On Encryption of Infinitesimal Neighbourhoods in Geometric Invariants of the Conic Structure on the Space of Nearby Submanifolds

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Darko Mrakovčić

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Abstract of the Dissertation

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The object of study here is an integrable conic connection defined on a general conic structure. This holomorphic second-order geometric structure was introduced in [12] and shown to be (naturally) equivalent to a double fibration inducing a holomorphic family of submanifolds; in this correspondence the underlying manifold of the geometric structure is the parameter space of the family. The following problems are considered: characterization of those conic structures which are induced by families of submanifolds, examination of the 'degree of reconstructibility' of the family from the

conic structure alone, construction of an apparatus for translation of the invariants of an embedding into differential invariants of the induced conic structure, introduction of analogous invariants (namely fattenings of certain manifolds) even in the case of conic structures not induced by families of submanifolds, construction of distinguished connections etc. In connection with the above translation problem, we restrict our attention to the case of infinitesimal neighbourhoods of low order, but the method we develop seems to constitute the rudiments of a general approach to such problems more direct than the method used in [5]. Furthermore, we obtain a generalization and reinterpretation in the context of conic structures of some of the results from [13] on locally complete parameter spaces of Legendrian submanifolds. Finally, the above general results are applied to the 'hypersurface-directional' conic structures equivalent to G_n-structures (in terminology of [4]; the indicated structural group is a quotient of $GL(2)$). Among these applications are a generalization to arbitrary n of a theorem from [4]) on the spaces of Legendrian rational curves and G_n-structures, the description of a family of rational curves in a surface in terms of mutually compatible G_n-structure and projective structure, and determination of the values of the self-intersection index n for which the G_n-structure alone (subject to certain restrictions) suffices for that purpose. (These results generalize from the cases $n = 1, 2$ to the general case the description of such families in [8].) Furthermore, we study the relationship between the intrinsic torsion (or torsion of the Cartan connection when the latter is defined) of a G_n-structure and the infinitesimal neighbourhoods of the rational curves. Apart from the theory of G_n-structures, we also show how some simple conic-structural invariants provide a tool for proving a result stated in [16] involving the first-order infinitesimal neighbourhood of an anti-self-dual Kaehler surface in its twistor space.

Mojoj majci i u spomen mojem ocu

Contents

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Introduction

The original objective of this thesis was to generalize Hitchin's 'minitwistor correspondence' to arbitrary self-intersection mumber. This essentially reversible correspondence was established in [9] in the following way: any locally complete holomorphic family of rational curves embedded with self-intersection two is encoded in the corresponding Einstein-Weyl structure on the parameter space of the family.

However, in the process of accomplishing the above objective, other results of more general character and of independent importance have been obtained. Indeed, in the general case of self-intersection n, the analogous geometric structure on the parameter space also ultimately admits a description in terms of standard geometric concepts: it turns out to be either a 'G_n-structure' (in R. Bryant's terminology [3]) subject to some conditions, or, in case of low self-intersection index, a G_n-structure together with a compatible projective structure. (The group G_n is the quotient of $GL(2)$ defined below.)

That notwithstanding, in order to establish this correspondence and derive its further properties, we were naturally lead to view such a structure as a special case of a more general geometric structure. Later we recognized the latter as a conic structure, or, in case of low self-intersection, a conic connection, introduced by Yu. Manin in [13].

In fact, many of the results we have derived in order to study families of rational curves in surfaces were of a more general nature; more precisely, their natural context was the general theory of conic structures and conic connections. Therefore, the objective of the thesis has gradually evolved into an investigation of general properties of conic structures. In this context the G_n-structures serve only as a basic example, which, albeit relatively simple, in comparison with paraconformal conic structures exhibits certain generic features of conic structures more faithfully.

To explain the original motivation for the study of conic structures, we quote Yu. Manin's own description of the role played by conic structures and conic connections (v. [13]): '... These concepts arose when the geometrical data in the self-duality theory of the Yang-Mills and Einstein equations (and also the Yang-Mills supersymmetry equations) were axiomatized.' Indeed, the conic structures of paraconformal type (or 'Grassmanian spinor structures' in the original terminology of [13]) have been rather exhaustively studied in the above quoted work, and also by T. Bailey and M. Eastwood in [2]. There also exists a vast literature devoted to the more special cases of conic structures associated to self-dual and quaternionic manifolds.

G_n-structures

The following part of the introduction will be devoted to a preliminary and rather formal exposition (without proofs) of some of the results on G_n-structures which will in Chapter V be obtained by applying the more general and more consequential results of this thesis. The purpose of this exposition is to make the special results more readily accesible, i.e. to formulate them using only the bare minimum of the rather elaborate terminology and notation we introduce later on. In other words, we will attempt to formulate less complex results involving these special conic structures and conic connections also in an alternative form without reference to these concepts.

Unless otherwise stated, all the objects we consider will belong to the holomorphic category (although many of the results also hold in the real-analytic category).

Definition 0.1 Let $n \in \mathbf{N}$. We will identify the n-th symmetric power $(\mathbf{C}^2)^{\odot n} := \odot^n \mathbf{C}^2$ with $\mathbf{C}^{n+1}$ by means of the basis of $(\mathbf{C}^2)^{\odot n}$ built in the usual way from the standard basis e_1, e_2 of $\mathbf{C}^2$. The (closed) subgroup $G_n \subset GL(n+1)$ is by definition the image of $GL(2)$ under its natural action on the n-th symmetric tensor power (i.e. on the space of n-th degree polynomials in e_1, e_2). A G_n**-structure** on a $(n+1)$-dimensional complex manifold M is a reduction P_{G_n} with the structural group G_n of the linear-frame bundle $P_{GL(n+1)}$ on M.

Definition 0.2 In the case of even (resp. odd) n the **underlying conformal structure** (resp. **underlying conformal symplectic structure**) of a G_n-structure is defined by transfering the standard structure of a conformal euclidean (resp. conformal symplectic) vector space on the symmetric tensor power $(\mathbf{C}^2)^{\odot n}$ to the tangent spaces $T_m(M)$ via arbitrary distinguished frames. The above mentioned standard structure on $(\mathbf{C}^2)^{\odot n}$ is constructed from the (unique due to 2-dimensionality) structure of a conformal symplectic vector plane on $\mathbf{C}^2$ by means of tensor multiplication of inner products.

Remark 0.3 The above definition could be rephrased in a more formal language as follows: Let us denote by $v_0, ..., v_n$ the standard basis of $\mathbf{C}^{n+1}$; thus

$v_i = e_1^{\odot i} \odot e_2^{\odot n-i}$. Furthermore, in the case of even (resp. odd) n we consider on this vector space the euclidean (resp. symplectic) inner product defined by $< v_i, v_{n-j} >= \delta_{ij}$ (resp. by $< v_i, v_{n-j} >= (-1)^{n-i}\delta_{ij}$). We claim that $G_n$is contained in the automorphism group $CO(n+1)$ (resp. $CSp((n+1)/2)$) of the conformal class of the above euclidean (resp. symplectic) inner product, and we define the underlying conformal (resp. conformal symplectic) structure of a G_n-structure by requiring that the latter be a reduction of the former.

Proposition 0.4 Let

$$S \xleftarrow{\nu} R \xrightarrow{\mu} M \tag{0.1}$$

be a *locally complete* (holomorphic) family of non-exceptional (i.e. with non-negative sef-intersection) embedded rational curves (of the form $\nu(\mu^{-1}(m))$ for $m \in M$) in a surface S. We will suppose that the self-intersection index $n\,(\geq 0)$ of such a curve is independent of the curve. (This is obviously true as soon as M is connected). Then the map ν is open and submersive onto its image, i.e. (0.1) becomes a double fibration if S is replaced by this image. Furthermore, *in this situation M has dimension $n+1$ and comes equiped with a naturally associated G_n-structure*, which will be referred to as the **canonical** G_n-structure on the parameter space of rational curves in a surface. (The explicit construction of this structure will be given in 0.13.)

Definition 0.5 For two families

$$S \xleftarrow{\nu} R \xrightarrow{\mu} M \quad \text{and} \quad S' \xleftarrow{\nu'} R' \xrightarrow{\mu'} M$$

of submanifolds with common parameter space M we define an **isomorphism over parameter space** as an isomorphism of the families inducing identity on the parameter space; explicitly, such an isomorphism consists of a pair of biholomorphisms $\phi : R \to R', \psi : S \to S'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} S & \xleftarrow{\nu} & R & \xrightarrow{\mu} & M \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle id} \\ S' & \xleftarrow[\nu']{} & R' & \xrightarrow[\mu']{} & M \end{array} \tag{0.2}$$

A class of families isomorphic over their parameter space M will be called a **structure on M of a parameter space** of (a family of) submanifolds. Furthermore, for a structure of a parameter space as in 0.4 on M the canonical G_n-structure on M relative to a representative family is virtually by definition (v. 0.13) independent of the latter, and will be called the **underlying G_n-structure** of the parameter-space structure.

Lemma 0.6 Consider on a manifold M a structure of a **completely geometric** locally complete parameter space of non-exceptional rational curves in a surface; the reasonably weak condition of being 'completely geometric', which is always fulfilled locally for sufficiently restricted families, means by definition that for any point of the surface the manifold of parameters of incident rational curves (i.e. rational curves through that point) is non-empty and connected. (In particular connectedness of the surface is in view of 0.4 equivalent to that of M.) Then the family representative of this structure is determined up to a *unique* isomorphism over parameter space; more precisely, for two families of rational curves in surfaces (with common parameter space M) realizing (i.e. representative of) this structure, the isomorphism over parameter space between these families is unique. (In other words, the natural association of completely geometric parameter-space structures to completely geometric families is an equivalence of categories, i.e. the family is functorially reconstructible from the parameter-space structure.)

Proposition 0.7 Let $n \geq 5$. Consider a G_n-structure which can be realized as the canonical G_n-structure on the parameter space of some completely geometric (locally complete) family

$$S \xleftarrow{\nu} R \xrightarrow{\mu} M$$

(of rational curves of sef-intersection n in a surface S). *Then this family is determined (by the G_n-structure) up to a unique isomorphism over parameter space*, i.e. for any other such family

$$S' \xleftarrow{\nu'} R' \xrightarrow{\mu'} M$$

there exist unique biholomorphisms $\phi : R \to R', \psi : S \to S'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} S & \xleftarrow{\nu} & R & \xrightarrow{\mu} & M \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle id} \\ S' & \xleftarrow[\nu']{} & R' & \xrightarrow[\mu']{} & M \end{array} \tag{0.3}$$

Remark 0.8 In view of 0.6, the above proposition could be restated as follows: If $n \geq 5$, a structure of a completely geometric locally complete parameter space of embedded rational curves of self-intersection n is uniquely determined by the underlying G_n-structure. In other words, with the appropriate identification the set of these parameter-space structures on M is a subset of the set set of G_n-structures on M.

Remark 0.9 It is well-known (cf. [9]) that an analogue of the above proposition for $n = 2$ does not hold. (Indeed, a G_2 -structure on a 3-fold is easily seen to be simply a conformal structure, and, according to [9], we need a connection in addition to that structure in order to reconstruct the family.) We will consider the case $n \leq 4$ later in this chapter.

Definition 0.10 Consider a general G_n-structure on a manifold M. A non-zero subspace V of a tangent space $T_m(M)$ is said to be an α-**subspace** if for some (or, equivalently, every) frame p at m belonging to the structure there exists a non-zero vector $e \in \mathbf{C}^2$ such that

$$V = p(e^{\odot n-i} \odot (\mathbf{C}^2)^{\odot i}).$$

Of course, in this equality i is precisely $dim\, V - 1$. (In particular, α-lines are precisely vector lines in $T_m(M)$ consisting of 'perfect powers' since here $i = 0$.) In other words, if tangent vectors are by means of a structural frame thought of as homogeneous polynomial functions on $\mathbf{C}^{2*}$, then an α-subspace consists of all functions for which a fixed point in $\mathbf{P}(\mathbf{C}^{2*})$ is a zero with multiplicity not less than a fixed integer. Furthermore, a submanifold $X \subset M$ is said to be an α-**submanifold** if all its tangent spaces are α-spaces.

Lemma 0.11 Let M be an $(n+1)$-dimensional manifold.

(i) For a G_n-structure on M the disjoint union of the sets of α-line directions (resp. α-hyperplane directions) at various points is the total space of a holomorphic subfamily of the projectivized tangent bundle $\mathbf{P}(T(M))$ (resp. of the projectivized cotangent subbundle $\mathbf{P}(T^*(M))$) whose fibers are the images of Veronese embeddings of rational curves into the projective spaces constituting the projectivized bundle. (We will say more briefly that those fibers are Veronese rational curves.) Clearly, this subfamily is in fact a subbundle.

(ii) For a given G_n-structure on M the underlying conformal inner-product structure establishes a biholomorphism of the manifold of all α-line directions (at various points of M) and the manifold of all α-hyperplane directions as a restriction of the induced isomorphism of the projetivized tangent and cotangent bundles. (In other words, thus obtained perpendiculars of α-lines are α-hyperplanes and vice versa.) Furthermore, all the above directions are null relative to this underlying structure. (The last statement is tautological in the conformal symplectic case, of course.)

(iii) The correspondence established in (i) between G_n-structures on M and subfamilies of the projectivized tangent (resp. cotangent) bundle formed by Veronese rational curves is bijective.

[In the case of α-lines the assertion (iii) is on the level of fibers stated in [3]. For a proof of the lemma see Subsection III.3.1. Incidentally, the only assertion which is not completely straightforwrd is surjectivity in (iii).]

Definition 0.12 A curve-directional (resp. hypersurface-directional) **Veronese conic structure** on a manifold M is a (holomorphic) subfamily of the projectivized tangent (resp. cotangent) bundle of M consisting of Veronese rational curves. According to 0.11 such a structure is completely 'equivalent' to a G_n-structure and the subfamily is actually a subbundle. Furthermore, these structures are in terminology of [13] special conic structures. (Cf. the second part of this introduction.)

Proposition 0.13 Let us again consider the situation of 0.4. The canonical G_n-structure is explicitly defined in the the following way: For any point in the surface S the set of parameters of incident rational curves is a hypersurface in M. Claim: The jets of the such hypersurfaces through any point of M form a Veronese rational curve and the disjoint union of these rational curves is a hypersurface-directional Veronese conic structure on M. In particular, there exists a unique G_n-structure for which all such hypersurfaces are α-hypersurfaces; this structure is called the canonical G_n-structure on the parameter space of the family. Furthermore, every α-hyperplane direction (at every point in M) is realized as the (α-) hypersurface of parameters of the rational curves incident with a unique point of S.

Proposition 0.14 Let M be a manifold of arbitrary dimension $n+1$. There exists a canonical bijective correspondence between the following objects:

(a) Structures on M of a completely geometric parameter space (v. 0.6) of rational curves (of self-intersection n) in a complex surface;

(b) **Admissible** pairs formed by a G_n-structure and a **compatible** projective structure, where the indicated two properties are defined as follows: Compatibility means that for every α-hyperplane direction (at every point of M) the (obviously unique) tangent (totally) geodesic (relative to the projective structure) hypersurface exists and is in fact an α-hypersurface; Admissibility means that the foliation of the space R of all α-hyperplane directions by canonical lifts of the maximal geodesic immersed α-hypersurfaces is in fact a fibration, and that these hypersurfaces are immersed injectively.

(c) **Admissible** integrable conic connections (in terminology of [13]) on hypersurface-directional Veronese conic structures, where admissibility is the reasonably weak condition on maximal geodesic immersed hypersurfaces of the preconnection analogous to the admissibility in (b).

In fact, the immersed hypersurfaces in (b) and (c) are the same in both contexts relative to the bijective correspondence between these objects.

Proposition 0.15 Suppose that the mutually corresponding objects in 0.14 have been chosen. We will call a connection on M **complementary** (relative to these objects) if the geodesic α-hypersurfaces of the projective structure are *autoparallel* relative to the connection. In the language of conic structures, a connection is complementary iff it induces the given conic connection.

Claim: 'Localized' complementary connections at a point form a (non-empty) $(3n+1)$-dimensional affine space (embedded into the affine space of all localized connections). More precisely, its vector space is G_n-invariant and isomorphic to the G_n-module $V_n \oplus V_n \oplus V_{n-2}$, where V_k denotes the irreducible modul of degree k (i.e. of dimension $k+1$. Furthermore, the conic structure in question is *symmetric* in the sense that the space of torsionless complementary connections is non-empty. More precisely, it consists precisely of the torsionless connections inducing the given projective structure. (In particular, it is of dimension $n+1$. Therefore, the torsions of complementary connections form a G_n-module isomorphic to $V_n \oplus V_{n-2}$.

Remark 0.16 Finally, we very briefly indicate some of the further applications of general conic-structural theory in the case of Veronese conic structures: G_n-structures with vanishing 'intrinsic pretorsion' precisely encode families of Legendrian rational curves; here the intrinsic pretorsion is an invariant obtained from the intrinsic torsion of the G_n-structure by an explicit linear-algebraic procedure. In particular, since for $n = 3$ this invariant actually coincides with the intrinsic torsion, Legendrian rational curves with the appropriate normal bundle can serve to construct 1-flat G_3-structures; this is a theorem proved by a different method in [3]. Furthermore, when the Legendrian curves are actually lifts of rational curves in a surface, i.e. under the assumptions of the previous proposition, the 'conjunctive intrisic pretorsion' (which is finer than the intrinsic pretorsion) and 'conjunctive intrisic precurvature' vanish. In this situation the geodesic α-curves exist through all α-line directions, and in the case of even n they are actually null-geodesics of the underlying conformal structure. Under the same assumption, a still finer invariant constructed from the intrisic torsion precisely corresponds to the first-order infinitesimal neighbourhood of the rational curve in the surface. If this invariant vanishes, intrinsic torsion also encodes the 'transverse component' of the second-order infinitesimal neighbourhood.

General Conic Structures

Before an intuitive exposition of the general results a justification for their inclusion into the thesis, as well as arguments for further development of the theory of general (i.e. not only paraconformal) conic structures, would be in order. Although the twistor correspondences producing paraconformal conic structures are at present undoubtedly of greatest interest, it should be noticed that a given first-order geometric structure can often be thought of as a conic structure *in different ways*, and each of these associated conic structures may give rise (in a way described further in this introduction) to a set of invariants of some importance for the geometric structure in question. [For example, an oriented conformal structure on a complex 4-manifold can be given by the total space of one-dimensional null-directions, or (alternatively) hypersurface null-directions, or (two-dimensional totally null) α-directions, or β-directions.] Thus, even in the study of paraconformal structures it may be useful to consider the interaction with other associated conic structures; in particular, even if a paraconformal conic structure does not arise from a double fibration, the geometric structure in question may still be encoded in a 'general twistor space' by means of some associated general conic structure; cf. the role played by the ambitwistor space in the study of general, not only non-self-dual, conformal 4-manifolds.

Although these general twistor double fibrations do not necessarily solve important PDEs of mathematical physics (due to their abundance), the solutions of the PDEs in question may still admit interesting characterizations in terms of the general twistor spaces. [Cf. the elegant characterization of general (not only self-dual) Einstein conformal structures on complex 4-manifolds in terms of the ambitwistor space in [11].] Since the conditions imposed on the conic structure by the PDE are of finite order, it seems plausible that it is possible to obtain such characterizations involoving only sufficiently high-order infinitesimal neighbourhoods of the given manifolds within the general twistor space. (Indeed, in view of the analytic context and the usual local completeness assumption, the 'Kodaira's main theorem' from [17] implies that solutions can be characterized at least in terms of germs of embeddings; on the other hand, in [11] only finite-order Legedrian fattenings are used for that purpose.) In accomplishing this objective it is clearly desirable to have a more direct and more general method designed to translate the invariants of the infinitesimal neighbourhoods into differential invariants of the conic structure. The lack of such a method at present might be explained by the non-genericity of many paraconformal conic structures in the sense that the first two infinitesimal neighbourhoods are tautologically trivial; consequently the first stages of this translation (which might due to their simplicity serve as a guide for the later stages) are degenerate for such conic structures. In this thesis these first stages are carried out in detail. (In this context the G_n-structures, although perhaps not very interesting in themselves, acquire some importance at least by pointing the way to carry out the necessary cohomological computations.)

Another possible argument for a further development of the general theory of conic structures is a peculiar case of interaction between two mutually associated conic structures (cf. Chapter 4): with each 'hypersurface-directional' conic structure one may associate a 'derived' conic structure which is more likely to arise from a double fibration; in fact there is often a canonical integrable conic connection on the latter conic structure and the submanifolds from the family thus obtained are Legendrian with respect to a canonical contact structure. (A special case of this construction produces a family of Legendrian quadrics in the ambitwistor space of an arbitrary conformal manifold, cf [10],[11].) This family could be thought of as a 'family of canonical lifts of hypersurfaces in a general twistor space for the original conic structure (and thus of lower dimension) given up to contact equivalence'; here the 'general twistor space' is not just unspecified but often completely virtual; however, the thickenings of those 'hypersurfaces' realized by this 'virtual space' might be well-defined and serve as a guide for the study of the comparatively coarser Legendrian fattenings of the 'canonical lifts of these hypersurfaces' into the contact manifold. (In Chapter 4 we give a generalization and reinterpretation from the viewpoint of the theory of conic structures of certain related results of S. Merkulov in [14].)

In the remainder of the introduction we give a self-contained intuitive (and thus non-rigorous) explanation of the main ideas involved in the method we use to study conic structures.

We first describe Manin's correspondence between families of submanifolds and certain second-order geometric structures, namely integrable (second order) *conic connections.* These structures are essentially determined by certain first-order geometric structures called (first-order) *conic structures.* In our terminology the expression 'conic connections' has been replaced with the term *'preconnections'* in order to emphasize that these objects are neither connections nor necessarily even classes thereof, in particular to avoid confusion with the concept of conic-structure-preserving connections.

Consider a (complex) manifold M and integers $p \leq dim\, M$ and k. Let JM denote the manifold of k-jets of p-dimensional ('unparametrized') submanifolds of M. [In other words, JM is the k-th-order contact manifold of M. When $k = 1$, this is simply the total space of the Grassmanian bundle of TM.] Then by a **generalized conic structure** of order k and jet-dimension p we simply mean a submanifold $R \subset JM$. The k-jets belonging to the generalized conic structure R are called **integral**, and, similarly, an x-dimensional submanifold of M is said to be **integral** if all its k-jets are integral (i.e. if its canonical k-th order lift is contained in the generalized conic structure R).

Thus, a k-th order generalized conic structure, or a **'PDE given up to point transformations'** in traditional terminology (v. [1]), could be described as a 'completely invariantly' defined PDE, where the notion of dependent and even independent variable is meaningless. For example one obtains such a structure from a standard PDE (on sections of a fibre bundle) by 'forgetting' not only the codomain of a solution, but also the domain (i.e. the fibration of the total total space).

Conic structures and 'sufficiently regular' preconnections are special cases of generalized conic structures of respectively first and second order. More precisely, a generalized conic structure of first order is simply called a **conic structure** if the restriction $\pi : R \to M$ of the canonical projection $JM \to M$ is a proper submersion. In other words, a conic structure is the total space of a holomorphic subfamily (conceivably not a subbundle) of the Grassmanian bundle of M.

Let us now consider a holomorphic family

$$S \longleftarrow R' \longrightarrow M$$

of submanifolds of S. Recall that this implies submersivity and properness of the indicated mapping of the incidence-relation manifold R onto the parameter space M; in particular, its fibers, i.e. the submanifolds of S from the family, are compact. Such a family can be encoded in a preconnection on its parameter space (or base) M if and only if it is subject to an additional reasonably weak condition; we will refer to the latter as **first-order geometric amenability**. This condition includes first of all the requirement that the family be double-fibrational, i.e. that the other mapping (into the ambient manifold S) also be submersive. In order to explain the concept of first-order geometric amenability, we assume that the family is double-fibrational and first construct the **conjugate family** of possibly non-compact submanifolds: these are the submanifolds of the parameter space M (for the original family) which are obtained by selecting a point $s \in S$ and considering the parameter space (which is easily seen to be a submanifold of M of the same codimension as the submanifolds of S) of all submanifolds from the original family through the point s. Since the conjugate family of possibly non-compact submanifolds obviously consists of the same diagram with the roles of parameter space and ambient space reversed, the result of the application of the same procedure to the conjugate family is simply the original family: parameters of submanifolds from the conjugate family through a given point m are precisely the points of the submanifold from the original family with parameter m. In particular, we obtain a mapping of the submanifold of S with parameter m into the the space of 1-jets through m (i.e. into a certain Grassmanian space). We say that the original family is **(first-order) geometrically amenable** or, more briefly, **geometrical**, if this mapping is always an embedding. (Since the domain of this mapping was assumed to be compact, and the space of $(k+1)$-jets tangent to a given k-jet is an affine space, this is indeed a reasonably weak condition.)

A (first-order) geometrical parameter space M of submanifolds of a manifold S inherits a canonical (first-order) conic structure: indeed, its total integral-jet space is naturally defined to be simply the manifold of all 1-jets of ('unparametrized') submanifolds from the conjugate family. It is not difficult to see that the obvious canonical mapping (cf. the above definition of first-order geometric amenability) of the incidence manifold R into the the total-integral jet space of the conic structure is then a biholomorphism over M; in this way these two manifolds will henceforth be identified.

With the obvious generalizations of these definitions, first-order geometric amenability of a family of possibly non-compact submanifolds is simply the condition that the induced generalized conic structure be of first order; the generalized conic structure induced by the conjugate family (on the other base of the double fibration) could be of arbitrary order. [E.g. in the special case of a family of rational curves in a surface S, the induced conic structure on the parameter space M is easily seen to be of first order, while the order of the generalized conic structure on the surface S (induced by the conjugate family) equals the self-intersection number of the rational curves. Incidentally, in this case the conic structure on M, which will be referred to as a **Veronese conic structure** in this thesis, is precisely one of the conic structures (namely the 'hypersurface-directional' one) associated to (and equivalent to) a G_n-structure. The former term is motivated by the fact that the integral hypersurface 1-jets through a given point form a Veronese curve in the Grassmanian (i.e. in the projectivized cotangent space)].

A *preconnection* which is *integrable* in a certain sense is precisely the geometric (i.e. infinitesimally defined) structure which will 'by design' be equivalent to the (a priori not differential-geometric) structure of a geometrical parameter space of submanifolds. In order to motivate its definition, it suffices to recall that the total integral-jet space of the conic structure induced on the parameter space of submanifolds has been canonically identified with the 'incidence manifold' of the double fibration. Thus it comes equiped with two fibrations; however, one of them does not contain new information (relative to the conic structure) since its fibers are just spaces of integral 1-jets through fixed points. Of course, the other fibration suffices to reconstruct the family of submanifolds; on the other hand, it is clearly reconstructible from the associated distribution. Therefore, a preconnection on the given conic structure is defined simply as a distribution of a certain type (v. Chapter 2 for more details) on the total integral-jet space, and its integrability as the integrability of the distribution. In fact, preconnections satisfying a condition weaker than integrability can be thought of as second-order generalized conic structures such that for each integral (relative to the underlying conic structure) 1-jet there exists precisely one tangent integral (relative to the preconnection) 2-jet. (Thus, we have actually obtained a generalization of the concept of a 'normal second second-order PDE given up to point transformations'.) It goes without saying that higher-order 'possibly non-proper' preconnections could be defined in an analogous way.) In particular, projective structures (i.e. sprays given up to reparametrization of integral curves) are curve-directional preconnections such that the underlying conic structures are 'omnidirectional' (i.e. trivial).

The main principle in the study of conic structures and connections is to view these objects as ordinary geometric structures, or 'fields of geometric quantities' in terminology of Alekseevski, Vinogradov and Lychagin in [1], by carrying out 'globalization' at a point of the basic manifold; of course, the globalization is not relative to the basic manifold, but a subspace of the incidence manifold, i.e. the space of integral 1-jets through a given point. (Cf. the method used by Yu. Manin in [13] to investigate existence and degree of uniqueness of a connection realizing a given 'omnidirectional' preconnection.) More precisely, by requiring that the integral-jet spaces at fixed points be compact submanifolds of Grassmanians and to be 'isomorphic' in an obvious sense at various points, we can consider an 'infinitesimally homogeneous' conic structure as a (holomorphic) field (i.e. section) in a certain bundle, i.e. as a field of **localized conic structures** (or 'conic-structural geometric quantities'). The 1-jet of this field will be called a **conic-structural (1-)jet**. Such structural jets are again local geometric structures (of *second* order and not necessarily isomorphic at various points) and their classification according to the general theory of geometric structures amounts to the study of intrinsic torsion of a conic structure. [Recall that we can define 'tangentiality' (or 'structure-preserving') of a 'localized connection' relative to a given structural jet as the requirement that the structural jet be horizontal for the localized connection; now the intrinsic torsion could be defined as the class of all torsions of tangential localized connections]. It turns out that the study of such (i.e. second-order) invariants of (not local, but 'expanded') conic structures involves not only classical facts regarding prolongability (of a generalized conic structure at a given jet), but also non-trivial sheaf-cohomological arguments at a given point of the basic manifold applied to the integral-jet space at that point. In the case when the conic structure arises from the structure of a parameter space of submanifolds, the principle expounded above precisely corresponds to the fundamental idea of R. Penrose that the local geometric data at a point of such a space are stored in the 'vicinity' of the submanifold with that parameter, i.e. as the complex-differential-topological properties of the corresponding infinitesimal neighbourhood.

The guiding idea in our study of conic structures and preconnections is to apply the above mentioned principle to various auxiliary naturally defined affine and vector bundles over the integral-jet space at a point. In order to illustrate such constructions, let us recall that the vector space of local **connectors** at a point m, by which we will mean the vector space of the affine space of localized connections, is a triple tensor product (namely $T \otimes T^* \otimes T^*$, where T denotes the tangent space at m). At a given integral 1-jet we can form seven distinct quotient affine spaces of the space of localized connections (since a subspace of T, and thus also of T^*, is distinguished by the jet, and we can apply quotient maps in various combinations of the three factors.) A localized conic structure at m will therefore determine seven quotient affine bundles (of the trivial one) over the integral-jet space. Although these bundles, being completely determined by a linear-algebraic structure (i.e. a localized conic structure on a single tangent space), are not in themselves geometrically significant (e.g. they all admit sections), in the presence of a conic structural 1-jet they provide the starting point for many useful constructions: we can form their distinguished subbundles by imposing various conditions of 'tangentiality' relative to the structural jet, symmetry with respect to (the obvious analogues of) what A. Besse calls the 'canonical involution' in [18]; furthermore, we can form certain fiberwise amalgamated products of the affine bundles thus obtained, and construct some geometricaly important affine bundles from sections of simpler affine bundles. While the possible isomorphism classes of the various affine bundles thus obtained can be read off from the intrinsic torsion of the conic-structural jet, a careful analysis of these constructions renders the twistorial interpretation of these bundles: sections of one of these bundles correspond to 'integrable localized preconnections' (and, consequently, their existence is a necessary condition for realizability of the conic-structural jet by a family of submanifolds); furthermore, two of the above bundles (constructible under certain conditions from such a section) can be thought of as (respectively) first-order and transverse second-order infinitesimal neighbourhoods of the submanifold with parameter m from any family inducing the given conic-structural jet.

In other words, the information on the (potential) low-order infinitesimal neigbourhoods is stored in certain obvious (at least from a formal viewpoint) auxiliary objects (namely affine bundles) which are naturally constructed from a conic-structural 1-jet and thus easily classified in terms of intrinsic torsion. However, in order to extract this information it is crucial to find an intuitive interpretation of these constructions and relate it to an equally intuitive interpretation of infinitesimal neighbourhoods.

The method outlined above essentially consists in the study of the close interaction of two 'PDEs given up to point equivalence' associated with a conic structure: the first of them is the 'Lie PDE of complete flatness' (which is defined for general geometric structures), and the second is the conic structure itself. Our choice of terminology reflects this interaction in a consistent way: we modify the well-established terms pertaining to the Lie equation by adding certain prefixes in order to obtain corresponding terms pertaining to the conic structure.

As has already been observed, by considering higher-order conic-structural jets and connections on higher-order frame bundles, this method seems extensible to the study of higher-order infinitesimal neighbourhoods; however, we do not dwell on such an extension in this thesis. We only mention that, for example, in the case of conic structures of finite type the study of interaction of the 'Lie equation of complete flatness' and the conic structure enables one to 'read off' the infinitesimal neighbourhoods from the curvature of the Cartan connection. In fact, the Cartan connections appear in this context in a very natural way: they can be thought of as first-order 'pseudo-prolongations' of the Lie equations. (What is more, Cartan connections of the simpler type are actual prolongations.) This (apparently not quite standard) interpretation of Cartan connections leads to a direct generalization of our method in this context.

Chapter I

Conic Structures

I.1 Conic Structures on Geometrical Parameter Spaces of Submanifolds

Definition I.1 A **conic structure on a vector space**, or a **vectorial conic structure** (or simply a **conic structure** if there is no possibility of confusion) is an ordered pair $T_{cs} := (T, J^{\varepsilon})$ consisting of a (finite-dimensional complex) vector space T, which will be called the **underlying vector space**, and a compact submanifold J^{ε} of the Grassmanian space $J := Gr(x, T)$ (for some $x \in \mathbf{N}$). The (x-dimensional) directions in T (i.e. points of the Grassmanian space, not necessarily of projective space) belonging to J^{ε} are said to be **integral** relative to T_{cs}, while the manifold J^{ε} is called the **integral-direction space**. Thus a vectorial conic structure is simply a vector space in which some directions, which constitute a compact manifold, are distinguished. Since Grassmanian spaces are naturally associated to vector spaces, vectorial conic structures are obviously vector spaces equiped with a structure (in the sense of category theory). Consequently, they form a category, where their mappings (or morphisms) are defined to be structure-preserving linear isomorphisms. The automorphism group of T_{cs} will be denoted by $G^{cs} (\subset G := AutT)$. The vector bundles $T^{\alpha}J^{\varepsilon}$ and $T^{/\alpha}J^{\varepsilon}$ on J^{ε} whose fibers (resp.) T^{α} and $T^{/\alpha}$ over a direction j are defined to be (resp.) the subspace in that direction and the corresponding quotient space are called (resp.) the **integral-tangent** (or 'tautological') and **integral-transverse** bundle associated to T_α. (The above notation for bundles, subspaces and quotient spaces is explained in the Appendix.)

Definition I.2 (a) A **localized conic structure** (or simply a conic structure if there is no possibility of confusion) consists of a manifold M (called the *underlying manifold*) equiped with a distinguished point m and a vectorial conic structure on the tangent vector space T at m. The integral directions of the localized conic structure will also be called its integral (1-) jets since jets will always be understood to be *jets of ('unparameterized') submanifolds* (unless otherwise specified). Such a localized conic structure is said to be *located at the point* m and *on the manifold* M. Clearly, localized conic structures also form a category (more precisely a category of 'first-order localized manifolds equiped with a structure') equivalent to the category of vectorial conic structures if we define the mappings (or morphisms) of localized conic structures to be simply the mappings of the underlying vectorial conic structures (i.e. special 1-jets of mappings of manifolds).

(b) An **expanded conic structure** (or simply a conic structure if there is no possibility of confusion) with x-dimensional integral jets consists of a manifold M (called the *underlying manifold*) and a 'set-theoretical field' [1] on M of localized conic structures with x-dimensional integral jets such that the set $J^\epsilon.M \subset J.M := Gr(x,T).M$, defined as the disjoint union of integral-jet spaces of these localized conic structures, is a submanifold of the total space $J.M$ (of the bundle of Grassmanian spaces) and projects submersively onto the base M. More succinctly, the manifolds J^ϵ form a (holomorphic) subfamily (not necessarily a subbundle) $J^\epsilon M$ of the bundle JM. The total space $J^\epsilon.M$ of this family is called the **expanded integral-jet space** (or simply the integral-jet space, if there is no possibility of confusion). An x-dimensional submanifold of M (or, more generally immersed manifold) is said to be **integral** (relative to the conic structure) if all its 1-jets are integral. The vector bundles $T^\alpha J^\epsilon.M$ and $T^{/\alpha} J^\epsilon.M$ (on the integral-jet space) defined in the

[1] More precisely, this means a system of localized conic conic structures with the parameter space M such that the localized conic structure with parameter m is located at the point m.

obvious way are called (resp.) the **(expanded) integral-tangent bundle** and the **(expanded) integral-transverse bundle** of the conic structure. Conic structures on manifolds are clearly manifolds equiped with a structure (in the sense of category theory), and thus form a category.

Definition I.3 *Let us consider an expandedconic structure, where notation is as in I.2(b). A submanifold of M is said to be integral (relative to the conic structure) if it is x-dimensional and all its jets are integral. More generally, one defines an* **integral immersion** *by the same condition. The conic structure is said to be* **integrable** *at an integral (1-) jet j through a point $m \subset M$ if there exists an integral submanifold with jet j. Furthermore,* **integrability at the point** *m is defined as integrability at all integral jets through m. Similarly, the conic structure on M is said to be* **integrable** *if it is integrable at each point of M.*

Remark I.4 Compound conic structures on vector spaces and manifolds are defined in a similar way, just the maps $J^e \to J$ are not required to be injective or immersive. Thus, a compound conic structure on a manifold M is determined by a (holomorphic) map over base from a family J^eM into the Grassmanian bundle JM. While expanded conic structures are 'geometric structures of first order' (in terminology of [1]) and will often be G-structures (or 'holomorphic infinitesimally homogeneous geometric structures'), expanded compound conic structures are *compound* (or 'generalized' in terminology of [1]) *geometric structures* of first order on the underlying manifold, and will usually be 'compound G-structures'. (These general concepts will be rigourously introduced in Subsection I.2.1 only in the case of (ordinary, not compound) geometric structures; for the sake of completeness we only mention that a 'compound G-structure' of order k is essentially defined as a reduction of the k-frame bundle relative to a conceivably non-injective mapping of Lie groups). In particular, a compound conic structure on a manifold is conceivably not a manifold equiped with a structure in the sense of category theory (in contrast with a conic structure).

Lemma I.5 Let $\rho : T \to H^0(T_S^{/\alpha} S^\alpha)$ be a linear system (with parameter vector space T) of sections (of a vector bundle $T_S^{/\alpha} S^\alpha$). (According to the notational conventions from the Appendix we do not assume here that an embedding of the base manifold S^α is given or that the fibers $T_S^{/\alpha}$ are represented as quotients of tangent spaces to some manifold, although this notation is usually used in such a situation.) We say that ρ is a **compound-geometrically amenable**, or, more briefly, **geometrical**, linear system of sections if for each point $s^\alpha (\in S^\alpha)$ the associated 'evaluation' map

$$T \to T_S^{/\alpha},\ v \mapsto (\rho v) s^\alpha \tag{I.1}$$

is surjective, or, equivalently, that through each vector in the bundle there passes a section parameterized by some vector from T. Let us denote the kernels of maps (I.1) by T^α; thus for each s^α the corresponding map (I.1) descends to an isomorphism $T^{/\alpha} := T/T^\alpha \to T_S^{/\alpha}$. Claim: The resulting (set-theoretical) family of subspaces $T^\alpha \subset T$ with parameter space S^α gives a (holomorphic) map $S^\alpha \to J$, i.e. it constitues a *compound conic structure on the vector space T with possibly non-compact compound integral-jet space.* (This is the motivation for the term 'compound-geometrical'.) In fact, the converse is also true, i.e. a compound-geometrical linear system of sections is in this way determined up to a unique isomorphism by a *general* compound conic structure with possibly non-compact integral-jet space. [The rigourous proof of holomorphicity of the map $S^\alpha \to J$ will be given in Subsection II.2.2. For the time being it will suffice to know the obvious weaker set-theoretical version of this lemma.]

Remark I.6 This construction is an obvious generalization of the construction of a mapping of the base of a line bundle into a projective space if a 'sufficiently rich' or 'base-point free' parameter vector space of sections is given; being a compound-geometrical parameter vector space of sections precisely means this 'sufficient richness'.

Definition I.7 A linear system of sections (of a vector bundle) is said to be **geometrically amenable**, or, more briefly, **geometrical**, if it is compound-geometrical and in notation of I.5 the map $S^\alpha \to J$ constructed there is an embedding. In other words, geometrical linear systems of sections are precisely those compound-geometrical linear systems of sections which in the sense of the I.5 correspond to those compound conic structures with possibly non-compact compound integral-jet spaces which are in fact *conic structures* with possibly non-compact integral-jet spaces. (In classical terminology, a geometrical parameter vector space of sections is said to be a parameter vector space of sections which 'distinguishes between points of the base' and induces an immersion of the base into the Grassmanian.)

In the following definition we review some standard concepts which are of fundamental importance in this thesis.

Definition I.8 A **(holomorphic) family**

$$S \leftarrow R \rightarrow M$$

of submanifolds S^α of a manifold S with parameter space M consists of a (holomorphic) family of manifolds $(R \rightarrow M) =: BM$, i.e. a proper holomorphic submersion with total space $R := B.M$ and base M, and a holomorphic map $R \rightarrow S$ whose restrictions $B \rightarrow S$ to the fibers of the family are embeddings; therefore the manifolds B from the family will usually be identified with their images S^α (via the restrictions of the map $R \rightarrow S$), and the family BM will consequently often be denoted by $S^\alpha M$. The total space R will in this context be called the **incidence manifold**. The **canonical linear system of normal-vector fields** along the submanifold S^α with parameter m (or of 'tangent vectors' to the space of all submanifolds at the submanifold S^α with parameter m) is the linear mapping $T \rightarrow H^0(T_S^{/\alpha} S^\alpha)$ of the tangent space at m ('parameter vector space') into the vector space of sections of the normal bundle of the submanifold S^α defined in the following way: with a vector v one associates the (obviously well-defined) field of normal vectors which are obtained by mapping (under the differential of the map $(R \rightarrow S)$) arbitrary lifts of v to R.

Next we introduce some non-standard terminology:

Definition I.9 A family of possibly non-compact submanifolds S^α of a manifold S with parameter space M is said to be **compound-geometrically amenable**, or, more briefly, **compound-geometrical** if at each point m the tangent space T is a compound-geometrical parameter vector space of normal-vector fields along the submanifold S^α with parameter m (with respect to the canonical linear system associated to m). Intuitively this condition means that each normal vector of the submanifold S^α is 'realized' by some 'tangent vector' at the submanifold (to the space of all submanifolds) which arises from a tangent vector at its parameter m. It is a well-known (and trivial) fact that a family is compound-geometrical iff it is a **double fibration** of the incidence-relation manifold R. (This by definition means that the projection of the incidence manifold to the manifold S is also submersive.) Therefore, in the case of a compound-geometrical family, the (set-theoretical) family of subsets $M^\alpha \coloneqq \mu(\nu^{-1}s)$ of M (where $s \in S$) with parameter space S is in fact a family of possibly non-compact submanifolds. We will refer to the latter as the **conjugate family** of the original family. (It is clear that the conjugate family of the conjugate family is precisely the original family.)

Proposition I.10 Consider a compound-geometrical family of submanifolds, where notation is as in I.9.

(i) According to the weak version of I.5, the tangent vector space T at any point m in this situation comes equiped with a set-theoretical compound conic structure. Claim: The set-theoretical map $S^\alpha \to J := Gr(x, T)$ which constitues this localized structure can alternatively be defined by mapping a point of $s \in S^\alpha$ into the direction at m of the submanifold M^α (from the conjugate family) with parameter s. (In particular, subspaces T^α in integral directions are in fact tangent spaces to submanifolds M^α from the conjugate family.)

[Proof of (i): According to the very definition of T^α (v. I.5) the evaluation map (I.1) descends to a canonical isomorphism of the integral-transverse space $T^{/\alpha}$ (more precisely, the transverse space to the integral jet with parameter s) with the normal space $T_S^{/\alpha}$ to the manifold S^α at s. On the other hand, in this situation the above evaluation map has been defined by descent of the obvious surjective mapping of the tangent space T_R at $r := s.r$ into $T_S^{/\alpha}$. The kernel T_R^α of the latter map is clearly the direct sum of the tangent spaces to the fibers $M^\alpha.s$ and $S^\alpha.m$ of the two fibrations (constituting the double fibration). It remains to observe that $M^\alpha.s \subset R$ projects onto $M^\alpha \subset M$. QED]

(ii) The set-theoretical localized compound conic structures from (i) are induced by a distinguished expanded compound conic structure: by 'assembling' the maps from (i) for various m we obtain a mapping of the incidence-relation manifold $R = S^\alpha.M$ into the contact manifold $J.M$ of (x-dimensional) 1-jets in M), and this is in fact a map over base $S^\alpha M \to JM$ (between families of manifolds). We will refer to the structure constructed in this way (i.e. consisting of this map) as the **compound conic structure induced by the compound-geometrical family of submanifolds**. (In particular, the above set-theoretical localized conic structures on the manifold M are in fact localized compound conic structures.)

[Proof of (ii): Holomorphicity of the above map follows from the holomorphicity of the tangent distribution to the fibration $M^\alpha S$ of R and the fact that the differential of the projection $R \to M$ is a morphism of holomorphic vector bundles. QED]

Corollary I.11 A family of possibly non-compact submanifolds is said to be **geometrically amenable** or, more briefly, **geometrical**, if the canonical linear systems of normal-vector fields along all submanifolds of the family are geometrical (v. I.8 and I.7 for definitions of these concepts). It is easy to see that geometric amenability of a family of (compact!) submanifolds is equivalent to the requirement that the family be compound geometrical (v. I.9) and the induced (expanded) compound conic structure on the parameter space (v. I.10) be in fact a conic structure. [Indeed, let us suppose geometric amenability. The tangent space s to the parameter space, being geometrical parameter vector spaces of sections thus inherit vectorial conic structures (cf. I.7). Therefore the map $R \to J.M$ (in notation of I.10) which defines the above compound conic structure is an injective immersion. The fact that this situation the above evaluation map has been defined by descent of the obvious surjective mapping of the tangent space T_R at $r := s.r$ into $T_S^{/\alpha}$. The kernel T_R^α of the latter map is clearly the direct sum of the tangent spaces to the fibers $M^\alpha.s$ and $S^\alpha.m$ of the two fibrations (constituting the double fibration). It remains to observe that $M^\alpha.s \subset R$ projects onto $M^\alpha \subset M$. QED]

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Intuitively, the fairly mild condition that a family be geometric means that 'the conjugate family of submanifolds of M is well-defined (compound-geometric amenability) and the submanifolds from that family have no first-order contact with each other', i.e., that the submanifolds of the original family be 'rendered as spaces of some 1-jets in M through their parameters'.

Definition I.12 It is clear that a submanifold for which there exists a geometrical family of submanifolds containing it, must itself be 'geometrical' in the following sense: a submanifold is said to be **geometrically amenable** or **geometrical** if its normal bundle is an 'geometrical vector bundle', where **geometric amenability of a vector bundle** is in turn defined as existence of a geometrical linear system of sections. It is clear that the latter requirement on the bundle means simply that its entire space of sections is a geometrical parameter vector space of sections (and consequently is canonically endowed with a vectorial conic structure).

Definition I.13 Clasically normality of a compound conic structure on the vector space T given by a map $J^\varepsilon \to J$ (v. I.5) is defined by the requirement that the corresponding canonical linear system of sections of the compound integral-tranverse vector bundle $T^{/\alpha}J^\varepsilon$ (with parameter space T) be *complete* (meaning bijective). (In other words, T is simply required to be the entire vector space of sections of that bundle.) Although this terminology is strictly speaking unambiguous, avoiding confusion with the notion of a normal bundle (which will often appear in the same context), requires some concentration. Therefore we will replace the word 'normal' by **'completive'** when it is used in the first meaning. For instance, what is clasically called 'a normal embedding of a manifold into a projective space' will be called 'a completive embedding'. Similarly we will have a **completive localized (resp. expanded) conic structure.** (Explicitly, completivity of an expanded conic structure is defined as completivity of the induced localized conic structures at all points.)

I.2 Conic Structures on Special Geometrical Parameter Spaces of Submanifolds

I.2.1 Homogeneal Conic Structures on Homogeneally Geometrical Parameter Spaces of Submanifolds

Since an expanded conic structure has been defined as a special 'expanded geometric structure', i.e. a manifold equiped with a 'set-theoretical field of localized geometric structures' (or 'geometric quantities' in [1]) in the sense of the theory of G-structure s, the concept of its *'infinitesimal homogeneity'* (or *'homogeneality'* in our terminology) is well-defined as an obvious necessary condition for its homogeneity (cf. [1]). Before the more detailed rigorous introduction of this concept, we recall some obvious facts about general geometric structures and apply them to the case of conic structures:

Remark I.14 Let us consider a category of $\hat{m}$-dimensional vector spaces equiped with a structure. In this remark we will refer to its objects as *specific structures on vector spaces.* Recall that this category can alternatively be defined by a natural association (i.e. functor) of sets U to $\hat{m}$-dimensional vector spaces T: in this context U is simply the set of (parameters of) all specific structures with the underlying vector space T. (In other words, a specific structure with the underlying vector space T can be identified with a pair (T, u), where $u \in U$.)

We will say that the given category (i.e. the category of specific structures) has been organised into a category of holomorphic structures on vector spaces if we have made a 'simultaneous' choice of structures of complex manifolds U_{mf} on the sets U of specific conic structures on $\hat{m}$-dimensional vector spaces T so that two conditions are fulfilled:

(a) The natural association of the sets U to vector spaces T becomes a natural asociation of *manifolds* U_{mf} to vector spaces T;

(b) The above natural association of manifolds to vector spaces is *holomorphic* in the following sense: for each vector space T of that dimension the naturally induced action of the automorphism group $G := Aut\, T$ (of the vector space T) on the manifold U_{mf} is holomorphic. (Notice that in case of transitivity of the action of G on T this condition alone uniquely determines the structure of a complex manifold on U).

Finally, let us suppose that all specific structures are *isomorphic*, i.e. that the for at least one (or, equivalently, every) T the natural action of the group $G = Aut\,T$ upon U is transitive. We will now review the proof of the following standard fact: *The category of specific structures can be organized into a category of holomorphic structures on vector spaces in at most one way, and a precise criterion for this is closedness of the automorphism group G^{cs} of at least one (or, qeuivalently, every) specific conic structure (T, u) in the automorphism group G of (the underlying vector space) T.* Indeed, uniqueness follows already from the condition (b) alone and even for for each T independently since the actions of the groups G on the sets U are by assumption transitive. In order to check the criterion, it suffices to recall that the automorphism group G^{cs} is simply the stabilizer of the point u relative to the above transitive action and to apply the already established uniqueness.

Lemma I.15 Let us consider a *'set-theoretical' conic structure* on a manifold M, meaning a pair formed by M and a set-theoretical field of localizedconic structures on M. We will say that this structure is **infinitesimally homogeneous** or, more briefly, **homogeneal**, if all the induced localized conic structures (i.e. conic structures on the tangent vector spaces T at various points m), are mutually isomorphic. (In other words, the category of homogeneal 'set-theoretical' conic structures is a special case of a category of homogeneal 'set-theoretical' geometric structures, cf. [1].) Suppose the given structure is homogeneal. In particular, an isomorphy class of localized conic structures is fixed (namely the class with the above mentioned representative structures). In this lemma we will call structures of this class *localized specific conic structures* and denote the dimension of the underlying manifolds by $\hat{m}$. Thus, the given structure is a *'set-theoretical' (expanded) specific conic structure* in the sense that all induced localized conic structures are specific.

(i) Let us first observe that the automorphism group G^{cs} of any (in particular specific) conic structure on any vector space T is a closed subgroup of the automorphism group G of T; indeed, the action of G on the Grassmanian J is holomorphic, the subgroup consists by definition of those elements which fix the integral-jet space J^ε, and the latter is due to its compactness a closed subset of J. Therefore, the category of localized *specific* conic structures can according to I.14 in a unique way be organized into a category of holomorphic localized specific conic structures. QED

(ii) In view of (i) and the general principles the theory of geometric structures, for the given manifold M we obtain (owing to its $\hat{m}$-dimensionality) a *bundle* UM of spaces of localized specific conic structures. (Of course, its total space, i.e. the structure of a complex manifold on the set $U.M$ is constructed by means of the 1-frame bundle of M.) Therefore, we can define **holomorphicity** of the given 'set-theoretical' conic structure as *holomorphicity of the constituent set-theoretical field of localized specific conic structures* in the bundle UM. (In other words, the category of holomorphic homogeneal conic structures is a special case of a category of *holomorphic homogeneal geometric structures*; note that holomorphicity of inhomogeneal set-theoretical conic structures can not be defined in this way.) According to the well-known general properties of general holomorphic *homogeneal* geometric structures, the given set-theoretical conic structure is holomorphic iff it is a G-structure (whose Lie group is the automorphism group G^{cs} of the specific localized conic structure); recall that the latter condition by definition means that the given 'set-theoretical reduction' of the 1-frame bundle of M is actually holomorphic.

(iii) Note that both homogeneal conic structures and holomorphic homogeneal conic structures have been defined as set-theoretical fields of localized conic structures satisfying some additional conditions. We claim that the latter structures are special former structures. Indeed, it is clear from (ii) that *for a holomorphic homogeneal conic structure the associated set-theoretical family $J^\varepsilon M$ of integral-jet spaces is not only a holomorphic subfamily, but even a subbundle of the Grassmanian bundle JM.* (Here we have used the obvious structure of a *bundle* of spaces of localized specific conic structures, which is defined on the set-theoretical family of spaces of localized specific conic structures constituting the holomorphic expanded specific conic structure; notice that this by no means implies complete flatness of the latter in spite of the local triviality of the bundle.) Incidentally, we will later give some reasonably weak sufficient conditions on the category of localized specific conic structures in order for the converse inclusion to be valid.

I.2.2 Completive Conic Structures on Locally Complete Parameter Spaces of Geometrical Submanifolds

Definition I.16 A parameter space M of submanifolds S^α of a manifold S is said to be **locally complete** if at each point m the canonical linear system (v. I.8) of normal-vector fields along the submanifold S^α of S with parameter m is complete (i.e. bijective). (In other words, the tangent space T at m is a *complete* parameter vector space of such fields relative to the canonical linear system.)

Remark I.17 The motivation for the above terminology comes from deformation theory: locally complete families are universal in the sense that, roughly speaking, for any other family containing a given submanifold from the locally complete family, a nearby submanifold from the arbitrary family 'locally' has a unique and holomorphically dependent parameter relative to the locally complete family.

Proposition I.18 Let M be a parameter space of submanifolds S^α of a manifold S.

(i) Suppose M is geometrical (relative to the given family). In this situation M is conceivably not a locally complete parameter space of submanifolds. Claim: Local completeness is equivalent to the following condition:

The expanded conic structure M_{csm} induced by the geometrical family of submanifolds is completive (v. I.13 for definition of completivity). [This is an obvious consequence of definitions.QED]

(ii) Suppose submanifolds S^α of S from the family are geometrical and M is locally complete. In view of I.12 these assumptions obviously imply geometric amenability of the family. QED

(iii) In view of (i) and (ii) we infer that the concept of a locally complete parameter space of geometrical submanifolds is equivalent to the concept of a geometrical parameter space with completive induced conic structure. QED

(iv) Suppose M is locally complete and geometrical. Then for a given point m the isomorphy class of the localized conic structure at m and the isomorphy class of the normal bundle of the submanifold S^α with parameter m completely determine each other. Explicitly, they clearly correspond to each other relative to the obvious bijective correspondence between isomorphy classes of completive vectorial conic structures and isomorphy classes of amenable vector bundles; more precisely, the above correspondence is induced by associating with a completive vectorial conic structure its integral-transverse bundle. QED

Remark I.19 An obvious analogue I.18 could be stated in the more general context of *compound*-geometrical families. In this version some of the concepts appearing in I.18 would have to be replaced by resp. 'compound-geometrical parameter space of submanifolds', 'compound-geometrical submanifold', 'compound conic structure' and 'compound integral-transverse bundle'. QED

Proposition I.20 Suppose we are given an isomorphy class of amenable vector bundles. In other words, we are given an isomorphy class of completive vectorial conic structures (v. the assertion (iv) of P I.18).

Claim: For a locally complete space M of submanifolds S^α of a manifold S such that their normal bundles are all of the given isomorphy class, the induced conic structure on M is obviously well-defined (since M is according to I.18 a geometrical parameter space). Furthermore, this conic structure is homogeneal, where the (common) isomorphy class of the induced localized conic structures coincides with the given class.
[In order to prove this, it suffices to apply assertion (iv) of I.18. QED]

I.2.3 Homogeneal Completive Conic Structures on Locally Complete Parameter Spaces of Normally Rigid Geometrical Submanifolds

Remark I.21 Let us consider a completive expanded conic structure with *connected* underlying manifold, rigid localized integral-jet spaces and rigid localized integral-transverse bundles (at all points m). Then by the main properties of rigid manifolds and rigid vector bundles, the conic structure is homogeneal. QED

Remark I.22 Suppose we are given a connected locally complete parameter space M of normally rigid geometrical submanifolds S^α of a manifold S; here **(first-order) normal rigidity** of an embedding is defined as simultaneous rigidity of the manifold being embedded and the normal bundle of the embedding. According to I.18 the given parameter space is (due to its local completeness and geometric amenability of the submanifolds) geometrical and comes equiped with a completive expanded conic structure. Claim: The latter is homogeneal, i.e. (v. I.18) the normal bundles of the submanifolds of the family are isomorphic. [Indeed, I.21 is applicable in view of the normal rigidity assumption. QED]

Proposition I.23 Suppose we are given an isomorphy class of manifolds and an integer y such that any y-codimensional non-exceptional embedding of a manifold of that class gives rise to a normally rigid geometrical submanifold. (Explicitly, manifolds of that class are rigid, and all fiberwise y-dimensional vector bundles admitting nontrivial sections over manifolds of that class are amenable and rigid.) Our objective is to describe the first-order geometric structure induced on arbitrary locally complete connected parameter spaces M of y-codimensional submanifolds S^α of the given isomorphy class. Claim: Such a parameter space of submanifolds is geometrical and the induced conic structure (which is well-defined for that reason) has the following two properties, which are equivalent for any conic structure on a connected manifold:
(1) The conic structure is completive, its integral jets are y-codimensional and its localized integral-jet spaces are of the given isomorphy class.
(2) The conic structure is *homogeneal* and completive, its integral jets are y-codimensional and its localized integral-jet spaces are of the given class.

[Proof of (i): We first prove the equivalence of (1) and (2) for any expanded conic structure: by the assumptions on the given isomorphy class of manifolds, for an expanded conic structure with the property (1) the integral-transverse bundles are all rigid; this, along with I.21 shows that (1) and (2) are indeed equivalent.

Now it remains to observe that M is indeed geometrical according to I.18. QED]

Chapter II

Preconnections

II.1 Geometric Description of the Structure of a Geometrical Parameter Space of Submanifolds

II.1.1 Geometrization of Double Fibrations by Preconnections

We have seen in previous sections that a locally complete space M of normally rigid geometrical y-codimensional submanifolds S^α of a manifold S comes equiped with a homogeneal expanded conic structure with y-codimensional integral jets (or G^{cs}-manifold for some G^{cs}). Often the structure on a manifold M of a parameter space of submanifolds, which induces a given conic structure, is uniquely determined. However, this is not always the case. More precisely, from the conic structure alone, the manifold S and the locally complete family of submanifolds in it can in general not be reconstructed (not even up to isomorphism type, so all the more they can not be naturally associated). Thus the question arises of what additional geometric structure on the expanded conic structure M is needed to recover the manifold S and the manner in which M parametrizes submanifolds of S.

In order to answer this question, we will consider the more general situation from the last chapter: instead of locally complete families of normally rigid geometrical submanifolds we will study more general families, namely *geometrical* (v. I.11) families (possibly not locally complete) of submanifolds (possibly not mutually isomorphic as complex manifolds). The reason for doing so is twofold: On the one hand, this is a more natural framework for the above problem since (as was seen in previous sections) a geometrical parameter space of submanifolds also comes equiped (essentially by the very definition) with a conic structure (but this is conceivably not homogeneal, i.e. the induced localized conic structures are possibly not isomorphic for various points m). In addition to that, it will turn out that even in the theory of complete spaces of rational curves in surfaces certain associated non-complete geometrical families of submanifolds will play an important role.

In accordance with that, let us consider an arbitrary geometrical family

$$S \longleftarrow R \longrightarrow M \tag{II.1}$$

of submanifolds S^α of S with parameter space M. Our objective is to find a geometric structure on M from which this family can be 'reconstructed'. Of course, we expect the conic structure induced by the family (v. I.11) to be a constituent part of this geometric structure. We first recall that as a result of the geometric amenability assumption, the family (II.1) is in fact a double fibration. Thus, it can also be interpreted as a family of (also y-codimensional) possibly non-compact submanifolds M^α of M parametrized by points of S. Next we recall how this conjugate family induces the expanded conic structure.

Let us denote by $S^\alpha M$ the family of manifolds underlying the family (II.1) of submanifolds of S (in other words, this is one of the two fibrations of the incidence-relation manifold $R := S^\alpha.M$ that appear in the double fibration, namely the fibration over M). Consider the mapping of the incidence-relation manifold R into the space $J.M$ ($=: Gr(x,T).M$) of all x-dimensional jets in M, determined by the double fibration (II.1) (in the way described in previous sections; explicitly, r is mapped into the direction at the corresponding m of the submanifold M^α parametrized by the corresponding s). We have seen in previous sections that this map is actually an embedding (this in particular implies that manifolds M^α corresponding to different parameters s are different – what is more, they have no contact of first-order). The image, denoted by

$J^\epsilon M$, of the family $S^\alpha M$ in JM under that map is the subfamily of that bundle which constitutes the conic structure on M induced by the family of submanifolds of S (by the very definition of the induced conic structure). As in Chapter I, we will identify $R \coloneqq S^\alpha.M$ with its image in $J.M$, while for each m the corresponding submanifold S^α will be identified with the space $J^\epsilon \subset J$ of integral jets at m. Furthermore, S will be identified with the standard quotient manifold of R.

With these identifications the family (II.1) obviously becomes a manifold equiped with a structure (in the sense of category theory) with the underlying manifold M.

Definition II.1 For two families

$$S_1 \xleftarrow{\nu_1} R_1 \xrightarrow{\mu_1} M_1 \quad \text{and} \quad S_2 \xleftarrow{\nu_2} R_2 \xrightarrow{\mu_2} M_1$$

of submanifolds with common parameter space M_1 we define an **isomorphism over parameter space** as an isomorphism of the families inducing identity on the parameter space; explicitly, such an isomorphism consists of a pair of biholomorphisms $\phi : R_1 \to R_2, \psi : S_1 \to S_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc} S_1 & \xleftarrow{\nu_1} & R_1 & \xrightarrow{\mu_1} & M_1 \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle id} \\ S_2 & \xleftarrow{\nu_2} & R_2 & \xrightarrow{\mu_2} & M_1 \end{array} \qquad \text{(II.2)}$$

Furthermore, a class of families isomorphic over parameter space M_1 will be called a **structure on M_1 of a parameter space** of (a family of) submanifolds.

Proposition II.2 Consider on a manifold M_1 a structure of a geometrical locally complete parameter space of submanifolds such that the (conceivably non-compact) submanifolds (of M_1 from the conjugate family are non-empty and connected. Let us choose two families of submanifolds (with common parameter space M_1) realizing (i.e. representative of) this structure. Then the isomorphism over parameter space between these families is unique.

Of course, this 'double fibration structure on M' is not yet (differential-) geometric in the sense that it is not given by 'local data', i.e. it is not defined as a 'field of geometric quantities' in terminology of [1]. Thus, our objective will be to 'localize' this structure or, more precisely, to see what kind of localized geometric structures ('geometric quantities') could serve to encode the expanded structure as a field of such localized geometric structures.

Let us observe that with the above identifications the fibration $S^\alpha M$ (i.e. the one of the two constituent fibrations of the double fibration, which has M as its base), is precisely given by restricting to $R\,(= J^\epsilon.M)$ the projection of $J.M$ onto M. The structure of a parameter space of submanifolds on the expanded conic structure is obviously determined precisely by the other fibration of $R \subset J.M$, namely the fibration $M^\alpha S$ (with base S). It is now clear that the structure can indeed be 'localized' (what is more, not only with respect to M, but even with respect to R) as soon as the fibers M^α of this fibration are connected: then the fibration clearly coincides with the foliation which is induced by the integrable distribution on R formed by tangent spaces to the fibers $M^\alpha.s \subset R$. For this reason we now impose on the original geometrical family of submanifolds the slightly stronger condition from the following definition:

Definition II.3 A family of submanifolds is said to be **completely geometrical** if it is geometrical and the conjugate family (v. I.9) consists of (conceivably non-compact) non-empty *connected* submanifolds.

We have shown that the structure of a parameter space of submanfolds is owing to this additional hypothes completely determined by the above distribution. Let us denote fibers of the latter (at various points r) briefly by $F^\tau \subset T_R \subset T_{JM}$ (in other words, the distribution is denoted by $F^\tau R$; the use of the upper index, which in our notation usually suggests an embedding will soon be justified). The localization *with respect to M* can explicitly be accomplished in the following way: the appropriate localized geometric structure at the point m by this definition consists of the localized conic structure ans the system of subspaces F^τ of the tangent spaces to the manifold $J.M$ at integral 1-jets r through m (i.e. jets belonging to the space $S^\alpha.m \subset R \subset J.M$). Let us observe that this system actually actually constitutes a vector subbundle $F^\tau J^\varepsilon$ of $T_{JM}J^\varepsilon$ (which satisfies certain additional conditions, as we shall soon see). This is clearly a second-order localized geometric structure, and we have seen that the field of these localized structures indeed encodes the double fibration.

Before introducing suitable terminology, let us examine the obvious properties of the localized geometric structure $F^\tau J^\varepsilon$ considered above.

The x-dimensional subspaces F^τ of the tangent spaces $T_{J.M}$ at various jets $r = j.m \in R \subset J.M$ are of a very special type: under the differential of the projection $J.M \to M$ such a space clearly projects bijectively precisely onto the subspace T^α of the tangent spaces T in direction j, (i.e. the subspace whose parameter is the direction j). Let us equip the manifold $J.M$ with the canonical structure of a contact manifold. (By the latter we mean the obvious higher-codimensional generalization of the concept of a **'hypersurface-contact manifold'**; v. [5] for a precise definition.) Furthermore, let us denote by $F\,J.M$ the constituent distribution on $J.M$ of the contact structure. (Explicitly, its fiber $F \subset T_{JM}$ over the jet $j.m$ is the preimage of the space T^α in direction j under the differential of the projection $(J.M \to M)$.) Thus we have shown that spaces F^τ are necessarily direct complements in integral-tangent spaces F of the contact manifold of the vertical spaces T_J. This observation motivates the following definitions:

Definition II.4 *Let $F\,J.M$ denote (as above) the integral-tangent bundle of the canonical contact manifold $J.M$ of x-dimensional (1-) jets in a manifold M.*

Fix a jet j through a point m. The direction $c_\pi \in J_F := Gr(x, F) \subset J_{J.M} := Gr(x, T_{JM})$ of a direct complement F^τ of the vertical space T_J in the (contact-structural integral-tangent) space F (at $j.m$) will be called an **elementary preconnnection** *on the manifold M at the jet $j.m$.* (Thus F^τ is in this context the tangent space to the elementary preconnection.) The space of elementary preconnections on M at $j.m$ wil be denoted by $C_\pi(\subset J_F \subset J_{J.M})$. Thus we clearly get bundle $C_\pi J$, which we call the **localized elementary-preconnnection bundle** (on the localized jet space) of the manifold M. This is clearly a restriction of a bundle $C_\pi J.M$, which we call the **expanded elementary-preconnnection bundle** (on the expanded jet space) of the manifold M.

Remark II.5 If there is no possibility of confusion with the concept of a 'conjoint preconnection' (which is to be defined shortly), we will often supress the adjective in the expression 'elementary preconnection'. (For example, if no conic structures are considered, this causes no ambiguity.) *The motivation for the above (non-standard) term is the strong analogy and interaction between the Spencer complex associated to the 'Lie equation of complete flatness of a conic structure' and the Spencer complex associated to the conic structure thought of as a 'PDE up to point transformations'. (We will soon clarify these concepts.) Since the terminology for the former complex is well-established, we will carry it over to the latter complex by adding the prefix 'pre'. Thus the role of a connection (resp. intrinsic torsion, 'conic-structural jet' etc.) in the theory of geometric structures is analogous to the role played by (both 'elementary' and 'conjoint') preconnections (resp. 'intrinsic pretorsion', 'conic-structural prejet' etc.).* In fact, this analogy is twofold: if a *general* homogeneal geometric structure is thought of as a 'generalized conic structure' (cf. II.11) on a certain product by means of the above Lie equation, then all the standard concepts from the theory of geometric structures coincide with the their 'elementary preforms' in the theory of 'generalized conic structures'. In order to indicate the interaction between the two sets of concepts, we mention that a 'connection on the manifold at a point m in a given direction j', which will be defined precisely later, has an elementary preconnnection as an essential part.

Definition II.6 *A* **localized (resp. expanded) conjoint preconnection** *(or simply a preconnection if there is no possibility of confusion) on a given manifold M consists of a localized (resp. expanded) conic structure on M and a field of elementary preconnections c_π defined along the integral-jet space*

$J^\varepsilon \subset J$ *(resp.* $R := J^\varepsilon.M \subset J.M$*) of the conic structure.* [(It is implied that this field is a section of the **localized (resp. expanded) elementary-preconnection bundle** $C_\pi J^\varepsilon$ **(resp.** $C_\pi R$**) of the conic structure**, where this bundle is by definition the restriction of the localized (resp. expanded) elementary-preconnection bundle $C_\pi J$ (resp. $C_\pi J.M$) of the manifold M.] Of course, this field can equivalently be given by a distribution $F^\tau J^\varepsilon$ (resp. $F^\tau R$) in the manifold $J.M$ defined along the submanifold J^ε (resp. R) such that directions in T_{JM} of its fibers F^τ are elementary preconnections on M. [In particular, this distribution is contained in the restriction FJ^ε (resp FR) of the constituent ditribution $FJ.M$ of the contact structure.] The above distribution will be called the **tangent distribution** of the given preconnection.

Remark II.7 Clearly, localized preconnections form a category, more precisely, they are second-order localized geometric structures (i.e. 'second-order localized manifolds' equiped with a structure.) Similarly, the expanded preconnections form a category of manifolds equiped with a structure. In fact, these are also geometric structures in view of the following trivial observation: an expanded conjoint preconnection (or simply a preconnection if there is no possibility of confusion) on a manifold M can alternatively be given by a field of localized preconnections on M which is holomorphic in the following sense: the corresponding field of localized conic structures forms an expanded conic structure, and the set-theoretical field of elementary preconnections c_π defined along the expanded integral-jet space R simply by 'assembling' the constituent elementary-preconnection fields of the above mentioned localized preconnections, is holomorphic. (In other words, it is a section of the restriction $C_\pi R$ of the expanded elementary-preconnection bundle $C_\pi J.M$ to the expanded integral-jet space of the conic structure.

Definition II.8 We say that an *expanded* preconnection is **tangential** if the fibers F^τ of the tangent distribution (of the preconnection) are (in notation of II.6) tangent to the submanifold R of JM (more precisely, for each point r the corresponding space $F^\tau \subset T_{JM}$ is contained in the space $T_R \subset T_{JM}$). In other words, tangentiality simply means that the tangent distribution $F^\tau R$ (in the manifold JM along R) is actually a distribution in R. (It should be noted that an analogous condition does not make sense for a localized preconnection.)

It is clear that with this terminology at our disposal, the geometric structure which was constructed before on a geometrical parameter space of submanifolds, and which was shown to completely encode the family of submanifolds, can now briefly be described as a preconnection whose underlying conic structure is precisely the one associated to the family. In order to characterize those preconnections which actually correspond to families of submanifolds, let us recall how the family of submanifolds was reconstructed from the associated preconnection: The fibers M^α.s of the fibration of R over S were precisely the leaves (i.e. maximal connected integral submanifolds) of the distribution $F^\tau R$. In addition to that, the very definition of the conic structure on M induced by the family of submanifolds clearly implies that these leaves are precisely the canonical lifts to the contact manifold $J.M$ ($\supset R$) of the manifolds M^α. This fact indicates that the latter manifolds are 'integral manifolds' of the preconnection thought of as an 'invariantly defined' second-order PDE or a 'PDE given up to point transformations'; however, before a precise statement of this result we digress to clarify these concepts in the following subsection.

II.1.2 Symmetric Preconnectors, Second-Order Jets and Generalized Conic Structures

The objective of this subsection is to interpret the concept of a preconnection in the language of the theory of 'PDEs given up to point transormations', or 'generalized conic structures' in our terminology. We first give an informal description of these structures. (A precise exposition will follow shortly.) 'Generalized conic structures' are often obtained as certain structures underlying a standard PDE on a fiber bundle, namely structures obtained by 'forgetting' not only the trivial connection on the bundle (defining the notion of codomain of integrals), but also the fibration (defining the notion of domain) itself. In other words, such a 'generalized conic structure' is obtained from a PDE by keeping only the totalspace and the set of 'permissible' k-jets of its eventual integral submanifolds; integrals are still defined as immersed manifolds with 'permissible' k-jets at each point. For the time being we are interested only in '*second-oder* generalized conic structures' (or 'generalized conic 2-structures').

Remark II.9 We will use some non-standard terminology and notation in order to make the exposition of the theory of conic structures more systematic and the terminology more suggestive. In particular, constructions related to the natural association of vector spaces with affine spaces will be reflected in our terminology and notation in a simple and consistent way. Thus, the objects which are usually called 'differences of connections' or 'connection coefficientts' or 'contorsions', namely the elements of the vector space of an affine space of connections, will be called 'connection vectors' or, more briefly, **connectors**; the precise definition of the 'localized' and 'expanded' versions of that concept will be reviewed in Remark III.1. (The motivation for the term 'connection vectors' is the etymology of the term 'vector': Latin 'vehere' means 'to carry', thus vectors from the vector space of a given affine space are the objects which 'carry' or 'translate' one affine point into another. In the case considered above the affine points are connections. The shorter term 'connector' could be justified in the same way: by means of a connector we can generate new connections from a given, e.g. trivial, one.) Analogous terminology will be applied to other affine spaces: thus we will introduce e.g. 'tangential connectors', 'preconnectors', 'tangential symmetric preconnectors' and 'structural directors' as vectors of certain affine points called resp. 'tangential connections', 'preconnections', 'tangential symmetric elementary preconnnections', 'structural directions'.

Next we review the standard invariant description of 2-jets (of submanifolds) as 'symmetric' elementary preconnnections, and expound some elementary but not quite standard related facts concerning the tensor-type invariants of elementary preconnnections.

Remark II.10 (i) Let us first observe that *the space C_π of elementary preconnections at a given 1-jet in the manifold M comes equiped with the obvious structure of an affine space.* The associated vector space is clearly

$E_\pi := T^{/\alpha} \otimes T^{*/\alpha \otimes 2}$, where the notation for the integral-tangent and integral-transverse space is as in Definition I.2 and $T^{*/al}$ denotes according to our notational conventions (v. Appendix) the space $T^{\alpha *}$ ($= T^*/T^{*\alpha}$, where $T^{*\alpha}$ is the perpendicular of T^α in T^*). These 'elementary preconnnection vectors' will be briefly called **elementary preconnnectors**. (V. Remark II.9 for the motivation of this terminology). Furthermore, notice that the elementary preconnnection space is canonically represented as a product affine space: indeed, the elementary preconnnector space has a canonical decomposition into the direct sum of spaces of (resp.) **symmetric** an **antisymmetric** elementary preconnnectors, where the symmetry and antisymmetry are defined relative to the last two indices. We introduce the following notation for these spaces:

$$E_\pi^{sa} := E_{\pi\, \bullet(**)}$$

$$E_\pi^{as} := E_{\pi\, \bullet[**]}$$

where lowered parentheses (resp. brackets) indicate symmetry (resp. anti antisymmetry). Of course, the corresponding quotient spaces, e.g.

$$E_\pi^{/sa} \coloneqq T^{/\alpha} \otimes T^{*/\alpha \wedge 2} = E_{\pi\, \bullet[**]}$$

will be identified with appropriate subspaces. For reasons expounded in (b), the vectors belonging to the above quotient will be called **elementary pretorsions**. QED

(ii) It is well-known that the 2-jet of a submanifold M^α of M is completely determined by its 1-jet and the 1-jet (which is by definition the direction of the tangent space F^τ) of its canonical lift to the contact manifold of 1-jets. (It is implied the 1-jet in the contact manifold is taken precisely at the above mentioned 1-jet in M.) Such an 1-jet in the contact manifold is according to Definition II.4 clearly an elementary preconnnection (at the 1-jet in M). However, in general not all elementary preconnnections at a given 1-jet in M are 2-jets in M: the latter form an affine subspace of the affine space C_π of elementary preconnnections at the given 1-jet in M, where the associated vector subspace consists precisely of the symmetric connectors. For this reason

2-jets tangent to the given 1-jet will also be called **symmetric elementary preconnections** and the space they form will be denoted by C_π^{sa}.

[The proof of (ii) exploits the notion of **the canonical (affine) mapping of elementary preconnnections into elementary pretorsions**, which is defined in the following way: with a given elementary preconnnection we associate the elementary pretorsion obtained (with respect to the obvious identifications) by restricting the Frobenius tensor of the contact manifold to the tangent space F^τ of the elementary preconnnection. It is not difficult to see that this is indeed an affine map with obvious associated linear map and that the elementary preconnnections which determine 2-jets are characterized by vanishing of the associated elementary pretorsion (i.e. by isotropy relative to the Frobenius tensor).QED]

Remark II.11 (i) A **generalized pseudo-conic 2-structure** consists of a manifold M (the *underlying manifold*) and a submanifold $C_\pi^\varepsilon\,.\,J.M$ (called the **expanded integral elementary-preconnection space**) of the total space $C_\pi\,.\,J.M$ of the expanded elementary-preconnection bundle of M; here $C_\pi^\varepsilon\ J.M$ denotes the set-theoretical family of the obviously defined **(jet-localized) integral elementary-preconnnection sets** C_π^ε. (The latter are conceivably not submanifolds of elementary-preconnnection (affine) spaces C_π.) If the subset R of $J.M$ formed by points with non-empty integral elementary-preconnection sets defines a conic (1-) structure (called *the underlying conic structure*) and the restriction $C_\pi^\varepsilon R$ of the above set-theoretical family is in fact a (holomorphic) subfamily with possibly non-compact fibers, then we will omit the adjective 'generalized'. Similarly if the integral elementary preconnnections are all symmetric (i.e. 2-jets), we will omit the prefix 'pseudo' and call $C_\pi^\varepsilon\,.\,J.M$ the **expanded integral 2-jet space of the generalized conic 2-structure.**

An **integral** immersed manifold in M (of the given generalized pseudo-conic 2-structure is defined by the requirement that its obviously defined *canonical lift into the (second-order) contact manifold of 2-jets* $J^2.M(=C_\pi^{sa}\ J.M\subset C_\pi\ J.M)$ be contained in the expanded integral elementary-preconnnection space. (Note that this lift is formed by all first-order jets of the canonical lift into the first-order contact manifold.) **Integrability** of the given generalized pseudo-conic structure is defined as existence of integral submanifolds tangent to any integral elementary preconnnection. It is clear that the given structure must actually be a generalized *conic* 2-structure in order to be integrable. QED

(ii) It is clear that preconnections are second-order generalized pseudo-conic structures of a very special type, their main feature being that there exists at most one integral elementary preconnection at a given 1-jet jet. Therefore, we define an **integral** immersed manifold for a given preconnection M_{dm} as an immersed manifold which is integral relative to the preconnection viewed as a generalized pseudo-conic 2-structure. Explicitly, this is an integral immersed manifold for the underlying expanded conic 1-structure (v. I.2) such that its canonical lift to the contact manifold is an integral immersed manifold in R for the distribtion $F^{\tau}R$ (in the manifold $J.M$ along R) or, explicitly, that lift has the property that its tangent space at a 1-jet r in M is precisely $F^{\tau} \subset T_{JM}$ coresponding to r. In fact, from a well-known property of the constituent distribution $FJ.M$ of the contact structure on the 1-jet space it is clear that any x-dimensional immersed integral manifold of the distribution $F^{\tau}R$ is locally the lift of a unique submanifold of M, which is therefore integral.

Similarly, an expanded preconnection on M is said to be **integrable** if it is integrable when viewed as a generalized pseudo-conic 2-structure. Explicitly, for each point m and each integral (relative to the underlying conic structure) 1-jet j at m, there exists at least one integral manifold in M_{dm} tangent to j (this integral manifold is clearly locally unique since its lift has to be an integral manifold in R for a distribution of the same dimension). It is clear that tangentiality is a necessary condition for integrability. Furthermore, an integrable preconnection could alternatively be described as an expanded conic structure equiped with an integrable distribution in the integral-jet space R (or equivalently foliation of R) such that directions of its fibers F^{τ} are elementary preconnections in M. (Indeed, from the observation made above it follows that the leaves are locally canonical lifts.) In particular, *tangential* curve-directional preconnections (and even general tangential curve-directional pseudo-conic 2-structures) are always integrable. QED

(iii) The **pretorsion** at the point $m \in M$ of a preconnection is by definition the field of elementary pretorsions of the integral elementary preconnections at the point m. Thus, a preconnection is pretorsion-free iff it is **symmetric**, which by definition means that it consists of 2-jets (or symmetric elementary preconnections). Thus, a symmetric preconnection is a *conic* 2-structure whose main feature is that there exists at most one integral 2-jet tangent to any given 1-jet. Similarly, an integrable preconnection on a given (expanded) conic 1-structure roughly corresponds to the classical notion of a complete integral of a generalized conic 1-structure. QED

Remark II.12 A conic structure with x-dimensional integral directions will be said to be **omnidirectional** (or 'full' in terminology of [13]) if all x-dimensional jets are integral. Of course, this conic structure is trivial in the sense that it is completely determined by the underlying manifold and the integer x. (It is also completely flat as a geometric structure.) In the case $x = 1$ (resp. $x > 1$) this structure will be called the **projective** (resp. **Grassman) conic structure**. We say that a preconnection is **omnidirectional** (resp. **Grassman, projective**) if the underlying expanded conic structure has that property. Let us observe that omnidirectionality of a preconnection in a trivial way implies its tangentiality. In fact, an omnidirectional symmetric preconnection on a manifold precisely corresponds to the classical notion of 'a normal second-order PDE given up to point transformations'. Since a projective preconnection is tangential and curve-directional, it is according to II.11(iii) integrable. It is clear that a **projective structure** is by its very definition precisely a projective preconnection which is 'connection-induced' in a certain sense. In fact, it turns out that (owing to the holomorphicity assumption) any projective preconnection is necessarily connection-induced, i.e. a projective structure. Futhermore, we will show that the property of being connection-induced also makes sense for preconnections with higher-dimensional integral 1-jets. What is more, with a connection on a manifold we will associate preconnections with x-dimensional integral 1-jets in two different ways; the two preconnections thus obtained (which will often coincide) will be called 'the canonically associated preconnection' and 'the canonically asssociated symmetric preconnection'; the integral manifolds of the former (resp. latter) are called *autoparallel* (resp. *geodesic*) submanifolds relative to the connection. We will dwell on these and similar issues later on, since for the time being those results are not necessary.

II.1.3 Conclusion

With the terminology and facts from the previous subsection at our disposal, we are able to formulate concisely the results already deduced in this section as the first assertion in the following fundamental proposition. Let us again consider the tangent distribution $F^{\tau}R$ of the preconnection induced on a geometrical parameter space M of submanifolds. Since the spaces F^{τ} were defined as tangent spaces to the manifolds $M^{\alpha}.s$, and these are obviously the canonical lifts to the contact manifold of submanifolds M^{α} of M, the directions (of the spaces F^{τ}) are in fact elementary preconnections of a very special type, namely **2-jets** of x-dimensional submanifolds of M. (This is an obvious consequence of the Remark II.10.)

Proposition II.13 Let M be an arbitrary manifold.

(i) *There is a canonical injective correspondence between*

(a) Structures on M of a completely geometrical (v. II.3) parameter space of submanifolds (recall that these structures have been defined by II.2 as classes of completely geometrical families of submanifolds isomorphic over the parameter space M)

and

(b) Admissible integrable preconnections on M, where **admissibility** *is defined as the conjunction of the following two conditions (in notation of Definition II.6):*

(b1) The foliation of R by leaves of the integrable distribution $F^{\tau}R$ is actually a fibration (whose base S is therefore clearly the space of maximal connected integral immersed manifolds in M).
(b2) The projection of this fibration maps the submanifolds $J^{\varepsilon}.m$ of R injectively into the base S.

More precisely, the correspondence is defined in the way described in Subsection II.1.1: For a family as in (a) compound-geometric amenability means precisely that it is in fact a double fibration. Thus, the given family of submanifolds of S can also be interpreted as the conjugate family of submanifolds M^{α} of M with parameter space S. The corresponding preconnection as in (b) is now built from the conic structure induced by the family and the distribution in its integral-jet space R which determines one of the two fibrations of R appearing in the double fibration, namely the fibration over S. In other words, the preconnection is defined by the requirement that the submanifolds M^{α} of M from the conjugate family be integral.

[As remarked above, the assertion (i) has already been proved earlier in this section.QED]

(ii) The injective correspondence considered in (i) is in fact bijective. What is more, the inverse corespondence can explicitly be described in the following way: Consider an admissible integrable preconnection on M. If we denote by S the base of the fibration of R formed by the leaves of the tangent distribution $F^{\tau}R$ (of the preconnection), the corresponding geometrical family of manifolds in S is defined to be the diagram

$$S \longleftarrow R \longrightarrow M$$

formed by the obvious projections. [The proof of (ii) is equally straightforward, so we will provide only a few salient details: This diagram is obviously a double fibration and thus indeed defines a compound-geometrical family with parameter space M of submanifolds S^{α} of S which can clearly be identified with the spaces J^{ε}. Next we prove geometric amenability of the family, i.e. that for every point m the canonical (relative to the family) linear system $T \rightarrow H^0(T_S^{/\alpha} S^{\alpha}$ of normal-vector fields along the submanifold S^{α} with parameter m, is geometrical. What is more, we would like to see that the localized conic structure obtained in this way coincides with the one underlying the given localized preconnection. All this follows easily from the fact -proven in I.10- that for an arbitrary compound-geometrical family of submanifolds the map $S^\alpha \to J := Gr(x, T)$ which constitues the induced localized structure can alternatively be defined by mapping a point of $s \in S^\alpha$ into the direction at m of the submanifold M^α (from the conjugate family) with parameter s. QED]

Remark II.14 The previous proposition could be formulated (and proved in exactly the same way) in a somewhat more general version. More precisely, we could consider a larger class of integrable preconnections, namely, we do not have to require that the spaces J^ε be mapped injectively into the space S of maximal connected integral immersed manifolds in M. The twistorial description of this geometric structure turns out to be the following: the underlying manifold is equiped with the structure of a geometrical parameter space of possibly non-injectively immersed manifolds $J^\varepsilon = S^\alpha$ in an arbitrary manifold S. However, from the point of view of local differential geometry, this version is in fact not more general: as has already been said, in the next remark we will see that even a general integrable preconnection is locally admissible.

Remark II.15 Consider an integrable preconnection on amanifold M. A straifghtforward reasoning shows that for each point m there exists an open neighbourhood M' such that the restricted (obviously integrable) preconnection on M' is admissible, i.e. essentially the structure of a completely geometrical parameter space of submanifolds. [Explicitly, with notation analogous to the one from Definition II.6 the foliation of R' formed by the leaves of the distribution $F^\tau R'$ in R' is in fact a fibration and its projection maps the localized integral-jet spaces of the restricted conic structure injectively into its base manifold S'. Thus, according to the previous proposition, M' is a geometrical parameter space of submanifolds of S' with respect to a canonical parametrization, namely the one described in the assertion (ii) of the previous proposition.]

Remark II.16 In this section we have accomplished the process of 'geometrization' of the structure of a geometrical parameter space of submanifolds. It is not difficult to see that the condition of geometrical character is indispensable in this context: even the structure of a compound-geometrical parameter space of submanifolds can in general not be 'localized' in the sense that it can not be thought of as a field of localized geometric structures (which can be 'transferred' from point to point by jets of biholomorphisms). What is more, the latter structure is a priori not even a k-th order 'compound' (or 'generalized' in terminology of [1]) geometric structure; this by definition means a 'fiberwise built' (or first-order 'bundle-founded geometric') structure on a k-th order compound G-structure (i.e. on a reduction of the k-frame bundle relative to a conceivably non-injective mapping of Lie groups, cf. I.4). In fact, the structure of a geometrical parameter space of submanifolds is usually a *second*-order expanded '*bundle*-founded geometric structure' (defined in a plausible way) on a first-order compound G-structure, more precisely on a 'holomorphic homogeneal compound conic structure' (cf. I.15). In particular, a compound geometric structure (namely the compound conic structure) is only its constituent part.

II.2 Non-degeneracy of a Double Fibration and Geometric Amenability

Geometric amenability of a compound-geometrical family of possibly non-compact submanifolds has been defined as the strongest 'non-degeneracy' condition, i.e. 1-regularity, on the conjugate family. In this section we express the 'differential version' of this condition as a somewhat weaker non-degeneracy condition on the original family. In this context 'complete degeneracy' of the original family will intuitively mean that submanifolds from the family through a given point do not differ from each other (although their parameters may form a manifold of positive dimension):

Definition II.17 In the situation of I.9 we say that the given compound-geometrical family of possibly non-compact submanifolds of S is **completely degenerate** (resp. **differentially completely degenerate**) if for each point $s \in S$ the submanifolds S^α from the original family through s (whose parameters form the submanifold M^α from the conjugate family with parameter s) coincide with each other (resp. belong to the same germ of submanifolds).

Remark II.18 Clearly, an example of a completely degenerate compound-geometrical family is given by the obvious *double fibration of the total space of a family of possibly non-compact products of manifolds.* More precisely, such a double fibration is obtained from fibrations (or families of possibly non-compact manifolds) $S^\alpha B$ and $M^\alpha B$ (with the same base B) in the following way: we define $S := S^\alpha.B$ (the totalspace of the first fibration), $M := M^\alpha.B$, $R := S^\alpha \times M^\alpha .M$ (the totalspace of the fibration by products of fibers of the given fibrations, i. e. the 'Whitney product' of the given fibrations) and take as projections of the incidence manifold R onto S and M the obvious maps. In fact, it is not difficult to show that a general completely degenerate family of possibly non-compact submanifolds is essentaially of this form: indeed, it suffices to observe that the conjugate family is also completely degenerate.

II.2.1 Frobenius Tensor of a Double Fibration

Proposition II.19 Consider the situation of I.11. (In other words, we are given a general compound-geometrical family of possibly non-compact submanifolds.)

(i) Notice that the spaces $T^\alpha_R \subset T_R$ at various points r form a distribution $T^\alpha_R R$ (on the incidence manifold R of the given double fibration), namely the sum of the distributions associated to the fibrations $S^\alpha M$ and $M^\alpha S$ of R (i.e. the fibrations forming the given double fibration; notice that the sum distribution is well-defined due to intersectional transversity of these fibrations). The Frobenius tensor of this sum distribution will be referred to as the **Frobenius tensor of the double fibration**.

(ii) Choose incident points m and s. The fiber T_R^α of the sum distribution at the point $r := s.m$ is (by definition) the direct sum of the tangent spaces T_S^α and T^α to the submanifolds (respectively) $S^{al}\,(\ni m)$ and $M^\alpha\,(\ni s)$ with parameters (respectively) m and s. In order to simplify notation we identify these two spaces with their images in the direct sum. Let us denote by $fr \in Hom(T_R^{\alpha\wedge 2}, T^{/\alpha})$ the Frobenius tensor of the double fibration at the point r. (Recall that the transverse space $T_R^{/\alpha} R$ of the sum distribution at r is according to I.10 canonically isomorphic to the normal spaces $T_S^{/\alpha}$ and $T^{/\alpha}$ of the submanifolds resp. S^α and M^α.) Note that the restrictions of this tensor to subspaces $T_S^{\alpha\wedge 2}$ and $T^{\alpha\wedge 2}$ vanish (since the summands of the sum distribution were integrable distributions). Therefore the Frobenius tensor can be recovered from its restriction to the subspace $T_S^\alpha \wedge T^\alpha \approx T_S^\alpha \otimes T^\alpha \approx T_S^\alpha \wedge T^\alpha$.

Claim: The 'essential part' of the Frobenius tensor of the double fibration, i.e. the above restriction, admits the following alternative interpretation: The mapping

$$T_S^\alpha \to Hom(T^\alpha, T^{/\alpha}),\; v_S^\alpha \mapsto (v^\alpha \mapsto fr(v_S^\alpha, v^\alpha))$$

is precisely the differential of the mapping $S^\alpha \to J$ constituting the family-induced compound conic structure on T with possibly non-compact integral-jet space . [The proof is a straightforward application of the main properties of the 'canonical involution from [18]. Alternatively, one can consider the flow of vertical vector fields. QED]

II.2.2 Flat Conic Structures

Lemma II.20 Let us consider a compound conic structure on a vector space T, where notation is as in I.5. Then there is an up to a unique isomorphism determined *expanded* integrable compound preconnection on (the underlying manifold of) T such that the corresponding integral submanifolds are precisely the affine subspaces of T whose vector spaces are compound-itegral spaces of the given compound conic structure. This compund preconnection could alternatively be characterized in terms of the associated family of submanfolds: This turns out to be precisely the linear system of (the images of) sections (as submanifolds of $(T_S^{/\alpha} S^\alpha)$ given by the compound amenable linear system. This preconnection and the underlying conic structure is called **flat**. [The proof is straightforward. QED]

II.2.3 Frobenius Tensor and Geometric Amenability

The next straightforward lemma is in fact an analysis of the immersivity condition for a map into a Grassmanian space. Incidentally, assertion (ii) of this lemma, namely the case of a projective space, is stated (without proof) in [19].

Lemma II.21 Let us again consider the situation from I.10. (In other words, we are given a vectorial compound conic structure.) The differential $Dj_{s^\alpha} \in Hom(T_S^\alpha, T_J)$ of the map $(s^\alpha \mapsto j)$ at a given point s^α can be explicitly described in terms of the given parameter vector space of sections in the following way:

We first recall that $T_J = Hom(T^\alpha, T^{/\alpha})$. Therefore we have the following sequence of canonical isomorphisms:

$$Hom(T_S^\alpha, T_J) = Hom(T_S^\alpha, (Hom(T^\alpha, T^{/\alpha})) \approx Hom(T^\alpha, Hom(T_S^\alpha, T^{/\alpha}))$$

Claim: The element of $Hom(T^\alpha, Hom(T_S^\alpha, T^{/\alpha}))$ corresponding to Dj_{s^α} is precisely the negative of the map which assigns to a vector $v^\alpha \in T^\alpha$ the differential at s^α of the section $\{v^{/\alpha}\}_{S^\alpha}$ with parameter v^α. More concisely, the above mentioned element is the map

$$(T^\alpha \to Hom(T_S^\alpha, T^{/\alpha})) := ((T^\alpha \to Hom(T_S^\alpha, T^{/\alpha}),\ v \mapsto D(\,v^{/\alpha}\,)_{s^\alpha}).$$

Here $D(\,v^{/\alpha}\,)_{s^\alpha}$ denotes the (completely invariantly defined) covariant differential (of a section vanishing at the given point) with respect to the canonical fiber-transverse space at a zero vector in the total space of a vector bundle; in view of the fact that T^α precisely consists of parameters of those sections which vanish at s^α this differential is in our case well-defined .

[Proof: (See II.22 for an alternative proof. Since the present proof does not invoke any facts about double fibrations, it is somewhat longer. However, it is simpler and one obtains an additional insight from its main idea, namely the construction of a 'regular' local trivialization of the compound integral integral-transverse vector bundle.) Let T^{β} be a direct complement of T^{α}. It is a standard (and obvious) fact that the open subspace J^{ω} of J consisting of all directions $j \in J$ transverse to T^{β} has a canonical structure of an affine space with associated vector space $Hom(T^{/\alpha}, T^{\beta})$ and that the trivialization of the tangent bundle of J^{ω} induced by that chart on J is compatible with the obviously defined isomorphism

$$T_J := Hom(T^{/\alpha}, T^{/\alpha}) \approx Hom(T^{\alpha}, T^{\beta}).$$

In order to simplify notation, we will assume that the quotients $T^{/\alpha}$ and $T^{/\alpha}$ have been identified with the subspaces T^{β} and T^{α} respectively by means of the obvious isomorphisms. (In particular $T_J = Hom(T^{\alpha}, T^{\beta})$ and the isomorphism in the above formula is the identity.)

Furthermore, the space T^{β} is clearly a direct complement of tangent spaces $T^{\alpha\prime}$ to directions $j' \in J$ associated to all $s^{\alpha\prime}$ sufficiently close to s^{α}. Thus, all the corresponding quotient spaces $T^{/\alpha\prime}$ (i.e. fibers of the bundle $T^{/\alpha}S^{\alpha}$) are canonically isomorphic to T^{β}; in other words a trivialization of the bundle is given in a neighbourhood of s^{α}. In order to simpify notation, we will assume that those fibers are identified with T^{β} by means of this trivialization (this is clearly in agreement with our first notational convention).

Let $v^{\alpha} \in T^{\alpha}$ and $v^{\alpha}_S \in T^{\alpha}_S$ be arbitrary elements. Let us denote by $\{v^{/\alpha}\}_{S^{\alpha}}$ the section of $T^{/\alpha}S^{\alpha}$ with parameter v^{α} (we know this section is vanishing at s^{α}). Its restriction to a sufficiently small neighbourhood of s^{α} is according to the above notational convention a map into T^{β}. In addition to that the invariantly defined differential of that section at s^{α} coincides with the differential of that map, i.e. with the differential relative to the trivial connection induced by the trivialization defined above. (Indeed, the horizontal space at a zero vector relative to any connection on a vector bundle is precisely the tangent space to the zero section.) In view of this and our notational conventions, it is clear that our objective is simply to prove that

$$((Dj_{s^{\alpha}})v^{\alpha}_S)v^{\alpha} = -D(v^{/\alpha})_{s^{\alpha}}v^{\alpha}_S \in T^{\beta}.$$

But the left-hand side is clearly $D((j'-j)v^{\alpha'})_{s^{\alpha'}=s^\alpha}v^\alpha_S$, where $s^{\alpha'}$ is a point sufficiently close to s^α and j' is the associated direction in T. Now it remains to observe that $(j'-j_o)v^\alpha = -v^{/\alpha'}$, where $v^{/\alpha'} \in T^\beta$ denotes the vector at the point s^α from the field with parameter v^α. (This is a consequence of the fact that $v^{/\alpha'}$ is by definition of the map $(s^\alpha \mapsto j)$ precisely the image of v^α in the quotient $T^{/\alpha'} \coloneqq T/T^{\alpha'}$, and of the rule used to identify such quotient spaces with T^β.) QED]

Remark II.22 We now expound the above mentioned alternative proof of the preceding lemma. (The reason for its inclusion is the insight it offers into the interrelationship of various concepts considered in this chapter.) Let us consider the double fibration constructed in II.20 from the given flat compound conic structure with possibly non-compact localized integral jet spaces. In view of II.19 and II.20 the differential $Dj_{s^\alpha} \in Hom(T^\alpha_S, T_J)$ can in an obvious way be expressed in terms of the Frobenius tensor fr of the double fibration (defined in II.19). More concretely, this differential is precisely the map $(v^\alpha_S \mapsto fr(v^\alpha_S, v^\alpha)$. Therefore, we simply have to prove that

$$D(v^{/\alpha})_{s^\alpha}v^\alpha_S = fr(v^\alpha, v^\alpha_S)\,.$$

Again according to II.19 the right-hand side 'measures the rate of rotation' of the jets at s of the sections with parameters in T_α. It remains to apply the Schwarz theorem and linearity of the evaluation maps $(T \to T^{/\alpha'})$.

Proposition II.23 *Let a vector space T be equiped with a compound conic structure with possibly non-compact integral-jet space. In other words, we again consider the situation from II.20, i.e. T is a compound-geometrical parameter vector space of sections of a fiberwise y-dimensional vector bundle $T^{/\alpha}S^\alpha$.*

(i) *For any point s^α immersivity at s^α of the map $(s^\alpha \mapsto j)$ is equivalent to the following condition on the parameter space T of sections: the image of T^α under the map $(T^\alpha \to Hom(T^\alpha_S, T^{/\alpha}))$ defined in II.21 is not contained in a proper subspace of its codomain of the form $Hom(T^{\alpha/\rho}_S, T^{/\alpha})$, where $T^{\alpha/\rho}_S$ is some quotient space of T^α_S.*

(Proof of (i): It suffices to observe that in view of II.21 a vector v_S^α belongs to the kernel of the differential $Dj_{s^\alpha} \in Hom(T_S^\alpha, Hom(T^\alpha, T^{/\alpha})$ iff the image of the map $(T^\alpha \to Hom(T_S^\alpha, T^{/\alpha}))$ is contained in the space $Hom(T_S^{\alpha/p}, T^{/\alpha})$, where $T_S^{\alpha/p}$ is the quotient space of T_S^α by the subspace spanned by v_S^α.) QED

(ii) *In the case when $y = 1$ (i.e. $J = \mathbf{P}^*$) assertion (i) assumes a much simpler form:*

For any point s^α immersivity at s^α of the map $\{j\}_{S^\alpha}$ of S^α into J is equivalent to the following condition on the parameter space T of sections: the map $(T^\alpha \to Hom(T_S^\alpha, T^{/\alpha})$ defined in II.21 is surjective. Less formally, the condition is that any fibre-transverse 1-jet through the point s^α (identified with the corresponding zero-vector in the total space $T^{/\alpha}.S^\alpha$) be realized by the graph of some section from T (vanishing at s^α, of course).

(Proof of (ii): In view of (i), it suffices to prove that any subspace of $Hom(T_S^\alpha, T^{/\alpha}))$ is of the form $Hom(T_S^{\alpha/p}, T^{/\alpha})$, where $T_S^{\alpha/p}$ is some quotient space of T_S^α. But this is trivial since $T^{/\alpha}$ is by assumption a vector line. QED

II.3 Geometric Description of the Structure of a Special Geometrical Parameter Space of Submanifolds

II.3.1 Geometric Description of the Structure of a Locally Complete Parameter Space of Geometrical Submanifolds

In view of the results of Section II.1, and of the fact (proved in the Chapter I) that a locally complete parameter space of geometrical submanifolds of a given manifold is geometrical, the objective of this section could be rephrased as characterization of those integrable preconnections which are essentially locally complete parameter spaces of geometrical submanifolds.

Proposition II.24 Consider an admissible integrable preconnection on M (v. II.13 for definition of admissibility). In other words (according to Proposition II.13), the manifold M is equiped with a structure of a geometrical parameter space of (a family of) submanifolds S^α of a manifold S. *Local completeness*

of the parameter space M is equivalent to completivity of the (expanded) conic structure underlying the preconnection. (This condition was defined as completivity of the vectorial conic structures on the tangent spaces at all points m, i.e. as completivity or 'normality' of the embeddings of manifolds J^{ε} into the Grassmanian manifolds J.)
[This is an obvious consequence of the equivalent description in Proposition II.13 of the above conic structure (as the conic structure induced by the family) and of I.18. QED]

Proposition II.25 Suppose we are given an isomorphy class of fiberwise y-dimensional amenable vector bundles. In other words, we are given an isomorphy class of completive vectorial conic structures with y-codimensional integral jets (v. the assertion (iv) of Proposition I.18; the way in which these two classes correspond to each other has also been precisely described there).

Claim: A locally complete space M of submanifolds S^{α} (of an arbitrary manifold S) such that their normal bundles are all of the given class, can equivalently (in the sense of Proposition II.13) be described as an admissible integrable preconnection on M with the following property: The underlying conic structure is homogeneal, where the (common) isomorphy class of the induced localized conic structures coincides with the given class.
[In order to prove this, it suffices to invoke I.18(iv) and the proof of II.24. QED]

II.3.2 Geometric Description of the Structure of a Locally Complete Parameter Space of Normally Rigid Geometrical Submanifolds

Proposition II.26 A connected locally complete family of normally rigid (v. I.22) geometrical submanifolds can equivalently (in the sense of Proposition II.13, v. the next remark for a precise formulation) be described as an admissible (v. II.13) integrable preconnection on a connected manifold with the below stated properties (a), (b).

(a) The underlying expanded conic structure is completive.

(b) The underlying expanded conic structure has any of the following two properties, which are according to I.21 equivalent for any completive expanded conic structure on a connected manifold:
(1) All its localized integral-jet spaces J^ε and localizedintegral-transverse bundles $T^{/\alpha}J^\varepsilon$ are rigid (manifolds resp. vector bundles).
(2) The expanded conic structure is homogeneal, where for the induced (mutually isomorphic) localized conic structures the integral-jet spaces J^ε and integral-transverse bundles $T^{/\alpha}J^\varepsilon$ are rigid (manifolds resp. vector bundles). *QED*

Remark II.27 Of course, the above proposition could be more rigourously restated as follows: The invariant bijective correspondence defined in II.13 between structures on a manifold M of geometrical parameter manifolds of submanifolds and admissible integrable preconnections on M, restricts to a bijective correspondence between structures of locally complete parameter spaces of normally rigid geometrical submanifolds, and of admissible integrable preconnections satisfying (a), (b) .

Corollary II.28 Consider an integrable preconnection on M satisfying the conditions (a), (b) from II.26. Then for any point m there exists an open neighbourhood M' such that for the restricted (obviously integrable) preconnection on M' is admissible. (Thus the restricted preconnection is according to II.26 essentially a structure on M of a locally complete parameter space of normally rigid geometrical submanifolds.)
[This is an obvious consequence of II.15 and II.26.QED]

Proposition II.29 Suppose we are given an isomorphy class of manifolds and an integer y such that any y-codimensional non-exceptional embedding of a manifold of that class gives rise to a normally rigid geometrical submanifold . (Explicitly, manifolds of that class are rigid, and all fiberwise y-dimensional vector bundles admitting nontrivial sections over manifolds of that class are amenable and rigid.) Our objective is to give a geometric description of locally complete families of y-codimensional submanifolds of the given isomorphy class.

(i) Such a family can equivalently (in the sense made precise in II.13) be described as an admissible (v. II.13) integrable preconnection on a connected manifold with the following property: the underlying expanded conic structure M_{csm} has any of the two below stated properties, which are according to I.23 from the previous chapter equivalent for any conic structure on a connected manifold:

(1) The conic structure is completive, its integral jets are y-codimensional and its localized integral-jet spaces are of the given isomorphy class.

(2) The conic structure is *homogeneal* and completive, its integral jets are y-codimensional and its localized integral-jet spaces are of the given isomorphy class.

[Proof of (i): It suffices to apply I.23 and II.13. QED]

(ii) Let us consider an admissible integrable preconnection on a connected manifold with the above property (1). Then for completive localized conic structures belonging to the isomorphy class defined in (2) the integral-jet spaces and integral-transverse bundles are obviously rigid and this class can according to (i), II.24 and I.23 alternatively be characterized by the following property: the corresponding (relative to the bijective correspondence from I.22(iv)) isomorphy class of amenable completely rigid vector bundles is precisely the (common) isomorphy class of the normal bundles of the submanifolds S^α of S belonging to the given space M. QED

Remark II.30 Let a manifold M be equiped with an admissible integrable preconnection with a completive underlying conic structure, i.e. (according to II.24) with a structure of a locally complete geometrical parameter space of submanifolds S^α of a manifold S. Furthermore, suppose that for some point m both the submanifold S^α of S with that parameter and its normal bundle $T_S^{/\alpha} S^\alpha$ are rigid .

Claim: After replacing (if necessary) M by a sufficiently small open neighbourhood of m (again denoted by M) and appropriately restricting the family of (unchanged) submanifolds S^α of the (unchanged) manifold S, the preconnection is in fact the structure of a locally complete parameter space of *nor-*

mally rigid geometrical submanifolds. In particular, the underlying expanded conic structure is homogeneal (i.e. the induced localized conic structures are mutually isomorphic for various points m).

[This follows immediately from I.21 (i.e. essentially from the main properties of rigid manifolds and rigid vector bundles.) QED]

Chapter III

Second-Order Invariants of Conic Structures

In the second chapter we have already studied the most basic (meaning first-order) differential invariant of a locally complete parameter space M of geometrical submanifolds S^α of a given manifold S. Indeed, in the process of determining its equivalent geometric description, we have identified the differential invariant which 'precisely corresponds' to the normal bundle of a submanifold S^α from the given space M: this invariant turned out to be simply a localized conic structure at the parameter m of that submanifold (i.e. a vectorial conic structure on the tangent vector space T at m). We have also seen that the Frobenius tensor of the double fibration is encoded into this first-order invariant. The isomorphy class of this localized conic structure, i.e. of the normal bundle of the submanifold S^α, was seen to be independent on the submanifold from the family under reasonably weak rigidity conditions. E.g. in Chapter V, we shall see that in the case of a locally complete parameter space of rational curves in a surface this isomorphy class is completely determined by the self-intersection number x (of the curves S^α in S) or by $dim\, M$ $(= x + 1)$.

In the remanining chapters the structure on M of a locally complete parameter space of geometrical submanifolds, i.e. an admissible integrable preconnection on M, will be investigated in more detail. In fact, the main object of study will be the higher-order differential invariants, i.e. invariants of localized preconnections (which are of second order) and 'preconnectional jets' (which will be higher-order localized geometric structures). Just like in the first chapter, most arguments will for the sake of clarity be carried out in the appropriate more general (and more natural) context, namely in that of geometrical (v. I.11) parameter spaces of submanifolds. (We will impose additional conditions only when they yield results which are either more difficult to prove or without obvious meaning in the above general context; e.g. an entire chapter will be devoted to the much more special context of geometrical parameter spaces of *hypersurfaces* and *hypersurface-directional* conic structures since the study of Cartan distributions is greatly simplified in the case of 1-codimensional distributions.) This level of generality will not only make the proofs more transparent, but will also lead to results interesting in their own right.

Let us observe that *integrability* of the *localized* preconnections is well-defined since they were by definition (v. Remark II.11) special localized generalized pseudo-conic structures, and for the latter we define **integrability** in the obvious way, namely as the existence of a local (i.e. defined on a neighbourhood) integrable expanded structure inducing the given localized structure. This definition of integrability will also apply to 'preconnectional jets'. Generally speaking, our main objective will be to obtain a possibly coarse, but effective classification of the above mentioned not necessarily integrable localized geometric structures. More rigourously, we would like to be able to distinguish between them as finely as possible by means of their 'effectively manageable' invariants; by the latter we mean principally the invariants of order two or higher (meaning invariants of geometric structures of that order) which are themselves geometric structures and are of order one when considered as such. (Of course, most interesting such invariants will be tensor-type invariants.) However, we will also be interested in other invariants, e.g. distinguished connections or classes of connections, since they often generate tensor-type invariants, e.g. torsions, curvatures and classes thereof). One of the purposes of this classification will be strengthening the basic result quoted at the beginning of this chapter, namely reading off the appropriate tensor-type invariants of the higher order localized geometric structures from the higher order infinitesimal neighbourhoods of submanifolds belonging to the given parameter space. Of course, a more immediate purpose of the classification will of course be to investigate integrability, i.e. to 'eliminate' by identifying the tensor-type obstructions those localized structures which are not integrable (or, equivalently, which can not actually be realized by localizing the structure of a geometrical parameter space of submanifolds). In fact, one of these tensor-type obstructions is clearly the invariant of a localized preconnection we have already introduced in II.11, namely the *pretorsion*, while the other is the '*precurvature* of a preconnectional 1-jet' (cf. [13]).

In the above context the purpose of this chapter is the following: Since the structure of a geometrical parameter space of submanifolds is essentially an integrable preconnection, and this in turn includes a weaker (i.e. underlying) first-order geometric structure (namely the underlying conic structure), it is natural to first carry out the classification of the localized conic structures and of conic-structural jets (since, of course, the differential invariants of the weaker structure are also differential invariants -in a certain sense the most basic ones- of the stronger structure). In addition to that, since integrability of a preconnection is a fairly strong condition and the (expanded!) conic structures themselves have a rich geometry, it is reasonable to expect that there are few integrable preconnections on a given conic structure. In fact, we will show even uniqueness of such a preconnection under reasonable conditions. (The conditions we impose are less restrictive than those usually considered in the sense that we do not require the of '1-prolongation' of the conic structure to be nothing more than a preconnection, i.e. we do not assume 'sparsity' of the '1-prolongation' already at the level of jets.) Therefore, the differential geometry of geometrical parameter spaces of submanifolds can often litterally be reduced to the study of conic (1-) structures. In particular, all the (above mentioned) invariants of the integrable localized preconnections and integrable 'preconnectional jets' are then expressible in terms of the invariants of appropriate 'conic-structural jets'. In accordance with that, this chapter will be devoted to an investigation of conic structures on manifolds in their own right, in particular of their second-order invariants. More precisely, we will give a precise definition and a reasonably fine classification of 'conic-structural 1-jets' (i.e. localized second-order geometric structures 'expanding' localized conic structures).

As the case of Einstein-Weyl 3-manifolds (i.e. locally complete spaces of embedded rational curves of self-intersection two, cf. [9]) shows, there do exist examples where all the information is *not* stored in a conic structure and we actually need to study the richer geometry of preconnections. Of course, in this general case we will be interested in how relevant tensor-type conic-structural invariants determine the set of isomorphy classes of all possible compatible infinitesimal neighbourhoods (meaning those asssociated with various integrable preconnectional jets compatible with the given conic-structural jet). In other words, we will measure the extent to which such an isomorphy class is reconstructible already from the tensor-type conic-structural invariant.

III.1 Obstructions to Conjunctive Integrability

As suggested at the beginning of this chapter, the most obvious purpose of the classification of '1-jets' of expanded conic structures will be elimination of some of those structural jets which can not be obtained by localization from the underlying conic structure of some integrable expanded preconnection. Therefore, the starting point will be the following two questions:

(a) Are there any obvious necessary conditions on an expanded conic structure in order for it to admit an integrable preconnection?

(b) Under what circumstances does uniqueness of the integrable preconnection from (a) hold?

As for the question (a), we are immediately naturally lead to a (still rather strong, i.e. of high order) necessary condition: there has to be an 'abundance of integral submanifolds' of M_{csm}, meaning that the expanded conic structure is *integrable* (v. I.3). [Indeed, an integral submanifold through a given integral jet is furnished for instance by the unique integral submanifold with that property relative to the integrable preconnection.]

In view of this, it is natural to introduce for the (much stronger) requirement from (a) on an expanded conic structure the following name: **conjunctive integrability** of an expanded conic structure is defined as the existence of integrable preconnections (with that underlying conic structure). Similarly, we define **conjunctive integrability at a point** as conjunctive integrability of the restriction of the conic structure to some neighbourhood.

In the search of a necessary condition for conjunctive integrability (i.e. a condition as in (a)) which is weaker (more concretely 'of lower order' or 'more localized') and thus more useful than integrability, we are now equally naturally lead to the classical condition of (first-order) prolongability (or first-order formal integrability) of a (first-order) generalized conic structure:

Prolongability of an arbitrary generalized conic structure on M at an integral jet j through a point m is defined as the following condition (which is obviously weaker than integrability): there exists a *tangential* (relative to the expanded conic structure) 2-jet j^2 (of a submanifold of M) tangent to the 1-jet j, i.e. a tangential symmetric elementary preconnnection at j. (V. the definition of an elementary preconnection in II.4; recall that 2-jets with a given tangent 1-jet are precisely symmetric elementary preconnections at that 1-jet and that tangentiality of a 2-jet or, more generally, of an elementary preconnection means by definition that as an 1-jet in the contact manifold $J.M$ it is actually an 1-jet in the expanded integral-jet space of the generalized conic structure $R \subset J.M$.) Futhermore, **prolongability at a point** and **prolongability** of an expanded generalized conic structure are defined in the obvious way (cf. the above definitions of various concepts of integrability). Likewise, the **prolongation** of a prolongable generalized conic structure is defined as generalized *conic* 2-structure (i.e. symmetric generalized *pseudo*-conic 2-structure) formed by tangential 2-jets.

As indicated above, the prolongability of a conic structure on a manifold M is an obvious necessary condition as in question (a) and is a priori weaker than integrability.

Let us observe that there is also a third obvious necessary condition for conjunctive integrability, which is, like integrability, stronger than prolongability, and, like prolongability, 'of a low order', and therefore of a rather different nature than integrability. Indeed, we define **conjunctive (1-) prolongability at a point** m of an expanded conic structure as the existence of a tangential symmetric (relative to the conic structure) localized preconnection at m; less formally, we require the existence of tangential 2-jets tangent to various integral 1-jets through m *'simultaneously'* (with the implication of holomorphicity). The **conjunctive prolongability** of a conic structure on a manifold, which we define as conjunctive prolongability at each point, is according to the results from the previous chapter a third condition as in question (a).

As for the question (b), i.e. the uniqueness of an integrable preconnection with a given underlying conic structure, for the time being suffice it to say that we will obtain reasonably weak sufficient conditions for uniqueness even of a tangential symmetric preconnection. Of course, such results will follow from the study of conjunctive prolongability, i.e. of the 'richness' of the prolongation of the conic structure.

Before proceeding with rigorous exposition, we give an outline of the content of the rest of this section.

As suggested above, our objective will be to find tensor-type invariants of the conic structure on the manifold such that some of the above mentioned conditions as in (a) could be expressed in terms of these invariants. Since these invariants will turn out to be of second order, i.e. to depend only on the so called 'conic-structural 1-jets', it will follow that in the homogeneal case they can be expressed explicitly in terms of the **intrinsic torsion** of the conic structure. Analogously, in the general case they will be expressible in terms of *pretorsions*. Such second-order invariants will be called *'intrinsic pretorsions'* (of various kinds) since they, just like the *intrinsic* torsion, depend only on the *expanded* geometric 1-structure (namely the conic structure), and not on finer expanded geometric 2-structures of genuinely second order (like preconnections or connections).

In fact, it will turn out there are two succesively defined second-order 'jointly' precise second-order obstructions to the conjunctive prolongability of a conic structure on a manifold (succesiveness means that the second one is defined only if the first one vanishes). Vanishing of the first of those invariants at a point will be precisely the condition of prolongability of the conic structure. We will call this obstruction **'the intrinsic pretorsion'** of the 'conic-structural jet'. This invariant could roughly be described as the most obvious 'common part' of the pretorsions of tangential localized preconnections.

In the case of a prolongable (i.e. intrinsic pretorsion-free) conic-structural jet, the second, much finer, invariant is defined at a given point m as an element of a certain Cech cohomology group, more precisely the 'class' of a geometrically defined affine bundle, which we will call the affine bundle of spaces of 'tangential' symmetric elementary preconnections of the 'conic-structural jet' . Vanishing of this class means (as has already been stated) precisely the conjunctive prolongability of the conic structure. We will obtain an explicit relation between this class (i.e. the finer invariant) and intrinsic torsion, namely the class will be identified with the so called **'conjunctively intrinsic pretorsion'**. (It could roughly be described as the precise 'common part' of the pretorsions of tangential localized preconnections.)

As has already been mentioned, the investigation of conjunctive prolongability will also result in very concrete answers to the question (b) . For instance, in a subsequent chapter we will apply straighforwardly this result to Veronese conic structures and it will turn out that for most values of the self intersection number x (more precisely for $x \geq 5$) uniqueness does hold; globally this will immediately imply that for those x the structure of a locally complete parameter set of rational curves in a surface coincides simply with the structure of a Veronese-expanded conic structure with certain properties (in particular, it is a first-order geometric structure). In other words, we prove that a Veronese conic structure in most dimensions arises from at most one structure of a locally complete parameter set of embedded rational curves. (The analogous statement is standard in the case of paraconformal structures, but the proofs are essentially different: instead of consideringing prolongation at (or 'of') a single 1-jet, we must apply a cohomological argument to the totality of all integral jets at a point in order to describe explicitly the 'affine-bundle class of the 'tangential symmetric elementary-preconnnection bundle'). This result indicates that in the theory of spaces of rational curves in surfaces the associated Veronese conic (1-) structures (on these spaces) play a more prominent role than, for instance, the underlying projective (2-) structures which will be defined in a subsequent chapter, or the underlying (1-) structures of conformal manifolds (euclidean or symplectic, depending on the parity of x). (That notwithstanding, the conformal and projective invariants are of course important for the above mentioned reason, namely they are also invariants of the structure of a locally complete parameter set of curves in a surface; the fact that we have a large class of manifolds equiped simultaneously in a natural way with conformal and projective structures is an indication of the complexity of the theory.)

We now begin the rigorous exposition of the above desribed results.

III.1.1 Intrinsic Torsion of a Structural Jet

Remark III.1 In order to make the exposition more self-contained and to motivate some further constructions and terminology, we review briefly the main poperties of intrinsic torsion of general first-order holomorphic geometric structures (v. Proposition I.15 for precise definitions).

Let us consider a first-order holomorphic geometric structure 'of type cs' at a point m of a manifold M. Here the simbol cs is a marker of an arbitrary category of vectorial geometric structures. (In our applications the category in question will consist of vectorial conic structures of a given isomorphy class.) In other words, a vectorial cs-structure T_{cs} is given on the tangent space T. The automorphism group (resp. its Lie algebra) of T_{cs} will be denoted by $G^{cs} \subset G \coloneqq Aut\, T$ (resp. $\mathbf{G}^{cs} \subset \mathbf{G}$). The manifold parametrizing cs-structures on the tangent vector space T (or localized cs-structures on the localized manifold (M, m)) is denoted by U, and the parameter of the given localized structure by $u \in U$. According to Proposition I.15 the *bundle* UM of spaces of localized cs -structures is well-defined. In this situation we can make the following observations:

(a) We have the commutative diagram of affine spaces formed by the upper two (slanted) rows in Figure III.1 and the associated diagram of vector spaces is indicated in the same figure using the folowing convention: for an affine space in the diagram the associated vector space in parentheses is indicated above; if we allow the posssibility that the affine space in question is empty, that is indicated by the symbol for an empty set next as a left lower index. The spaces from the diagram which have not been introduced yet are defined in the folowing way:

C is the space of (linear) connections on the manifold M at the point m. Explicitly, such a localized connection c is given in the following way: if the frame bundle associated to the vector bundle TM and the 'model' vector space $T_\mu (= \mathbf{C}^{x+1})$ is denoted by PM, and its structural group (i.e. $Aut(T_\mu)$) by G_μ, then the (localized) connection could be thought of as a G_μ-invariant distribution $T_{PM}^{mp}P$ of fiber-transverse spaces in the total space $P.M$ (defined) along the fiber P.

C^{sa} is its subspace consisting of symmetric localized connections.

J_{UM} is the space of **structural 1-jets** expanding the localized structure u; these structural jets are by definition just (1-) jets of sections of the bundle UM at the point $u.m$, i.e. fiber-transverse jets of submanifolds of the total space $U.M$. Since UM is a bundle of spaces of localized (holomorphic) geometric 1-structures, localized *cs*-structures equiped with the above structural jets are clearly localized geometric 2-structures. We will refer to them as **holomorphic structural 1-jets expanding the** *cs* **-structure** on the tangent vector spaces T with parameter u, or less precisely as *cs*-**structural 1-jets** . Of course, this definition is motivated by the fact that each **expansion of the localized structure**, by which we mean an expanded *cs*-structure defined on a neighbourhood and inducing the given localized structure, determines the obvious structural 1-jet (namely the 1-jet of the given field of localized structures).

$C^{/te}$ is the space of structural (1-) jets expanding the given *cs* -structure u **through isomorphic** *cs*-**structures** on tangent vector spaces at nearby points of M, or, more briefly, **homogeneal structural jets**. The motivation for the latter term is obvious: in order for a structural jet to be realized by a homogeneal (or 'infinitesimally homogeneous') expanded structure (on a neighbourhood), it must be homogeneal. (Notice that the concept of a homogeneal extended homogeneal cs-structural jet is well-defined.) This is the quotient affine space of C (as the notation suggests) relative to the middle vertical map in this diagram; the latter is defined as the standard association of structural jets expanding the point u with (localized) connections; the structural jet associated to a given connection is said to be **horizontal relative to the connection** (or structural jet through 'covariantly constant' or 'parallel' localized structures).

We will also introduce the following non-standard terminology: a connection will be said to be **tangential** *to a given structural jet if the latter and the horizontal structural jet are identical (i.e. tangent to each other). Clearly, tangentiality of an* expanded *connection to an expanded (holomorphic) cs-structure, which is defined in an analogous way (i.e. as tangency of the given section of UM to the distribution of horizontal spaces) is equivalent to the requirement that the connection be* **structure-preserving**; *therefore, we will for a general (possibly not holomorphic) expanded geometric structure replace the term* structure-preserving connection *by the shorter* **tangential connection**.

Incidentally, if all localized cs-structures are mutually isomorphic, than both inclusions in this diagram are in fact equalities.

$C^{/te\,sa_*}$ is the space of flat structural jets.

(Incidentally, it is claimed that the above spaces are in an obvious way affine with the indicated associated vector spaces.)

The second map in the nether row is defined by associating to a localized connection its torsion. The second map in the upper row is defined by commutativity of the diagram .

FigureIII.1

$(E^{/sa_* te} \rightleftharpoons \frac{T\otimes T^{*\otimes 2}}{\mathbf{G}^{cs}\otimes T^* + T\otimes T^{*\odot 2}})$				
$C^{/sa_* te}$				
$\uparrow$	$\nwarrow$			
$(E^{/sa} \rightleftharpoons T\otimes T^{*\wedge 2})$		$(E^{/te} \rightleftharpoons \mathbf{G}^{/cs}\otimes T^*)$		
$C^{/sa}$		$C^{/te}$		
$\uparrow$	$\nwarrow$	$\uparrow$	$\nwarrow$	
$(E^{/sa\, sa_* te} \rightleftharpoons \mathbf{G}^{cs}_{\bullet[*} \otimes T^*_{*]})$		$(E \rightleftharpoons \mathbf{G}\otimes T^* = T\otimes T^{*\otimes 2})$		$(E^{/te\, sa_* te})$
$C^{/sa\, sa_* te} \rightleftharpoons C^{te\,/sa_\bullet te}$		C		$C^{/te\, sa_* te}$
	$\nwarrow$	$\uparrow$	$\nwarrow$	$\uparrow$
		$(E^{te} \rightleftharpoons \mathbf{G}^{cs}\otimes T^*)$		$(E^{sa} \rightleftharpoons T\otimes T^{*\odot 2})$
		C^{te}		C^{sa}
			$\nwarrow$	$\uparrow$
				$(E^{sa_\bullet te} \rightleftharpoons \mathbf{G}^{cs}_{\bullet(*} \otimes T^*_{*)})$
				$C^{sa_\bullet te} \rightleftharpoons C^{te}\cap C^{sa}$

(b) On the level of vector spaces we can enlarge the diagram from (a) by adding the lowest row, which is precisely **one of the sequences appearing in the Spencer complex for the structural group** G^{cs}. Here the upper (resp. lower) abstract indices are replaced by dots (resp. asterisks) and [. . .] means (as usual) antisymmetrization, while lowered (. . .) suggests tensors which are symmetric in the indicated indices. Furthermore, in the enlarged diagram the middle row is an extension of vector spaces. (This means exactness, injectivity of the first map and surjectivity of the second.) Furthermore, under the assumption that all cs -structures on a vector space are isomorphic the middle column is also an extension and, consequently, all rows and columns are extensions.

(c) Let us assume that a structural jet expanding the given localized structure u is given. Then the diagram from (a) can be enlarged also on the affine level, i.e. we obtain the whole diagram in the Figure III.1. Here the new spaces are defined as follows:

C^{te} is the affine space of tangential (or 'structure-preserving') localized connections. Therefore the vectors from the associated vector space, namely the space $E^{te} \coloneqq E\mathbf{G}^{cs} \otimes T^*(\subset E)$ introduced in (b) will be called tangential-connection vectors, or simply **tangential connectors** (v. (a) and Remark II.9 for the justification of terminology). As we have already observed in (b) *this subspace of E is independent of the structural 1-jet expanding the localized cs-structure, i.e. determined by that structure alone. Clearly, it occurs in the position* (1, 1) *in the Spencer diagram for the structural group* G^{cs}.

$C^{te_\bullet sa}$ is the affine space of tangential symmetric localized connections.

$C^{/sa\, sa_* te}$ is the affine space consisting of torsions of tangential connections, or, briefly, the affine space of **permissible torsions**.

From the diagram it follows in particular that the **intrinsic torsion of the given structural jet**, i.e. the element of the space

$$C^{/sa_* te} \coloneqq E^{/sa_* te} \coloneqq \frac{T \otimes T^{*\otimes 2}}{\mathbf{G}^{cs} \otimes T^* + T \otimes T^{*\odot 2}}$$

of **intrinsic torsions of the given localized structure**, defined in the usual way (namely as the parameter of the space of permissible torsions) is the precise obstruction to the **flatness** of the structural jet (i.e. to the non-emptiness of the only possibly empty affine space in the diagram). Indeed, the latter affine space was defined as the intersection of two affine subspaces of the affine space C of localized connections, and the obstruction to the non-emptiness of this intersection is according to the elementary theory of affine spaces an element of the quotient vector space of E by the vector space obtained as the span of the vector spaces of the two affine subspaces.

III.1.2 Conic-Structural Prejets

Remark III.2 In the following we will repeatedly need certain concepts regarding affine bundles for which no standard terminology seems to be established, although they play a prominent role in the Cech cohomology theory. Therefore, we introduce the following conventions: an isomorphism over base of two affine bundles with the same vector bundle will be said to be **over vector bundle** if the associated automorphism of the vector bundle is the identity (cf. Remark V.3). Thus, for a given vector bundle VM the space of classes of affine bundles isomorphic over the vector bundle VM (more rigourosly the space of classes of affine bundles with the vector bundle VM and isomorphic over vector bundle) is precisely the cohomology group $H^1(VM)$. The elements of this space (i.e. parameters of classes of affine bundles isomorphic over the vector bundle VM) will be called the **affine-bundle classes** (or classes of affine bundles) **on the vector bundle** VM. Furthermore, an affine bundle with vector bundle VM admits a global section if and only if its class (on the vector bundle VM) is the zero-element of the vector space $H^1(VM)$. If the affine bundle has this property, it will be said to be **vectorial**. (In other words, the class of an affine bundle is the precise obstruction to its vectoriality.)

Of course, the above space of affine-bundle classes on VM is mapped surjectively (but possibly non-injectively!) into the space $H^1(G_{\mathbf{A}}M)$ of classes of affine bundles isomorphic over the base M; here $\mathbf{A}M$ is a trivial affine bundle and $G_{\mathbf{A}}$ is the automorphism group of the 'model' affine space. By means of this surjection the space $H^1(G_{\mathbf{A}}M)$ (of classes of affine bundles isomorphic over the base M) can be thought as the space of equivalence classes for the induced equivalence relation on the vector space $H^1(VM)$ (of affine-bundle classes on VM). Of course, this is precisely the space of orbits for the obvious action of the group $H^0(G_V M)$ (defined by 'transferring' an affine-bundle class by means of a vector-bundle automorphism).

Next we define a 'formal (1-) jet of an (expanded) conic structure'. In fact, we will use for this object the somewhat technical name *'full prejet of a conic structure'* which will soon turn out to be more suggestive within the terminological system built on principles from II.5. For the time being we will only work with this rather primitive version of the concept of an ordinary 'conic-structural 1-jet' (defined as an actual jet of fields of localized structures, v. III.1), where no holomorphic structure or even topology on the space of localized conic structures is introduced. The equivalence of the two definitions will be proved later; in fact, we will define conic-structural jets, just like the holomorphic conic structures, only under the assumption of homogeneality (by means of finite-dimensional Lie groups). (However, this will be satisfactory since, firstly, we will also obtain reasonably weak sufficient conditions on a localized conic structure for the homogeneality of all compatible expanded conic structures, and, secondly, even when this condition is not fulfilled, the homogeneal case is still the most interesting one.)

Proposition III.3 (i) The general concept of a 'structural 1-jet' will be defined for an obvious generalization of the vectorial conic structures, namely *'vectorial aggregational structures'*, which are at the same time more general than holomorphic vectorial geometric structures. We now expound their definition:

Let us consider a holomorphic natural association (v. Proposition I.15) of manifolds denoted by J with vector spaces T of a given dimension. J will be called the space of **vectorial atomary structures** on T; thus a vectorial atomary structure consists of a vector space T and a point $j \in J$. (Here J

is not necessarily the Grassmanian space of T, although this will be the most important case in our applications; in this case the atomary structures will be called **directional structures**). Since these are special holomorphic vectorial geometric structures, the **(holomorphic) expanded atomary structures** are well-defined. (E.g. the expanded directional structures are obviously just distributions on manifolds.)

Furthermore, let us define a **vectorial aggregational structure** as a vector space T equiped with a submanifold J^ε of the associated parameter space J of atomary structures; this submanifold will be called the **integral atomary-structure space**. (E.g. in the special case consisered above a vectorial aggregational structure is precisely a vectorial conic structure.)

Similarly, we define **expanded aggregational structures** *as the obvious generalization of expanded conic structures; explicitly, such a structure consists of a manifold M and a (holomorphic) subfamily of the bundle $J^\varepsilon M$ (in our case the bundle of Grassmanians) naturally associated to the tangent bundle.* (Incidentally, all considerations in this proposition could be carried out without any modifications on the level of the more elementary and obviously defined '*bundles of vectorial aggregational structures*'.)

(**(ii)**) Intuitively, a 'full aggregational-structural (1-) prejet' will consist of a localized aggregational structure as in (i) and the 'first-order part' of an actual local (i.e. defined on a neighbourhood) expanded aggregational structure inducing the given localized aggregational structure; by this we mean the restriction of the tangent bundle of the expanded integral atomary-structure space to the localized integral atomary-structure space. However, we will not require that this vector bundle actually arise in this way, except for the obvious conditions on its individual fibers. Therefore, the precise definition will be the following:

Let $J^\varepsilon(\subset J)$ be the integral space of atomary structures for a localized aggregational structure at the point m of a manifold M (i.e. of an aggregational structure on the tangent space T at m). A **full (aggregational-) structural (1-) prejet** expanding this localized aggregational structure is by definition a localized geometric 2-structure at m consisting of the localized aggregational structure and a vector subbundle (called the **tangent distribution** of the full structural prejet) $T^{rh}_{JM}J^{\varepsilon}$ of the restricted tangent bundle $T_{JM}J^{\varepsilon}$ such that each of its fibers T^{rh}_{JM} is relative to the bundle JM a fiber-transverse space **modulo** $T^{\varepsilon}_{\mathfrak{J}} := T^{\varepsilon}_{J}$ **over** T; here by the last condition we mean that it has the following two properties:

(a) It intersects the vertical space T_J precisely in the space $T^{\varepsilon}_{\mathfrak{J}}$.

(b) It projects under the differential of the projection in the bundle of Grassmanians onto the space T.

Furthermore, for a given expanded aggregational structure the **induced full aggregational-structural prejet** at a point m is defined as explained above.

((**iii**)) Our next objective is to show that tangentiality (or 'structure-preserving') of an expanded connection relative to a given expanded aggregational structure (which was defined in Remark III.1 by means of the parallel transport) is still expressible as a certain tangency also in this more general context; this will be a further justification for the term *tangential connection*.

First we define the above mentioned tangency condition. We say that a connection c at the point m (v. Remark III.1) is **tangential** relative to a given conic-structural 1-jet at m if the distribution $T^{mj}_{JM}J^{\varepsilon}$ of horizontal spaces in the manifold $J.M$ on the fiber J defined by the given connection (v. the same remark) is a vector subbundle of the constituent vector bundle $T^{rh}_{JM}J^{\varepsilon}$ of the full structural prejet.

Claim: Let a manifold M be equiped with a conic structure and an arbitrary connection. Then the folowing two conditions are equivalent:

(a) The induced localized connections (at various points of M) are tangential relative to the conic-structural 1-jets induced by the given expanded conic structure.

(b) The expanded connection is tangential relative to the expanded conic structure. (Recall that this this requirement explicitly means that for each curve joining two points $m, \bar{m}$ the parallel transport $T \to \bar{T}$ induced by the connection is an isomorphism of vectorial aggregational structures; e.g. in the case when atomary structures are directional structures and aggregational structures are conic structures, this means that both the parallel transport and its inverse map vector subspaces in integral directions into vector susapaces in integral directions).

[The proof of this proposition is a straightforward 'tautological' reasoning and will thus be omitted. Incidentally, this proposition would also hold for real-differentiable connections on the tangent bundles viewed as real-differentiable complex vector bundles, with the proviso that we only allow real curves in the manifold (in order for the parallel transport to be well-defined).QED]

(iv) The simplest genuinely second-order invariant of an full aggregational-structural prejet can be constructed in the following way:

As we have already observed in the special case of conic structures, any expanded aggregational structure on M determines a possibly non-trivial (holomorphic) family of manifolds, namely the subfamily $J^{e}M$ (formed by integral atomary-structure spaces) of the bundle JM. In other words any etension of a given localized aggregational structure (into a neighbourhood) induces a deformation of the integral atomary-structure manifold. In particular, the **Kodaira-Spencer map** $KS \in Hom(T, H^1(T^e_J J^e)$ of this deformation is well-defined. We will call the linear maps belonging to the above space **infinitesimal deformation of the manifold** J^e (with parameter space M).

Claim: An infinitesimal deformation of the integral atomary-structure manifold can be associated in a canonical way (described in the proof) with each structural 1-jet expanding a localized aggregational structure. Furthermore, the infinitesimal deformation of the integral atomary-structure manifold associated to the full prejet of a local expanded aggregational structure, is precisely the Kodaira-Spencer map of the associated deformation. (More succinlty, the Kodaira-Spencer map of the deformation induced by an expanded structure depends only on its full prejet and it can be defined for arbitrary full structural prejets).

[Proof of (iv): Both statements follow easily from the following (not standard, but obvious) invariant reformulation of the definition of the Kodaira-Spencer KS map of an arbitrary deformation BM of a manifold B: KS assigns to a tangent vector v at m precisely the affine-bundle class (v. Remark

III.2) of the affine bundle on B whose fiber at the point b is defined as the inverse image of v in the tangent space $T_{B.M}$ at the point $(b.m)$ with respect to the differential of the projection (in the family BM).QED]

Let us observe that the infinitesimal deformation of the integral-jet space associated to a given full structural prejet vanishes as soon as either of the following reasonably weak conditions is fulfilled: rigidity of the integral-jet space or existence of (localized) connections tangential relative to the full structural prejet. Furthermore, the latter of these conditions is clearly weaker tha homogeneality of the full structural prejet.

In the next proposition we introduce the concept of a 'conic-structural prejet', which is related to the concept of a full conic-structural prejet, but of a more specific character, i.e. not defined for general aggregational structures. Indeed, a peculiarity of conic structures is that they can be 'expanded only in various integral-tangent directions at a point':

Proposition III.4 In the situation of the Definition III.3 let us denote (as in previous chapters) by $FJ.M.$ the structural vector bundle of the canonical contact structure on the manifold $J.M$. In other words, for a jet j at m the corresponding fiber F is the structural vector space for the contact manifold at $j.m$, which was defined as the inverse of the vector subspace T^{α} of T with direction j relative to the differential of the projection.

We define an **elementary structural prejet** expanding the given (localized) conic structure at an integral jet j as a direction in the contact-distribution fiber F at j such that the vector subspace F^{α_o} in that direction is relative to the bundle of Grasssmanians a fiber-transverse space modulo T^{ε}_{J} over T^{α}.

In other words, if we introduce notation $F^{\varepsilon} := T^{\varepsilon}_{J}$ and $F^{jx} := T_J$, elementary structural prejets are obviously in bijective correspondence with direct complements $F^{/\varepsilon\, \alpha_o}$ of $F^{/\varepsilon\, jx}$ in $F^{/\varepsilon}$, and thus form an affine space, which we denote $C^{/te}_{\pi} \subset Gr(x, F^{/\varepsilon}) \subset Gr(x+b, F)$, with associated vector space $Hom(F^{/jx}, F^{/\varepsilon\, \alpha_o})$, i.e. $F^{/\varepsilon\, \alpha_o} \otimes F^{*jx}$.

Let us observe that a **structural prejet** of this conic structure, which we define as a (holomorphic) field of elementary structural prejets, i.e. a field valued in the above described affine bundle) determines, and is determined by, a distribution $F^{\alpha_\circ}J^\varepsilon$ (in R along J^ε and contained in the restricted constituent distribution of the contact structure on $J.M$), such that each of its fibers $F^{\alpha_\circ}$ is relative to the bundle of Grasssmanians a fiber-transverse space modulo T^ε_J over T^α. $F^{\alpha_\circ}J^\varepsilon$ will be called the **tangent distribution of the structural prejet**.

Claim: Each structural (1-) jet induces a structural prejet by intersecting the constituent vector bundle $T^{\tau h}_{J.M}J^\varepsilon$ with the restricted integral-tangent vector bundle FJ^ε of the contact manifold.
[Indeed, the intersections of fibers of these vector subbundles of the restricted tangent bundle $T_{J.M}J^\varepsilon$ are clearly of constant rank.QED]

Remark III.5 When a conic structure is given on a manifold, the concept of tangentiality of an elementary preconnection at a given jet by its very definition depends only on the elementary structural prejet expanding the localized conic structure associated to the given conic structure. Furthermore, for every localized conic structure and elementary preconnection at an integral jet there obviously exists precisely one elementary structural prejet expanding the localized conic structure such that the given elementary preconnection is tangential to the full structural prejet.

Similarly, tangentiality of a localized preconnection depends only on the prejet at m of the given (expanded) conic structure. Furthermore, for every localized preconnection there obviously exists precisely one structural prejet expanding the underlying localized conic structure such that the given preconnection is tangential to the full structural prejet.

Remark III.6 If a given elementary conic-structural prejet expanding at an integral 1-jet j a given localizedconic structure can be realized by a local (i.e. defined on a a neighbourhood) expanded conic structure, then prolongability of the latter at j (which was defined as the existence of tangential 2-jets) according to the previous remark depends only on the given elementary structural prejet. For this reason we define **prolongability** of an elementary conic-structural prejet as the existence of tangential 2-jets.

III.1.3 Intrinsic Pretorsion and Prolongability

Proposition III.7 Let us consider a localized conic structure, where notation is as in III.3.

(i) We have the commutative diagram of affine bundles formed by the bundles in the upper two (slanted) rows of Figure III.7 and the associated diagram of vector bundles is indicated in the same figure using the convention from Remark III.1. The only bundle from the diagram which has not been introduced yet, namely $C_\pi^{/\!\!le\, sa_*} J^\varepsilon$ is the affine subbundle of $C_\pi^{/\!\!le} J^\varepsilon$ defined in the folowing way:
At an integral jet $j \in J^\varepsilon$ its fiber $C_\pi^{/\!\!le\, sa_*}$ is by definition the space of *prolongable* elementary structural prejets (cf. Proposition III.4).

The maps in this diagram are defined as follows: The middle vertical map is the surjection defined by associating with a given elementary preconnection the elementary structural prejet expanding the conic structure for which the elementary preconnection is tangential (cf. Proposition III.3). The second map in the nether row is the surjection defined by associating to an elementary preconnection its elementary pretorsion (in the way described in Remark II.10, i.e. by restricting the Frobenius tensor of the contact manifold). The leftmost vertical map and the second maping in the first row are defined to be the quotient maps relative to the two structures of a quotient affine bundle (on the codomain bundle) constructed by means of the second and third isomorphism theorem . The first map in either row of this diagram is defined to be the inclusion. The rightmost vertical map is defined by commutativity of the diagram . (In particular, it is claimed that the above spaces are in an obvious way affine with the indicated associated vector spaces.)

FigureIII.7

$(E_\pi^{/sa_* te} J^\varepsilon \risingdotseq \frac{T^\alpha \otimes T^{*/\alpha\otimes 2}}{\mathbf{G}^{cs} \otimes T^* + T^\alpha \otimes T^{*/\alpha\odot 2}} J^\varepsilon)$

$C_\pi^{/sa_* te}$

$\uparrow$ $\nwarrow$

$(E_\pi^{/sa} J^\varepsilon \risingdotseq T^\alpha \otimes T^{*/\alpha\wedge 2} J^\varepsilon)$ $(E_\pi^{/te} J^\varepsilon \risingdotseq \mathbf{G}^{/ds\,/cs} \otimes T^{*/\alpha} J^\varepsilon)$

$C_\pi^{/sa} J^\varepsilon$ $C_\pi^{/te} J^\varepsilon$

$\uparrow$ $\nwarrow$ $\uparrow$ $\nwarrow$

$(E_\pi^{/sa\, sa_* te} J^\varepsilon \risingdotseq \mathbf{G}_{\bullet[*}^{/ds\, cs} \otimes T_{*]}^{*/\alpha} J^\varepsilon)$ $(E_\pi J^\varepsilon \risingdotseq \mathbf{G}^{/ds} \otimes T^{*/\alpha} J^\varepsilon = T^\alpha \otimes T^{*/\alpha\otimes 2} J^\varepsilon)$ $(E_\pi^{/te\, sa_* te} J^\varepsilon)$

$C_\pi^{/sa\, sa_* te} J^\varepsilon = C_\pi^{te\,/sa_\bullet te} J^\varepsilon$ $C_\pi J^\varepsilon$ $C_\pi^{/te\, sa_* te} J^\varepsilon$

$\nwarrow$ $\uparrow$ $\nwarrow$ $\uparrow$

$(E_\pi^{te} J^\varepsilon \risingdotseq \mathbf{G}^{/ds\, cs} \otimes T^{*/\alpha} J^\varepsilon)$ $(E_\pi^{sa} \risingdotseq T^\alpha \otimes T^{*/\alpha\odot 2} J^\varepsilon)$

$C_\pi^{te} J^\varepsilon$ $C_\pi^{sa} J^\varepsilon$

$\nwarrow$ $\uparrow$

$(E_\pi^{sa_\bullet te} J^\varepsilon \risingdotseq \mathbf{G}_{\bullet(*}^{/ds\, cs} \otimes T_{*)}^{*/\alpha} J^\varepsilon)$

${}_\phi C_\pi^{sa_\bullet te} J^\varepsilon \risingdotseq C_\pi^{te} \cap C_\pi^{sa} J^\varepsilon$

(ii) On the level of vector bundles we can enlarge the diagram from (i) by adding the lowest row. Furthermore, in the enlarged diagram the middle row and the middle column are extensions of vector spaces. (Cf . Remark III.1.) Consequently, all rows and columns are extensions.

[Proof of (i) and (ii): At a given integral jet the affine space $C_\pi^{/te}$ of elementary structural prejets has already been shown to be a quotient affine space of the affine space C_π of elementary preconnections. Similarly, according to the Appendix, the vector space of elementary pretorsions is a quotient affine space of the affine space of elementary preconnections, where the distinguished fiber (i.e. the fiber over zero) is precisely the space of symmetric elementary preconnections, i.e. of second-order jets with the given associated tangent first-order jet. All the other statements follow immediately.QED]

(iii) Let us assume that a structural prejet expanding the given (localized) conic structure is given. Then the diagram from (i) can oviously be enlarged also on the affine level, i.e. we obtain the whole diagram in the Figure III.7. Here the fibers over a fixed integral jet j of the affine bundles are defined as follows:

C_π^{te} is the space of tangential elementary preconnections. (This is conceivably not a quotient affine space of the space of tangential connections; in order for that to be true, it is first of all necessary for the latter space to be non-empty; this will obviously be the case for a structural prejet realized by a holomorphic homogeneal expanded conic structure.)

$C_\pi^{sa\,te\bullet}$ is the space of tangential second-order jets in M (or tangential symmetric elementary preconnections). (As above, this is possibly not a quotient affine space of the space of tangential symmetric connections.)

$C_\pi^{/sa\,te\bullet}$ is the space consisting of the elementary pretorsions of tangential elementary preconnections, or, more briefly, **permissible elementary pretorsions.**

Claim: At the given integral jet j the vector from the space

$$C_\pi^{/sa_*te} \risingdotseq E_\pi^{/sa_*te} \risingdotseq \frac{T^{/\alpha} \otimes T^{*/\alpha\otimes 2}}{T_J^\varepsilon \otimes T^{*/\alpha} + T^{/\alpha} \otimes T^{*/\alpha\odot 2}} \qquad \text{(III.1)}$$

associated to the given elementary structural prejet by the map from the diagram is the precise obstruction to the prolongability of that elementary structural prejet (which was defined as the existence of tangential second-order jets), i.e. to the non-emptiness of the only possibly empty affine space in the diagram . For this reason vectors from the space III.1 will be called **intrinsic elementary pretorsions. The moral here is that the obstruction to the prolongability is encoded in the space of elementary pretorsions** (as the affine subspace formed by permissible elementary pretorsions, i.e. as the intrinsic elementary pretorsion which is the parameter of this subspace).

In conclusion, the field of intrinsic elementary pretorsions (i.e. field valued in the vector bundle in the leftmost uppermost position in the diagram) associated to the given structural prejet is (as an element of the vector space of fields) a precise obstruction to its prolongability, i.e. to the existence - meaning the non-emptiness of all fibers- of the affine bundle (*over* J^{ε}) in the rightmost nethermost position in the diagram (or, explicitly, to the existence of tangential second-order jets tangent with arbitrary integral jets through m). [Proof: The reasoning from the proof of the analogous statement in Remark III.1 obviously applies to this situation as well.QED]

Remark III.8 In order to simplify the terminology, we will refer to the fields valued in (i.e. sections of) affine and vector bundles from the diagram in Figure III.7 as **'pre-'** elements of corresponding spaces from the diagram in the Figure III.1. Clearly, we have already applied this convention to connections, torsions and structural jets, tangential connections, symmetric connections etc. Similarly, we define **permissible pretorsions** (as fields of permissible elementary pretorsions) and **intrinsic pretorsions**. Thus, we have obtained in (the previous) Proposition III.7 the **intrinsic pretorsion associated with a given structural prejet** as a precise obstruction to its prolongability, or, equivalently, to the fiberwise non-emptiness (meaning non-emptiness of all fibers) of the affine bundle of spaces tangential symmetric elementary preconnections. (This is the affine bundle in the rightmost nethermost position in the diagram from Proposition III.7.)

III.1.4 Conjunctively Intrinsic Pretorsion and Conjunctive Prolongability

Proposition III.9 Let us consider a localized conic structure, where notation is as in Proposition III.3.

(i) Our present objective is to study fields valued in bundles from the diagram in part (i) of Proposition III.7. If we consider spaces of fields valued in those bundles, we obtain the commutative diagram formed by the affine spaces (of fields) from the upper two (slanted) rows in Figure III.9 and the associated diagram of vector spaces (of fields) is indicated in the same figure using the convention from Remark III.1. (In particular, it is claimed that the above spaces are in an obvious way *non-empty* affine spaces with the indicated associated vector spaces.)

[Proof of (i): The only delicate point in the proof is the non-emptiness of these affine spaces of fields, i.e. vectoriality of the corresponding affine bundles. First let us observe that preconnections always exist, i.e. that the affine bundle of spaces of elementary preconnections is always a vectorial affine bundle: indeed, even the affine bundle of spaces of symmetric elementary preconnections (which is an affine subbundle of the former bundle) is always a vectorial affine bundle since any symmetric connection (not necessarily tangential) gives rise to a symmetric preconnection (i.e. a field of second-order jets). Furthermore, a symmetric preconnection gives rise to a prolongable structural prejet, which

FigureIII.9

$(H^0(E_\pi^{\prime sa_* te} J^\varepsilon))$					
$H^0(C_\pi^{\prime sa_* te} J^\varepsilon$					
$\uparrow$	$\nwarrow$				
$(H^0(E_\pi^{\prime sa} J^\varepsilon))$		$(H^0(E_\pi^{\prime te} J^\varepsilon))$			
$H^0(C_\pi^{\prime sa} J^\varepsilon)$		$H^0(C_\pi^{\prime te} J^\varepsilon)$			
$\uparrow$	$\nwarrow$	$\uparrow$	$\nwarrow$		
$(H^0(E_\pi^{\prime sa\, sa_* te} J^\varepsilon))$		$(H^0(E_\pi J^\varepsilon))$		$(H^0(E_\pi^{\prime te\, sa_* te} J^\varepsilon))$	
$H^0(C_\pi^{\prime sa\, sa_* te} J^\varepsilon)$		$H^0(C_\pi J^\varepsilon)$		$H^0(C_\pi^{\prime te\, sa_* te} J^\varepsilon)$	
	$\nwarrow$	$\uparrow$	$\nwarrow$	$\uparrow$	
		$(H^0(E_\pi^{te} J^\varepsilon))$		$(H^0(E_\pi^{sa} J^\varepsilon)))$	
		$H^0(C_\pi^{te} J^\varepsilon)$		$H^0(C_\pi^{sa} J^\varepsilon)$	
			$\nwarrow$	$\uparrow$	
				$(H^0(E_\pi^{sa_* te} J^\varepsilon))$	
				${}_\phi H^0(C_\pi^{sa_* te} J^\varepsilon)$	

(ii) On the level of vector spaces we can enlarge the diagram from (i) by adding the lowest row. However, in the enlarged diagram the middle row and the middle column are not necessarily extensions of vector spaces since the second maps are not necessarily surjective (their images are kernels of certain maps from long exact sequences associated to extensions of affine bundles). QED

(iii) Let a structural prejet expanding the given (localized) conic structure be given. Then an enlargement of the diagram from (i) is determined also on the affine level, i.e. one obtains the whole diagram in the Figure III.7 (cf. Proposition III.7). Let us observe that the structural prejet is an element of the affine space in the middle uppermost position in this diagram . In this situation we can actually claim the existence of a tangential preconnection (i.e. non-emptiness of the affine space in the middle nethermost position) under a mild additional assumption: clearly, it suffices that the structural prejet be induced by some (first-order) full structural prejet with trivial associated infinitesimal deformation (with parameter space M) of the integral-jet space J^{ε}; v. Proposition III.3 for definitions. (We have already seen in that proposition that this assumption holds as soon as either of the following conditions is fulfilled: rigidity of the integral manifold of jets or existence of localized connections tangential relative to the structural prejet.)

Let us observe that *the intrinsic pretorsion of the given structural prejet was constructed in Proposition III.7 by applying to the structural prejet the second map in the uppermost row of this diagram. Furthermore, the prolongability of the given structural prejet is obviously tantamount to the condition that the structural prejet actually be an element of the affine subspace in the rightmost uppermost position.* (On the other hand, the intrinsic pretorsion was seen in Proposition III.7 (iii) to be the precise obstruction to prolongability. This result could also be deduced from the fact we have just observed and the exactness in the uppermost row. Of course, this exactness resulted from a simple 'fiberwise' argument, and the proof of Proposition III.7 (iii) essentially exploited the same argument.)

If the space of permissible pretorsions (in the leftmost nethermost position) is non-empty (e.g. if the above mild assumption is made), *the intrinsic pretorsion can obviously be equivalently obtained by applying to this space the second map in the leftmost column. Consequently, the prolongability of the structural prejet is then equivalent to the equality of the space* $H^0(C_\pi^{/sa\, sa_* te} J\varepsilon)$ *(of permissible pretorsion vectors) and its vector space (of permissible pretorsion vectors) as affine subspaces of the vector space of pretorsions.* (This is also an essentially fiberwise argument: we have exploited the fact that prolongability means the equality of the affine bundle $C_\pi^{/sa\, sa_* te} J^\varepsilon$ (of spaces of permissible elementary pretorsions) and its vector bundle $E_\pi^{/sa\, sa_* te} J^\varepsilon \coloneqq T^\varepsilon_{J\bullet[*} \otimes T^{*/\alpha}_{*]} J^\varepsilon$ (of spaces of permissible elementary pretorsion vectors.) QED

(iv) Conjunctive prolongability of a structural prejet expanding the given conic structure is defined as the existence of a field of tangential symmetric elementary preconnections (i.e. as existence of tangential symmetric preconnections, i.e. as non-emptiness of the affine space in the rightmost nethermost position.) In other words, conjunctive prolongability of a structural prejet means that the bundle (in the rightmost nethermost position) of affine spaces of tangential symmetric elementary preconnections is a vectorial affine bundle on J^ε with non-empty fibers. (In particular, prolongability, i.e. non-emptiness of those fibers, is a weaker condition.) The conjunctively prolongable full structural prejets clearly form precisely the image of the second map in the rightmost column. In particular, they form an affine subspace of the space of prolongable structural prejets. Furthermore, the vector space of conjunctively prolongable structural predirectors is the image (i.e. quotient) of the vector space of symmetric preconnectors.

Claim: For full structural prejets admitting tangential preconnections (cf. (iii)) there is an obvius precise obstruction to conjunctive prolongability defined in terms of pretorsions: Let us define the vector space of **conjunctively intrinsic pretorsions** determined by the given (localized) conic structure as the quotient vector space

$$\frac{H^0(C_\pi^{/sa}J^\varepsilon)}{H^0(E_\pi^{te}J^\varepsilon)}(\risingdotseq \frac{H^0(E_\pi^{/sa}J^\varepsilon)}{H^0(E_\pi^{te}J^\varepsilon)} \risingdotseq \frac{H^0(T^\alpha \otimes T^{/*\alpha\wedge 2}\, J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^{*/\alpha}\, J^\varepsilon)}) \tag{III.2}$$

(of the vector space of pretorsions by the vector space of conjunctively permissible pretorsion vectors). The pretorsions of preconnections tangential relative to a given structural prejet will be said to be **conjunctively permissible** (relative to the structural prejet). Furthermore, for a full structural prejet admitting tangential pretorsions, the parameter of the affine space of conjunctively permissible pretorsions in the vector space (III.2) of conjunctively intrinsic pretorsions will be called the **conjunctively permissible** (relative to the given structural prejet) conjunctively intrinsic pretorsion, or simply the **conjunctively intrinsic pretorsion associated with the given structural prejet.**

Claim: For a structural prejet admitting tangential preconnections conjunctive prolongability is equivalent to vanishing of the associated conjunctively intrinsic pretorsion.

[Proof of this claim is a straightforward application of Proposition III.7. QED]

The remainder of this proposition will be devoted to a more precise interpretation of the conjunctively intrinsic pretorsion. Such considerations (in a somewhat more complex setting) will play an important role in the investigation of the first infinitesimal neighbourhood in the next section.

(v) (Here we do not assume that a fixed structural prejet is given.) Since the bundle

$$C_\pi^{/te\, sa_*}J^\varepsilon \text{ (= the quotient affine bundle in this context} \tag{III.3}$$

(of spaces of prolongable elementary structural prejets) is by its definition a quotient affine bundle of the bundle $C_\pi^{sa}J^\varepsilon$ (of spaces of symmetric elementary preconnections), we can form the corresponding long exact sequence of sheaf cohomology spaces. (Indeed, affine-bundle extensions, i.e. surjective base-preserving mappings of affine bundles, give rise to long exact sequences just like the vector-bundle extensions; of course, the long exact sequences thus obtained differ from the long exact sequences induced by the associated vector-bundle extensions only in 0-dimensional cohomology.) This sequence obviously begins with the sequence formed by the non-empty affine spaces in the rightmost column of the diagram. Since the above quotient affine bundle is according to (i) vectorial, we clearly obtain in the long exact sequence the connecting mapping of the (non-empty!) affine space in the rightmost uppermost position (i.e. the space of prolongable structural prejets) into (the affine space underlying) the vector space

$$H^1(E_\pi^{/sa_\bullet te} J^\varepsilon) \risingdotseq H^1(T^\varepsilon_{J\bullet_{(*}} \otimes T^{*/\alpha}_{*)} J^\varepsilon) \tag{III.4}$$

(of affine-bundle classes on the tangential symmetric elementary preconnector bundle). The explicit description of this map is obvious: it associates with a prolongable structural prejet the affine-bundle class of the associated affine bundle of spaces of tangential (relative to that structural prejet) symmetric elementary preconnections. We will refer to this map as the **connecting mapping of prolongable structural prejets** or the **connecting association of affine-bundle classes with prolongable structural prejets**. Let us observe that the image of this map is naturally identified with the quotient *affine* space

$$\frac{H^0(C_\pi^{/te\,sa_*} J^\varepsilon)}{H^0(E_\pi^{sa} J^\varepsilon)} \risingdotseq \frac{H^0(C_\pi^{/te\,sa_*} J^\varepsilon)}{H^0(T^{/\alpha} \otimes T^{*/\alpha\odot 2} J^\varepsilon)}$$

(of the affine space of prolongable structural prejets by the space of conjunctively prolongable structural predirectors). Note that this image (i.e. the affine space consisting of the canonical affine-bundle classes of prolongable full structural prejets) would not contain the zero affine-bundle class if the affine space $H^0(C_\pi^{sa} J^\varepsilon)$ (of symmetric preconnections) were empty. However, since $H^0(C_\pi^{sa} J^\varepsilon)$ is (according to (i)) non-empty, the above image coincides with its vector space as an affine subspace of the space (III.4) (of affine-bundle classes on the tangential symmetric elementary-preconnnector bundle). Clearly the zero affine-bundle class is (as a point of the quotient affine space) precisely the parameter of the space of conjunctively prolongable structural prejets. Furthermore, the linear map associated to the connecting mapping of prolongable structural prejets is clearly the first connecting map in the long exact sequence induced by the vector-bundle extension in the rightmost column. We will refer to it as the **connective mapping of prolongable structural predirectors**. According to the above consideration its image coincides with the image of the connecting affine map. Since the vector-bundle extension is clearly independent of the full structural prejet expanding the given localized conic structure, the same applies to the connective mapping.

With the above terminology we can reformulate the assertion (iv) in the following way: For any structural prejet the following properties are mutually eqivalent:

(a) The structural prejet is conjunctively prolongable;

(b) The conjunctively intrinsic pretorsion of the structural prejet (i.e. the conjunctively permissible one) vanishes;

(c) Both succesively defined obstructions to conjunctive prolongability, namely the intrinsic pretorsion and the canonical affine-bundle class of the structural prejet (i.e. the class of the tangential symmetric elementary-preconnnection bundle), vanish.

In this situation the vectors of tangential symmetric preconnections are according to the diagram (in Figure III.9) precisely the tangential symmetric preconnectors. Let us observe that the vector space of these fields does not depend on the structural prejet, but only on the (localized) conic structure.

(vi) Our next objective is to interpret the (above defined) connecting mapping of prolongable structural prejets in terms of pretorsion. (More precisely, the objective is to find a method which would enable one to reconstruct the affine-bundle class of a prolongable structural prejet from the localized conic structure and some set of pretorsions distinguished by the full structural prejet.)

Suppose an arbitrary structural prejet is given (i.e. we again consider the situation from (iii)).

(vi.1) Let us apply the argument from (v) to the affine bundle $C^{te}_{\pi}J^{\varepsilon}$ (of spaces of tangential elementary preconnections) and its quotient affine bundle $A^{pi}_{\pi}J^{\varepsilon}$:

$\coloneqq C_\pi^{/sa\,sa_*te} J^\varepsilon$ (of spaces of permissible elementary pretorsions); thus we obtain the long exact sequence which begins with the sequence formed by non-empty affine spaces in the nethermost row. The resulting connecting map, i.e. the mapping of the (possibly empty) affine space $H^0(C_\pi^{/sa\,sa_*te} J^\varepsilon)$ of **permissible pretorsions** into the vector space $H^1(E_\pi^{/sa_\bullet te} J^\varepsilon) \coloneqq H^1(T^\varepsilon_{J_{\bullet(*}} \otimes T^{*/\alpha}_{*)} J^\varepsilon)$ (of affine-bundle classes on the tangential symmetric elementary-preconnnector bundle) will be referred to as the **connecting mapping of permissible pretorsions** or the **connecting association of affine-bundle classes with permissible torsions**. According to the general properties of affine-bundle extensions (already applied in (v)), this map descends to an affine injection of the quotient affine space

$$\frac{H^0(C_\pi^{/sa\,sa_*te} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \coloneqq \frac{H^0(C_\pi^{/sa\,sa_*te} J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^{*/\alpha} J^\varepsilon)} \tag{III.5}$$

(of the affine space of permissible pretorsions by the vector space of conjunctively permissible pretorsion vectors). This quotient affine space is, on the other hand, clearly also injectively mapped into the vector space of conjunctively intrinsic pretorsions (defined in (iv)). The latter affine injection will be thought of as inclusion, and its domain (i.e. the quotient (III.5) will in accordance with that be called the space of **permisssible** *conjunctively intrinsic pretorsions.* Accordingly, the former injection will be referred to as the **canonical (affine) injection of permissible conjunctively intrinsic pretorsions** or **canonical injective association of affine-bundle classes with permissible conjunctively intrinsic pretorsions**. **(vi.2)** The linear map associated to the affine connecting map from (vi.1), i.e. the connective map from the long exact sequence for the associated quotient vector space clearly has analogous properties, and we introduce analogous terminology in the obvious way. (Explicitly, the connective map descends to a linear injection of the space of **permissible** (relative to the given localized conic structure) conjunctively intrinsic pretorsion vectors

$$\frac{H^0(E_\pi^{/sa\,sa_*te} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \coloneqq \frac{H^0(T^\varepsilon_{J_{\bullet[*}} \otimes T^{*/\alpha}_{*]} J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^{*/\alpha} J^\varepsilon)} \tag{III.6}$$

into the vector space $H^1(E_\pi^{/sa\bullet te} J^\varepsilon) \coloneqq H^1(T^\varepsilon_{J\bullet(*} \otimes T^{*/\alpha}_{*)} J^\varepsilon)$ of affine-bundle classes on the tangential symmetric elementary-preconnnector bundle). Let us observe that this 'vectorial component' of the construction in (vi.1) is independent of the full structural prejet expanding the given localized conic structure (i.e. completely determined by the localized conic structure alone).

(vi.3) Since the domain of the affine injection from (vi.1) (of permissible conjunctively intrinsic pretorsions) is an affine subspace of a vector space (namely of the space of conjunctively intrinsic pretorsions), and the codomain is a vector space (of affine-bundle classes), a special caution is needed when we identify its domain with its image by means of the affine injection. More precisely, if we do so, it is often impossible to do the same for the associated linear injection in a consistent way; in other words, if either the domain and its vector space (which are parallel affine subspaces of their ambient vector space) coincide, or the image and its vector space (which are in a similar relation) coincide, it could be impossible to carry out both identifications simultaneously. However, it is desirable to carry out at least one of them in order to simplify terminology and notation. Since the linear injection is more canonical (in the sense made precise in (vi.2)), we will think of it as an inclusion (i.e. carry out the identification of the vector subspaces by means of the linear injection). In other words, under this convention, the affine-bundle class (linearly) injectively associated to a permissible conjunctively intrinsic pretorsion vector coincides with it, but the analogous statement is conceivably not true of the affine-bundle class (affinely) injectively associated to a permissible conjunctively intrinsic pretorsion.

(vi.4) In view of (iii), the domain of the affine injection from (vi.1) and its vector space clearly coincide (as affine subspaces of the vector space of conjunctively intrinsic pretorsions) iff the structural prejet is *prolongable.* In fact, the space of permissible conjunctively intrinsic pretorsions is clearly the preimage of the intrinsic pretorsion of the structural prejet in the vector space of conjunctively intrinsic pretorsions, and on the vector level the space of permissible conjunctively intrinsic pretorsion vectors is precisely the space of intrinsic-pretorsion-free conjunctively intrinsic pretorsions (by which we mean the preimage of the zero intrinsic pretorsion in the vector space of conjunctively intrinsic pretorsions).

Claim: If we asssume prolongability of the full structural prejet, the canonical affine-bundle class of the prolongable structural prejet (which was defined in (v)) coincides with the canonical affine-bundle class of the zero permisssible intrinsic pretorsion *[i.e. it is equal to the image of the zero conjunctively intrinsic pretorsion via the* (affine) *canonical injection of permisssible conjunctively intrinsic pretorsions (not pretorsion vectors)].* (This image is well-defined since the zero intrinsic pretorsion is due to prolongability not only a permissible conjunctively intrinsic pretorsion vector, but also a permisssible conjunctively intrinsic pretorsion. Notice that the assertion (vi.4) does not yet accomplish the goal of (vi).)

[Proof of (vi.4): The canonical affine-bundle class of the prolongable structural prejet is by definition represented by the tangential symmetric elementary-preconnnection bundle (which has non-empty fibers due to prolongability). It remains to recall that for a tangential elementary preconnector symmetry is equivalent to vanishing of elementary pretorsion, and to apply the definition of the connecting mapping of permissible pretorsions. QED]

(vi.5) From the general properties of the affine-bundle extensions (already exploited in (v)) we infer the equivalence of the following two conditions:

(a) Existence of tangential (relative to the given structural prejet) preconnections, i.e. vectoriality of the tangential-elementary preconnnection bundle, i.e. existence of conjunctively permissible pretorsions;

(b) Equality of the image of the affine injection of permissible conjunctively intrinsic pretorsions and its vector space (as affine subspaces of the vector space $H^1(E_\pi^{/sa\bullet te} J^\varepsilon) \coloneqq H^1(T^\varepsilon_{J\bullet_{(*}} \otimes T^{*/\alpha}_{*)} J^\varepsilon)$ of affine-bundle classes on the tangential symmetric elementary-preconnnector bundle).

Furthermore, if these conditions are fulfilled, the conjunctively intrinsic pretorsion (introduced in (iv) as the parameter of the affine space of conjunctively permissible pretorsions in the vector space (III.2) of conjunctively intrinsic pretorsions) is a well-defined *permissible* conjunctively intrinsic pretorsion (in fact clearly the only one) mapped into zero by the affine injection of permissible conjunctively intrinsic pretorsions.

(vi.6) Let us now assume both prolongability and existence of tangential preconnections (in other words, we consider a full structural prejet for which the conditions of (vi.5) and (vi.6) are simultaneously fulfilled).

Thus the affine space (**??**) (of permissible conjunctively intrinsic pretorsions) contains two distinguished points, namely the conjunctively intrinsic pretorsion of the full structural prejet and the zero conjunctively intrinsic pretorsion. In order to achieve the objective of (vi), it suffices to apply (vi.5) and (vi.6): indeed, according to these assertions the affine injection of permissible conjuntively intrinsic pretorsions maps these distinguished points into respectively zero affine-bundle class and the canonical affine-bundle class of the prolongable structural prejet. Therefore the associated linear injection, i.e. the linear injection of permissible conjunctively intrinsic pretorsion vectors (which depends only on the localized conic structure!), maps the vector joining these points into the vector joining their images. In other words, it maps the conjunctively intrinsic pretorsion of the structural prejet into the negative of the canonical affine-bundle class of the structural prejet. Under the convention from (vi.4) we obtain the following conclusion:

The canonical affine-bundle class of the prolongable structural prejet is precisely the negative of the conjunctively intrinsic pretorsion of the structural prejet.

In summary, the objective of (vi) has been accomplished for the prolongable structural prejets satisfying the above mild additional condition (namely the existence of tangential preconnections).

(vii) According to (vi.1) the vector space $H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon) \risingdotseq H^0(T^\varepsilon_{J\bullet[*} \otimes T^{*/\alpha}_{*]} J^\varepsilon)$ (of permissible pretorsion vectors) has a canonical (determined by the localized conic structure alone) structure of an extension, namely the one constructed in (vi), where the quotient space is the space $\frac{H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \risingdotseq \frac{H^0(T^\varepsilon_{J\bullet[*} \otimes T^{*/\alpha}_{*]} J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^{*/\alpha} J^\varepsilon)}$ of permissible conjunctively intrinsic pretorsion vectors, and the subspace is obviously a quotient of the space $H^0(E_\pi^{te} J^\varepsilon) \risingdotseq H^0(T^\varepsilon_J \otimes T^{*/\alpha} J^\varepsilon)$

(of tangential preconnectors), namely the space

$$\frac{H^0(E_\pi^{te} J^\varepsilon)}{H^0(E_\pi^{te_\bullet sa} J^\varepsilon)} = \frac{H^0(T_J^\varepsilon \otimes T^{*/\alpha} J^\varepsilon)}{H^0(T_{J_{\bullet(*}}^\varepsilon \otimes T_{*)}^{*/\alpha} J^\varepsilon)}$$

of conjunctively permissible pretorsion vectors. We will record this fact briefly in the following way:

$$H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon) = \frac{H^0(E_\pi^{te} J^\varepsilon)}{H^0(E_\pi^{te_\bullet sa} J^\varepsilon)} \dot{+} \frac{H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \tag{III.7}$$

or, more concretely,

$$H^0(T_{J_\bullet[*}^\varepsilon \otimes T_{*]}^{*/\alpha} J^\varepsilon) = \frac{H^0(T_J^\varepsilon \otimes T^{*/\alpha} J^\varepsilon)}{H^0(T_{J_{\bullet(*}}^\varepsilon \otimes T_{*)}^{*/\alpha} J^\varepsilon)} \dot{+} \frac{H^0(T_{J_\bullet[*}^\varepsilon \otimes T_{*]}^{*/\alpha} J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^{*/\alpha} J^\varepsilon)}. \tag{III.8}$$

In other words, the right-hand 'extensional summand' will be understood to be a quotient space, not a subspace (at least not in an invariant way relative to the localized conic structure, i.e. not G^∞ -invariantly). However, for specific conic structures it will often be be possible to realize this quotient invariantly as a direct complement to the subspace.

QED

Proposition III.10 We say that a conic structure on an arbitrary vector space (e.g. a localized conic structure on a manifold) is **of pretype one** if it is 'free of tangential symmetric preconnectors' in the sense that the associated vector space of tangential symmetric preconnectors is the zero-space.

Suppose a localized conic structure of pretype one is given. Let us observe that for any structural prejet expanding this conic structure (e.g. a structural prejet induced by any full structural prejet expanding this conic structure) the quotient mapping of the affine space of tangential preconnections onto the affine space of permissible pretorsions is bijective. (This follows immediately from Proposition III.9 (vi).) The same reasoning shows that there can exist at most one tangential symmetric preconnection on the given structural prejet. Of course, its existence is even without the above assumption on pretype equivalent to vanishing of the conjunctively intrinsic pretorsion (i.e. to vanishing of the intrinsic pretorsion and the affine-bundle class of the tangential symetric elementary preconnection bundle).

The global version of this result (for conic structures on manifolds or for germs thereof) has the following straightforward consequence: a global symmetric preconnection (which is a second-order geometric structure), e.g. an integrable expanded preconnection, such that all the induced localized conic structures are of pretype one, can be reconstructed from the underlying expanded conic structure (which is a first-order geometric structure). More precisely, on a given expanded conic structure with that property there can exist at most one tangential symmetric (expanded) preconnection. (Of course, in order to reconstruct the preconnection pointwise, it does not suffice to consider localized conic structures: we had to make an essential use of their structural prejets, and these are 2-structures).

III.2 Complementally Intrinsic Pretorsion and 1-Fattennings

We have seen in the previous section that the prolongability properties of a structural prejet expanding a (localized) conic structure are naturally expressed in terms of two first-order invariants, namely the intrinsic pretorsion and the somewhat finer conjunctively intrinsic pretorsion, which were both defined in terms of pretorsions. In this section we will construct subtler invariants of a (first-order) full structural prejet expanding a conic structure which are still expressible in terms of pretorsions. More concretely, our present objective includes e.g. a translation of the properties of the first-order infinitesimal neighbourhood of an embedded manifold into localized differential-geometric invariants of a parameter space of 'nearby' submanifolds.

The starting point in this section will be an investigation of the space of pretorsions from the algebraic viewpoint. (Since every torsion gives rise to a pretorsion, this space will usually be related to the space of torsions, but we will postpone the pertinent investigation until a later section devoted to the applications to 'homogeneal' structural prejets). In fact, since we already have a geometric interpretation of the above mentioned invariants and are ultimately interested in invariants of integrable preconnections, we will mostly concentrate our attention on the subspace in the extension (III.7), namely the space (associated to a given localized conic structure)

$$\frac{H^0(E^{te}_{\pi} J^{\varepsilon})}{H^0(E^{te_{\bullet}sa}_{\pi} J^{\varepsilon})} \coloneqq \frac{H^0(T^{\varepsilon}_{J} \otimes T^{*/\alpha}\, J^{\varepsilon})}{H^0(T^{\varepsilon}_{J_{\bullet(*}} \otimes T^{*/\alpha}_{*)}\, J^{\varepsilon})}$$

of conjunctively permissible pretorsion vectors (or pretorsion vectors free of conjunctively intrinsic pretorsion vector). Our immediate objective is to give an as explicit as possible straightforward formal description of the above space (which would enable one to 'compute' it for concrete conic structures). By interpreting geometrically the various entries in this description, we will later identify an invariant of conjunctively prolongable conic structures (more precisely of their first-order full structural prejets) related to tangential (i.e. structure-preserving) connections with an additional condition of compatibility. (Just like prolongability properties from the previous section, this condition on tangential connections does not make sense for more general geometric structures, i.e. it is peculiar to the theory of conic structures).

Since the above space (of conjunctively permissible pretorsion vectors) is defined as a quotient of the space $H^0(E^{te}_{\pi} J^{\varepsilon}) \coloneqq H^0(T^{\varepsilon}_{J} \otimes T^{*/\alpha}\, J^{\varepsilon})$ of tangential preconnectors, we will actually first investigate the latter space. The most natural way to obtain its explicit description is the following:

By definition tangential preconnectors are fields valued in the bundle $E^{te}_{\pi} J^{\varepsilon} :$ $= T^{\varepsilon}_{J} \otimes T^{*/\alpha}\, J^{\varepsilon}$ of vector spaces of tangential elementary preconnectors. In order to obtain information on the space of these fields, it is natural to look for a vector-bundle extension such that the associated long exact sequence includes this vector space of fields. We have already expressed the above bundle as a quotient vector bundle, namely as $E^{te\,/np}\, J^{\varepsilon}$. However, this bundle has additional two obvious simpler and more useful structures of a quotient bundle since it is by definition the tensor product of two quotient bundles:

Indeed, the first tensor factor $T^{\varepsilon}_{J} J^{\varepsilon}$ can be thought of as the quotient $\mathbf{G}^{/ds\,\varepsilon}\, J^{\varepsilon} = \mathbf{G}^{\varepsilon\,/ds}\, J^{\varepsilon} = \frac{\mathbf{G}^{\varepsilon}}{\mathbf{G}^{ds}} J^{\varepsilon}$, where $\mathbf{G}^{ds}$ denotes the stabilizer of the jet j relative to the action on the manifold J of the Lie algebra $\mathbf{G} \coloneqq T \otimes T^{*}$ of the general linear group G of T. (In other words, $\mathbf{G}^{ds}$ is defined as the Lie algebra of the automorphism group G^{ds} of the 'directional structure' j on the vector space T, and $\mathbf{G}^{/ds}$ is consequently the tangent space T_J of the space J of 'directional structures'.)

Similarly, the second tensor factor is the quotient $T^{*/\alpha}J^\varepsilon = \frac{T^*}{T^{*\alpha}}J^\varepsilon$.

Of course, in our investigation of the space of tangential preconnectors, we will use the simpler of the above mentioned structures of the quotient bundle, namely the one arising from the quotient structure on the second tensor factor. In this connection, we first introduce some suitable terminology:

The vector space of **full elementary preconnectors** at a given integral jet j is defined as the quotient vector space $E_{\pi_1} := T_J \otimes T^* = \mathbf{G}^{/ds} \otimes T^*$ of the space E of connectors. According to a convention introduced before, fields of full elementary preconnectors (defined on the integral-jet space of a localized conic structure) will be called **full preconnectors**. (Roughly speaking, these concepts differ from the corresponding concepts without the term *full* in the sense that at a given jet we now consider a **larger** -meaning 'finer', i.e. with smaller equivalence classes- quotient space than before.) Furthermore, the space of **tangential** full elementary preconnectors is by definition the subspace $E^{te}_{\pi_1} := T^\varepsilon_J \otimes T^* = \mathbf{G}^{/ds\,\varepsilon} \otimes T^*$ of E_{π_1}. On the other hand, an elementary-preconnector-free full elementary preconnector, i.e. a full elementary preconnector belonging to the vector space $E^{np}_{\pi_1} := T_J \otimes T^{*\alpha} = \mathbf{G}^{/ds} \otimes T^{*al}$, will be called **complementary**. (The motivation for this terminology will be given in the next proposition.) Similarly, a full preconnector is said to be **tangential** if it consists of tangential full elementary preconnectors and **complementary** if it consists of complementary full elementary preconnectors.

The above mentioned simple quotient structure on the bundle $E^{te}_\pi J^\varepsilon$ of vector spaces of tangential elementary preconnectors is now constructed as follows: Since at a given integral jet j obviously

$$E_\pi = E^{np}_{\pi_1} \ \text{ and } \ E^{te}_\pi = E^{np\,te_*np}_{\pi_1} \ ,$$

i.e. the space $E^{te}_\pi := T^\varepsilon_J \otimes T^{*/\alpha}$ of tangential elementary preconnectors is precisely the image of the space $E^{te}_{\pi_1} := T^\varepsilon_J \otimes T^*$ of tangential full elementary preconnectors, the above bundle is by the second isomorphism theorem obviously the quotient

$$E^{te\,/te_{\bullet}np}_{\pi_1} := \frac{E^{te}_{\pi_1}}{E^{te_{\bullet}np}_{\pi_1}} J^{\varepsilon} := \frac{T^{\varepsilon}_{J} \otimes T^{*}}{T^{\varepsilon}_{J} \otimes T^{*\alpha}} J^{\varepsilon}$$

of the tangential full-elementary preconnnector bundle, by the bundle $E^{te_{\bullet}np}_{\pi_1} J^{\varepsilon} := T^{\varepsilon}_{J} \otimes T^{*\alpha} J^{\varepsilon}$ of the vector spaces of complementary tangential full elementary preconnectors.

Lemma III.11 In conclusion, for a given localized conic structure we have obtained the above described vector-bundle extension. From the long exact sequence of sheaf cohomology spaces associated to that vector-bundle extension we conclude that the vector space of tangential preconnectors is itself canonically an extension of this form:

$$H^0(E^{te}_{\pi} J^{\varepsilon}) = \frac{H^0(E^{te}_{\pi_1} J^{\varepsilon})}{H^0(E^{te_{\bullet}np}_{\pi_1} J^{\varepsilon})} \dot{+} \frac{H^0(E^{te}_{\pi} J^{\varepsilon})}{H^0(E^{te}_{\pi_1} J^{\varepsilon})} \tag{III.9}$$

or, more concretely,

$$H^0(T^{\varepsilon}_{J} \otimes T^{*/\alpha}\, J^{\varepsilon}) = \frac{H^0(T^{\varepsilon}_{J} \otimes T^{*}\, J^{\varepsilon})}{H^0(T^{\varepsilon}_{J} \otimes T^{*\alpha}\, J^{\varepsilon})} \dot{+} \frac{H^0(T^{\varepsilon}_{J} \otimes T^{*/\alpha}\, J^{\varepsilon})}{H^0(T^{\varepsilon}_{J} \otimes T^{*}\, J^{\varepsilon})}. \tag{III.10}$$

(Here we again use the notation for extensions introduced in connection with (III.7) from the last section.) Let us observe that the quotient space in this extension is by definition a vector subspace of $H^1(E^{te_{\bullet}np}_{\pi_1} J^{\varepsilon})$, and thus consists of the **canonical affine-bundle classes of tangential preconnectors**, which we define as the affine-bundle classes (for which we have not yet found a geometric interpretation in terms of conic structures or preconnections) 'connectingly associated' to tangential preconnectors. The subspace in this extension obviously consists of **complementable** tangential preconnectors, which we define as those tangential preconnectors which are associated to tangential full preconnectors. (The motivation for these terms will also be given in the next proposition.)

Furthermore, the description of the space of permissible pretorsion vectors given in the equality (III.7) from the last section, assumes in view of (III.9) the following more explicit form:

$$H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon) = \frac{\frac{H^0(E_{\pi_1}^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te_\bullet np} J^\varepsilon)} + \frac{H^0(E_\pi^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te} J^\varepsilon)}}{H^0(E_\pi^{te_\bullet sa} J^\varepsilon)} + \frac{H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \qquad \text{(III.11)}$$

or, more concretely,

$$H^0(T^\varepsilon_{J\bullet[*} \otimes T^{*/\alpha}_{*]} J^\varepsilon) = \frac{\frac{H^0(T^\varepsilon_J \otimes T^* J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^{*\alpha} J^\varepsilon)} + \frac{H^0(T^\varepsilon_J \otimes T^{*/\alpha} J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^* J^\varepsilon)}}{H^0(T^\varepsilon_{J\bullet(*} \otimes T^{*/\alpha}_{*)} J^\varepsilon)} + \frac{H^0(T^\varepsilon_{J\bullet[*} \otimes T^{*/\alpha}_{*]} J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^{*/\alpha} J^\varepsilon)}. \qquad \text{(III.12)}$$

(Let us observe that due to the above mentioned convention on the meanings of the two summands in a sum, this equality involves two G^{∞} -invariant extensions recorded as sums, where the two quotient vector spaces are embedded into certain first Cech cohomology spaces.) In particular, the space of conjunctively permissible pretorsion vectors has been (rather formally) 'computed' to be the first summand on the right hand side of (III.9). For concrete conic structures the above 'computation' often gives an explicit identification of the isomorphy classes of the simple (or irreducible) components of this G^{∞} -module. However, this would not by itself be particularly useful without the geometric interpretation of the various entries expounded in the following propositions. QED

Proposition III.12 Let us consider a full structural prejet expanding a (localized) conic structure at the point m of a manifold M (v. Proposition III.3; the concepts introduced in this Proposition do not make sense if only a structural prejet is given). Then there exists a naturally defined affine bundle for which the associated vector bundle is the (above defined) bundle $E_{\pi_1}^{te} J^\varepsilon = T^\varepsilon_J \otimes T^* J^\varepsilon$ of vector spaces of tangential full elementary preconnectors; indeed, such an affine bundle is obviously the affine bundle of spaces of **tangential full elementary preconnections**, which is defined in the following way:

First we introduce the bundle $C_{\pi_1} J^\varepsilon := J_{JM} J^\varepsilon$ of the affine spaces of **full elementary preconnections**; in other words, such an affine space C_{π_1} at an integral jet j is defined simply as the quotient affine space of the space C (of localized connections) formed by distributional (1-) jets expanding that integral jet (v. Remark III.1), i.e. by the fiber-transverse jets at that point in the bundle JM of Grassmanians; the associated vector space is according to the same remark the space $E_{\pi_1} J^\varepsilon \coloneqq T_J \otimes T^* = \mathbf{G}^{/ds} \otimes T^*$ of full elementary preconnectors. A full elementary preconnection c_{π_1} at an integral jet j is called **tangential** (relative to the given full structural prejet) if the fiber-transverse subspace of T_{JM} in direction c_{π_1} is contained in the subspace determined by the full structural prejet (cf. Proposition III.3). Spaces of tangential elementary preconnections at various integral jets j clearly form an affine subbundle of $C_{\pi_1} J^\varepsilon$ which we denote by $C^{te}_{\pi_1} J^\varepsilon$. The associated vector bundle is clearly the bundle $E^{te}_{\pi_1} J^\varepsilon \coloneqq T^\varepsilon_J \otimes T^* J^\varepsilon$ of spaces of tangential full elementary preconnectors (which in view of that was appropriately termed).

(i) Let us associate with a full elementary preconnection c_{π_1} at the jet j an elementary preconnection c_π 'by intersecting' with the structural space F of the contact structure of JM, more precisely by means of the requirement that the vector subspaces T^{mj}_{JM} and F^τ of T_{JM} in directions respectively c_{π_1} and c_π satisfy the equality

$$F^\tau = T^{mj}_{JM} \cap F\,.$$

In this way we clearly obtain a well-defined surjective map over the base

$$C_{\pi_1} J^\varepsilon \to C_\pi J^\varepsilon \tag{III.13}$$

of affine bundles with non-empty fibers; what is more, this map obviously preserves tangentiality in the sense that it restricts to a map

$$C^{te}_{\pi_1} J^\varepsilon \to C^{te}_\pi J^\varepsilon \tag{III.14}$$

of affine subbundles. Clearly, for this restriction the associated mapping of vector budles is precisely the surjective map defined in (previous) Proposition III.11. Furthermore, surjectivity on the vector level clearly implies surjectivity on the affine level, i.e. $C^{te}_\pi J^\varepsilon$ is (relative to (III.14) a quotient affine bundle of $C^{te}_{\pi_1} J^\varepsilon$, or, in other words, (III.14) is an **affine-bundle extension**. Therefore we actually obtain the associated long exact sequence the (which has already been constructed on the vector level):

$$H^0(C^{te}_{\pi_1}J^\varepsilon) \to H^0(C^{te}_{\pi}J^\varepsilon) \to$$

$$\to H^1(E^{te\bullet np}_{\pi_1}J^\varepsilon) \to H^1(E^{te}_{\pi_1}J^\varepsilon) \to H^0(E^{te}_{\pi}J^\varepsilon) \to \; ...$$

(Note that the two affine spaces in that sequence are possibly empty.) Q.E.D.

(ii) Let us now suppose that tangential (relative to the given full structural prejet) preconnections exist, i.e. that the affine space in the leftmost nethermost position of the long exact sequence from (i) is non-empty. (This is a reasonably mild condition, as we have already seen in Proposition III.9: indeed, it is weaker than vanishing of the associated infinitesimal deformation with parameter space M of the integral-jet space; the latter condition in turn is weaker than either rigidity of the integral manifold of jets or the existence of tangential localized connections.) The **boundary (affine) mapping of tangential preconnections** or the boundary (affine) association of affine-bundle classes with tangential preconnections is defined (cf. Proposition III.9) as the appropriate connecting map from the above long exact sequence. (A more explicit description of this map will be given soon.) We wil identify its image, which is an affine subspace of the vector space $H^1(E^{te\bullet np}_{\pi_1}J^\varepsilon) \coloneqq H^1(T^\varepsilon_J \otimes T^{*\alpha}J^\varepsilon$ (of affine-bundle classes on the tangential complementary full-elementary preconnnector bundle) with the quotient affine space

$$\frac{H^0(C^{te}_{\pi}J^\varepsilon)}{H^0(E^{te}_{\pi_1}J^\varepsilon)} \coloneqq \frac{H^0(C^{te}_{\pi}J^\varepsilon)}{H^0(T^\varepsilon_J \otimes T^*J^\varepsilon)}$$

(of the affine space of tangential preconnections by the vector space of complementable tangential preconnectors) by means of the canonical isomorphism. The vector space of this affine subspace is the subspace determined by the localized conic structure (i.e. independent of its full structural prejet) since it obviously consists of the canonical affine-bundle classes of tangential *preconnectors* (according to terminology introduced in Proposition III.11). The terminology from Proposition III.11 is obviously compatible with the one just introduced in the following sense: the vector space of the affine space of the canonical affine-bundle classes of tangential preconnections is precisely the vector space of the canonical affine-bundle classes of tangential preconnectors. In particular, this vector space is also independent of the full structural prejet expanding the given localized conic structure. It is also obvious from the exact sequence that the distinguishing property of the affine-bundle classes from $H^1(E_{\pi_1}^{te_\bullet np} J^\varepsilon)$ which are canonically associated to tangential preconnections is the following: they induce (by means of the approriate map from the exact sequence) the infinitesimal deformation with parameter space M of the integral-jet space J^ε (i.e. the affine-bundle class from $H^1(E_{\pi_1}^{te} J^\varepsilon)$) associated to the given full structural prejet (v. Proposition III.3 for definitions). QED

(iii) Let us now suppose that tangential (relative to the given full structural prejet) *full* preconnections exist, i.e. that both affine spaces in the long exact sequence from (i) are non-empty. Although this is a slightly stronger condition on the full structural prejet than the condition from (ii), it is still reasonably mild. Indeed, as we have seen in Proposition III.3, it is *equivalent* to vanishing of the associated infinitesimal deformation (with parameter space M) of the integral-jet space J^ε, and thus still weaker than either rigidity of the integral manifold of jets or the existence of tangential localized connections. In particular, the affine space

$$\frac{H^0(C_\pi^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te} J^\varepsilon)}$$

(consisting of canonical affine-bundle classes of tangential preconnections) is now a **vector** subspace of $H^1(E_{te_\bullet np} J^\varepsilon)$. (cf. the reasoning in Proposition III.9).

In other words, that quotient affine space has been organized into a vector space as the vector quotient vector space

$$\frac{H^0(C_\pi^{te} J^\varepsilon)}{H^0(C_{\pi_1}^{te} J^\varepsilon)} \approx \frac{H^0(E_\pi^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te} J^\varepsilon)} \risingdotseq \frac{H^0(T_J^\varepsilon \otimes T^{*/\alpha} J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^* J^\varepsilon)}$$

of the affine space $H^0(C_\pi^{te} J^\varepsilon)$ of tangential preconnections by the affine subspace consisting of **complementable** tangential preconnections, which we define as the image

$$\frac{H^0(C_{\pi_1}^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te_\bullet np} J^\varepsilon)} \risingdotseq \frac{H^0(C_{\pi_1}^{te} J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^{*\alpha} J^\varepsilon)} \qquad \text{(III.15)}$$

of the affine space $H^0(C^{te}_{\pi_1}J^\varepsilon)$ (of tangential full preconnections).

(In other words, the role of the zero vector in the vector quotient of the affine space $H^0(C^{te}_{\pi}J^\varepsilon)$ of tangential preconnections is by definition played by the affine subspace consisting of the complementally permissible tangential preconnections. This quotient vector space is by its very definition canonically isomorphic -as indicated above- to the vector space of the quotient affine space, namely the quotient vector space of vector spaces, or the space of the canonical affine-bundle classes of *preconnectors*. Furthermore, this isomorphism is due to our conventions precisely the identity.)

In summary, the image of the connecting mapping of preconnections coincides with the image of the connecting mapping of preconnectors, and is therefore independent on the full structural prejet expanding the localized conic structure. QED

(iv) In order to describe explicitly the connecting mapping of tangential preconnections, we first analyse the map (III.13) on the fiber level. Let us consider an arbitrary tangential (relative to the given full structural prejet) elementary preconnection c_π at some integral jet j. The preimage of this elementary preconnection in the affine space C_{π_1} of full elementary preconnections at j (relative to (III.13) will be called the space of **complementary** (relative to c_π) full elementary preconnections and denoted by $C^{np}_{\pi_1}$. This is in view of (i) clearly a (non-empty) affine subspace of C_{π_1} with the associated vector space $E^{np}_{\pi_1} \coloneqq T_J \otimes T^{*\alpha} J^\varepsilon$ (consisting of complementary full elementary preconnectors).

Similarly, the space of complementary *tangential* full elementary preconnections is denoted by $C^{te\bullet np}_{\pi_1}$ since this, being a preimage relative to (III.14) from (i), is clearly either an affine subspace of $C^{te}_{\pi_1}$ with the associated vector space $E^{te\bullet np}_{\pi_1} \coloneqq T^\varepsilon_J \otimes T^{*\alpha}$ (consisting of complementary tangential full elementary preconnectors) or empty, depending on whether the elementary preconnection c_π is tangential (relative to the given full structural prejet) or not. QED

(v) Before we give the explicit description of the connecting mapping of tangential preconnections, let us consider an arbitrary (localized) preconnection on the given (localized) conic structure. The preimage of this preconnection in the affine space $H^0(C_{\pi_1} J^\varepsilon)$ of arbitrary full preconnections will be called the space of **complementary** (relative to the given preconnection) full preconnections. This is in view of (iii) clearly the space $H^0(C^{np}_{\pi_1} J^\varepsilon)$ of fields valued in the (indicated) affine bundle of spaces of complementary full elementary preconnections. Thus it is either an affine subspace of $H^0(C_{\pi_1} J^\varepsilon)$ with the associated vector space $H^0(E^{np}_{\pi_1} J^\varepsilon) := H^0(T_J \otimes T^{*\alpha} J^\varepsilon)$ (consisting of complementary preconnectors) or empty.

Similarly, the space of complementary *tangential* full preconnections is either the affine subspace $H^0(C^{te\bullet np}_{\pi_1} J^\varepsilon)$ of $H^0(C^{te}_{\pi_1} J^\varepsilon)$ (consisting of fields in the indicated well-defined affine bundle of subspaces of complementary tangential full elementary preconnections) with the associated vector space $H^0(E^{te\bullet np}_{\pi_1} J^\varepsilon) :$ $= H^0(T^\varepsilon_J \otimes T^{*\alpha} J^\varepsilon)$ (consisting of complementary tangential preconnectors) or empty.

Furthermore, tangentiality of the given preconnection is in view of (iv) clearly equivalent to the non-emptiness of all spaces of complementary full elementary preconnections (i.e. to the existence of $C^{te\bullet np}_{\pi_1} J^\varepsilon$ as an affine bundle on the integral-jet space with non-empty fibers). This is a priori only a necessary condition for the existence of a complementary tangential full preconnection. QED

(vi) The explicit description of the connecting mapping of tangential preconnections is now obvious: Let us consider again the situation from (v) and suppose that the preconnection is indeed tangential. (Thus it is a point of the affine space in the second in the above affine exact sequence.) According to (v) this exact seqence can now be enlarged by inserting a new entry at the beginning:

$$H^1(E^{te\bullet np}_{\pi_1} J^\varepsilon) \to H^0(C^{te}_{\pi_1} J^\varepsilon) \to H^0(C^{te}_{\pi} J^\varepsilon) \to$$
$$\to H^1(E^{te\bullet np}_{\pi_1} J^\varepsilon) \to H^1(E^{te}_{\pi_1} J^\varepsilon) \to H^0(E^{te}_{\pi} J^\varepsilon) \to \ ...$$

The canonical affine-bundle class (in $H^1(E^{np}_{\pi_1} J^\varepsilon) := H^1(T_J \otimes T^{*\alpha} J^\varepsilon)$ of the given tangential preconnection is by the very definition of the long exact sequence precisely the affine-bundle class of the (above mentioned) bundle $C^{te\bullet np}_{\pi_1} J^\varepsilon$ (of affine spaces of tangential complementary full elementary precon-

nections) on the associated vector bundle. In view of exactness (which is proved by precisely this argument), the (tangential) preconnection is complementable if and only if this affine-bundle class vanishes. QED

(vii) Let us again consider the situation from (vi) and suppose that there exist tangential full preconnections (i.e. that the assumption from (iii) is fulfilled). (In this case the statement from (vi) to the effect that the canonical affine-bundle class of the given tangential preconnection is the precise obstruction to complemental permissibility follows also from its definition as the 'distance' between the tangential preconnection and the space of complementally permissible tangential preconnections.) In view of (vi), the vector space of the canonical affine-bundle classes of tangential preconnectors (which is determined by the localized conic structure alone) is precisely equal to the affine space (a priori dependent on the full structural prejet!) of the canonical affine-bundle classes of tangential preconnections [more explicitly the classes (on the vector bundle $E^{te\bullet np}_{\pi_1} J^{\varepsilon} \risingdotseq T^{\varepsilon}_{J} \otimes T^{*\alpha}$) of the affine bundles (of spaces of complementary tangential full elementary preconnections) associated to tangential (relative to the full structural prejet) preconnections]. (As we have already observed, without the assumption on the existence of tangential full preconnections, the latter space of affine-bundle classes would a priori only be affine and dependent on the full structural prejet.) QED

Proposition III.13 Suppose that in the situation of III.12 the given full structural prejet expanding the localized conic structure is induced by (the underlying global conic structure of) an admissible integrable global preconnection, i.e. by the structure of a geometrical parameter space of submanifolds of a manifold S. (In particular, it satisfies the condition from (ii) on the existence of tangential preconnections). Let S^{α} be the manifold with parameter m. According to the second chapter, its normal bundle $T^{/\alpha}_{S} S^{\alpha}$ is canonically isomorphic to the integral-transverse bundle $T^{/\alpha} J^{\varepsilon}$ on the integral-jet space of the given localized conic structure.

Let us recall that an **1-fattening** of a given manifold is by definition an extension of its tangent bundle (i.e. a vector bundle containing the tangent bundle as a subbundle), and that the quotient bundle is in this context called the **normal bundle** of the fattening. (The obvious 1-fattening of a manifold associated to an embedding thereof, namely the restricted tangent bundle of the ambient manifold, is called the **first-order infinitesimal neighbourhood.**) An isomorphism of two 1-fattenings (i.e. an isomorphism of extensions whose restriction to the subbundles is precisely the differential of the associated isomorphism of the bases) of the same manifold with the same normal bundle wil be said to be **over normal bundle** if the associated ('descended') automorphism of the normal bundle is the identity; in particular the associated ('restricted') automorphism of the subbundle is also the identity since it has to be the differential of the identity automorphism of the basic manifold . It is easily seen that 1-fattenings under such isomorphisms are 'equivalent' to affine bundles under isomorphisms over associated vector bundles. More precisely, by associating with an 1-fattening the affine bundle of spaces of direct complements of the tangent spaces (in their given extensions), we obtain an equivalence of these categories. [Indeed, 1-fattenings under isomorphisms over normal bundles are by definition equivalent to vector-bundle extensions under 'isomorphisms over embedded and quotient bundles', and for the latter such equivalence is well-known. QED] In particular, the space of **1-fattening classes** on a given vector bundle, which we define as the space of classes of 1-fattenings isomorphic over that normal bundle (or 'classes constituting structures of a normal bundle of an 1-fatteninng'), is a special case of a (similarly defined) space of **extensional classes** on a pair of vector bundles, and the latter can be identified with the *vector* space of affine-bundle classes on the obvious associated vector bundle.

We now apply these definitions to the submanifold S^α. The given embedding determines an 1-fattening class on its normal bundle. It is clear that this class is the precise obstruction to the realizability of the normal bunlde as a direct complement of the restricted tangent bundle, and the first-order obstruction to the existence of a tubular neighbourhood; indeed, this class is precisely the affine-bundle class (on the appropriate vector bundle) induced by the bundle -which we will denote by $J^\kappa_S S^\alpha$ - of affine spaces of direct complements in the tangent spaces T_S (of the ambient manifold S) to the tangent spaces T_S^α (of the submanifold). More precisely, the above class is an affine-bundle class on the vector bundle $Hom(T_S^{/\alpha}, T_S^\alpha)S^\alpha$ (namely the vector bundle associated to the affine bundle $J_S^\kappa S^\alpha$), i.e. an element of $H^1(Hom(T_S^{/\alpha}, T_S^\alpha)S^\alpha)$. On the other hand, this bundle is in view of the above quoted result (on isomorphy of normal and integral-transverse bundles) canonically isomorphic to the vector bundle $E_{\pi_1}^{te\bullet np} J^\varepsilon = T_J^\varepsilon \otimes T^{*\alpha} J^\varepsilon$.

Claim: With respect to this canonical isomorphism, the above 1-fattening class is associated precisely with the canonical affine-bundle class of the given tangential (localized) preconnection (i.e. with the affine-bundle class induced by the tangential complementary full-elementary preconnnection bundle). (More succinctly, **the canonical affine-bundle class of a tangential preconnection is the precise geometric counterpart** *-defined even for non-symmetric (in particular non-integrable) tangential preconnections-* **of the first-order infinitesimal neighbourhood***). What is more, there exists a canonical isomorphism (defined explicitly in the proof) between the bundle $J_S^\kappa S^\alpha$ (of affine spaces of direct complements of the tangent spaces of the submanifold) and the bundle $C_{\pi_1}^{te\bullet np} J^\varepsilon$ (of affine spaces of complementary tangential full elementary preconnections) such that the associated isomorphism of the associated vector bundles coincides with the above canonical isomorphism.*

[Proof: We will use notation from the second chapter pertaining to the double fibration

$$S \leftarrow R \rightarrow M.$$

Let j be an arbitrary integral jet at the point m and s the associated point of the submanifold S^α (relative to the canonical biholomorphism of the integral-jet space J^ε and S^α). By the definition of the dual family the submanifold M^α of M with parameter s is the image relative to the projection $(R \rightarrow M)$ (in the fibration $S^\alpha M = J^\varepsilon M$ of the preimage $M^\alpha.s$ relative to the projection $(R \rightarrow S)$ (in the fibration $M^\alpha S$) of the point s. The direction (denoted by c_π) of the tangent space (denoted by F^τ) of this preimage was by definition of the induced preconnection precisely the elementary preconnection constituent for the localized preconnection at m.

Let us observe that the preimage in T_R (relative to the differential of the projection) of any direct complement of T_S^{α} in T_S is a direct complement in T_R of the vertical space for the fibration $J^{\epsilon}M$ (namely of the space $F^{\varepsilon} \approx T_J^{\varepsilon} \approx T_S^{\alpha}$). (Indeed, it suffices to notice that for any surjective linear map which is injective on a subspace, the preimage of a direct complement of the image is a direct complement.) Furthermore, this preimage clearly contains F^{τ}. Consequently, its direction is a complementary (relative to the given tangential elementary preconnection) tangential full elementary preconnection. But the map $J_S^{\kappa} \to C_{\pi_1}^{te\bullet np}$ is clearly an affine isomorphism with the associated linear isomorphism equal to the canonical one. The system of these isomorphisms for various integral jets clearly constitutes the required affine-bundle isomorphism.QED]

III.13 suggests that for an *arbitrary* full structural prejet the canonical affine-bundle classes of *arbitrary* tangential preconnections should be viewed as special 1-fattening classes (constituting normal-bundle structures) on the integral-transverse bundle of the localized conic structure. In the next proposition we will carry out the pertinent considerations in more detail. A close examination of the degree of dependence of the related constructions on the full structural prejet expanding the given conic structure (cf. assertion (vii) of the last proposition) will result in a description of these classes in terms of pretorsion. Crucial in this context will be the observation on the independence of the 'vectorial component' of such constructions (just like in assertion (vii) of the last proposition) and the consequent reconstructibility of the isomorphy classes of full structural prejets from invariants expressible in terms of pretorsion.

Proposition III.14 Let us consider a (localized) conic structure at the point m of a manifold M.

(i) The associated vector space

$$\frac{H^0(E_{\pi}^{te}J^{\varepsilon})}{H^0(E_{\pi_1}^{te}J^{\varepsilon})} = \frac{H^0(T_J^{\varepsilon} \otimes T^{*/\alpha}J^{\varepsilon})}{H^0(T_J^{\varepsilon} \otimes T^{*}J^{\varepsilon})}.$$

of the canonical affine-bundle classes of tangential preconnections is by definition (cf. Proposition III.12 (ii)) a subspace of the vector space $H^1(E_{\pi_1}^{te\bullet np}J^{\varepsilon})$:

$= H^1(T^\varepsilon_J \otimes T^{*\alpha})$ of affine-bundle classes on the vector bundle $E^{te\bullet np}_{\pi_1} J^\varepsilon \coloneqq T^\varepsilon_J \otimes T^{*\alpha} J^\varepsilon$ of spaces of complementary tangential elementary preconnectors. Since the latter vector bundle has a distinguished structure of a tensor-product bundle (i.e. of a homohorphism bundle), the affine-bundle classes on it are actually extensional classes. (V. III.13 for definitions.)

Claim: More specifically, these classes are precisely 1-fattening classes (constituting normal-bundle structures) on the integral-transverse bundle $T^{/\alpha} J^\varepsilon$ of the localized conic structure.

[Proof of (i): It suffices to observe that the homomorphisms from the fibers of the bundle $E^{te\bullet np}_{\pi_1} J^\varepsilon \coloneqq T^\varepsilon_J \otimes T^{*\alpha} J^\varepsilon = Hom(T^{/\alpha}, T^\varepsilon_J) J^\varepsilon$ are defined on integral-transverse spaces and assume values in the tangent spaces of the base.QED]

(ii) Suppose an arbitrary full structural prejet expanding the conic structure is given. According to Proposition III.12 (ii) this choice determines a (possibly empty) affine subspace of the space of 1-fattening classes (constituting normal-bundle structures) on the integral-transverse bundle, namely the subspace consisting of the canonical 1-fattening classes of tangential preconnections (by means of the full structural prejet); here we have slightly modified the terminology in accordance with (i)). By the same assertion, if tangential preconnections exist an equivalent distinguishing property of the 1-fattening classes belonging to this affine subspace is the compatibility with the same infinitesimal deformation (with parameter space M) of the integral-jet space J^ε.

Claim: A distinguished representative 1-fattening (i.e. vector-bundle extension) for the canonical 1-fattening class of a given tangential preconnection can be constructed in the following way: Its fiber at an integral jet $j \in J^\varepsilon$ is by definition the quotient vector space

$$T_S := \frac{T_R}{F^\varepsilon}$$

together with the obvious embedding of the space $T^\varepsilon_J =: T^\alpha_S$ into T_S, and with the obvious surjection $T_S \to T^{/\alpha} =: T^{/\alpha}_S$.

(Here T_S is *not* defined as the tangent space of a manifold, and the notation of Proposition III.3 is implied.)

[Proof of (ii): Clearly we indeed obtain an extension $T_S J^\varepsilon$ of the tangent bundle $T_J^\varepsilon J^\varepsilon$ (of the integral-jet space) over the integral-transverse bundle $T^{/\alpha}J^\varepsilon$ (of the localized conic structure), i.e. an 1-fattening of J_ε with normal bundle $T^{/\alpha}J^\varepsilon$. It remains to verify that the associated 1-fattening class coincides with the canonical 1-fattening class of the given tangential preconnection by means of the boundary afine mapping of tangential preconnections. According to Proposition III.12 (vi), the latter 1-fattening class can be characterized in the following way: as an affine-bundle class (v. Proposition III.13) it is represented by the tangential complementary full-elementary preconnnection bundle. Similarly, the former 1-fattening class is represented by the affine bundle $J_S^\kappa J^\varepsilon$ of spaces of direct complements of the tangent spaces T_J^ε in the extended tangent spaces T_S. However, the argument from the proof of III.13 clearly also applies in this more general situation and gives a canonical isomorphism of the two affine bundles (explicitly, with a direct complement we associate a complementary full elementary preconnection by taking the preimage in T_R.QED]

(iii) Suppose the given localized conic structure is constituent for the underlying global conic structure of an admissible integrable global preconnection, i.e. of a structure on M of a geometrical parameter space of submanifolds S^α of a manifold S. From the induced tangential preconnection on the induced full structural prejet expanding the given localized conic structure we can construct an 1-fattening of J^ε by the procedure given in (ii). It is obvious from the construction of III.13 that this 1-fattening is canonically isomorphic to the first-order infinitesimal neighbourhood of the submanifold S^α with paramenter m. (Informally speaking, the construction of (ii) is a generalization of the construction of III.13.)

(iv) Consider an arbitrary full structural prejet expanding the given localized conic structure. Our present objective is to study the **complemental prolongability** of the given full structural prejet, which we define as the existence of simultaneously complementable and symmetric tangential (localized) preconnections. (Of course, a necessary condition is conjunctive prolongability, i.e. the conjunctively intrinsic pretorsion is a rough obstruction.)

Any such preconnection can due to complementability be lifted to a tangential full preconnection. Such a lift will obviously be a field of tangential full elementary preconnections with the following additional property: it will actually consist of **symmetric** *full elementary preconnections*, where the latter are defined as full elementary preconnections map into symmetric elementary preconnections. In other words, the above lift will actually be a field in the affine subbundle $C^{sa\bullet te}_{\pi_1} J^{\varepsilon}$ of the affine bundle $C^{te}_{\pi_1} J^{\varepsilon}$ (of spaces of full elementary preconnections) defined as the bundle of spaces of tangential *symmetric* full elementary preconnections. (Let us observe that in general the above bundle is well-defined and fiberwise non-empty precisely when the full structural prejet is prolongable; some of those fibers would be empty if the full structural prejet were not prolongable). Incidentally, the tangential symmetric full-elementary preconnnector bundle (i.e. the vector bundle of the affine subbundle) is clearly equal to $E^{sa\bullet te}_{\pi_1} J^{\varepsilon} := T^{\varepsilon}_{J\bullet(*} \otimes T^{*}_{*)} J^{\varepsilon}$, where the parentheses indicate symmetry of the associated tensors from $T^{\varepsilon}_{J} \otimes T^{/*\alpha} \subset T^{/\alpha} \otimes T^{*/\alpha\otimes 2}$. In particular, this vector bundle is independent of the full structural prejet expanding the localized conic structure.

Conversely, it is clear that in the case of prolongability (i.e. existence and fiberwise non-emptiness of the above affine subbundle) any field in the that subbundle gives rise to a complementable tangential symmetric preconnection; of course, the given field is in this context a concrete complementary full preconnection. Thus we have proved the following statement:

The **canonical affine-bundle class of a prolongable full structural prejet**, *which is by definition the affine-bundle class of the affine bundle of spaces of tangential symmetric full elementary preconnections, is the precise obstruction to complemental prolongability of the full structural prejet.*

(v) Just like the canonical affine-bundle class of a prolongable structural prejet (defined in the last section), the just defined invariant (i.e. the canonical affine-bundle class of a prolongable full structural prejet) is expressible in terms of pretorsion. Indeed, the whole argument of the Proposition III.9 can be carried out in this context with slight modifications. The objective of (v) will be to accomplish this.

Let us consider an arbitrary full structural prejet.

(v.1) We first observe that the quotient affine bundle of the affine bundle

$$C^{te}_{\pi_1} J^{\varepsilon} \text{ (= the extended affine bundle in this context)} \tag{III.16}$$

of spaces of tangential full elementary preconnections by the vector bundle $E^{sa_{\bullet}te}_{\pi_1} J^{\varepsilon} \coloneqq T^{\varepsilon}_{J_{\bullet}(*} \otimes T^{*}_{*)} J^{\varepsilon}$ (of spaces of tangential symmetric full elementary preconnectors) is precisely the underlying affine bundle of the vector bundle of spaces of elementary pretorsions. (This follows from the fact that the affine bundle $C^{te}_{\pi} J^{\varepsilon}$ is a 'larger' quotient of $C^{te}_{\pi_1} J^{\varepsilon}$, and the distinguished vector subbundle of $E^{te}_{\pi_1} J^{\varepsilon}$ is defined as the pre-image of the subbundle $E^{sa_{\bullet}te}_{\pi} J^{\varepsilon}$ of the 'larger' quotient vector bundle.)

The connecting map from the long exact sequence associated to the affine-bundle extension (III.16), namely the mapping of the space $H^0(C^{/sa\, sa_{*}te}_{\pi} J^{\varepsilon})$ of permissible pretorsions into the space $H^1(E^{sa_{\bullet}te}_{\pi_1} J^{\varepsilon})$ of affine-bundle classes will be called the **complemental connecting mapping of permissible pretorsions** or the **complemental connecting association of affine-bundle classes with permissible pretorsions**. Similar terminology is introduced for the associated linear map (cf. Proposition III.9). Furthermore, since the vector-bundle extension associated to the affine-bundle extension (III.16) is according to (v.1) independent of the full structural prejet expanding the given conic structure, the same is true of the complemental connecting mapping of permissible pretorsion vectors. In order to relate the images of the complemental connecting and linear map with pretorsions, it is natural to introduce the following concepts (cf. Proposition III.9 (vi)):

Let us define the space of **complementally intrinsic pretorsions** for the given localized conic structure as the quotient vector space

$$\frac{H^0(E^{/sa}_{\pi} J^{\varepsilon})}{\frac{H^0(E^{te}_{\pi_1} J^{\varepsilon})}{H^0(E^{te_{\bullet}np}_{\pi_1} J^{\varepsilon})}} \coloneqq \frac{H^0(T^{\alpha} \otimes T^{/*\alpha\wedge 2}\, J^{\varepsilon})}{\frac{H^0(T^{\varepsilon}_{J} \otimes T^{*} J^{\varepsilon})}{H^0(T^{\varepsilon}_{J} \otimes T^{*\alpha} J^{\varepsilon})}} \tag{III.17}$$

of the vector space of pretorsions by the vector space of **complementally permissible pretorsion vectors** (where the latter are, as indicated, by definition the conjunctively permissible pretorsion vectors belonging to the image of the space of complementally permissible tangential preconnectors). Furthermore, let us define its *affine* subspace consisting of **permissible** complementally intrinsic pretorsions as the quotient affine space

$$\frac{H^0(C_\pi^{/sa\,te_*}J^\varepsilon)}{\frac{H^0(E_{\pi_1}^{te}J^\varepsilon)}{H^0(E_{\pi_1}^{te_\bullet np}J^\varepsilon)}} \risingdotseq \frac{H^0(C_\pi^{/sa\,te_*}J^\varepsilon)}{\frac{H^0(T_J^\epsilon \otimes T^* J^\epsilon)}{H^0(T_J^\epsilon \otimes T^{*\alpha} J^\epsilon)}} \quad \text{(with vector space } \frac{\mathrm{H}^0(\mathrm{T}^\varepsilon_{\mathrm{J}\bullet[*} \otimes \mathrm{T}^{*/\alpha}_{*]}\,\mathrm{J}^\varepsilon)}{\frac{\mathrm{H}^0(\mathrm{T}_\mathrm{J}^\epsilon \otimes \mathrm{T}^* \mathrm{J}^\epsilon)}{\mathrm{H}^0(\mathrm{T}_\mathrm{J}^\epsilon \otimes \mathrm{T}^{*\alpha} \mathrm{J}^\epsilon)}}) \tag{III.18}$$

(of the vector space of pretorsions by the vector space of complementally permissible pretorsion vectors)

With these definitions it is clear that the complemental connecting mapping of permissible pretorsions descends to an affine injection of the space of permissible complementally intrinsic pretorsions into the space $H^1(E_{\pi_1}^{sa_\bullet te}J^\varepsilon)$ *(of affine-bundle classes on the tangential symmetric full-elementary preconnector bundle).* This injection will be referred to as the **affine injection of permissible complementally intrinsic pretorsions** or the **affine injective association of affine-bundle classes with permissible complementally intrinsic pretorsions.** Its image obviously consists precisely of affine-bundle classes compatible with the given infinitesimal deformation of the integral-jet space.

(v.2) As in Proposition III.9 (vi.2), the linear map associated to the (just defined) complemental connecting mapping of permissible pretorsions is also a connecting map and we introduce analogous terminology. Similarly, the complemental connective mapping of permissible pretorsion vectors and the linear injection of complementally intrinsic pretorsion vectors are obviously independent of the full structural prejet expanding the localized conic structure.

(v.3) The reasoning from Proposition III.9 (vi.3) applies here and consequently we adopt the analogous convention: the linear injection of complementally intrinsic pretorsions will be thought of as an inclusion.

(v.4) As in Proposition III.9 (vi.4), we infer that the domain of the affine injection of complementally intrinsic pretorsions coincides with its vector space iff the full structural prejet is prolongable. Similarly, if we assume prolongability, the canonical affine-bundle class of the given prolongable full structural prejet (i.e. represented by the affine bundle $C_{\pi_1}^{sa_\bullet te}J^\varepsilon$ of spaces of tangential symmetric full elementary preconnections), clearly coincides with the (well-defined due to prolongability) affine-bundle class (on the tangential symmetric full elementary-preconnector bundle) associated to the zero permissible pretorsion. (Notice that the zero permissible torsion will not necessarily be mapped into zero since the map is only affine; cf. Proposition III.9.) QED

(v.5) As in Proposition III.9 (vi.5) we obtain the equivalence of the following two conditions:

(a) Triviality of the induced infinitesimal deformation of the integral-jet space (in other words existence of tangential full preconnections, or vectoriality of the tangential full-elementary preconnnection bundle, or existence of complementally permissible pretorsions);

(b) Equality of the image of the affine injection of permissible complementally intrinsic pretorsions and its vector space (as affine subspaces of the vector space $H^1(E_{\pi_1}^{/sa_\bullet te} J^\varepsilon) \risingdotseq H^1(T^\varepsilon_{J_{\bullet(*}} \otimes T^*_{*)} J^\varepsilon)$ of affine-bundle classes on the vector bundle of spaces of tangential symmetric full elementary preconnecotors).

Furthermore, if these conditions are fulfilled, the **complementally intrinsic pretorsion of the full structural prejet**, which we define as the parameter of the affine space of complementally permissible pretorsions in the vector space of complementally intrinsic pretorsions) is a well-defined *permissible* complementally intrinsic pretorsion (in fact clearly the only one) mapped into zero by the affine injection of permissible conjunctively intrinsic pretorsions. QED

(v.6) Let us now assume both prolongability of the full structural prejet and triviality of the induced deformation of the integral-jet space (in other words, we consider a full structural prejet for which the conditions of (v.5) and (v.6) are simultaneously fulfilled).

Thus the affine space of permissible complementally intrinsic pretorsions contains two distinguished points, namely the complementally intrinsic pretorsion of the full structural prejet and the zero complementally intrinsic pretorsion. In order to achieve the objective of (v), it suffices to apply (v.5) and (v.6): indeed, according to these assertions the affine injection of permissible complementally intrinsic pretorsions maps these distinguished points into respectively zero affine-bundle class and the canonical affine-bundle class of the prolongable full structural prejet. Therefore the associated linear injection, i.e. the linear injection of permissible complementally intrinsic pretorsion vectors (which depends only on the localized conic structure!), maps the vector joining these points into the vector joining their images. In other words, it maps the complementally intrinsic pretorsion of the full structural prejet into the negative of the canonical affine-bundle class of the full structural prejet. Under the convention from (vi.4) we obtain the following conclusion:

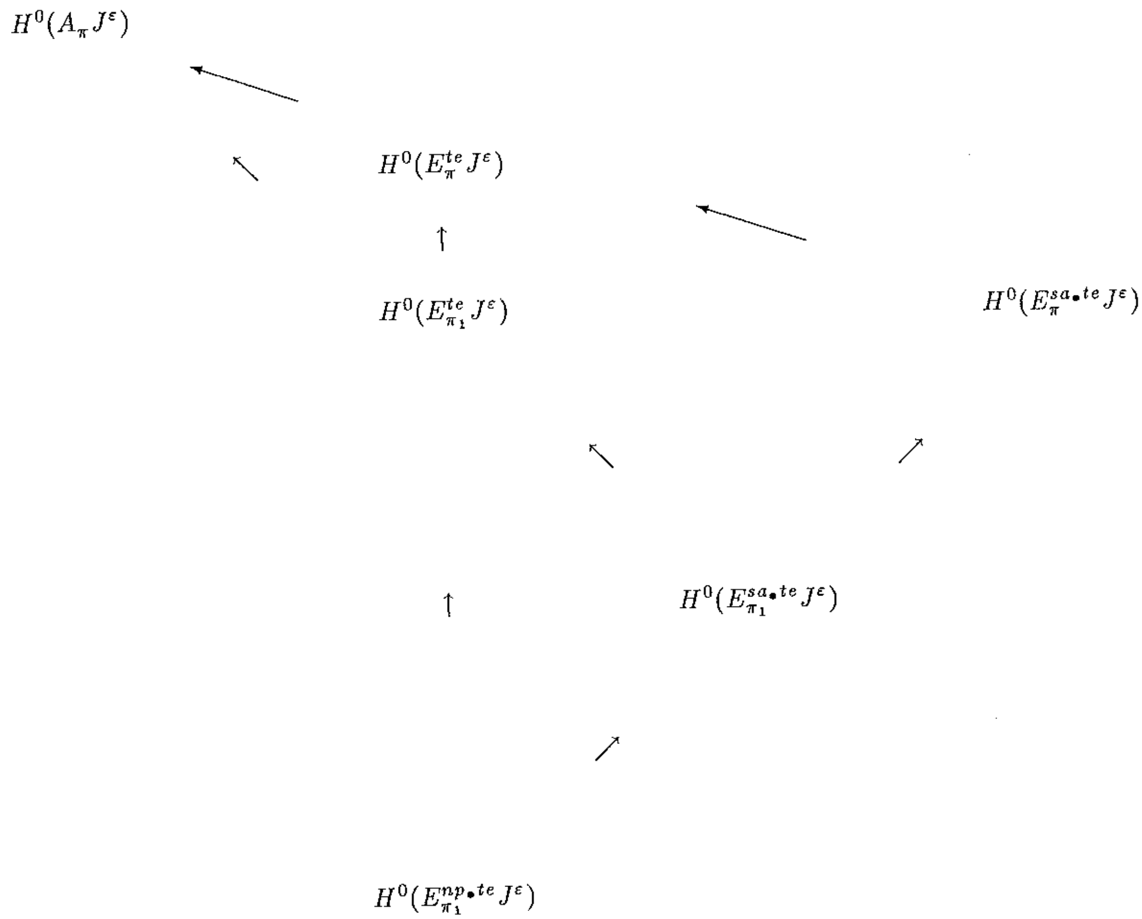
The canonical affine-bundle class of the prolongable full structural prejet is precisely the negative of the complementally intrinsic pretorsion of the full structural prejet. In particular, the latter is a precise obstruction to complemental prolongability.

In summary, the objective of (v) has been accomplished for the prolongable full structural prejets satisfying the above reasonably mild additional condition (namely the triviality of the induced deformation of the integral-jet space).

(vi) Let us again consider an arbitrary full structural prejet expanding the given localized conic structure. For similar reasons as in (v), the vectorial component of all 'affine constructions' in (vi) will depend only on the localized conic structure and we will be tacitly extending terminology to the vectorial constructions as before.

Our next goal is to investigate the interaction of the two analogous affine extensions (with common associated quotient bundle $E_{\pi}^{/sa} J^{\varepsilon}$) considered in (v), namely the extensions (III.3) (from Proposition III.9 (vi)) and (III.16). In fact, since we are given (as we have already observed) a 'chain' of two quotients of the tangential full-elementary preconnnection (affine!) bundle $C_{\pi_1}^{te} J^{\varepsilon}$, we obtain two additional obvious *vector*-bundle extensions with common associated vector subbundle $E_{\pi_1}^{np\bullet te} J^{\varepsilon}$; these four vector-bundle extensions (more precisely the corresponding sheaf cohomology in dimension k) are indicated in Figure III.14(vi). For the same reason, one of these two additional vector-bundle extensions is in fact associated to a canonical affine-bundle extension which fits into an analogous diagram; in our case this is precisely the extension (III.14) from the last proposition. The same general arguments show that after the choice of a field valued in the smaller (i.e. coarser) quotient affine bundle of the tangential elementary preconnection bundle, the same holds for the other additional vector-bundle extension (i.e. it is also associated to a canonical affine extension fitting into an analogous diagram). As we have already seen, in our case such a choice (namely the zero permissible pretorsion) is canonically given if the full structural prejet is prolongable; in fact, the bundles occuring in this second additional affine-bundle extension have already been considered (and named) in (iv),(v.4) and Proposition III.9 (vi.4). In accordance with that, we will refer to this extension as the **extensional structure on the tangential symmetric full-elementary preconnnection bundle** (i.e. on the affine bundle $C^{sa_\bullet te}_{\pi_1}J^\varepsilon$. However, for the time being, we do not assume prolongability.

FigureIII.14



(vi.1) By general cohomological arguments, the connecting maps in the long exact sequences associated to the above extensions also satisfy obvious commutativity. In particular, the linear mapping

$$H^1(E_\pi^{sa\bullet te} J^\varepsilon) \longrightarrow H^1(E_\pi^{sa\bullet te} J^\varepsilon)$$

(induced by means of the surjective vector-bundle map from the extensional structure on the tangential symmetric full-elementary preconnnector bundle) restricts (in the obvious sense) to the **canonical affine surjection**

$$\frac{H^0(C_\pi^{/sa} J^\varepsilon)}{H^0(E_{\pi_1}^{/te} J^\varepsilon)} \longrightarrow \frac{H^0(C_\pi^{/sa} J^\varepsilon)}{H^0(E_\pi^{/te} J^\varepsilon)} \qquad \text{(III.19)}$$

of permissible complementally intrinsic pretorsions onto permissible conjunctively intrinsic pretorsions, which we define as the canonical surjection of the larger quotient of the space of tangential full elementary preconnectors onto the smaller quotient; note that this surjection is a restriction of the similarly defined and *full structural prejet-independent* **canonical linear surjection of complementally intrinsic pretorsions onto conjunctively intrinsic pretorsions**.

More precisely, for a given permissible complementally intrinsic pretorsion, the following associated affine-bundle classes on the tangential symmetric elementary preconnector bundle coincide:

(a) The **affine-bundle class on the tangential symmetric elementary-preconnnector bundle which is associated to the given permissible complementally intrinsic pretorsion**, where this class is defined as the affine-bundle class injectively associated to the permissible *conjunctive* intrinsic torsion which is associated to the given permissible complementally intrinsic pretorsion;

(b) The affine-bundle class associated by means of the surjective vector-bundle map from the extensional structure on the tangential symmetric full-elementary preconnnector bundle with the affine-bundle class which is injectively associated to the given permissible complementally intrinsic pretorsion.

(vi.2) Let us observe that the (possibly empty) quotient affine space

$$\frac{\dfrac{H^0(C_\pi^{te} J^\varepsilon)}{H^0(E_\pi^{te\bullet sa} J^\varepsilon)}}{\dfrac{H^0(E_{\pi_1}^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te\bullet np} J^\varepsilon)}} \risingdotseq \frac{\dfrac{H^0(C_\pi^{te} J^\varepsilon)}{H^0(T_{J\bullet(*}^\varepsilon \otimes T_{*)}^{*/\alpha} J^\varepsilon)}}{\dfrac{H^0(T_J^\varepsilon \otimes T^* J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^{*\alpha} J^\varepsilon)}} \quad \text{(with vector space } \frac{\dfrac{H^0(T_J^\varepsilon \otimes T^{*/\alpha} J^\varepsilon)}{H^0(T_{J\bullet(*}^\varepsilon \otimes T_{*)}^{*/\alpha} J^\varepsilon)}}{\dfrac{H^0(T_J^\varepsilon \otimes T^* J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^{*\alpha} J^\varepsilon)}}\text{)} \tag{III.20}$$

(of the space of conjunctively permissible pretorsions by the vector space of complementally permissible pretorsion vectors) is contained in the space of *permissible* complementally intrinsic pretorsions. The points of this subspace will be called **conjunctively permissible** complementally intrinsic pretorsions. (Of course, non-emptiness of the affine space formed by these points is equivalent to the existence of tangential preconnections.)

From (i) we immediately obtain the following criterion in terms of affine-bundle classes for this conjunctive permissibility:

A permissible complementally intrinsic pretorsion is conjunctively permissible iff the associated affine-bundle class on the tangential symmetric elementary-preconnnector bundle (defined in (v.1) (b)) vanishes.

In other words, the affine space of *conjunctively* permissible complementally intrinsic pretorsions is precisely the pre-image of the conjunctively intrinsic pretorsion (of the full structural prejet) relative to the *affine* map (III.19).

(vi.3) The **canonical linear mapping of 1-fattening classes** determined by the localized conic structure (independent of the full structural prejet) is defined as the mapping

$$H^1(E_{\pi_1}^{te\bullet np} J^\varepsilon) \longrightarrow H^1(E_{\pi_1}^{sa\bullet te} J^\varepsilon)$$

(into affine-bundle classes on the tangential symmetric full-elementary preconnnector bundle) defined by means of the obvious vector-bundle mapping, namely inclusion. (In other words, this map occurs in the long exact sequence associated to the extensional structure on the tangential symmetric full-elementary preconnection bundle.)

Claim: The above linear map restricts to an affine surjection

$$\frac{H^0(C_\pi^{te} J^\varepsilon)}{H^0(E_{\pi_1}^{te} J^\varepsilon)} \longrightarrow \frac{H^0(C_\pi^{/sa} J^\varepsilon)}{H^0(E_{\pi_1}^{te} J^\varepsilon)}$$

of the (possibly empty) affine space of the canonical 1-fattening classes of tangential preconnections onto the space of conjunctively *permissible complementally intrinsic pretorsions.*

(Informally speaking, if we assume existence of tangential preconnections, the conjunctively permissible complementally intrinsic pretorsions are in a certain sense 'rudimentary' 1-fattening classes compatible with the given deformation of the integral-jet space.) Furthermore, this surjection, which will be referred to as the **canonical affine surjection of the canonical 1-fattening classes of tangential preconnections** could equivalently be obtained by *descent of the affine association of conjunctively permissible pretorsions with tangential preconnections.* In fact, the space (III.20) (of conjunctively permissible complementally intrinsic pretorsions) is the largest (i.e. 'finest') quotient of the space of conjunctively permissible pretorsions such that the quotient mapping of the space of tangential preconnections onto the space of conjunctively permissible pretorsions descends to a mapping of canonical 1-fattening classes of tangential preconnections into that quotient.

[Proof of (vi.3): The statement folows easily from a commutativity property of affine connecting maps analogous to (vi.1) and the fact that both distinguished subspaces have been defined as images of affine connecting maps.QED]

Theorem III.15 We again consider the situation of III.14. **(i)** Let us now assume assume prolongability of the full structural prejet. As we have already observed, we obtain the fourth *affine* bundle extension, namely the extensional structure on the tangential symmetric full-elementary preconnnection bundle. The corresponding connecting map will be called the **connecting mapping of symmetric tangential preconnections** or the **connecting association of 1-fattenings with tangential symmetric preconnections**. Note that non-emptiness of its domain amounts to *conjunctive* prolongability. This map coincides with the restriction of the connecting mapping of tangential preconnections: indeed, the 1-fattening class connecting-associated to a given tangential symmetric preconnection is (by the very definition of the connecting maps in general) simply the affine-bundle class of the corresponding tangential complementary full-elementary preconnnection bundle.

In conclusion, the affine space of 1-fattening classes connecting-associated to tangential symmetric preconnections (i.e. the affine space formed by classes of the tangential complementary full-elementary preconnnection affine bundles of tangential symmetric *preconnections) coincides with the space of 1-fattening classes which are canonically mapped into the canonical affine-bundle class of the prolongable full structural prejet (i.e. into the class of the tangential symmetric full-elementary preconnnection affine bundle; recall that this class is according to III.14(iv) the precise obstruction to complemental prolongability). In addition to that, non-emptiness fo this sapce is equivalent to conjunctive prolongability of the full structural prejet. Furthermore, the vector space of the latter affine space is the space of the canonical 1-fattening classes of tangential symmetric preconnectors, i.e. (due to exactness) the quotient vector space*

$$\frac{H^0(E_{\pi}^{sa_\bullet te}J^{\varepsilon})}{H^0(E_{\pi_1}^{sa_\bullet te}J^{\varepsilon})}. \tag{III.21}$$

QED

(ii) Let us assume (as in III.14(v.6)) both prolongability of the full structural prejet and triviality of the deformation of the integral-jet space induced by the full structural prejet.

In this situation the (above mentioned) affine space of the canonical 1-fattening classes of tangential symmetric preconnections is **encoded in the complementally intrinsic pretorsion** *(associated to the full structural prejet). Indeed, this space can be reconstructed from the localized conic structure and the complementally intrinsic pretorsion in view of the following fact: it consists precisely of those 1-fattening classes which are canonically mapped into the negative of the complementally intrinsic pretorsion. (Here we asume the convention from (v.3); recall that the canonical mapping of 1-fattening classes is independent of the full structural prejet expanding the localized conic structure). Furthermore, non-emptiness of that space (which according to (vi) amounts to conjunctive prolongability) is equivalent to vanishing of the conjunctively intrinsic pretorsion of the full structural prejet (i.e. canonically surjectively associated to the complementally intrinsic pretorsion, v.*

III.14(vi.2)).

[Proof of (ii): It suffices to apply (v) and the previous assertions of III.14(vi).QED]

(iii) Let us consider the situation of (ii) with the additional assumption that the given (localized) conic structure is of *pretype one* (i.e. free of tangential symmetric preconnectors, cf. Proposition III.10). From the exact sequence associated to the extensional structure on the tangential symmetric full-elementary preconnnection bundle (more concretely from vanishing of the kernel (III.21)), we infer that the canonical mapping of 1-fattenings is in this case injective. In particular, the canonical surjection of the 1-fattennigs associated to tangential preconnections (i.e. compatible with the given infinitesimal deformation of the integral-jet space) onto conjunctively permissible complementally intrinsic torsions (i.e. points from (III.20) is now bijective (being a restriction of the former map). (Incidentally, this could alternatively be seen from bijectivity (observed in Proposition III.10) of the canonical surjection of the space of tangential preconnections onto the space of conjunctively permissible pretorsions.) In particular, the canonical affine-bundle class of the (unique) tangential symmetric preconnection can be reconstructed from the complementally intrinsic pretorsion of the given conjunctively prolongable full structural prejet expanding the localized conic structure simply by applying the inverse mapping to the negative of the latter. QED

III.3 Intrinsic Torsion and Twistorial Invariants of Homogeneal Conic Structures

The assumption of homogeneality (or infinitesimal homogeneity, v. I.15) of an arbitrary first-order geometric structure (in particular of a conic structure) implies the existence of tangential (or structure-preserving) connections, and thus enables one to apply the apparatus of connection theory (cf. Remark III.1 from the last chapter) and construct the corresponding 'Lie equation of complete flatness'. A peculiar feature of (homogeneal) conic structures is the relation between on the one hand the Spencer complex associated to this 'Lie equation' [and the resulting sequence of succesively defined obstructions to the *unlimited finite-order flatness of the geometric structure, i.e. to the unlimited finite-order prolongability (or formal integrability) of the Lie equation - cf. the above mentioned remark] and, on the other hand, a peculiar conic-structural construction, namely the Spencer complex for the conic structure itself (and the resulting sequences of succesively defined obstructions to the various degrees of its prolongability of various orders; the latter are well-defined even without the assumption of homogeneality and usually have a natural twistorial interpretation).*

Of course, besides this amenability to classical differential-geometric methods and the correspondence between the two generalized conic structures (namely the Lie equation and the conic structure itself), the importance of homogeneal conic structures also derives, of course, from the fact that many such structures (e.g. paraconformal, in particular complexified quaternionic, structures) naturally arise in different areas of differential geometry.

In this section we will study the obstructions to various degrees of 1-prolongability (to second-order generalized conic structures) of a conic-structural 1-jet. (Such an obstruction is e.g. the associated complementally intrinsic pretorsion, which was shown to be precisely -in case of conjunctive prolongability at least- the negative of the parameter of the affine space of 1-fattening classes connecting-associated to tangential symmetric preconnections; as we have already seen, the latter invariant is of an explicitly twistorial character.) In particular, we will show how such invariants of a conic-structural 1-jet, which were constructed in previous sections, can in case of infinitesimal prolongability be explicitly 'read off' from the intrinsic torsion, and to what extent they determine the isomorphy clas of the structural 1-jet.

The former of these goals will be easily achieved since those invariants have already been encoded in terms of pretorsion; indeed, it only remains to investigate the obvious correspondence between torsions and pretorsions. The latter goal will be accomplished in this process; this is not unexpected in view of the already made (in Remark III.1) observation to the effect that a (holomorphic) structural 1-jet expanding a homogeneal (holomorphic) geometric structure can be reconstructed up to isomorphy class from the associated intrinsic torsion;

more precisely this observation consisted in an interpretation of the intrinsic torsion as the structural-jet class on the tangent vector space, *by which we mean the finest structure on the tangent vector space formed by a class of structural 1-jets. Indeed, an immediate consequence (also observed in that remark) was that each tensor-type invariant of the structural 1-jet (i.e. each second-order tensor-type invariant of the geometric structure) is in fact an invariant of the intrinsic torsion (on the tangent vector space). Notice that the latter observation was of a rather theoretical character: according to the proof of that fact, the procedure for recovering this invariant from the intrinsic torsion was through the construction of a full structural prejet inducing the given intrinsic torsion. Of course, for concrete invariants this procedure can usually be accomplished directly (i.e. without reference to differential-geometric concepts); it goes without saying that the first of above mentioned goals (realized in this section) implies such a direct procedure.*

III.3.1 Jets of Conic Structures

In this subsection we will show that (the primitive general version of) the concept of a conic-structural (or, more generally, aggregational-structural, v. Proposition III.3) 1-jet is often equivalent to the concept of a holomorphic 1-jet.

Lemma III.16 Consider an aggregational structure on a vector space T, where notation is as in Proposition III.3. Furthermore, let us suppose that the submanifold J^e (i.e. the integral atomary-structure space) of J is closed (as a subset of the topological space J). Owing to the latter condition, the construction we have already carried out in the special case of conic structures in Proposition I.15 can obviously be repeated in this more general context; therefore the **vectorial specific aggregational structures** obtained by this process are *holomorphic* in a unique way.

(i) Let us observe that the manifold U of the specific aggregational structures on T is the parameter space of a canonical (holomorphic) family of submanifolds of J, where the structure of a complex manifold on the set $J^e.U$ is well-defined as a submanifold of the product $J \times U$.

[Proof of (i): The *bundle* of aggregational-structural spaces associated to the trivial vector bundle $T \times U$ is clearly the trivial bundle $U \times U$ and its diagonal section determines a reduction of the structural group of the (trivial) frame bundle $P \times U$ of $T \times U$ to the automorphism group of the aggregational structure. Clearly, the set theoretical family $J^{\varepsilon}U$ is associated to this reduction and consequently inherits the structure of a holomorphic bundle. What is more, the latter is obviously a subbundle of the trivial bundle $J \times U$. QED]

(ii) U is a **locally effective** parameter space of submanifolds of J in the folowing sense: At each point u the tangent space T_U is an *effective* parameter vector space of normal-vector fields along the integral atomary-structure space $J^{\varepsilon} \subset J$ (i.e. sections of the normal bundle $T_J^{/\varepsilon} J^{\varepsilon}$ or 'tangent vectors' in the space of submanifolds at the submanifold J^{ε}) relative to the canonical linear system of the family from (i).

[Proof of (ii): Otherwise there would exist a non-zero tangent vector v_U at some point u inducing the zero normal-vector field along of the submanifold J^{ε}. However, v_U can be realized (relative to the given action of G on U) by some element from the Lie algebra of G, and its multiples clearly preserve $J^{\varepsilon} \subset J$ relative to the exponential action. This is a contradiction with the assumption $v_U \neq 0$ since the map $(J^{\varepsilon}.U \to J)$ could obviously be alternatively defined by descent of the restriction $(J^{\varepsilon} \times G \to J)$ of the action. QED]

Proposition III.17 Consider a localized aggregational structure on a manifold (or, more generally, a 'localized aggregational structure on an arbitrary bundle TM with a structural group'). Again, notation from Proposition III.3 will be implied.

(i) The **canonical (affine) mapping**

$$C^{/te} \longrightarrow C^{/te}_{\pi_1}$$

of holomorphic homogeneal aggregational-structural jets (i.e. holomorphic specific aggregational-structural jets) into homogeneal full elementary aggregational-structural prejets is correctly defined by descent of the (affine) mapping of connections into their horizontal homogeneal atomary-structural jets. Furthermore, the asociated linear mapping admits the explicit description given in the proof. In addition to that, the above affine mapping admits an obvious equivalent description in terms of expanded specific aggregational structures with the given (aggregational-structural) 1-jet and their bundles of integral atomary structures.

[Proof of (i): Clearly, it suffices to prove the analogous statement on the vector level. It remains to observe that the mapping

$$E^{/te} \coloneqq T_U \otimes T^* = \mathbf{G}^{/cs} \otimes T^* \longrightarrow E^{/te}_{\pi_1} \coloneqq T^{/\varepsilon}_J \otimes T^* \supset \mathbf{G}^{/ds\,/\varepsilon} \otimes T^*$$

is indeed well-defined since $\mathbf{G}^{cs} \cdot j \subset T^{\varepsilon}_J$. QED]

(ii) We define an affine mapping

$$C^{/te} \longrightarrow H^0(C^{/te}_{\pi_1}\, J^{\varepsilon})$$

of holomorphic homogeneal aggregational-structural jets into full aggregational-structural prejets (or formal aggregational-structural jets) formed by *homogeneal* full elementary aggregational-structural prejets by assigning to a holomorphic structural jet $c^{/te}$ the field of canonically associated full elementary structural prejets (defined in (i)) at various atomary-structures; in other words, for a given holomorphic structural jet $c^{/te}$ the affine space of all tangential connections is in view of (i) contained in the affine space of connections tangential relative to a unique full structural prejet; by definition we associate the latter with the given holomorphic structural jet.

Claim: For every structural jet as above the two affine spaces of tangential connections actually coincide, i.e. the above affine mapping is injective. Therefore, we will identify holomorphic homogeneal aggregational-structural jets with correspanding aggregational-structural jets by means of this affine injection.

[Proof of (ii): The assertion follows from the obvious injectivity on the vector level:

$$E^{/te} \coloneqq T_U \otimes T^* = \mathbf{G}^{/cs} \longrightarrow H^0(E^{/te}_{\pi_1}\, J^{\varepsilon})$$

$$\coloneqq H^0(T^{/\varepsilon}_J \otimes T^*\, J^{\varepsilon}) = H^0(T^{/\varepsilon}_J\, J^{\varepsilon}) \otimes T^* \supset \text{(in view of III.17(ii))} \supset T_U \otimes T^*.$$

(Alternatively, we could use Proposition III.3, i.e. the characterization of tangentiality of a connection relative to an expanded aggregational structure in terms of preserving the structure.) QED

(iii) Suppose the parameter space U of submanifolds of J is locally complete, i.e. the inclusion $U \subset H^0(T_J^{/\varepsilon} J^{\varepsilon})$ is in fact an equality. Then the affine injection from (ii.2) is clearly bijective, i.e. all full aggregational-structural prejets consisting of homogeneal full elementary aggregational-structural prejets are holomorphic homogeneal aggregational-structural jets. QED

di(iv) In the situation of (iii) suppose that all atomary structures are isomorphic (i.e. the action of G upon J is transitive). Then obviously all full aggregational-structural prejets are holomorphic homogeneal aggregational-structural jets.

III.3.2 Intrinsic Torsion and Intrinsic Pretorsion. Space of Complementary Connectors

Proposition III.18 Our present objective is to investigate a naturally defined mapping of localized connections into elementary preconnections. (This will be the first step towards establishing the geometric significance of the obvious algebraic relation between intrinsic torsions and intrinsic pretorsions on a vectorial conic structure.)

Let us consider a (1-) jet j through a point m of a manifold M. (For the purposes of this proposition we do not need the choice of a localized conic structure.) Consider the diagram formed by the spaces from the middle row in the figure from Remark III.1, which is analogous, as we have already observed, to the diagram formed by the fibers over j of the bundles from the middle row in the figure from Proposition III.7. This analogy was reflected in the fact that (localized) connections on M can (after the choice of an 1-frame) be thought of as special elementary preconnections *in the product space* $M_\mu \times M$, or as 2-pseudo-jets of biholomorphisms $M \to M_\mu$, where M_μ is a manifold furnished with a distinguished chart (the 'model' manifold).

(i) Now we further strenghten this analogy by observing that connections on M are also related to elementary preconnections *on the same* manifold M: the points of affine spaces from the latter of the above diagrams are 'rudimental points' of corresponding affine spaces from the former diagram in the following sense: Each affine space from the former diagram is (surjectively) mapped in a canonical way into the corresponding affine space from the latter diagram, and the combined diagram thus obtained (formed by all six of these affine spaces) is commutative. (In other words, we obtain a surjective mapping of affine-space extensions.) More precisely, for one of these maps, namely the mapping of the extended space, we take the **canonical affine mapping of the affine space C of (localized) connections onto the space C_π of elementary preconnections** (at the given jet), which is defined in the following way:

Suppose a (localized) connection c on the manifold M is given at m (v. Proposition III.3). Let us denote the horizontal space (of the given connection) at the given point $j.m$ of the contact manifold $J.M$ by T^{mj}_{JM} and the contact-structural integral-tangent space at the same point by F. The proconnection c_π canonically associated to the connection is now defined simply by the following requirement: the space $F^\tau \subset T_{J.M}$ in that direction (namely c_π) is given as the intersection $T^{mj}_{JM} \cap F$.

Clearly, this is indeed a well-defined affine mapping, and the other two affine maps constituting the mapping of extensions are then uniquely determined by commutativity of the above combined diagram; they are also surjective and we introduce analogous terminology for them. (For the proof of their existence we use the main properties of elementary pretorsions associated to elementary preconnections.)

Furthermore, the linear maps associated to these affine maps are invariants of the vectorial directional structure (T, j) for obvious category-theoretical reasons and we introduce analogous terminology for them; what is more, they are obviously the *canonical surjections of tensor products onto tensor products of quotients.* In addition to that, the main properties of elementary pretorsions associated to elementary preconnections also show that the canonical affine mapping of torsions into elementary pretorsions is linear (i.e. coincides with the canonical surjection onto the product of quotients).

Similarly, we define the **symmetric proconnection canonically associated to the (localized) connection** as the symmetric proconnection associated to the above defined preconnection. QED

(ii) The assertion (i) clearly has an analogue on the level of *full* 'preforms'; more precisely, we can replace the terms 'elementary preconnnection' and 'symmetric elementary preconnection' by resp. 'full elementary preconnection' and 'symmetric full elementary preconnection' (v. Proposition III.12 for definitions), where the diagram from the Proposition III.7 is also replaced by the corresponding diagram defined in Proposition III.12. QED

(iii) The diagrams from (i) and (ii) (i.e. the two maps of affine-space extension $C^{sa} \to C \to C^{/sa} (\coloneqq E^{/sa} \coloneqq T \otimes T^{*\wedge 2})$ together with the canonical mapping of the first codomain affine-space extension into the second codomain affine-space extension defined in Proposition III.12 (i.e. with the canonical affine surjection of full elementary preconnections onto elementary preconnections) clearly combine into a commutative diagram. QED

Proposition III.19 Let us consider a conic structure at a point m of a manifold M, where notation is as in Chapter 1. The primary objective of this proposition is to obtain information on the relation of the associated space of intrinsic torsions (v. Remark III.1) with the associated spaces of resp. intrinsic, conjunctively intrinsic and complementally intrinsic pretorsions. Since the space $E^{/sa} (\coloneqq T \otimes T^{*\wedge 2})$ of torsions (which will of course be identified with the space E^{as} of connectors *antisymmetric* in the last two indices) will in this context be fundamental, we will denote it more briefly by the symbol A. The space $E^{/sa\, sa_* te}$ of *permissible* torsion vectors will accordingly be denoted by A^{pi}, or, in other words, the space $E^{/te_* sa}$ of *intrinsic* torsions will be denoted by $A^{/pi}$. Analogous obvious notation will be introduced on the level of elementary pretorsions.

(i) Let us fix an integral jet j.

(i.1) Our immediate aim is to see how arbitrary localized connections and induced structural jets (of the given localized conic structure) are related to elementary preconnections and elementary structural prejets. Obviously we can now carry out the reasoning from the Proposition III.18 (i) for suitably enlarged diagrams; indeed, now we can include also the uppermost rows of the two figures, i.e. the rows corresponding to resp. the space of homogeneal structural jets and the space of elementary structural prejets expanding the localized conic structure. We also introduce the analogous terminology. E.g. we obtain the **canonical affine mapping of homogeneal structural jets into elementary structural prejets** and the **canonical affine mapping of intrinsic torsions into intrinsic elementary pretorsions**, where the latter mappng is actually linear. (More explicitly, we have the following statement: Consider the diagram formed by the spaces from the upper two rows in the figure from Remark III.1 (i.e. the diagram from Remark III.1 (a)), which is analogous, as we have already observed, to the diagram formed by the fibers over j of the bundles from the upper two rows in the figure from Proposition III.7 (i.e. the bundles from the diagram from Proposition III.7 (i)). Just like in the previous proposition, the analogy between, on the one hand, connections and homogeneal structural jets, and, on the other, elementary preconnections and elementary structural prejets (in the sense that the former are a special case of the latter) can be strenghtened. Indeed, the points of affine spaces from the latter diagram are 'rudimental points' of corresponding affine spaces from the former diagram in the following sense:

Each affine space from the former diagram is mapped in a canonical way into the fiber over j of the corresponding affine bundle from the latter diagram, and the combined diagram thus obtained (formed by all twelve of these affine spaces) is commutative; in particular, for each of the two rows we obtain a mapping of an affine-space extension. More precisely, the mapping of the extension in the middle row is taken to be the one from the previous proposition, i.e. essentially the canonical affine mapping of the affine space C of connections onto the space C_π of elementary preconnections at the given integral jet; clearly, all the other maps are then uniquely determined by commutativity of the above combined diagram.) QED

(i.2) The present aim is to see how arbitrary connections and induced structural jets are related to *full* elementary preconnections and *full* elementary structural prejets. Clearly Proposition III.18 (ii) can now also be modified in the same way in which Proposition III.18 (i) has been modified in (i.1); more explicitly, we obtain a similar diagram (formed by twelve spaces) which contains a surjective mapping of affine-space extensions in each of the two rows. Furthermore, thus obtained canonical affine mapping of connections into full elementary preconnections is by its very definition the canonical affine mapping of connections into atomary-structural jets. Therefore, thus obtained canonical mapping of homogeneal structural jets into full elementary structural prejets is a special case of the canonical affine mapping (introduced in the Proposition III.17) of aggregational-structural homogeneal structural jets into full elementary aggregational-structural prejets.

(i.3) Our next objective is to study the relation between the diagrams from (i.1) and (i.2). Again, the reasoning from the Proposition III.18 can be carried out for an enlarged diagram. More precisely, by combining these two diagrams with the diagram from Proposition III.12 we clearly obtain a *commutative* diagram. E.g., the canonical affine map (defined in (i.1)) of homogeneal structural jets into elementary structural prejets is precisely the composition of the canonical affine map (defined in (i.2)) of homogeneal structural jets into full elementary structural prejets and the canonical affine map (defined in that proposition) of full elementary structural prejets into elementary structural prejets.

(ii) By applying the three assertions of (i) simultaneously at all integral jets, we get three corresponding assertions concerning *pre*connections, structural prejets, full preconnections and full structural prejets (i.e. structural jets); these assertions are completely analogous, except for the absence of the statements on surjectivity. E.g. we obtain the **canonical affine mapping of structural jets into structural prejets** and the **canonical affine mapping of intrinsic torsions into intrinsic pretorsions**, where the latter mapping is actually linear. Also, we have proved that thus defined canonical affine mapping of homogeneal structural jets into structural jets (i.e. full structural prejets) is precisely the iclusion (which was defined as a special case of the affine inclusion of homogeneal aggregational-structural jets into aggregational-structural jets i.e. fields of full elementary aggregational-structural prejets). *In particular, according to that proposition, it is actually the identity mapping under fairly weak conditions.*

[Proof of (ii): The only slightly less trivial point involves holomorphicity of the 'set-theoretical preconnections' (resp. structural prejets) thus obtained. Suppose a connection c at m is given; explicitly, in the notation of Proposition III.3 the connection is determined by a G_μ -invariant distribution $T^{mp}_{PM}P$ of fiber-transverse spaces on the total space $P.M$ along the fiber P. Let us denote the distribution of horizontal spaces in the contact manifold $J.M$ on the fiber J of the given connection by $T^{mj}_{JM}J$ and the restriction of the structural distribution of the contact manifold by FJ (as in Proposition III.4). In the statement of (i.2) we have defined the a priori 'set-theoretical' **(localized) preconnection canonically** (with respect to the given localized conic structure) **associated to the connection** (by means of the following requirement: the integral-tangent set-theoretical distribution $F^\tau J^\varepsilon$ (on the contact manifold along the integral-jet space) of the set-theoretical preconnection (which is well-defined since the latter was defined as a special field of directions in the contact manifold) has to satisfy the equality

$$F^\tau J^\varepsilon = (\,T^{mj}_{JM} \cap H\,)J^\varepsilon\,.$$

It is immediate that the above intersection of vector bundles is indeed a vector bundle and that it indeed gives a preconnection - the intersections F^τ of fibers T^{mj}_{JM} and F have constant rank since they are obviously fiber-transverse over the space T^α in direction j. The statement involving the **symetric preconnection associated to the conic structure and the connection** is now obvious.QED]

(iii) Let us call a preconnection (resp. pretorsion, structural prejet, intrinsic pretorsion) **homogeneal**, if it belongs to the image of the canonical mapping of connections (resp. torsions etc.). Similarly, we define **homogeneally symmetric preconnections** and **homogeneally prolongable structural**

prejets. We introduce similar terminology on the vector level, and also for full 'preforms'; notice that for full (conic-) structural prejets (i.e. structural jets) the notions of homogeneality and homogeneal prolongability coincide by definition with resp. homogeneality and 1-flatnessof the structural jet.

(iii.1) In the same spirit, when a structural jet expanding the given localized conic structure is given, we define **homogeneally tangential preconnections** (resp. **homogeneally permissible pretorsions, homogeneally symmetrically tangential preconnections**) as points in the image of the obvious affine mapping with possibly empty domain. The obviously defined counterparts of these concepts on the vector level are clearly independent of the structural jet, and they form non-empty (vector) spaces. *It is easy to see that homogeneally permissible pretorsion vectors are necessarily complementally permissible.* The quotient of the space of pretorsions by this subspace will be called the vector space of **homogeneally intrinsic pretorsions**. For a given *homogeneal* structural jet the (necessarily homogeneal) vector in that space parametrizing the (non-empty due to homogeneality) affine space of homogeneally permissible pretorsions will be called the **homogeneally intrinsic pretorsion of the homogeneal structural jet**.

This invariant of the homogeneal structural jet is clearly finer than the complementally intrinsic pretorsion of the structural jet in the sense that it is mapped precisely into the latter invariant via the obvious linear map (of the 'larger' quotient into the 'smaller' quotient). (In a less formal language, for a homogeneal structural jet the associated homogeneally intrinsic pretorsion constitutes an invariant which contains more information than the complementally intrinsic pretorsion; e.g. complemental prolongability does not necessarily imply vanishing of homogeneally intrinsic pretorsion.) Intuitively, the homogeneally intrinsic pretorsion is the invariant which contains as much information on the intrinsic torsion as can possibly be stored in a pretorsion. Notice, however that this invariant has not been interpreted as an affine-bundle class. Instead, it can obviously be interpreted as a precise obstruction to the **weak homogeneal prolongability** of the structural jet, which we define as the existence of symmetric homogeneally tangential preconnections (or, explicitly, existence of tangential **pre-symmetric** connections, where the latter property by definition means symmetry of the canonically associated preconnection. (Notice that according to the above definitions the concept of a 'weakly homogeneally prolongable structural jet' is more special than that of a 'homogeneal (and) prolongable structural jet', but more general than that of a 'homogeneally prolongable (i.e. 1-flat) structural jet'; the latter concept was defined as existence of tangential *symmetric connections.*)

(iii.2) The vector space of **homogeneal conjunctively intrinsic pretorsions (resp. homogeneal complementally intrinsic pretorsions)** is defined as the image of the vector space of homogeneal pretorsions in the space of conjunctively intrinsic pretorsions (resp. complementally intrinsic pretorsions) relative to the quotient (linear!) mapping. *Let us observe that homogeneal intrinsic (resp. homogeneal conjunctively intrinsic, homogeneal complementally intrinsic) pretorsions are obviously precisely those which are associated to homogeneal structural 1-jets.* (This is yet another justification for the use of the term *homogeneal.*)

(iii.3) Furthermore, we introduce the following concepts and notation:

E^{np} is the vector space of connectors which are **complementary**; this by definition means vanishing of the canonically associated preconnector.

For a fixed preconnection we denote by C^{np} the affine space of connections which are **complementary**; this by definition means belonging to the preimage of the given preconnection relative to the canonical affine mapping of connections into preconnections. Clearly, the associated vector space is precisely the above space and non-emptiness of this affine space is equivalent to the homogeneality of the fixed preconnection.

A^{np} is the vector space of **complementary** torsion vectors; this space is defined as the kernel of the canonical linear mapping of torsions into pretorsions. For a fixed pretorsion we denote by A^{np}_{af} the affine space of **complementary** torsions; this is by definition the subspace of the underlying affine space A_{af} of A defined as the preimage of the given pretorsion relative to the canonical linear maping of torsions into pretorsions. Clearly, the associated vector space is precisely the above space and non-emptiness of this affine space is equivalent to the homogeneality of the fixed pretorsion.

Let us observe that the space E^{np} of complementary connectors is clearly symmetric with respect to the permutation of the second and the third index, i.e. the direct sum of its symmetric-tensor and antisymmetric-tensor parts:

$$E^{np} = E^{np\bullet sa} \oplus E^{np\bullet as};$$

in particular $A^{np} \coloneqq E^{np\bullet as} = E^{/sa\, sa_* np}$

(iii.4) Let us now introduce the following vector subspaces of the space A (of torsions), which clearly all contain the spaces A^{pi} (of permissible torsion vectors) and A^{np} (of complementary torsion vectors), and form a decreasing chain of subspaces:

A^{pi_o} or the space of **pre-permissible torsion vectors** is (defined to be) the pre-image of the space of permissible pretorsions (relative to the canonical linear mapping of torsions into pretorsions);

A^{pi_j} or the space of **conjunctively pre-permissible torsion vectors** is the pre-image of the space of conjunctively permissible pretorsos;

A^{pi_p} or the space of **complementally pre-permissible torsion vectors** is the pre-image of the space of complementally permissible pretorsion vectors;

A^{pi_h} or the space of **homogeneally pre-permissible torsion vectors** is the pre-image of the space of homogeneally permissible pretorsion vectors, in other words the sum $A^{pi_* np} \coloneqq A^{pi} + A^{np}$.

For intrinsic torsions we introduce the analogous terminology (e.g. $A^{pi_p/pi}$ is the space of complementally pre-permissible intrinsic torsion vectors). For a given structural jet, we define similarly affine spaces with these associated vector spaces.

It is immediately obvious that the quotient $A^{pi_o/np}$ (resp. $A^{pi_j/np}$, $A^{pi_p/np}$, $A^{pi_h/np}$) can canonically be identified with the vector space of homogeneal permissible (resp. homogeneal conjunctively permissible, homogeneal complementally permissible, homogeneally permissible) pretorsions. Observe that almost by definition all homogeneally permissible pretorsions are homogeneal; in addtion to that the vector space $A^{pi_h/pi}$ (of homogeneally pre-permissible intrinsic torsions) can by the second isomorphism theorem be canonically identified with the quotient $A^{np/pi\bullet np}$ (of the space of complementary torsion vectors by the space of complementary permissible torsion vectors).

Therefore, the space A of torsions admits a canonical (relative to the localized conic structure) structure of a succesive vector space extension (with the convention from (III.7) on the meaning of the summation sign $\dot{+}$)

$$A = A^{pi} \dot{+} \frac{A^{pi_h}}{A^{pi}} \dot{+} \frac{A^{pi_p}}{A^{pi_h}} \dot{+} \frac{A^{pi_j}}{A^{pi_p}} \dot{+} \frac{A^{pi_0}}{A^{pi_j}} \dot{+} \frac{A}{A^{pi_0}}$$

or, more compactly,

$$A = A^{pi} \dot{+} A^{pi_h/pi} \dot{+} A^{pi_p/pi_h} \dot{+} A^{pi_j/pi_p} \dot{+} A^{pi_0/pi_j} \dot{+} A^{/pi_0}, \qquad \text{(III.22)}$$

where the 'extensional summands' are precisely the spaces of respectively permissible torsions, homogenally ?pre-permissible intrinsic torsions, homogeneal complementally permissible homogeneally intrinsic pretorsions, homogeneal conjunctively permissible complementally intrinsic pretorsions, homogeneal permissible conjunctively intrinsic pretorsions and homogeneal intrinsic pretorsions.

Theorem III.20 Let us a again cosider the situation of III.19. The invariants of a structural jet defined under mild conditions (homogeneality being the strongest) in terms of pretorsion and interpreted as succesive obstructions to various degrees of prolongability, namely the intrinsic torsion, conjunctively intrinsic pretorsion, complementally intrinsic pretorsion and homogeneally intrinsic pretorsion, are in the case of homogeneality of the structural jet necessarily homogeneal, i.e. vectors from the spaces resp. $A^{/pi_0}$, $A^{/pi_j}$, $A^{/pi_p}$ and $A^{/pi_h}$.

Thus they are explicitly expressible as images of the intrinsic torsion via the **canonical linear maps** into intrinsic pretorsions of various types ; these maps are defined by descent of the canonical linear mapping of torsions into pretorsions. A close examination of these canonical maps, in particular the explicit determination of their kernels (i.e. essentially the 'denominators' in (III.22)), for specific conic structures enables one to identify the invariant components of the space of intrinsic torsions which carry the information on the obstructions to the various degrees of prolongability.

In particular, for a homogeneal conjuntively prolongable structural jet the problem of explicit determination of the (affine space of) 1-fattening classes associated to tangential symmetric preconnections from the (conjunctively pre-permissible) intrinsic torsion of the structural jet alone (without reference to the structural jet, v. the introduction to this section) is solvable by means of the (surjective) canonical linear map

$$A^{pi_j/pi} \longrightarrow A^{pi_j/pi_p}$$

(of conjuntively pre-permissible intrinsic torsions into homogeneal conjunctively permissible complementally intrinsic pretorsions). More precisely, the above problem is thus reducible to the explicit procedure of Proposition III.15 (for which the initial information is the conjunctively permissible complementally intrinsic pretorsion). For this reason the above linear map will play an important role in the study of general geometrical parameter spaces of submanifolds.

Furthermore, the vector space of complementally pre-permissible intrinsic torsions, which is obviously of particular importance in the study of geometrical parameter spaces of submanifolds with vanishing associated 1-fattening classes, has the structure of an extension of the following form:

$$A^{pi_p/pi} = A^{pi_h/pi} \dot{+} A^{pi_p/pi_h}$$

(where the subspace consists of homogeneally permissible intrinsic torsions, and the quotient of homogeneal complementally permissible homogeneally intrinsic pretorsions).

Although this quotient vanishes in many important cases (as we will see in (iv.ii)), the subspace does not. (What is more, we will construct (in Chapter V) examples of conjunctively integrable, weakly homogeneally -thus all the more complementally- prolongable, and simultaneously non-1-flat expanded conic structures. In particular, this will show that the first infinitesimal neighbourhood of a submanifold from a given geometrical parameter space is not always sufficient to reconstruct the isomorphy class of the structural 1-jet expanding the localized conic structure).

Proposition III.21 We again cosider the situation of III.20. **(i)** Our next objective is to study the canonical linear mapping of torsions into pretorsions. Since this is a restriction of the canonical linear mapping of connectors into preconnectors, and we are also interested in homogeneally pre-permissible torsions, our investigation will actually be more comprehensive in the sense that it will include the whole of the latter mapping. By exploiting its factorization introduced in (ii) (i.e. the factorization via complete preconnections) the problem is obviously reduced to the study of the two factor maps. Therefore, let us consider the obvious diagram including all these three maps, namely the diagram from the Figure III.21 (associated to the given localized conic structure), where the heretofore not introduced spaces are defined as follows (and the maps other than the ones just mentioned are defined in the obvious canonical way):

$E^{ui} := E^{vd'' * vd'} \coloneqq E^{vd'} + E^{vd''}$ is the sum of spaces of first trace-part and second trace-part connectors $e \in \dot{E} \coloneqq T \otimes T^{*\otimes 2}$. (The motivation for our notational conventions has been explained precisely on this example in the Appendix.) Notice that these spaces are independent of the given conic structure on the

FigureIII.21

$$T^{/\alpha} \otimes T^{*/\alpha\otimes 2}$$

$$\|$$

$$E_\pi$$

$$E^{vd'/vd'_\bullet vd''} = E^{/vd''\,ui} \longrightarrow E^{/vd''} \longrightarrow \quad E^{/ui} \quad \to \quad E^{/np} \quad \to \quad H^0(\overbrace{E^{/np}_{\pi_1}}\, J^\varepsilon) \quad \supset \quad H^0(E^{te}_\pi J^\varepsilon)$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\mathbf{G}^{vd} \otimes T^* = \quad E^{vd'} \quad \longrightarrow \quad E \quad \longrightarrow \quad E^{/vd'} \quad = \quad \mathbf{G}^{/vd} \otimes T^* \quad \to \quad H^0(E_{\pi_1} J^\varepsilon) \quad \supset \quad H^0(E^{te}_{\pi_1} J^\varepsilon)$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$0 \qquad = E^{vd'_\bullet vd''} \longrightarrow E^{vd''} \xrightarrow{\approx} \quad E^{/vd'\,ui} \quad \overset{(2')}{\to} \quad H^0(\underbrace{E^{np\,vd''}_{\pi_1}}\, J^\varepsilon) \quad \overset{(2)}{\to} \quad H^0(\underbrace{E^{np}_{\pi_1}}\, J^\varepsilon) \quad \supset H^0(E^{te_\bullet np}_{\pi_1} J^\varepsilon)$$

$$(T^{/\alpha} \otimes T^{*/\alpha} \otimes T^{*\alpha})^{vd''} \quad T^{/\alpha} \otimes T^{*/\alpha} \otimes T^{*\alpha}$$

$$\uparrow\approx$$

$$\uparrow$$

$$T^* \qquad \to \qquad T^{*/\alpha}$$

(i.i) E^{ui} is clearly contained in the space E^{np} (of complementary connectors) and will therefore be said to consist of **universal complementary connectors**. (This term is meant to suggest the obvious fact that these complementary connectors are common to all conic structures on T. In fact, if $dim\, T \geq 3$, universal complementary connectors could be characterized by this property in view of the following result proved in [13]: For an omnidirectional vectorial conic structure with at least 2-dimensional integral directions there are no complementary connectors other than the universal ones. However, we will not use this fact.) Similarly, the vectors from the quotient $E^{np/ui}$ will be called **inherent complementary connectors** (since they are peculiar to the conic structure in question).

Furthermore, the space of universal complementary connectors is, just like the space of all complementary connectors (v. (iii.3)), obviously symmetric relative to the permutation of the last two indices. Hence it follows (by the same argument as in (iii.3)) that its antisymmetric part can be expressed as $A^{ui} := E^{ui\bullet as} = E^{/sa\, sa_* ui}$. In other words, the space of **universal complementary torsions**, which we define as the space A^{ui} (of antisymmetric universal complementary preconnectors) is equal to its ambient space consisting of antisymmetrized universal complementary preconnectors. Furthermore, the latter space is clearly equal to its subspace consisting of antisymmetrized first trace-part connectors. As above, the vectors from the quotient $A^{np/ui}$ will be called **inherent complementary torsions**.

On the other hand, we have the obvious inclusion $E^{vd'} \subset E^{te}$, i.e. first trace-part connectors are tangential relative to the localized conic structure. In view of these facts the following inclusion holds: $A^{ui} \subset A^{pi\bullet np}$. *In conclusion, the space $A^{pi_h/pi}$ (of homogeneally ?pre-permissible intrinsic torsions), which has already been identified with the quotient $A^{np/np\bullet pi}$ (of the space of complementary torsion vectors by the space of complementary permissible torsion vectors) is a smaller quotient of the space A^{np} (of complementary torsions) than the space $A^{np/ui}$ (of inherent complementary torsions; in particular, the former quotient is also a quotient of the latter quotient, i.e. we have de-*

fined a map which will be called the **canonical surjection of the inherent complementary torsions onto homogeneally ?pre-permissible intrinsic torsions**.

We introduce analogous terminology on the level of full preconnectors: the vectors in the image $H^0(E^{np}_{\pi_1}J^{\varepsilon})^{ui}$ (resp. $H^0(E^{np}_{\pi_1}J^{\varepsilon})^{ui_h}$) of the map (2) (resp. (2)') from the diagram, i.e. the vectors which are in the obvious sense (uniquely) precovector-induced (resp. covector-induced), will be called **universal complementary full preconnectors** (resp. **homogeneally universal complementary full preconnectors**), while the quotient $H^0(E^{np}_{\pi_1}J^{\varepsilon})^{/ui}$ (resp. $H^0(E^{np}_{\pi_1}J^{\varepsilon})^{/ui_h}$) will be said to consist of **inherent complementary full preconnectors** (resp. **homogenally inherent complementary full preconnectors**). It is obvious from the diagram that the **canonical linear mapping of inherent complementary connectors into universal homogeneally inherent complementary full preconnectors** is well-defined (and its image consists precisely of the homogeneal vectors).

QED

(i.ii) Suppose every full preconnector is homogeneal (i.e. associated to some connector); in other words the map (1) in the diagram below is surjective. (It is obvious from the diagram that this condition is fulfilled as soon as the integral-transverse bundle is a direct sum of line bundles, the conic structure is completive, amd its integral-jet space is (biholomorphic to) a projective space. These assumptions are clearly fulfilled in the case of quaternionic paraconformal conic structures and Veronese conic structures.)

Claim: Complemental prolongability of a homogeneal structural jet expanding the given localized conic structure implies weak homogeneal prolongability. What is more, for the given localized conic structure the space A^{pi_p/pi_h} *(of homogeneal complementally permissible homogeneally intrinsic pretorsions) vanishes.*

[Proof of (i.ii): The first statement follows easily from the fact (proved in Proposition III.3) that a connection inducing tangential *full* preconnection is itself tangential (i.e. structure-preserving).

As for the second statement, consider a complementally pre-permissible torsion

$a(\in A^{pi_p})$. The canonically associated pretorsion $a^{/np}$ is therefore complementally permissible, i.e. it is associated (via antisymmetrization) with some tangential full preconnector. By assumption the latter is canonically associated to some connector e, which is (according to the above mentioned fact from Proposition III.3) also tangential. (This is the only step in the proof which does not follow directly from definitions.) The (permissible) torsion a^{pi} associated (via antisymmetrization) with the tangential connector e differs from a by a complementary torsion a^{np} (since they both induce the same pretorsion $a^{/np}$). It remains to recall that $A^{pi_h} = A^{pi} + A^{np}$.QED]

(i.iii) Suppose the conic structure is hypersurface-directional and completive, and its integral-jet space is (biholomorphic to) a projective space. (In other words, the assumption on the integral-transverse bundle in the remark from (iv.ii) has been replaced by the stronger assumption that the conic structure be hypersurface-directional.) In this situation the map (1) in the diagram is easily seen to be not only surjective, but also bijective. In other words, under those circumstances the vector space of full preconnectors is (by means of this mapping) a quotient space of the space $E \coloneqq T \otimes T^{*\otimes 2}$ (of connectors) canonically isomorphic to (and will therefore be identified with) the quotient space $E^{/vd'}(= E^{dv'})$ formed by first-trace-free connectors. In particular, the map (2)' in the diagram is injective, i.e. the surjection of covectors onto homogeneally universal complementary full preconnectors is bijective. (Incidentally, let us observe that the following related statement with weaker assumptions and conclusion can be easily proved by a completely different method: If the conic structure is hyperplane-directional and the integral-jet space is not 0-dimensional, the map (2)' is injective.) QED

(i.iv) If the conic structure is hyperplane-directional, the injection (2) in the diagram is bijective. In other words, every complementary full preconnector is universal (i.e. uniquely precovector-induced). (Here we imply the correspondence between precovectors and complementary full preconnectors which is defined fiberwise in the way indicated in the diagram .)

[Proof: It suffices to note that any complementary full elementary preconnector is universal in the sense that it is elementary precovector-induced (i.e. a second-trace-part full elementary preconnector) due to the fact that $T^{/\alpha}$ are vector lines.QED]

(i.v) If the conic structure has all the properties assumed in (i.iii), an explicit description of the space E^{np} of complementary connectors is possible owing to the statement on injectivity therefrom. More precisely, the canonical linear mapping of inherent complementary connectors into homogeneally inherent complementary full preconnectors is bijective, and its codomain is canonically isomorphic (in a way explicitly described in the proof) to the vector space of the affine bundle classes on the integral-tangent covector line bundle:

$$E^{np\,/ui}(\coloneqq E^{np\,/vd''{}_*vd'} \coloneqq \frac{E^{np}}{E^{vd''{}_*vd'}} \coloneqq \frac{E^{np}}{E^{vd'} + E^{vd''}} = E^{/vd'\,np\,/ui}) \approx H^0(E^{np}_{\pi_1} J^\varepsilon)^{/ui_h} \approx H^1(T^{*\alpha} J^\varepsilon). \tag{III.23}$$

In particular, the subspace $A^{np\,/ui}$ of $E^{np\,/ui}$ (consisting of inherent complementary torsions and according to (iv.i) related to the space of homogeneally permisssible intrinsic torsions) is also embedded into the above space of affine-bundle classes.

[Proof: The bijectivity follows from the obvious fact that the space of complemetary connectors is precisely the preimage of the space of full complementary preconnectors and the assertions (iv.ii), (iv.iii) and (iv.iv), while the second isomorphism is obtained from the long exact sequence associated to the vector-bundle extension T^*J^ε (: = the trivial bundle) with the subbundle $T^{*\alpha}J^\varepsilon$ (formed by the vector spaces of integral-tangent or 'complementary' covectors) and the quotient bundle $T^{*/\alpha}J^\varepsilon$ (formed by the vector spaces of integral-transverse covectors, i.e. elementary precovectors).QED]

(i.vi) If every full preconnector is homogeneal (as in (iv.ii)) and the integral-jet space is (biholomorphic to) a projective line (e.g. if the assumptions of the remark in (iv.ii) are fulfilled), then every preconnector is (III.20) also homogeneal (i.e. connector-induced). [This assertion easily follows from the diagram. QED]

(ii) Suppose the given localized conic structure is of pretype one (v. Proposition III.10). Then the conic structure is also of type one (i.e. the space $E^{sa_\bullet te}$ of tangential symmetric connectors vanishes) as soon as the conic structure is hypersurface directional and the space of inherent complementary *symmetric* preconnectors vanishes (i.e. there are no other complementary symmetric connectors other than universal ones).
[Proof of (ii): Let us observe that the canonical linear mapping of connectors into preconnectors restricts to a map

$$E^{sa_\bullet te} \rightarrow H^0(E_\pi^{sa_\bullet te} J^\varepsilon),$$

which we call **canonical linear mapping of tangential symmetric connectors into tangential symmetric preconnectors**. Clearly, it suffices to prove the injectivity of this map under the above assumptions.

The kernel of the above map obviously consists precisely of *complementary elements.* By the last of the above assumptions these must also be universal. In other words, it suffices to show that the connectors from the space $E^{ui_\bullet np}(= E^{vd'}_{\bullet(**)} := \delta \otimes T^*_{\bullet(**)}$ (of symmetric universal complementary connectors) can not be tangential. This is easily seen since the 1-jets in the Grassmanian tangent to the integral-jet space are according to the Remark III.22 below 'axially decomposable' (in the sense made precise therein). QED]

Remark III.22 Consider an integral jet j of a *hypersurface-directional* vectorial conic structure, where notation is as in Definition I.2. Since the integral-transverse space $T^{/\alpha}$ is a vector line, the tangent space T^ε_J to the integral-jet space J^ε at j must be a subspace of

$$T_J = Hom(T^\alpha, T^{/\alpha}) = T^{/\alpha} \otimes T^{*/\alpha}$$

of the form

$$Hom(T^{\alpha\,/\alpha'}, T^{/\alpha}) = T^{/\alpha} \otimes T^{*/\alpha\,\alpha'}$$

where $T^{\alpha'}$ is the subspace of T^α uniquely determined by the space T^ε_J and called the **axis** thereof. (Of course, an alternative characterization of the axis is the following: the 'linearly' embedded projective space of the perpendicular to the axis is tangent to the integral-jet space. On the other hand, the projective space of the axis is by elementary projective-geometric arguments precisely the intersection of projective hyperplanes parametrized by points of the former projective space. This justifies the term 'axis' and gives it an intuitive interpretation.)

Thus the tangent 1-jet j_J of the integral-jet space is a **decomposable** 1-jet in the Grassmanian J, where the latter concept is defined (for arbitrary Grassmanians, not necessarily projective spaces) by the following requirement: the subspace of the tangent space T_J in direction j_J decomposes into a tensor product of subspaces relative to the canonical structure of a tensor product on T_J. What is more, we have seen that j_J is an **axially decomposable** 1-jet in the Grassmanian, meaning that in the above tensor product of subspaces the left factor is not a proper subspace.

Remark III.23 Let us observe that in case $y = 1$, $dim\, T = 2$ the space

$$E^{ti_\bullet as} \risingdotseq E^{vd''_* vd'_\bullet as} \risingdotseq (\, E^{vd'} + E^{vd''}\,)_{\bullet[**]} = (\, E^{vd'} + E^{vd''}\,)_{\bullet[**]}$$

(of antisymmetric or, equivalently, antisymmetrized sums of first-trace-part and second-trace-part connectors) is equal to the whole of the vector line $E^{as} \risingdotseq T \otimes T^{\wedge 2}$ of (antisymmetric connectors). (On the other hand, it is clear that for any curve-directional conic structure all antisymmetric connectors are complementary.)

Chapter IV

Hypersurface-Directional Conic Structures

When the general theory of the previous chapters is applied to conic structures and preconnections with integral directions of codimension one (i.e. to the case $y = 1$), the invariants introduced in the general case assume a more concrete form. Such consideratons will actually exploit only the already observed (in Remark III.22) fact that in the case $y = 1$ (but not only then) the tangent spaces to the integral-jet spaces J^ϵ have a very simple description (more precisely, their directions are axially decomposable jets in the Grassmanian, v. the above cited remark).

However, a more important feature of the case $y = 1$ is that some new peculiar constructions arise naturally. More precisely, the theory of integrable conic structures, which is clearly more basic (since no 2-structure has been chosen) and more general (since the conic structure is possibly not conjunctively *integrable) than the theory of integrable preconnections (i.e. structures of geometrical parameter manifolds of submanifolds), turns out to have by itself in this case a twistorial counterpart. In fact, the generality of this construction is enhanced by the well-known fact that integrability of a* hypersurface-directional *generalized conic structure follows already from 1-prolongability. (However, in our exposition this fact will not be taken for granted - its proof will be given in the process of the construction of the twistorial equivalent of the 1-prolongable* *conic structure.)*

For these reasons, as well as for the fact that e.g. conformal and Veronese structures on manifolds can be thought of as hypersurface-directional conic structures, this chapter will be devoted to the case of codimension $y = 1$. In accordance with that, the terms 'conic structure' and 'preconnection' will in this chapter mean the more special notions 'hypersurface-directional conic structure' and 'hypersurface-directional preconnection', unless otherwise specified. The objective of this chapter will be to carry out the above mentioned twistorial constructions and obtain in this fashion the twistorial counterpart of (1-) prolongable conic structures. It will turn out that the latter is simply an 'geometrical' parameter space of Legendrian submanifolds of a contact manifold. *As in the case of integrable dispersions, reversiblility of this construction will enable us to build a structure with rich differential geometry from relatively simple holomorphic data, more precisely from a single* **compact Legendrian submanifold of a (complex) contact manifold** *(satisfying certain mild conditions). This could be considered as a generalization and simplification of the results obtained by S. Merkulov in [14] (with stronger restrictions on the geometric structures in question and a more abstract description thereof) and R. Bryant (in a very special case of the situation studied by Merkulov, but with a concrete description of the geometric structure; v. Section 2 for more details.) Our construction in part relies on results from these articles: we construct a family of Legendrian submanifolds from a single submanifold by means of the 'Legendrian analogoue' of 'Kodaira's main theorem' proved in [14] (in full generality) and [3] (in a special case). However, our translation of the holomorphic data into geometric language is accomplished by completely different and more general methods of Manin's theory of conic structures.*

Our investigation in Section 1, which is of the most general character, will proceed through an application of invariantly formulated general principles of the theory of first-order PDEs (i.e. essentially generalized conic structures). More precisely, we will first reduce in a standard (elegant) way the integrability problem for a possibly not hypersurface-directional conic structure to the investigation of the Cartan distribution on the total integral manifold of jets *(i.e. on the submanifold R of the contact manifold $J.M$ determined by the conic structure). Then we introduce the assumption $y = 1$. As a result of that, we will be naturally led to reformulate (in a somewhat less standard way) prolongability and, more importantly, to deduce a twistorial interpretation of a prolongable (hypersurface-directional!) conic structure on a manifold. It will turn out that the latter* **is essentially the structure of 'a geometrical parameter space of (compact) Legendrian submanifolds'** *of a 'twistor space' Z (which will be defined as the quotient contact manifold of the above mentioned total integral manifold of jets R, where the fibers are the characteristics).*

As a result of that, the structure of a parameter space of Legendrian submanifolds is more elementary than the structure of a parameter space of arbitrary hypersurfaces. Indeed, a prolongable conic structure (which, as asserted above, is essentially the structure of a parameter space of Legendrian submanifolds) can under the much stronger assumption of conjunctive integrability *be realized, perhaps in many different ways, as a constituent part of an integrable pre-connection (which, according to Chapter 1 is essentially the structure of a parameter space of arbitrary hypersurfaces). The twistorial manifestation of that realization will turn out to be the following: the original family of Legendrian manifolds is simply the family of canonical lifts to the jet contact manifold of of the hypersurfaces from the subsequently chosen family. More succinctly, a prolongable conic structure can often be identified with the structure of an* **'geometrical' parameter space of canonical lifts of hypersurfaces 'given up to contact equivalence'.** *Of course, this fact has been long known (cf. [10], [9]) in the case of the structure of a 3-dimensional conformal manifold (which can in principle be realized as a constituent part of the structure of an Einstein-Weyl manifold, i.e. of a locally complete parameter set of embedded rational curves of self-intersection $x = 2$), in the case of the structure of an arbitrary conformal manifold (v. [11], [12]), and (a relatively recent result from [3] in the case of the structure of an 1-flat (or, equivalently - as we will see in Chapter V, prolongable) Veronese conic structure on a 4-manifold (which can in principle be realized as a constituent part of the structure of a locally* *complete parameter set of embedded rational curves of self-intersection $x = 3$).*

IV.1 Prolongable Hypersurface-Directional Conic Structures and Geometrical Parameter Spaces of Legendrian Submanifolds

In this section the starting point will be two questions which arise quite naturally from the results of previous sections, more concretely from the fact that the structure of a geometrical parameter space of submanifolds is essentially the same as an integrable preconnection (not necessarily a hypersurface-directional preconnection), and from the fact that the conic structure (not necessarily hypersurface-directional) underlying an integrable preconnection is prolongable. The questions, which make sense for conic structures of arbitrary codimension y, but are most interesting in the case $y = 1$, are as follows:

(a) Is there a 'twistorial equivalent' of a manifold equiped with a (not necessarily hypersurface-directional) prolongable manifold?

(b) If the answer to question (a) is positive, how does this twistorial equivalent in the situation when an integrable preconnection on the expanded conic structure is chosen relate to the twistorial equivalent of the integrable preconnection-equiped manifold (namely to the geometrical family of submanifolds) ? (More precisely, what additional structure on the twistorial counterpart of the prolongable expanded conic structure is needed in order to construct the geometrical family of submanifolds?)

The most important result of this section, namely Theorem IV.5, will completely resolve question (a) in the case of hypersurface-directional conic structures: it asserts that prolongability of a (hypersurface-directional) conic structure is equivalent to an equally natural, although slightly more complex, condition on the associated Cartan distribution, and that a prolongable (hypersurface-directional) conic structure on a given manifold is equivalent to the structure of an **'geometrical parameter space of Legendrian submanifolds '** *(of a contact manifold), which is to be defined precisely shortly. The question (b)* *will be addressed later on in this section - it will turn out that the additional structure is an open embedding of the contact manifold into the 'standard' jet contact manifold $J_S S (= \mathbf{P}(T^*_S) S)$ and that the geometrical family of Legendrian submanifolds is obtained from the geometrical family of hypersurfaces of S simply by lifting the hypersurfaces to the contact manifold .*

Our immediate objective is a rigourous statement of Theorem IV.5. However, before this we will have to define the above mentioned concepts, as well as some concepts pertaining to the apparatus necessary for the proof of the theorem .

Remark IV.1 Consider the situation of Remark VI.1 from the last chapter, but with the additional assumption that $y = 1$; in other words, we assume that M_{csm} is a (hypersurface-directional-) expanded conic structure. (In particular $J = \mathbf{P}(\mathbf{T}^*)$.) Let us observe that now the general facts $dim J = xy$ and $b, h \leq xy$ from that remark imply that $dim\ J = x$ and $b, h \leq x$. According to the same remark, the Cartan distribution $F^{\alpha_o} R$ on R is now (fiberwise) one-codimensional . Thus its Frobenius tensor $fr_R \in Hom((F^{\alpha_o})^{\wedge 2}, T^{/\alpha})$ at a point r is a line valued alternate bilinear map on F^{α_o}, i.e. the structure on F^{α_o} of a conformal possibly degenerate symplectic vector space. Similarly, F is in this situation a conformal symplectic vector space with respect to the Frobenius tensor $fr \in Hom(F^{\wedge 2}, T^{/\alpha})$ of the structural distribution in the contact manifold JM at the point r. In particular, F is canonically its own conformal dual (i.e. dual modulo a choice of a base vector of the line $T^{/\alpha}$). For this reason the symbol F^* will be reserved for this conformal dual (thus $F^* = F$). According to (the above cited) Remark VI.1, the subspace F^{α_o} of F is h -codimensional and consequently its perpendicular $F^{*\alpha_o}$ in the conformal dual F^* (which is of course well-defined, i.e. independent of a base vector of the line $T^{/\alpha}$) is h -dimensional .

In the statement of Theorem IV.5 we will also need some concepts occuring in the standard structural theorem of 'constant rank' distributions of tangent hyperplanes. Therefore we next recall the statement of this theorem (the proof of which is long, but as straightforward as the proof of the Frobenius theorem;

in fact both proofs are most naturally carried out together):

Remark IV.2 i) Suppose a manifold R is equiped with a fiberwise one-codimensional distribution FR. The Frobenius tensor $fr_R \in Hom(F^{\wedge 2}, T_R/F)$ of this distribution at a point r is a line valued alternate bilinear map on F, i.e. the structure on F of a conformal possibly degenerate symplectic vector space. The rank of the given distribution at the point r is defined as the rank ρ of this conformal possibly degenerate symplectic vector space, i.e. the rank of the Frobenius tensor (of course. this is an even number). Recall that the defining property of the rank is the following: if the kernel (or 'radical') of the Frobenius tensor is denoted by $F^{\delta\rho}$, then the quotient $F^{/\delta\rho} := F/F^{\delta\rho}$, which is endowed with the structure of a symplectic vector space defined by descent of fr, has dimension ρ (in particular $dimF = \delta + \rho$, where δ denotes the 'defect', i.e. the dimension of the kernel).

(ii) In the situation of (i) suppose that the distribution is of a constant rank ρ. Therefore the kernels (or 'radicals') $F^{\delta\rho}$ of the Frobenius tensor for various points r form a vector subbundle $F^{\delta\rho}R$ of FR, which is called the **characteristic distribution** on R for the given distribution. This distribution is always integrable. Its leaves are called **characteristics** of the distribution FR. When the foliation by characteristics is a fibration (which is always true locally, of course), the quotient manifold Z (i.e. the base of the fibration) has a unique structure of a contact manifold such that the original distribution FR is precisely the pull back of the structural distribution $T_Z^{\alpha}Z$ of tangent hyperplanes in the contact manifold Z. Furthermore, the Frobenius tensors of FR and $T_Z^{\alpha}Z$ are related as described in (i).

Next we proceed with the definitions of other concepts which will occur in the statemaent of Theorem IV.5:

Definition IV.3 *The* **submersed normal bundle** *of a Legendrian immersed manifold Z^{τ} in a (general possibly not hypersurface-) contact manifold Z is by definition the restriction $T_Z^{/\alpha}Z^{\tau}$ to Z^{τ} of the contact-structural integral-transverse bundle $T_Z^{/\alpha}Z$ (i.e. the quotient of the tangent bundle by the contact-structural integral-tangent bundle $T_Z^{\alpha}Z$). Let us observe that the submersed* *normal bundle is in a canonical way a quotient of the normal bundle (since it is a coarser quotient of the restricted tangent bundle of Z), where the kernel of the projection to the quotient bundle is precisely the distinguished subbundle*

$$\frac{T_Z^\alpha}{T_Z^\tau} Z^\tau = T_Z^{\alpha/\tau} Z^\tau$$

of the normal bundle.

Definition IV.4 *A* **geometrically amenable (or geometrical) family of Legendrian immersed manifolds** Z^τ **in a** y **-codimensional- contact manifold** Z *with parameter space* M *is a family of immersed manifolds with the following two properties:*

(a) The immersed manifolds in Z *from the family are Legendrian.*

(b) For each point m *we first construct the* **canonical representation of the tangent space** T **at** m **as a parameter vector space of sections for the submersed normal bundle** $T_Z^{/\alpha} Z^\tau$ **of the manifold** Z^τ **with parameter** m *from the canonical representation of* T *as a parameter vector space of sections for the normal bunlde of* Z^τ *in the obvious way (i.e. using the fact that the submersed normal bundle is a quotient of the normal bundle); the requirement is that for each* m *the tangent space* T *be a geometrical parameter vector space of sections for the submersed normal bundle; thus the vector space* T *inherits a conic structure.*

These vectorial conic structures (with y *-codimensional integral directions) on the tangent spaces at various points of* M*, obviously form a conic structure on* M*. (The fact that we indeed obtain an embedding of the incidence-relation manifold* $R = Z^{sn}M =:$ *(by def)* $J^e M$ *into* JM *is proved like Proposition I.11, using compactness of the manifolds* J^e*.) We say this structure is* **induced** *by the given geometrical family of Legendrian submanifolds.*

Now we are at last in position to state Theorem IV.5 rigourously:

Theorem IV.5

(i) *Let us recall the following fact proved in Chapter 3: For any point* $m \in M_{csm}$ *of a manifold equiped with a possibly not hypersurface-directional conic* *structure, whose underlying manifold is denoted by M and (global) integral-jet space by R, the (below stated) conditions (b) and (c) are mutually equivalent, and they are weaker than the (below stated) condition (a).*

(a) Integrability at m (in the sense defined in previous chapters, i.e. as 'abundance of integral submanifolds through m';)

(b) Prolongability at m (this means by definition that arbitrary integral jet j is associated as the tangent first-order jet with some tangential *second-order jet; explicitly, for each point $r = j.m \in J^{\epsilon}.m \subset J^{\epsilon}M = R$ (i.e. each integral jet in M_{csm} at the point m) there exists a second oder jet $j^{(2,1)}$ at m of a hypersurface of M such that as a jet in the contact manifold JM, $j^{(2,1)}$ is in fact a jet in the submanifold R at the point r);*

(c) Intrinsic-pretorsion freedom at m.

(ii) *For any (*hypersurface-directional*) conic structure on a manifold the following conditions are equivalent:*

(a) Integrability (at each point of M)

(b) Prolongability (at each point of M)

(iii) *For any prolongable (hypersurface-directional) conic structure on a manifold the Cartan distribution on the integral-jet space has constant rank (according to Remark IV.2 this implies that the foliation of the integral-jet space by characteristics is well-defned). More precisely, the value of the rank equals $2m$, where b is the dimension of the integral-jet space (i.e. of the space of all integral jets at an arbitrary point).*

(iv) *Let M be an arbitrary manifold . There is an invariant bijective correspondence between*

(a) Structures on M of a prolongable (hypersurface-directional-) expanded conic structure M_{csm} such that the foliation (which is well-defined according to (iii)) of the total integral manifold of jets R by the characteristics of the Cartan distribution is in fact a fibration and its projection maps the submanifolds $J^{\epsilon}.m$ of R injectively into its base (i.e. the space of characteristics)

and

(b) Structures on M of a geometrical parameter space of Legendrian submanifolds (of an arbitrary contact manifold).

More precisely, this correspondence will be described in Proposition IV.10, while the inverse correspondence is precisely the one introduced immediately in the definition of Legendre geometric amenability (namely Definition IV.4).

[The proof of this theorem is postponed - essentially it will be given as a sequence of simpler propositions. QED]

The above theorem could be less rigourously restated as follows: a geometrical parameter space of Legendrian submanifolds is essentially the same as a prolongable expanded conic structure M_{csm} with a certain global property, namely the space of characteristics has to be a well-defined (Hausdorffœ50z) manifold and the jet space at an arbitrary point is mapped injectively into that manifold. We will soon see that any prolongable expanded conic structure is locally of that form.

Remark IV.6 Just like Proposition II.13 from Chapter 2 (on twistorial interpretation of integrable preconnections), the above theorem could be formulated (and proved in exactly the same way) in a somewhat more general version. More precisely, we could consider a larger class of prolongable manifolds, namely, we do not have to require that the spaces $J^e.m$ be mapped injectively into the space of characteristics. The twistorial description of this geometric structure turns out to be the following: the underlying manifold is equiped with the structure of a geometrical parameter space of Legendrian immersed manifolds in an arbitrary contact manifold S.

However, from the point of view of local differential geometry, this version is in fact not more general: as we have already observed in Chapter 2, any prolongable conic structure locally satisfies even the stronger conditions from the previous theorem .

Proposition IV.7 *Let us consider the situation of Remark VI.1. (In other words, we are given a y-codimensional conic structure on a manifold, where y is arbitrary.) The Frobenius tensor $fr_{_R} \in Hom((F^{\alpha_o})^{\wedge 2}, T^{/\alpha})$ of the Cartan* *distribution in the manifold R at a given point r is the restriction of the Frobenius tensor $fr \in Hom(F^{\wedge 2}, T^{/\alpha})$ of the structural distribution in the contact manifold JM at the same point R.*

Proof: *We first observe that the inclusion map $R \to JM$ is in fact a mapping of distribution-equiped manifolds (meaning that at each point of the domain the differential maps the fibre of the distribution on the domain into the fibre of the distribution on the codomain; in our case this is simply the obvious fact $F^{\alpha_o} \subset F$). It remains to apply this general fact:*

If $f : (M_1, T_1^\alpha M_1) \to (M_2, T_2^\alpha M_2)$ is a mapping of distribution-equiped manifolds, then the Frobenius tensors $fr_i \in Hom(T_i^{\alpha\wedge 2}, T_i^{/\alpha})$ at points m_i ($i = 1, 2$) such that $f(m_1) = m_2$ are related in the following way:

$$f_* fr_1(v_1, v_{r1}) = fr_2(f_* v_1, f_* v_{r1}),$$

where both the differential $T_1 \to T_2$ and its action $T_1^{/\alpha} \to T_2^{/\alpha}$ are denoted by f_*. (Of course, this fact is an easy consequence of the definition of the Frobenius tensor and the standard fact about the Lie bracket of vector fields which are f -related to some fields in the domain.)

QED

Definition IV.8 *A (hypersurface-directional) conic structure on a manifold M is said to be* **coisotropically prolongable** *at a point m if at each integral jet j in M through m (i.e. for each $r = j.m \in J^c.m \subset JM$ in notation of Remark IV.1) the corresponding fibre of the Cartan distribution is coisotropic as a subspace of the contact-structural integral-tangent conformal symplectic vector space. In notation of Remark IV.1 this condition explicitly means that for all such r, F^{α_o} is coisotropic in F, i.e. its perpendicular $F^{*\alpha_o}$ is a (totally) isotropic space or, equivalently (in view of finite-dimensionality), F^{α_o} contains its perpendicular $F^{*\alpha_o}$. We will soon see that the term 'coisotropically prolongable' is justified (meaning suggestive) in the sense that this is actually a property of the first-order structural jet expanding the localized conic structure (to a second-order localized geometric structure), and it is weaker than flatness of this structural jet.*

Proposition IV.9 *Let us consider the situation of Remark IV.1; in other words, we are given a (hypersurface-directional) conic structure on a manifold . We will also use notation and facts expounded in the more general Remark IV.2.*

(i) *At each point $r \in R$ the corresponding space F^{α_o} (which was defined as the fiber of the Cartan distribution), $F^{*\alpha_o}$ (which was defined as its perpendicular in the standard conformal symplectic dual $F^* = F$) and the kernel $F^{\delta\rho}$ of the conformal possibly degenerate symplectic vector space F^{α_o} (defined in Remark IV.2) are related in the following way:*

$$F^{\delta\rho} = F^{\alpha_o} \cap F^{*\alpha_o}.$$

In particular, we have the following upper bound of the defect δ (which, just like the rank ρ, in general depends on the point r) of the Frobenius tensor fr_R at r:

$$\delta \leq h \leq (\textit{according to Remark IV.1}) \leq xy = x \leq b + x.$$

(In other words, the defect has a finer upper bound -namely h - than $b + x$. This bound is clearly 'universal' in the sense that it does actually not depend on the concrete expanded conic structure, but only on the dimensions of M and J^ε.)

[Proof of (i): The assertion follows almost immediately from (previous) Proposition IV.7: indeed, according to that proposition, the conformal possibly degenerate symplectic product fr_R of F^{α_o} is simply the restriction of the conformal symplectic product fr of F; it remains to recall that the kernel of fr_R is by definition the perpendicular of the whole of F^{α_o} with respect to fr_R. QED]

(ii) *In order to obtain an estimate of the genre $\delta + \frac{\rho}{2}$ of the Frobenius tensor fr_R at some point r let us consider the following sequence of trivial equalities:*

$$\textit{dim } F = xy + x = 2x = b + x + h = \delta + \rho + h = \delta + \rho + \delta + (d - \delta) = 2(\delta + \frac{\rho}{2}) + (d - \delta). \tag{IV.1}$$

Hence we conclude that the non-negative (according to (i)) integer $d - \delta$ is even. Therefore we have

$$u = \frac{1}{2}(d - \delta) + \delta + \frac{\rho}{2} \quad \textit{and} \quad x \geq \delta + \frac{\rho}{2}. \tag{IV.2}$$

(In particular the genre $\delta + \frac{\rho}{2}$ has a finer upper bound -namely x - than $b + x$. Clearly, this bound is also 'universal' in the sense described in (i).) Hence and from the equality $b + x = \delta + \rho$ we infer

$$c \leq \frac{\rho}{2}. \tag{IV.3}$$

[The proof of (ii) has already been carried out.QED]

(iii) *Let us recall from the Remark IV.2 that for any one-codimensional distribution the rank ρ of the Frobenius tensor at a point is lower than at another (i.e. the defect δ is higher) iff the genre $\delta + \frac{\rho}{2}$ is higher. In view of this fact and assertions (i), (ii), it is clear that for a point $r \in R$ the following four properties are equivalent:*

(a) The genre $\delta + \frac{\rho}{2}$ at r precisely equals its ('universal') upper bound x.

(b) The defect δ at r precisely equals its ('universal') upper bound h.

(c) The fiber F^{α_o} of the Cartan distribution at r is coisotropic in the conformal symplectic vector space F, i.e. $F^{*\alpha_o} \subset F^{\alpha_o}$.

(d) The rank ρ at r precisely equals its ('universal') lower bound $2m$.

[**(iii)** has already been proved .QED]

(iv) *The perpendicular $F^{*\alpha_o}$ of the fiber F^{α_o} of the Cartan distribution at a point r (in the standard conformal dual $F^* = F$ of $F \subset T_{JM}$) is transverse to the tangent space T_{J^ϵ} at r of the corresponding fiber J^ϵ hookrightarrow$R = J^\epsilon M$. Explicitly*

$$F^{*\alpha_o} \cap T_J^\varepsilon = 0.$$

Proof of (iv): Is is a standard fact that $J.m$ is a Legendrian submanifold of the contact manifold JM. Therefore $T_J = F^{jx}$ is a Lagrangian subspace of the conformal symplectic vector space F, i . e it coincides with its perpendicular F^{*jx}. In particular, the conformal duality of $F^* = F$ and F descends to a conformal duality of the subspace T_J and the quotient space $F^{/jx} := \frac{H}{T_J} =$ (obviously) $= T^\alpha$. Thus no vector from T_J^ε different from zero annihilates the whole of T^α. It remains to observe that the image of F^{α_o} in the quotient space T^α is the whole of T^α (since $T_R \subset T_{JM}$ projects onto the whole of T).

(v) *For a point m and a point $r \in J^\epsilon.m \subset J^\epsilon M = R$, coisotropy of the corresponding fiber F^{α_o} of the Cartan distribution (or any of the other three* *equivalent properties of r from (iii), of course) is equivalent to the existence of a tangential symmetric elementary preconnection $j^{(2,1)}$ at m in contact with r (explicitly, this tangentiality means that as a jet at r in the contact manifold JM, $j^{(2,1)}$ is in fact a jet in the submanifold R).*

[Proof of (v): First we recall the fundamental fact which relates the symmetric elementary preconnections and the canonical structure of a conformal symplectic vector space on the contact-strucural hyperplane $F \subset T_{JM}$ at r (v. Appendix):

A subspace F^{τ} of F whose direction is an elementary preconnection $j^{(2,1)}$ (i.e. a x -dimensional subspace transverse to the fiber J) is isotropic in F iff $j^{(2,1)}$ is a symmetric elementary preconnection .

Therefore our present objective is to investigate when there exists such a space F^{τ} which is also contained in T_R (or, equivalently, in F^{α_o}). With this purpose we recall the assertion (i) from Remark IV.2, namely the interpretation of the genre $\delta + \frac{\varrho}{2}$ as the dimension of the maximal isotropic spaces in F^{α_o}.

From this and assertion (iii) it obviously follows that coisotropy of F^{α_o} is a necessary condition for existence of a symmetric elementary preconnection with prescribed properties.

In order to prove sufficiency, we assume F^{α_o} is coisotropic . We only have to prove that among maximal (clearly x -dimensional) isotropic spaces of F^{α_o} there exists at least one transverse to the fiber J, or, equivalently, transverse to $J^{\varepsilon} \subset J$. To see this, we will need some additional facts from Remark IV.2 (i):

The conformal possibly degenerate symplectic product fr_R on F^{α_o} descends to a symplectic vector product on the quotient $F^{\alpha_o}/H^{*\alpha_o} =: T_Z^{\alpha}$ (which clearly has dimension $2m$ in our case) and the maximal isotropic spaces F^{τ} in F^{α_o} are precisely the preimages of the Lagrangians (which are obviously b -dimensional in our case).

In view of these facts, the proof of existence of F^{τ} with required properties has clearly been reduced to the proof of existence of a Lagrangian subspace $T_Z^{\alpha\,kc}$ (of the quotient symplectic space T_Z^{α}) whose preimage F^{τ} in F^{α_o} is transverse to T_{J^ϵ}. In order to find a Lagrangian $T_Z^{\alpha\,\tau}$ satisying this condition, we first observe that such a space should necessarily be transverse to the image of T_{J^ϵ}. We claim that this necessary condition on a Lagrangian $T_Z^{\alpha\,kc}$ (namely transversity to the image of T_{J^ϵ}) is in fact also sufficient: indeed, if this necessary condition is fulfilled, the preimage of $T_Z^{\alpha\,kc}$ obviously intersects T_{J^ϵ} precisely in $T_{J^\epsilon} \cap F^{*\alpha_o}$, which according to (iv) is zero . Thus, it remains to prove existence of a Lagrangian $T_Z^{\alpha\,kc}$ satisfying the above necessary condition (i.e. transversity to the image of T_{J^ϵ}).

In order to accomplish that we first observe that as a consequence of (iv) (i.e. of transversity of T_{J^ϵ} and $F^{*\alpha_o}$) the space T_{J^ϵ} projects bijectively onto its image which we denote by $T_Z^{\alpha\,\tau}$ in the quotient symplectic space F^ζ. Since T_J is Legendrian in F (v. Appendix), T_{J^ϵ} is isotropic in F^{α_o} and consequently $T_Z^{\alpha\,\tau}$ is isotropic in T_Z^α; moreover, it is Lagrangian because of its dimension (or, more conceptually, since it obviously coincides with its perpendicular). Therefore there indeed exists at least one Lagrangian $T_Z^{\alpha\,kc}$ in T_Z^α (actually a whole affine space of them, as we shall see later on) transverse to the Lagrangian $T_Z^{\alpha\,\tau}$. QED

(vi) *For a point m and a point $r \in J^\epsilon.m \subset J^\epsilon M = R$, the below stated condition (a) implies (as we have already observed at the beginning of this section) the below stated condition (b) . Claim: the converse is true under the additional assumption of constancy of rank of the Cartan distribution in a neighbourhood of r in R.*

(a) There exists an integral submanifold of the expanded conic structure through the point m in contact with r.

(b) Coisotropy of the fiber F^{α_o} at r of the Cartan distribution or, equivalently (according to (v)), existence of a tangential symmetric elementary preconnection $j^{(2,1)}$ at m in contact with r.

[Proof of (vi): Suppose the rank of the Cartan distribution $F^{\alpha_o}R$ (i.e. the rank of the Frobenius tensor $(fr_R)_{r'}$) is constant in some neighbourhood of $r \in R$ and the condition (b) is fulfilled. We recall that according to the Remark IV.2, constancy of the rank implies the following:

A sufficiently small open neighbourhood R^ω of R is fibrated by characteristics, i.e. leaves of the characteristic distribution (which is by definition formed by kernels of the Frobenius tensor at various points). The base Z of this fibration has dimension equal to the rank and carries a unique structure of a contact manifold such that the Cartan distribution $F^{\alpha_o}R$ is the pull-back of the structural distribution $T_Z^{\alpha}Z$ (of hyperplanes) in the contact manifold . This implies that the conformal symplectic vector space T_Z^{α} defined as the structural hyperplane of the contact manifold at the image z of r equiped with the Frobenius tensor, is precisely the quotient conformal symplectic vector space of the conformal possibly degenerate symplectic vector space F^{α_o} (whose structure is by definition the Frobenius tensor of the Cartan distribution). The highest-dimensional integral manifolds of the Cartan distribution on R^{ω} have dimension equal to the genre and they are precisely open subsets of the preimages in R^{ω} of Legendrian submanifolds of Z.

As a result of the condition (b) and constancy of the rank, the characteristic distribution coincides with the distribution $F^{*\alpha_o}R^{\omega}$ of perpendiculars of the fibers of the Cartan distribution. (This follows easily from the inclusion of the kernels of the Frobenius tensor in those perpendiculars -v. assertion (i) - and the obvious fact that the characteristic distribution on R^{ω} is fiberwise h -dimensional.) In addition to that, it is obvious that Legendrian submanifolds of Z are b -dimensional and their preimages in R^{ω} are x -dimensional . In view of this and Remark IV.1, it remains to prove existence of a Legendrian submanifold Z^{α} through the image z of r in Z such that its preimage admits an open in R^{uj} containing r and projecting biholomorphically onto a submanifold M^{α} of M. (Indeed, M^{α} will then be an integral submanifold of the expanded conic structure with required properties.)

Obviously, the above condition on a Legendrian submanifold Z^{α} through z can be formulated more simply like this: its preimage in R^{ω} has to be transverse to the fiber $J^{\varepsilon}\, inR$ at the point r. By definition, i.e. on the level of tangent spaces , this condition precisely means that the tangent space $T_Z^{\alpha\,\tau}(uc)$ of Z^{α} at z (which is clearly a Lagrangian in the quotient symplectic vector space T_Z^{α} of F^{α_o}) has the following property: its preimage F^{τ} in F^{α_o} has to be transverse to $T_{J^{\varepsilon}}$. However, in the proof of (v) we saw that this condition on the Lagrangian $T_Z^{\alpha\, kc}$ is equivalent to its transversity to the Lagrangian $T_Z^{\alpha\,\tau}$ (which was defined there as the image of T_{J^ϵ} in T_Z and proven to be Lagrangian).

Therefore, it remains to prove existence of a Legendrian submanifold Z^α through z whose tangent space $T_Z^{\alpha\, kc}$ at z is transverse to the Lagrangian $T_Z^{\alpha\,\tau} \subset F_{zeta} \subset T_Z$. But this folows easily from the fundamental Darboux theorem on the local structure of a contact manifold .

(Incidentally, in this situation $Z^\tau := J^\epsilon \cap R^\omega$ is immersed into Z (because of (iv)) and thus it is a Legendrian immersed manifold in Z through z with tangent space $T_Z^{\alpha\,\tau}$. Therefore the above condition on the Legendrian submanifold Z^α is precisely transversity at z to the Legendrian immersed manifold Z^τ in Z.) QED

Proof *of Theorem IV.5 (i), (ii), (iii):*
To prove assertion (i) (i.e. equivalence of existence of compatible symmetric elementary preconnections at a given point in all integral directions and of intrinsic-intrinsic-pretorsion freedom at that point) it suffices to apply assertion (v) of the previous proposition to each integral direction at the given point.

Assertion (iii), namely the fact that in the case of a prolongable expanded conic structure the rank of the Frobenius tensor of the Cartan distribution has the constant value $2m$*, follows immediately from assertion (iii) of the previous proposition.*

In order to prove assertion (ii), namely equivalence (globally) of integrability and prolongability, we first notice that in view of (i) integrability is (even on the local level) a stronger condition. Therefore, let us suppose prolongability of the expanded conic structure M_{csm} *(at each point). According to (iii) the rank of the Frobenius form of the Cartan distribution is constant. In particular, for any point* $r \in R$*, it is locally constant at* r*, and thus we may apply assertion (vi) of the previous proposition to* r*. QED*

Now we are at last in position to construct a geometrical family of Legendrian submanifolds from a prolongable expanded conic structure satisfying *certain global conditions (i.e. to define the correspondence from (the main) Theorem IV.5). In fact, the results proved so far make it possible to prove with relatively little effort a weaker version (namely assertion (v) of the following proposition) of the main and the only not yet proved assertion in Theorem IV.5 (namely assertion (iv) of that theorem):*

Proposition IV.10

(i) *Let us consider the situation of Remark IV.1 with the additional assumption of prolongability. In other words, we are given a prolongable (hypersurface-directional-) expanded conic structure, where notation is as in that remark .*

Since the rank of the Frobenius tensor equals $2m$ *at each point (by the assertion (iii) of the theorem), Remark IV.2 implies existence of the foliation of* R *by characteristics. According to the assertion (iv) from the previous proposition, the characteristics are transverse to the fibers of* $J^{\varepsilon}M = R$*. Consequently, they are (with respect to the restricted projection into* M*) immersed topologically possibly non-countable* h *-dimensional manifolds in* M*. In accordance with that, the foliation (or 'set-theoretical fibration') of* R *by characteristics will be denoted by* $M^qZ(=R)$*. Explicitly, such a characteristic will be denoted by* M^q*, while the 'set-theoretical space' of characteristics will be denoted by* Z*.*
[Proof of assertion (i) has already been given.QED]

(ii) *Let us consider the situation of (i) with this additional assumption: the foliation* M^qZ *of* R *is in fact a fibration. (Thus,* Z *has the structure of a quotient manifold of* R*, while* M^q *are immersed manifolds in* M*.) The above quoted assertion on transversity (from the previous proposition) precisely means that the fibrations* $J^{\varepsilon}M$ *and* M^qZ *of* R *form an immersional double fibration. In other words, the family (with parameter space* Z*) of possibly non-compact immersed manifolds* M^q *in* M *can alternatively be interpreted as a family (with parameter space* M*) of immersed manifolds* Z^{τ} *in*

Z, where Z^τ is defined to be J^ϵ (i.e. the fibration $J^\epsilon M$ has been denoted more suggestively by $Z^\tau M$).

Let us recall that (according to Remark IV.2) Z also has a unique structure of a contact manifold such that the Cartan distribution $F^{\alpha_o}R$ on R is the pull-back of the structural distribution $T_Z^\alpha Z$ in the contact manifold Z.

Claim: The above family (with parameter space M) of immersed manifolds Z^τ in Z is Legendre-geometrical and the conic structure on M induced in this way (v. Definition IV.4) coincides with the given conic structure. Explicitly, if m is an arbitrary point and T denotes the tangent space at m, then the compact immersed manifold Z^τ in Z with parameter m is Legendrian, the parameter vector space T of sections of the integral-transverse (line) bundle $T_Z^{/\alpha} Z^\tau$ of Z^τ is geometrical (with respect to the canonical representation introduced in Definition IV.4 of that space as a parameter vector space of sections), and the vectorial conic structure on T induced by this representation is precisely the one constituent for the given structure of an odered manifold .

[Proof of (ii): The fact that Z^τ is Legendrian in Z has already been observed at the end of the proof of the assertion (vi) of Proposition IV.9: indeed, this stuation is a special case (namely the case when $R^\omega = R$) of the situation considered there. The remaining statements to be proved can clearly be formulated still more explicitly like this: for any given $r \in J^\epsilon.m \coloneqq Z^\tau.m$ the inverse in T_R (with respect to the differential of the projection) of the contact-strucural hyperplane $T_Z^\alpha \subset T_Z$ is mapped by the differential of the projection $R \to M$ precisely onto the hyperplane $T^\alpha \subset T$ in direction r. But this is indeed the case since that inverse is clearly $F^{\alpha_o} \coloneqq F \cap T_R$. QED

(iii) *Let us now further specialize the situation of (i): in addition to the assumption from (ii) we make the following assumption: the immersions of manifolds Z^τ are injective (for all m), i.e. (because of their compactness) embeddings. In other words (according to . ..), we suppose that the immersional double fibration of R defined above (i.e. formed by the fibrations $R^q Z$ and $Z^\tau M$) is in fact a double fibration. In particular, it can be thought of as a* *family of of possibly non-compact* $(y+d)$ *-codimensional manifolds* $M^{\alpha'}$ *in* M *or as a family of* $(y+d)$ *-codimensional manifolds* Z^{τ} *in* Z. *The latter family is by assertion (ii) in fact a geometrical family of Legendrian manifolds in the contact manifold* Z.

[Proof of (iii) has already been given.QED]

(iv) *Let* M *be an arbitrary manifold .*

In (iii) we have clearly defined an invariant corespondence between

(a) Structures on M *of a prolongable (hypersurface-directional-) expanded conic structure* M_{csm} *such that the foliation of the total integral manifold of jets* R *by the characteristics of the Cartan distribution is in fact a fibration and its projection maps the submanifolds* $J^{\epsilon}.m$ *of* R *injectively into its base (i.e. the space of characteristics)*

and

(b) Structures on M *of a geometrical parameter space of Legendrian submanifolds (of an arbitrary contact manifold).*

On the other hand, immediately in the definition of Legendre geometric amenability (namely Definition IV.5) we had introduced an invariant correspondence between structures on M *of a geometrical parameter space of Legendrian submanifolds and structures of an expanded conic structure. Clearly, in (iii) it is in fact claimed that the latter correspondence is a left inverse of the former correspondence. In particular, the former correspondence is injective.*

[As indicated above, (iv) has already been proved .QED]

(v) *(A weaker version of Theorem IV.5 (iii))*

When we restrict the codomain of the invariant injective correspondence from (iv) to its image, we obtain a (clearly) bijective invariant correspondence between *(a) Structures on M of a prolongable (hypersurface-directional-) expanded conic structure M_{csm} such that the foliation (which is well-defined according to Theorem IV.5 (iii)) of the total integral manifold of jets R by the characteristics of the Cartan distribution is in fact a fibration and its projection maps the submanifolds $J^{\varepsilon}.m$ of R injectively into its base (i.e. the space of characteristics)*

and

(b) Structures on M of a geometrical parameter space of Legendrian manifolds Z^{τ} in an arbitrary contact manifold Z satisfying the below stated condition (b1), which is stronger than the below stated condition (b2)

(b1) The given family of manifolds in Z is compound-geometrical (clearly, this condition does not involve the given structure of a contact manifold on Z).

(b2) The induced conic structure on M is prolongable.

Furthermore, the inverse correspondence is the appropriate restriction of the correspondence introduced in the definition of Legendre geometric amenability (namely Definition IV.4).

[Proof of (v): Clearly, the only new assertion in (v) is the characterization of those structures of geometrical parameter spaces of Legendrian submanifolds which belong to the image of the correspondence from (iii) by the condition (b1). In fact, we know already that this condition is necessary. Therefore it remains to prove its sufficiency.

Let us assume that (b1) holds. Define on the total integral manifold of jets $R = Z^{\tau}M$ a distribution $F^{\alpha_\circ}R$ by pulling back from Z the structural hyperplane distribution $T^{\alpha}_{Z}Z$ in the contact manifold (this is a well-defined distribution on R of indicated fiber dimension since the projection $R \to Z$ is submersive and the dimension of $R = Z^{\tau}M$ is $b + dim\ M = b + x + y$). The structure on M of an expanded conic structure induced by the given family of Legendrian submanifolds has been defined essentially by the requirement (v. Definition IV.4) that this distribution $F^{\alpha_\circ}R$ be its Cartan distribution, more precisely that the space $T^{\alpha} \subset T$ parametrized by a point r be the image of F^{α_o} (relative to the differential of the projection). Since the quotient Z of the distribution-equiped manifold R is a contact manifold, we conclude (v. Remark IV.2) that the Cartan distribution has constant rank and its characteristics are precisely the fibers $M^q.z$ of $M^qZ = R$. According to the same remark, the tangent spaces to these caracteristics are precisely the kernels of the Frobenius tensor. Since their dimension h (i.e. the defect of the Frobenius tensor) equals the 'universal upper bound for the defect' (v. Proposition IV.9), we conclude that the expanded conic structure M_{csm} is indeed prolongable and that the corresponding structure on M of a geometrical parameter space of Legendrian submanifolds is precisely the original one. QED

Clearly, the only difference between Theorem IV.5 and its above weaker version (i.e. assertion (v) of the last proposition) consists in the following: while in the theorem it is claimed that structures of prolongable expanded conic structures are in a bijective correspondence with structures of arbitrary *geometrical parameter spaces of Legendrian submanifolds, according to the weaker version they are in a bijective correspondence with structures of those geometrical parameter spaces of Legendrian submanifolds, which are also compound-geometrical. Therefore, the proof of the theorem will be completed when we prove Proposition IV.12 (which is to follow soon), namely the fact that compound-geometric amenability is a consequence of Legendre geometric amenability. However, for the proof of this proposition we will need a rather non-trivial lemma, although its proof is intuitively clear. Informally speaking, its purpose will be to investigate how infinitesimal variations (i.e. certain sections of the normal bundle) of a Legendrian submanifold within the space of ('unparametrized') Legendrian submanifolds (or, briefly, the* **Legendrian infinitesimal variations***) are distinguished among infinitesimal variations within the space of all submanifolds, and how the Legendrian infinitesimal variations are related to the induced sections of the submersed normal bundle. It turns out that these two problems are related: the Legendrian infinitesimal variations are characterized by their 'reconstructibility' from the induced sections of the submersed normal bundle. This 'reconstructibility' is precisely defined in terms* *of the invariant isomorphism (constructed in [11]) of the normal bundle and the jet vector bundle of the submersed normal bundle: the condition is simply that the section of the normal bundle be 'holonomic', i.e. that its graph be the canonical lift (to the appropriate jet contact manifold) of the graph of the induced section, of the submersed normal bundle. In the process of proving these facts one also obtains an intuitive (but 'less invariant') interpretation of the above mentioned completely invariant isomorphism: When the ambient contact manifold is locally represented as the jet contact manifold of some 'base manifold', then to a jet of (a section which is) an infinitesimal variation of the distinguished hypersurface in the base manifold this isomorphism assigns the normal vector (at the approprate point) belonging to the (plausibly defined) 'canonically lifted infinitesimal variation'.*

In fact, since we will be interested only in jets of sections at the points of their zero-loci (or, equivalently, in normal vectors belonging to the distinguished subspace), we will only partially prove the above stated facts (i.e. we will only prove results related to this particular type of jets). However, the proofs of the stronger versions could be carried out in a very similar fashion.

Lemma IV.11 *Let Z^τ be a Legendrian submanifold of a contact manifold Z. Notation from Definition IV.3 will be implied.*

(i) *Let us denote the (first-order) jet vector bundle of the submersed normal bundle $T_Z^{/\alpha} Z^\tau$ of Z^τ by $(\mathbf{P}^*_{(\sigma)} T_Z^{/\alpha})Z^\tau$. In other words, $\mathbf{P}^*_{(\sigma)}$ is the open submanifold of $\mathbf{P}^*_\sigma := J_{T_Z^{/\alpha} Z^\tau}$ consisting of those hypersurface directions at a generic point of the total space $T_Z^{/\alpha} Z^\tau$ which are transverse to the fibers, i.e. of jets of sections through that generic point.) We recall that its integral subbundle $\mathbf{P}^*_{(\sigma)} Z^\tau$ defined as the restriction of the bundle $\mathbf{P}^*_{(\sigma)}(T_Z^{/\alpha} Z^\tau)$ to the zero section $Z^\tau \subset T_Z^{/\alpha} Z^\tau$ (i.e. the subbundle whose total space consists of the jets of sections at the points of their zero-loci) is invariantly isomorphic to the bundle $Hom(T_Z^\alpha, T_Z^{/\alpha})Z^\tau$, where the isomorphism is given bythe (invariantly defined, cf. II.21) covariant differential of such a jet.*

Claim: This subbundle $\mathbf{P}^*_{(\sigma)} Z^\tau = Hom(T_Z^\alpha, T_Z^{/\alpha})Z^\tau$ is invariantly isomorphic to the distinguished subbundle $T_Z^{\alpha/\tau} Z^\tau$ of the normal bundle $T_Z^{/\alpha}$ (recall *that here $h = b$ and $T_Z^{\alpha/\tau}$ was defined as $T_Z^\alpha / T_Z^{\alpha\,\tau}$, where $T_Z^{\alpha\,\tau}$ is an alternative notation for the tangent space T_Z^τ of Z^τ). More precisely, this isomorphism is defined in the way described in the proof . Informally speaking, to a normal vector from the domain we assign the 'conformal linear covector' determined by that normal vector by means of the Frobenius tensor.*

[Proof of (i): We recall that at a given point $z \in Z^\tau$ the Frobenius tensor $fr_\zeta \in Hom(T_Z^{\alpha\,\wedge 2}, T_Z^{/\alpha})$ is a structure on T_Z^α of a conformal symplectic vector space (i.e. it is non-degenerate in a certain sense). Of course, the conformal duality between $T_Z^{\alpha\,*} = T_Z^\alpha$ and T_Z^α (i.e. the Frobenius tensor) descends to a conformal duality between some quotient $T_Z^{\alpha\,*/\tau}$ of $T_Z^{\alpha\,*}$ and the subspace F^{ck} of F tangent to Z^τ, where the kernel of the quotient map is precisely the perpendicular of the subspace F^{ck}. However, this perpendicular coincides with the subspace since the subspace, being tangent to the Legendrian submanifold Z^τ, is Lagrangian. Therefore the quotient $T_Z^{\alpha\,*/\tau}$ coincides with the quotient $T_Z^{\alpha/\tau}$. In other words, the descended conformal duality is a bilinear map belonging to the space

$$Hom(T_Z^{\alpha/\tau} \otimes T_Z^{\alpha\,\tau}, T_Z^{/\alpha}) \approx Hom(T_Z^{\alpha/\tau}, Hom(T_Z^{\alpha\,\tau}, T_Z^{/\alpha})).$$

Thus it gives rise to a map $T_Z^{\alpha/\tau} \longrightarrow Hom(T_Z^{\alpha\,\tau}, T_Z^{/\alpha})$. The latter is an isomorphism due to non-degeneracy of the bilinear map . It is clear that such isomorphisms for various points $z \in Z^\tau$ give rise to an isomorphism of bundles.QED]

(ii) *Suppose that Z is represented as the jet contact manifold $\mathbf{P}^*_S S$ of some manifold S such that Z^τ projects biholomorhically onto a hypersurface S^α in S (in particular Z^τ is the canonical lift of S^α by the main property of Legendrian manifolds in jet contact manifolds). Let us observe that in this situation the submersed normal bundle of Z^τ is canonically isomorphic to (and thus will be identified with) the normal bundle $T_S^{/\alpha} S^\alpha$ of S^α. (It should also be pointed out that the contact manifold $\mathbf{P}^*_\sigma T_Z^{/\alpha} Z^\tau$ from (i) is a special -namely 'flat'- case of $\mathbf{P}^*_S S$; indeed this is the case when $S = T_Z^{/\alpha} Z^\tau$.)*

Furthermore, suppose we are given a parameter space M of possibly non-compact hypersurfaces in S such that S^α belongs to that space. Let $m \in M$ be a parameter of S^α. The canonical lifts of hypersurfaces from the given family *obviously form a family of possibly non-compact Legendrian submanifolds of Z. It is also clear from the very definitions that the canonical (with respect to the family of possibly non-compact submanifolds of Z) representation of the tangent space T at m as a parameter vector space of sections for the submersed normal bundle of Z^τ coincides with the canonical (with respect to the family of possibly non-compact hypersurfaces of S) representation of T as a parameter vector space of sections for the normal bundle of S^α (i.e. a parameter vector space of infinitesimal variations of S^α).*

Claim: Let v be a tangent vector at m and s^α a point at which the infinitesimal variation of S^α with parameter v (i.e. the section $\{v_S^{/\alpha}\}_{S^\alpha}$ with parameter v of the normal bundle $T_S^{/\alpha} S^\alpha$) vanishes. According to the last observation the normal vector $v_Z^{/\alpha}$ at the corresponding point z^α belonging to the infinitesimal variation of Z^τ with parameter v, is contained in the distinguished subspace $T_Z^{\alpha/\tau}$ of the normal space $T_Z^{/\alpha}$. The assertion is that its image under the ('completely') invariant isomorphism from (ii) is precisely the jet at s of the infinitesimal variation of S^α.

(Thus the isomorphism from (i) has been given an alternative interpretation in the presence of a generic at least local identification of the contact manifold with a jet contact manifold . It is remarkable that this isomorphism does not depend on this identification.)

The proof of assertion (ii) will be postponed until after the statemnt of the proposition. QED]

(iii) *Let $\{v_Z^{/\alpha}\}_{Z^\tau}$ be an infinitesimal variation of Z^τ. Claim: In order for it to be Legendrian (i.e. to be an infinitesimal variation within the space of Legedrian submanifolds), the following necessary condition must be fulfilled:*

Let $\{v_Z^{/\alpha}\}_{Z^\tau}$ be the corresponding section of the submersed normal bundle (obtained by applying the quotient map from the normal bundle onto the submersed normal bundle). Furthermore, let $z \in Z^\tau$ be a point at which the section of the submersed normal bundle vanishes, i.e. for which the normal vector $v_Z^{/\alpha}$ belongs to the subspace $T_Z^{\alpha/\tau}$ of the normal space $T_Z^{/\alpha}$. The condition is that the image of that normal vector under the invariant isomorphism $T_Z^{\alpha/\tau} \to Hom(F^{ck}, T_Z^{/\alpha})$ (from (i)) be precisely the jet at z of the section of *the submersed normal bundle.*

[Proof of (iii): Let us consider a family of Legendrian manifolds in Z such that there exists a parameter m of the given Legendrian submanifold Z^τ for which the given infinitesimal variation belongs to the induced parameter vector space T of infinitesimal variations of Z^τ. Furhermore, let $v \in T$ be a parameter of that infinitesimal variation.

Since the above condition on the given infinitesimal variation is of a local nature, we may assume without loss of generality that Z is the jet contact manifold of some manifold S such that Z^τ projects biholomorhically onto a hypersurface S^α in S. For the same reason, the above introduced family of submanifolds may be assumed to coincide with the family of canonical lifts of hypersurfaces of S from some (clearly unique) family. Now it suffices to apply assertion (ii).QED]

[Proof of (ii): Since the assertion to be proved is of a local nature (with respect to the point $s \in S$), we may without loss of generality assume that the 'structure' on S of a product manifold $Y \times B$ can be chosen; in fact, a certain condition (to be specified soon) can be imposed on this 'structure' .

We will adopt the following conventions in order to simplify notation: The dimension of Y will be y, but will henceforth in this proof be denoted by $\mathbf{y}$ since the symbol y will be reserved for the points of Y. (In other words, when a manifold Y of dimension y is given, the symbol $\mathbf{y}$ will replace the symbol y with the meaning $dimY$.) Of course, the same conventions will hold for other symbols.

The above mentioned condition (which can clearly be fulfilled) on the product 'structure' is the following: If the points y and b are defined by the requirement $s^\alpha = (y, b)$, the slice $y \times B$ coincides (locally) with the submanifold S^α. In order to simplify the notation, we will in this way identify B with S^α in the obvious way (in particular the given point s^α coincides with b).

Since we have chosen on S the 'structure' of the total space of a fibration $YC := Y \times B$ over B, we can define an affine bundle $\mathbf{P}^*_{(S)}S = \mathbf{P}^*_{(S)}(YC)$ embedded as an open subbundle (of manifolds) into the projective bundle

$\mathbf{P}^*_S S = Z$; this affine bundle consists by definition of directions transverse to the fibers. The vector bundle associated to this affine bundle is clearly $Hom(T_B, T_Y)S$. In fact, since we have chosen a transverse fibration as well, this affine bundle has a distinguished section, and will thus be identified with the associated vector bundle. For instance, the direction of S^α at any of its points $s^\alpha_1 = (y, b_1)$ is clearly $(\mathbf{p}^*_{(S)_1}.(y, b_1)$, where $\mathbf{p}^*_{(S)_1}$ is the zero mapping.

Let us also observe that the possibly not submersive projection (defined by the given family of hypersurfaces) $(R = S^\alpha M \to S)$ of the integral-jet space, when composed with the given coordinate projection $(S \to C)$ of S gives rise to a submersion $(R \to C)$; indeed; the submanifolds $Z^\tau.m$ of R are mapped biholomorphically onto B (again, we have restricted S if necessary). What is more, this same argument clearly shows that this submersion $R \to S)$ and the projection $(R = S^\alpha M \to M)$ form a pair of coordinate projections on the manifold R (i.e. they define on R a 'structure' of the product of the manifolds B and M). For this reason, R will also be denoted by $B \times M$. Furthermore, we will denote the Y -component of the projection $(R \to S)$ by $(R = B \times M \to Y)$; of course, this will imply that it is alternatively (on the element level) denoted by $(r = (b, m) \to y)$. It is clear that in view of these conventions (in particular of the definition of $(R \to C)$), the coordinate description of the projection $(R \to S)$ is given by the following equality of maps:

$$(R \to S) = ((b, m) \to \ (y, b))$$

.

Now we describe in coordinates the first of the two objects for which the stated relationship is to be proved: The infinitesimal variation of S^α with parameter v is a section $\{v^{/\alpha}_S\}_{S^\alpha}$ of the normal bundle $T^{/\alpha}_S S^\alpha$. This bundle has obviously been trivialized, i.e. expressed as the product $T_Y \times B$. Thus, this infinitesimal variation will also be denoted by $(B \to T_Y)$. Its coordinate description in terms of the projection $(R \to S)$ is obviously given by the equality

$$(B \to T_Y) = (b \to \ D_m y_{(b,m)} v)$$

where D_m denotes the partial differential with respect to the m -coordinate on R. (This follows from the very definition of the canonical representation of T as a parameter vector space of infinitesimal variations). The covariant differential $D(v_S^{/\alpha})_{s^\alpha} \in Hom(T_S^\alpha, T_S^{/\alpha})$ (which is invariantly defined since the infinitesimal variation vanishes at $s^\alpha = b$), i.e. the jet at b of this infinitesimal variation, is in this situation simply the differential

$$D(v_Y)_b = D(D_m y_{(b,m)} v))_b \in Hom(T_B, T_Y)$$

of that map (i.e. section) $B \to T_Y$ (since we have a trivialization of the normal bundle of S^α).

Our objective is to relate that jet with the 'second object', namely the normal vector $v_Z^{/\alpha}$ at z^α (belonging to the infinitesimal variation of Z^τ with parameter v). In order to describe explicitly that normal vector in terms of the projection $((b,m) \to (y,b))$, we will first do the same for the map $((b,m) \to \mathbf{p}^*_{(S)}.(y,b)) := (r \to \mathbf{p}^*_{(S)}.s) := (r \to z) :=$ the projection $R \to Hom(T_B, T_Y)S = \mathbf{P}^*_{(S)}S \subset \mathbf{P}^*_S S = Z$ defined by the family of canonical lifts of hypersurfaces. According to this definition, for any fixed m' the image of the composite map $(b \to (b,m') \to \mathbf{p}^*_{(S)}.(y,b))$ of B into $Hom(T_B, T_Y)S$ is the canonical lift of the image of the composition $(b \to (b,m') \to (y,b))$. But the latter image is obviously the graph of the composition $(b \to (b,m') \to y)$. Therefore $(b \to (b,m') \to \mathbf{p}^*_{(S)}.(y,b)) = (b \to D_b y_{(b,m')}.(y.b) \in Hom(T_B, T_Y).(y,b))$. More succinctly, the above projection of the integral-jet space into the contact manifold has in coordinates the folowing description:

$$((b,m) \to \mathbf{p}^*_{(S)}.(y,b)) = ((b,m) \to D_b y_{(b,m)}.(y,b) \in Hom(T_B, T_Y).(y,b)).$$

As a result of the assumption that the normal vector $v_Z^{/\alpha}$ at s^α vanishes, the normal vector $v_Z^{/\alpha}$ at z^α belongs to the distinguished subspace $T_Z^{\alpha/\tau}$ of the normal space $T_Z^{/\alpha}$. When we identify this subspace with the distinguished direct complement $T_J = \mathbf{P}^*_{(S)} = Hom(T_B, T_Y)$ of T_Z^τ in T_Z^α, the normal vector $v_Z^{/\alpha}$ at z^α is clearly (by definition of the representation of T as a parameter space of infinitesimal variations of Z^τ) the differential at m evaluated on v of the composite map $(m \to (b,m) \to \mathbf{p}^*_{(S)}.(y.b))$ of M into Z. In conclusion

$$v_Z^{/\alpha} = D_m(\, D_b y_{(b,m)}.(y.b)\,)_{(b.m)} v \in Hom(T_B, T_Y) \subset T_Z.$$

In order to obtain more information on these second-order differentials, we will introduce suitable charts. More precisely, we choose an arbitrary product chart on Y and B at the point $s = (y, b)$. In fact, again due to the local nature of the problem, we may assume that global charts (i.e. 'structures' of open subsets of affine spaces) can be chosen on manifolds Y and S. The vector spaces associated to these affine spaces will be denoted by T_Y and T_B respectively. Clearly, since a chart on S is given, we also have a trivialization of the bundle $\mathbf{P}_S^* S = Z$. What is more, since it is a product chart we actually have a trivialization of the above defined affine bundle $\mathbf{P}_{(S)}^* S = Hom(T_B, T_Y) S \subset \mathbf{P}_S^* Z$. In other words, this affine bundle is simply the product $Hom(T_B, T_Y) \times Y \times B$.

Now the differential $D(\, v_Y\,)_b = D_b(\,(\, D_m y_{(b,m)} v\,).y\,)_{(b,m)} \in Hom(T_B, T_Y)$ can obviously be expressed more simply as $D_b(\,(\, D_m y\,) v\,)_{(b,m)}$.

Similarly, the normal vector $v_Z^{/\alpha} = D_m(\, D_b y_{(b,m)}.(y.b)\,)_{(b.m)} v \in Hom(T_B, T_Y) \subset T_Z$ has a simplified interpretation, namely it is equal to $D_m(\, D_b y_{(b,m)}\,)_{(b.m)} v$.

Now let us observe that by the 'invariant version' of the Schwarz Theorem we obtain the equality

$$D_b(\,(\, D_m y\,) v\,)_{(b,m)} = D_m(\, D_b y_{(b,m)}\,)_{(b.m)} v$$

which is clearly tantamout to assertion (iii). (In the preceding argument no charts on M or R have been used . If we had applied the classical version of the Schwarz theorem, we would have had to choose a chart at m as well.) QED

Proposition IV.12 *Any geometrical family of Legendrian submanifolds is a compound-geometrical family of submanifolds (i.e. a double fibration).*

[Proof: Let us introduce notation as in Definition IV.4. Our objective is to prove that for a fixed point m the canonical representation of the tangent space T at m as a parameter vector space of infinitesimal variations of the Legendrian submanifold Z^τ with parameter m (i.e. a parameter vector space of sections for the normal bundle of Z^τ) is compound-geometrical (i.e. 'base-point-free'). Explicitly, this means (by definition) that for any point $z \in Z^\tau$ the evaluation mapping of T into the normal space $T_Z^{/\alpha}$ at z is surjective.

Let us observe that the canonical representation of T as a parameter vector space of sections for the submersed normal bundle is geometrical as a result of the assumption of Legendre geometric amenability (by the very definition of that property). In particular, this representation is compound-geometrical. Thus, we know that the evaluation map of T into the submersed normal space $T_Z^{/\alpha}$ is surjective. Since the submersed normal space is the quotient of the normal space by the subspace $T_Z^{\alpha/\tau}$ (and the two evaluation maps are related in the obvious way), it is clear that it suffices to prove that the image of the former evaluation map (of T into the normal space) contains that subspace of the normal space.

In other words, it suffices to prove surjectivity of the restriction of the former evaluation map to a map of the preimage of the space $T_Z^{\alpha/\tau}$ into that space. But this preimage is precisely the hyperplane T^α in direction z (here it is implied that $Z^\tau =: J^\epsilon$ has been embedded in J by means of the geometrical representation of T as a parameter vector space of sections for the submersed normal bundle; thus T^α consists of parameters of sections of the submersed normal bundle which vanish at z). In accordance with that, the above mentioned restricted evaluation map will be denoted by $(T^\alpha \to T_Z^{\alpha/\tau})$.

Since the manifolds from the given family are by assumption Legendrian, T is a parameter space of Legendrian infinitesimal variations of the Legendrian submanifold Z^τ. According to assertion (iii) of Lemma IV.11, this implies that the mapping $(T^\alpha \to Hom(T_Z^{\alpha\tau}, T_Z^{/\alpha}))$ constructed from $(T^\alpha \to T_Z^{\alpha/\tau})$ by means the canonical isomorphism (v. Lemma IV.11 (i)) of their codomains could alternatively be characterized in the following way: it assigns to a tangent vector v precisely the jet at z of the section of the submersed normal bundle with parameter v.

Therefore, it remains to prove that each jet at z of a section vanising at z of the submersed normal bundle (i.e. each direction in the total space at the zero vector which is transverse to the fiber) can be realized as the jet of a section from the parameter vector space T^α (of sections). But this follows immediately from the assumption that the parameter vector space of sections for the submersed normal bundle (which means that the induced mapping $(Z^\tau \to J^\varepsilon)$ is an embedding) and the characterization of imersiveness of $(Z^\tau \to J^\varepsilon)$ given in II.21. QED]

Remark IV.13 In the proof of the last proposition we have only used the assumption of immersiveness of the maps $(Z^\tau \to J^\varepsilon)$, and not of their injectivity. Furthermore, the assumption that the contact manifold is not of a more general type (i.e. locally isomorphic to a x -dimesional-jet contact manifold with arbitrary x) was essential .

Proof of Theorem IV.5: *As has already been remarked, in order to complete the proof ot the theorem, it suffices to observe that its weak version (i.e. assertion (v) of Proposition IV.10) in view of the last proposition implies the theorem . QED]*

IV.2 Special Prolongable Hypersurface-Directional Conic Structures and Special Geometrical Parameter Spaces of Legendrian Submanifolds

In this section we will specialize the general results from section 1 by introducing ideas of uniform geometric amenability and completeness. More concretely, we specialize to the cases of 'uniformly geometrical parameter space of Legendrian submanifolds' (Subsection 1), or 'complete parameter space of geometrical Legendrian submanifolds' (Subsection 2), or 'locally complete parameter space of normally rigid geometrical Legendrian submanifolds' (Subsection 3). These additional assumptions will be shown to be tantamount to the requirements that the associated prolongable conic structure on the parameter space be respectively homogeneal (Subsection 1), or completive (Subsection 2), or homogeneal completive of a very special type (Subsection 3). (In particular, the additional assumptions of Subsection 3 turn out to imply the conjunction of the additional assumptions of Subsections 1 and 2.)

IV.2.1 Homogeneal Prolongable Hypersurface-Directional Conic Structures and Uniformly Geometrical Parameter Spaces of Legendrian Submanifolds

The importance of the situation studied in this subsection stems from the following simple observation: since the prolongable (expanded) conic structure is then a homogeneal first-order (expanded) geometric structure, the affine space of tangential (i.e. conic structure-preserving) connections is well-defned (and non-empty, at least locally); moreover, it essentially even encodes the conic structure on the manifold.

IV.2.2 Completive Prolongable Hypersurface-Directional Conic Structures and Locally Complete Parameter Spaces of Geometrical Legendrian Submanifolds

The considerations of this subsection form perhaps the most interesting part of the general theory developed in this chapter. Namely, the present subject is in a sense at the confluence of, on the one hand, simple classic ideas from the completely invariantly formulated theory of first-order PDEs (as developed in Section 1 in a rather less classic way), and, on the other hand, the main twistor-theoretical idea of encoding (by means of Kodaira's theory) a differential geometric object by specifying a single compact submanifold in a given (complex!) manifold. *Indeed, according to the result (the 'Legendrian analogue' of the 'Kodaira's main theorem') proved in different ways in [14] and (in a somewhat more special situation) in [3] , any compact Legendrian submanifold (of a contact manifold) which is 'Kodaira-regular' (as a Legendrian submanifold) in a certain sense (v. Subsection 2) belongs to a (unique, of course) locally complete parameter space of Legendrian submanifolds. When the mild additional condition of geometric amenability of the Legendrian manifold is imposed, a completely equivalent differential geometric desription of this situation follows from our result from Section 1 .*

IV.2.3 Homogeneal Completive Prolongable Hypersurface-Directional Conic Strucures and Locally Complete Parameter Spaces of Normally Rigid Geometrical Legendrian Submanifolds

In this subsection the assumptions from Subsections 1 and 2 are considered simultaneously (as already stated). (An important simplification will be that, due to normal rigidity, geometric amenability of one of the Legendrian submanifolds implies geometric amenability of all.) This is essentially the situation studied in [14], except for the following differences: there is no assumption of geometric amenability of the Legendrian submanifolds in that article, but a condition on the Legendrian submanifolds equivalent essentially to 1-flatness of the conic structure (and thus stronger than prolongability) is imposed . However, our situation (considered in Subsection 4) seems to be a more appropriate context for the study of geometry of paramater spaces of Legendrian submanifolds for the following reason: under the assumption of 1-flatness the above mentioned affine space of tangential (i.e. conic-structure-preserving) connections has a distinguished (non-empty) subspace, namely the subspace consisting of symmetric connections: this is precisely the affine space of symmetric connections constructed in a completely different way in [14]; a salient difference between the two contexts is that we are able to **reverse this construction** *(i.e. to recover the structure of the parameter space of Legendrian submanifolds from the geometric structure) owing to our additional condition of geometric amenability (even -as we already stated- under more general circumstances of section 1, when completeness does not necessarilly hold and no connections are distinguished). It would not be difficult to see that without that crucial additional condition (which is in a certain sense equivalent to geometric amenability from the prevoius chapters), the structure of a parameter space of Legendrian submanifolds still has a first-order geometric structure as a constituent part (that is why a class of connections, namely the one investigated in [14], is distinguished), but it also has another 'component', which is 'non-geometric' in the sense of Remark II.16 (that is why the construction can not be reversed). It should be emphasized that our approach (and* *proofs) is substantially different from the one in [14]: our starting point is not a contact manifold Z with a distinguished space of Legendrian submanifolds, but the manifold M equiped with a geometric structure and Z is constructed naturally as the quotient contact manifold of a distribution-equiped manifold . There seem to be some further obvious advantages with this approach: in our interpretation the geometric structure is manifestly of* **first-order** *(i.e. each tangent space is simply a equiped with a conic structure), the distinguished connections are more easily constructed - they are simply those which preserve this structure, and, most importantly,* **we do not impose the too restrictive condition of 1-flatness**, *but just the natural weaker condition of prolongability (similarly, the connections for which the 'characteristic' submanifolds of the paramater space are totally geodesic do not have to be symmetric as in [14], but only 'partially symmetric'). This last freedom will play an important role in the theory of Veronese conic structures (or G_x -structures in terminology of R . Bryant):*

In [3] it is proven that a locally complete parameter space of Legendrian rational curves (in a contact 3-manifold) such that their 'submersed normal bundles' (which will be defined as the distinguished quotient line bundles of their normal bundles) have Chern class $x = 3$, is essentially the same as a 1-flat 4 -dimensional Veronese (expanded) conic structure. On the other hand an application (v. [3]) of Berger's criteria for holonomy groups shows that an 1-flat Veronese conic structure is necessarily completely flat as soon as the dimension is ≥ 5 (i.e. $x \geq 4$). Therefore, for such x there are no nontrivial complete spaces of Legendrian rational curves whose induced Veronese conic structures are 1-flat. However, by specializing (Chapter V) our general results to Veronese conic structures we prove that a locally complete parameter space of Legendrian rational curves (in a contact 3-manifold) is always (regardless of the type of the normal bundles) essentially the same as a prolongable (i.e. ' **partially** *1-flat') Veronese-expanded conic structure. (Such manifolds are not of no interest - for instance despite their possible first-order non-flatness in the case of even self-intersection number x there is an underlying non-trivial conformal structure (which is 1-flat, of course). We will also reprove the above* *quoted Bryant's result in a rather different way, namely by showing that for* $x = 3$ *prolongability implies 1-flatness.*

Chapter V

Veronese Conic Structures

V.1 Veronese Conic Structures on Locally Complete Parameter Spaces of Rational Curves in Surfaces

Remark V.1 In view of the well-known classification of line bundles over a projective line and standard facts concerning the Veronese embedding, it is immediate that a non-exceptional rational curve in a surface is a geometrical submanifold, i.e. that the space of sections of the normal bundle inherits a conic structure (which has to be hypersurface-directional since the normal bundle is a line bundle): indeed, the integral-direction space will be precisely the image of a certain Veronese embedding of the projective line into the projectivized dual space $J = \mathbf{P}(T^*)$. Furthermore, the complete rigidity of a line bundle on a rational curve (which is a well-known fact) implies (by definition) normal rigidity of a rational curve in a surface.

Remark V.2 The objective of this section will be to give an explicit characterization of the inherent localized first-order geometric structures at a point of a locally complete parameter set of non-exceptional rational curves in a surface. From the previous remark it follows that the isomorphy class of rational curves and the integer 1 satify the hypothesis of the Proposition I.23. Hence we infer that such a space of embedded curves is geometrical and thus comes equiped with a hypersurface-directional conic structure. Therefore, the objective of this section could be rephrased more rigourously as an explicit description of localized conic structures constituent for the conic structures $R \subset J.M \coloneqq \mathbf{P}(T^*).M$ canonically induced on complete spaces M of rational curves in surfaces.

From the Proposition I.23 it also folows that this objective amounts to a classification (i.e. an explicit description of the isomorphy classes) of **vectorial Veronese conic structures**, which we define as vectorial completive hypersurface-directional conic structures T_{ϖ} for which the integral-direction spaces $J^{\varepsilon} \subset J :$ $=$ (since $y = 1$) $= J$ are rational curves.

Another application of Proposition I.23 shows that this classification is equivalent to a classification of line bundles over rational curves admitting nontrivial sections. More precisely, the isomorphy classes considered above correspond bijectively to the isomorphy classes of line bundles over rational curves which admit nontrivial sections. (For a concrete locally complete parameter space of rational curves in a surface the above mentioned isomorphy class of completive vectorial conic structures corresponds according to Proposition I.23 precisely to the common isomorphy class of the normal line bundles of those rational curves.)

These in turn are well-known to correspond bijectively to nonnegative integers, where the correspondence is defined by Chern classes of line bundles. (For a rational curve in a surface the Chern class of the normal line bundle was proved to coincide with the self-intersection number x of the curve.)

The explicit description of the isomorphy class of vectorial Veronese conic structures determined by a given $x \geq 0$ is very simple: it consists of the structures T_{ϖ} such that the underlying vector space T is $(x+1)$-dimensional. (In other words, such a structure consists of the $(x+1)$-dimensional vector space T and a rational curve J^{ε} in $J \coloneqq \mathbf{P}(T^*)$ embedded completively; incidentally, for a rational curve in J completivity of the embedding is known to be equivalent to the maximality of its degree.)

On the other hand, we observe that for a given $(x+1)$-dimensional vector space T the (vectorial) Veronese conic structures on T correspond naturally and bijectively to the **(vectorial) Veronese structures on** T, which are

defined in the following way: Let us denote the 'model' vector space $\mathbf{C}^{x+1}$ by T_μ and its automorphism group (i.e. $GL(x+1)$) by G_μ. Veronese structures on $T$are defined as $G_\mu^{\odot}$ -structures, where $G_\mu^{\odot}$ is the subgroup of G_μ obtained by effectivization of the action of $GL(2)$ on $\mathbf{C}^{2\,\odot x}$, where the latter is identified with T_μ (v. [3]). (Intuitively, a Veronese structure on a vector space T is given by 'a class (on T) of an x-th ymmetric tensor power of a vector plane'; v. the next remark for a rigorous definition.) The above mentioned natural correspondence is defined by associating to a vectorial Veronese structure $T_\odot$ the vectorial Veronese conic structure $T_{\mathfrak{cs}}$ whose integral hyperplanes are by definition precisely the perpendiculars of 'simple covectors' in T^* – meaning covectors which are tensor powers of a single covector.
(Of course, bijectivity of the above natural correspondence amounts to equality $G_\mu^{\mathfrak{cs}} = G_\mu^{\odot}$, where the left-hand side is by definition the automorphism group of the Veronese conic structure on T_μ associated to the 'model' Veronese structure. Thus, Veronese conic structures on T can also be thought of as $G_\mu^{\odot}$-structures on T.)
In conclusion, the conic structure induced on a locally complete parameter space M of rational curves of self-intersection number x in a surface has been seen to be a Veronese (hypersurface-directional) conic structure on an $(x+1)$-dimensional manifold, which can be identified (in the way described precisely above) with an expanded Veronese structure $M_{\odot mp}$ (i.e. with a $G_\mu^{\odot}$ -manifold).

Remark V.3 Although we will not need the above mentioned more intuitive definition of a Veronese structure (or a $G_\mu^{\odot}$-structure) as a 'vector space equiped with a class of symmetric tensor powers of planes', we give it here in a precise form for the sake of completeness. An isomorphism between two (symmetric) tensor powers with the same tensor-power vector space will be said to be **over tensor-power vector space** if the associated automorphism of the tensor-power vector space is the identity. A **class of symmetric tensor powers** on a vector space T is defined as a class of symmetric tensor powers isomorphic over the space T. These are indeed structures on vector spaces since they can be transferred by means of isomorphisms of vector spaces in an obvious way.

Moreover, the automorphism group of the canonical such structure on the 'model' space $T_\mu \coloneqq \mathbf{C}^{x+1}$ is obviously the above defined group $G^{\odot}_{\odot}$. Therefore these structures may be thought of as $G^{\odot}_{\mu}$ -structures. Let us observe that for such a structure the 'tensor-root' vector plane is given only up to isomorphism inducing indentity in the tensor power.

V.2 Geometric Description of the Structure of a Locally Complete Parameter Space of Rational Curves in a Surface

In view of the observation (made in the previous section) that a non-exceptional rational curve in a surface is a normally rigid geometrical submanifold, and of Proposition II.29 (which can according to the above observation be applied to complete spaces of rational curves in surfaces), it is clear that the objective of this section will be accomplished as soon as we obtain a classification (i.e. an explicit description of the isomorphy classes) of completive hyperplane-directional conic structures on vector spaces T_{ϖ} *for which the integral-direction spaces* $J^{\varepsilon} \subset J \coloneqq (\text{since } y = 1) = J$ *are rational curves. But that classification has already been given in section 1 .*

Proposition V.4 For a given $(x+1)$ -dimensional manifold M there is a bijective correspondence between

(a) Structures on M of a locally complete parameter space of non-exceptional rational curves S^{α} in a surface S (i.e. equivalence classes defined in an obvious way of complete families with parameter space M of such submanifolds)

and

(b) Integrable preconnections on M with the following two properties:
(b1) Admissibility; (Explicitly, in notation of Definition II.6 the foliation of R by the leaves of the distribution $F^{\tau}R$ is in fact a fibration and its projection maps the submanifolds $J^{\varepsilon}.m$ of R injectively into its base.)

(b2) The preconnection on M is a **Veronese preconnection**, which is defined as a hypersurface-directional preconnection such that the underlying conic structure on M is a Veronese conic structure, i.e. a Veronese structure on the manifold M.

More precisely, this correspondence is obtained by restricting the correspondence defined in Proposition II.13.

[As remarked earlier in this section, this proposition is an obvious consequence of Proposition II.29 and Remark V.2.QED]

Remark V.5 In the following we will see that a locally complete parameter space of rational curves on a surface admits a different geometric description: instead of a hypersurface-directional preconnection on a Veronese-expanded conic structure, we could take a projective structure (i.e. a curve-omnidirectional preconnection) which is in a certain way compatible with the Veronese conic structure.

V.3 Prolongable Veronese Conic Structures and Geometric Description of the Structure of Locally Complete Parameter Spaces of Legendrian Rational Curves

Proposition V.6 (i) In view of the general facts from the chapters on second-order invariants and hypersurface-directional conic structures, a locally complete parameter space M of Legendrian rational curves in a (3-dimensional) contact manifold comes equiped with a prolongable Veronese conic structure, or, equivalently, a Veronese conic structure with pre-permissible intrinsic torsion. QED

(ii) According to the same general results the construction from (i) is reversible: the structure of a parameter space can be reconstructed from the conic structure. What is more, the converse of (i) is true for a sufficiently small neighbourhood of any given point of the underlying manifold M of the conic structure. (In other words, an arbitrary 'germ of a prolongable Veronese conic structure' is obtained by the construction from (i)).

(iii) Suppose a single Legendrian rational curve in a contact 3-manifold is given. Since it is a geometrical normally rigid Legendrian submanifold with extendable (infinitesimal) 1-variations, we obtain by the general method of Chapter 4 (on hypersurface-directional conic structures) a locally complete parameter space of Legendrian manifolds as in (i). *Thus simple holomorphic data encode a solution of an (a priori non-trivial) first-order quasiaffine invariant PDE (of the type considered by Bryant in [4]) on natural bundles over manifolds, more precisely an integral of the* **quasiaffine PDE of prolongable Veronese conic structures** *on a manifold M. (Recall that this equation is naturally associated to the manifold M, i.e. that the above mentioned invariance is implied to be under the biholomorphism group Aut M. What is more, these equations are, as we have already noted,* geometrically *invariant, i.e. their construction was completely local.)* (We will soon see that the condition we have imposed on the structural 1-jets of the localized conic structures is quite severe, although weaker than 1-flatness; indeed, while the space of pre-permissible intrinsic torsions will turn out to have at most six irreducible components, the number of the irreducible components of the space of all intrinsic pretorsions increases rapidly with the degree x of the Veronese conic structure.)

V.4 Vectorial Structures and Invariant Spaces of Tensors

V.4.1 Elementary Pretype and Intrinsic Pretorsions

Proposition V.7 Consider a Veronese conic structure on a vector space T, where notation is as in Definition I.2 (in particular $dim\ T = x + 1$).

(i) Let j be an integral jet. As we have already observed in Remark III.22 for general hypersurface-directional conic structures, the tangent 1-jet of the integral-jet space J^e (at j) must be **axially decomposable** (v. the above quoted remark; explicitly, the tangent space T_J^ε to the integral-jet space J^ε at j must be a subspace of

$$T_J = Hom(T^\alpha, T^{/\alpha}) = T^{/\alpha} \otimes T^{*/\alpha}$$

of the form

$$Hom(T^{\alpha\,/\alpha'}, T^{/\alpha}) = T^{/\alpha} \otimes T^{*/\alpha\,\alpha'}$$

where $T^{\alpha'}$ is the subspace of T^α uniquely determined by the space T_J^ε and called the **axis** thereof. (V. Remark III.22 for its intuitive interpretation and justification for this terminology.) Furthermore, since the integral-jet space is in our case a curve (i.e. one-dimensional), the quotient $T^{\alpha\,/\alpha'}$ must be a vector line as well. (In fact, by explicit differentiation of the natural parametrization of the integral-jet space - i.e. of the rational normal curve in $J := \mathbf{P}(T^*)$, we could easily prove that the axis consists precisely of all the symmetric tensor products which have at least two factors from the distinguished vector line in the tensor root vector plane; however, we will not use this fact.)

(ii) Veronese conic structures are of infinite elementary pretype. [Indeed, this is true of general axially decomposable conic structures since the space of tangential symmetric k-elementary preconnnectors can obviously be expressed in the following way:

$$E_\pi^{k\,sa_\bullet te} == Hom(T^{\alpha\,/\alpha'\odot k}, T^{/\alpha}) = T^{/\alpha} \otimes T^{*/\alpha\,\alpha'\odot k}.$$

QED

(iii) Let us fix a compatible structure on T of a symmetric tensor power of a vector plane T_r (which will be referred to as the 'tensor-root plane'). (Compatibility of this structure and the Veronese conic structure has been defined in Proposition V.2.) Furthermore, let us choose a symplectic product (i.e. a volume form) on the tensor-root plane.

Claim: The tangential symmetric elementary-preconnnector bundle

$$E_\pi^{sa_\bullet te} J^\varepsilon \coloneqq T_J^\varepsilon \otimes T^{*/\alpha}$$

is canonically isomorphic to the power $T_r^{/\alpha\,\odot 4-x} J^\varepsilon$ (less precisely denoted by $(4-x)$) of the transversal line bundle (1) on the projectivized dual space J_r of the tensor root plane T_r.

[Proof of (ii): Since the transversal spaces $T^{/\alpha}$ are one-dimensional, the equality

$$T_J^\varepsilon = T^{/\alpha} \otimes T^{*/\alpha\,\alpha'}$$

(from (i)) implies

$$T_J^\varepsilon \otimes T^{*\alpha} = T^{*/\alpha\,\alpha'}. \qquad \text{(V.1)}$$

On the other hand, we can express fibers of the tangential symmetric elementary-preconnnector bundle as follows:

$$E_\pi^{sa_\bullet te} = (\text{according to (ii)}) = Hom(T^{\alpha\,/\alpha'\,\odot 2}, T^{/\alpha}) = T^{/\alpha} \otimes T^{*/\alpha\,\alpha'\odot 2} =$$

$$= (\text{due to one} - \text{dimensionality of } T^{*/\alpha\,\alpha'}) = T^{/\alpha} \otimes T^{*/\alpha\,\alpha'\,\otimes 2} = (\text{by (V.1)}) =$$

$$= T^{/\alpha} \otimes T_J^{\varepsilon\,\otimes 2} \otimes T^{*\alpha\,\otimes 2} = T_J^{\varepsilon\,\otimes 2} \otimes T^{*\alpha} \approx T_r^{\;/\alpha\,\odot 4-x}.$$

Finally, in order to prove the statement on canonicity, it suffices to observe that the last isomorphism of fibers is canonical since a symplectic product on the tensor root vector plane T_r is fixed; indeed, $T_J^\varepsilon \approx T_r^{\;/\alpha\otimes 2}$ canonically (relative to the symplectic structure). QED]

V.4.2 Pretype and Permissible Conjunctively Intrinsic Pretorsions

Proposition V.8 Consider a vectorial Veronese conic structure on a vector space T, where notation is as in Proposition III.9.

(ii) *The G^{cs}-module $H^0(E_\pi^{sa_\bullet te} J^\varepsilon)$ (of tangential symmetric preconnectors) is isomorphic to $T_r^{\;(\odot 4-x)}$, where the latter symbol denotes $T_r^{\;\odot 4-x}$ if $4 \geq x$ and 0 otherwise.* (More conceptually, this symbol denotes the space $H^0(T_r^{\;/\alpha\,\odot 4-x} J^\varepsilon)$ of sections of the indicated power (less precisely denoted by $(4-x)$) of the transversal line bundle (1) on the projectivized dual space J_r of the tensor root vector plane T_r.) In particular, the Veronese conic structure is of pretype one iff $x \geq 5$. Furthermore, according to the well-known general properties of G^{cs}-modules, the indicated isomorphism of irreducible modules is unique up to a scalar factor. In addition to that, this isomorphism becomes canonical (i.e. its 'scale is chosen') as soon as a symplectic structure onthe tensor-root vector plane is chosen.

[In order to prove (ii), it suffices to apply the explicit description of the tangential symmetric elementary preconnnection bundle obtained in Proposition V.2. QED]

(ii) By examining in this case one of the long exact sequences from Proposition III.9, namely the sequence beginning with the nethermost row of the diagram, we conclude that the $G^{\mathcal{cs}}$-module of permissible conjunctively intrinsic pretorsions either vanishes or is isomorphic to ${T_r}^{(\odot x-6)}$. (Indeed, we know from Proposition III.9 that this module is embedded into the module $H^1(E_\pi^{sa\bullet te} J^\varepsilon)$ of affine-bundle classes on the tangetial symmetric elementary-preconnnector bundle; on the other hand the latter module is in view of the explicit description of that bundle in Proposition V.7 clearly isomorphic to the *irreducible* module ${T_r}^{(\odot x-6)}$.) However, by means of an appropriate exact sequence and one-dimensionality of the integral-jet space, the space

$$H^1(E_\pi^{te} J^\varepsilon) \risingdotseq H^1(T_J^\varepsilon \otimes T^{*/\alpha}\, J^\varepsilon)$$

is easily seen to be vanishing (the exact sequence in question is the one obtained by viewing the second tensor factor as a quotient). Therefore, the latter of the above alternatives in fact always occurs, i.e. we obtain

$$\frac{H^0(E_\pi^{/sa} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \risingdotseq \frac{H^0(T^\alpha \otimes T^{/*\alpha\wedge 2}\, J^\varepsilon)}{H^0(T_J^\varepsilon \otimes T^{*/\alpha}\, J^\varepsilon)} \approx {T_r}^{(\odot x-6)}. \tag{V.2}$$

In summary, the $G^{\mathcal{cs}}$-module of permissible conjunctively intrinsic pretorsions is isomorphic to ${T_r}^{(\odot x-6)}$, where T_r denotes a tensor-root symplectic plane, and the meaning of the parentheses is the same as in (i). Furthermore, the conclusions from (i) regarding canonicity of the isomorphism of irreducible modules are obviously also valid in this case. QED

V.4.3 Conjunctively Permissible Complementally Intrinsic Pretorsions

Proposition V.9 Consider a vectorial Veronese conic structure on a vector space T, where notation is as in Proposition III.14. For the sake of convenience, the diagram pertaining to that proposition has been reproduced here.

(i) The space $H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon)$ (of permissible pretorsion vectors), which was expressed in (III.11) in terms of extensions as

$$\frac{\frac{H^0(E_{\pi_1}^{te} J^\epsilon)}{H^0(E_{\pi_1}^{te\bullet np} J^\epsilon)} \dot{+} \frac{H^0(E_\pi^{te} J^\epsilon)}{H^0(E_{\pi_1}^{te} J^\epsilon)}}{H^0(E_\pi^{te\bullet sa} J^\varepsilon)} \dot{+} \frac{H^0(E_\pi^{/sa\, sa_* te} J^\varepsilon)}{H^0(E_\pi^{te} J^\varepsilon)} \tag{V.3}$$

has a decomposition into irreducible G^{∞}-modules of the following form:

$$\frac{\frac{T_r{}^{\odot x-2} \oplus T_r{}^{\odot x} \oplus T_r{}^{\odot x+2}}{T_r{}^{(\odot 2-x)}} \dot{+} T_r{}^{(\odot x-4)}}{T_r{}^{(\odot 4-x)}} \dot{+} T_r{}^{(\odot x-6)}. \tag{V.4}$$

Furthermore, these decompositions are compatible in the sense of equality of entries of (V.3) and (V.4) in analogous positions. Of course, this module is completely reducible, and consequently the quotients in the indicated extensions can be realized *in a unique way* as (invariant!) direct summands. (Uniqueness follows from their irreducibility and non-existence of isomorphic irreducible modules contained in the submodules.) Furthermore, the conclusions from Proposition V.8 regarding canonicity of the isomorphism of irreducible modules can also be drawn in this case. (Explicitly, the indicated isomorphisms of irreducible modules are unique up to a scalar factor, and they become canonical if a symplectic structure on the tensor-root vector plane is chosen.)
[Proof of (i) will be carried out gradually through the proofs of other assertions of this proposition; the latter are more elementary than (i) and together imply (i), but also contain more detailed information on effective construction of the isomorphisms whose existence is asserted in (i). QED]

(ii) The investigation of the tangential symmetric elementary-preconnnector bundle in Propositions V.7 and V.8 has already resulted in the determination of the isomorphy classes of the quotient (i.e. the space of permissible conjunctively intrinsic pretorsions) from the extension (V.3) and of the 'denominator' in the subspace (namely the space of tangential symmetric preconnectors). Likewise, the issue of how canonical the isomorphisms are has already been addressed in those propositions. Let us now observe that their proofs also give explicit procedures for the construction of canonical isomorphisms. QED

(iii) Now we focus our attention on the space of tangential preconnectors, i.e. the 'numerator' in the subspace from the extension (V.3). The proof of the corresponding statements from (i) easily follows by application of the method used in (ii) to the long exact sequence defined in Proposition III.12, computation of the first two cohomology groups pertaining to the tangential complementary-elementary preconnnector bundle and observation of the distinguished decomposition of the space of tangential preconnectors:

$$E^{te} = E^{vd'} \oplus E^{te_{\infty}} = T_r^{\odot x} \oplus T_r^{\odot x-2} \oplus T_r^{\odot x} \oplus T_r^{\odot x+2}.$$

Here the spaces $E^{vd'}$ and $E^{te_{\infty}}$ consist by definition of resp. the first-trace-part connectors and tangential connectors for the associated obviously defined **scaled conic structure**, while the (unique due to distinctness of the irreducible components) decomposition of the latter space into irreducible components has been obtained from Clebsch-Gordon formula. (Cf. [3]). QED

V.4.4 Complementally Permissible Homogeneally Intrinsic Pretorsions, Homogenally Permissible Intrinsic Torsions and Complementary Connectors

In order to obtain information on the relation of the space of full preconnectors and the space of preconnectors with the space of connectors, we wil apply straightforwardly the relevant assertions of Theorem III.20 to the case of Veronese conic structures.

Proposition V.10 Let us consider a Veronese conic structure on a vector space T, where notation is as in Theorem III.20. The diagram pertaining to that theorem has for the sake of convenience been reproduced here.

(i) Since a vectorial Veronese conic structure on a is hypersurface-directional, completive and its integral-jet space is a projective space (being a rational curve), the map (1) in the diagram is according to III.21 (i.iii) bijective. In other words, the vector space of full preconnectors is (by means of this mapping) a quotient space of the space $E \coloneqq T \otimes T^{*\otimes 2}$ (of connectors) canonically isomorphic to (and will therefore be identified with) the quotient space

$E^{/vd'}(= E^{dv'})$ of first-trace-free connectors. In particular, all full preconnectors are homogeneal (i.e. connector-induced). According to III.21 (i.ii), this implies *vanishing of the space of complementally permissible homogeneally intrinsic pretorsions.*

Furthermore, according to III.21 (i.iii), the map (2)' is also injective, i.e. the surjection of the space of covectors onto the space of homogeneally universal complementary full preconnectors is bijective. QED

(ii) Since a Veronese conic structure on a vector space is hyperplane-directional, the map (2) in the diagram is according to III.21 bijective. In other words, every complementary full preconnector is universal, i.e. uniquely precovector-induced. (Here we imply the correspondence between precovectors and complementary full preconnectors which was defined fiberwise in the way indicated in the diagram.)

(iii) Since a Veronese conic structure has all the properties enumerated in (i), the space of *inherent* complementary full preconnectors (which was defined as the quotient of the space of precovector-induced complementary full preconnectors by the space of the universal ones) admits according to III.21 the following description: The canonical linear mapping of inherent complementary connectors into homogeneally inherent complementary full preconnectors is bijective, and its codomain is canonically isomorphic to the vector space of the affine bundle classes on the vector bundle of the lines of integral-tangent (or 'complementary') covectors:

$$E^{np\,/ui}(\risingdotseq E^{np\,/vd''{}_*vd'} \risingdotseq \frac{E^{np}}{E^{vd''{}_*vd'}} \risingdotseq \frac{E^{np}}{E^{vd'}+E^{vd''}} = E^{/vd'\,np\,/ui}) \approx H^O(E^{np}_{\pi_1}J^{\varepsilon})^{/ui_h} \approx H^1(T^{*\alpha}J^{\varepsilon}). \tag{V.5}$$

QED

(iv) The vector space $H^1(T^{*\alpha}J^{\varepsilon})$ (on the right-hand side of (V.5)) of affine-bundle classes on the vector bundle of spaces of integral-tangent (i.e. 'complementary') covectors is *irreducible* as a G^{cs}-module; more precisely, it becomes canonically isomorphic to the module ${T_r}^{*(\odot x-2)}$ (of degree $x-2$ or zero) when a root symplectic plane T_r is chosen (v. Proposition V.4).

Proof of (v): According to Proposition V.4, the (above mentioned) line bundle $T^{*\alpha} J^{\varepsilon}$ is after such a choice canonically isomorphic to the bundle $T_r{}^{*\alpha\odot x} J_r \eqqcolon (-x)$, where $T_r{}^{/\alpha} J_r \eqqcolon (1)$ is the hyperplane-divisor line bundle on the projective line J_r. QED

(v) Since the irreducible module from (vi) is according to (iv) (canonically isomorphic to) the quotient module (V.5) (of inherent complementary connectors) and it is clearly not isomorphic to the two components of the submodule $E^{ui} \coloneqq E^{vd'} + E^{vd''}$ of E^{np} (which are both isomorphic to T^* and thus irreducible), the (clearly completely reducible) module E^{np} (of complementary connectors) has a unique *submodule* forming a direct complement to $E^{ui} \coloneqq E^{vd'' * vd'}$. We will identify the quotient module with this submodule and denote the latter by E^{iu}. It will also be identified with its image $E^{/vd'\, it_* vd'}$ (in the module $E^{/vd'\, np}$), which is clearly the unique submodule forming a direct complement of the submodule $E^{/vd'\, ui}$ (of homogeneally universal complementary full preconnectors). QED

(vi) Our present objective is to investigate the symmetry of the space

$$E^{np\,/ui} = E^{np\, iu} \tag{V.6}$$

(of inherent complementary connectors) as a *subspace* of the space E^{np} (of complementary connectors). According to III.21, the latter space is symmetric (with respect to the permutation of the second and the third index), i.e. the direct sum of its symmetric-tensor and antisymmetric-tensor parts:

$$E^{np} = E^{np \bullet sa} \oplus E^{np \bullet as}.$$

Furthermore, it was already noticed there that the symmetry automorphism of that space (obtained by restriction) descends to the above *quotient* space (V.6). Claim: The space (V.6) is also symmetric as a *subspace* of E^{np}. [Indeed, it suffices to observe that its image under the symmetry automorphism is an isomorphic G^{cs} -module and that there is only one submodule of that isomorphy class in E^{np}. QED] In particular, it splits into a sum of its symmetric-tensor and antisymmetric-tensor parts. Since these are again G^{cs} -invariant and their sum is an irreducible module, we infer that one of them has to vanish . In conclusion, the space (V.6) either consists of symmetric or antisymmetric tensors (with respect to the permutation of the last two indices). QED]

(viii) Since the jet space of a Veronese conic structure is (biholomorphic to) a projective line and every full preconnector is homogeneal (i.e. connector-induced), every preconnector is according to Theorem III.21 also homogeneal (i.e. associated to some connector). QED

Lemma V.11 The space E^{iu} from Proposition V.10 (i.e. the space of inherent complementary connectors for a localized Veronese conic structure) consists of antisymmetric connectors. (In particular, the only symmetric complementary connectors are the universal ones, while the space $A^{np/np \bullet ui}$ of inherent complementary torsions is non-vanishing.)

Proof: We will give only an outline of the (rather long but natural) algebraic proof since we will later show independently by a somewhat circuitous differential-geometric construction that $A^{np/np \bullet ui}$ is non-vanishing; this will in view of Proposition V.10 amount to an alternative proof.

For an arbitrary hypersurface-directional completive conic structure it is possible to give an explicit canonical construction (and thereby prove the existence) of a connector in the preimage of a given *universal* (i.e. pre-covector induced) full preconnector. By its very construction this connector turns out to be antisymmetric (in the last two indices) and to depend linnearly (and G^{∞}-invariantly) on the given full-preconnector. (Antisymmetry follows easily from the following formula for the component $v^{/\alpha}$ in an arbirary integral-transverse space $T^{/\alpha}$ of the value $v \in T$ of the connector on a pair (v_1, v_2):

$$v^{/\alpha} = < v^{*/\alpha}, v_2 - \frac{v_2^{/\alpha}}{v_1^{/\alpha}} v_1^{/\alpha} > v_1.$$

Here $v^{*/\alpha}$ denotes the procovector from the given precovector at the integral jet j, and $v_i^{/\alpha}$ is the image of v_i in the integral-transverse space; furthermore, we consider only those j for which the denominator is non vanishing; corectness of the definition follows from the completivity assumption.)

In the case of Veronese conic structures the image of this invariant map must be precisely E^{iu} by invariance and uniqueness of such an irreducible module in E^{np}. (What is more, one could prove in this manner that E^{iu} consists of first-trace-free connectors.)

QED

Proposition V.12 (i) For a Veronese vectorial conic structure the space A (of torsions), which was expressed in Theorem III.20 as the succesive extension

$$A^{ui}\dot{+}A^{pi/ui}\dot{+}A^{pi_h/pi}\dot{+}A^{pi_p/pi_h}\dot{+}A^{pi_j/pi_p}\dot{+}A^{pi_0/pi_j}\dot{+}A^{/pi_0}, \tag{V.7}$$

or the succesive extension

$$A^{ui}\dot{+}A^{np/ui}\dot{+}(\,A^{pi_h/np}\dot{+}A^{pi_p/pi_h}\dot{+}A^{pi_j/pi_p}\,)\dot{+}A^{pi_0/pi_j}\dot{+}A^{/pi_0}, \tag{V.8}$$

has a decomposition into (irreducible except for the last entry) G^{cs}-modules of the following form:

$$T_r{}^{\odot x}\dot{+}T_r{}^{(\odot x-2)}\dot{+}\frac{\frac{T_r{}^{\odot x-2}\oplus T_r{}^{\odot x}\oplus T_r{}^{\odot x+2}}{T_r{}^{(\odot 2-x)}}\dot{+}0\dot{+}T_r{}^{(\odot x-4)}}{T_r{}^{(\odot 4-x)}}\dot{+}T_r{}^{(\odot x-6)}\dot{+}A^{/pi_0}. \tag{V.9}$$

Furthermore, the *latter* two expressions are compatible in the sense of equality of entries of (V.7) and (V.9) in analogous positions. (It is implied that the expression within the brackets corresponds to the 'total fraction'). Of course, this module is completely reducible, and consequently the quotients in the indicated extensions of (V.7) can be realized as (invariant!) direct summands; in addition to that, such a realization is unique except for the case of homogeneally permissible pretorsion vectors. (Uniqueness follows from their irreducibility and non-existence of isomorphic irreducible modules contained in the submodules.) Furthermore, the conclusions from Proposition V.8 regarding canonicity of the isomorphism of irreducible modules can also be drawn in this case. (Explicitly, the indicated isomorphisms of irreducible modules are unique up to a scalar factor, and they become canonical if a symplectic structure on the tensor-root vector plane is chosen.) [Proof of (i): All these facts follow immediately from previous propositions in this sections and III.21.QED]

(ii) In the case $x \geq 4$ the assertion (i) implies equality of the space $A^{np\bullet pi}$ (of complementary permissible torsions) and its subspace A^{ui} (consisting of universal complementary torsions): Indeed, it suffices to observe that the space

$A^{te\mathfrak{sc}}$ (consisting of tangential connectors of the scalar conic structure) is then mapped injectively into the space of pretorsions since it does not contain components isomorphic to either ${T_r}^{(\odot 4-x)}$ or ${T_r}^{(\odot 2-x)}$. *Consequently, the canonical surjection of inherent complementary torsions (which have been canonically identified with inherent complementary torsions a^{iu}) into homogeneally permissible intrinsic torsions (defined in III.21) is then bijective.*

(iii) The ($G^{\mathfrak{cs}}$-invariant) canonical linear map

$$E^{sa_\bullet te} \to H^0(E_\pi^{sa_\bullet te} J^\varepsilon)$$

(of tangential symmetric connectors into tangential symmetric preconnectors) is injective. In view of Proposition V.8 (on the pretype of Veronese conic structures), this assertion has the following consequences:

If $x \geq 5$, the Veronese conic structure is also of type one (i.e. the space $E^{sa_\bullet te}$ of tangential symmetric connectors vanishes). Furthermore, if $x = 2$ (resp. 3,4) the latter space has dimension at most 3 (resp. 2,1). In fact, in the case $x = 2$ it is 3-dimensional (since the Veronese conic structure is then clearly equivalent to the conformal structure, cf. [9]); in particular, the structure is not of type one and the above map is bijective. Furthermore, by a straightforward tedious algebraic computation it could be shown that in cases $x = 3, 4$ the conic structure is actually of type one; however, we will not use this fact.

[Proof of (i): The above injectivity follows from the III.21 (ii) since the conic structure is hypersurface-directional and the space of inherent complementary *symmetric* connectors vanishes according to Lemma V.11. (Alternatively, we could exploit the fact that an either euclidean or symplectic conformal structure underlies the given conic structure and use the well-known explicit descriptions of the space of tangential symmetric connectors for this underlying structure.) QED] QED

V.5 Differential-Geometric Implications

V.5.1 Prolongability of Veronese Conic Structures

Proposition V.13 (i) The PDE from Proposition V.6 (on the geometric description of a locally complete parameter space M of Legendrian rational curves in a contact manifold), i.e. the PDE of prolongable Veronese conic structures, (or, equivalently, Veronese conic structures with pre-permissible intrinsic torsions) can now be formulated in a more explicit way. Indeed, the space A^{pio} of pre-permissible torsions has been explicitly described in the last section as the 'extensional sum' of the entries of (V.9) with the last one excluded. In other words, a decomposition of the space $A^{pio\,/pi}$ of pre-permissible *intrinsic* torsions into irreducible components is given by

$$\frac{T_r^{\odot x} \dot{+} T_r^{(\odot x-2)} \dot{+} \frac{\frac{T_r^{\odot x-2} \oplus T_r^{\odot x} \oplus T_r^{\odot x+2}}{T_r^{(\odot 2-x)}} \dot{+} 0 \dot{+} T_r^{(\odot x-4)}}{T_r^{(\odot 4-x)}} \dot{+} T_r^{(\odot x-6)}}{\frac{T_r^{\odot x} \oplus T_r^{\odot x-2} \oplus T_r^{\odot x} \oplus T_r^{\odot x+2}}{E^{sa_\bullet te}}}. \tag{V.10}$$

(Cf. the decomposition of the space E^{te} of tangential connectors given in Proposition V.9.) In particular, this space indeed has at most seven irreducible components, as we had already announced. As a consequence of this relative 'sparseness' of the pre-permissible intrinsic torsions (among arbitary intrinsic torsions), i.e. of the 'richness' of the space of (necessarily homogeneal) intrinsic pretorsions, the above PDE is overdetermined: Indeed, on the one hand it is defined on the Veronese conic-structural bundle UM, whose fibers $U \coloneqq G^{/cs} :! = G/G^{cs}$ are clearly of dimension $(x+1)^2 - 4$ ('the number of dependent variables'); on the other hand the 'number of equations' equals the dimension of the space $A^{/pio}$ of homogeneal intrinsic pretorsions, or, explicitly, the dimension $(x+1)^3$ of the space A of torsions decreased by the dimension of (V.10), i.e. by at most $4(x+1) - 12$. (Cf. Proposition V.12). QED

(ii) Let us consider the case $x = 3$. Claim: The space of pre-permissible torsions is now equal to its subspace consisting of permissible torsion. (In other words, the space of pre-permissible intrinsic torsions vanishes.) *In particular, 1-prolongability of a conic structure is in this case equivalent to 1-flatness. Therefore the method of Proposition V.6(iii) produces in this case 1-flat conic structures, i.e. solutions of an a priori even more restrictive PDE. (This fact has been proved by a somewhat different method by R. Bryant in [3].)*

In summary, the present proposition is the proper generalization of the just quoted result in the following sense: while for Veronese conic structures of higher degrees x the PDE of 1-flatness has -as proved in the same article- only trivial integrals, the same is not true of the PDE of (1-) prolongability: according to Proposition V.6, its integrals are precisely given by the same twistorial construction.) QED

[Proof of (ii): Direct computation easily shows that no other component of (V.10) is non-zero.QED]

V.5.2 Conjunctive Prolongability of Prolongable Veronese Conic Structures

Proposition V.14 Consider an expanded Veronese conic structure with underlying manifold M.

(i) Suppose the conic structure is of degree $x \geq 5$. Since this conic structure is of type one (according to Proposition V.8), the assumptions of Proposition III.10 are fulfilled in this situation. In particular, *all the conclusions therefrom apply to the case of Veronese conic structures of degree at least five.*
(Explicitly, there exists at most one conjunctively 1-prolongable -in particular at most one conjunctively integrable- preconnection on the given conic structure. Therefore, the structure of a locally complete parameter space of rational curves of self-intersection $x(\geq 5)$ is a first-order geometric structure. However, recall that these conic structures are not of *elementary pretype* one, as opposed to e.g. paraconformal structures, and consequently conjunctive prolongability is possibly not equivalent to ordinary prolongability. Similarly, the argument we have used included the consideration of the tangential symmetric elementary-preconnnection bundle (defined on the whole integral-jet space J^{ϵ} at a point, not only the tangential symmetric elementary preconnnections at one integral jet.) QED

(ii) Let us suppose the given conic structure is 1-prolongable, i.e. a solution of the PDE from the previous subsection. The conjunctively intrinsic pretorsion is then an element of the module ${T_r}^{(\odot x-6)}$ from (V.10). (Notice that this module is by definition a smaller quotient of the 'numerator' in (V.10) then the quotient (V.10)).

Incidentally, we know this invariant can be interpreted as the negative of the affine-bundle class associated to (fiberwise non-empty due to prolongability) tangential symetric-elementary preconnnector bundle and is thus the precise obstruction to conjunctive 1-prolongability (i.e. the second-order obstruction to conjunctive integrability).

In particular, if we specialize to the case $x \leq 5$, this invariant always vanishes, i.e. tangential symmetric preconnections (on the given *prolongable* expanded conic structure) always exist.

The further specialization to the case $x = 5$, which is at the confluence with the case from (i), enables one to claim both existence and uniqueness of a tangential symmetric preconnection. QED

(iii) Suppose the structure is conjunctively prolongable. Our next objective is an investigation of the **conjunctive-integrability PDE** of the given conic structure, which is defined as the first-order quasiaffine PDE on the tangential symmetric *pre*connection bundle $H^0(C^{sa\bullet te}J^{\varepsilon})M$ (over the manifold M) whose integrals are precisely the integrable preconnections on the given conic structure. In this context **conjunctive 2-prolongability** of the conic structure, i.e. vanishing of the **conjunctively intrinsic precurvature**, is a necessary condition for conjunctive integrability, more precisely a necessary condition for the existence of a PDE-integral jet through every point of the above affine bundle.

In the case $x \geq 5$ the fibers of the above affine bundle consist of single points (since the pretype is zero according to Proposition V.8). Thus it has only one section (namely the unique tangential symmetric expanded preconnection), and conjunctive integrability of the conic structure is clearly equivalent to conjunctive 2-prolongability, i.e. to the vanishing of the conjunctive intrinsic precurvature.(Recall that the latter is by its very definition a third-order invariant of the conic structure.)

In the cases resp. $x = 2, 3, 4$ the fibers of the above affine bundle have already been seen in Proposition V.12 to be of dimensions resp. 3,2,1. Therefore the study of conjunctive integrability of the conic structure in these cases involves the study of various properties (e.g. prolongability) of the above PDE. Notice that the conjunctive 3-pr olongability of the conic structure is a priori only a necessary condition for the 1-prolongability of the PDE.

V.5.3 Complemental Prolongability of Conjunctively Prolongable Veronese Conic Structures

Proposition V.15 Consider a conjunctively (1-) prolongable expanded Veronese conic structure.

(i) According to Proposition V.9 at any given point m the associated complementally intrinsic pretorsion belongs (being conjunctively permissible) to the quotient of the module $T_r{}^{(\odot x-4)}$ from (V.9).

Incidentally, we know from Proposition III.12 (vi.4) that this invariant can be interpreted as the negative of the affine-bundle class associated to (fiber-wise non-empty due to prolongability) tangential symmetric *full*-elementary preconnnection bundle and is thus the precise obstruction to complemental prolongability. In addition to that, we saw in that proposition that its preimage relative to the canonical mapping of 1-fattening classes (determined by the localized conic structure) consists precisely of the 1-fattening classes associated to tangential *symmetric* preconnections.

(ii) Suppose the degree x of the Veronese conic structure is at least five. In accordance with Proposition V.8 the pretype is one, and thus the above preimage consists of a single 1-fattening class, namely the one associated wiht the (unique) tangential symmetric preconnection; in other words, the canonical mapping of affine-bundle classes is bijective. The explicit description of this mapping is in this case facilitated by the (already observed) fact that isomorphism of two irreducible G^{cs}-modules is unique up to a constant factor.

(iii) Suppose $x = 4$. It is clear from (V.9) that the space of conjunctively permissible complementally intrinsic pretorsions vanishes. (Indeed, it suffices to observe that in the 'total denominator' in (V.9) the only two 1-dimensional modules must be 'cancelled'.) Therefore the conic structure is always complementally prolongable and all 1-fattening classes (which form a vector line) can be obtained from tangential symmetric preconnections (which also form a line). More succinctly, the canonical mapping of tangential symmetric preconnections into 1-fattening classes is a bijection of vector lines. **(iv)** Suppose $x \leq 3$. Then both the space of 1-fattening classes and conjunctively permissible complementallyintrinsic pretorsions vansih.

Chapter VI

Appendix

VI.1 Generic Notation

(a) EM *= A bundle (or the total space thereof) over* M *whose fibers will be denoted by* E*, or, more precisely,* E_m*. Pull-backs (in particular restrictions) of that bundle will most often have fibers denoted by* E *as well; in other words, such a pull-back relative to a map* $M_1 \to M$ *will usually be denoted by* EM_1*. However, this always has to be stated in the definition of* EM_1 *(which is thought of as a single -although 'composite'- symbol whose meaning is not completely determined by the meanings of* E *and* M_1*; thus, the bundle denoted by* EM_1 *does not have to be a pull-back even if the bundle* EM *and a map* $M_1 \to M$ *are given.*

(b) When the distinction between a bundle EM *and its total space has to be made in notation, the latter will be denoted by* $E.M$*. A section of that bundle (which will also be called 'a field valued in that bundle') will be denoted by* $\{e\}_M$ *(this symbol could be thought of as an abbreviation for the system* $\{e_m\}_{m \in M}$*), and its values by* e_m *or simply* e*.*

(c) If a point m *in the base of a bundle* EM *and a point* e *of the fiber* E *over* m *are given, the corresponding point of the total space is denoted by* $e.m$ *or (if the fibers* E *already are disjoint) it is taken to be* e*. Similarly, the 'disjoint copy' (a submanifold of the total space) of the fiber* E *over* m *is denoted by*

$E.m(\subset E.M)$.

(d) M^α will usually denote a subspace of the space (meaning manifold or vector space etc.) M.

(e) A quotient space of M will usually be denoted by $M^{/\alpha}$.

(f) If T, T^* is a pair of dual vector spaces and $x + y = dimT$, we will often identify the Grassmanian spaces $Gr(x,T)$ and $Gr(y,T^*)$ (by requiring that two perpendicular subspaces have the same parameter) and denote this space by $J(x,y)$ of just J. In this context the two pairs of dual spaces determined by a direction $j \in J$ will usually be denoted by T^α, $T^{*/\alpha}$ and $T^{/\alpha}$, $T^{*\alpha}$. (Explicitly, $T^{*\alpha}$ is the perpendicular of T^α and $T^{/\alpha}$ is the complemenary quotient T/T^α to the given subspace etc.). Furthermore, the subspace $T^\alpha \cap T^\alpha$ (resp. $T^\alpha + T^\alpha$) will be denoted by $T^{\alpha \bullet \alpha}$ (resp. $T^{\alpha * \alpha}$). (Consequently, $T^{*\alpha} \cap T^{*\alpha}$ is denoted by $T^{\alpha * \alpha}$ etc.)

(g) The tangent space at a point s of a manifold S will be denoted by T_S or, more precisely, by $(T_S)_s$. Usually the manifold of a central importance will be denoted by M and its tangent spaces T_M simply by T.

(h) If a submanifold M^α of M is given, its tangent, normal, cotangent and conormal bundles will usually in accordance with (d), (e), (f) and (g) be briefly (but still suggestively) denoted by resp. $T^\alpha M^\alpha$, $T^{/\alpha} M^\alpha$, $T^{*/\alpha} M^\alpha$ and $T^{*\alpha} M^\alpha$. Similarly, if S^α is a submanifold of S, its normal bundle will be usually denoted by $T_S^{/\alpha} S^\alpha$.

(i) In certain situations the upper indices (usually Greek letters) described in (d) - (f) will be double, i.e. they will be formed by two Latin letters, where the two parts will serve as markers of the two mutually complementary pairs of dual spaces from (f) as illustrated in the following example: Let $\mathbf{G}$ denotes the Lie algebra of the automorphism group G of a vector space T. In the following this Lie algebra will be the 'main' ambient space. We will usually denote by $\mathbf{G}^{vd}$ the vector subspace consisting of trace-free linear endomorphisms. Here the letters 'v' and 'd' suggest 'volume forms' and 'directions' in the following sense: With respect to the main ambient space (not with respect to its dual!)

the second part of the double index will be the marker of the quotient space, *namely of the space* $\mathbf{G}^{/vd}$ *of infinitesimal automorphisms of the Grassmanian space; the letter 'd' has been chosen for this purpose in order to suggest that the elements of the quotient space 'act on directions'.* In accordance with the above connection, the first part of the double index will be the marker of the subspace, *which in this case 'acts on volume forms'. Of course, for the dual ambient space* $\mathbf{G}^*$ *the conventions will be opposite. In other words, it is implied that for a subspace of* $\mathbf{G}$ *(resp.* $\mathbf{G}^*$*), the first part (resp. the second part) of the double index is indicative of the pair of dual spaces consisting of the subspace and its dual.*

Since in this case there is a distinguished complementary subspace *to* $\mathbf{G}^{vd}$, *namely the space* $\mathbf{C} \cdot id$ *of 'trace-part elements', we will denote it by* $\mathbf{G}^{dv}$. *(However, such a convention will not always be applicable.)*

(j) For a parameter space of objects with a certain property the parametrization does not have to be bijective. If it is injective (resp. bijective), such a parameter space is said to be effective or simply a space of objects with that property *(resp. complete or simply* space of all objects with that property*). For instance, a transformation group is a parameter transformation group such that the corresponding action (which is a homomorphism of that group and the group consisting of all transformations) is injective.*

VI.2 Cartan Distribution

In this section we will expound the standard reduction ('by lifting to the contact manifold') of the problem of integrating generalized conic structures (or 'substratal PDEs') to the problem of finding certain possibly non-equidimensional integral submanifolds of the associated 'Cartan distribution' on the integral-jet space. Although this construction can be carried out in a succesive way for generalized conic structures of arbitrary order, we will concentrate our attention on first-order generalized conic structures. (The second-order conic structures of the greatest importance in the geometriztion of double fibrations are general preconnections, i.e. those which are already distributions on the *structural global 1-jet space. The associated Cartan distribution on the total integral 2-jet space essentially coincides with that distribution, i.e. lifting to the second-order contact manifold produces nothing new.)*

We begin by reviewing the relevant facts and outlining their (conceptual and intuitively very clear) proofs.

Remark VI.1 Suppose M_{csm} is a x -dimensional and y -codimensional-expanded conic structure, where y is not necessarily 1 . Let us introduce notation as in Chapter 1 . In particular, the total integral manifold of jets is denoted by $R = J^{\varepsilon}M \subset JM$, while $T^{\alpha}R$ denotes the restriction of the (tautological) integral-tangent bundle $T^{\alpha}J.M$, and $T^{/\alpha}R$ denotes the restriction of the integral-transverse bundle $T^{/\alpha}J.M =: \frac{T}{T^{\alpha}}J.M$. Furthermore, as in Chapter 1, the (fiberwise) y -codimensional structural distribution of the contact manifold JM (consisiting of all y -codimensional jets) is denoted by $FJ.M$.

Let us observe that the total integral manifold of jets R is span-transverse to this distribution (since at each point r already the spaces T_J and T_R span T_{JM}). For this reason the problem of solving the first-order generalized conic structure determined by M_{csm} is reducible (just like in the case of more general generalized conic structures with that property) to the investigation of possibly non-equidimensional integral submanifolds of a certain distribution-equiped manifold . More precisely, we define the **Cartan distribution** for M_{csm} as the distribution formed by induced elementary structural prejets (of localized conic structures) at various jets $j.m$ (cf. Proposition III.4); explicitly, it is defined as the distribution $F^{\alpha_o}R$ on R formed by intersections F^{α_o} of the subspaces F and T_R of T_{JM}; these intersections indeed form a (holomorphic) distribution because of transversity, i.e. since the set-theoretical distribution $F^{\alpha_o}R$ is the kernel of the surjective mapping of vector bundles

$$T_R R \longrightarrow \frac{T_{JM}}{F}R = \frac{T}{T^{\alpha}}R =: T^{/\alpha}R$$

What is more, this argument also shows that the Cartan distribution is fiberwise y -codimensional . Similarly, if we introduce notation

$b := dim\ J^{\varepsilon}$ *and* $d := dim\ J - b$ (= *the codimension of manifolds* J^{ε} *in* J), the obvious fact

$$dim\frac{T_{JM}}{T_R}\ (= \text{'}\textit{the number of equations' in classical language}) = h$$

and an argument analogous to the one just pursued shows that the fibers F^{α_o} of that distribution are h -codimensional in the $(dim\, J + x)$ -dimensional spaces F; in particular, the Cartan distribution $F^{\alpha_o}R$ is (fiberwise) $(b + x)$ -dimensional (since by definition of $h \quad dim\ J = b + h$) and thus

$$b + x = dim\ R - y = dim\ F - h.$$

Let us also observe that here $b, h \leq xy$ since $dim\ J = xy$.

The above mentioned reduction of the problem of solving the first-order generalized conic structure consists in the following standard and almost evident fact: the x -dimensional submanifolds M^α of M which solve the equation correspond bijectively to those x -dimensional integral submanifolds R^{uj} of the Cartan distribution-equiped manifold R which project biholomorphically onto submanifolds of M. More precisely, for a given R^{uj} the corresponding M^α is simply its image under the projection, while for a given M^α the corresponding (relative to the inverse correspondence) R^{uj} is precisely its canonical lift to the contact manifold JM. For the sake of completeness we sketch the (simple and intuitively very clear) proof:

By the very definition of a generalized conic structure an integral submanifold is characterized by the requirement that its jets in M (at all of its points) be integral, i.e. that its lift be contained in R. Thus we only have to characterize those submanifolds of R which are lifts. Now we recall that by the main property of the (general possibly not hypersurface-) contact manifold JM (v. Appendix) the submanifolds of JM which are lifts are precisely those x -dimensional integral (relative to the contact structure) submanifolds which project biholomorphically onto submanifolds of M. There remains only to take into account the definition of the Cartan distribution.

In addition to this, the integral submanifolds of the Cartan distribution-equiped manifold R which are only transverse to the fibers J^e of $J^eM = R$ clearly give rise locally to integral submanifolds of M_{csm} and are therefore called 'multi-valued' integral submanifolds of M_{csm}.

Finally let us recall that the Frobenius tensor fr_R of the Cartan distribution at a point r is an alternate bilinear map on F^{α_o} with values in $\frac{T_R}{F^{\alpha_o}} = T_R^{/\alpha} = T^{/\alpha}$. More succinctly, $\mathit{fr}_R \in Hom((F^{\alpha_o})^{\wedge 2}, T^{/\alpha})$.

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