Fundamental Groups of Riemannian manifolds, Sigma Constant and Scalar Curvature

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Abstract of the Dissertation

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This thesis is divided into two parts. In the first part, we look into some propeties of the fundamental groups of Riemannian manifolds satisfying some geometric conditions. We show the fundamental groups for a class of compact Riemannian manifolds whose volume is uniformly bounded from below, diameter is bounded from above and the Ricci curvature is almost non-negative must be almost abelian and in the low dimension case, the same property holds for a class of compact Riemannian manifolds with almost non-negative sectional curvature without volume condition if we restrict the growth of fundamental group by its dimension. Key tools are Gromov-Hausdorff distance, equivariant Hausdorff approximation in the sense of Fukaya , splitting theorem and some geometric and algebraic group theory related with finitely generated nilpotent group.

In the second part, we prove the sigma constant of a compact manifold surgered by a sphere embedded in it is greater than or equal to that of given manifold if the sigma constant is nonpositive. We use Gromov-Lawson technique and maximum principle as key tools.

To my parents and my family, Hyangsoo, Yura

Contents

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Chapter 1

Introduction

One of the main topics in Riemannian geometry is to study the relation of geometric features like curvature, diameter and volume with the topology of manifolds. The simplest topological invariants of Riemannian manifolds are homotopy and homology groups of them. Among other higher homotopy groups, the first homotopy group, called the fundamental group, is well-known and developed much more. The reason is the following. Let (M, g) be a compact Riemannian manifold and $(\widetilde{M}, \tilde{g})$ its universal Riemannian covering with covering metric $\tilde{g}$. Then the fundamental group $\pi_1(M)$ of M acts on $\widetilde{M}$ isometrically and the action is properly discontinuous and fixed point free if we fix base points in M and $\widetilde{M}$. In other words, we can consider $\pi_1(M)$ as a subgroup of $Isom(\widetilde{M})$, the isometry group of $\widetilde{M}$.

One of the important tools in studying the property of the fundamental group of a Riemannian manifold is to measure how large it is, namely, the growth of group. It is well-known that the growth of the fundamental group is very much related with the growth of volume of geodesic balls in the universal covering.

Let us recall the definition of the growth of a finitely generated discrete group. Let Γ be a finitely generated discrete group with generators $\{\gamma_1, \ldots, \gamma_l\}$, $\gamma_i \in \Gamma$. Then every element of Γ can be expressed as a word in $\{\gamma_1, \ldots, \gamma_l\}$. Now define the growth function Φ as

$$\Phi(r) := \#U(r)$$

where $\#U(r)$ is the number of distinct words in Γ of length $\leq r$

Γ is said to be of *polynomial growth* of order k if there exists a positive constant C such that

$$\#U(r) \leq Cr^k$$

It is well-known ([23]) that the property of polynomial growth is independent of the generating set $\{\gamma_i\}$. Obviously a free abelian group has polynomial growth. More generally, it is known that a finitely generated nilpotent group has polynomial growth ([16], [30]). A group Γ is called *nilpotent* if denoting $[\Gamma, \Gamma] = \Gamma^{(1)}$ and $[\Gamma, \Gamma^{(i)}] = \Gamma^{(i+1)}$ inductively, then we have $\Gamma^{(m)} = \{1\}$ for some m, where 1 is the identity element.

M. Gromov proved the following remarkable result:

Theorem 1.0.1 (M. Gromov [16]) *A finitely generated discrete group has polynomial growth iff it is almost nilpotent, i.e., it contains a nilpotent subgroup of finite index.*

Example 1 **(Heisenberg Group)** Let N be the Heisenberg group and Γ the integer lattice :

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \Gamma = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

Then the quotient space $M = N/\Gamma$ is orientable compact 3-dimensional manifold and the fundamental group is Γ, i.e., $\pi_1(M) = \Gamma$. It is easy to see that

$$[\Gamma, \Gamma] = \mathbb{Z} = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : c \in \mathbb{Z} \right\}.$$

and so

$$\Gamma/[\Gamma, \Gamma] = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad b_2(M) = 2.$$

It is also known ([23]) that the growth is

$$growth(\Gamma) = 4 > 3 = dim(M)$$

♠

There are several fundamental results for the fundamental groups of Riemannian manifolds. We first state the Milnor's theorem.

Theorem 1.0.2 (J. Milnor [23]) *Let (M, g) be a complete Riemannian manifold of non-negative Ricci curvature. Then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of order $\leq dim(M)$.*

In fact, he proved this theorem by using the relation of the growth of the fundamental group of a manifold with the growth of the volume of geodesic balls in the universal cover. J. Cheeger and D. Gromoll improved Milnor's result by using their splitting theorem which is one of the fundamental structural theorem in Riemannian geometry.

Theorem 1.0.3 (J. Cheeger and D. Gromoll [10]) *If (M, g) is a compact Riemannian manifold of non-negative Ricci curvature, then the fundamental group $\pi_1(M)$ is almost abelian,i.e., it contains an abelian subgroup of finite index.*

One of the main problems in Riemannian geometry is to extend the properties on manifolds of non-negative curvature to manifolds of almost non-negative curvature. Recently, G. Wei ([29]) proved that the fundamental group $\pi_1(M)$ of a Riemannian manifold M with almost non-negative Ricci curvature, diameter $diam(M) \leq D$ and volume $vol(M) \geq v$ is of polynomial growth of order $\leq n$.

Recall (Theorem 1.0.1) that a finitely generated group is of polynomial growth if and only if it is almost nilpotent. So, together with this, $\pi_1(M)$ is almost nilpotent in the Wei's theorem. In this thesis, we proved the following theorem which improved the Wei's theorem.

Theorem 1.0.4 *Given n and $D, v > 0$, there exists a positive number $\epsilon = \epsilon(n, D, v) > 0$ such that if a closed n-manifold M satisfies*

$$Ric(M) \geq -\epsilon, \qquad diam(M) \leq D, \qquad vol(M) \geq v$$

then $\pi_1(M)$ is almost abelian.

In chapter 4, we consider fundamental groups of manifolds of almost non-negative curvature. Namely, we will show the same property of the fundamental groups of manifolds of almost non-negative sectional curvature without volume condition. In [13] , K. Fukaya and T. Yamaguchi have proved that there exists a positive small number ϵ, depending only on the dimension, such that the fundamental groups of manifolds satisfying $K_M diam(M)^2 \geq -\epsilon$ must be almost nilpotent. If we restrict the growth of the fundamental group by dimension, we get the same result as Theorem 1.0.4 for the low dimension.

Theorem 1.0.5 *Let $n \leq 4$. There exists a positive number $\epsilon = \epsilon(n)$ such that if M is a closed Riemannian n-manifold satisfying*

(1) $K_M diam(M)^2 \geq -\epsilon$

(2) *$\pi_1(M)$ has polynomial growth of degree $\leq n$,*

then $\pi_1(M)$ is almost abelian.

The proof of this theorem depends on the fibration theorem and classification theorem of almost non-negative curvature manifolds of dimension 3 due to K. Fukaya and T. Yamaguchi ([13], [33], [34]).

In chapter 5, we will discuss the scalar curvature and relation of it with topology of manifolds focused on the sigma constant. For the purpose, we consider an invariant $\sigma(M)$ of a smooth closed manifold M, which is called the *sigma constant* of M, defined as the supremum of $\mu(M,\mathcal{C})$ over all conformal classes $\mathcal{C}$ of Riemannian metrics on M,

$$\sigma(M) = \sup_{\mathcal{C}} \mu(M,\mathcal{C})$$

where $\mu(M,\mathcal{C})$ is the Yamabe constant defined as

$$\mu(M,\mathcal{C}) = \inf_{g\in\mathcal{C}} \frac{\int_M s_g \, dv_g}{\left(\int_M dv_g\right)^{\frac{n-2}{n}}}, \tag{1.1}$$

where s_g is the scalar curvature of g and $n = dim(M)$.

O. Kobayashi has proved the following theorem.

Theorem 1.0.6 (O. Kobayashi [20]) *(a) If M_1 and M_2 are compact manifolds of the same dimension $n \geq 3$, then*

$$\sigma(M_1 \# M_2) \geq \begin{cases} -(|\sigma(M_1)|^{n/2} + |\sigma(M_2)|^{n/2})^{2/n} & \text{if } \sigma(M_1) \leq 0 \text{ and } \sigma(M_2) \leq 0 \\ \\ min\{\sigma(M_1), \sigma(M_2)\} & \text{otherwise} \end{cases}$$

(b) If M is an S^{n-1} bundle over S^1 with $n \geq 3$, then

$$\sigma(M) = \sigma(S^n) = n(n-1)vol(S^n)^{2/n}.$$

Note that S^{n-1} bundle over S^1 can be obtained from surgery on 0-sphere in S^n. Gromov-Lawson and Schoen-Yau have proved ([17], [27]) that if M is a compact manifold which carries a Riemannian metric of positive scalar curvature, then any manifold which can be obtained from M by performing surgeries in *codim* ≥ 3 carries a metric of positive scalar curvature.

So, we can consider the same property as Kobayashi's result for the surgeries on *p-sphere*, $p \leq n-3$ and we will show the following similar result for the manifolds with non-positive sigma constants.

Theorem 1.0.7 *Let M be a compact smooth manifold and assume $\sigma(M) \leq 0$. Let M_s be a manifold obtained from M by surgery on a homotopically trivial S^p , for $p \leq n-3$. Then* $\sigma(M_s) \geq \sigma(M)$.

The one of the crucial steps is to show that for given metric g, we can control the volume of surgered part, namely, we have to verify we can make the volume of surgered part sufficiently small so that the total volume of M_s is almost equal to the volume of (M, g).

Now let M and N be two smooth closed manifolds of the same dimension n and $S^p, p \leq n$ be a homotopically trivial p-sphere embedded in M and N, respectively. Take off tubular neighborhood $D^{n-p} \times S^p$ in M and N and then glue them along their boundary. We denote the resulting manifold by $M * N$. Then we can show the following which is similar to the above theorem.

Theorem 1.0.8 *If both sigma constants of M and N are nonpositive and S^p is homotopically trivial, then*

$$\sigma(M * N) \geq min\{\sigma(M), \sigma(N)\}$$

Chapter 2

Gromov-Hausdorff Distance and Related Results

In this chapter, we state some notions and related results for the convergence of metric spaces. Gromov has introduced a notion of Hausdorff distance on the set of isometry class of all metric spaces and some equivalent definitions are known. This concept makes it possible to produce several important results in Riemannian geometry.

Definition. A (not necessarily continuous) map $f : X \to Y$ between metric spaces is called an ϵ - *Hausdorff approximation* if

(1) $|d(f(x), f(y)) - d(x, y)| < \epsilon$ for all $x, y \in X$

(2) The ϵ-neighborhood of $f(X)$ covers Y.

Then the *Hausdorff distance* $d_H(X, Y)$ is defined as the infimum of ϵ such that there exist ϵ-Hausdorff approximations from X to Y and from Y to X.

The Hausdorff distance actually defines a distance on the set of all compact metric spaces. For unbounded spaces, this metric is not useful, but the notion of pointed Hausdorff distance is effective. For pointed metric spaces (X,p) and (Y,q), the *pointed* Hausdorff distance $d_{p.H}((X,p),(Y,q))$ is defined as the infimum of ϵ such that there exist ϵ-Hausdorff approximations $f : B_p(1/\epsilon, X) \to B_q(1/\epsilon + \epsilon, Y)$ and $g : B_q(1/\epsilon, Y) \to B_p(1/\epsilon + \epsilon, X)$ between metric balls with $f(p) = q$ and $g(q) = p$.

The following theorem due to M. Gromov is the standard precompactness theorem.

Theorem 2.0.9 (M. Gromov [15]) *Let k be an arbitrary real number and $D > 0$. Then*

(1) *The set of all closed n-dimensional Riemannian manifolds M with $Ricci_M \geq k$ and $diam_M \leq D$ is relatively compact with respect to the Hausdorff distance.*

(2) *The set of all pointed complete n-dimensional Riemannian manifolds (M,p) with $Ricci_M \geq k$ is relatively compact with respect to the pointed Hausdorff distance.*

Example 2 (Continued) In the example 1, N is a nilpotent Lie group and so M admits a left invariant Riemannian metric. In fact, with respect to the standard coordinate x, y, z, the metric $g = dx^2 + dy^2 + (dz - xdy)^2$ is a left invariant metric with curvature $-1 \leq K_g \leq 1$. For every $\epsilon > 0$, define the left invariant metric g_ϵ on M by

$$\begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \in T_eN, \qquad \left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|^2 = u^2 + v^2 + \epsilon^2 w^2,$$

where e is the identity element of N. Then the sectional curvature and diameter of g_ϵ satisfy $|K_{g_\epsilon}| \leq 24\epsilon^2, \quad diam(M) \leq 2$ and (M, g_ϵ) converges to a flat torus T^2 with respect to the Hausdorff distance. ♠

Next we define the concept of *equivariant Hausdorff distance* due to K. Fukaya ([12]).

Definition. We say that a triple (X, Γ, p) belongs to $\mathcal{M}_{eq}$ if every metric ball in X is relatively compact, $p \in X$ and if Γ is a closed subgroup of $Isom(X)$, the group of isometries of X. For $r > 0$, we put

$$\Gamma(r) = \{\gamma \in \Gamma \mid |d(\gamma(p), p) < r\}$$

For $(X, \Gamma, p), (Y, \Lambda, q) \in \mathcal{M}_{eq}$, we say that a triple (f, φ, ψ) represents an ϵ-*pointed equivariant Hausdorff approximation* from (X, Γ, p) to (Y, Λ, q) if

(1) $f : B_p(1/\epsilon, X) \to B_q(1/\epsilon + \epsilon, Y)$ is an ϵ-Hausdorff approximation with $f(p) = q$.

(2) $\varphi : \Gamma(1/\epsilon) \to \Lambda$ and $\psi : \Lambda(1/\epsilon) \to \Gamma$ satisfy the following:

(2.1) If $\gamma \in \Gamma(1/\epsilon)$ and $x, \psi(\mu)(x) \in B_p(1/\epsilon, X)$, then

$$d(f(\gamma x), \varphi(\gamma)(fx)) < \epsilon.$$

(2.2) If $\mu \in \Lambda(1/\epsilon)$ and $x, \psi(\mu)(x) \in B_p(1/\epsilon, X)$, then

$$d(f(\psi(\mu)(x)), \mu(fx)) < \epsilon.$$

We remark that it is assumed neither that f is continuous nor that φ, ψ are homomorphisms.

Now the pointed equivariant Hausdorff distance $d_{p.e.H}((X,\Gamma,p),(Y,\Lambda,q))$ is defined as the infimum of ϵ such that there exist ϵ-pointed equivariant Hausdorff approximations from (X,Γ,p) to (Y,Λ,q) and from $(Y,,\Lambda,q)$ to (X,Γ,p).

Example 3 Let γ_i be the isometry of $\mathbb{R}^3$ defined by $\gamma_i(x,y,z) = R_{1/i}(x,y), z+ 1/i^2)$, where R_θ denotes the rotation on the $(x,y)-$plane around the origin with angle θ. Let Γ_i be the group generated by γ_i. Then $(\mathbb{R}^3, \Gamma_i, 0)$ converges to $(\mathbb{R}^3, S^1 \times \mathbb{R}, 0)$. Note that the limit depends on the choice of reference points. For instance, if we take p_i with $dist(0,p_i) = i$ as the reference points, then $(\mathbb{R}^3, \Gamma_i, p_i)$ converges to $(\mathbb{R}^3, \mathbb{R} \times \mathbb{Z}, 0)$. ♠

K. Fukaya and T. Yamaguchi proved the following theorems which play an essential role in proving the Theorem 1.0.5 which is one of the main theorem.

Theorem 2.0.10 ([13], [33], [34]) *Let $(X_i, \Gamma_i, p_i) \in \mathcal{M}_{eq}$ and assume that (X_i, p_i) converges to (Y, q) with respect to the pointed Hausdorff distance. Then there exists a closed subgroup Λ of Isom(Y) such that for a subsequence (X_i, Γ_i, p_i) converges to (Y, Λ, q) with respect to the pointed equivariant Hausdorff distance.*

Theorem 2.0.11 ([13], [34]) *Let (X_i, Γ_i, p_i) converge to (Y, G, q) and G' a normal subgroup of G, and suppose that*

(1) *G/G' is discrete.*

(2) *Y/G is compact.*

(3) *X_i is simply connected and the action of Γ_i is free and properly discontinuous.*

(4) *There exists a positive number R_o such that G' is generated by $G'(R_o)$.*

Then G/G' is finitely represented and there exists a normal subgroup Γ_i' of Γ_i such that

(5) *(X_i, Γ_i', p_i) converges to (Y, G', q) for a subsequence.*

(6) *Γ_i/Γ_i' is isomorphic to G/G' for sufficiently large i.*

(7) *For every $\epsilon > 0, \Gamma_i'$ can be generated by $\Gamma_i'(R_o + \epsilon)$ for sufficiently large i.*

In the case when G' is the identity component of G, the group Γ_i' constructed above is called the *collapsing part* of Γ_i. For example, in the convergence $(\mathbb{R}^3, \Gamma_i, 0) \to (\mathbb{R}^3, \mathbb{R} \times \mathbb{Z}, 0)$ in Example 3, the group Γ_i' generated by γ_i^i is the collapsing part.

Chapter 3

Ricci Curvature and Fundamental Group

3.1 Preliminaries

We start with the Wei's theorem. That is,

Theorem 3.1.1 (G. Wei [29]) *Given n and $D, v > 0$, there exists a positive number $\epsilon = \epsilon(n, D, v)$ such that if a closed n-manifold M satisfies*

$$Ric(M) \geq -\epsilon, \qquad diam(M) \leq D, \qquad vol(M) \geq v$$

then $\pi_1(M)$ has polynomial growth of order $\leq n$.

As we said in chapter 1, $\pi_1(M)$ is almost nilpotent in the above theorem.

On the other hand, note that for any nilmanifold N^n which is not a torus, $\pi_1(N)$ has polynomial growth of order $> n$ (See [23], [30]). This shows that the conclusion of Wei's theorem may be extended to a stronger result and this thesis gives an affirmative answer.

To extend the well-known results for manifolds of almost non-negative sectional curvature to manifolds of almost non-negative Ricci curvature, a main problem is the *splitting phenomena* conjectured by K. Fukaya and T. Yamaguchi ([13]). Recently, J. Cheeger and T. Colding have announced that they have solved this conjecture.

Theorem 3.1.2 (Splitting Theorem [8]) *Let (X, p) be the pointed Hausdorff limit of a sequence (M_i, p_i) of complete n-manifolds with $Ric(M_i) > -\epsilon_i \to 0$. Then the splitting theorem holds for X, i.e., if X contains a line, then X splits $\mathbb{R} \times X'$ isometrically .*

Now we state the following fundamental results about estimating the lengths of closed geodesics and finiteness theorem for fundamental groups due to M. Anderson.

Theorem 3.1.3 (M. Anderson [1]) *In the class of compact n-dimensional Riemannian manifolds M such that*

$$(\#) \qquad Ric(M) \geq -(n-1)k^2, \quad vol(M) \geq v, \quad diam(M) \leq D,$$

there are only finitely many isomorphism classes of $\pi_1(M)$.

Theorem 3.1.4 (M. Anderson [1]) *Let M be a compact n-manifold satisfying the bounds (#). If γ is a curve in M with $[\gamma]^p \neq 0$ in $\pi_1(M)$ for all $p \leq N \equiv v_k(2D)/v$, then the length of γ satisfies*

$$l(\gamma) \geq \frac{Dv}{v_k(2D)},$$

where $v_k(r)$ denotes the volume of a geodesic ball of radius r in the space form of constant curvature $-k$.

Using these theorems together with other results, we can prove the following theorem which improves the Wei's thoerem :

Theorem 3.1.5 *Given n and $D, v > 0$, there exists a positive number $\epsilon = \epsilon(n, D, v)$ such that if a closed n-manifold M satisfies*

$$Ric(M) \geq -\epsilon, \qquad diam(M) \leq D, \qquad vol(M) \geq v$$

then $\pi_1(M)$ is almost abelian, i.e., $\pi_1(M)$ contains an abelian subgroup of finite index.

3.2 Algebraic Lemmas

In this section, we prove several algebraic lemmas which are needed. First let us describe a result which is related with the splitting phenomena and isometry group. Let Y be a compact metric space, and G a closed subgroup of the group of isometries of the product $\mathbb{R}^k \times Y$. Since G preserves the splitting $\mathbb{R}^k \times Y$, the projection $\phi : G \to Isom(\mathbb{R}^k)$ is well defined. Then we have the following lemma.

Lemma 3.2.1 ([13]) *For each $\delta > 0$, there exists a normal subgroup G_δ of G such that*

(1) *G/G_δ is discrete.*

(2) *There exists an exact sequence :*

$$1 \to G_\delta \to G \to \Lambda \to 1$$

where Λ *contains a finite index free abelian subgroup of rank not greater than* $dim(\mathbb{R}^k/\phi(G))$.

Proof. Let us remind just how to construct the normal subgroup G_δ. For a complete proof, see [13], lemma 6.1, p.29.

Let K be the kernel of ϕ, which acts on Y. We let

$$K_\delta := \langle \{ \gamma \in K \mid d(\gamma(x), x) < \delta, \quad for \quad all \quad x \in Y \} \rangle$$

where $\langle \quad \rangle$ denotes the group generated by the set inside. Then K_δ is a normal subgroup of G and the natural projection $\pi : G \to G/K_\delta$ is well-defined. Moreover, we can show that G/K_δ is a Lie group.

We now define

$$G_\delta = \pi^{-1}((G/K_\delta)_o),$$

where $(G/K_\delta)_o$ denotes the identity component of G/K_δ. Q.E.D.

Remark 1 In particular, note that if $G_\delta = \{1\}$, then G is almost abelian.

Lemma 3.2.2 *Let* Γ *be a finitely generated group, and* Λ *a subgroup of finite index. Then* Λ *is also finitely generated.*

Proof. Let $[\Gamma : \Lambda] = m$ and let $S = \{\gamma\Lambda\}$ be the set of all left cosets so that $|S| = [\Gamma : \Lambda] = m$ is finite.

Define

$$\tau : \Gamma \rightarrow \Sigma(S) \cong Sym(m)$$

by $\tau(\gamma) := \tau_\gamma$, where $\tau_\gamma : \Sigma(S) \rightarrow \Sigma(S)$ is a map defined by $\tau_\gamma(\gamma'\Lambda) = \gamma\gamma'\Lambda$ and $Sym(m)$ denotes the symmetric group. Then it is easy to check that everything is well defined and the kernel of τ is the core of Λ,i.e.,

$$ker(\tau) = \bigcap_{\gamma \in \Gamma} \gamma\Lambda\gamma^{-1}.$$

Moreover , by the first isomorphism theorem, $\Gamma / ker(\tau)$ is finite. So since Γ is finitely generated, so is $ker(\tau)$. In particular since $ker(\tau) \subset \Lambda$, by viewing the following exact sequence

$$0 \rightarrow ker(\tau) \rightarrow \Lambda \rightarrow \Lambda / ker(\tau) \rightarrow 0$$

we can see Λ is finitely generated. Q.E.D.

A solvable group γ is called *polycyclic* if there is a normal series

$$\Gamma = \Gamma_o \supset \Gamma_1 \supset \ldots \supset \Gamma_k = \{1\}$$

with every quotient Γ_i / Γ_{i+1} finite or infinite cyclic.

Lemma 3.2.3 *Let Γ be a finitely generated group of polynomial growth. Then Γ contains a torsion free nilpotent subgroup of finite index.*

Proof. Theorem 1.0.1 implies that Γ is *almost nilpotent*. That is, there is a nilpotent subgroup Λ of finite index. And then Λ is also finitely generated by Lemma 3.2.2.

Now, it follows from [24] that every subgroup of Λ is finitely generated. Thus, by [30] (Theorem 4.1 and 4.4) , Λ is polycyclic and so contains a torsion free subgroup of finite index. Q.E.D.

In the next section, we prove the main theorem.

3.3 Manifolds of Almost Non-negative Ricci Curvature

We are now in position to prove the main theorem.

Theorem 3.3.1 *Given $n, v > 0$ and $D > 0$,there exists $\epsilon = \epsilon(n, v, D) > 0$ such that if a closed n-manifold M satisfies that*

$$Ric(M) \geq -\epsilon, \quad vol(M) \geq v, \quad diam(M) \leq D,$$

then $\pi_1(M)$ is almost abelian.

Proof. By Wei's theorem 3.1.1, there is a positive $\epsilon_o = \epsilon_o(n, v, D)$ such that if a closed n-manifold M satisfies that

$$Ric(M) \geq -\epsilon_o, \quad vol(M) \geq v, \quad diam(M) \leq D$$

then $\pi_1(M)$ is finitely generated group of polynomial growth of order $\leq n$.

Suppose the theorem does not hold. Then there is a sequence of closed n-manifolds M_i with

$$Ric(M_i) \geq -\epsilon_i \to 0, \quad \epsilon_i < \epsilon_o, \quad vol(M_i) \geq v, \quad diam(M_i) \leq D,$$

and that $\pi_1(M)$ is not almost abelian.

By Lemma 3.2.3, there is a torsion free nilpotent subgroup Γ_i of $\pi_1(M_i)$ of finite index. Then Γ_i is also not almost abelian. Moreover, by Theorem 3.1.3, we may assume that

(*) the index $[\pi_1(M_i) : \Gamma_i]$ is uniformly bounded.

Passing to a subsequence, we may assume that M_i converges to a compact length space X. Since X might not be a manifold, we need to consider the action of Γ_i on the universal cover $\widetilde{M_i}$. For p_i in $\widetilde{M_i}$, consider $(\widetilde{M_i}, \Gamma_i, p_i)$.

Then Theorem 2.0.10 shows that there exist a length space (Y, q) and a closed subgroup G of $Isom(Y)$ such that $(\widetilde{M_i}, \Gamma_i, p_i)$ converges to a triple (Y, G, q) with respect to the pointed equivariant Hausdorff distance.

Splitting theorem 3.1.2 shows that Y is isometric to a product $\mathbb{R}^k \times Y_o$, where Y_o is compact.

On the other hand, by Theorem 3.1.4, there exists a positive number $\delta = \delta(n, D, v, \epsilon_o) > 0$ such that $\Gamma_i(\delta) := \{\gamma \in \Gamma_i : d(p_i, \gamma(p_i)) < \delta\} = \{1\}$, which implies that $G(\delta/2) = \{\gamma \in G : d(\gamma(q), q) < \delta/2\} = \{1\}$ and so the corresponding normal subgroup $G_{\delta/2}$ in Lemma 3.2.1 above is trivial in viewing of the construction of G_δ. Therefore, by Lemma 3.2.1, $G/G_{\delta/2} = G$ is discrete and G contains a finite index free abelian subgroup of $rank \leq n$.

Moreover, by (*), $\mathbb{R}^k \times Y_o/G$ is compact. In fact, $diam(\widetilde{M}_i/\Gamma_i) \leq 2(diam(M_i) + (max.index) \cdot diam(M_i)) \leq 2(1 + max.index)D$.

Thus, by Theorem 2.0.11, Γ_i is isomorphic to G for large i, a contradiction to the fact that Γ_i is not almost abelian. Q.E.D.

Chapter 4

Sectional Curvature and Fundamental Group

4.1 Preliminaries

We say that a manifold M is almost flat if for any $\epsilon > 0$, there exists a metric g on M such that

$$|K_M|^2 diam(M)^2 < \epsilon.$$

M. Gromov classified the almost flat manifolds. Namely,

Theorem 4.1.1 (M. Gromov [7], [14]) *If M satisfies $|K_M|^2 diam(M)^2 < \epsilon_n$ for a positive number ϵ_n, then a finite covering space of M is diffeomorphic to a quotient of a simply connected nilpotent Lie group by its lattice.*

If a manifold M satisfies $K_M \cdot diam(M)^2 > -\epsilon$, then we say that M is of ϵ-nonnegative curvature. We also say that a closed manifold M is of almost nonnegative curvature if for any $\epsilon > 0$, M has a metric of ϵ-nonnegative curvature. K. Fukaya and T. Yamaguchi generalized the almost flat manifold theorem in the π_1-level.

Theorem 4.1.2 ([13]) *There exists a positive number $\epsilon = \epsilon(n) > 0$, depending only on n such that if the curvature and diameter of a compact Riemannian n-manifold M satisfy $K_M \cdot diam(M)^2 > -\epsilon$, then $\pi_1(M)$ is almost nilpotent.*

Next we state the fibration theorem and classifying theorem of dimension three for the almost non-negative curvature manifolds due to Fukaya-Yamaguchi.

Theorem 4.1.3 ([34]) *Let M be an ϵ_n-nonnegative curvature with infinite fundamental group. Then a finite covering of M fibers over S^1.*

Theorem 4.1.4 ([13], [34]) *There exists a positive number ϵ such that if a closed three-dimensional manifold M is of ϵ-nonnegative, then a finite covering of M is either homotopic to S^3 or diffeomorphic to one of $S^1 \times S^2$, a nilmanifold or a torus.*

Finally we need simple algebraic lemmas.

Lemma 4.1.1 *Let $A = B = \mathbb{Z}$ and let Γ a torsion free nilpotent group satisfying the following exact sequence:*

$$1 \to A \to \Gamma \to B \to 1$$

Then Γ is abelian and so

$$\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. We can consider $A \cong \mathbb{Z}$ as a subgroup of Γ. Since $\Gamma/A \cong \mathbb{Z} = B$ is abelian, for any $\alpha, \beta \in \Gamma$,

$$(\alpha A)(\beta A) = (\beta A)(\alpha A)$$

i.e.,

$$\alpha\beta\alpha^{-1}\beta^{-1} \in A$$

So $[\Gamma, \Gamma] \subset A$. If $[\Gamma, \Gamma] = [1]$, then nothing to prove . Suppose that

$$[\Gamma, \Gamma] \neq [1].$$

Then there exist $\alpha, \beta \in \Gamma$ such that $\alpha\beta \neq \beta\alpha$.

Let $\Gamma_1 =< \alpha, \beta >\cong \mathit{Heisenberg} \quad \mathit{group} \subset \Gamma$. Then

$$\mathbb{Z} \cong [\Gamma_1, \Gamma_1] \subset [\Gamma, \Gamma] \subset A \cong \mathbb{Z}$$

Therefore, all are the same,i.e.,

$$\mathbb{Z} = [\Gamma_1, \Gamma_1] = [\Gamma, \Gamma] = A \cong \mathbb{Z}$$

Hence

$$\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_1/[\Gamma_1, \Gamma_1] \subset \Gamma/[\Gamma, \Gamma] \cong \Gamma/A \cong B \cong \mathbb{Z},$$

a contradiction. Q.E.D.

Lemma 4.1.2 *Let a group* Γ *admits an exact sequence :*

$$1 \to \Lambda \to \Gamma \to \mathbb{Z} \to 1$$

where Λ *is a finite group. Then* Γ *is almost abelian. In fact,* Γ *contains* $\mathbb{Z}$ *as a subgroup and the index of it is finite.*

Proof. It is trivial since $\Gamma/\Lambda \equiv \mathbb{Z}$ Q.E.D.

4.2 Manifolds of almost non-negative sectional curvature

In this section, we state and prove the following theorem which is one of the main theorems:

Theorem 4.2.1 *Let $n \leq 4$. There exists a positive number $\epsilon = \epsilon(n)$ such that if M is a closed Riemannian n-manifold satisfying*

(1) $K_M diam(M)^2 \geq -\epsilon$

(2) *$\pi_1(M)$ has polynomial growth of degree $\leq n$,*

then $\pi_1(M)$ is almost abelian.

Proof. If dimension is less than or equal to 3, then it is trivial. In fact, by the Theorem 4.1.2 and Lemma 3.2.3 , $\pi_1(M)$ contains a finitely generated torsion free nilpotent subgroup Γ of finite index. Then $growth(\Gamma) \leq n \leq 3$. So, if Γ is not abelian, it contains a Heisenberg group as a subgroup and the growth of Heisenberg group is exactly 4 by Example 1. Therefore Γ must be abelian,i.e., $\pi_1(M)$ is almost abelian.

Now consider case dim(M) = n = 4. Fibration theorem 4.1.3 implies that there exists a finite cover M^* of M such that

$$F \to M^* \to S^1$$

is a fibration with fiber F.

In particular,

$$dim(F) = dim(M^*) - 1.$$

Moreover, we have an exact sequence:

$$1 \to \pi_1(F) \to \pi_1(M^*) \to \mathbb{Z} \to 1.$$

Since

$$growth(\pi_1(M^*)) = growth(\pi_1(M)) \leq n = 4,$$

Gromov's splitting theorem ([16]) implies that

$$growth(\pi_1(F)) = growth(\pi_1(M^*)) - 1 \leq 3.$$

Thus we get

$$growth(\pi_1(F)) \leq dim(F) = 3.$$

Therefore, $\pi_1(F)$ is almost abelian since F admits a metric (pull-back metric) of ϵ-nonnegative as above.

On the other hand, by classifying theorem 4.1.4, there exists a finite cover F^* of F such that

(i) $F^* \sim S^3$ (homotopic)

(ii) $F^* \cong S^1 \times S^2$ (diffeomorphic)

(iii) $F^* \cong T^3$ (diffeomorphic)

Note that the *nilmanifold* cases are excluded because the fundamental group of F is almost abelian.

case i) $F^* \sim S^3$ (homotopic)

Then $\pi_1(F^*) = 0$ and so $\pi_1(F)$ is finite. Recall the following exact sequence:

$$1 \to \pi_1(F) \to \pi_1(M^*) \to \mathbb{Z} \to 1.$$

Note that

$$\pi_1(M^*)/ker = \pi_1(M^*)/\pi_1(F) \cong \mathbb{Z}$$

Since $\pi_1(F)$ is finite, $\pi_1(M^*)$ is almost abelian and so is $\pi_1(M)$.

case ii) $F^* \cong S^1 \times S^2$

Then we have

$$\pi_1(F) = \mathbb{Z} \oplus \Lambda,$$

where Λ is a finite group.

Recall we get

$$1 \to \mathbb{Z} \oplus \Lambda \to \pi_1(M^*) \to \mathbb{Z} \to 1$$

and $\pi_1(M^*)$ is almost abelian.

Let Γ be a torsion free nilpotent subgroup of $\pi_1(M^*)$ of finite index. Then $\Gamma \cap 0 \oplus \Lambda = \emptyset$. Therefore, we get the following exact sequence:

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z} \to 1.$$

So, by Lemma 4.1.1, Γ is abelian and so $\pi_1(M^*)$ is almost abelian and so is $\pi_1(M)$.

case iii) $F^* \cong T^3$

In this case, we have

$$b_1(F) = b_1(F^*) = 3$$

and so $F \cong T^3$ by Yamaguchi's theorem ([33]). Thus we have

$$T^3 \to M^* \to S^1$$

and

$$1 \to \mathbb{Z}^3 \to \pi_1(M^*) \to \mathbb{Z} \to 1.$$

Then we have

$$growth(\pi_1(M^*)) = 4$$

Let Γ be a torsion free nilpotent subgroup of $\pi_1(M^*)$ of finite index. We claim that Γ is abelian. Note that $growth(\Gamma) = 4$. So if Γ is not abelian, then there exist $\alpha, \beta \in \Gamma$ such that $\alpha\beta \neq \beta\alpha$. Let $\Gamma_1 =< \alpha, \beta >\cong Heisenberg \quad group \subset \Gamma$. Then $growth(\Gamma_1) = growth(\Gamma)$ implies that $\Gamma = \Gamma_1$. Also note that $\Gamma = \Gamma_1$ does not contain $\mathbb{Z}^3$ as a subgroup. But $\mathbb{Z}^3 \subset \pi_1(M^*)$ and since Γ is torsion free, Γ contains $\mathbb{Z}^3$, impossible. This is a contradiction and so Γ is abelian. Q.E.D.

Remark 2 If $n \geq 5$ then the theorem is false. For example, take a left-invariant metric g_ϵ on a nilpotent Lie group N with the sectional curvature $K_{(N,g_\epsilon)} \geq -\epsilon$ and canonical metric g_o on 2-sphere S^2. Then consider the product $M = N \times S^2$ with product metric $g = g_\epsilon \times g_o$. Then (M, g) satisfies the above conditions in the theorem, but the fundamental group of M is not almost abelian. However, in higher dimension case, I believe that such a splitting is the only possibility. Namely, if M satisfies the above conditions, then maybe $\pi_1(M)$ is almost abelian or M is a bundle with nilmanifold fiber over a compact space S with finite fundamental group.

Chapter 5

Scalar Curvature and Sigma Constant

5.1 Basic Facts on the Yamabe Problem

In 1960, H. Yamabe ([32]) proposed and attempted to solve the following problem using techniques of calculus of variations and elliptic partial differential equations.

The Yamabe Problem. *Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$, find a metric conformal to g with constant scalar curvature*

Suppose (M, g) is a compact Riemannian manifold of dimension $n \geq 3$. Any metric conformal to g can be written $\tilde{g} = u^{\frac{4}{n-2}} g$, where u is a positive smooth function on M. If s_g and $\tilde{s}$ denote the scalar curvatures of g and $\tilde{g}$, respectively, they satisfy the transformation law:

$$\frac{4(n-1)}{n-2}\Delta u - s_g u + \tilde{s} u^{\frac{n+2}{n-2}} = 0 \tag{5.1}$$

where Δ is the Laplacian in the g metric. Thus, $\tilde{g} = u^{\frac{4}{n-2}} g$ has a constant scalar curvature $\tilde{s}$ iff f satisfies the *Yamabe equation*:

$$Lu = \tilde{s} u^{\frac{n+2}{n-2}} \tag{5.2}$$

where $L = -\frac{4(n-1)}{n-2}\Delta + s_g$, called *conformal Laplacian*. This is a sort of 'non-linear eigenvalue problem'. Yamabe observed that equation (5.1) or (5.2) is the Euler-Lagrange equation for the functional restricted to conformal classes

$$\mathcal{S}(g) = \frac{\int_M s_g \, dv_g}{\left(\int_M dv_g\right)^{\frac{n-2}{n}}}.$$

Hence, by viewing the equations (1.1) and (5.1), we have

$$\mu(M, \mathcal{C}) = \inf_{\phi \in C^\infty(M), \phi > 0} \frac{\frac{4(n-1)}{n-2} \int_M (|d\phi|^2 + s_g \phi^2) \, dv_g}{\left(\int_M \phi^{\frac{2n}{n-2}} \, dv_g\right)^{\frac{n-2}{n}}}.$$

where g is a base metric contained in $\mathcal{C}$. It is now known by contribution of several people ([4], [25], [28], [32]) that the Yamabe problem is always solvable. In other words, for given constant $\tilde{s}$, there always exists a solution $u > 0$ satifying the equation (5.1). Now by using the standard minimax procedure, we can take the supremum, i.e.,

$$\sigma(M) = \sup_{\mathcal{C}} \mu(M, \mathcal{C})$$

We will call such a geometric invariant the *sigma constant*.

There are very few manifolds for which the sigma constant is known. In fact, the only ones known to the author are given by the following.

Theorem 5.1.1 (T. Aubin [4], [5]) *$\sigma(S^n) = n(n-1)vol(S^n(1))^{2/n}$, and for any n-manifold M, $\sigma(M) \leq \sigma(S^n)$.*

Theorem 5.1.2 (R. Schoen, O Kobayashi [21],[26],[Theorem 1.0.6])

$$\sigma(S^{n-1} \times S^1) = \sigma(S^n), \quad \sigma(\#_k(S^{n-1} \times S^1)) = \sigma(S^n).$$

Also it is reasonable to expect c.f. [6], that $\sigma(M)$ is a critical value of $\mathcal{S}$, i.e., any metric g_o with unit volume such that $s_{g_o} = \mu(M, [g_o]) = \sigma(M)$ is an Einstein metric. This remains unknown for the positive case. But for the negative case, we have

Theorem 5.1.3 ([6], [26]) *If $\sigma(M) \leq 0$, then any metric g with unit volume realizing $\sigma(M)$ is Einstein.*

5.2 Sigma Constant of Surgered Manifolds

Gromov-Lawson and Schoen-Yau have showed the following theorem:

Theorem 5.2.1 ([17],[27]) *Let M be a compact manifold which carries a Riemnnian metric of positive scalar curvature. Then any manifold which can be obtained from M by performing surgeries in codimension ≥ 3 also admits a metric of positive scalar curvature.*

Due to Yamabe, Trudinger, Elíasson, and Aubin, it is known that there is no obstruction to constant negative scalar curvature. Namely, any manifold of dimension ≥ 3 admits a metric of constant negative scalar curvature. However, there is a topological implication of scalar curvature which provides an obstruction to positive scalar curvature for certain special manifolds. In fact, we have

Theorem 5.2.2 ([22], [31]) *If the scalar curvature is positive on a compact even-dimensional spin manifold, then Hilzebruch $\hat{A}$ genus must vanish.*

Thus, if $\hat{A}$ genus, $\hat{A}(M) \neq 0$ for a compact even-dimensional spin manifold M, then M does not carry metrics of positive scalar curvature. So it is valuable to consider when or under what condition, manifold does not admit metrics of positive scalar curvature. $\sigma(M) \leq 0$ for a compact manifold M implies by solution of Yamabe problem that M does not admit metrics of positive scalar curvature. Theorem 1.0.6 due to O. Kobayashi can be viewed as a quantitative version of (a special case of) Theorem 5.2.1. which gives us a relation of the sigma constant of connected sum of two manifolds with each sigma constant. In particular, roughly speaking, in the negative case, we can see the sigma constant of connected sum of two manifolds is greater than or equal to negative of sum of absolute value of each sigma constant up to power related with dimension. On the same line, we can show the following theorem (Theorem 1.0.7).

Theorem 5.2.3 *Let M be a compact smooth manifold and assume $\sigma(M) \leq 0$. Let M_s be a manifold obtained from M by surgery on a homotopically trivial S^p , for $p \leq n-3$. Then $\sigma(M_s) \geq \sigma(M)$.*

Proof. Given $\epsilon > 0$, we can take a metric g on M with unit volume, i.e., $vol(M,g) = 1$ such that

$$s_g := scal(g) = \mu(M,[g]) = \sigma(M) - \epsilon$$

where $scal(g) = s_g$ denotes the *scalar curvature* of the metric g on M. Then, using Gromov-Lawson technique (see [17]), we can construct a metric g_s on M_s such that

$$scal(g_s) \geq scal(g) - \tau,$$

where $\tau = \tau(\epsilon) \to 0$ as $\epsilon \to 0$.

We claim that we can construct M_s so that

$$vol(M_s, g_s) \approx vol(M, g) = 1.$$

We start the embedded submanifold $S^p(\rho) \times D^{n-p}(\delta) \hookrightarrow M$ where δ is sufficiently small positive real number which will be decided later. The metric g on $S^p(\rho) \times D^{n-p}(\delta)$ is given by

$$g \approx dr^2 + r^2 ds^2_{S^{n-p-1}} + f^2(r,x) ds^2_{S^p}$$

and

$$f \approx 1 \quad \text{as} \quad r \to 0 \tag{5.2}$$

So the volume of $S^p(\rho) \times D^{n-p}(\delta)$ is

$$\begin{aligned} vol(S^p(\rho) \times D^{n-p}(\delta)) &\approx \int_{S^p(\rho)\times D^{n-p}(\delta)} f^p(r,x) dv_g \\ &\geq (1-\delta)^p vol(S^p(\rho)) vol(D^{n-p}(\delta)) \qquad \text{by (5.2)} \\ &= (1-\delta)^p \rho^p \delta^{n-p} vol(S^p(1)) vol(S^{n-p-1}(1)) \qquad (5.3) \end{aligned}$$

Now recall the Gromov-Lawson construction again (see [17]) :

$$N := \{(y, x, t) : (\|x\|, t) \in \gamma\} \subset S^p(\rho) \times D^{n-p}(\delta) \times \mathbb{R}, \qquad r(x) = \|x\|$$

We can choose $\theta_o > 0$ sufficiently small so that for $t \geq \delta/2$ (This implies that $\gamma(t) = \delta_o << \delta$ for $t \geq \delta/2), \partial(S^p(\rho) \times D^{n-p}(\delta_o))$ can be homotoped to the product metric on $S^p(\rho) \times D^{n-p}(\delta_o)$. Then cut-off N along $S^p(\rho) \times D^{n-p}(\delta_o)) \times \{t = \delta/2\}$ and glue $S_+^{p+1}(\rho) \times S^{n-p-1}(\delta_o)$, where $S_+^{p+1}(\rho)$ denotes the upper-hemi sphere in $S^{p+1}(\rho)$.

Let N_s denote the resulting manifold. Then

$$\begin{aligned} vol(N_s, g_s) &\approx \int_0^{\delta/2} vol\Big(S^p(\rho) \times S^{n-p-1}(\gamma(t))\Big)\, dt \\ &+ \ vol\left(S_+^{p+1}(\rho) \times S^{n-p-1}(\delta_o)\right) \end{aligned} \tag{5.4}$$

Since, on the level set of N, the metric is product, so the metric g_s on the $(S^p(\rho) \times S^{n-p-1}(\gamma(t))$ is given by

$$g_s \approx f^2(\gamma(t), x) ds^2_{S^p} + ds^2_{S^{n-p-1}(\gamma(t))}$$

Thus,

$$\begin{aligned} vol \quad & (S^p(\rho) \times S^{n-p-1}(\gamma(t)), g_s) \\ \approx \quad & \int_{S^p(\rho)\times S^{n-p-1}(\gamma(t))} f^p \\ \leq \quad & (1+\delta)^p \rho^p vol(S^{n-p-1}(\gamma(t))) vol(S^p(1)) \qquad \text{by (5.2)} \\ = \quad & (1+\delta)^p \rho^p (\gamma(t))^{n-p-1} vol(S^{n-p-1}(1)) vol(S^p(1)) \\ \leq \quad & (1+\delta)^p \rho^p (-2(1-\delta_o/\delta)t + \delta)^{n-p-1} \omega_p \omega_{n-p-1} \end{aligned}$$

where ω_p denotes the volume of p-sphere, S^p. Since we can choose θ_o and δ_o sufficiently small so that $\delta_o \leq \delta/2$ for given δ, it is easy to see that

$$\begin{aligned} \int_0^{\delta/2} (-2(1-\delta_o/\delta)t + \delta)^{n-p-1}\, dt &= \frac{1}{2(n-p)} \frac{\delta}{\delta - \delta_o} (\delta^{n-p} - \delta_o^{n-p}) \\ &\leq \frac{1}{(n-p)} (\delta^{n-p} - \delta_o^{n-p}) \leq \frac{\delta^{n-p}}{n-p}. \end{aligned}$$

So

$$\begin{aligned}
&\int_0^{\delta/2} vol\Big(S^p(\rho)\times S^{n-p-1}(\gamma(t))\Big)\,dt\\
\le\ & (1+\delta)^p\rho^p\omega_p\omega_{n-p-1}\int_o^{\delta/2}(-2(1-\delta_o/\delta)t+\delta)^{n-p-1}\,dt\\
=\ & \frac{(1+\delta)^p\delta^{n-p}}{(n-p)\rho^p}\omega_p\omega_{n-p-1}
\end{aligned}$$

and the second term in (5.4) is

$$vol(S_+^{p+1}(\rho)\times S^{n-p-1}(\delta_o))=\frac{1}{2}\rho^{p+1}\delta_o^{n-p-1}\omega_{p+1}\omega_{n-p-1}$$

Hence

$$\begin{aligned}
vol(N_s,g_s)\ &\le\ \left(\frac{\omega_p(1+\delta)^p\delta^{n-p}}{(n-p)}\rho^p+\frac{1}{2}\rho^{p+1}\delta_o^{n-p-1}\omega_{p+1}\right)\omega_{n-p-1}\\
&=\ \left(\frac{\omega_p(1+\delta)^p\delta^{n-p}}{(n-p)}+\frac{1}{2}\rho\delta_o^{n-p-1}\omega_{p+1}\right)\rho^p\omega_{n-p-1}
\end{aligned}$$

Now since $n-p\ge 3$, for fixed $\rho>0$, we can choose $\delta>0$ sufficiently small so that

$$\frac{(1+\delta)^p}{(n-p)}\le\frac{1}{2}(1-\delta)^p$$

and then choose $\delta_o>0$ so small that

$$\rho\delta_o^{n-p-1}\omega_{p+1}\le(1-\delta)^p\omega_p\delta^{n-p}$$

Then by (5.3), we have

$$vol(N_\delta,g_s)\le vol(S^p(\rho)\times D^{n-p}(\delta),g)$$

This proves the claim.

If then, by the maximum principle, we have

$$\sigma(M_s)\ge\mu(M,[g])-\tau\ge\sigma(\dot{M})-\epsilon-\tau$$

In fact, if $\mu(M_s, [g_s]) \geq 0$, then there is nothing to prove. If $\mu(M_s, [g_s]) < 0$, then there exists *unique* metric $\bar{g} \in [g_s]$ such that

$$\bar{s} := scal(\bar{g}) = \mu(M_s, [g_s]) < 0$$

and

$$vol(M_s, \bar{g}) = 1$$

This implies that there is a positive function $u \in C^\infty(M_s)$ such that

$$\int_{M_s} u^{\frac{2n}{n-2}}\, dv_{g_s} = 1 \tag{5.5}$$

and

$$\begin{aligned} u^{\frac{n+2}{n-2}}\bar{s} &= -\frac{4(n-1)}{n-2}\Delta_{g_s}u + scal(g_s)u \\ &\geq -\frac{4(n-1)}{n-2}\Delta_{g_s}u + (scal(g) - \tau)u \end{aligned}$$

Consider maximum of u on M_s and let $\max_{M_s} u = u(x_o)$. Then $\Delta u(x_o) \leq 0$. Since $vol(M_s, g_s) \approx 1$, we can see $u(x_o) \gtrsim 1$ from (5.5).

Note that $\bar{s}$ and $scal(g) - \tau$ are negative constants. Hence we have

$$\bar{s} = scal(\bar{g}) \geq scal(g) - \tau$$

Therefore,

$$\sigma(M_s) \geq \mu(M_s, [\bar{g}]) = \bar{s} \geq scal(g) - \tau = \mu(M, [g]) - \tau = \sigma(M) - \epsilon - \tau$$

Since ϵ is arbitrary, we have

$$\sigma(M_s) \geq \sigma(M)$$

Q.E.D.

Next we will prove the Theorem 1.0.8.

Theorem 5.2.4 *Let M_1 and M_2 be compact manifolds of same dimension ≥ 3. If both sigma constants of M_1 and M_2 are nonpositive and S^p is homotopically trivially embedded $p-$sphere in both M_1 and M_2, then*

$$\sigma(M_1 * M_2) \geq min\{\sigma(M_1), \sigma(M_2)\}$$

Proof. As theorem 5.2.3, given $\epsilon > 0$, we can take metrics g_i on $M_i (i = 1, 2)$ with unit volumes such that

$$s_{g_i} = \mu(M_i, [g_i]) = \sigma(M_i) - \epsilon$$

Then the same argument implies that there exist metrics $\tilde{g}_i$ such that

$$\tilde{s}_i \geq s_{g_i} - \delta$$

for some small number $\delta > 0$.

If $\mu(M_1 * M_2, [\tilde{g}_1 \cup \tilde{g}_2]) \geq 0$, then we have obviously

$$\sigma(M_1 * M_2) \geq \mu(M_1 * M_2, [\tilde{g}_1 \cup \tilde{g}_2]) \geq \min\{\sigma(M_1), \sigma(M_2)\}$$

If $\mu(M_1 * M_2, [\tilde{g}_1 \cup \tilde{g}_2]) < 0$, then there exists unique metric $\bar{g} \in [\tilde{g}_1 \cup \tilde{g}_2]$ such that

$$\bar{s} = s_{\bar{g}} = \mu(M, [\tilde{g}_1 \cup \tilde{g}_2]) < 0.$$

As theorem 5.2.3, we may assume that $vol(M_1 * M_2, \bar{g}) \leq 1$ and this implies that there is a positive function $u \in C^\infty(M)$, where $M = M_1 * M_2$, such that

$$\begin{aligned} u^{\frac{n+2}{n-2}} \bar{s} &= -\frac{4(n-1)}{n-2} \Delta u + scal(\tilde{g}) u \qquad where \quad \tilde{g} = \tilde{g}_1 \cup \tilde{g}_2 \\ &\geq -\frac{4(n-1)}{n-2} \Delta_{g_s} u + \left(\min\{s_{g_1} - \delta, s_{g_2} - \delta\} \right) u \end{aligned}$$

Considering the maximum of u on M , we get ,similarly as above,

$$\bar{s} \geq \min\{s_{g_1} - \delta, s_{g_2} - \delta\} = \min\{\sigma(M_1), \sigma(M_2)\} - \epsilon - \delta$$

Therefore,

$$\sigma(M) = \sigma(M_1 * M_2) \geq \min\{\sigma(M_1), \sigma(M_2)\}$$

Q.E.D.

Remark 3 By viewing theorem 1.0.6 and some examples, we also can expect that the same property holds for the positive case or nontrivial sphere surgery. Namely, we conjecture $\sigma(M_s) \geq \sigma(M)$ for any compact smooth manifold M and surgered manifold M_s.

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