchy-Riemann equations near an equation with two solutions on the standard torus T^2. For simplicity we will consider operators on a trivial complex line bundle over a torus T^2 with the standard complex structure, however our computations apply with small adjustments to non-trivial line bundles over surfaces of arbitrary genus $g \geq 1$. Let $a(\lambda) = a_0 + A(\lambda_1, \ldots, \lambda_k)$, and $B(\lambda) = b_0 + B(\lambda)$ be k-parameter, smooth families of perturbations of

$(0,1)$ forms on T^2, $a_0 = a(0)$, and $b_0 = b(0)$. We assume that the operator $u \to \overline{\partial} u + a_0 u + b_0 \overline{u}$, where u is a complex valued function, has two dimensional kernel. The following propositions will be proved:

Proposition 3.2.1 *For an open (but not dense) (in the C^∞-Whitney topology) set of two-dimensional deformations $(a(\lambda_1, \lambda_2), b(\lambda_1, \lambda_2))$ of a_0, b_0 of $\overline{\partial}_{a(0)b(0)}$ all operators $\overline{\partial}_{a(\lambda)b(\lambda)}$ (perturbations of $\overline{\partial}_{a(0)b(0)}$) have trivial kernel, except for $\lambda = 0$.*

Proposition 3.2.2 *There is an open set of deformations $(a(\lambda_1, \lambda_2), b(\lambda_1, \lambda_2))$ of (a_0, b_0) such that $\overline{\partial}_{a(\lambda)b(\lambda)}$ have one dimensional kernel for λ in a one dimensional subset of the space of parameters.*

Proposition 3.2.3 *Let $(a(\lambda_1, \lambda_2, ..., \lambda_k), b(\lambda_1, \lambda_2, ..., \lambda_k)$ be a generic k-dimensional deformation of an operator $\overline{\partial}_{a(0)b(0)}$ with two dimensional kernel. Parameters $(\lambda_1, \cdots, \lambda_k)$ corresponding to operators with one dimensional kernel form a codimension one (non-linear) cone with vertex at the origin.*

We will now prepare the turf for the proofs of the propositions. Denote by F, and G the functions spanning the kernel of $\overline{\partial} u + a_0 u + b_0 \overline{u}$. Let v, and w be two functions spanning the orthogonal complement of the image of $\overline{\partial}_{a_0,b_0} u = \overline{\partial} u + a_0 u + b_0 \overline{u}$ .Here the orthogonal complement refers to the L^2 inner product Re($\int_{T^2} f \overline{g}\, dx \wedge dy$). We will assume that functions v, and w are orthonormal. We will use the Lyapunov-Schmidt reduction to study the space of solutions of $\overline{\partial} u + a u + b \overline{u} = 0$. First we make an observation that $\overline{\partial} u + a u + b \overline{u} = \overline{\partial} u + a_0 u + b_0 \overline{u} + Au + B\overline{u}$ with $A(0) = 0$,and $B(0) = 0$.

Since the kernel of $\overline{\partial}_{a_0,b_0}$ is two dimensional it follows that there is a family of functions $u = u(x, y, \lambda)$ depending on the parameter λ and two real variables (x, y) such that

$$\overline{\partial}_{a_0,b_0}(xF + yG + u) + A(xF + yG + u) + B(\overline{xF + yG + u}) = 0 \operatorname{mod} \operatorname{Im}(\overline{\partial}_{a_0,b_0})$$

and $u(x, y, 0) = 0$, $u(0, 0, \lambda) = 0$. Such a family $u(x, y, \lambda)$ is very special; if $u_1 = u_1(\lambda)$, and $u_2 = u_2(\lambda)$ are two k-parameter families of functions on T^2 such that

$$\overline{\partial}_{a_0,b_0}u_1 + A(F + u_1) + B(\overline{F + u_1}) = 0 \operatorname{mod} \operatorname{Im}(\overline{\partial}_{a_0,b_0})$$

and

$$\overline{\partial}_{a_0,b_0}u_2 + A(G + u_2) + B(\overline{G + u_2}) = 0 \operatorname{mod} \operatorname{Im}(\overline{\partial}_{a_0,b_0})$$

then $u(x, y, \lambda) = xu_1(\lambda) + yu_2(\lambda)$. To continue with our analysis we use functions $f_1(\lambda), g_1(\lambda), f_2(\lambda), g_2(\lambda)$ such that:

$$\begin{aligned}\overline{\partial}_{a_0,b_0}u_1 + A(F + u_1) + B(\overline{F + u_1}) &= f_1 v + g_1 w\\ \overline{\partial}_{a_0,b_0}u_2 + A(G + u_2) + B(\overline{G + u_2}) &= f_2 v + g_2 w\end{aligned}$$

A parameter λ represents an operator with non-trivial kernel if for some $(x, y) \neq 0$

$$\begin{aligned}xf_1 + yf_2 &= 0\\ xg_1 + yg_2 &= 0\end{aligned}$$

i.e if the matrix $\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}$ has rank 1 or 0. More precisely, if rank of the matrix is one then the corresponding Cauchy-Riemann operator $\overline{\partial}_{a(\lambda),b(\lambda)}$ has one dimensional kernel, and if the rank is zero the kernel is two dimensional. The space of 2×2 matrices is four-dimensional; the subset of matrices of rank one is a cone over a two dimensional torus, therefore of dimension three. The vertex of this cone is the zero matrix, the only 2×2 matrix of rank zero. To see this notice that the equation

$$\det \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}x_{22} - x_{12}x_{21} = 0$$

can be transformed into

$$|z_1|^2 = |z_2|^2$$

in a suitable system of complex co-ordinates (z_1, z_2). The set of points satisfying this equation is a cone over a torus $|z_1|^2 = 1$, $|z_2|^2 = 1$. Notice that there are plenty of 2-dim planes disjoint from the cone (such as $z_1 = 0$ etc.) except at the origin, as well as planes (passing through the origin) that intersect the cone transversally. Both sets are open in the Grassmanian of two planes.

We will now supply proofs of the propositions listed at the beginning of this section.

Proof of Proposition 3.2.1.: We will show "by hand " how to choose an appropriate perturbation. Let $<,>$ denote the standard inner-product in the space of square integrable functions on a torus. Since for $\lambda = 0$ we have $A(0) = B(0) = u_1(0) = u_2(0) = 0$

$$\frac{\partial}{\partial \lambda}(xf_1 + yf_2) = < \frac{\partial}{\partial \lambda}(A)\,(xF + yG) + \frac{\partial}{\partial \lambda}(B)\,(\overline{(xF + yG)})\,, v_1 >$$

and similarily

$$\frac{\partial}{\partial\lambda}(xg_1 + yg_2) =< \frac{\partial}{\partial\lambda}(A)\,(xF + yG)\, + \,\frac{\partial}{\partial\lambda}(B)\,(\overline{(xF + yG)})\,, v_2 >$$

all this at $\lambda = 0$. Choose α_1^1 , α_1^2, α_2^1 , α_2^2 , β_1^1 , β_1^2 , β_2^1 , β_2^2 so that for $\lambda = 0$ the following holds modulo $\mathrm{Im}\,\overline{\partial}_{a_0,b_0}$

$$\begin{aligned}
\frac{\partial}{\partial\lambda_1}(A)F + \frac{\partial}{\partial\lambda_1}(B)\overline{F} &= \alpha_1^1 v_1 + \alpha_1^2 v_2 \\
\frac{\partial}{\partial\lambda_1}(A)G + \frac{\partial}{\partial\lambda_1}(B)\overline{G} &= \beta_1^1 v_1 + \beta_1^2 v_2 \\
\frac{\partial}{\partial\lambda_2}(A)F + \frac{\partial}{\partial\lambda_2}(B)\overline{F} &= \alpha_2^1 v_1 + \alpha_2^2 v_2 \\
\frac{\partial}{\partial\lambda_2}(A)G + \frac{\partial}{\partial\lambda_2}(B)\overline{G} &= \beta_2^1 v_1 + \beta_2^2 v_2
\end{aligned}$$

Then $\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}$ is equal to $\begin{bmatrix} \alpha_1^1\lambda_1 + \alpha_2^1\lambda_2 & \beta_1^1\lambda_1 + \beta_2^1\lambda_2 \\ \alpha_1^2\lambda_1 + \alpha_2^1\lambda_2 & \beta_2^1\lambda_1 + \beta_2^2\lambda_2 \end{bmatrix}$ modulo terms of second order $O(|\lambda|^2)$. The determinant of the last matrix is nonzero (except at the origin $(\lambda_1, \lambda_2) = 0$) for a choice of numbers α_1^1 through β_2^2 from a nonempty open set in $\mathbb{R}^4$ (to see that such a choice is at all possible, take $\alpha_2^1 = \alpha_2^2 = \beta_1^1 = \beta_1^2 = 0$, then the determinant gets reduced to $\lambda_1\lambda_2(\alpha_1^1\beta_2^2 - \alpha_2^1\beta_2^1)$ which can be made non-zero outside the origin). Finally choose the functions $a(\lambda)$, $b(\lambda)$ according to a choice of α_1^1 through β_2^2 . □

Proof of Proposition 3.2.3: When considering generic k-dimensional perturbations $a(\lambda_1, \lambda_2, ..., \lambda_k)$, $b(\lambda_1, \lambda_2, ..., \lambda_k)$ we choose $\alpha_j^1, \alpha_j^2, \beta_j^1, \beta_j^2$ so that

$$\frac{\partial}{\partial\lambda_j}(A)F + \frac{\partial}{\partial\lambda_j}(B)\overline{F} = \alpha_j^1 v_1 + \alpha_j^2 v_2$$

$$\frac{\partial}{\partial \lambda_j}(B)G + \frac{\partial}{\partial \lambda_j}(B)\overline{G} = \beta_j^1 v_1 + \beta_j^2 v_2$$

modulo the $\operatorname{Im} \overline{\partial}_{a_0 b_0}$, for $j = 1, ..., k$. $\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}$ is equal to $\begin{bmatrix} \Sigma\alpha_j^1\lambda_j & \Sigma\beta_j^1\lambda_j \\ \Sigma\alpha_j^2\lambda_j & \Sigma\beta_j^2\lambda_j \end{bmatrix}$ mod terms of second order $O(|\lambda|^2)$. This matrix is a linear combination of matrices $\begin{bmatrix} \alpha_j^1 & \beta_j^1 \\ \alpha_j^2 & \beta_j^2 \end{bmatrix}$ If $k \leq 4$ we can choose those matrices to be in general position by choosing $\frac{\partial A(\lambda)}{\partial \lambda_j}$ at $\lambda = 0$ appropriately (notice that "being in general position" is an open condition). Therefore we can guarantee that the inverse image of the cone of rank 1 matrices is itself a cone of codimension one, with vertex at the origin. Hence the set of operators with one dimensional kernel will also be a cone. If $k > 4$ then choose the perturbation so that the first four matrices form a basis, then the inverse image of the cone of rank one matrices will be a product of a cone by a linear $k-4$ dimensional space, hence itself a cone. The set of operators with two-dimensional kernel is homeomorphic to a linear $k-4$- dimensional space. That is enough to prove Proposition 3.2.3. □

Proposition 3.2.2 can be proved by choosing the first two matrices so that the vector space spanned by them cuts the cone of matrices of rank one transversally.

We will now consider generalized Cauchy-Riemann operators on non-trivial line bundles over Riemann surfaces. Let $\overline{\partial} + A$ be a generalized Cauchy-Riemann operator on a holomorphic line bundle $L \to \Sigma$, where $A \in \Omega^{0,1} \otimes$

$\mathrm{End}_{\mathbb{R}}(L)$. In this section we will assume that the index of $\overline{\partial}+A$ is non-negative. In what follows we will need special bases of kernels of $\overline{\partial}+A$, and its adjoint $(\overline{\partial}+A)^*$.

Definition 3.2.4 *Let f be a pseudo-holomorphic section of L. If f vanishes at point p then in local co-ordinates centered around p, $f(z) = az^n + h.o.t.$ We define the degree $\deg f(p) = n$, and $\deg f(p) = 0$ if f does not vanish at p.*

Definition 3.2.5 *Let $L \to \Sigma$ be a line bundle, $\overline{\partial}+A$ a Cauchy-Riemann operator and $(\overline{\partial}+A)^*$ its adjoint. Let $\{g_i\}_{i=1}^m$, and $\{w_j\}_{j=1}^n$ be linearly independent subsets of kernels of $\overline{\partial}+A$ and its adjoint.*

A point $p \in \Sigma$ will be called generic if there are linearly independent subsets $\{f_i\}_{i=1}^m$, and $\{v_j\}_{j=1}^n$ of the kernels of $\overline{\partial}+A$, and its adjoint such that:

(0) *The vectors $\{f_i\}$, and $\{v_j\}$ are linear combinations of $\{g_i\}$, and $\{w_j\}$.*

(1) *The vectors $f_1(p)$, $f_2(p)$, and $v_1(p)$, $v_2(p)$ are pairwise independent (over $\mathbb{R}$) in the fiber L_p.*

(2) $\deg f_i(p) \leq \deg f_{i+1}(p)$, *and* $\deg v_i(p) \leq \deg v_{i+1}(p)$.

(3) $|\deg f_i(p) - \deg f_{i+1}(p)| \leq 1$, *and* $|\deg v_i(p) - \deg v_{i+1}(p)| \leq 1$.

(4) *If $\deg f_i(p) = \deg f_{i+1}(p)$ then, in local coordinates:*

$$f_i(z) = a_i z^{\deg f_i(p)} + \cdots \quad f_{i+1}(z) = a_{i+1} z^{\deg f_{i+1}(p)} + \cdots$$

and a_i, $a_{i+1} \in \mathbb{C}$ are independent over $\mathbb{R}$. Analogously, if $\deg v_i(p) = \deg v_{i+1}(p)$ then, in local coordinates:

$$v_i(z) = \{b_i z^{\deg v_i(p)} + \cdots\}\overline{dz} \quad v_{i+1}(z) = \{b_{i+1} z^{\deg v_{i+1}(p)} + \cdots\}\overline{dz}$$

where b_i, b_{i+1} are independent over $\mathbb{R}$.

Lemma 3.2.6 *For any choice of linearly independent subsets* $\{g_i\}_{i=1}^m$, *and* $\{w_j\}_{j=1}^n$ *of the kernel of* $\overline{\partial} + A$, *and its adjoint, the set of generic points is open, dense and non-empty.*

Proof: The proof (by induction on m, and n) follows easily from the following observations. The conditions (1) – (4) (with the exception of (2) which can be satisfied by replacing sections g_i, and w_j by their linear combinations, and by renumbering them) define the set of generic points as a complement of closed subset of Σ cut out by a finite number of equations (see below). As it can be seen by inspection the complement of the set of generic points is a union of isolated points, and one-dimensional arcs. Moreover, it is enough to consider one operator at a time, and then take intersection of both sets of generic points. We will consider $\overline{\partial} + A$. Let $\{g_i\}_{i=1}^m$ be a basis of the kernel of this operator. The proof proceeds by induction on m. If $m = 1$ then observe that the complement of the generic set is simply $g_1^{-1}(0)$, and, by the Similarity Principle, is a collection of isolated points. For $m = 2$ consider a point $p \in \Sigma - g_1^{-1}(0) - g_2^{-1}(0)$. If $g_1(p)/g_2(p) \notin \mathbb{R}$ then p is generic. If, on the other hand, $g_1(p)/g_2(p) = t \in \mathbb{R}$, then consider a complex valued function $\eta = \frac{g_1 - tg_2}{g_2}$. This function satisfies an equation of Cauchy-Riemann type, and $\eta(p) = 0$ hence, in local co-ordinates, $\eta(z) = az^k + \cdots$. It follows that $\eta^{-1}(\mathbb{R})$ is a set of codimension 1 in Σ. The complement of the union of those arcs (including the zero locus of both sections) is the required set of generic points. Here one can set $f_1 = g_1$, $f_2 = g_2$. The remaining part of the proof proceeds in similar fashion. □

Corollary 3.2.7 *If* $\text{index}(\overline{\partial}+A) > 0$ *then* $\dim_{\mathbb{R}} \ker(\overline{\partial}+A) \leq \text{index}(\overline{\partial}+A)+2g$.

Proof: Suppose that $\dim_{\mathbb{R}} \ker(\overline{\partial} + A) \geq \text{index}(\overline{\partial} + A) + 2g + 1$. Choose a basis $\{f_j\}_{j=1}^{m}$ of the kernel of $\overline{\partial} + A$, and let p be a generic point. Then $\deg f_m \geq \text{index} + g + 1$, and therefore $c_1(L) \geq \text{index} + g + 1$. This contradicts however the assumption that the index of $\overline{\partial} + A$ is positive. □

Let $L \to \Sigma$ be a holomorphic vector bundle of rank one on a Riemann surface Σ. Let $\mathcal{CR}$ denote the space of generalized Cauchy-Riemann operators on L of the form $\overline{\partial} + A$, where $\overline{\partial}$ is the Cauchy-Riemann operator defining the holomorphic structure of L, and A is an endomorphism of the bundle. All such operators have the same index, $\text{index} = 2(1-g) + 2c_1(L)$. Let $\mathcal{NR}(s)$ be a subspace of operators with s-dimensional cokernel.

Proposition 3.2.8 *If the index of the operator* $\overline{\partial} + A$ *is at least* 2 *then the codimension of* $\mathcal{NR}(s)$ *is greater or equal to* $\left[\frac{s+1}{2}\right] \times (s + \text{index})$.

Proof: We will consider only the case $s = 2$. The proof for $s = 1$, and $s \geq 3$ is similar. Let $\overline{\partial} + A$ be a generalized Cauchy-Riemann operator on a line bundle $L \to \Sigma$, $g(\Sigma) \geq 1$, $A \in \Omega^{0,1} \otimes End_{\mathbb{R}}(L)$. Let s be the dimension of the cokernel of $\overline{\partial} + A$ and let $r = \dim_{\mathbb{R}}(\ker(\overline{\partial} + A))$. Let $\overline{\partial} + A + B(\lambda)$, $\lambda \in \mathbb{R}^n$ be a perturbation of the given operator such that $\left[\frac{s+1}{2}\right] \times (s + \text{index}) \leq n$. We need to show that for a generic perturbation $B(\lambda)$ the set of values of the parameter λ for which the operator $\overline{\partial} + A + B(\lambda)$ has s-dimensional cokernel has co-dimension $\left[\frac{s+1}{2}\right] \times (s + \text{index})$ or higher. It is equivalent to consider parameters such that the kernel of $\overline{\partial}+A+B(\lambda)$ is r-dimensional. Recall that the space of generalized Cauchy-Riemann operators on L is an affine space modelled on $\Omega^{0,1}\otimes \mathrm{End}_{\mathbb{R}}(L) = \Omega^{0,1}\otimes \mathrm{End}_{\mathbb{C}}(L)+\Omega^{0,1}\otimes \mathrm{End}_{\overline{\mathbb{C}}}(L)$. We let $B(\lambda)=\Sigma\lambda_i A_i + h.o.t$ be the perturbation term with $B(\lambda)\in\Omega^{0,1}\otimes \mathrm{End}_{\mathbb{R}}(L)$. Let $f_1,...,f_r$ span the kernel of $\overline{\partial}+A$, and v_1, v_2 be two sections complementary to the image of $\overline{\partial}+A$. To find the kernel of the perturbed operator we solve equations $(\overline{\partial}+A)u_j + B(f_j+u_j) = 0 \bmod \mathrm{Im}(\overline{\partial}+A)$, for sections u_j in a fixed complement of the kernel of $\overline{\partial}+A$. Let then $\mu_{j,s}\in\mathbb{R}$ be such that $(\overline{\partial}+A)u_j+B(f_j+u_j)=\mu_{j,1}v_1+\mu_{j,2}v_2$. Then a linear combination $\Sigma x_j(f_j+u_j)$ with real coefficients $x_1, x_2, ...$ is in the kernel of the perturbed operator if

$$(\overline{\partial}+A)(\Sigma x_j(f_j+u_j))+B((\Sigma x_j(f_j+u_j))=(\Sigma x_j\mu_{j,1})v_1+(\Sigma x_j\mu_{j,2})v_2=0,$$

i.e when $\Sigma_{j=1}^{r} x_j\mu_{j,s}=0$ for $s=1,2$. The perturbed operator has r, $r-1$ or $r-2$-dimensional kernel if the rank of the matrix $C(\lambda)=[\mu_{j,s}]_{j=1,s=1}^{j=r,s=2}$ is 0, 1, or 2 respectively. The perturbation will be called generic if at least $\left[\frac{s+1}{2}\right]\times(s+\text{index})$ entries of $C(\lambda)$ are linearly independent (as functions of λ) at the origin $\lambda=0$. We will now show that generic perturbations exist. We will insist that $B(\lambda)$ be anti-linear since such endomorphisms arise from perturbations of almost-complex structures on M. Recall the definition of μ_j, and differentiate with respect to $\lambda=\lambda_k$:

$$\begin{aligned}(\overline{\partial}+A)u_j+B(\lambda)(f_j+u_j) &= \mu_{j,1}v_1+\mu_{j,2}v_2\\ (\overline{\partial}+A)\frac{\partial}{\partial\lambda}u_j+(\frac{\partial}{\partial\lambda}B(\lambda))(f_j+u_j) &= \frac{\partial}{\partial\lambda}\mu_{j,1}v_1+\frac{\partial}{\partial\lambda}\mu_{j,2}v_1\end{aligned}$$

The last equation implies that at $\lambda = 0$

$$\frac{\partial}{\partial \lambda} B(0)(f_j) = \frac{\partial}{\partial \lambda} \mu_{j,1}(0) v_1 + \frac{\partial}{\partial \lambda} \mu_{j,2}(0) v_1 \mathrm{mod}\, \mathrm{Im}\, (\overline{\partial} + A).$$

Let $\frac{\partial}{\partial \lambda_k} B(0) = B_k \in \Omega^{0,1} \otimes \mathrm{End}_{\mathbb{R}}(L)$. To find a generic perturbation B we consider a generic point p of v_1, v_2 and $f_1, \cdots, f_r$. We may assume that in local coordinates, and a local trivialisation of the bundle L, $v_1(p) = 1$, and $v_2(p) = i$. We arrange notation so that:

$$\deg(f_1)(p) \geq \deg(f_2)(p) > \deg(f_3)(p) \geq \deg(f_4)(p) > \ldots$$

and if $\deg f_j(p) = \deg f_k(p)$, then in local coordinates near p:

$$\begin{aligned} f_j(z) &= a_j z^{m_j} + h.o.t \\ f_k(z) &= a_k z^{m_k} + h.o.t \\ a_j &\perp a_k \quad \text{in} \quad \mathbb{C} \end{aligned}$$

We work in local co-ordinates, and look for anti-linear B_k's with small support, so that $B_k(f_j) = \{b_k(z)(\overline{a_j z^{m_j} + h.o.t})\}\overline{dz}$ for some complex-valued functions b_k with a small support. The proposition will follow if we observe that functions $\overline{a_j z^{m_j} + h.o.t}(v_1(z)/(\overline{dz}))$ are linearly independent in L^2 on any open neighborhood of the generic point p. It is then possible to choose the functions b_k so that r of the entries of C are independent (as functions of λ). The general case is proved in exactly the same way if we work in a neighborhood of a generic point, and use bases provided in the lemma 3.2.6. □

In the borderline case of line bundles of degree $g - 1$ one can prove a proposition analogous to Proposition 5. Assume that $\overline{\partial} + A$ is a generalized

Cauchy-Riemann operator on a line bundle L, $\deg L = g - 1$, with two dimensional kernel. Then

Proposition 3.2.9 *For a generic $3 \leq k$-parameter perturbation $\overline{\partial} + A(\lambda)$ of $\overline{\partial} + A$ there is a codimension one cone in the space of parameters corresponding to operators with one dimensional kernel. The vertex of the cone is at the origin which corresponds to $\overline{\partial} + A$. If $k \geq 4$ the space of operators with two-dimensional kernel is homeomorphic to $\mathbb{R}^{k-4}$.*

Proof: The proof is exactly as that of Proposition 3.2.3.

3.3 Estimates of codimensions of singular curves, and curves with self-intersections.

We will now prove some estimates of the codimension of J-holomorphic curves with triple self-intersections. The proofs, much like in the previous section rely on the notion of transversality.

Definition 3.3.1 *Let $u : \Sigma \to M$ be a J-holomorphic curve. A point $P \in M$ is a triple self-intersection point of u if there are three distinct points $z_1, z_2, z_3 \in \Sigma$ such that $u(z_1) = u(z_2) = u(z_3) = P$.*

Proposition 3.3.2 *For a generic almost-complex structure J (i.e in a set which is a countable intersection of open, dense sets) the evaluation mapping* ev :

$$\mathrm{ev} : \overline{\mathcal{M}} \times_{\mathcal{G}} (\Sigma \times \Sigma \times \Sigma) \;\; \to \;\; M \times M \times M \qquad (\S 3)$$

$$(u, z_1, z_2, z_3) \rightarrow (u(z_1), u(z_2), u(z_3)) \qquad (\S3)$$

(where $\overline{\mathcal{M}}$ denotes the set of parametrized curves, and $\mathcal{G}$ the reparametrization group) has the following property. The inverse image of "the triple diagonal" i.e the set$\{(u, z_1, z_2, z_3)$: none of the z_i's is a singular point, and $u(z_1) = u(z_2) = u(z_3)\}$ is of real codimension 8. For a generic J, if $\mathcal{M}$ contains an immersed curve then it also contains an immersed curve without triple self-intersection points.

Proof: Let $\mathcal{J}$ denote the space of almost-complex structures on M. Recall that for a generic J (in a set of second category) the moduli space $\mathcal{M}$ is a manifold, it will be enough to work with such almost-complex structures. We will work with an extended evaluation map:

$$\overline{\text{ev}} : \mathcal{J} \times \overline{\mathcal{M}} \times_{\mathcal{G}} (\Sigma \times \Sigma \times \Sigma) \rightarrow M \times M \times M. \qquad (\S3)$$

Let Δ be the triple diagonal in $M \times M \times M$. The proposition will follow if we can show that $\overline{\text{ev}}$ is transversal to Δ. Let P be a triple point, and (z_1, z_2, z_3) three non-singular points which are mapped onto P by a parametrized curve u. We will use local charts on Σ, and on M. Those can be chosen so that $J = \begin{bmatrix} j_1 & 0 \\ 0 & j_2 \end{bmatrix}$, and $J = J_0$ at the origin. Moreover we can ensure that $u_1(z) = (z, 0)$. Let $u_2 = (v_2, w_2)$, and $u_3 = (v_3, w_3)$. After choosing local co-ordinates near z_1, z_2, z_3 it can be assume that v's, and w's are defined on a common disc in $\mathbb{C}$. The point now is to manufacture enough perturbations of J which will move the three branches of u through P in all directions transversal to Δ. For that we consider a small annulus $A : (\delta/2 < |z_1| < \delta) \times \mathbb{C}$. If $(v_2, w_2) \in A$, then: $|w_2(z)| \geq c|z|^{p_2/q_3} > c\delta_1$, where all constants come from the expansion: $v_2(z) = a_2 z^{q_2} + h.o.t$, and $w_2(z) = b_2 z^{p_2} + h.o.t$. If $v_2 \equiv 0$, we replace the above inequalities by $|w_2(z)| \geq c|z|$. Similar inequalities hold for u_3, in $A : |w_3(z)| \geq c|z|^{p_3/q_3} > c\delta_1$, or ,if $v_3 \equiv 0$, take $p_3 = q_3$ in the last inequality. We will now consider a particular perturbation $u_{1\epsilon}$, $\epsilon \in \mathbb{R}$, of u, and show it is a path of J_ϵ-holomorphic curves (for an appropriate perturbation J_ϵ of J). Consider $\overline{u}_{1\epsilon} = (z, \epsilon)$, and connect it to u_1 to obtain $u_{1\epsilon} = (z, \eta(|z|) + \epsilon(1 - \eta(|z|)))$. Here η is a cut-off function chosen so that (for small ϵ) $u_{1\epsilon}$ is immersed, near P, and without extra intersection points, and so that the connecting annulus lies in the region $\delta/2 < |z_1| < \delta$, $|z_2| < c\delta_1$. Such a choice guarantees that the connecting neck is disjoint from u_2, u_3. Now perturb J to J_ϵ which makes $u_{1\epsilon}$ a pseudo-holomorphic curve. (This construction is taken from [McD94].) Now repeat the same with u_2, and u_3 in place of u_1 to show that if (z_1, z_2, z_3) are non-singular, and $v \in T_P M$ then for $i = 1, 2, 3$ there exists a 1-parameter family J_t, and u_t such that $\frac{d}{dt}u_t(z_i) = v$, $u_t(z_j) = u_{z_j}$, $j \neq i$. The proposition follows by standard arguments of perturbation theory. □

We will now show that for generic almost-complex structures singular curves have codimension at least two in the moduli space of pseudo-holomorphic curves.

Definition 3.3.3 *A mapping $u : \Sigma \to M$ is called singular if there is a point $p \in \Sigma$ such that the differential* $du(p)$ *does not have the maximal rank. The* *point p is then referred to as a singular point of the mapping u. In particular, immersed curves are not considered to be singular in this section.*

Proposition 3.3.4 *Let $A \in H(M, \mathbb{Z})$ be an arbitrary homology class that can be represented by curves of genus g. For a generic almost-complex structure J (i.e in a set of second category) the set of singular curves $u \in \mathcal{M}(J, A, g)$ is of real codimension 2 or higher in $\mathcal{M}(J, A, g)$.*

Proof: Here is an outline of the proof. Consider the space of parametrized curves $\widetilde{\mathcal{M}}(\mathcal{J}, A, 1)$ and a partial differential Ψ of the evaluation mapping:

$$\begin{aligned} \Psi : \widetilde{\mathcal{M}}(Jj, A, 1) \times_{G_1} T\Sigma &\rightarrow TM \\ (J, u, z, v) &\rightarrow (u(z), du(v)). \end{aligned}$$

where G_1 denotes the reparametrization group of T^2. Let $\mathcal{O}$ denote the zero section of TM. The proposition follows from standard arguments in perturbation theory if we can show that Ψ is transversal to $\mathcal{O}$. That in turn follows immediately from arguments either in Sikorav's article ([Sik94]) or in an article by McDuff ([McD94]). We will use a combination of both. Let u be a curve singular at a point $p \in \Sigma$, i.e $\mathrm{d}u(p) = 0$. Let v be an arbitrary vector in $T_{u(p)}M$. We need to find a 1-parameter family (J_t, u_t) of pseudo-holomorphic curves such that $(J_0, u_0) = (J, u)$, and $\mathrm{d}u_t(0) = vt$. A theorem due to J.C Sikorav ([Sik94]):

Theorem 3.1.1. [Sikorav]. Let (V, J_0) be an almost complex manifold of class C^r for some $r > 1$. Fix a point $v \in V$. Then:

(i) For every $X \in T_v V$ small enough there exists a J_0-holomorphic map $f : (D,0) \to (V,v)$ such that $\mathrm{d}f(0) \cdot 1 = X$.

(ii) Let $f_0 : D \to V$ be a J_0-holomorphic map. Then there exists $\alpha > 0$, a neighborhood $calU$ of J_0 in $\mathcal{J}^r(V)$ and a map of class $\mathcal{C}^r$:

$$F : \mathcal{U} \times D(\alpha) \to V$$

such that $F(J_0, \cdot) = f_0$ and $F(J, \cdot)$ is J-holomorphic.

gives us a family $(J_t, \overline{u_t})$ of pseudo-holomorphic maps $\overline{u_t} : D^2 \to M$ such that $\mathrm{d}\overline{u_t}(0) = vt$. These can be extended to maps $u_t : \Sigma \to M$, which are J_t-holomorphic for some almost-complex structures which agree with J on a neighborhood of $u(p) = u_t(p)$. Now recall that there are only finitely many singular points on u, and repeat the above for all of them, and for $v = v_1, v_2$- a basis of TM at an image of a critical point. That is enough to show transversality of Ψ to $\mathcal{O}$. It follows that for J in a set of second category the restriction of Ψ:

$$\widetilde{\mathcal{M}} \times_{G_g} T\Sigma \quad \to \quad TM$$

is transversal to the zero section, and hence the set of singular curves in $\mathcal{M}$ has co-dimension 2 (or higher). □

Chapter 4

Regularity of almost-complex structures in dimension four.

In this section we prove a simple but useful criterion of regularity of almost-complex structures on 4-dimensional symplectic manifolds (Proposition 4.3.2). Having that application in mind we formulate the criterion for singular as well as immersed pseudo-holomorphic curves.

4.1 Set-up

Our working space is the space $\mathrm{Map}^{k,p}(\Sigma, \mathrm{M})$ of maps of a closed Riemann surface Σ into M which are somewhere injective, and locally of class (k, p). We fix k, and p once and for all so that $k - \frac{2}{p} > 2$. This guarantees that all our maps are at least of class C^2. If the genus g of Σ is greater or equal to one, we allow for variations of complex structures on the surface. Let then $\mathcal{T}_g$ denote the Teichmuller space of Σ, which parametrizes the space of such complex structures (up to modular group). There is a smooth map $\tau \to j_\tau$ from the

Teichmuller space to the space of smooth complex structures on Σ (McDuff-Salamon [MS94]). Given an almost complex structure J on M and a homology class $A \in H_2(M, Z)$ we define the space of parametrized pseudo-holomorphic curves

$$\overline{\mathcal{M}}(J, A, g) = \{(u, \tau) \in \mathrm{Map}^{k,p}(\Sigma, \mathrm{M}) \times \mathcal{T}_g : \overline{\partial}_J(u) = du \circ j_\tau - J \circ du = 0\}.$$

The spaces of unparametrized curves $\mathcal{M}(J, A, g)$ are obtained by dividing by relevant reparametrization groups G_g ([MS94]).

The operator $\overline{\partial}_J(u)$ can be thought of as a section of a bundle over $\mathrm{Map}^{k,p} \times \mathcal{T}_g$. An almost complex structure J is called regular if the linearisation D_u of $\overline{\partial}_J(u)$:

$$D_u = D\overline{\partial}_J(u) : W^{k,p}(u^*TM) \to \Omega^{0,1}(u^*TM)$$

is surjective for every curve $u \in \mathcal{M}(J, A, g)$. For applications we need an explicit formula for D_u. If we fix a connection on M such that $\nabla J = 0$ then

$$D_u = D\overline{\partial}_J(u) : W^{k,p}(u^*TM) \to W^{k-1,p}(T^{0,1}\Sigma \otimes u^*TM)$$

$$D_u(\xi) = \frac{1}{2}(\nabla\xi + J(u)\nabla\xi \circ j_\tau + \frac{1}{2}tor(\partial_J(u), \xi))$$

at a point (u, τ). Here $tor(\partial_J(u), \xi) = Tor(\partial_J(u), \xi) + J(u)Tor(\partial_J(u), \xi)$, and Tor is the torsion of ∇. Note that the connection ∇ induces holomorphic structures on u^*TM hence the usual operator $\overline{\partial}$. The operator D_u is of the form $\overline{\partial} + A$ where A is a linear zero order term, i.e a $(0, 1)$-form with values in the bundle of endomorphisms $A \in W^{k-1,p}(T^{0,1}\Sigma \otimes u^*TM)$ hence a generalized Cauchy-Riemann operator.

4.2 Reduction of D_u to the normal bundle.

The regularity of an complex structure J means that the operator D_u is surjective. By the very definition this is an operator on a bundle of rank two or higher. It is easier, however, to deal with operators on line bundles. By allowing variations of complex structures on Σ we can reduce D_u to $N_\sigma = u^*TM/T\Sigma$, the normal bundle to Σ in M, provided that the curve u is immersed (or with mild singularities). The tangent bundle u^*TM is equipped with a holomorphic structure so that $u^\star : T\Sigma :\longrightarrow TM$ is holomorphic, therefore N_σ becomes a holomorphic line bundle, and D_u a generalized Cauchy-Riemann operator on a line bundle.

We will now outline the reduction process. Let u be a pseudo-holomorphic curve. For simplicity we assume that u is embedded, but the construction goes through in the immersed case as well. Let g be a J-compatible hermitian metric on (TM, J). We then choose a J-compatible connection ∇ on TM which makes the embedding $T\Sigma \rightarrow TM$ holomorphic. Namely, let ∇ be a hermitian connection on TM with additional properties:

a) $\bar{\partial}_\nabla = \bar{\partial}$ on $T\Sigma$

b)$Tor_\nabla = \frac{1}{4}\mathrm{Nij}_\nabla$

Where Nij_∇ is the Nijenhuis tensor of J. To find such a connection let v_Σ be the g-normal bundle to $T\Sigma$, so that $TM|_\Sigma = T\Sigma \oplus v_\Sigma$. Extend the splitting to a neighborhood U of Σ, $TM|_U = L_1 \oplus L_2$, where L_i are complex line bundles, g-orthogonal extending $T\Sigma$ and v_Σ. Let ∇_1 be the unique torsion-free, hermitian connection on Σ extended to L_1, and ∇_2 be a hermitian connection on L_2. We define $\nabla = \nabla_1 \oplus \nabla_2$ on U, and extend it across M. By subtracting components of torsion of ∇ we can obtain b). The $\bar{\partial}_\nabla$ operator on $TM|_\Sigma$ defines a holomorphic structure on $TM|_\Sigma$ (which is in fact canonical, see Taubes [Tau95]), and by property a) above the embedding $T\Sigma \to TM|_\Sigma$ is holomorphic. We define the holomorphic structure on the normal bundle $N_\Sigma = TM/T_\Sigma$ by an exact sequence of holomorphic bundles

$$0 \to T\Sigma \to u^*TM \to N_\Sigma \to 0.$$

Our operator D_u composed with projection onto N_Σ descends to an operator

$$D_u : W^{k,p}(N_\Sigma) \to W^{k-1,p}(T^{0,1}\Sigma \otimes N_\Sigma).$$

(This follows from properties a) and b) of ∇. The second property guarantees that the zero order part of D_u vanishes on $T\Sigma$ because the Nijenhuis tensor of an almost-complex structure on a surface is always zero). Since we allow for variations of complex structures on Σ, surjectivity of D_u restricted to N_Σ is equivalent to surjectivity of D_u on u^*TM.

4.3 Regularity of almost-complex structures.

To state Proposition 4.3.2 we recall the definition of a multiplicity of a singular point. Let (M, J) be an almost-complex manifold, and $u : \Sigma \to M$ a J-holomorphic curve. Such a curve has a finite number of singular points (see McDuff [McD92b]), let $z_0 \in \Sigma$ be one of them. There are local coordinate systems z, and (z_1, z_2) on Σ and M such that:

$$u(z) = (u_1, u_2) = (z^{k_0}, 0) + (a, b)z^{k_0+1} + O(z^{k_0+2}) \qquad (\S 3)$$

and $J = J_0$ at the origin (J_0 being the standard integrable complex structure on $\mathbb{C}^2$).

Definition 4.3.1 *Let $u : \Sigma \to M$ be a J-holomorphic curve, with a singular point $z_0 \in \Sigma$. The multiplicity of z_0 is the number k_0 in equation §3 above.*

Proposition 4.3.2 *Let (M, J) be a 4-dim manifold with an almost complex structure J, and let A be a 2-dim homology class in $H^2(M, Z)$.*

(a) *If $c_1(A) \geq 1$ then J is regular for all immersed J-holomorphic curves in class A.*

(b) *J is regular for all singular curves u such that $c_1(A) - \sum(k_i - 1) \geq 1$, where $\{k_i\}$ is the set of multiplicities of all singular points of u.*

Here $c_1 = c_1(TM, J)$ is the first Chern class of (M, J).

Proof: First we consider the case of immersed curves. The regularity of an complex structure J means that the operator D_u is surjective. As explained in Section 4.2 by allowing variations of complex structures on Σ we can reduce the bundle u^*TM to N_Σ , the normal bundle to Σ in M. The surjectivity of D_u on u^*TM is equivalent to surjectivity of the reduction $D_u|_{N_\Sigma}$. N_Σ is a holomorphic line bundle, and if J is integrable the operator D_u is exactly the Cauchy-Riemann (or the Dolbeaut) $\bar{\partial}$ operator on N_σ, and the proof is reduced (via Serre duality) to a simple fact: a holomorphic line bundle $N_\Sigma^* \otimes K_\Sigma$ with negative first Chern number has no holomorphic sections. If $N_\Sigma^* \otimes K_\Sigma$ had a holomorpic section ξ then its intersection with the zero section would be positive which implies positivity of the degree. Exactly the same proof works in the non-integrable case if one can show that similar positivity of intersections holds. More precisely, surjectivity of D_u is equivalent to injectivity of the adjoint operator $D_u^* : \Gamma(N_\Sigma) \otimes K_\Sigma \to \Gamma(N_\Sigma)$. In local coordinates this is an operator of the form $-\partial + B$. It is then enough to show that if this operator has a non-trivial kernel then the Euler class of $N_\Sigma \otimes T^{0,1}$ is negative, which would contradict positivity of $c_1(A)$. It will be more convenient to use complex-conjugate bundle $\overline{N_\Sigma \otimes T^{0,1}}$ since upon identification of this bundle with $N_\Sigma \otimes T^{0,1}$ the conjugate operator assumes the more familiar form $\overline{\partial} + B$, with $B \in \mathrm{End}_{\mathbb{R}}(N_\Sigma \otimes T^{0,1}, N_\Sigma)$. Finally we are reduced to a simple fact.

Lemma 4.3.3 *Let L_1, L_2 be two holomorphic line bundles over a Riemann surface Σ, and let $\mathcal{D} : \Gamma(L_1) \to \Gamma(L_2)$ be a generalized Cauchy-Riemann operator, i.e in local trivialisations $\mathcal{D} = \overline{\partial} + B$, where $B(z) \in \mathrm{End}_{\mathbb{R}}(\mathbb{C})$. If the kernel of $\mathcal{D}$ is non-trivial then the degree of L_1 is non-negative.*

Proof: We will reduce the proof of the lemma to a local statement about pseudo-analytic functions. Let $B \in C^\infty(D^2, \mathrm{End}_{\mathbb{R}}(\mathbb{C}))$ be a function on a disc $D^2 \subset \mathbb{C}$ (centered at the origin) with values in the space of real endomorphisms of $\mathbb{C}$. Let $f(z, \bar{z})$ be a complex valued function in the kernel of a generalized Cauchy-Riemann operator:

$$\frac{\overline{\partial}}{\partial \bar{z}} f(z, \bar{z}) + B(z, \bar{z}) f(z, \bar{z}) = 0. \qquad (\S 3)$$

First observe that the function $f(z, \bar{z})$ has only isolated zeroes, all of finite order. This follows directly from the similarity principle. Assume now that f has only one zero in the center of D^2 which we take to be the origin.

Lemma 4.3.4 *The graph* Γ_f *of* f *in* $\mathbb{C}\times\mathbb{C}$ *has positive intersection with* $\mathbb{C}\times 0$ *in* $H_2(\mathbb{C}\times\mathbb{C}, \partial(\mathbb{C}\times\mathbb{C}))$.

Proof: If the differential $\mathrm{d}f|(0) \neq 0$ the intersection is transversal, and we can compute the intersection index directly. Write $f = u+iv$, and $\frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\partial_t + i\partial_s)$, where $z = t + is$. Since $\overline{\partial}f|_{(0)} = 0$, $\mathrm{d}f(\partial_t) = (u_t, v_t)$, and $\mathrm{d}f(\partial_s) = (u_s, v_s)$. Then

$$\mathrm{d}f|_{(0)}(\partial_t \wedge \partial_s) = ((u_t)^2 + (u_s)^2)\partial_t \wedge \partial_s \neq 0$$

which proves positivity of the intersection.

If $\mathrm{d}f|_{(0)} = 0$ we need to put the graph of f in a transversal position. Let k be the order of f at 0 i.e $j^{(k-1)}f(0) = 0$, but $j^k f(0) \neq 0$ then:

$$f(z,\bar{z}) = az^k + h.o.t$$

(this is because f satisfies the elliptic system §3). The following two estimates follow readily. For appropriate constants c, δ :

$$|f(z,\bar{z})| > |a|\,|z|^k - |h.o.t| > |a|\,|z|^k - c|z|^{k+1} > |z|^k\frac{|a|}{2}$$

if $|z| < 2\delta$. Therefore:

$$|f(z,\bar{z})| > \delta^k\frac{|a|}{2}.$$

The second estimate:

$$|f(z,\bar{z})| < 2|a||z|^k < \frac{2|a|\delta^k}{3^k}$$

holds for $|z| < \frac{\delta}{3}$ and a sufficiently small δ. Let now η be a positive number $2|a|\frac{\delta^k}{3^k} < \eta < |a|\frac{\delta^k}{2}$, and $\rho(t)$ any non-negative cut-off function such that:

(a) $\rho(t) = \eta$ if $t < \frac{10}{6}\delta$

(b) $\rho(t) = 0$ if $t > \frac{11}{6}\delta$

We define a perturbation of f :

$$\phi(z, \bar{z}) = f(z, \bar{z}) - a\rho(|z|).$$

Then $\phi = 0$ implies that $\frac{\delta}{3} < |z| < \frac{11\delta}{2}$. Since ρ is constant on this annulus we have that $d\phi = df$ at any point where $\phi = 0$. Now $df(\partial_s \wedge \partial_t) = (|\frac{\partial f}{\partial z}|^2 - |\frac{\bar{\partial} f}{\bar{\partial} z}|^2)dt \wedge ds$. Since

$$\frac{\partial f}{\partial z} = kaz^{k-1} + O(|z|^k)$$

and

$$\frac{\bar{\partial} f}{\bar{\partial} z} = O(|z|^k)$$

we see that

$$|\partial_z f|^2 - |\partial_{\bar{z}} f|^2 > 0$$

provided that $|z|$ is small enough, what we can guarantee by choosing δ sufficiently small. That is enough to prove the Lemma 4.3.4. $\square$

To prove Lemma 4.3.3 suppose that $\mathcal{D}(\eta) = 0$. Apply Lemma 4.3.4 to every zero of η to conclude that the first Chern class $c_1(L_1) > 0$, or that $c_1(L_1) = 0$ if η never vanishes. $\square$

That is enough to prove the non-singular case. We will now extend the proof to the case of singular curves. The first step is to define N_Σ, and D_u :

$N_\Sigma \to N_\Sigma \otimes T^{0,1}$. For this one needs to replace $u^*T\Sigma$ by a line subbundle of u^*TM. Recall that a pseudo-holomorphic curve $u : \Sigma \to M$ has a finite number of singular points, and that given a singular point $z_0 \in \Sigma$ there are local holomorphic coordinate systems on Σ, and M such that:

$$u(z) = (u_1, u_2) = (z^{k_0}, 0) + (a, b)z^{k_0+1} + O(z^{k_0+2}). \qquad (\S 3)$$

Let now $\{z_1, ..., z_r\}$ be the collection of all singular points, $\{k_1, ..., k_r\}$ their multiplicities. The embedding $u^*T\Sigma \to u^*TM$ is well defined on the complement of singular points. It extends across the singular set since according to the equation §3 the vector field:

$$\frac{1}{kz^{k-1}}\frac{\partial u}{\partial z} = (1, 0) + (a, b)\frac{k+1}{k}z + O(z^2)$$

is of class C^1. That allows us to define a line subbundle of u^*TM. More precisely if we let $L = T\Sigma \otimes_{i=1}^r \mathcal{O}(z_i^{k_i-1})$ then there is a C^1 embedding ψ of L into u^*TM. If J is integrable then of course the embedding is holomorphic. We will use $\psi(L)$ instead of $u^*T\Sigma$. Our analysis will be done in the bundles u^*TM, and $T^{0,1}\Sigma \oplus u^*TM$ hence the fact that $\psi(L)$ is only C^1, but not smooth will not be an obstruction. The reduced bundle $T^{0,1}\Sigma \oplus u^*TM/\psi(L)$ will be used only to extend Lemma 4.3.3, where C^1-smootheness is sufficient. Suppose now that D_u is not surjective. Then there is a section η of $T^{0,1}\Sigma \otimes u^*TM$ in the kernel of D_u^*, which is orthogonal (with respect to L^2-inner product) to the image of D_u. Now D_u is the linearisation of the equation $\mathrm{d}u + J \circ \mathrm{d}u \circ j_\tau$ where j_τ is a complex structure on Σ. Since we allow variations of such structures, let $y \in \mathrm{End}_{\mathbb{R}}(T\Sigma)$, $y \circ j_\tau = j_\tau \circ y$. Then $D_u(y) = J \circ \mathrm{d}u \circ y \in \Gamma(T^{0,1}\Sigma \otimes u^*TM)$.

Sections of this form are dense in $\Gamma(T^{0,1}\Sigma \otimes u^*TM)$ (in L^2 topology). It follows that η is pointwise orthogonal to $T^{0,1}\Sigma \otimes T\Sigma_{,}$, and hence to $T^{0,1} \otimes \psi(L)$. Therefore η can be canonically considered a section of $T^{0,1}\Sigma \otimes u^*TM/\psi(L)$. Suppose now that $c_1(u^*TM) - \Sigma_{i=1}^r(k_i - 1) > 0$, so that $c_1(T^{0,1}\Sigma \otimes u^*TM/\psi(L)) > 0$. We will show, as before, that existence of η implies that the Euler class of the quotient bundle $T^{0,1}\Sigma \otimes u^*TM/\psi(L)$ is in fact non-positive. If η has no zeroes then the quotient bundle is trivial with zero Euler class. Therefore assume that η has zeroes. There is only a finite number of them. If w is a zero of η but not a singular point of u then our analysis of the "immersed case" applies and we see that indeed w contributes negatively to the Euler characteristic of $T^{0,1}\Sigma \otimes u^*TM/\psi(L)$. Suppose now that $\eta(z_i) = 0$ for some singular point z_i, say z_1. From the local formula for D_u^* (on the "full" bundle $T^{0,1} \oplus u^*TM$) we see that in local coordinates on M, and Σ :

$$\partial\eta + B(\eta) = 0$$

where B is a C^∞-function with values in the space of real endomorphisms of $\mathbb{C}^2$. It then follows as in Lemma 4.3.3 that:

$$\eta = (\eta_1, \eta_2) = (\overline{z}^l(0, d) + O(z^{l+1}))\overline{\mathrm{d}z}.$$

(Here we are using the similarity principle applied to η thought of as being a section of $T^{0,1} \oplus u^*TM$.) This implies that z_1 is an isolated zero of η. Recall that $\psi(L)$ is locally trivialised by $\frac{1}{kz^{k-1}}\frac{\partial u}{\partial z} = (1,0) + (a,b)\frac{k+1}{k}z + O(z^2)$ so that $T^{0,1}\Sigma \otimes u^*TM/\psi(L)$ is trivialised by $((0,1) + O(z)) \otimes \overline{\mathrm{d}z}$ and the projection of η is of the form $\overline{z}^l d\,[(0,1) + O(z)] \otimes \overline{\mathrm{d}z}$. As in Lemma 4.3.3 this is enough to claim that z_1 contributes negatively to $c_1(T^{0,1}\Sigma \otimes u^*TM)$. We can then claim that the condition $c_1(u^*TM) - \Sigma_{i=1}^{r}(k_i - 1) > 0$ implies genericity of J.

Chapter 5

Non-generic almost-complex structures in symplectic manifolds of dimension four.

In this chapter we study the space of ω-tame almost-complex structures J on a compact symplectic 4-manifold (M, ω) by looking at the associated moduli spaces of J-holomorphic curves. If the almost-complex structure J is sufficiently generic then the space of all J-holomorphic curves of genus g in a homology class $A \in H_2(M, \mathbb{Z})$ is a smooth manifold $\mathcal{M}(J, A, g)$ of dimension equal to the virtual dimension. If $c_1(A) \geq 1$ (where c_1 is the first Chern class of (TM, J)), any j is generic in this sense, provided that we restrict to immersed curves ([Gro85], [HLJ94]) or curves with controlled singularities (i.e such that $c_1(A) > \sum(m_i - 1)$, where the sum is taken over all singular points, and m_i denotes the multiplicity of a singular point, see Chapter 3). In that case, if it also happens that the unparametrized moduli space $\mathcal{M}(J, A, g)$ is compact, all moduli spaces $\mathcal{M}(J, A, g)$ are diffeomorphic (for all J, but keeping g, A fixed). In particular, if for one J the moduli space is non-empty, then it is non-empty for all almost-complex structures.

Here we consider the simplest non-generic case. The condition $c_1(A) \geq 1$ is equivalent to $c_1(N_C) \geq 2g-1$ (here N_C denotes the normal bundle of a curve $C \subset M$), so we see that the simplest case violating that condition is of tori (genus $g = 1$) in a class A with $A \cdot A = 0$. Now, in contrast to the generic case, the moduli spaces $\mathcal{M}(J, A, 1)$ depend a great deal on J. It is not difficult to see, using the adjunction formula ([McD92b], [McD91b]) , that if $\mathcal{M}(J, A, 1)$ contains one embedded torus, then all J-holomorphic tori are embedded. On the other hand, if there is an immersed, or singular torus in $\mathcal{M}(J, A, 1)$ then the virtual dimension of the moduli spaces is at most -2. Hence, in this case, for a generic J, as well as for a generic path J_t of almost-complex structures the moduli spaces are empty (Section 4.4). It is then reasonable to consider only the case of embedded tori, where the virtual dimension of $\mathcal{M}(J, A, 1)$ is zero. However, as we argue in Section 4.1, the actual dimension of the moduli spaces $\mathcal{M}(J, A, 1)$ can be $0, 1$, or 2 but no higher. We then stratify the space of almost-complex structures. We denote by $\mathcal{J}_{i,j}$ the space of all J's such that for all $u \in \mathcal{M}(J, a, 1)$, $\dim \ker(D_u) \leq i$ (with equality for some u), and so that $\dim \mathcal{M}(J, A, 1) = j$. (Here D_u is the Cauchy-Riemann operator cutting out the moduli space of pseudo-holomorphic curves, see Chapter 3) . Our main results are in Section 4.4, where we show that all but two strata have codimension two or higher:

Proposition 5.0.5 *The set $\mathcal{J}_{0,0}$ is open, and dense but with infinitely many connected components. The sets $\mathcal{J}_{2,0}$, $\mathcal{J}_{2,1}$, and $\mathcal{J}_{1,1}$ all have codimension 2 (or higher), while $\mathcal{J}_{1,0}$ has codimension one. Therefore two generic (in $\mathcal{J}_{0,0}$)*

almost complex structures can be connected by a path $\{J_t\} \subset \mathcal{J}_{0,0} \cup \mathcal{J}_{1,0}$.

Here a set $S \subset \mathcal{J}$ *has codimension 2 etc. if a generic one parameter family* $\{J_t\}$ *avoids* S.

We show also that all strata are non-empty, give examples of generic almost complex structures with arbitrarily large number of J-holomorphic tori, as well as examples of generic almost-complex structures which are non-homotopic (through the space of generic structures).

Remark 5.0.6 In this chapter we deal only with pseudo-holomorphic tori. However, all examples, as well as the above proposition are easily extendable (with appropriate changes) to curves of higher genus g. In general, the borderline case, in which the regularity of almost-complex structures is not automatic, is that of embedded curves C in class A with $A \cdot A = 2g - 2$. The Cauchy-Riemann operator D_u on the normal bundle N_C admits at most $4g-2$ linearly independent (over $\mathbb{R}$) sections, and we can stratify $\mathcal{J} = \bigcup_{i,j=0}^{4g-2} \mathcal{J}_{i,j}$. An analogue of Proposition 5.0.5 would say that all strata except $\mathcal{J}_{0,0}$, and $\mathcal{J}_{1,0}$ have codimension two or higher. Any two generic almost-complex structures $J_0, J_1 \in \mathcal{J}_{0,0}$ can be connected by a path $\{J_t\} \subset \mathcal{J}_{0,0} \cup \mathcal{J}_{1,0}$.

The reduction of D_u to the normal bundle N_Σ works if the curve $u : \Sigma \to M$ is non-singular (or if it has mild singularities). It is therefore necessary to show that for a generic almost-complex structure J singular curves form a subset of high codimension. Let $\mathcal{J}$ be the space of all ω-tame almost-complex structures on M, and $\mathcal{M}(\mathcal{J}, A, 1)$ the universal moduli space of all pseudo-holomorphic curves. Furthermore, denote by $\widetilde{\mathcal{M}}(\mathcal{J}, A1)$ the space of all parametrized pseudo-holomorphic tori in M. We need only to quote Proposition 3.3.4:

Proposition 5.0.7 *Let $A \in H(M, \mathbb{Z})$ be an arbitrary homology class that can be represented by curves of genus g. For a generic almost-complex structure J (i.e in a set of second category) the set of singular curves $u \in \mathcal{M}(J, A, g)$ is of real codimension 2 or higher in $\mathcal{M}(J, A, g)$.*

Before proceeding to the next section we recall two lemmas, and a proposition needed later. The following two elementary lemmas deal with Cauchy-Riemann operators acting on the space of complex functions on the standard two dimensional torus T^2. They were proved in Chapter 3, and used to give an example of a 1-dimensional moduli space of pseudo-holomorphic tori. We consider equations of the form: $\overline{\partial}u + \overline{u}\, a d\overline{z} = 0$ where u is a complex valued function $u \in W^{k,2}, k > 2$, and $a d\overline{z}$, $b d\overline{z}$ are smooth forms of type $(0,1)$. We have Lemma 3.1.1:

Lemma 5.0.8 *The equation $\overline{\partial}u + \overline{u}\, a d\overline{z} = 0$ has no solution if $a\overline{z}$ is a non-zero $(0,1)$-form with constant coefficients.*

The second Lemma 3.1.2:

Lemma 5.0.9 *If $c d\overline{z}$ is a non-zero $(0,1)$-form with constant coefficients then*

$$\overline{\partial}u + (\overline{u} - u) c d\overline{z} = 0$$

has exactly one solution (up to multiplication by real constants).

Finally, Proposition 3.1.3:

Proposition 5.0.10 *Let $\overline{\partial} + A$, $A \in \Omega^{0,1}(\mathrm{End}_{\mathbb{R}}(\mathbb{C}))$, be a Cauchy-Riemann operator on $T^2 \times \mathbb{C}$. Then the index of $\overline{\partial} + A$ is 0, while the real dimension of its kernel can be* $0, 1, 2$.

implies that we can expect 2, 1, or 0-dimensional spaces of tori embedded in the manifold $T^2 \times \mathbb{C}$.

5.1 Stratification of almost-complex structures. Examples of almost-complex structures with non-generic properties.

Let (M^4, ω) be a symplectic manifold with an almost-complex structure J. Let $A \in H_2(M, \mathbb{Z})$ be a homology class with $A \cdot A = 0$. The main purpose of this section is to stratify (coarsely) the space $\mathcal{J}$ of ω-tame almost-complex structures according to the dimension of $\mathcal{M}(A, J, g)$ and to the dimension of the kernel of the Cauchy-Riemann operator D_u. To clarify the statements we make a few observations.

(i) If $u : T^2 \to M$ is an immersed (but not an embedded) or a singular J-holomorphic curve then by the adjunction formula ([McD92b], [McD91b]) the first Chern class of the normal bundle N_{Σ} is negative. Hence, the virtual dimension of $\mathcal{M}(A, J, 1) = 2(c_1(N_{\Sigma}) + 1 - g) \leq -2$ is negative, and generically we do not expect to have immersed or singular curves. Moreover, the same remains true of (generic) paths $\{J_t\}$, since the dimension $\dim \bigcup_t \mathcal{M}(J_t, A, 1) = \dim \mathcal{M}(A, J, 1) + 1 < 0$, and so $\bigcup_t \mathcal{M}(J_t, A, 1)$ is empty.

(ii) In view of the above we may assume that A can be represented by embedded tori. In that case it follows from the adjunction formula that all tori in $\mathcal{M}(J, A, 1)$ (for any J) are embedded (observe that $c_1(A) = 0$ if there is an embedded torus in class A, but that $c_1(A) < 0$ if A can be represented by a singular or immersed torus).

(iii) Let now $u : \Sigma \to M$ be an embedded torus in $\mathcal{M}(A, J, 1)$. In Section 3, Proposition 3.1.3 we made an observation to the effect that the dimension of D_u is either $0, 1$, or 2. It then follows that the moduli space $\mathcal{M}(A, J, 1)$ can be zero , one or two dimensional.

We can now stratify $\mathcal{J}$.

Definition 5.1.1 *Let $\mathcal{J}_{i,j}$ denote the space of all ω-tame almost-complex structures J such that:*

(i) *For all $u \in \mathcal{M}(J, A, 1)$ the operator D_u has kernel of real dimension less or equal to i. Moreover, we require that for some u, $\dim \ker(D_u) = i$.*

(ii) *The real dimension of $\mathcal{M}(J, A, 1)$ is less or equal to j. There exist components of the moduli space of dimension j.*

In particular, $\mathcal{J}_{0,0}$ is the set of generic almost-complex structures. For $J \in \mathcal{J}_{0,0}$ the moduli space $\mathcal{M}(J, A, 1)$ consists of a finite number of embedded tori.

Proposition 5.1.2 *The set $\mathcal{J}_{0,0}$ is open, and dense but with infinitely many connected components. The sets $\mathcal{J}_{2,0}$, $\mathcal{J}_{2,1}$, and $\mathcal{J}_{1,1}$ all have codimension 2 (or higher), while $\mathcal{J}_{1,0}$ has codimension one. Therefore, two generic (in $\mathcal{J}_{0,0}$) almost complex structures can be connected by a path $\{J_t\} \subset \mathcal{J}_{0,0} \cup \mathcal{J}_{1,0}$.*

Here a set $S \subset \mathcal{J}$ has codimension 2 etc. if a generic one parameter family $\{J_t\}$ avoids S.

By way of comparison, if $A \cdot A = 1$ (or is greater then one) then any J is regular for all embedded tori. As above, if one torus in $\mathcal{M}$ is embedded then all of them are. If A can be represented by a singular, or immersed torus then the virtual dimension of $\mathcal{M}$ is negative, hence for a generic J the moduli space is empty (same holds for generic paths $\{J_t\}$). We will postpone the proof until the end of the section, and first provide examples illustrating the proposition.

5.2 Non-homotopic regular almost-complex structures.

Here we will construct examples of almost-complex structures on $T^2 \times S^2$ which are regular (for tori in the homology class $A = [T^2 \times \text{pt}]$), but which can not be joined by a path of regular almost-complex structures, i.e they belong to different components of $\mathcal{J}$. In fact, there are infinitely many connected components of regular complex structures on $T^2 \times S^2$. An almost-complex structure J on $T^2 \times S^2$, regular for tori in $[T^2 \times \text{pt}]$ admits a finite number of J-holomorphic tori. If two such structures J_1, and J_2 admit different number of pseudo-holomorphic tori, then they are non-homotopic through regular almost-complex structures. The example of an almost-complex structure on $T^2 \times S^2$ whose description follows is a warm up to a slightly more general almost-complex structures on $T^2\Sigma$, where Σ is an arbitrary Riemann surface.

Given a vector field $\vec{v}$ on S^2, we define an almost-complex structure $J = J_V$ on $T^2 \times S^2$ standard in the direction of S^2 (so that all spheres pt $\times S^2$ are J-holomorphic) but such that no $T^2 \times$ pt is J-holomorphic unless $\vec{v}(\text{pt}) = \vec{0}$.

Definition 5.2.1 Let $z = x_1 + iy_1$ be coordinates on T^2, and ∂_{y_1}, ∂_{x_1} the vector fields spanning $T(T^2)$. Then define

$$J_V(\partial_{x_1}) = \partial_{y_1} + \vec{v}, \text{and} \quad J_V(\partial_{y_1}) \quad = \quad -\partial_{x_1} - J_0\vec{v}$$

If $\vec{w} \in T(S^2)$, put $J_V(\vec{w}) = J_0$. Here J_0 denotes the standard complex structure on S^2, as well as on T^2. In other words

$$J_V = \begin{bmatrix} J_0 & 0 \\ B & J_0 \end{bmatrix},$$

where B (in local coordinates) is a matrix with the first column $\vec{v}$, and the second $J_0(\vec{v})$.

$T^2 \times S^2$ is a symplectic manifold with a standard product symplectic form ω_0. If $\vec{v}$ is sufficiently small (pointwise) then $J_{\vec{v}}$ is ω_0-tame.

In the following lemma $\vec{v}$ will be holomorphic, i.e in a local holomorphic coordinate: $\vec{v} = (h+ig)\partial_z = \frac{1}{2}\left[h\partial_x - g\partial_y + i(h\partial_y + g\partial_x)\right]$, with $h+ig$ a holomorphic function. Under the standard identification of $T_{\mathbb{R}}S^2 \equiv T^{1,0}$ $\vec{v} = \frac{1}{2}(h\partial_x - g\partial_y)$.

Lemma 5.2.2 *Assume that $\vec{v}$ is a holomorphic vector field on S^2. Let j be an almost-complex structure on T^2, and*

$$u : (T^2, j) \to (T^2 \times S^2, J_V), \qquad u = (u_1, u_2)$$

be a $J_{\vec{v}}$-holomorphic torus in the class $[T^2 \times \mathrm{pt}]$. Then u_2 is constant, and $\vec{v}(u_2) = 0$. Also, j is isomorphic to the standard complex structure j_0.

Proof: If, at a point $p \in S^2$, $\vec{v}(p) = 0$, then J_V is equal to the standard product complex structure (j_0, j_1) along the torus $T^2 \times p$, which is then J_V-holomorphic. Since the self-intersection number of $[T^2 \times \mathrm{pt}]$ is zero, and distinct pseudo-holomorphic curves have positive intersections, it follows that all distinct J_V-holomorphic tori in the given class are disjoint. Since the Euler characteristic of S^2 is 2, our vector field $\vec{v}$ vanishes at least once. We can remain in the complement of a zero of $\vec{v}$, and work entirely in $T^2 \times \mathbb{C}$ by introducing a local chart on S^2. Let (z_1, z_2) denote coordinates on $T^2 \times \mathbb{C}$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$. Let $\vec{v} = -h\partial_{x_2} + g\partial_{y_2}$. Then

$$J_V = \begin{bmatrix} J_0 & 0 \\ B & J_0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} -h & g \\ g & h \end{bmatrix}.$$

If $u = (u_1, u_2)$ is a $j - J_V$-holomorphic torus for some complex structure j on T^2 , then $du + J_V \circ du \circ j = (du_1 + J_0 \circ du_1 j,\ du_2 + B \circ du_1 \circ j + J_0 \circ du_2 j) = 0$. Since $du_1 + J_0 \circ du_1 \circ j = 0$ the complex structure j is isomorphic to the standard complex structure J_0, and by composing u with the inverse of u_1 it can be assumed that $u(z, \overline{z}) = (z, u_2(z, \overline{z}))$. The above equation is then equivalent to a system of equations:

$$\partial_x \mathrm{Re} u_2 - \partial_y \mathrm{Im} u_2 = -g(u_2)$$

$$\partial_x \mathrm{Im} u_2 + \partial_y \mathrm{Re} u_2 = -h(u_2)$$

or simply to a single equation:

$$\overline{\partial}_z u_2 = F(u_2)$$

where $F = -(g + ih)$, so that F is holomorphic. To prove that u_2 is constant we need to show that F vanishes on the image of u_2. It is enough to show that F vanishes at a single point p in the image of u_2, for then $T^2 \times \{p\}$ is J-holomorphic and intersects u. By the positivity of intersection of pseudo-holomorphic curves the intersection number $[T^2 \times \{\mathrm{pt}\}]^2$ would be positive which is false unless $F(u_2)$ were identically 0. Suppose, to the contrary, that F does not vanish at any point on the image of u_2. Let G be a function on $T^2 \times \mathbb{C}$ such that $\overline{\partial} G = \frac{1}{F}$, then $\overline{\partial}(G(u)) = 1$. Therefore $G(u) = \overline{z} + \phi(z)$, for some holomorphic function $\phi : T^2 \to \mathbb{C}$. However, no complex valued function on a torus can have this form, hence we reach a contradiction. F vanishes on the image of u_2, and u_2 is constant. □

Next we prove that under mild assumptions on $\vec{v}$ the almost-complex structure J_V is regular. To prove regularity of J_V one only needs to consider the restriction $\overline{\partial}_u$ of D_u to the (canonically topologically trivial) normal bundle of $J_V$holomorphic torus $u : T^2 \to T^2 \times S^2$. We need to prove that the kernel of $\overline{\partial}_u$ is trivial. We use notation as in the proof of the previous lemma. We will work in local holomorphic coordinates which are normal at a fixed point on the sphere S^2. Let $\vec{v} = -h\partial_{x_1} + g\partial_{x_2}$ in those coordinates. We choose the local chart so that $u = (u_1, 0)$.

Lemma 5.2.3 *If, for every $J_{\vec{v}}$-holomorphic torus, the determinant*

$$\det \begin{bmatrix} \partial_{x_2} h & \partial_{y_2} h \\ \partial_{x_2} g & \partial_{y_2} g \end{bmatrix} (0)$$

is not an integer then the almost-complex structure is regular.

Proof: The proof follows from a direct computation of $\overline{\partial}_u$ in local coordinates. Let ∇ be a connection on $\mathbb{C} \times T^2$ preserving J_V, e.g:

$$\nabla = d - \frac{1}{2} J_V d J_V = d - \frac{1}{2} \begin{bmatrix} 0 & 0 \\ J_0 \circ \mathrm{d}B & 0 \end{bmatrix},$$

where d denotes the trivial connection, and $\mathrm{d}B = \begin{bmatrix} -\mathrm{d}h & \mathrm{d}g \\ \mathrm{d}g & \mathrm{d}h \end{bmatrix}$ is a matrix of 1-forms. The bundle normal to u is trivial, hence its section can be thought of as a complex valued function. The operator $\overline{\partial}_u$ becomes a generalized Cauchy-Riemann operator on complex-valued functions on T^2. Let $\eta = s + it$ be a section, then a computation shows that the equation $\overline{\partial}_u \eta = 0$ becomes, in local coordinates:

$$\begin{aligned} \partial_{x_1} s - \partial_{y_1} t + \frac{1}{4} s(\partial_{x_2} h - \partial_{x_2} g) + \frac{1}{4} t(\partial_{y_2} h - \partial_{y_2} g) &= 0 \\ \partial_{x_1} t + \partial_{y_1} s - \frac{1}{4} s(\partial_{x_2} h + \partial_{x_2} g) - \frac{1}{4} t(\partial_{y_2} h + \partial_{y_2} g) &= 0 \end{aligned}$$

This is a system of equations with constant coefficients (recall that u_2 is constant). Such a system has no non-trivial solutions if the determinant of the zero-order terms is not an integer. A computation shows that this determinant is equal to $-\frac{1}{8}\left[(\partial_{x_2} h)(\partial_{y_2} g) - (\partial_{x_2} g)(\partial_{y_2} h)\right]$, hence the lemma follows. □

There is a useful corollary to Lemma 5.2.3.

Corollary 5.2.4 *Suppose that along each J-holomorphic torus $\vec{v} = \operatorname{grad} H$ is the gradient of a function (with respect to the standard metric on the sphere). Then J_V is regular if the determinant of the Hessian of H is not an integer (in particular such an H is a Morse function).*

After this warm-up we will construct more general almost-complex structures (of the same type $J_{\vec{v}}$) on $T^2 \times \Sigma$, where (Σ, J) is a fixed Riemann surface. In the previous example we utilised holomorphic vector fields on S^2. To use the same tool here, we first choose a Morse function H on Σ, and cut the surface into regions containing critical points of H (topologically pairs of pants or discs) and "transitional" regions between the critical points (topologically cylinders). The cutting circles D_l (boundaries of all regions) must lie on the level sets of H. Denote the regions sorrounding critical points by $\{P_j\}_{j=1}^n$, and the transitional cylinders by $\{T_k\}$. Our second step is to construct a vector field $\vec{v}$ on Σ which is holomorphic on P_j's and "spirals" in the cylinders T_k. Also, $\vec{v}$ will be tangent to all circles D_l. There are three cases. Suppose that P_j is a disc. Identify P_j conformally with a disc in the plane $\mathbb{C}$. Define $\vec{v} = X_G$, where $G(z, \overline{z}) = \epsilon_j \cdot |z|^2$. Here ϵ_j is a constant, (positive or negative depending on whether H has a minimum or a maximum in P_j) and X_G denotes the Hamiltonian of G with respect to the standard symplectic structure on $\mathbb{C}$. The point is that since G is harmonic the vector field $\vec{v}$ is holomorphic. In fact, $\vec{v} = -4i\epsilon_j z\partial_z$. Next, suppose that P_j is a pair of pants, conformally equivalent to a disc in $\mathbb{C}$ with two smaller discs removed. For simplicity, we assume that P_j has been identified with $D(0,1) - D(e,r) - D(-e,r)$, where e is real, positive. Here $D(a,r)$ denotes a disc with center at a, and radius r. It will be clear that the assumption is non-essential. Define $G(z, \overline{z}) = \epsilon j \cdot \ln |\frac{z-e}{1-ze}|^2 |\frac{z+e}{1+ze}|^2$, and $\vec{v} = X_G$. Since G is harmonic the vector field $\vec{v}$ is holomorphic. We note in passing that G has a unique critical point at the origin $z = 0$, which will give rise to a $J_{\vec{v}}$ holomorphic vector field. Since G is constant along the outer boundary $\vec{v}$ is tangent to $\partial D(0,1)$. To make it tangent to all components of the boundary we redefine the domain of $\vec{v}$. Instead of circles $D(e,r)$, and $D(-e,r)$ we will remove region $\{z : G(z, \overline{z}) > r'_j\}$. For large r' this region is a disjoint union of two topological discs, so that we still deal with a pair of pants. Finally consider a cylinder T_l. Let $\partial T_l = D_1 \cup D_2$. The vector field $\vec{v}$ is already defined on D_1, and D_2. Extend it to the cylinder T_l in such a way, that if $\vec{v} = a\partial_\theta + b\partial_r$ (in polar coordinates $\{\theta, r\}$) then $b \geq 0$, with equality only along the boundary. Since we want $\vec{v}$ to be smooth, the above construction requires some smoothing out which is better left implicit. There is one important property of $\vec{v}$ which will be needed. It is possible to choose the constants ϵ_j, and r'_j that were used above so that $\vec{v}$ has no closed orbits on circles D_l with rational period. We will assume that such a choice has been made. It is also possible to arrange that that v and H have the same critical points.

Finally we can go back to $T^2 \times \Sigma$. Recall that Σ comes equipped with a complex structure J. Let ω_0 be the standard symplectic structure on T^2, and ω_1 a symplectic form on Σ compatible with J. Then $\omega = \omega_0 \times \omega_1$ is a symplectic form on $T^2 \times \Sigma$.

Lemma 5.2.5 *Choose a Morse function H on Σ and let $\vec{v}$ be a vector field on as described above. Define the almost-complex structure $J_{\vec{v}}$ on $T^2 \times \Sigma$. If $\vec{v}$ is small enough that J_v is ω-tame and regular for all tori in the homology class* $[T^2 \times \text{pt}]$. *Moreover $J_{\vec{v}}$-holomorphic tori in that class correspond precisely to critical points of H (these are the same as critical points of $\vec{v}$).*

Proof: First of all we observe that (as in Lemma 5.2.2) every critical point p of $\vec{v}$ (or H) gives rise to a $J_{\vec{v}}$-holomorphic torus $T^2 \times \text{p}$. It follows from Lemma 5.2.3 that for small $\vec{v}$ all those tori are regular. In fact this is true for generic $\vec{v}$ not necessarily small. Next observe that if $u : T^2 \to \Sigma$ is a $J_{\vec{v}}$-holomorohic torus in class $[T^2 \times \text{pt}]$ then the image of u does not intersect any of the circles D_l. This is because the three dimensional tori $T^2 \times D_l$ are foliated by $J_{\vec{v}}$-cylinders. Specifically, let $\phi_t(\text{q})$ denote the flow of $\vec{v}$, then the cylinder $(x, y, \phi_y(\text{q})$ is $J_{\vec{v}}$-holomorphic. This cylinder does not closes to a torus since the orbit of $\vec{v}$ on D_l has irrational period. The foliation defines a current, hence a homology class, say $[a]$ in $T^2 \times \Sigma$. It easy to see that the intersection number $[T^2 \times \mathrm{pt}] \cdot [a] = 0$ so that by positivity of intersections of pseudo-holomorphic curves the leaves of foliation and the J_{vecv}-holomorphic tori we are looking at do not intersect. Therefore, the image $u(T^2)$ lies either in one of the $P_j's$, or one of the $T_j's$. If it is embedded in a region P_j we can repeat the proof of Lemma 5.2.2 to show that $u(T^2) = T^2 \times \mathrm{p}$, with p a critical point of $\vec{v}$. If, on the other hand, $u(T^2) \subset T_j$ then observe first that it lies away from the boundary ∂T_j. However, on a subcylinder S_j compactly embedded in T_j the vector field $\vec{v}$ is diffeomorphic to a holomorphic vector field $z \cdot \partial_z$ on a cylinder in S^2. Therefore, it can be assumed that once again we are in the situation of the Lemma 5.2.2, and use that proof. That is enough to conclude the proof of the lemma. □

The above lemma allows us to construct almost-complex structures with different numbers of pseudo-holomorphic tori. As we mentioned at the beginning of this section such almost-complex structures are non-homotopic in the space of regular almost-complex structures. The following proposition should now be clear.

Proposition 5.2.6 *Let H_1,and H_2 be two Morse functions on the sphere and $J_{\vec{v_1}}$, and $J_{\vec{v_2}}$ two almost complex structures as defined above. Suppose that H_1, and H_2 have different number of critical points. Then the induced complex structures $J_{\vec{v_1}}$, and $J_{\vec{v_2}}$ on $T^2 \times S^2$ are regular (for small v_1, and v_2) for tori in the homology class $[T^2 \times \mathrm{pt}]$, but cannot be joined by a path of* *regular almost-complex structures.*

Remark 5.2.7 The above example can be modified to an example of almost-complex structures J_1, J_2 which are non-homotopic through regular almost-complex structures, and yet the moduli spaces of J_1, J_2-holomorphic tori have positive dimensions. Consider e.g $X = T^2 \times S^2 \times S^2$, and let A be the homology class $A = [\mathrm{pt} \times T^2 \times \mathrm{pt}] + 2\,[\mathrm{pt} \times \mathrm{pt} \times S^2]$. Let, as before, $\vec{v}$ be a vector field on S^2 and consider almost-complex structure $\widetilde{J_V} = \begin{bmatrix} J_V & 0 \\ 0 & i \end{bmatrix}$ (where J_0 is the standard complex structure on S^2). The expected complex dimension of the moduli space is 4 (since $c_1(A) = 4$). All tori can be found explicitely. Let $u : T^2 \to X$ be a $\widetilde{J_V}$-holomorphic torus, $u = (u_1, u_2, u_3)$. Then (as in Lemma 5.2.2) it can be assumed that $u_1(z) = z$, and that the complex structure on T^2 is the standard one. The third component, u_3 can be thought of as a meromorphic function on the torus with two poles. The proof of Lemma 5.2.2 applies here, and we conclude that the necessary, and sufficient condition for the mapping u to be pseudo-holomorphic is that u_2 be constant, and that $\vec{v}(u_2) = 0$. Finally we notice that the space of meromorphic functions with two poles on T^2 is (complex) four-dimensional (the location of poles provides two complex parameters, the residues gives one more parameter, and the value of the function at a lattice point gives the final parameter). Choosing a small (or generic) $\vec{v}$ related to a Morse function H, as in the aabove construction one can produce regular almost-complex structures with arbitrary (even) number of components of moduli spaces of tori. As before they are not homotopic through regular almost-complex structures.

5.3 Examples of non-regular almost-complex structures.

We now turn to examples of somewhat pathological almost-complex structures which admit 1-parameter families of pseudo-holomorphic tori.

Example 5.3.1 In the first example the ambient space $X = T^2 \times T^2$, the almost-complex structure has the familiar form

$$J = \begin{bmatrix} J_0 & 0 \\ B & J_0 \end{bmatrix},$$

where

$$B = \begin{bmatrix} -h & g \\ g & h \end{bmatrix}.$$

We will be looking for J-holomorphic tori in the class $[T^2 \times \mathrm{pt}]$. Take $g = 0$ and h vanishing on two circles which cut T^2 in two strips. Let $z = x + iy$ be a complex coordinate on the torus. We will assume that $h = h(y)$ is a function of one variable, that it vanishes at two points only, and that these points are the only inflection points. If those assumptions hold, we will show that each of the circles parametrizes a component of the moduli space of pseudo-holomorphic tori, and that all J-holomorphic tori appear in that way. Let $u : T^2 \to X$ be a J-holomorphic torus, $u = (u_1, u_2)$. We need to show that u_2 is constant, and that $h(u_2) = 0$. Notice that u_2 is a map between tori of degree zero, and that it satisfies a system of equations:

$$\begin{aligned} \partial_x \mathrm{Im}\, u_2 &+ \partial_y \mathrm{Re}\, u_2 = -h(\mathrm{Im}\, u_2) \\ \partial_y \mathrm{Im}\, u_2 &- \partial_x \mathrm{Re}\, u_2 = 0 \end{aligned} \qquad (\S3)$$

where $z = x + iy$ is a local coordinate on T^2.

Lemma 5.3.2 *If $u_2 : T^2 \to T^2$, is a degree zero map, which satisfies the system (§3), then u_2 is constant, and h vanishes along u_2.*

Proof: Again, we will use the Hopf's maximum principle. First we notice that $h(u_2)$ has to vanish somewhere. If not, i.e if u_2 stays away from the zero locus of h then we can think of $\mathrm{Im}u_2$ as a (single-valued) function, which satisfies a second order elliptic equation (differentiate both equations in the above system, and add them up):

$$\partial_x{}^2 \mathrm{Im}\, u_2 + \partial_y{}^2 \mathrm{Im}\, u_2 \approx 0 \text{ modulo } \partial_y \mathrm{Im}\, u_2,\ \partial_x \mathrm{Im}\, u_2 \qquad (\S3)$$

It follows from the Hopf's maximum principle that any function satisfying the equation (§3) is constant. Now go back to equations (§3) to see that if $\mathrm{Im}u_2$ is constant, then in fact, $h(u_2) = 0$, and $\mathrm{Re}u_2$ is constant too. □

It follows that u is a pseudo-holomorphic curve if and only if u_2 is constant, and $h(u_2) = 0$. That is enough to prove our claims about the moduli space.

Example 5.3.3 We now give another example of an almost-complex structure admitting a one-parameter family of tori. This time we consider J on $X = T^2 \times S^2$, and require that for every J-holomorphic torus appearing in a 1-dimensional family the kernel of D_u is 1-dimensional (over $\mathbb{R}$). It means that J is as regular as an almost-complex structure admitting a 1-parameter family of curves can be. In the notation of the previous section, we take $J = J_{\mathrm{grad}\, H}$, where H is a Morse function satisfying the condition of Corollary 5.2.4, except for a circle of critical points. For simplicity we will assume that in a local holomorphic chart $T^2 \times \mathbb{C} \subset T^2 \times S^2$ we have $H(x_2, y_2) = c(x_2^2 + y_2^2 - 1)^2$ on an neighbourhood of the unit circle. Lemma 5.2.2 shows that that the moduli space of pseudo-holomorphic tori in the class $[T^2 \times \mathrm{pt}]$ consists of a finite number of isolated tori (which correspond to isolated critical points of h), and a 1-parameter familly of tori parametrized by the unit circle. A computation shows that $\overline{\partial}_u$ acting on a section of the normal bundle of a J-holomorphic torus is given by:

$$\begin{aligned} \overline{\partial}_u \xi &= \overline{\partial} \xi + \frac{1}{2} c(x_2 - i y_2) \left[(x_2 - y_2) - i(x_2 + y_2) \right] \xi \\ &\quad + \frac{1}{2} c(x_2 + i y_2) \left[(x_2 - y_2) - i(x_2 + y_2) \right] \overline{\xi} \end{aligned}$$

where as before the section ξ is thought of as a complex-valued function. If we change variables $\eta = \xi \frac{1}{2} c(x_2 - i y_2) i$, then the equation $\overline{\partial}_u \xi = 0$ is transformed into: $\overline{\partial} \eta + (\eta - \overline{\eta}) a \overline{z} = 0$, where a is a non-zero constant. It is proved in the Lemma 3.1.2 that such an equation has exactly one solution (up to multiplication by real constants).

Example 5.3.4 Next we construct examples of regular almost-complex structures which are locally non-homotopic in the space of regular almost-complex structures. More precisely, let $X = T^2 \times \overline{D}^2$ with its standard symplectic structure, and let $\mathcal{J}_0$ be the set of almost-complex structures on X equal to the standard almost-complex structure $J_0 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ along the central torus $T^2 \times \{(0,0)\}$. Each such structure $J \in \mathcal{J}_0$ admits at least one pseudo-holomorphic curve, namely the central torus. We will show that there are almost-complex structures in $\mathcal{J}_0$ which are regular for the central torus, but which cannot be joined by a path $\{J_t\} \subset \mathcal{J}_0$ of almost-complex structures remaining regular for $T^2 \times \{(0,0)\}$.

Let $J \in \mathcal{J}_0$, and let $\overline{\partial}_j$ be the induced generalized Cauchy-Riemann operator $\overline{\partial}_j$ on the normal bundle of the central torus. The normal bundle is trivial, and the equation $\overline{\partial}_j \xi = 0$, ξ being a section of the normal, is equivalent to the system:

$$\begin{aligned} \partial_x s - \partial_y t + s a_{11} + t a_{12} &= 0 \\ \partial_x t + \partial_y s + s a_{21} + t a_{22} &= 0 \end{aligned}$$

where $\xi = s + it$. Here $a_{11}, ..., a_{22}$ are real valued functions, which will be be referred to as the zero-order coefficients of $\overline{\partial}_j$. We will denote the determinant $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ by $\det(J)$.

Proposition 5.3.5 *Let $J_1, J_2 \in \mathcal{J}_0$ be two almost-complex structures such that the operators $\overline{\partial}_{j_1}$, and $\overline{\partial}_{j_2}$ have constant coefficients. Suppose that* $\det(J_1) < 0$, *and* $0 < \det(J_1)$. *Then both almost-complex structures are regular, but they are non-homotopic in $\mathcal{J}_0$.*

Proof: The proof follows immediately from the following lemma.

Lemma 5.3.6 *Let s, and t be real valued functions on T^2. The system of equations with constant coefficients:*

$$\begin{aligned} \partial_x s - \partial_y t + s a_{11} + t a_{12} &= 0 \\ \partial_x t + \partial_y s + s a_{21} + t a_{22} &= 0 \end{aligned}$$

has a non-trivial solution if and only if there are integers n, m such that:

$$\begin{aligned} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= n^2 + m^2 \\ n(a_{22} + a_{11}) + m(a_{21} - a_{12}) &= 0. \end{aligned}$$

Proof: To prove the lemma expand both functions in their Fourier series. □

□

We will now return to Proposition 5.1.2, and supply its proof.

Proof of Proposition 5.1.2: Assume that A is a homology class with $A \cdot A = 0$, which can be represented by embedded pseudo-holomorphic tori. Let J be an almost-complex structure, and C_0 a J-holomorphic torus (the image of $u : T^2 \to M$). C_0 is embedded, and there is a convenient model for the space of pseudo-holomorphic tori which are C^1-close to C_0 (see Gromov [Gro85],and for a detailed presentation, which we will follow here, Hofer-Lizan-Sikorav [HLJ94]). We recall the description from [HLJ94] in a set-up adapted to our needs. Let N denote the normal bundle to C_0, and D_u the Cauchy-Riemann operator on N (section 2.1). If D_u has a non-trivial kernel then choose a pseudo-holomorphic section η. Since N is topologically trivial, η never vanishes (each zero would contribute positively to the Euler characteristic of N) and we trivialize $N = T^2 \times \mathbb{C}$ so that η becomes the constant section 1. If the kernel of D_u is trivial then trivialize N by any section of norm one (recall that TM, hence N, is equipped with a metric). Let ∇ denote the trivial connection on $T^2 \times \mathbb{C}$. ∇ induces a splitting $TN = TC_0 \oplus N$, and we define a canonical almost-complex structure $J_0(p)$:

$$J_0(p) = \begin{pmatrix} j(x) & 0 \\ 0 & i \end{pmatrix}.$$

Here x is the point in the base underlying p, and j is the complex structure on $C_0 = T^2$ induced by J. In general J_0 is equal to J only along C_0, but there is an endomorphism Φ of TN such that: $J(p) = \Phi(p)^{-1} J_0(p) \Phi(p)$. Φ breaks into blocks:

$$\Phi(p) = \begin{pmatrix} \alpha(p) & \beta(p) \\ \gamma(p) & \delta(p) \end{pmatrix}$$

where:

$$\alpha(p) \in \mathrm{End}_{\mathbb{R}}(TC_0), \qquad \beta(p) \in \mathrm{Hom}_{\mathbb{R}}(N, TC_0)$$

$$\gamma(p) \in \mathrm{Hom}_{\mathbb{R}}(TC_0, N), \qquad \delta(p) \in \mathrm{End}_{\mathbb{R}}(N).$$

A surface C close to C_0 is a graph of a section $\phi \in \Gamma(N)$. C is J-holomorphic if and only if ϕ satisfies a non-linear equation: $\partial_\nu(\phi) = 0$. Here ∂_ν is a first-order non-linear operator, which takes sections of N to $\Omega^{0,1}(N)$. ∂_ν is defined on a neighbourhood U of the zero section by:

$$\partial_\nu = P(\phi) \circ j - i \circ P(\phi)$$

where:

$$P(\phi) = (\gamma(\phi) + \delta(\phi)\nabla_\phi) \cdot (\alpha(\phi) + \beta(\phi)\nabla_\phi)^{-1}$$

so that $P(\phi) : TC_0 \to N$ is an endomorphism. Finally, we have a hands-on formula for $D_u : \Gamma(N) \to \Omega^{0,1}(N)$, obtained by linearisation of ∂_ν at the zero section. It is given by $D_u = \overline{\partial} + a$, where:

$$a(\psi) = (\mathrm{d}\gamma(0)) \cdot \psi \circ j - i \circ (\mathrm{d}\gamma(0) \cdot \psi), \qquad a \in \Omega^{0,1}(N)$$

where $\overline{\partial}$ is the standard Cauchy-Riemann operator on $T^2 \times \mathbb{C}$. Note that $\overline{\partial}$ depends on the complex structure j on $T^2 = C_0$.

The proposition will follow from the following elementary lemma which shows that curves $u : T^2 \to M$ such that D_u has two dimensional kernel can be always avoided. Denote by $\mathcal{CR}$ the space of generalized operators on the trivial bundle $T^2 \times \mathbb{C}$ of the form $\overline{\partial}_j + a$. Here $\overline{\partial}_j$ depends on a complex structure j on T^2, so that there is a fibration $\mathcal{CR} \to \mathcal{T}_1$. The above construction gives us a mapping from the universal moduli space $\mathcal{M}(\mathcal{J}, A, 1)$ to the space of generalized Cauchy-Riemann operators $\mathcal{CR}$. More precisely, since the construction is local we can cover $\mathcal{M}(\mathcal{J}, A, 1)$ with a countable family of sets $\{U_\alpha\}$ so that there is a smooth mapping $\Psi : U_\alpha \to \mathcal{CR}, \quad u \to D_u = \overline{\partial}_j + a$. We choose the sets so that their subsets $V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$ cover $\mathcal{M}(\mathcal{J}, A, 1)$. Abusing notation however, we will write $\Psi : \mathcal{M}(\mathcal{J}, A, 1) \to \mathcal{CR}, \quad u \to D_u = \overline{\partial}_j + a$. Similarly, for a given $(J, u) \in \mathcal{M}(\mathcal{J}, A, 1)$ let $\mathcal{J}_0$ denote the set of all almost-complex structures which agree with J along $u(T^2)$. By restriction, we have a smooth mapping: $\Psi : \mathcal{M}(\mathcal{J}_0, A, 1) \to \mathcal{CR}$. Denote by $\mathcal{CR}(2)$ the space of operators with two dimensional kernel, and by $\mathcal{CR}(1)$ operators whose kernel is one dimensional.

Lemma 5.3.7 *$\mathcal{CR}(2)$ has codimension 4, and $\mathcal{CR}(1)$ codimension 1 in $\mathcal{CR}$. For every U_α the mapping Ψ is transversal to both subspaces.*

Proof: We will describe a local model for $\mathcal{CR}(2)$ near an operator with two dimensional kernel. To this end we fix a complex structure j on T^2, and use $\overline{\partial}$ in place of $\overline{\partial}_j$. This is sufficient for our purposes because of the fibration $\mathcal{CR} \to \mathcal{T}_1$. Let $\xi \to \overline{\partial}\xi + a_0\xi + b_0\overline{\xi}$ be such an operator, and let $\xi \to \overline{\partial}\xi + a\xi + b\overline{\xi}$ be its perturbation. Here $a = a_0 + A(\lambda) \in \Omega^{0,1}$, and similarily $b = b_0 + B(\lambda) \in \Omega^{0,1}$, $\lambda \in \Lambda$ which is a finite space of parameters. Let η_1, η_2 span the kernel of the unperturbed operator, and let v, and w be two orthonormal $(0,1)$ forms spanning the orthogonal complement of the image of $\overline{\partial}u + a_0 u + b_0 \overline{u}$. We'll use the Lyapunov-Schmidt reduction to find out about the space of solutions of $\overline{\partial}u + a\,u + b\,\overline{u} = 0$. There is a family of functions $\eta = \eta(x, y, \lambda)$ depending on the parameter λ and two real variables (x, y) such that

$$\overline{\partial}_{a_0,b_0}(x\eta_1 + y\eta_2 + \eta) + A(x\eta_1 + y\eta_2 + \eta) + B(\overline{x\eta_1 + y\eta_2 + \eta}) = 0$$

Modulo the image of $\overline{\partial}_{a_0,b_0}$ with $\eta(x, y, 0) = 0$, $\eta(0, 0, \lambda) = 0$. Such a family $\eta(x, y, \lambda)$ is very special; if $\xi_1 = \xi_1(\lambda)$, and $\xi_2 = \xi_2(\lambda)$ are two families of functions on T^2 such that

$$\overline{\partial}_{a_0,b_0}\xi_1 + A(\eta_1 + \xi_1) + B(\overline{\eta_1 + \xi_1}) = 0 \quad \text{modulo the Image of } \overline{\partial}_{a_0,b_0}$$

and

$$\overline{\partial}_{a_0,b_0}\xi_2 + A(\eta_2 + \xi_2) + B(\overline{\eta_2 + \xi_2}) = 0 \quad \text{modulo the Image of } \overline{\partial}_{a_0,b_0}$$

then $\eta(x, y, \lambda) = x\xi_1(\lambda) + y\xi_2(\lambda)$. Let now $f_1(\lambda), g_1(\lambda), f_2(\lambda), g_2(\lambda)$ be functions such that:

$$\begin{aligned}
\overline{\partial}_{a_0,b_0}\xi_1 + A(\eta_1 + \xi_1) + B(\overline{\eta_1 + \xi_1}) &= f_1 v + g_1 w \\
\overline{\partial}_{a_0,b_0}\xi_2 + A(\eta_2 + \xi_2) + B(\overline{\eta_2 + \xi_2}) &= f_2 v + g_2 w
\end{aligned}$$

A parameter λ represents an operator with non-trivial kernel if for some $(x, y) \neq 0$:

$$\begin{aligned}
x f_1 + y f_2 &= 0 \\
x g_1 + y g_2 &= 0
\end{aligned}$$

i.e if the matrix $\begin{bmatrix} f_1(\lambda) & f_2(\lambda) \\ g_1(\lambda) & g_2(\lambda) \end{bmatrix}$ has rank 1 or 0. If the rank of the matrix is one then the corresponding Cauchy-Riemann operator $\overline{\partial}_{a(\lambda),b(\lambda)}$ has one dimensional kernel, and if the rank is zero the kernel is two dimensional. The space of 2×2 matrices is four-dimensional; the subset of matrices of rank one is a cone over a two dimensional torus, therefore of dimension three. The vertex of this cone is the zero matrix, the only 2×2 matrix of rank zero. (To see this notice that the equation

$$\det \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}x_{22} - x_{12}x_{21} = 0$$

can be transformed into $|z_1|^2 = |z_2|^2$ in a suitable system of complex coordinates (z_1, z_2). The set of points satisfying this equation is a cone over a torus $|z_1|^2 = 1, |z_2|^2 = 1$.) Suppose that Λ is a k-dimensional perturbation, i.e $A(\lambda) = A(\lambda_1, \lambda_2, ..., \lambda_k)$, $B(\lambda) = B(\lambda_1, \lambda_2, ..., \lambda_k)$, and choose real constants $\alpha_j^1, \alpha_j^2, \beta_j^1, \beta_j^2$ so that

$$\begin{aligned} \frac{\partial}{\partial \lambda_j}(A)\eta_1 + \frac{\partial}{\partial \lambda_j}(B)\overline{\eta_1} &= \alpha_j^1 v_1 + \alpha_j^2 v_2 \\ \frac{\partial}{\partial \lambda_j}(B)\eta_2 + \frac{\partial}{\partial \lambda_j}(B)\overline{\eta_2} &= \beta_j^1 v_1 + \beta_j^2 v_2 \end{aligned}$$

modulo the image of $\overline{\partial}_{a_0 b_0}$. $\begin{bmatrix} f_1 & f_2 \\ g_1 & g_2 \end{bmatrix}$ is equal to $\begin{bmatrix} \Sigma\alpha_j^1\lambda_j & \Sigma\beta_j^1\lambda_j \\ \Sigma\alpha_j^2\lambda_j & \Sigma\beta_j^2\lambda_j \end{bmatrix}$ modulo terms of second order $O(|\lambda|^2)$. Now this matrix is a linear combination of matrices $\begin{bmatrix} \alpha_j^1 & \beta_j^1 \\ \alpha_j^2 & \beta_j^2 \end{bmatrix}$ If $k \geq 4$, and any four of the above matrices are in general position then the set of parameters λ corresponding to operators with two dimensional kernel forms a codimension 2 submanifold. The set of λ giving operators with one dimensional kernel is a cone of codimension one.

To complete the proof of the lemma we only need to observe that the mapping Ψ can yield such generic perturbations of D_u. Any k-parameter family of almost-complex structures $\lambda \to J(\lambda) = \Phi^{-1}J_0\Phi$ gives rise to a k-parameter family $\lambda \to D_u = \overline{\partial} + a(\lambda)$. Here $\Phi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, and γ is arbitrary as long as it vanishes along C_0. Recall now the definition of $D_u\xi = \overline{\partial}\xi + (\mathrm{d}\gamma(0)) \cdot \psi \circ j - i \circ (\mathrm{d}\gamma(0) \cdot \xi)$, and of the entries α_j^i, β_j^i of the above matrices:

$$\begin{aligned} \alpha_j^i &= \int_{T^2} (\frac{\partial}{\partial\lambda_j}(a) \cdot \eta_1) \cdot \overline{v}_i \\ \beta_j^i &= \int_{T^2} (\frac{\partial}{\partial\lambda_j}(a) \cdot \eta_2) \cdot \overline{v}_i. \end{aligned}$$

Since γ is arbitrary it can be seen that an appropriate choice would yield matrices $\begin{bmatrix} \alpha_j^1 & \beta_j^1 \\ \alpha_j^2 & \beta_j^2 \end{bmatrix}$ in general position.

That is enough to prove the lemma for the case of $\mathcal{CR}(2)$. The case of $\mathcal{CR}(1)$ is analogous. □

Corollary 5.3.8 *It follows from the previous lemma by standard arguments of perturbation theory that:*

(i) *A generic path $\{J_t\}$ does not meet $\mathcal{J}_{2,0}$, $\mathcal{J}_{2,1}$, and $\mathcal{J}_{2,2}$, i.e those strata have codimension 4 (or higher).*

(ii) *For a generic path $\{J_t\}$ the restriction $\Psi : \bigcup_t \mathcal{M}(J_t, A, 1) \to \mathcal{CR}$ is transversal to $\mathcal{CR}(1)$. The set of pairs (J, u) for which D_u has one dimensional kernel is discrete in the moduli space $\bigcup_t \mathcal{M}(J_t, A, 1)$.*

(iii) *For a generic path $\{J_t\}$ the space $\bigcup_t \mathcal{M}(J_t, A, 1)$ is a 1-dimensional manifold. Therefore, it follows from (ii) above, that $\{J_t\}$ avoids $\mathcal{J}_{1,1}$.*

To complete the proof of our theorem we need only to prove that all strata are non-empty, and that the stratum $\mathcal{J}_{0,0}$ has infinitely many components. The first statement is shown in examples in Sections 4.1 and 4.2. To see the latter we find regular almost-complex structures with a different number of pseudo-holomorphic tori in class A. As observed in Section 4.1, such almost-complex structures can not homotopic through regular structures. Let C_0 be a fixed embedded symplectic torus. A small neighborhood U of C_0 is symplectomorphic to $T^2 \times \mathbb{C}$ with the standard product symplectic structure. Let (z_1, z_2) denote the complex coordinates on the product, and J_0 be the standard complex structure. All tori $T^2 \times \{\mathrm{pt}\}$ are holomorphic. Choose a finite number of them, $T^2 \times \{\mathrm{pt}_k\}$ $(k = 1, \cdots, N)$ and perturb J_0 to a regular J but so that all of the tori remain J-holomorphic. Each perturbation can be written as $\Phi^{-1} J_0 \Phi$. To obtain regular J we choose $\Phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ so that $\gamma(z_1, \overline{z}_1, z_2, \overline{z}_2) = \gamma(\overline{z}_2)$, and so that $\frac{\partial}{\partial \overline{z}_2} \gamma$ is constant, and non-zero. It follows from the explicit form of D_u (see above), and Lemma 3.1.1 that such J is regular along the tori $T^2 \times \mathrm{pt}_k$. Now perturb it further to obtain a regular J. It is clear that one obtains J's with arbitrarily many pseudo-holomorphic tori.

□

Chapter 6

An example of a moduli space of pseudo-holomorphic curves with orientation reversing evaluation mapping.

We give an elementary construction of a generic almost-complex structure J on a four-dimensional manifold with a rather unpleasant property. An evaluation mapping defined on an moduli space of J-holomorphic curves is not orientation preserving. This happens for all values of the first Chern class for which the moduli space is generically of positive dimension. Such behavior is in contrast to the case of integrable complex structures where evaluation maps are holomorphic, hence preserve orientation.

We will now sketch both constructions, and introduce the main "tools" of the proof i.e homogenous almost-complex structures and generalized Cauchy-Riemann operators. For the most part we work in a local model of an embedded pseudo-holomorphic curve, i.e we specialize to the case where $M = E(L)$, the total space of a complex line bundle L on a surface Σ, and B is the homology class of the zero section of L. This is because the property of pseudo-holomorphic curves we want to establish is local and it is easy to prove our main theorems once we carry out local constructions. This reduction allows us to introduce homogenous almost-complex structures on $E(L)$ i.e the ones which are invariant under multiplication by real scalars in the fibers of L (section 2). The advantage in using a homogenous almost-complex structure J (rather than an arbitrary almost-complex structure) is that the moduli space $\mathcal{M}$ of J-holomorphic curves becomes simply a linear subspace of the space of sections of L. There are two consequences of this fact. The first is that every J-holomorphic curve has a canonical parametrization. As we shall see in section 5, this considerably simplifies the evaluation map ev. The second is that we can use effectively our main tool i.e $\mathbb{R}$-linear generalized Cauchy-Riemann operators (section 3). More precisely, $\mathcal{M}$ is equal to its own tangent space $T_{\sigma_0}\mathcal{M}$ at the zero section σ_0 . This is a finite dimensional subspace of the space of sections of L, and the kernel of a generalized Cauchy-Riemann operator $\overline{\partial}_\nu$ defined on L. The construction of the operator $\overline{\partial}_\nu$ (which, in general, is non-linear) due to Gromov [Gro85] and Hofer et al.[HLJ94] is presented in section 4. For a homogenous almost-complex structure J the operator $\overline{\partial}_\nu$ turns out to be linear (over $\mathbb{R}$), and the differential of ev (at the zero section σ_0) splits into a complex-linear part and $\overline{\partial}_\nu$. Our construction is based on a simple observation that $\overline{\partial}_\nu$ is $\mathbb{R}$-linear but not $\mathbb{C}$-linear, and that for such operators, on a Riemann surface of genus $g \geq 1$, the Carleman Similarity Principle does not hold. (It does hold on S^2 which is the reason why pseudo-holomorphic spheres behave quite like holomorphic spheres.) We show (Proposition 6.3.1) that ev will not preserve orientation if $\overline{\partial}_\nu$ has the following property. There are N ($2N = \dim_{\mathbb{R}} \mathcal{M}$) points $z_1, ..., z_N$, a section $\sigma \in \ker(\overline{\partial}_\nu)$ with simple zeros at $z_1, ..., z_N$ such that σ is the only (up to multiplication by real scalars) section vanishing at $z_1, ..., z_N$. We give elementary constructions (sections 6 and 7) of almost-complex structures J together with a section σ with just such property.

Remark 6.0.9 As we mentioned above, regularity of the constructed almost structures is achieved by stipulating $c_1(B) \geq 1$. We will outline an example of a moduli space of tori in a class B where $c_1(B) = 0$ for which the evaluation map ev does not preserve orientation. Let $M = \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P^2}$ be $\mathbb{C}P^2$ blown up at nine points (with the standard integrable complex structure J extended from $\mathbb{C}P^2$), and B be the homology class of the proper transform of a torus passing through all points. Then $c_1(B) = 0$, and the moduli space $\mathcal{M}(J, B, 1)$ of pseudo-holomorphic tori in M is generically zero-dimensional and finite. Recall that moduli spaces $\mathcal{M}$ are canonically orientable, and in the current case of $\mathcal{M}(J, B, 1)$ the number of tori counted with signs is one because there is only one torus passing through nine generic points. Now take the nine points to lie on the intersection of two different tori. Then there are (at least) two J-holomorphic tori in the blow up (in fact the blow up is fibered by tori), and one can perturb J to a regular almost-complex structure J' for which these two tori are still pseudo-holomorphic. Since the algebraic number of tori is one, there are tori in $\mathcal{M}(J', B, 1)$ counted with negative signs. It can be seen that the evaluation mapping ceases to preserve orientation at these tori. For details see [MS94].

The following notation will be used:

- Σ will denote a Riemannian surface of genus g, $1 \leq g$,

- $\pi : L \to \Sigma$ a complex line bundle over Σ,

- $E = E(L)$ the total space of $L \to \Sigma$,

- 0_Σ or σ_0 the zero section of L.

We will not keep track of the ever changing complex structure on Σ. Unless stated otherwise we will work in the smooth (C^∞) category.

6.1 Homogenous almost-complex structures

It will be convenient to make use of an almost-complex structure J which is invariant under multiplication by scalars in L. Let $\mu_s : E \to E$ denote multiplication by a real number s in fibers of L.

Definition 6.1.1 An almost-complex structure J on E preserving the zero section 0_Σ and equal to multiplication by $\sqrt{-1}$ in the fibers will be called homogenous if

$$\mu_s^* J = J \text{ for every } s \in R$$

where $\mu_s^* J$ denotes the pull-back of J by the diffeomorphism μ_s.

Remark 6.1.2 The tangent space TE restricted to the zero section 0_Σ splits into $T\Sigma \oplus L$. J preserves both factors and in particular defines a complex structure j on Σ. We will always assume that the base Σ is equipped with this induced complex structure j. It is also worth observing that a homogenous almost-complex structure J tames a symplectic form ω (which depends on J) on a neighborhood of the zero section of L. The complex structure j on Σ tames a form ω_0 (on Σ). On the other hand by choosing a Hermitian metric h and a Hermitian connection ∇ we can introduce a holomorphic structure on L so that $E(L)$ becomes a complex manifold. Let $t(v) = h(v,v)\ \ v \in L$. Then $\omega_1 = \sqrt{-1}\partial\bar{\partial}t$ is a closed form on E tamed by J in the fibers. For k large enough $\omega = k\pi^*\omega_0 + \omega_1$ is a symplectic form tamed by J at least on a small neighborhood of the zero section.

We will now prove that homogenous almost-complex structures are fairly abundant.

Lemma 6.1.3 *Let J' be an arbitrary almost-complex structure on E preserving the zero section and fibers. Then*

$$\lim_{s \to 0} \mu_s^* J'$$

exists and defines a homogenous almost-complex structure.

Proof: Convergence of the limit follows easily from a computation in a local trivialisation $\psi : U \times \mathbb{C} \to L$. We will denote the tangent bundle to the first factor by H (for "horizontal"), and to the second by V (for "vertical"). We write

$$J' = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}$$

where $\alpha \in \mathrm{End}_{\mathbb{R}}(H)$, $\beta \in \mathrm{End}_{\mathbb{R}}(H, V)$, and $\gamma \in \mathrm{End}_{\mathbb{R}}(V)$. Let (z_1, z_2) be the co-ordinates on $U \times \mathbb{C}$. Note that along the zero section $\beta = 0$, i.e $\beta(z_1, z_2) = O(z_2)$. Then for a horizontal vector h at a point (z_1, z_2):

$$\mu_s^* J'(z_1, z_2)(h) = (\alpha(z_1, sz_2)(h),\ \frac{1}{s}\beta(z_1, sz_2)(h))$$

and the right-hand side converges to

$$(\alpha(z_1, 0)h, z_2 \frac{\partial \beta}{\partial z_2}(z_1, 0)h + \overline{z}_2 \frac{\partial \beta}{\partial \overline{z}_2}(z_1, 0)h).$$

For a vertical vector σ tangent at the point (z_1, z_2):

$$\mu_s^* J'(z_1, z_2)(v) = (0, \frac{1}{s}\gamma(z_1, sz_2)(sv))$$

which converges in turn to $(0, \gamma(z_1, 0)(v))$. That is enough to establish existence of J. □

In sections 3 and 6 we will need an explicit construction of homogenous almost-complex structures. Recall that L is a complex line bundle. We fix a Hermitian connection ∇ and define a holomorphic structure on L via ∇. Then the total space $E(L)$ becomes a complex manifold whose complex structure will be denoted by $J' = J'(\nabla)$. It follows from the very definition of $J' = J'(\nabla)$ that this complex structure is homogenous. The tangent bundle TE splits into horizontal and vertical subbundles $TE = H \oplus V$ both preserved by J'. We have immediate isomorphisms $H \equiv \pi^* T\Sigma$ and $V \equiv \pi^* L$. For any $\Phi \in \mathrm{End}_{\mathbb{R}}(TE)$ we can define an almost-complex structure J by:

$$J = \Phi^{-1} \circ J' \circ \Phi.$$

Φ will be called homogenous if $\Phi \circ \mu_s^* = \mu_s^* \circ \Phi$.

Lemma 6.1.4 *The almost-complex structure J is homogenous if and only if there exists a Hermitian connection ∇ and a homogenous endomorphism Φ such that $J = \Phi^{-1} \circ J'(\nabla) \circ \Phi$. Moreover one can arrange that Φ have the form:*

$$\Phi = \begin{pmatrix} id & 0 \\ \gamma & id \end{pmatrix}$$

where $\gamma \in \mathrm{End}_{\mathbb{R}}(H, V)$ is homogenous i.e $\gamma \circ \mu_s^* = \mu_s^* \circ \gamma$ for all $s \in \mathbb{R}$.

Proof: If J' and Φ are homogenous then so is J. To prove the other direction we simply note again that J induces a complex structure j on σ and that by introducing a connection ∇ on L we get an auxiliary complex structure $J' = J'(\nabla)$ on E. Since $J = J'$ along the zero section we can write $J = \Phi^{-1} \circ J' \circ \Phi$ for some Φ. $\Phi = \mathrm{Id}$ along the zero section. Now write

$$\Phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Here $\alpha \in \mathrm{End}_{\mathbb{R}}(H), \gamma \in \mathrm{End}_{\mathbb{R}}(H, V), \beta \in \mathrm{End}_{R}(V, H), \delta \in \mathrm{End}_{\mathbb{R}}(V)$. Along the zero section $\alpha = \mathrm{id}$, $\delta = \mathrm{id}$, $\beta = 0$, $\gamma = 0$. As in the previous lemma this is enough to show that the limit $\lim_{s\to 0} \mu_s^* \Phi$ exists and defines a homogenous endomorphism of TE. Since J is homogenous we have:

$$\begin{aligned} J = \lim_{s\to 0} \mu_s^* J &= \lim_{s\to 0} \left((\mu_s^*\Phi)^{-1} \circ J' \circ (\mu_s^*\Phi) \right) \\ &= (\lim_{s\to 0} \mu_s^*\Phi)^{-1} \circ J' \circ (\lim_{s\to 0} \mu_s^*\Phi) \end{aligned}$$

and we can replace our original choice of Φ by the homogenous $\Phi' = \lim_{s\to 0} \mu_s^* \Phi$. That is enough to establish the first part of the lemma.

If Φ is already homogenous and $\Phi = \mathrm{Id}$ along the zero section then it has a special form:

$$\Phi = \begin{pmatrix} \alpha & 0 \\ \gamma & \mathrm{id} \end{pmatrix}$$

and all entries of this matrix are homogenous under multiplication by real scalars. Upon making the identifications $H \equiv \pi^* T\Sigma$ and $V \equiv \pi^* L$, the homogeneity translates into the statement $\alpha(sp) = \alpha$, for all s, where $p \in L_z$. Hence $\alpha(p) = \alpha(z) = \mathrm{id}$. Note also that $\gamma(sp) = s\gamma(p)$ which is exactly what we wanted. □

Remark 6.1.5 A local version of this computation will be used to construct almost-complex structure J in section 6. Fix a local trivialisation $\psi : U \times \mathbb{C} \to L$ of L. Then take ∇ to be the trivial connection defined over U. The subbundles H, and V of TE are just the subbundles tangent to the factors of $\mathrm{U} \times \mathbb{C}$ and are trivial bundles with fibre $\mathbb{C}$. Let now $p = (x + iy)$ be a point in the fibre, define $\gamma(x+iy) = x\gamma_1 + y\gamma_2$, where γ_1, γ_2 are functions supported compactly in U and with values in $\mathrm{End}_{\mathbb{R}}(\mathbb{C})$. Finally let $\Phi = \begin{pmatrix} \mathrm{id} & 0 \\ \gamma & \mathrm{id} \end{pmatrix}$. Then $J = \Phi^{-1} \circ J' \circ \Phi$ is a well defined homogenous almost-complex structure.

6.2 Moduli spaces of pseudo-holomorphic curves.

In this section we describe the space $\mathcal{M}$ of pseudo-holomorphic curves for a homogenous almost-complex structure on the total space E of a line bundle L.

First, we define parametrized pseudo-holomorphic curves. Let (M, J) be a smooth manifold with a smooth almost-complex structure J.

Definition 6.2.1 A parametrized J-holomorphic curve in M in a homology class $B \in H_2(M, \mathbb{Z})$ is a mapping $u : \Sigma \to M$ such that

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u$$

for a complex structure j on Σ. We allow j to vary.

An unparametrized curve may be considered to be either the image $u(\Sigma)$ of a parametrized curve or an equivalence class of J-holomorphic maps u, and one could make corresponding definitions of the moduli space $\mathcal{M}$ of unparametrized curves (see D. McDuff, D. Salamon [MS94]). The case at hand, however, is simpler because all J-holomorphic curves are graphs of sections of L and hence have a distinguished parametrization. Therefore, we have the following characterisation of the moduli space $\mathcal{M}$.

Lemma 6.2.2 *Let J be a smooth (C^∞) almost-complex structure on $E(L)$ which preserves fibers and the zero section of L. Then the moduli space $\mathcal{M}$ of unparametrized J-holomorphic curves in the homology class $\left[0_\Sigma\right]$ is diffeomorphic to the space of C^∞-sections $\sigma \in \Gamma(L)$ such that the tangent space of the graph of σ is invariant under J.*

Proof: Let $u : \Sigma \to E$ be a J-holomorphic section in the class of the zero section. Since J is smooth it follows from elliptic regularity ([MS94]) that u is a smooth mapping. Furthermore, J preserves fibers (as it happens in the case of homogenous almost-complex structures) i.e fibers become smooth pseudo-holomorphic discs. It then follows from the intersection theory of pseudo-holomorphic curves that every pseudo-holomorphic u intersects every fiber only once, and the intersection is transversal i.e the intersection number is 1. (see Theorem 1.1 [McD91a]). In particular u can not be singular, since a singular point of u would contribute at least 2 to the intersection number of u and a fiber through that point. Hence the image $u(\Sigma)$ is a graph of a section of L. This establishes the isomorphism between the spaces. □

It turns out that for a homogenous J the moduli space is simply a linear space. In fact, when J is homogenous, it is the kernel of a generalized Cauchy-Riemann operator on L.

Proposition 6.2.3 *The moduli space $\mathcal{M}$ of J-holomorphic curves of class $[0_\Sigma]$ is canonically isomorphic to the kernel of a generalized Cauchy-Riemann operator $\overline{\partial}_\nu$ on L.*

Proof: First, following Gromov [Gro85] and Hofer, Lizan, Sikorav [HLJ94] we construct an operator $\overline{\partial}_\nu$ on L:

$$\overline{\partial}_\nu : \Gamma(L) \to \Gamma(\Lambda^{0,1} \otimes L)$$

whose kernel is the space of J-holomorphic curves in E for an arbitrary almost-complex structure J which preserves the zero section. In general $\overline{\partial}_\nu$

is non-linear. We will see, however, that if the almost-complex structure J is homogeneous the operator is $\mathbb{R}$-linear, and is a generalized Cauchy-Riemann operator as defined in the previous section. The result follows.

Let J be an almost-complex structure on $E(L)$ preserving the zero section, and j denote the induced complex structure on Σ (i.e J restricted to the zero section). As before we pick a Hermitian connection ∇ on L and split the tangent bundle to E into horizontal and vertical spaces, $TE = H \oplus V$. We have $H \equiv \pi^* T\Sigma$, and $V \equiv \pi^* L$. The connection ∇ defines a holomorphic structure on L , and the corresponding complex structure J_0 on TE has the form:

$$J_0 = \begin{pmatrix} j & 0 \\ 0 & i \end{pmatrix}$$

where i denotes the multiplication by $\sqrt{-1}$ pulled-back to V, and j is the complex structure on $T\Sigma$ pulled-back to H. Then for a point p close to the zero section

$$J(p) = \Phi^{-1}(p) J_0 \Phi(p)$$

where $\Phi \in \mathrm{End}_{\mathbb{R}}(TE)$. As before

$$\Phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Here $\alpha \in \mathrm{End}_{\mathbb{R}}(H)$, $\beta \in \mathrm{End}_{\mathbb{R}}(V, H)$, $\gamma \in \mathrm{End}_{\mathbb{R}}(H, V)$ and $\delta \in \mathrm{End}_{\mathbb{R}}(V)$. For a section $\sigma \in \Gamma(L)$ define an auxiliary operator $P(\sigma)$:

$$P(\sigma) : T\Sigma \to L$$

$$P(\sigma) = (\gamma(\sigma) + \delta(\sigma)\nabla\sigma)(\alpha(\sigma) + \beta(\sigma)\nabla\sigma)^{-1}$$

Here $\nabla\sigma, \gamma(\sigma)$ etc. are to be interpreted as $\mathbb{R}$-linear endomorphisms in $\mathrm{End}_{\mathbb{R}}(T\Sigma, L)$. We use the identifications $H \equiv \pi^* T\Sigma$, and $V \equiv \pi^* L$ to make sense of this interpretation. The reason for introducing $P(\sigma)$ is that σ defines a J-holomorphic curve only if the tangent spaces of the graph Γ_σ of σ are J-invariant. This is equivalent to $\Phi(T(\Gamma_\sigma))$ being J_0-invariant. However, it is not hard to check that $\Phi(T(\Gamma_\sigma))$ is equal to the graph of $P(\sigma)$ (in $T\Sigma \times L$), so that σ defines a J-holomorphic curve only if $P(\sigma)$ is complex linear. Now define the operator $\overline{\partial}_\nu$ by:

$$\begin{aligned} \overline{\partial}_\nu : \quad & \Gamma(L) \quad \to \quad \Lambda^{0,1} L \\ \overline{\partial}_\nu(\sigma) = \quad & P(\sigma) \circ j \quad - \quad i \circ P(\sigma) \end{aligned}$$

A section σ defines a J-holomorphic curve, i.e the tangent space to the graph of σ is preserved by J if and only if $\overline{\partial}_\nu(\sigma) = 0$. For more details see [HLJ94].

We now show that $\overline{\partial}_\nu$ is a linear operator if J is homogeneous. Recall from section 2 that if J is homogenous then we can arrange a simple Φ:

$$\Phi = \begin{pmatrix} \mathrm{id} & 0 \\ \gamma & \mathrm{id} \end{pmatrix}.$$

The endomorphism $\gamma \in \mathrm{End}_{\mathbb{R}}(H, V)$ is homogenous, i.e $\gamma \circ T_{s*} = T_{s*} \circ \gamma$, or (if we identify H with $\pi^* T\Sigma$, V with $\pi^* L$) $\gamma(sp) = s\gamma(p)$ for a point $p \in L$.

That gives: $P(\sigma) = \gamma(\sigma) + \nabla\sigma$. Moreover $\overline{\partial}_\nu(0) = 0$ and is homogeneous:

$$\begin{aligned}\overline{\partial}_\nu(s\sigma) &= P(s\sigma) \circ j - i \circ P(s\sigma) \\ &= (\gamma(s\sigma) + \nabla(s\sigma)) \circ j - i \circ (\gamma(s\sigma) + \nabla(s\sigma)) \\ &= s(\gamma(s\sigma) + \nabla(s\sigma)) \circ j - s(i \circ (\gamma(s\sigma) + \nabla(s\sigma)) \\ &= s\overline{\partial}_\nu(\sigma)\end{aligned}$$

It follows that $\overline{\partial}_\nu$ is equal to its linearisation along the zero section i.e:

$$\overline{\partial}_\nu(\sigma) = \overline{\partial}(\sigma) + (\mathrm{d}\gamma(0)\sigma) \circ j - i \circ (\mathrm{d}\gamma(0)\sigma)$$

where $\mathrm{d}\gamma(0)\sigma$ is the derivative of γ in the direction of σ ,and $\mathrm{d}\gamma(0)$ is understood as an endomorphism

$$\mathrm{d}\gamma(0) : \Gamma(L) \to \Gamma(Lambda^{0,1} \otimes L).$$

Since this is a $\mathbb{R}$-linear operator we have proved the proposition. □

6.3 The evaluation mapping

The lemma 6.2.2 gives a simplified point of view of the moduli space $\mathcal{M}$. For a homogenous J we identify $\mathcal{M}$ with the kernel of a $\mathbb{R}$-linear operator $\overline{\partial}_\nu$ on L, so that a J-holomorphic curve is simply thought of as a section σ of L. We can now define the evaluation mapping by:

$$\begin{aligned}\mathrm{ev} : \mathcal{M} \times \Sigma^N &\to E^N \\ \mathrm{ev}(\sigma, w_1, ..., w_N) &= (\sigma(w_1), ..., \sigma(w_N))\end{aligned}$$

where $2N = \dim_{\mathbb{R}} \ker(\overline{\partial}_\nu)$ (so that both sides of the above equations have equal real dimension $4N$). Since $\mathcal{M}$ is a vector space, it is a stable almost-complex manifold hence orientable. If J is integrable then $\overline{\partial}_\nu$ is complex linear, $\mathcal{M}$ is canonically a complex space and the evaluation map ev is complex linear. In particular, ev is orientation preserving. It is no surprise however that this fails if J is not integrable, and the surface Σ is not a sphere. If $\Sigma = S^2$ then, using the Carleman Similarity Principle, we can define an almost complex structure on $\mathcal{M}$ and use it to show easily that ev is orientation preserving). To construct a counter-example we first look at the derivative $\mathrm{ev}_{*,\underline{w}}$ at $(\sigma_0, \underline{w}) \in \mathcal{M} \times \Sigma^N$ where $\underline{w} = (w_1, ..., w_N)$ is apoint in Σ^N and σ_0 is the zero section . Let $\overline{\partial}_\nu$ be the linear generalized Cauchy-Riemann operator introduced in the previous section:

$$\overline{\partial}_\nu : \ \Gamma(L) \to \Gamma(\Lambda^{0,1} \otimes L).$$

The tangent space $T_{\sigma_0}\mathcal{M}$ is canonically identified with the kernel of $\overline{\partial}_\nu$ i.e with $\mathcal{M}$ itself. If $(\sigma_0, \underline{w})$ is point in $\mathcal{M} \times \Sigma^N$ then

$$\mathrm{ev}_{*,\underline{w}} : T_{(\sigma_0,\underline{w})}(\mathcal{M} \times \Sigma^N) \to TE^N.$$

Now $\mathrm{ev}_{*,\underline{w}}$ restricted to the factor $T_{\underline{w}}\Sigma^N$ is just the differential $d\sigma_0$ which is a complex linear mapping. If we identify $T_{\sigma_0}\mathcal{M}$ with the kernel of $\overline{\partial}_\nu$, and $T_{\sigma_0(\underline{w})}E^N$ with $L_{w_1} \oplus ... \oplus L_{w_N} \oplus T_{w_1}\Sigma \oplus ... \oplus T_{w_N}\Sigma$ then the composition

$$\overline{\mathrm{ev}_*} : T_{\sigma_0}\mathcal{M} \to T_{\sigma_0(w_1),...,\sigma_0(w_N)}E^N \to L_{\sigma_0(w_1)} \oplus ... \oplus L_{\sigma_0(w_N)} = L_{w_1} \oplus ... \oplus L_{w_N}$$

becomes simply the evaluation

$$\ker(\overline{\partial}_\nu) \rightarrow L_{w_1} \oplus ... \oplus L_{w_N}$$

$$\eta \rightarrow (\eta(w_1), ..., \eta(w_N))$$

Proposition 6.3.1 *Let L be a complex line bundle with $c_1(L) \geq 2g-1$, J a homogeneous almost-complex structure on L and $\mathcal{M}$ the moduli space of J-holomorphic curves in E in the homology class of the zero section. Let* $\dim_{\mathbb{R}}\mathcal{M} = 2N$. *The evaluation mapping* ev *does not preserve orientation at all points of $\mathcal{M} \times \Sigma^N$ if there are points $(z_1, ..., z_N)$ on the surface Σ such that there is a section $\eta \in ker(\overline{\partial}_\nu)$ with simple zeros at $\{z_1, ..., z_N\}$ and the space of pseudo-holomorphic sections (i.e in* $\ker(\overline{\partial}_\nu)$*) vanishing at $\{z_1, ..., z_N\}$ has real dimension 1.*

Proof: In view of the decomposition of ev_* it is enough to prove that $\overline{\text{ev}_{*,\underline{w}}}$ does not preserve orientation at all points $\underline{w}$. The assumption made in the lemma (that η is the only, up to multiplication by real scalars, pseudo-holomorphic section vanishing at $\{z_1, ..., z_N\}$) implies that $\overline{\text{ev}_{*,\underline{w}}}$ has rank $2N-1$ at $(z_1, ..., z_N)$. Let $\{\eta_1, ..., \eta_{2N-1}, \eta_{2N}\}$ be a basis of $T_{\sigma_0}\mathcal{M}$ such that η_{2N} is a section vanishing at $z_1, ..., z_N$. By changing indexing of $z_1, ..., z_N$ and fiddling with the basis we can assume that $\overline{\text{ev}_{*,\underline{w}}}$ restricted to the first $2N-1$ sections has rank $2N-1$, and that the matrix of $\overline{\text{ev}_{*,\underline{w}}}$ at $z_1, ..., z_N$ is of the form

$$\begin{bmatrix} id & 0 \\ \\ \star & 0 \end{bmatrix}$$

where id denotes the $(2N-1)\times(2N-1)$ identity matrix. This requires choosing trivialisations $L_{z_i}=\mathbb{R}^2$ which we extend over small neighborhoods of the z_i's. Then, in local co-ordinates w_i around z_i, we have:

$$\eta_{2N}(w_i) = a_i w_i + O(w_i^2), \;\; a_i \neq 0$$

as, if $w_i = s_i + it_i$, and $a_i = \alpha_i + i\beta_i$:

$$\eta_{2N} = (\alpha_i s_i - t_i\beta_i) + i(\beta_i s_i + \alpha_i t_i) + O(w_i^2).$$

Then at a point $(w_1, ..., w_N)$ close to $z_1, ..., z_N$ we have:

$$\overline{\mathrm{ev}_{*,\underline{w}}} = \left[\begin{array}{c|c} & \alpha_1 s_1 - t_1\beta_1 \\ & \beta_1 t_1 + \alpha_1 s_1 \\ A & \cdot \\ & \cdot \\ & \cdot \\ \hline \star & \beta_N s_N + \alpha_N t_N \end{array}\right] + O(w_i^2)$$

where A is a $(2N-1)\times(2N-1)$ matrix with a non-zero determinant, and the last column of $\overline{\mathrm{ev}_{*,\underline{w}}}$ is $(\alpha_1 s_1 - t_1\beta_1, \beta_1 s_1 + \alpha_1 t_1, ..., \alpha_N s_N - t_N\beta_N, \beta_N s_N + \alpha_N t_N)$. The determinant of $\overline{\mathrm{ev}_{*,\underline{w}}}$ is of the form

$$\det(\overline{\mathrm{ev}_{*,\underline{w}}}) = \beta_N s_N + \alpha_N t_N + f(s_1, t_1, ..., s_{N-1}, t_{N-1}) + h.o.t$$

where "h.o.t" denotes quadratic and higher order terms, and f is a linear form in s_i, t_i. It follows that $\det(\overline{\mathrm{ev}_{*,\underline{w}}})$ viewed as a function of the $2N$ real variables s_i, t_i has non-zero gradient at $z_1, ..., z_N$. Therefore the set where $\det(\overline{\mathrm{ev}_{*,\underline{w}}})$ vanishes is of codimension one and separates non-empty sets where the determinant is positive/negative. That proves the proposition. □

6.4 Construction of an almost-complex structure

Here we prove the first main theorem. The notion of an ω-tame almost-complex structure, and a positively symplectically immersed surface are explained in the Introduction.

Theorem 6.4.1 *Let (M, ω) be a 4-dimensional, compact, symplectic manifold and $B \in H_2(M, \mathbb{Z})$ a homology class such that $c_1(B) \geq 1$. Assume that there is a positively symplectically immersed surface Σ of genus $g \geq 1$ in the homology class B. Then there exist ω-tame almost-complex structures J such that the evaluation mapping* ev *defined on $\mathcal{M}(J, B, g)$ does not preserve orientation.*

Proof: We will indicate how to reduce the theorem to local results, i.e to Proposition 6.4.2, and Proposition 6.4.4. Let Σ be a positively symplectically immersed curve of genus $g \geq 1$. It will follow from the proofs of Proposition 6.4.2, and Proposition 6.4.4 that we can assume that Σ is embedded. Then by a version of Darboux theorem due to Weinstein ([MS94]) we can identify a small neighborhood of Σ with a total space E of a holomorphic line bundle L. Let $\tilde{J}$ denote the complex structure on E. We can arrange that $\tilde{J}$ extends to an ω-tame almost-complex structure on M. Since $c_1(B) \geq 1$ the degree of L is greater or equal to $2g - 1 \geq g$. We can furthermore assume that our line bundle L and the complex structure $\tilde{J}$ are constructed from a holomorphic line bundle on a sphere S^2 via the following cutting and pasting procedure. Start with a holomorphic line bundle $L' \to S^2$ of degree g, connect this line bundle with a copy of a trivial bundle $S^2 \times \mathbb{C}$ along two fibers to obtain a holomorphic line bundle of degree g over a surface of genus 1. Perform this operation g times to obtain a holomorphic line bundle of degree g on a surface Σ. If necessary form tensor products with divisor line bundles on Σ to increase the degree of the constructed line bundle.

We will show (Proposition 6.4.2,Proposition 6.4.4) how to construct an almost-complex structure J on E for which the evaluation mapping (defined on the moduli space of J-holomorphic curves in E) is orientation reversing at the zero section of L. By construction J will be a small perturbation of $\tilde{J}$. Since $\tilde{J}$ is well defined on M we can easily modify J away from the zero section and extend it across the whole manifold M. Since we preserve J near the zero section the resulting almost-complex structure on M , still denoted

J, will not lose the property that the evaluation mapping does not preserve orientation. We also note that J is close to $\tilde{J}$ hence is ω-tame. That is enough to prove our theorem. □

We now turn to the local results needed to establish the above main theorem. We will show how to construct an almost-complex structure J on the total space E of a line bundle L which satisfies the conditions in Proposition 6.3.1. It will be implicitely assumed that our line bundle L is constructed in the way which was outlined in the proof of Theorem 6.4.1. We will use two different methods. The first one (by cutting and pasting) is presented here, the next section contains an alternative construction via perturbations of complex structures.

First we construct a suitable almost-complex structure J on a line bundle L of degree g. Later we will show that by tensoring with bundles $\mathcal{O}(p)$, $p \in \Sigma$ one can increase the degree of L arbitrarily while extending J.

Recall that according to Proposition 2.0.13, and remark 2.0.14 a generalized Cauchy-Riemann operator $\overline{\partial} + A$ wich is sufficiently close to the standard Cauchy-Riemann operator $\overline{\partial}$ on a line bundle $L \to \Sigma$ of degree g with a generic holomorphic structure has a two dimensional kernel, spanned (over $\mathbb{R}$) by two sections. This applies to operators $\overline{\partial}_\nu$ associated to any almost-complex structure J on the total space E of L which is sufficiently close to a generic holomorphic structure. Proposition 6.3.1 applied to such an operator $\overline{\partial}_\nu$ says that the appropriate evaluation map ev will not be orienation preserving if we can find two sections, say σ and τ spanning $\ker_{\mathbb{R}}(\overline{\partial}_\nu)$ and a point q where σ

has a simple zero, and $\tau(q) \neq 0$. We will construct J with such property.

Proposition 6.4.2 *There is a complex line bundle $L \to \Sigma$, with $deg(L) = g$ and a homogenous almost-complex structure J such that the following holds. There are two sections σ and τ spanning $ker(\overline{\partial}_\nu)$ and a point q where τ does not vanish, but σ has a simple zero.*

We will need the following lemma about generalized Cauchy-Riemann operators on holomorphic line bundles over S^2.

Lemma 6.4.3 *For an arbitrary generalized Cauchy-Riemann operator $\overline{\partial} + A$ on a holomorphic line bundle $L' \to S^2$ ($deg(L') \geq 1$) there are two sections $\sigma, \tau \in ker(\overline{\partial} + A)$ with simple zeros and a sequence of pairwise distinct points $\{q, q_1, p_1, ..., q_g, p_g\}$ such that:*

a) $\sigma(q) = 0$, $\tau(q) \neq 0$,

b) the pairs of vectors $\{\sigma(q_j), \tau(q_j)\}$ and $\{\sigma(p_j), \tau(p_j)\}$ are linearily independent (over $\mathbb{R}$), and positively oriented in the fibres L'_{q_j} and L'_{p_j} respectively.

Proof: Start with a triple of holomorphic sections η, $i\eta$ and η' with simple zeros, and a point q where $\eta(q) = 0$ but $\eta'(q) \neq 0$. Choose η' to be close (in C^∞ toplogy) to $i\eta$. Use Proposition 2.0.11 to obtain pseudo-holomorphic sections σ, σ' and τ (corresponding to η, $i\eta$ and η') such that $\sigma(q) = 0$, and $\tau(q) \neq 0$. It follows from the proof of this proposition that all zeros of σ and τ are simple if the same is true about η and η'. It follows also that σ' and τ are pointwise close if η' and $i\eta$ are close. Now choose the points $\{q_i, p_i\}$ away from the zero locus of τ and σ'. If the sections σ' and τ are sufficiently (pointwise) close (and this can be guaranteed by an appropriate choice of η and η') then the points $\{q_i, p_i\}$ have the required properties. □

In particular, if J' is any homogenous almost-complex on L' then this lemma applies to the associated $\overline{\partial}_\nu$ operator.

Proof of proposition 6.2. Start with a holomorphic line bundle $L' \to S^2$ of degree g and an arbitrary homogeneous almost-complex structure J' on $E'(L')$. Denote by J'_0 the unique holomorphic structure on L'. Let σ, τ and $q, q_1, p_1, ..., q_g, p_g$ be as in the above lemma. Trivialize L' near points $q_1, p_1, ..., q_g, p_g$ using the sections σ and τ, let $\phi_{q_i} : U \times \mathbb{C} \to L'$ be the local trivialisation around q_i defined as follows:

$$\phi_{q_i}(z, x+iy) = x\sigma(\psi_{q_i}(z)) + y\tau(\psi_{q_i}(z)),$$

where $\psi_{q_i} : U \to \Sigma$ is a local holomorphic chart near q_i, $\psi_{q_i}(0) = q_i$. Since for all fixed x and $y \in \mathbb{R}$, $x\sigma + y\tau$ is a J'-holomorphic curve, and since J' preserves fibers we have

$$\phi_{q_i}{}^* J' = \begin{bmatrix} J'_1 & 0 \\ 0 & J'_2 \end{bmatrix}$$

It is then easy to modify the pull-back of J' by ϕ_{q_i} on a neighborhood of $0 \times \mathbb{C} \equiv L_{q_i}$ so that it becomes the standard $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ complex structure near the central fiber $0 \times \mathbb{C}$, and remains homogeneous throughout. For that simply write $J_1' = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $bc = -1 - a^2$. Then let $a \to 0$, and $c \to 1$. This defines an almost-complex J'' which is standard near the central fiber but not necessarily homogenous. To find a homogenous structure simply homogenize by taking the limit $\lim_{s\to 0} \mu_s^* J''$. Now we can form the fiberwise connected sum $L' \#_{q_i,p_i} S^2 \times \mathbb{C}$ along the fibers L'_{q_i}, L'_{p_i} and extend J' as if it were an integrable complex structure. We perform this cutting and pasting so that the genus of the underlying surface increases by one each time the surgery is performed. To form the connected sum we identify $S^2 \times \mathbb{C}$ with $U \times \mathbb{C}$ (where U as above is a neighborhood of the origin in $\mathbb{C}$) near a point $r \in S^2$ $(r = q_1, p_1, q_2, p_2, ..., q_g, p_g)$:

$$\psi : U \times \mathbb{C} \to S^2 \times \mathbb{C}.$$

The gluing map in local trivialisations has the form: $(z, x+iy) \to (\epsilon\frac{1}{z}, x+iy)$ where z is a local co-ordinate on $U \subset \mathbb{C}$ and $(x+iy)$ is a co-ordinate in the fibers of $U \times \mathbb{C}$. After performing this at all points $\{q_i, p_i\}_{i=1}^g$ one gets the desired bundle L. Notice that J' extends across to L as do both sections σ and τ (simply because in all trivialisations $\sigma(z) \approx (z, 1)$, and $\tau(z) \approx (z, i)$). We will denote the extension of J' by J, and the extensions of σ and τ by the same letters. The almost-complex structure J obtained is homogenous since the gluing mapping and the complex structure on $S^2 \times \mathbb{C}$ are homogenous. The extended sections σ and τ are J-holomorphic curves or, equivalently, are sections of L in the kernel of the associated operator $\overline{\partial}_\nu$. Also there is a point

q on the surface Σ where σ has a simple zero, but $\tau(q) \neq 0$. It follows from this construction that all the zeros of σ,and τ are simple since we started with sections with that property and did not add any extra zeros.

To ensure that σ, and τ span $\ker(\overline{\partial}_\nu)$ we need to apply Proposition 2.0.13. We will be able to apply this proposition if we can arrange for J to be arbitrarily close to a generic holomorphic structure J_0 which is fixed at the beginning of the construction (see Remark 2.0.14). This is easy to achieve. Observe that we could carry through our surgery with J_0' instead of J', and with holomorphic sections η, $i\eta$ (notation as in Lemma 6.4.3) in place of σ, and τ. We would obtain a holomorphic structure J_0 on L and a holomorphic section $\tilde{\eta}$ of L. If J_0' and J' are close then so are J_0, and J. Now L is the divisor line bundle of the zero divisor of $\tilde{\eta}$ which depends only on η, and is independent of J'. We can choose η so that the zero divisor of $\tilde{\eta}$ and the holomorphic structure J_0 on L will be generic in the sense of Proposition 2.0.13. Having fixed η we now choose J' to be (C^∞) close to J_0', and observe that σ, and τ will be close (also in C^∞ topology) to η and $i\eta$. It follows that the almost-complex structure J will be close to the generic holomorphic structure J_0 which is what we needed. That is enough to prove the proposition. □

To increase the degree of L to $g + r$ we form tensor products of L with divisor bundles $\mathcal{O}(p)$, $p \in \Sigma$.

Proposition 6.4.4 *Let σ, τ, q be as in the previous proposition. Let $\{c_i\}_{i=1}^{r} \in \Sigma$ be a sequence of distinct points on Σ such that $\sigma(c_i)$ and $\tau(c_i)$ are linearily independent in L_{c_i}. Then the almost-complex J and sections σ,and τ extend to* *the tensor product* $L \otimes \mathcal{O}(c_1) \otimes ... \otimes \mathcal{O}(c_r)$. *The extended almost-complex structure, still denoted by* J*, is homogenous and the extended sections, still denoted by* σ*, and* τ*, are pseudo-holomorphic sections for the associated operator* $\overline{\partial}_\nu$*. The section* σ *is the only section, up to real scalars, vanishing at* $(q, c_1, ..., c_r)$ *and has simple zeros at those points.*

Proof: As before we use sections σ and τ to trivialize L near c_i. Let

$$\begin{aligned} \phi_{c_i} : U \times \mathbb{C} &\rightarrow L \\ \phi_{c_i}(z, x+iy) &= (x\sigma(\psi_{c_i}(z)) + iy\tau(\psi_{c_i}(z))) \end{aligned}$$

be a local trivialisation, where $\psi_{c_i} : U \rightarrow \Sigma$ is a local chart. The pullback of J is diagonal, and as in the previous proposition can be modified on a neighborhood of L_{c_i} so that it becomes the standard integrable structure $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. After this is done we can form tensor products as if J were an integrable structure. J, σ and τ all extend to $L \otimes \mathcal{O}(c_1) \otimes ... \otimes \mathcal{O}(c_r)$ for $1 \leq r$. To ensure that this works we make the following simple observation. If we modify J as above then J restricted to $E|_{\phi_{c_i}(U)}$ is an integrable structure, namely a holomorphic structure on $L|_{\phi_{c_i}(U)}$ and it extends to the total space of $L \otimes \mathcal{O}(c_1) \otimes ... \otimes \mathcal{O}(c_r)|_{\phi_{c_i}(U)}$. On the other hand each bundle $\mathcal{O}(c_i)$ has a standard holomorphic section η_i vanishing only at c_i. By tensoring with $\eta_1 \otimes \eta_2 \otimes ... \otimes \eta_r$ we construct a linear (over $\mathbb{R}$) isomorphism

$$\psi : L \otimes \mathcal{O}(c_1) \otimes ... \otimes \mathcal{O}(c_r)|_{\Sigma - \{c_1, ..., c_r\}} \rightarrow L$$

with which we pull back J, i.e $\psi^* J$ is the extension of J on $\Sigma - \{c_1, ..., c_r\}$. Since ψ is holomorphic with respect to $J|\phi_{c_i}(U) - c_i$ and the holomorphic structure on $\mathcal{O}(c_i)$ for every $i = 1, ..., r$ both extensions agree. To extend the sections σ and τ just tensor them with $\eta_1 \otimes \eta_2 \otimes ... \otimes \eta_r$. The extended almost-complex structure J is still homogenous since ψ was a linear mapping. The new associated generalized Cauchy-Riemann operator operator $\overline{\partial}_\nu$ has $2(r+1)$ independent sections, but none of them vanishes at $\{q, c_1, ..., c_r\}$ except σ and real multiples thereof. For if ξ were such a section then by tensoring with all the η_i^{-1} we would get a J-holomorphic section of L vanishing at q, hence a multiple of σ. That would imply in turn that $\xi = sv$ for some $s \in \mathbb{R}$. We notice that by the construction σ has simple zeros at $\{q, c_1, ..., c_r\}$. That is enough to prove the proposition. □

This proposition applied with Proposition 6.3.1 concludes our first construction of the required almost-complex structure J.

6.5 Alternative construction.

In this section we prove our second main theorem.

Theorem 6.5.1 *Let (M^4, J) be an almost-complex (compact, smooth) manifold, $B \in H_2(M, \mathbb{Z})$ a homology class with $c_1(B) \geq 1$, and $u : \Sigma \to M$ an embedded J-holomorphic curve in class B of genus $g \geq 1$ such that J is integrable on a neighborhood of $u(\Sigma)$. Then there are almost-complex structures J' arbitrarily close to J (in the C^1 topology) for which the evaluation mapping* ev *on $\mathcal{M}(J', B, g)$ does not preserve orientation.*

Proof: We will outline how to reduce this theorem to a local statement. As in the proof of Theorem 6.4.1 we identify a neighborhood of the curve Σ with the total space $E(L)$ of a holomorphic line bundle L with a generic holomorphic structure. Although we cannot assume that the almost-complex structure J on M restricts to the linear complex structure on E we can arrange that both structures agree along the zero section. Now we introduce an auxiliary complex structure J' on M which is equal to J away from the zero section, and which is equal to a generic linear complex structure J_0 on a neighborhood of the zero section of L. Needless to say we can arrange that J and J' be C^1 close. Our local results presented below let us approximate J' on E by an almost-complex structure J'' with desired properties. J'' is also an approximation of J. Moreover since it is (C^1) close to J it can be extended to an ω-tame almost-complex structure on M. That is enough to prove the theorem. □

We now turn to local results, and prove how to perturb a given integrable generic complex structure J_0 on L (with $\deg(L) \geq g$) a little bit to find an almost-complex structure such that the appropriate evaluation mapping does not preserve orientation. We need to find an almost-complex structure J'' near J_0 to which we can apply Proposition 6.3.1. First we consider perturbations on the level of generalized Cauchy-Riemann operators. We will show that it is always possible to perturb $\overline{\partial}$ to get a generalized Cauchy-Riemann operator $\overline{\partial}+B$ which has the property of Proposition 6.3.1 i.e there are points $\{z_1, ..., z_N\}$ and a section $\sigma \in \ker(\overline{\partial} + B)$ vanishing at $\{z_1, ..., z_N\}$. Also we require that $\{z_1, ..., z_N\}$ are simple zeros of σ.

Let A be a $(0,1)$-form on Σ with values in the bundle of endomorphisms of L i.e a section of $\Lambda^{0,1} \otimes \mathrm{End}_R(L)$, and consider $\overline{\partial} + tA$ where t is a small real parameter . Choose a basis of the kernel of the unperturbed operator $\overline{\partial}$ of the form $\sigma_1, \sigma_2 = i\sigma_1, \sigma_3, ..., \sigma_{2N-1}, \sigma_{2N} = i\sigma_{2N-1}$. Since we assumed $\deg(L) \geq g$, and that J' is generic, the Dolbeault operator $\overline{\partial}$ is onto (this is a special case of the "geometric" version of the Riemann-Roch theorem, see [GH78]). If we fix a subspace complementary to the kernel of $\overline{\partial}$ then by the implicit function theorem we get the perturbed sections of L spanning the kernel of $\overline{\partial} + tA$:

$$\sigma_j(t) = \sigma_j + tb_i + O(t^2) \text{ where } b_j \in \Gamma(L).$$

Lemma 6.5.2 *If A is anti-holomorphic then $\sigma_{2j}(t) = i\sigma_{2j-1} - tib_{2j-1} + O(t^2)$.*

Proof: Let $\sigma_{2j} = \sigma_{2j} + t\sigma_{2j} + \cdots$, and $\sigma_{2j-1} = \sigma_{2j-1} + t\sigma_{2j-1} + \cdots$. Since the sections b_j are unique, in order to see that $b_{2j} = -ib_{2j-1}$ it is enough to check that $(\overline{\partial} + tA)(\sigma_{2j} - it\sigma_{2j-1})$ is of order $O(t^2)$. But that is immediate from the fact that A is anti-holomorphic. □

Proposition 6.5.3 *Assume that L is a holomorphic line bundle such that for a choice of points $z_1, ..., z_N$ there is a holomorphic section unique, up to multiplication by complex numbers, with simple zeros at $z_1, ..., z_N$. Then there is a dense open subset $\mathcal{A}$ in the space of anti-holomorphic endomorphisms such that for $A \in \mathcal{A}$ and a small t the operator $\overline{\partial} + tA$ has the following property: there are points $\{w_1, ..., w_N\}$ and a section σ in $\ker_{\mathbb{R}}(\overline{\partial} + tA)$ with simple zeros at $\{w_1, ..., w_N\}$ which is unique, up to multiplication by real scalars.*

Proof: We use the notation developed so far. We may assume that the section $\sigma_{2N}(t) = \sigma$ has N simple zeros at $z_1, ..., z_N$. Let $\sigma_{2i-1}(t) = \sigma_{2i-1} + t\, b_{2i-1} + O(t^2)$. Then according to the Lemma 6.5.2 (since A is anti-holomorphic)

$$\sigma_{2i}(t) = \sigma_{2i} + t\, b_{2i} + O(t^2) = i\sigma_{2i-1} - it\, b_{2i-1} + O(t^2).$$

Let $z_j(t)$ be the zero of $\sigma_{2N}(t)$ developing from z_j, i.e $z_j(0) = z_j$. In local co-ordinates we have $z_j(t) = z_j + td_j + O(t^2)$ with

$$d_j = -\frac{b_{2N}(z_j)}{\frac{\partial\sigma_{2N}}{\partial z_j}} = \frac{b_{2N-1}(z_j)}{\frac{\partial\sigma_{2N-1}}{\partial z_j}}.$$

It follows

$$\begin{aligned}\sigma_{2N-1}(t)(z_j(t)) &= \sigma_{2N-1}(z_j(t)) + tb_{2N-1}(z_j(t)) \\ &= td_j\tfrac{\partial\sigma_{2N-1}}{\partial z_j} + tb_{2N-1}(z_j) + O(t^2) \\ &= t\,[2b_{2N-1}(z_j)] + O(t^2).\end{aligned}$$

The assumption that σ_{2N} is the only section (up to complex scalars) vanishing at $z_1, ..., z_N$ implies (after possibly reordering $z_1, ..., z_N$) that the evaluation mapping

$$\mathrm{ev} : \ker(\overline{\partial}) \to L_{z_1} \oplus ... L_{z_{N-1}}$$

when restricted

$$\mathrm{ev}|_{span\{\sigma_1,...,\sigma_{2N-2}\}} : span\{\sigma_1, ..., \sigma_{2N-2}\} \to L_{z_1} \oplus ... L_{z_{N-1}}$$

is an isomorphism. To prove that $\sigma_{2N}(t)$ is the unique (up to real scalars) section vanishing at $\{z_j(t)\}_{j=1}^N$ we need to prove:

$$\mathrm{ev}_t : \ker(\overline{\partial} + tA) \to L_{z_1(t)} \oplus ... L_{z_N(t)}$$

is of rank $2N-1$. This is a computation similar to that done in Proposition 6.3.1. By writing down a matrix of ev_t we can easily see that this would follow if we could show that $\sigma_{2N-1}(t)(z_N(t))$ can be chosen independently of $\sigma_{2N-1}(t)(z_j(t))$ for $j < N$, or , that $b_{2N-1}(z_N)$ is independent of $b_{2N-1}(z_j)$ (for $j < N$). Since b_{2N-1} is only constrained to be a solution of

$$(\star) \quad \overline{\partial} b_{2N-1} + A(\sigma_{2N-1}) = 0,$$

we can choose b_{2N-1} to be holomorphic on a small neighborhood U of the zero divisor of σ_{2N-1} (in particular around $z_1, ..., z_N$) with arbitrary prescribed values at $z_1, ..., z_N$ and extend it smoothly across Σ (in order to do this we multiply b_{2N-1} by a smooth cut off function with support in U). There is then an anti-holomorphic endomorphism A such that $(\star)$ is satisfied. That proves existence of the perturbations we are after. We define $\sigma = \sigma_{2N}$. Note that b_{2N-1} depends linearly on A and, by adding perturbations, we can alter any given one perturbation so that the appropriate evaluation mapping ev_t has rank $2N-1$. This proves that there is a dense set of perturbations A such that $\overline{\partial} + tA$ has (for small t) the properties stated in the lemma. This set of perturbations A is obviously open. □

Let now J_0 denote the given linear complex structure on the total space of a holomorphic line bundle L.

Proposition 6.5.4 *There are arbitrarily small perturbations J of J_0 such that $\overline{\partial}_\nu$ has the property that there are points $\{w_1, ..., w_N\}$ and a section σ in $ker(\overline{\partial}_\nu)$ vanishing at $\{w_1, ..., w_N\}$, which is unique up to multiplication by* *real scalars. Moreover it can be arranged that such section has only simple zeros at* $\{w_1, ..., w_N\}$.

Proof: We will adapt the proof of Proposition 6.5.3, and show that the perturbation A that appeared in that proof can come from a perturbation of complex structures (and not just an abstract perturbation of operators). First change the holomorphic structure on L to ensure that there exist points $z_1, ..., z_N$ and a unique section σ with simple zeros at these points. We will look for an almost complex structure J'' of the form:

$$J'' = \begin{pmatrix} \mathrm{id} & 0 \\ \gamma & \mathrm{id} \end{pmatrix}^{-1} J_0 \begin{pmatrix} \mathrm{id} & 0 \\ \gamma & \mathrm{id} \end{pmatrix}$$

where γ is homogenous as in Proposition 6.2.3 . For such an almost-complex structure J'' we have an explicit formula for the associated operator $\overline{\partial}_\nu$ (see Proposition 6.2.3):

$$\begin{aligned} \overline{\partial}_\nu &= \overline{\partial} + a, \quad \text{where} \\ a\tau &= \mathrm{d}\gamma(0)\tau \circ j - i \circ \mathrm{d}\gamma(0)\tau. \end{aligned}$$

The equation $\star$ now reads

$$(\star') \; \overline{\partial} b_{2N-1} + a\sigma_{2N-1} = 0$$

where (in the notation of Proposition 6.5.3) $i\sigma_{2N-1} = \sigma_{2N} = \sigma$, and b_{2N-1} is a section of L. As before we choose b_{2N-1} to be holomorphic on a small neighborhood U of the zero divisor of σ_{2N-1} (which is equal to the zero divisor of σ) and, multiplying by a cut-off function, extend it across the surface Σ. For convenience we arrange that U be a disjoint union of co-ordinate charts centered around points in the zero divisor of σ_{2N-1}. Let p be a zero of σ_{2N-1} and $z \in U_p$ be local co-ordinates near that point. We will construct J'' by prescribing γ. If we trivialize L near p and $x+iy$ parametrizes fibers of L then we can write

$$\gamma(x+iy) = x\gamma_1 + y\gamma_2$$

for $\gamma_1, \gamma_2 \in \mathrm{End}_R(\mathbb{C})$. In addition we ask that γ_1, and γ_2 satisfy $i\gamma_1 = -\gamma_2$. Then for any local section τ of L we have

$$\begin{aligned} d\gamma(0)\tau &= d\gamma(0)(\tau_1 + i\tau_2) \\ &= \tau_1\gamma_1 + \tau_2\gamma_2 \\ &= (\tau_1 - i\tau_2)\gamma_1 \end{aligned}$$

It then follows

$$a\tau = (\tau_1 - i\tau_2)(\gamma_1 \circ j - i \circ \gamma_1) = \overline{\tau}\gamma_1^{0,1}$$

where $\gamma_1^{0,1}$ denotes the $(0,1)$-part of γ_1. Since $\overline{\partial} b_{2N-1}$ is a form of type $(0,1)$ supported in a collection of local charts we can apply this local computation, with $\tau = \sigma_{2N-1}$ in a local chart, to solve $(\star')$. We see immediately that our equation is solvable for γ_1 (recall that $\overline{\partial} b_{2N-1}$ vanishes on a neighborhood of the zero locus of σ_{2N-1}). Having done this we define $\gamma = x\gamma_1 - yi\gamma_2$, and observe that γ is globally well defined and prescribes an almost-complex structure J''. This discussion shows that the perturbation term A in the equation

(⋆) (Proposition 6.5.3) indeed comes from the required perturbation of the complex structure J_0. □

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