Associativity of Quantum Multiplication

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Abstract of the Dissertation
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We prove the associativity of the multiplication of quantum cohomology for a monotone compact symplectic manifold V for which $c_1(A) > 1$ for every effective class$A \in H_2(V)$. The same proof with a suitable modification also works for any positive compact symplectic manifold with $c_1(A) > 1$.

To my family

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Chapter 1

Introduction

Quantum cohomology was introduced by the physicists Vafa and Witten. The associativity of quantum multiplication and its relation with composition law was described by Witten in his influential paper [Wi]. It has many significant applications and remarkable relations with various branches of mathematics, such as applications in classical enumative algebraic geometry, and in counting rational curves in Calabi-Yau manifolds, and relation to Floer homology. It also has several important equivalent forms. For instance, it is equivalent to the fact that the related Gromov-Witten potential satisfies the WDVV-equation. This was used by Ruan and Tian in [RT2] to obtain a flat connection over the complexified quantum cohomology, which is a deformation of the trivial connection, but is not Gauss-Manin. It is a key step towards proving the mirror symmetry conjecture. Inside the subject of quantum cohomology itself, the law of associativity and composition form the basis of recent calculations of the quantum cohomology rings of several important geometric spaces, such as Grassmanianns and flag manifolds. A rigorous mathematical proof of associativity, however, was obtained only recently, first by Ruan and Tian in [RT2] as a corollary of a general composition law, and shortly after that by myself in [L] independently by a different method. This dissertation is devoted to give a detail presentation of the proof of associativity in [L].

A manifold V is called symplectic if there is a closed non-degenerate 2-form ω on V. Such a form ω is called a symplectic form. A diffeomorphism $f : V_1 \to V_2$ between two symplectic manifolds (V_1, ω_1) and (V_2, ω_2) is called symplectomorphism if $f^*\omega_2 = \omega_1$. Unlike Riemanian geometry where curvature is a local invariant, all symplectic manifolds look like the same locally. This is the context of the classic Darboux' theorem. Therefore, all symplectic invariants have to be global. For a given compact symplectic manifold (V, ω), there are two obvious global invariants.

The first is the cohomology class of the form ω. Note that the condition that ω is non-degenerate is equivalent to that the top wedge product ω^n is not zero at every point of V for a $2n$-dimensional symplectic manifold V. Together with the condition that $d\omega = 0$, this implies that the form ω gives rise to a non-zero cohomology class $[\omega] \in H^2(V, \mathbf{R})$.

The second is the homotopy class of the ω-compactible almost complex structures. It can be described as following:

Recall that a section J of the bunddle $Hom_R(TV)$ is called an almost complex structure if $J^2 = -Id$, and it is ω-compactible if $g_J(x, y) = \omega(Jx, y)$ is a J-invariant Riemannian metric on V. It can be proved that the set $\mathcal{J}(\omega)$ of all such J is a non-empty contractible set. Therefore, we have a well-defined homotopy class of ω-compatible almost complex structures, and their associated invariants, such as characteristic classes.

However a famous example of D. McDuff shows that there exists a compact manifold, on which there exsit two non-isomorphic symplectic forms with the same homotopy invariants. This suggests that more subtle global invariants are needed other than the above simple homotopy invariants. Such invariants were obtained in 1985 in Gromov's remarkable paper [G], in which Gromov defined the moduli space of J-holomorphic curves and its associated invariants. It was the initial point of the subsequently important developments in symplectic topology, Floer homology and quantum cohomology. Unlike those "soft" invariants coming from algebraic topology, the Gromov theory of J-holomophic curves has a "hard" nature. It has been successfully used to measure the symplectic rigidity, such as in Floer's proof of Arnold conjecture and Gromov's proof of non-squeezing theorem. It is also one of the cornerstones of quantum cohomology.

We will give an outline of this theory here.

(i) Moduli space of J-holomorphic curves.

As we mentioned before, the set $\mathcal{J}(\omega)$ of ω-compactible almost complex structures is a non-empty contractible set. For a $J \in \mathcal{J}(\omega)$ and the complex structure i of S^2, a map $f : (S^2, i) \to (V, J)$ is called J-holomorphic if $df \circ i = J \circ df$. This condition is equivalent to that $\overline{\partial}_J f = df + J \circ df \circ i = 0$. In a local conformal coordinate (s, t) of S^2, it has the form:

$$\frac{\partial f}{\partial s} + J(f)\frac{\partial f}{\partial t} = 0.$$

This is a quasi-linear elliptic equation. Its linearization $D\overline{\partial}_J(f)$ at a J-

holomorphic map f is an elliptic operator from the Sobolev space $L^p_1(f^*TV)$ to $L^p(\Omega^{0,1}(f^*TV))$ for some $p > 2$, given by

$$\xi \to \nabla\xi + J(f) \circ \nabla\xi \circ i + \frac{1}{4} N_J(\partial_J(f), \xi),$$

where the connection ∇ is a J-invariant connection with its torsion proportional to the Nijenhuis tensor N_J of J.

Definition. For a given almost complex structure $J \in \mathcal{J}(\omega)$ and $A \in H_2(V)$, the moduli space $\mathcal{M}(A, J)$ is defined as follows:

$$\mathcal{M}(A, J) = \{f \mid f : S^2 \longrightarrow V \textit{ is } J\textit{-holomorphic and simple}, \; [f] = A\}.$$

Here a J-holomorphic curve f is called simple if any factorization $f = f_1 \circ \pi$ with $\pi : S^2 \to S^2$ being holomorphic and $f_1 : S^2 \to V$ being J-holomorphic implies that π is biholomorphic. The reason for restricting to simple curves is to avoid the pathological phenomenon concerning transversality of the linearization $D\overline{\partial}_J$ of $\overline{\partial}_J$ at multiply covered curves.

Now for a generic choice of J, $D\overline{\partial}_J$ is surjective at every $f \in \mathcal{M}(A, J)$. Later on we will denote the set of such J by $\mathcal{J}_{reg}(\omega)$. It follows from the ellipticity, hence Fredholm property of $D\overline{\partial}_J$ and implicit function theorem that

Theorem 1.0.1 *For any $J \in \mathcal{J}_{reg}(\omega)$, $\mathcal{M}(A, J)$ is a smooth finite dimensional manifold. If the dimension of V is $2n$, the dimension of $\mathcal{M}(A, J)$ is $2(c_1(A) + n)$.*

The above moduli space $\mathcal{M}(A, J)$ depends not only on the symplectic form ω and class A, but also on a particular choice of $J \in \mathcal{J}_{reg}(\omega)$. In order to get invariants of ω, we need to know how $\mathcal{M}(A, J)$ depends on J. It is proved in [G] and [M]:

Theorem 1.0.2 *For any J_1 and J_2 in $\mathcal{J}_{reg}(\omega)$, $\mathcal{M}(A, J_1)$ and $\mathcal{M}(A, J_2)$ are in the same bordism class.*

According to this, for any $J \in \mathcal{J}_{reg}(\omega)$ the bordism class of $\mathcal{M}(A, J)$ is well-defined, depending only on ω and A. However, any manifold M is the boundary of the non-compact manifold $M \times [0, 1)$. Therefore, in order to make the above bordism class useful, we have to consider the compactness of $\mathcal{M}(A, J)$. But, due to the biholomorphic action of $SL(2, \mathbf{C})$ on S^2, which induces reparametrizations on J-holomorphic curves, $\mathcal{M}(A, J)$ is never compact. One may rule out this trivial non-compactness by factoring out the reparametrization group $SL(2, C)$ and defining the unparametrized moduli space

$$\widehat{\mathcal{M}}(A, J) = \mathcal{M}(A, J)/SL(2, \mathbf{C}).$$

Even then, this unparametrized moduli space $\widehat{\mathcal{M}}(A, J)$ is still not compact in general. But it has a natural compactification, i.e. the Gromov compactification $\bar{\mathcal{M}}(A, J)$ of $\widehat{\mathcal{M}}(A, J)$.

In order to formulate this, we need the notions of cusp curves and weak convergence of J-holomorphic curves.

A cusp curve $C = C_1 \cup \cdots \cup C_l$ is a union of J-holomorphic curves C_i with marking points $x_{i,j}$ in S^2 of the domain of C_i, $i = 1, \cdots, l$ and $j = 1, \cdots m_i$

and identifications of pairs of marking points such that $C_i(x_{i,n}) = C_j(x_{j,k})$ if $x_{i,n}$ and $x_{j,k}$ are identified with each other. The homology class $[C]$ of C is $\sum_{i=1}^{l}[C_i]$ and the area $Area(C)$ of C is $\sum_{i=1}^{l} Area(C_i)$. Note that there may exist repeated components $C_i = C_j$ for some $i \neq j$.

Now assume that $\{f_j\}$ is a sequence of J-holomorphic curves. Let $(f, x_1, \cdots, x_m)$ be a J-holomorphic curve with marking points $x_1, \cdots, x_m$. We say that $\{f_j\}$ is locally convergent to f outside the marking points if $\{f_j\}$ is C^∞-convergent to f on any compact subset of $S^2 - \{x_1, \cdots, x_m\}$. If $C = C_1 \cup \cdots \cup C_l$, we say that $\{f_j\}$ is weakly convergent to C if

(i)for each j, there are l elements $u_{j,1}, \cdots u_{j,l}$ of $SL(2, \mathbf{C})$ such that each sequence $\{f_{j,i}\} = f_j \circ u_{j,i}$, $i = 1, \cdots l$, is locally convergent to C_i outside the marking points of C_i.

(ii) $lim_{j \to \infty} Area(f_j) = Area(C)$.

This implies that the image of f_j is C^0-convergent to the image C and that the class $[f_j] = [C]$ for large j. With this preparation, we can state the Gromov's compactness theorem.

Theorem 1.0.3 *Let $\{f_j\}$ be a sequence of J-holomorphic spheres with $Area(f_j)$ $< C$ for some constant C for all j. Then there exists a subsequence of $\{f_j\}$ (we still denote it by $\{f_j\}$) such that f_j is weakly convergent to a cusp curve C.*

(ii) Quantum cohomology

In order to avoid further algebraic ramification, we will define quantum cohomology only for monotone symplectic manifold.

A symplectic manifold (V, w) is called monotone, if there exists a constant $\lambda > 0$ such that for any $f \in \pi_2(V)$, $\omega(f) = \lambda c_1(f^*TV)$. By rescaling the symplectic form ω, we may assume that $\lambda = 1$. The quantum cohomology $QH^*(V)$ of a monotone symplectic manifold is additively just the usual cohomology of V with coefficients in $\mathbf{R}[q]$, the polynomial ring in q. However, its multiplication is a deformation of the ordinary cup-product which can be described as follows. If a_1', a_2' are two cohomology classes in V, then we will define the quantum product $a_1' * a_2'$ by specifying its pairing with all classes $a_3' \in H^*(V, \mathbf{R})$. To define this triple index, we have to first introduce the evaluation maps of the moduli space $\mathcal{M}(A, J)$. The p-fold evaluation map

$$e_{A,J} : \mathcal{M}(A, J) \times (S^2)^{p-3} \to V^p$$

is given by:

$$e_{A,J} : (f, z_1, \cdots, z_{p-3}) \mapsto (f(0), f(1), f(\infty), f(z_1), \cdots, f(z_{p-3})).$$

Now assume that $dim(V) = 2n$ and that, for $i = 1, 2, 3$, a_i is a submanifold in V of codimension $2\alpha_i$, which is Poincaré dual to the class $a_i' \in H^*(V, \mathbf{R})$. (We will later deal with the fact that not every homology class has such a representative.) Fix a $J \in \mathcal{J}_{reg}(\omega)$, the triple index is defined as follows:

Definition.

$$< a_1' * a_2', a_3' >= \sum_{A \in H_2(V)} \#(e_{A,J}^{-1}(a_1 \times a_2 \times a_3)) q^{\omega(A)},$$

where the sum runs over all A such that $c_1(A) + n = \alpha_1 + \alpha_2 + \alpha_3$. Here $e_{A,J}$ is the 3-fold evaluation map and $\#(e_{A,J}^{-1}(a_1 \times a_2 \times a_3))$ is the oriented intersection number of $e_{A,J}$ and $a_1 \times a_2 \times a_3$,. (This makes sense since $\mathcal{M}_{A,J}$, V^3 and $a_1 \times a_2 \times a_3$ are all oriented.)

Remark.

(1) The dimension conditions are always chosen so that the sets which are being counted have dimension 0 for generic J.

(2) Using Gromov's compactness theorem and the mononotonicity assumption, one can show that the coefficient $\#e_{A,J}^{-1}(a_1 \times a_2 \times a_3)$ before $q^{\omega(A)}$ is finite for generic J.

(3) Since $\alpha_1 + \alpha_2 + \alpha_3 \leq 3n$, we have $\omega(A) = c_1(A) \leq 2n$. Therefore the triple index $< a_1' * a_2', a_3' >$ is in $\mathbf{R}[q]$.

(4) The zero order term of $a_1' * a_2'$ is just the ordinary cup product $a_1' \cup a_2'$. This can be seen as follows: The condition that $\omega(A) = c_1(A) = 0$ implies that f is a constant map for any $f \in \mathcal{M}_{A,J}$. Therefore

$$\sum_{A,\omega(A)=c_1=0} \#e_{A,J}^{-1}(a_1 \times a_2 \times a_3)$$

is nothing but the triple intersection number of a_1, a_2 and a_3.

In above definition, we have assumed that the cycles there are actually submanifolds, which is not true in general as is well-known. For the purpose of the above definition, we can get around the problem by using the fact that there is a basis of rational homology $H_*(V, \mathbf{Q})$ consisting of push-forward images of fundmental cycles of some bordism classes. However, there are two places where we would like to consider a representative as a submanifold. The first is at the beginning of chapter 3 where the local charts are given for the

Banach manifold of maps from S^2 with marked points to the manifold V with given cycles, and the second is in chapter 5 where we do the 'bubbling' analysis. Therefore some justification is needed. It is well-known that every cycle can be represented by a map from a pseudo-manifold (a stratified manifold with at most codimension 2 singularities) to V. There are two obvious differences between the pseudo- manifold representatives and submanifold representatives. The first is that in the former case one needs to use a stratified manifold structure, but this causes no further problems since we can work on each strata equally well. The second is that in the former case one cannot think of a representative as a subset of V, there is a map from the pseudo-manifold to V involved. Although for most part of this dissertation one could replace the map by its graph, for the basic Banach manifold setup with marked point case of Chapter 3 we have to require that the image of the map is at least a stratified submanifold of V so that the top strata can be throught as a geodesic submanifold under some suitable choice of metric on V. This can be done by using the fact that every compact smooth manifold has a smooth triangulation (see, [MI]). We can then use simplicial cycles as representatives which can be thought as maps from some pseudo-manifolds to V, whose image, the carrier of the simplicial cycle, is a a embedded stratified submanifold by the definition of the smooth triangulation. Note that we do not require that the carrier of the simplicial cycle is a submanifold with singularity of at least codimension 2, since what is really involved is only the top strata for all constructions of this dissertation except the property that the triple index in the above definition is well-defined, independent of any particular choice of $J \in \mathcal{J}_{reg}(\omega)$ which is already established in [MS1] and [R].

Therefore to simplify our presentation we will assume throughout this dissertation that the cycles in V can be represented by some submanifolds whenever we feel 'safe', and indicate proper modifications when problems arise.

The above definition of quantum multiplication obviously encodes important information about Gromov's moduli space of J-holomorphic curves, and makes it possible to organize the information in a nice way as long as some formal algebraic properties, such as associativity and the related composition law, hold.

It was pointed out by Witten in [Wi] that associativity can be deduced from a composition law for the 4-point Gromov-Witten invariant, which we now explain. Suppose given homology classes $a_1, a_2, a_3, a_4 \in H_*(V, Z)$ and $A, B, P \in H_2(V)$, and a "generic" point $z = (z_1, z_2, z_3, z_4) \in (S^2 - \infty)^4$. Let $M^z(P, a_1, a_2, a_3, a_4)$ be the moduli space of J-holomorphic curves f with $f(z_i) \in a_i$ and $[f] = P$, and $M(A, B; a_1, a_2; a_3, a_4)$ be the moduli space of (A, B)-cusp-curves (f_1, f_2) with a similar restraint plus the condition that $f_1(\infty) = f_2(\infty)$. For a suitable choice of classes P, A and B, and a generic J, the above two spaces are compact smooth manifolds of dimension zero. Gromov-Witten invariants $\Phi_P(a_1, a_2, a_3, a_4)$ and $\tilde{\Phi}_{A,B}(a_1, a_2; a_3, a_4)$ are defined as the number of points inside these two spaces, counted with sign. The composition law for 4-point Gromov-Witten invariants is the following equality:

$$\Phi_P(a_1, a_2, a_3, a_4) = \sum_{A+B=P} \tilde{\Phi}_{A,B}(a_1, a_2; a_3, a_4).$$

In order to prove this equality, we need to establish an oriented bijection

$$\#_z : \coprod_{A,B\in H_2(V):A+B=P} \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4) \to \mathcal{M}^z(P, J; a_1, a_2, a_3, a_4).$$

Such a map glues each (A, B)-cusp-curve into an $A + B$ curve. We prove that there is such a gluing map $\#_z$, when the cross ratio of z is large enough. This is the main technical part of this dissertation.

Gluing in symplectic geometry was initiated by Floer by adapting Taubes' gluing construction in gauge theory. He used this construction for gluing broken trajectories, which was crucial in Floer homology. However, there is a big difference between the gluing in our case and that in Floer homology. In Floer homology, the ends of each trajectory convergence to some periodic Hamiltonian orbits , which do not move. For a generic choice of Hamiltonian, the linearization of the Hamiltonian equation has no kernel. This implies that all trajectories will exponentially decay to some Hamiltonian orbits, and we only need to set up Fredholm theory by using the ordinary Sobolev L_1^p -norm for the relevant Banach manifolds and Banach bundles. The main estimate, which establishes the uniform invertibility of the linearization of the perturbed $\overline{\partial}_J$-operator, is relatively easy in this case.

Now consider an (A, B)-cusp-curve (f_1, f_2), which is a union of two J-holomorphic spheres f_1 and f_2 meeting at a cuspidal point. This cuspidal point can be thought as the limit of two cylindrical ends, one on each sphere, but now it can move in many different directions through movement of (f_1, f_2)

within the moduli space of (A, B)-cusp-curves. Closely related to this, the operator D, which is asymptotic to the linearized $\overline{\partial}_J$-operator E, has a non-trivial kernel. In this situation, in order to get a Fredholm theory for the linearized operator E, we have to make full use of the elliptic operator theory over non-compact manifolds with cylindrical ends (see [LM]) and to use the weighted Sobolev norms. However, under any reasonable choice of a weight, the operator D still has non-trivial kernel, which will prevent us from proving the desired uniform invertibility of the linearized $\overline{\partial}_J$-operator over the pre-gluing family.

In his unpublished note, Floer already noticed this difficulty. His idea for dealing with the problem was to split each section over the pre-gluing family into two parts. Roughly speaking, one part corresponds to the movement at 'infinity' which has finite dimension , and the other corresponds to the nearby curves with fixed point at 'infinity'. He then gave the component of the first part the usual norm for elements of a finite dimensional vector space and the component of the second part a weighted Sobolev norm. The analytic behavior of the second component is completely similar to the case of Floer homology. But the real problem is how to deal with that simple looking component in the first part, and how to deal with the problem of the inconsistency of these two norms due to their different weights. This is the problem which Floer failed to deal with in his note. The difficulty was overcome in my work through an indirect argument, which combines the theory of elliptic operators over manifolds with cylindrical ends with a "removal singularity" theorem for the solution of the linearized $\overline{\partial}_J$-operator with a singular point. The existence of the gluing map $\#_z$ is then established by applying Picard method. The injectivity of $\#_z$ is more or less obvious since the domain of $\#_z$ is finite. Surjectivity follows from the uniqueness part of Picard method, together with a detailed analysis how the corresponding J-holomorphic curves degenerate when the cross ratio of z tends to infinity. In this way, we finished the proof of associativity.

This dissertation is organised as follows. Chapter 2 is an outline of the proof of associativity. We first translate the problem about associativity into a decomposition rule for the Gromov-Witten invariant in four variables. Then we indicate the main steps in the proof of this special decomposition rule.

Chapter 3 describes the basic Sobolev space setup and establishes transversality properties of various evaluation maps. Most of the results in this chapter are already in [M] and [MS1]. Our observation here is simply that the technique developed in [M] and [MS1] enables us to achieve all required transversalities by "moving" only the almost complex structure J of V and leaving the pseudo-cycles in V fixed. This will be important in Chapter 5 where we analyze how a J-sphere degenerates when two of the marked points on the given cycles approach each other. In order to use a dimension counting argument in this bubbling analysis, we have to require that the 4-fold evaluation maps of all relevant moduli spaces of cusp-curves are transversal to the product $a = a_1 \times a_2 \times a_3 \times a_4$ of the given four cycles of V. If we also perturb this cycle a in V^4 to achieve transversality, the product structure of the cycle could be destroyed. This will cause a problem in the very last step of the proof of Proposition 5.0.4 This is a minor point, but it was overlooked in [MS1].

Chapter 4 is on Floer's gluing technique and is the main technical part of this dissertation. To simplify our presentation, we develop the technique only for zero-dimensional components of the moduli space of (A, B)-cusp-curves, since that is all we need in order to prove associativity. However, the proofs in this section can be adapted to apply to any compact part of a moduli space of cusp curves, with arbitrary genus, marked points and varying complex structures. Even though it appears here only as an intermediate step in the proof of the associativity, a result of this kind can be viewed as the converse of Gromov's compactness theorem for J-holomorphic curves, and hence has its own interest.

Chapter 5 analyzes how a J-holomorphic curve with four marked points lying on four given pseudo-cycles degenerates when two of the marked points approach each other. One of the consequences of this analysis is that as long as two of the marked points are very close to each other, there will be a cusp curve such that the C^0-distance between the pre-gluing of the cusp curve and the J-curve above is very small. Combining this with the gluing procedure in chapter 4 one can easily prove the special decomposition rule mentioned before, thereby finishing the proof of associativity.

Chapter 2

Outline of the Proof

To begin with, let us consider what is involved in proving associativity. According to our definition, in order to know $(a_1' * a_2') * a_3'$ one needs first to know what the Poincaré dual $(a_1' * a_2')'$ is. For generic J, consider the 2-fold evaluation map

$$e_{A,J} : \mathcal{M}(A, J) \to V^2, \quad f \mapsto (f(0), f(1)),$$

and define

$$\mathcal{M}(A, J; a_1, a_2) = e_{A,J}^{-1}(a_1 \times a_2).$$

This is a smooth manifold of dimension $2(c_1(A) + n - \alpha_1 - \alpha_2)$. If we define the 'pseudo-cycle' in the sense of MS[1] (see MS[1], page 90)

$$\Phi_{A,J}(a_1, a_2) : \mathcal{M}(A, J; a_1, a_2) \to V$$

by $f \mapsto f(\infty)$, then we have:

Lemma 2.0.1

$$(a_1' * a_2')' = \sum_{A \in H_2(V)} \Phi_{A,J}(a_1, a_2) q^{\omega(A)}.$$

Proof. (sketch) Note first that since

$$\dim \Phi_{A,J}(a_1, a_2) = 2(c_1(A) + n - \alpha_1 - \alpha_2) \leq 2n,$$

we have $\omega(A) = c_1(A) \leq \alpha_1 + \alpha_2 \leq 2n$. Therefore, by Gromov's compactness theorem, the summation in the RHS must be finite. Now, on the one hand, we know that

$$< a_1' * a_2', a_3' >= \#((a_1' * a_2')' \cap a_3).$$

On the other hand, by definition 1,

$$< a_1' * a_2', a_3' >= \sum_{A \in H_2(V)} \#e_{A,J}^{-1}(a_1 \times a_2 \times a_3) q^{\omega(A)},$$

the sum being taken over such A that $n + c_1(A) = \alpha_1 + \alpha_2 + \alpha_3$. But

$$\#e_{A,J}^{-1}(a_1 \times a_2 \times a_3)$$

can be obtained by first taking the inverse image of $a_1 \times a_2$ under the 2-fold evaluation map $e_{A,J}$ which is $\mathcal{M}(A, J; a_1, a_2)$, and then taking the intersection of its image under $\Phi_{A,J}(a_1, a_2)$ with a_3. In other words,

$$\#e_{A,J}^{-1}(a_1 \times a_2 \times a_3) = \#(\Phi_{A,J}(a_1, a_2) \cap a_3),$$

which concludes the proof of the lemma. □

Let

$$e_{A,B,J} : \mathcal{M}(A, J) \times \mathcal{M}(B, J) \to V^6$$

be the evaluation map given by $(f, g) \mapsto (f(0), f(1), g(0), g(1), f(\infty), g(\infty))$. Given two spherical classes $A, B \in H_2(V)$ and cohomology classes $a_i' \in H^*(V)$,

$i = 1, \cdots, 4$, with Poincaré duals a_i of codimension $2\alpha_i$, we define the moduli space of (A, B)-cusp-curves to be

$$\mathcal{M}(A, B, J; a_1, a_2; a_3, a_4) = e_{A,B,J}^{-1}(a_1 \times a_2 \times a_3 \times a_4 \times \Delta),$$

where Δ is the diagonal. This moduli space is a smooth manifold of dimension $2(c_1(A+B) + n - \sum \alpha_i)$ for generic J.

It is easy to see that when $c_1(A+B) + n = \sum \alpha_i$, the space

$$\mathcal{M}(A, B, J; a_1, a_2; a_3, a_4)$$

has dimension 0, and the number of points in it (counted with their orientations) equals the intersection number of the cycles $\Phi_{A,J}(a_1, a_2)$ and $\Phi_{B,J}(a_3, a_4)$. In short,

$$\#\mathcal{M}(A, B, J; a_1, a_2; a_3, a_4) = \#(\Phi_{A,J}(a_1, a_2) \cap \Phi_{B,J}(a_3, a_4)).$$

Lemma 2.0.2

$$< (a_1' * a_2') * a_3', a_4' >$$
$$= \sum_{A,B \in H_2(V)} \#(\Phi_{A,J}(a_1, a_2) \cap \Phi_{B,J}(a_3, a_4)) q^{\omega(A+B)},$$

the sum being taken over all A, B such that $c_1(A+B) + n = \sum_{i=1}^4 \alpha_i$.

Proof. By definition,

$$< (a_1' * a_2') * a_3', a_4' > = \sum_{A,B \in H_2(V),\, f \in \mathcal{M}(B,J)} sign(f) q^{\omega(B)} \cdot q^{\omega(A)},$$

where the sum is taken over all $f \in \mathcal{M}(B, J)$ such that

(i) $f(0)$ is in the component $\Phi_{A,J}(a_1,a_2)$ of

$$(a_1' * a_2')' = \sum_{A\in H_2(V)} \Phi_{A,J}(a_1,a_2) q^{\omega(A)},$$

(ii) $f(1) \in a_3$, and $f(\infty) \in a_4$, and

(iii) $2(c_1(B)+n) = codim\Phi_{A,J}(a_1,a_2) + 2\alpha_3 + 2\alpha_4$.

Geometrically, the image of $\Phi_{A,J}(a_1,a_2)$ is just the set of points which lie on A-curves $g(S^2)$ intersecting a_1 and a_2 at $g(0)$ and $g(1)$ respectively, and its dimension is $2(c_1(A)+n-\alpha_1-\alpha_2)$. Therefore, each B-curve f which is counted above determines an A-curve g, and so gives rise to an (A,B)-cusp-curve (g,f) which intersects the given 4-cycles and satisfies the condition that $c_1(A+B)+n = \alpha_1+\alpha_2+\alpha_3+\alpha_4$. In other words, each such f determines an element of $\mathcal{M}(A,B,J;a_1,a_2;a_3,a_4)$. The conclusion follows immediately. □

Now for any $P \in H_2(V)$ with $c_1(P)+n = \sum \alpha_i$ we define the extended Gromov-Witten invariant

$$\Psi'_{P,J}(a_1,a_2;a_3,a_4) = \sum_{A,B\in H_2(V):A+B=P} \#(\Phi_{A,J}(a_1,a_2)\cap \Phi_{B,J}(a_3,a_4)).$$

Then Lemma 2.0.2 can be restated as

$$< (a_1' * a_2') * a_3', a_4' >= \sum_{P\in H_2(V)} \Psi'_{P,J}(a_1,a_2;a_3,a_4) q^{\omega(P)},$$

the sum being taken over P such that $c_1(P)+n = \sum_{i=1}^4 \alpha_i$. Similarly,

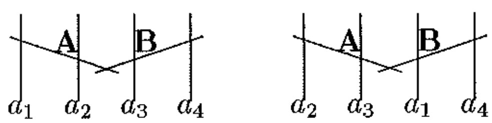
$$\begin{aligned} < a_1' * (a_2' * a_3'), a_4' > &= (-1)^{2\alpha_1(2\alpha_2+2\alpha_3)} < (a_2' * a_3') * a_1', a_4' > \\ &= (-1)^{2\alpha_1(2\alpha_2+2\alpha_3)} \sum_{P\in H_2(V)} \Psi'(P,J)(a_2,a_3;a_1,a_4) q^{\omega(P)}, \end{aligned}$$

the sum running over all P such that $c_1(P)+n = \sum \alpha_i$. Therefore, associativity will follow if we can prove that

$$\Psi'_{P,J}(a_1, a_2; a_3, a_4) = (-1)^{2\alpha_1(2\alpha_2+2\alpha_3)}\Phi'_{P,J}(a_2, a_3; a_1, a_4).$$

This is equivalent to the fact that $\Psi'_{P,J}$ is graded-commutative. It is obvious that $\Psi'_{P,J}$ is graded-commutative on its first two variables and last two too. But it is not obvious at all that it is also graded-commutative, for example, on its second and third variables.

Geometrically it is equivalent to the fact that the following two configuration spaces of cusp-curves have the same cardinality.



In order to prove the graded-commutativity of $\Psi'_{P,J}$, we will relate $\Psi'_{P,J}$ to the Gromov-Witten invariant $\Psi_{P,J}$ defined by Ruan in [R], which is graded-commutative by its definition. It can be defined as follows:

Fix $P, a_i, i = 1, \cdots, 4$ as above with $c_1(P) + n = \sum \alpha_i$. Consider the 4-fold evaluation map

$$e_{P,J} : \mathcal{M}(P, J) \times (S^2 - \{0, 1, \infty\}) \to V^4.$$

Then

$$\mathcal{M}(P, J; a_1, a_2, a_3, a_4) = e_{P,J}^{-1}(a_1 \times a_2 \times a_3 \times a_4)$$

is a smooth submanifold of $\mathcal{M}(P,J)\times(S^2-\{0,1,\infty\})$ of dimension 2. Consider the restriction of the projection

$$\pi_2 : \mathcal{M}(P,J) \times (S^2 - \{0,1,\infty\}) \to S^2 - \{0,1,\infty\}$$

to $\mathcal{M}(P,J;a_1,a_2,a_3,a_4)$ and denote it by π. Picking a generic point $z \in S^2 - \{0,1,\infty\}$, we define

$$\Psi_{P,J}(a_1,a_2,a_3,a_4) = \#(\pi^{-1}(z)).$$

Thus this invariant counts the number of curves which meet the given 4 cycles at the images of a fixed set of 4-points $\{0,1,\infty,z\}$. Later on we will use $\mathcal{M}^z(P,J;a_1,a_2,a_3,a_4)$ to denote $\pi^{-1}(z)$.

It is proved in [MS1] that, when $c_1(A) > 1$ for every effective class A, $\pi^{-1}(z)$ is finite for generic J and that the Gromov-Witten invariant Ψ_P is well-defined, and independent of the particular choices of z, J and representing cycles a_i. Here a class $A \in H_2(V)$ is called effective if it can be represented by a J-holomorphic curve for some ω-compactible J. It is also proved in [R], for example, that Ψ_P is graded-commutative. Therefore associativity will follow if one can prove the following special decomposition rule:

Theorem 2.0.4 *If V is monotone with $c_1(A) > 1$ for every effective class $A \in H_2(V)$, then we have*

$$\Psi_{P,J}(a_1,a_2,a_3,a_4) = \Psi'_{P,J}(a_1,a_2;a_3,a_4).$$

There is a heuristic argument in [MS1], Chapter 8 to explain why one could expect that such a relation holds. For a formal proof, we will construct a family of gluing maps with gluing parameter $z \in \mathbf{C}^*$:

$$\#_z : \coprod_{A,B \in H_2(V):A+B=P} \mathcal{M}(A,B,J;a_1,a_2;a_3,a_4) \to \mathcal{M}^z(P,J;a_1,a_2,a_3,a_4),$$

where $c_1(P) + n = \sum \alpha_i$ and $|z|$ is large enough. Thus this map associates to every (A,B)-cusp-curve through the α_i a unique $(A+B)$-curve which intersects the given cycles at the images of the fixed set of points $\{0,1,\infty,z\}$. The existence of these gluing maps is established in Chapter 5. The main problem here is to establish a uniform estimate for the inverse of the linearized $\overline{\partial}$-operator.

We will prove the special decomposition rule by showing that $\#_z$ is an orientation-preserving bijection when $|z|$ is large enough. The injectivity of $\#_z$ is more or less obvious, since the domain of $\#_z$ is a finite set. The surjectivity will follow from the uniqueness part of Lemma 4.0.16 (Picard's method), provided that one can prove that, when $|z|$ large enough, there is for any $f \in \mathcal{M}^z(P,J;a_1,a_2,a_3,a_4)$ an approximate J-holomorphic curve (or pre-gluing) $g = g_1\chi_z g_2$ which is made from the cusp-curve

$$(g_1,g_2) \in \mathcal{M}(A,B,J;a_1,a_2;a_3,a_4)$$

such that the C^0-distance between f and g is small. This can be shown by analyzing what can happen for a sequence

$$f_n \in \mathcal{M}^{z_n}(P,J;a_1,a_2,a_3,a_4)$$

when $|z_n|$ tends to infinity, which is done in chapter 5.

Chapter 3

Banach Manifolds of Mapping Spaces and Transversality

The space of maps from S^2 to V is a Banach manifold, over which there are two Banach bundles $\mathcal{W}$ and $\mathcal{L}$ which we need to deal with. In this setting, the $\overline{\partial}_J$-operator becomes a section of the bundle $\mathcal{L}$. We first give the local charts and trvializations for these Banach manifolds and bundles. Under these local charts and trivializations, the $\overline{\partial}_J$-operator becomes a nonlinear map between certain Banach spaces. There is a Taylor expansion for this map, which is Lemma 3.0.3. Then we will establish transversality results for the evaluation maps of spaces of cusp-curves. The main result here is Proposition 3.0.1, in which we prove that transversality can be achieved by 'moving' only J and leaving the given cycles fixed.

Pre-gluing

We start by discussing the smoothing of $S^2 \vee S^2$ and pre-gluing for cusp-curves.

The Riemann sphere (S^2, i) can be described as the union of two copies of $\mathbf{C}$, with coordinates w, w' respectively, with the identification $w = 1/w'$. Thus

$$S^2 = ((\mathbf{C}, w) \coprod (\mathbf{C}, w')) \,/\, (w = \frac{1}{w'}).$$

For $i = 1, 2$, let

$$S_i^2 = ((\mathbf{C}, w_i) \coprod (\mathbf{C}, w_i')) \,/\, (w_i = \frac{1}{w_i'}).$$

Then $S_1^2 \vee S_2^2$ is given by identifying $w_1' = 0$ in S_1^2 with $w_2' = 0$ in S_2^2. Let y be the cuspidal point of $S_1^2 \vee S_2^2$, and 0_L, 1_L and 0_R, 1_R be the points of $w_1 = 0, 1$ in S_1^2 and $w_2 = 0, 1$ in S_2^2 respectively.

When $|\tilde{z}|$ is small enough, the complex sphere $S_1^2 \#_z S_2^2$ with gluing parameter $z = \frac{2}{\tilde{z}^2} \in \mathbf{C}^*$ and four 'marked' points 0_L,1_L,0_R,1_R can be constructed as follows:

One first cuts out $|w_1'| \leq |\tilde{z}|/2$ and $|w_2'| \leq |\tilde{z}|/2$ from S_1^2 and S_2^2 respectively, and then glues the two remaining discs along the annuli

$$\frac{|\tilde{z}|}{2} < |w_1'| < |\tilde{z}|, \quad \text{and} \quad \frac{|\tilde{z}|}{2} < |w_2'| < |\tilde{z}|$$

by the formula $w_1' \cdot w_2' = \tilde{z}^2/2$.

Let $S_1^2 \#_z S_2^2$ denote the resulting complex sphere with four marked points 0_L, 1_L, 0_R, 1_R. It has a 'left' and a 'right' complex coordinate w_1, w_2 respectively with the relation

$$w_1 \cdot w_2 = \frac{2}{\tilde{z}^2}.$$

In the w_1-coordinate, the points 0_L,1_L,0_R,1_R have coordinates $w_1 = 0, 1, \infty, 2/\tilde{z}^2$. Therefore the cross-ratio of these four points is $2/\tilde{z}^2$. Since the cross-ratio is the only invariant for 4-tuples in S^2 under the action of $PSL(2, \mathbf{C})$, we may consider the moduli space $\mathcal{M}^z(P, J; a_1, a_2, a_3, a_4)$ equally as

$$\mathcal{M}^{\tilde{z}}(P, J; a_1, a_2, a_3, a_4) = \left\{ f : S_1^2 \#_z S_2^2 \to V \middle| \begin{array}{c} f \textit{ is } J\textit{-holomorphic and simple}, \\ f(0_L) \in a_1, f(1_L) \in a_2, f(0_R) \in a_3, f(1_R) \in a_4 \end{array} \right\}$$

where $z = 2/\tilde{z}^2$. Note that when $\tilde{z}$ varies, the domain $S_1^2 \#_z S_2^2$ of the second moduli space also varies. But it has four fixed marked points on it.

Here is another description of $S_1^2 \#_z S_2^2$, this time in cylindrical coordinates. This is what we need in order to do Floer gluing. Recall that y is the cuspidal point of $S^2 \vee S^2$. We may think of $S_i^2 - \{y\}$ as a union of a hemisphere H_i and a half infinite cylinder $\mathbf{R}^+ \times S^1$ with cylindrical coordinate (τ_i, t_i), $i = 1, 2$, with ∂H_i identified to $\{0\} \times S^1$.

In these coordinates the previous construction $S_1^2 \#_z S_2^2$ will become the following:

The part of $S_i^2 - \{y\}$, $i = 1, 2$, with cylindrical coordinate $\tau_i > -\log|\tilde{z}| + \log 2$ is cut off, and the remainder is glued along the collars of length $\log 2$ of the cylinders twisted with an angle $arg\tilde{z}^2$. To be more precisely, we have the following gluing formula:

$$(\tau_1 + \log|\tilde{z}|) + (\tau_2 + \log|\tilde{z}|) = \log 2, \quad t_1 + t_2 = -arg\, \tilde{z}^2.$$

We now define the pre-gluing $f_1\chi_z f_2$ of the cusp-curve (f_1, f_2). This is a parametrized sphere which is approximately J-holomorphic, and is used as the initial point of a Picard iteration which converges to a true J-holomorphic curve. It will be convenient to define this in cylindrical coordinates.

Definition. Let β be the 'bump' function with β' supported in $[0, \frac{\log 2}{2}]$

$$\begin{aligned}\beta(\tau) &= 1 \text{ when } \tau < 0,\\ &= 0 \text{ when } \tau > \frac{\log 2}{2},\end{aligned}$$

and let β_z be the shifting of β by the amount $-\log|\tilde{z}|$ where $z = 2/\tilde{z}^2$, i.e.

$$\beta_z(\tau) = \beta(\tau - \log|\tilde{z}|).$$

Then we set

$$f_1\chi_z f_2(w) = \begin{cases} f_i(w) & if\ \tau_i(w) < -\log|\tilde{z}|,\ i = 1,2,\\ \exp_x(\beta_z(\tau_1(w))\tilde{f}_1(w) + \beta_z(\tau_2(w))\tilde{f}_2(w)) & otherwise,\end{cases}$$

where (τ_i, t_i) is the cylindrical coordinate of $S_i^2 - \{y\}$, $x = f_i(y)$ and $\tilde{f}_i$ is the lifting $\exp_x^{-1} \circ f_i$ of f_i under the exponential map exp_x for $i = 1,2$. Here we assume that $|\tilde{z}|$ is so small that each f_i maps the cylinder $[-\log|\tilde{z}|, \infty) \times S^1$ into the range of the exponential map $\exp_x$.

The analytic set-up

Definition. Fix $p > 2$,and $A \in H_2(V)$. We define the mapping space $\mathcal{B}_{1,A}^p$ by

$$\mathcal{B}_{1,A}^p = \{f | f : S^2 \to V,\ [f] = A,\ \|f\|_{1,p} < \infty\},$$

where the Sobolev norm $\|f\|_{1,p}$ is measured using the standard metric on S^2 and some fixed metric on V. By making a suitable choice of a metric on V such that a_i is a totally geodesic submanifold of V for $i = 1, \cdots, 4$, we can similarly define

$$\mathcal{B}^p_{1,A}(z; a_1, a_2, a_3, a_4) = \left\{ f \middle| \begin{array}{l} f : S^2_1 \#_z S^2_2 \to V, \|f\|_{1,p} < \infty, \\ \\ f(0_L) \in a_1, f(1_L) \in a_2, f(0_R) \in a_3, f(1_R) \in a_4 \end{array} \right\}.$$

Remark. As we remarked before, to simplify our representation, we have assumed that the simplical cycles a_1, a_2, a_3 and a_4 are actually submanifolds of V. If we still use simplicial representatives, which is what we should, our mapping space in the marked point case will be a stratified Banach manifold. But note that the only purpose of this Banach manifold setup is to do the gluing for discrete cusp-curves, and in that case only the top strata will be involved.

To simplify our notation, we will often omit the subscript A in $\mathcal{B}^p_{1,A}$.

Definition. The fiber at f of the tangent bundle $T\mathcal{B}^p_1 = \mathcal{W}^p_1$ of $\mathcal{B}^p_1$ is the space of L^p_1-sections of the pull-back $f^*(TV)$. Thus

$$W^p_1(f) = \{\xi | \xi \in L^p_1(f^*TV)\}.$$

Similarly, the fiber at f of the tangent bundle

$$T\mathcal{B}^p_1(z; a_1, a_2, a_3, a_4) = \mathcal{W}^p_1(z; a_1, a_2, a_3, a_4)$$

is

$$W_1^p(f, z; a_1, a_2, a_3, a_4) = \left\{ \xi | \xi \in L_1^p(f^*TV), \begin{array}{ll} \xi(0_L) \in T_{f(0_L)}a_1, & \xi(1_L) \in T_{f(1_L)}a_2, \\ \xi(0_R) \in T_{f(0_R)}a_3, & \xi(1_R) \in T_{f(1_R)}a_4 \end{array} \right\}.$$

The bundle $\mathcal{L}^p$ over $\mathcal{B}_1^p$ (or $\mathcal{B}_1^p(z; a_1, a_2, a_3, a_4)$) is defined to be the bundle whose fiber $L^p(f)$ at f is the space of L^p 1-forms on S^2 of type $(0,1)$ and with values in $f^*(TV)$ with respect to some fixed ω-compatible almost complex structure J. Thus

$$L^p(f) = \{\eta | \eta \in L^p(\Omega^{0,1}(f^*TV))\}.$$

Note that when $p > 2$, $L_1^p(E) \hookrightarrow C^0(E)$ for any vector bundle E over S^2.

Since in the marked point case the formulae of local charts and trivializations for these Banach manifolds and bundles are similar to the non-marked point case, we will only deal with the latter case.

Let ι be the injectivity radius of a fixed metric on V. Consider

$$U_f = \{\xi | \xi \in W_1^p(f), \|\xi\|_\infty < \iota\}.$$

Then the maps

$$Exp_f : U_f \to Exp_f(U_f) \hookrightarrow \mathcal{B}_1^p$$

given by

$$\xi(\tau, t) \mapsto exp_{f(\tau,t)}(\xi(\tau, t))$$

form smooth local charts for $\mathcal{B}_1^p$. Their derivatives

$$D\,Exp_f : U_f \times W_1^p \to \mathcal{W}_1^p$$

given by

$$(D\,Exp(\xi,\eta))(\tau,t) = D\,exp_{f(\tau,t)}(\xi(\tau,t))(\eta(\tau,t))$$

will give local trivializations for $\mathcal{W}_1^p$.

Remark. Note that when giving the coordinate charts for the mapping space in the marked point case, we do need the fact that the carriers of the four simplicial cycles are stratified submanifolds of V.

The local trivializations for $\mathcal{L}_1^p$ can be obtained by using a J-invariant parallel transformation coming from a corresponding J-invariant connection which is described in detail in [M], sec.4 and [MS1], sec 3.3.

There they also showed that the connection ∇ can be chosen in such a way that $Tor = \frac{1}{4}N$ where N is the Nijenhuis tensor of J

Now the $\overline{\partial}_J$-operator can be thought as a section of the bundle $\mathcal{L}^p$ over $\mathcal{B}_1^p$ given by $f \mapsto df + J(f) \circ df \circ i$. Let $\overline{\partial}_{J,f} : U_f \to L^p$ be the corresponding non-linear map under the above local charts Exp_f of $\mathcal{W}_1^p$ and the local trivializations of $\mathcal{L}^p$ over U_f. Then we have

Lemma 3.0.3 $\overline{\partial}_{J,f}$ *has the following Taylor expansion:*

$$\overline{\partial}_{J,f}(\xi) = \overline{\partial}_J(f) + D\overline{\partial}_{J,f}(0)\xi + N(\xi)$$

where

(1) the first order term $D_f = D\overline{\partial}_{J,f}(0) : W_1^p(f^*TV)) \to L^p(\Omega^{0,1}(f^*TV)$ *is given by:*

$$D_f(\xi) = \nabla\xi + J(f) \circ \nabla\xi \circ i + \frac{1}{4}N_J(\partial_J(f), \xi),$$

where the connection ∇ *is a* J*-invariant connection with its torsion proportional to the Nijenhuis tensor* N_J*.*

(2) the non-linear part is of the form:

$$N(\xi) = L_1(\xi) \circ \nabla\xi + L_2(\xi) \circ \nabla\xi \circ i + Q_1(\xi) \circ du + Q_2(\xi) \circ du \circ i,$$

where L_i *and* Q_i *are linear and quadratic respectively, in the sense that there exists a constant* $C(f)$ *depending only on* f *and the 'geometry' of* V *such that* $\|L_i(\xi)\|_\infty \leq C(f)\|\xi\|_\infty$ *and* $\|Q_i(\xi)\|_\infty \leq C(f)\|\xi\|_\infty^2$ *for* $\|\xi\|_\infty < \iota$ *when* $i = 1, 2$*.*

Proof. See [M], sec. 4 and [F1], Section 2 for the proof of (1) and (2) respectively. □

Transversality

Our next goal in this chapter is to state results on transversality. In this connection, the result of [M] about the deformation of J-holomorphic curves plays a fundamental role. Following [MS1], we will state it in its linearized form.

Lemma 3.0.4 *Given* $J \in \mathcal{J}(V, \omega)$*, and a* J*-holomorphic sphere* $f : S^2 \to V$*, there exists a constant* δ *such that for every* $v \in T_{f(z_0)}V$ *and every pair* $0 < \rho < r < \delta$ *there exists a smooth vector field* $\xi(z) \in T_{f(z)}V$ *along* f *and an* *infinitesimal almost complex structure* $Y \in C^\infty(\mathrm{End}(TV, J, \omega))$ *such that the following hold.*

(1) $D_f(\xi) + Y(f) \circ df \circ i = 0,$

(2) $\xi(z_0) = v,$

(3) ξ *is supported in* $B_r(z_0)$ *and* Y *is supported in an arbitrary small neighbourhood of* $f(B_r(z_0) - B_\rho(z_0))$.

where $D_f = D\overline{\partial}_{J,f}(0)$

Proof. See [MS1], Chapter 6, Lemma 6.1.2.

□

In order to state the transversality results we need to give some formal definitions about cusp-curves and their evaluation maps.

First recall that cusp-curves are always connected, and that in order to describe the type D of a cusp-curve with n components we must not only specify their homology classes $A_i \in H_2(V), i = 1, \cdots, n$, but also prescribe how these components intersect. To do this we choose a 'framing' which consists of a sequence of integers $j_\nu \in \{1, \cdots, n-1\}, \nu = 2, \cdots, n$, with $j_\nu < \nu$. Thus the type D is

$$D = \{A_1, \cdots, A_N, j_2, \cdots, j_n\}.$$

Then the moduli space of simple cusp-curves of type D is defined as follows:

$$\mathcal{M}(D, J) = \left\{(f, w, z) | f \in \prod_{i=1}^{n} \mathcal{M}(A_i, J), w, z \in (S^2)^{n-1}, f_{j_\nu}(w_\nu) = f_\nu(z_\nu), \nu = 2, \cdots, n\right\},$$

where $w = (w_2, \cdots, w_n)$ and $z = (z_2, \cdots, z_n)$.

Note that if $A_i = A_j$, for some $i \neq j$ in the above definition, we require that $f_i \neq f_j \circ \phi$ for any $\phi \in PSL(2, C)$.

Given $T : \{1, \cdots, p\} \to \{1, \cdots, n\}$, the p-fold evaluation map $e_{D,T,J}$: $\mathcal{M}(D, J) \times (S^2)^p \to V^p$ is defined as follows:

$$e_{D,T,J}(f, w, z, m) = (f_{T(1)}(m_1), \cdots, f_{T(p)}(m_p)).$$

Thus T specifies how the components of the cusp-curve are distributed among the different factors of the product V^p.

Now we are ready to state the main result on the transversality of $e_{D,T,J}$.

Proposition 3.0.1 *Given $P \in H_2(V)$ and submanifolds $a_1, \cdots, a_p$ of V in general position, there exists a dense subset $\mathcal{J}_{reg}(\omega)$ of second category of $\mathcal{J}(\omega)$ such that for any $J \in \mathcal{J}_{reg}(\omega)$, $e_{D,T,J}$ is transversal to $a_1 \times \cdots \times a_p$ for all (D, T) with*

$$c_1(D) = c_1(A_1 + A_2 + \cdots + A_N) \leq c_1(P)$$

when restricted to the set of all $(f, w, z, m) \in \mathcal{M}(D, J) \times CP^p$ which satisfy the conditions that for any $i \in \{1, \cdots, p\}$,

(1) *$f_{T(i)}(m_i) \neq f_{T(i)}(z_{T(i)})$ and*

(2) *if $T(i) = j_\nu$ for some $\nu \in \{2, \cdots, n\}$, then $f_{T(i)}(m_i) \neq f_{T(i)}(w_\nu)$.*

Proof. The proof of Proposition 6.3.3 of [MS1] can be easily adapted here. We will only sketch the proof and refer reader to [MS1] for details.

Let

$$\mathcal{M}(A_i, \mathcal{J}) = \coprod_{J \in \mathcal{J}} \mathcal{M}(A_i, J)$$

denote the universal moduli space of class A_i, consisting of all J-holomorphic A_i-curves, for all ω-compatible J. This is a Banach manifold. (See, for example, [MS1], Chapter 3). Consider the evaluation map

$$e_{D,T} : \prod_{i=1}^{n} \mathcal{M}(A_i, \mathcal{J}) \times (S^2)^{2(n-2)} \times (S^2)^p \to V^{2n-2} \times V^p$$

given by:

$$(f, w, z, m) \mapsto (f_{j_2}(w_2), f_2(z_2), \cdots f_{j_n}(w_n), f_n(z_n), f_{T(1)}(m_1), \cdots f_{T(p)}(m_p)),$$

and the two associated evaluation maps $\pi_1 \circ e_{D,T}$ and $\pi_2 \circ e_{D,T}$, where $\pi_1 : V^{2n-2} \times V^p \to V^{2n-2}$ and $\pi_2 : V^{2n-2} \times V^p \to V^p$ are the two projections.

From Lemma 3.0.4, arguing similarly to [MS1], we conclude that $e_{D,T}$ is trans-versal to $\Delta^{n-1} \times (a_1 \times \cdots \times a_p)$ when restricted to the subset of its domain defined by the conditions (1) and (2) above. Similarly, $\pi_1 \circ e_{D,T}$ and $\pi_2 \circ e_{D,T}$ are transversal to Δ^{n-1} and $a_1 \times \cdots \times a_p$, respectively. By taking the inverse images of the above three submanifolds under the appropriate evaluation maps and then projecting to $\mathcal{J}$, we find a dense subset $\mathcal{J}_{reg}(\omega)$ of second category in $\mathcal{J}(\omega)$, which has the property that for any $J \in \mathcal{J}_{reg}(\omega)$, the above three evaluation maps, when restricted to

$$\prod_{1}^{n} \mathcal{M}(A_i, J) \times (S^2)^{2(n-2)} \times (S^2)^p,$$

are also transversal to the corresponding submanifolds as before.

Now $\mathcal{M}(D, J) = (\pi_1 \circ e_{D,T,J})^{-1}(\Delta^{n-1})$, and the claim of this lemma is just that $\pi_2 \circ e_{D,T,J}$ is transversal to $a_1 \times \cdots \times a_p$ when restricted to the open subset of $\mathcal{M}(D,J)$ detailed in the lemma. This is a consequence of the following elementary fact:

Let M_1, M_2 and M be three manifolds and $h_i : M \to M_i$,$i = 1,2$, be smooth maps. Assume that N_i is a submanifold of M_i such that h_i is transversal to N_i, $i = 1,2$ and (h_1, h_2) is transversal to $N_1 \times N_2$. Let $W_i = h_i^{-1}(N_i)$ $i = 1,2$, and $g_j = h_j|_{W_i}$, $j \neq i$. Then g_j is transversal to N_j, $j = 1,2$. The proof of this fact follows from the corresponding linear algebra lemma obtained by replacing everything above with its linearization. □

Remark. For those 'bad' points (f,w,z,m) in $\mathcal{M}(D,J) \times (S^2)^p$, at which at least one of the conditions in the lemma is not satisfied, the above argument does not apply. Since the two bad cases we need to deal with are similar, we only consider the case where the condition (1) is violated, i.e., we need to consider those points (f,w,z,m) at which $f_{T(i)}(z_{T(i)}) = f_{T(i)}(m_i)$ for some $i \in \{1, \cdots, p\}$. In this case if we assume, without loss of generality, that $i = 1, T(i) = 1$, then we can form the $(p-1)$-fold evaluation map $e_{D,T',J}$ from $e_{D,T,J}$ by deleting the $T(1)$ factor, where $T' : \{2, \cdots, p\} \to \{2, \cdots, n\}$ given by $T'(i) = T(i)$, $i = 2, \cdots, p$. Now we can require that for generic J, $e_{D,T',J}$ is transversal to $\Delta_{a_1} \times \Delta^{n-1} \times (a_2 \times \cdots \times a_p)$ in an appropriate domain similar to the one defined in the lemma above. Here $\Delta_{a_1} = \Delta \cap (a_1 \times a_1)$. Proceeding in this way inductively, we can form $(p-i)$-fold evaluation maps,$i = 1, \cdots, p$, and achieve transversality for generic J. It is easy to see that all the 'bad ' points could be covered by an inverse image at some stage.

Intuitively, the 'bad' points correspond to those m for which the images of some of its components under f lie on the cuspidal points of f. The above argument shows that transversality still holds even including these 'bad' points if we use the above evaluation maps. Thus, we have proved that the p-fold evaluation maps of cusp-curves are transversal to the given cycles $a_1 \times \cdots \times a_p$ by only 'moving' J in above modified sense, which is what we need in Chapter 5 in order to use the dimension counting argument. Now we will consider two special cases of the above lemma.

Lemma 3.0.5 *Given $P \in H_2(V)$ and submanifold a_i of V, $i = 1, 2, 3, 4$, consider all evaluation maps $e_{A,B,J} : \mathcal{M}(A, J) \times \mathcal{M}(B, J) \to V^6$ with $A + B = P$ of the form*

$$e_{A,B,J}(f, g) = (f(0), f(1), g(0), g(1), f(\infty), g(\infty)).$$

Then for generic J, every $e_{A,B,J}$ is transversal to $a_1 \times a_2 \times a_3 \times a_4 \times \Delta$.

As a corollary we have the following result about cusp-curves $f = (f_1, f_2)$ with marked points on the a_i. Here, as in Definition 3, we write

$$W_1^p(f, a_1, a_2; a_3, a_4) = \left\{ \xi \left| \begin{array}{ll} \xi \in L_1^p(f^*TV), & \\ \xi(0_L) \in T_{f_1(0)}a_1, & \xi(1_L) \in T_{f_1(1)}a_2, \\ \xi(0_R) \in T_{f_2(0)}a_3, & \xi(1_R) \in T_{f_2(1)}a_4, \\ \xi(\infty_L) = \xi(\infty_R) & \in T_{f_1(\infty)}V = T_{f_2(\infty)}V \end{array} \right. \right\}$$

for the tangent space to the moduli space of cusp-curves.

Corollary 3.0.1 *If* $f = (f_1, f_2) \in \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4)$*, then for generic* J $D_f = (D_{f_1}, D_{f_2}) : W_1^p(f, a_1, a_2; a_3, a_4) \to L^p(\Omega^{0,1}(f^*(TV)))$ *is surjective with kernel of dimension* $2(n + c_1(A + B) - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)$.

Proof. The proof of the surjectivity of D_f is basically the same as the one for D_g when $g \in \mathcal{M}(P, J)$, which can be found in [MS1], chapter 3, for example. The formula for the dimension of the kernel comes from Proposition 3.0.1 and the fact that $KerD_f = T_f\mathcal{M}(A, B, J; a_1, a_2, a_3, a_4)$. □

This is what we need in order to do basic gluing in Chapter 4. The corollary of the next lemma is what we need when we give a direct argument for gluing $(A, 0)$-type cusp-curves at the end of next chapter.

Lemma 3.0.6 *Given* $P \in H_2(V)$ *and submanifolds* a_i *of* V,$i = 1, 2, 3, 4$, *in general position, consider the evaluation map*

$$e_{P,J} : \mathcal{M}(P, J) \times (S^2 - \{0, 1, \infty\}) \to V^4$$

given by

$$e_{P,J}(\phi, z) = (\phi(0), \phi(1), \phi(\infty), \phi(z)).$$

Then for generic J, $e_{P,J}$ *is transversal to* $a_1 \times a_2 \times a_3 \times a_4$.

Corollary 3.0.2 *The domain of* $e_{P,J}$ *can be extended to* $\mathcal{M}(P, J) \times S^2$ *such that* $e_{P,J}$ *is transversal to* $a_1 \times a_2 \times a_3 \times a_4$ *for generic* J. *In fact the restriction of* $e_{P,J}$ *to* $\mathcal{M}(P, J) \times \{0, 1, \infty\}$ *is also transversal to* $a_1 \times a_2 \times a_3 \times a_4$

Proof. By Lemma 3.0.6 and the symmetry of the points $0, 1, \infty$, we only need to prove, for example, that for generic J, the map $e : \mathcal{M}(P, J) \to V^4$ given by

$$e(\phi) = (\phi(0), \phi(1), \phi(\infty), \phi(\infty))$$

is transversal to $a_1 \times a_2 \times a_3 \times a_4$ when $a_3 \cap a_4$ is not empty. This is true if and only if $e_1 : \mathcal{M}(P, J) \to V^3$ given by

$$e_1(\phi) = (\phi(0), \phi(1), \phi(\infty))$$

is transversal to $a_1 \times a_2 \times (a_3 \cap a_4)$, which is a very special case of Proposition 3.0.1.

Chapter 4

Gluing

In this chapter we will construct a gluing map

$$\#_z : \coprod_{A+B=P} \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4) \to \mathcal{M}^z(P, J; a_1, a_2, a_3, a_4)$$

when $|z|$ is large enough, J is generic, and $c_1(P) + n = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

Given a cusp-curve

$$f = (f_1, f_2) \in \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4)$$

and $z_\alpha \in \mathbf{C}^*$ we will use the following short notations:

$$S^2_\alpha = S^2_1 \#_{z_\alpha} S^2_2,$$

$$f_\alpha = f_1 \, \chi_{z_\alpha} \, f_2,$$

$$D_{f_\alpha} = D_\alpha = D\overline{\partial}_{J,f_\alpha}(0),$$

$$D_f = D_\infty = (D_{f_1}, D_{f_2}).$$

We will use $w_\alpha(s,t)$ or (s,t) with $(s,t) \in \mathbf{R} \times S^1$ to denote the cylindrical coordinate starting from the 'middle' of S^2_α and $(\tau_i, t) \in \mathbf{R}^+ \times S^1$, $i = 1,2$, to denote the one with $\tau_i = 0$ at the boundary ∂H_i of the hemisphereH_i. The cylindrical coordinates (τ_i, t) have been used in Chapter 3. The formula for coordinate changing between these cylindrical coordinates is the following:

$$
\begin{aligned}
t &= t, \\
\tau_1 &= s + \frac{\log(2z_\alpha)}{2}, \text{ where } -\frac{\log(2z_\alpha)}{2} < s < 0, \\
\tau_2 &= s - \frac{\log(2z_\alpha)}{2}, \text{ where } 0 < s < \frac{\log(2z_\alpha)}{2}.
\end{aligned}
$$

Here $\frac{\log(2z_\alpha)}{2} = -\log|\tilde{z}_\alpha| + \log 2$ is the length of the τ_i-direction of the cylindrical coodinates (τ_i, t), $i = 1,2$.

The crucial step is to define correct norms on $L^p(\Omega^{0,1}(f^*_\alpha(TV)))$ and $W^p_1(f_\alpha, a_1, a_2, a_3, a_4)$. If ξ_α is in $W^p_1(f_\alpha, a_1, a_2, a_3, a_4)$, then we define

$$\tilde{\xi}^0_\alpha = \int_{S^1} \xi_\alpha \circ w_\alpha(0,t)dt \in T_xV$$

where $x = f_1(\infty) = f_2(\infty)$. Now we switch to (τ_i, t) coordinates. Fix a bump function φ on S^2_α such that $\varphi(\tau_i) = 1$ for $\tau > 1$ and $\varphi(\tau_i) = 0$ for $\tau_i < 1/2$, $i = 1,2$, and define

$$
\begin{aligned}
\xi^0_\alpha(\tau, t) &= D\, exp_x(\tilde{f}_\alpha(\tau,t))(\varphi(\tau)\tilde{\xi}^0_\alpha) \\
\xi^1_\alpha &= \xi_\alpha - \xi^0_\alpha.
\end{aligned}
$$

Here $\tilde{f}_\alpha$ is the lifting $exp_x^{-1}(f_\alpha)$ of f_α under exp_x,

$$D\, exp_x(\tilde{f}_\alpha(\tau,t)) : T_{\tilde{f}_\alpha(\tau,t)}(T_xV) \longrightarrow T_{f_\alpha(\tau,t)}V$$

is the derivative of the map exp_x at the point $\tilde{f}_\alpha(\tau,t)$,and $\tilde{\xi}^0_\alpha$ is considered as a constant vector field over T_xV by the obvious identification of T_xV with $T_z(T_x)$ for any $z \in T_xV$, therefore as a vector field along $\tilde{f}_\alpha$ via the map $\tilde{f}_\alpha$. Thus ξ^0_α and ξ^1_α are in $W^p_1(f_\alpha, a_1, a_2, a_3, a_4)$ as ξ_α is. Note that in the above definition we have assumed implicitly that the whole cylindrical charts (τ_i, t) are taken into the range of exp_x, which is just the matter that where these charts start.

Definition. Let $0 < \varepsilon < 1$. For any η_α in $L^p(f_\alpha)$ and ξ_α in $W^p_1(f_\alpha, a_1, a_2, a_3, a_4)$, we define

$$\|\eta_\alpha\|_{X,0} = \|\eta_\alpha\|_{0,p;\varepsilon} = \|e^{\varepsilon\tau}\eta_\alpha\|_{0,p}, \quad \text{and}$$

$$\|\xi_\alpha\|_{X,1} = \|\xi^1_\alpha\|_{1,p;\varepsilon} + |\xi^0_\alpha| = \|e^{\varepsilon\tau}\xi^1_\alpha\|_{1,p} + |\xi^0_\alpha|,$$

where $|\xi^0_\alpha| = |\tilde{\xi}^0_\alpha|$.

Note that the metric on S^2_α which we used in the above definition of $\|\cdot\|$'s is the one induced from the cylindrical coordinate $w_\alpha(s,t)$ (or, equivalently induced from (τ_i, t)).

Remark. In the introduction we have explained roughly the idea behind the above norm. Now let us look at this more closely. Note that our main goal here is to prove the uniform invertibility of D_α for large α, for which we need to compare D_α with its limit D_∞ under some suitable choice of norms. It is quite nature to require that the norms used for sections over S_α is locally convergent to a limit norm for sections over $S^2 \vee S^2 - \{y\}$ under the cylindrical coordinates (τ, t) when $\alpha \mapsto \infty$. There are two obvious properties that this limit norm should have, which, in turn, will 'determine' the norms for the α's.

The first is that D_∞ should be Fredholm with respect to the norm, which suggests that a weighted Sobolev norm should be used as we already mentioned in the introduction. The second is that the norm should give rise to a induced metric on moduli space of cusp-curves. At the infinitesimal level this amounts to say that each tangent vector of the moduli space of cusp-curves which is in the kernel of D_∞ has a finite norm. Now due to the $2n$- dimensional movement of the cuspidal points of cusp-curves as we mentioned before, we can find elements in the kernel of D_∞ whose value is not zero at infinity. Therefore the two requirements contradict each other unless we split off a $2n$-dimensional component which corresponds to 'moving' at infinity. Note that for an arbitary section ξ over $S^2 \vee S^2 - \{y\}$, $\lim_{\tau \mapsto \infty} \xi(\tau, t)$ may have different values with respect to t. This suggests that we need to take its average , which leads us to the above definition of Floer's norm. The main estimate in this chapter is

Proposition 4.0.2 *Suppose given a generic J and a cusp-curve $f \in \mathcal{M}(A, B, J, a_1, a_2; a_3, a_4)$ with $c_1(A+B) + n = \sum \alpha_i$ such that for this J, by Lemma 3.0.5 and its corollary,*

$$D_f : W_1^p(f, a_1, a_2; a_3, a_4) \to L^p(\Omega^{0,1}(f^*(TV)))$$

is an isomorphism in spherical coordinates. Then there exists a constant $C(f)$, which is only dependent on f, independent of z_α such that for $|z_\alpha|$ large enough,

$$D_\alpha : W_1^p(f_\alpha, a_1, a_2, a_3, a_4) \to L^p(\Omega^{0,1}(f_\alpha^*(TV)))$$

with the norms above has a uniform inverse G_α such that

$$\|G_\alpha(\eta)\|_{\chi,1} \le C(f)\|\eta\|_{\chi,0}$$

for any $\eta \in L^p(\Omega^{0,1}(f_\alpha^*(TV)))$.

Proof. We only need to prove that when $|z_\alpha|$ is large enough, there exists a constant C such that $\|\xi_\alpha\|_{\chi,1} \le C\|D_\alpha\xi_\alpha\|_{\chi,0}$ for any $\xi_\alpha \in L_1^p(f_\alpha, a_1, a_2, a_3, a_4)$. If this is not true, then there exists a sequence

$$\{\xi_\alpha\} \in W_1^p(f_\alpha, a_1, a_2, a_3, a_4)$$

with $|z_\alpha| \to \infty$ such that

(i) $\|\xi_\alpha\|_{\chi,1} = \|\xi_\alpha^1\|_{1,p;\varepsilon} + |\xi_\alpha^0| = 1$

(ii) $\|D_\alpha\xi_\alpha\|_{\chi,0} = \|D_\alpha\xi_\alpha\|_{0,p;\varepsilon} \to 0$ when $\alpha \to \infty$.

We will prove that (i) and (ii) contradict each other. In the proof we will repeatedly use the following fact:

Lemma 4.0.7 *Let B be a Banach space with a norm $\|.\|_B$ and $\rho : B \to \mathbf{R}^+$ be a convex continuous function. If $\{x_i\}$ is a sequence in B such that $x_i \to x$ weakly for some x in B, then $\rho(x) \le \liminf_i \rho(x_i)$.*

We will apply this when ρ is a continuous semi-norm with respect to $\|.\|_B$. Until further notice we assume that the assumptions in Proposition 4.0.2 hold.

Lemma 4.0.8 *(i) and (ii) above imply that there exists a subsequence $\{\xi_\alpha^0\}$ such that $|\xi_\alpha^0| \to 0$, when $\alpha \to \infty$.*

Proof. By definition $|\xi^0_\alpha| = |\tilde{\xi}^0_\alpha|$. From (i) we know that $|\tilde{\xi}^0_\alpha| \leq 1$. This implies that there exists a convergent subsequence $\{\tilde{\xi}^0_\alpha\}$, with limit $\tilde{\xi}^0_\infty \in T_xV$. We only need to prove that $\tilde{\xi}^0_\infty = 0$. The idea of the proof is to construct an element

$$\xi = (\xi_1, \xi_2) \in W^p_1(f, a_1, a_2; a_3, a_4)$$

such that $D_f\xi = 0$, and $\xi_1(\infty) = \xi_2(\infty) = \tilde{\xi}^0_\infty$ in spherical coordinates. Because we have assumed that D_f is an isomorphism, this implies that $\xi = 0$. Therefore $\tilde{\xi}^0_\infty = 0$.

To this end, we set

$$\xi^0_\infty = D\, exp_x(\tilde{f}(\tau, t))(\varphi(\tau)\tilde{\xi}^0_\infty),$$

as before with ξ^0_α. Then $\xi^0_\infty \in \Gamma(f^*(TV))$. It is easy to see that as $\alpha \to \infty$, ξ^0_α is locally C^∞-convergent to ξ^0_∞ in $S^2_1 \vee S^2_2 - \{y\}$.

Given $R > 0$, let D_R be the domain in S^2_α (or in $S^2_1 \vee S^2_2$) which is the the union of the two half spheres at the two ends plus the cylindrical part up to $\tau = R$. From (i) we know that $\|\xi^1_\alpha\|_{1,p;\varepsilon} \leq 1$. This implies that for any $R > 0$, there exists a constant $C(R)$ depending on R such that $\|\xi^1_\alpha\|_{1,p} \leq C(R)$ for all α. Note that when R is fixed, all these $\xi^1_\alpha|_{D_R}$'s live in same space for large α. Therefore $\xi^1_\alpha|_{D_R} \to \xi^1_{\infty;R}$ weakly in L^p_1- space for some $\xi^1_{\infty;R} \in L^p_1(f|_{D_R})$. By letting $R \to \infty$ and taking a diagonal subsequence in the usual way, we conclude that all these $\xi^1_{\infty;R}$'s can be pasted together to give rise to a single section

$$\xi^1_\infty \in L^p_{1,\mathrm{loc}}(f, a_1, a_2, a_3, a_4)$$

such that

$$\xi^1_\alpha|_{D_R} \to \xi^1_\infty|_{D_R} = \xi^1_{\infty;R}$$

weakly in L^p_1-space. Here we have used the fact that the weak limit $\xi^1_{\infty;R} = \xi^1_{\infty;R_1}|_{D_R}$, if $R < R_1$, which can be proved by a standard Sobolev embedding argument.

Let $\xi_\infty = \xi^0_\infty + \xi^1_\infty$. Then $\xi_\alpha|_{D_R} \to \xi_\infty|_{D_R}$ weakly in L^p_1- space. Therefore

$$D_\alpha\xi_\alpha|_{D_R} \to D_\infty\xi_\infty|_{D_R}$$

weakly in L^p-space. Here we have used the fact that $D_\alpha = D_\infty$ on D_R when α is large enough. Our assumption (ii), which says that

$$\|D_\alpha\xi_\alpha\|_{0,p;\varepsilon} \to 0 \quad \text{as} \quad \alpha \to \infty,$$

implies that

$$\|(D_\alpha\xi_\alpha)|_{D_R}\|_{0,p} \to 0 \quad \text{as} \quad \alpha \to \infty$$

for any fixed R. From Lemma 4.0.7 we conclude that

$$\|(D_\infty\xi_\infty)|_{D_R}\|_{0,p} \leq \liminf_{\alpha} \|(D_\alpha\xi_\alpha)|_{D_R}\|_{0,p} = 0.$$

This implies that $(D_\infty\xi_\infty)|_{D_R} = 0$ for any R, therefore $D_\infty\xi_\infty = 0$.

In spherical coordinates, this gives us a solution of $D_\infty\xi = 0$ with a singularity at the cuspidal point y. We will soon prove that this singularity is removable, i.e. that ξ_∞ can be extended over y such that the restriction of ξ_∞ to each half of $S^2 \vee S^2$ is smooth.

Suppose that this is true. Note that ξ^0_∞ is already smooth. Going back to the cylindrical coordinate (τ, t), we conclude that as $\tau \to \infty$, all these three sections, ξ_∞, ξ^0_∞ and ξ^1_∞ of the bundle f^*TV over $S^2_1 \vee S^2_2 - \{y\}$ are convergent uniformly with respect to t under the trivalization $D\, exp_x$ of TV near x. Combining this with the fact that

$$\int_{R^+ \times S^1} e^{p\varepsilon\tau} |\xi^1_\infty|^p d\tau dt \leq \|\xi^1_\infty\|_{0,p;\varepsilon} \leq \liminf_\alpha \|\xi^1_\alpha\|_{0,p;\varepsilon} \leq 1,$$

which comes from (i) and Lemma 4.0.7, we conclude that $\lim_{\tau\to\infty} \xi^1_\infty = 0$. Therefore

$$\lim_{\tau\to\infty} \xi_\infty(\tau, t) = \lim_{\tau\to\infty} \xi^0_\infty(\tau, t) = \tilde{\xi}^0_\infty.$$

This proves that in spherical coordinates ξ_∞ has the required properties, and therefore finishes the proof of the lemma modulo a removable singularity lemma.

Proof that the singularity of ξ is removable

Let (u, v) be coordinates on one of the spheres near the cuspidal point y where y is $u = v = 0$. From Lemma 3.0.3, by contracting with $\frac{\partial}{\partial u}$, the equation $D_f \xi = 0$ becomes

$$\nabla_u \xi_i + J(f_i)\nabla_v \xi_i + N(\frac{\partial f_i}{\partial u} - J(f_i)\frac{\partial f_i}{\partial v}, \xi_i) = 0, i = 1, 2. \qquad (1)$$

We know that in (u, v)-coordinates, ξ_∞ satisfies (1) except at the point where $u = v = 0$. By elliptic regularity, we only need to prove that ξ_∞ is a weak solution of (1) near the origin.

To this end, let us denote the left side of (1) by $E_i\xi_i$,set $E = (E_1, E_2)$ and think of the differentials in E_i as weak derivatives. Then for any 'test function' $\phi = (\phi_1, \cdots, \phi_{2n})$ with $\phi_i \in C_0^\infty(D^2), i = 1, \cdots, 2n$, we have

$$\begin{aligned}\int_{D^2} (E\xi_\infty) \cdot \phi &= \int_{D^2} \xi_\infty \cdot (\tilde{E}\phi) \\ &= \int_{D^2} (\xi_\infty^0 + \xi_\infty^1)(\tilde{E}\phi) \\ &= \int_{D^2} (\hat{E}\xi_\infty^0) \cdot \phi + \int_{D^2} \xi_\infty^1 (\tilde{E}\phi).\end{aligned}$$

Here we consider E as taking weak derivatives, and have written $\tilde{E}$ for the operator induced by E on test functions and $\hat{E}$ for E considered as a differential operator on smooth functions. Note that because ξ_∞^0 is smooth everywhere, the first term of the last equality makes sense.

Therefore in order to prove that ξ_∞ is a weak solution of (1), we only need to prove that

$$\int_{D^2} \xi_\infty^1 \cdot (\tilde{E}\phi) = \int_{D^2 - \{0\}} (\hat{E}\xi_\infty^1) \cdot \phi.$$

Since

$$\hat{E}\xi_\infty^1 = \frac{\partial \xi_\infty^1}{\partial u} + J(f)\frac{\partial \xi_\infty^1}{\partial v} + N(\frac{\partial f}{\partial u} - J(f)\frac{\partial f}{\partial v}, \xi_\infty^1),$$

we only need to deal with the first order terms

$$\frac{\partial \xi_\infty^1}{\partial u}, \quad J(f)\frac{\partial \xi_\infty^1}{\partial v}.$$

and the latter case can be reduced to $\frac{\partial \xi_\infty^1}{\partial v}$. By symmetry of the two cases, we therefore only need to prove that

$$\int_{D^2 - \{0\}} \frac{\partial \xi_\infty^1}{\partial u} \cdot \phi = - \int_{D^2 - \{0\}} \xi_\infty^1 \cdot \frac{\partial \phi}{\partial u}. \tag{2}$$

In order to prove this, we need the following three lemmas.

Lemma 4.0.9 $\|\xi_\infty^1\|_{1,p;\varepsilon} \leq 1$.

Proof. From the assumption (i), we have $\|\xi^1_\alpha|_{D_R}\|_{1,p;\varepsilon} \le 1$ for any $R > 0$.

By Lemma 4.0.7 we have

$$\|\xi^1_\infty|_{D_R}\|_{1,p;\varepsilon} \le \liminf_{\alpha\to\infty} \|\xi^1_\alpha|_{D_R}\|_{1,p;\varepsilon} \le 1$$

for any $R > 0$. Therefore $\|\xi^1_\infty\|_{1,p;\varepsilon} \le 1$. □

Lemma 4.0.10

$$\int_{D^2-\{0\}} |\frac{\partial \xi^1_\infty}{\partial u}| dudv < \infty, \quad and \quad \int_{D^2-\{0\}} |\xi^1_\infty| dudv \le \infty.$$

Proof. The coordinates (u, v) and (τ, t) are related by

$$\begin{cases} u = e^{-\tau}\cos t \\ v = e^{-\tau}\sin t. \end{cases}$$

Hence $du \wedge dv = -e^{-2\tau} d\tau \wedge dt$, and

$$\begin{pmatrix} \frac{\partial\xi}{\partial\tau} \\ \frac{\partial\xi}{\partial t} \end{pmatrix} = -e^{-\tau} \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} \frac{\partial\xi}{\partial u} \\ \frac{\partial\xi}{\partial v} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial\xi}{\partial u} \\ \frac{\partial\xi}{\partial v} \end{pmatrix} = -e^{\tau} \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} \frac{\partial\xi}{\partial\tau} \\ \frac{\partial\xi}{\partial t} \end{pmatrix}.$$

Therefore

$$\begin{aligned}\int_{D^2-\{0\}} |\frac{\partial \xi_\infty^1}{\partial u}| du dv &\leq \int_{R^+\times S^1} e^{-\tau}(|\frac{\partial \xi_\infty^1}{\partial \tau}| + |\frac{\partial \xi_\infty^1}{\partial t}|) d\tau dt \\ &\leq (\|\xi_\infty^1\|_{0,p}) \cdot \left(\int_{R^+\times S^1} e^{-q\tau} d\tau dt\right)^{\frac{1}{q}} \\ &\leq C\|\xi_\infty^1\|_{1,p;\varepsilon} < \infty.\end{aligned}$$

The proof for $\int_{D^2-\{0\}} |\xi_\infty^1| du dv$ is similar. □

Let Γ_δ, Γ_δ^L, and Γ_δ^R be the three regions in

$$\{(u,v) | \, |v| \leq \delta, u^2 + v^2 \leq 2\delta^2\}$$

cut by $u = \pm\delta$, and let γ_δ^L and γ_δ^R be the two arc boundaries of Γ_δ^L and Γ_δ^R respectively.

Lemma 4.0.11 *There exists a sequence $\{\delta_i\} \to 0$ with $\delta_i > 0$ such that*

$$\lim_{\delta_i \to 0} \int_{\gamma_{\delta_i}^L \cup \gamma_{\delta_i}^R} |\xi_\infty^1| = 0.$$

Proof. We prove the stronger statement that there exists a sequence$\{r_i\} \to 0$ with $r_i > 0$ such that

$$\int_0^{2\pi} |\xi_\infty^1(r_i, \theta)| d\theta < 1.$$

This implies that

$$\int_{\gamma_{r_i}^L \cup \gamma_{r_i}^R} |\xi_\infty^1| < \int_0^{2\pi} |\xi_\infty^1(r_i, \theta)| r_i d\theta < r_i \to 0.$$

Suppose that the statement is not true. Then there exists a constant δ such that for any $0 < r < \delta$,

$$\int_0^{2\pi} |\xi_\infty^1(r, \theta) d\theta| \geq 1.$$

Therefore

$$\int_0^\delta \int_0^{2\pi} \frac{|\xi^1_\infty(r,\theta)|}{r^{1+\varepsilon}} d\theta dr > \int_0^\delta \frac{1}{r^{1+\varepsilon}} dr = \infty.$$

But

$$\begin{aligned}\int_0^\delta \int_0^{2\pi} \frac{|\xi^1_\infty(r,\theta)|}{r^{1+\varepsilon}} d\theta dr &= \int_{D^2_\delta - \{0\}} \frac{|\xi^1_\infty|}{r^{2+\varepsilon}} du dv \\ &\leq \int_{\mathbb{R}^+ \times S^1} e^{\varepsilon\tau} |\xi^1_\infty| d\tau dt,\end{aligned}$$

and this is bounded by Lemma 4.0.9. We get a contradiction. □

Proof of (2):

Choose a sequence $\{r_i\} \to 0$ with $r_i > 0$ such that Lemma 4.0.11 holds. By Lemma 4.0.10

$$\lim_{r_i \to 0} \int_{\Gamma_{r_i} - \{0\}} |\frac{\partial \xi^1_\infty}{\partial u}| du dv \to 0, \quad \lim_{r_i \to 0} \int_{\Gamma_{r_i} - \{0\}} |\xi^1_\infty| du dv \to 0.$$

Therefore

$$\begin{aligned}&| \int_{D^2 - \{0\}} \frac{\partial \xi^1_\infty}{\partial u} \cdot \phi + \int_{D^2 - \{0\}} \xi^1_\infty \cdot \frac{\partial \phi}{\partial u} | \\ &= \lim_{r_i \to 0} | \int_{D^2 - \Gamma_{r_i}} (\frac{\partial \xi^1_\infty}{\partial u} \cdot \phi + \xi^1_\infty \cdot \frac{\partial \phi}{\partial u}) | \\ &= \lim_{r_i \to 0} | \int_{D^2 - \Gamma_{r_i}} \frac{\partial}{\partial u} (\xi^1_\infty \phi) du dv | \\ &= \lim_{r_i \to 0} | \int_{\gamma^L_{r_i}} \xi^1_\infty \phi - \int_{\gamma^R_{r_i}} \xi^1_\infty \phi | \\ &\leq \lim_{r_i \to 0} \left\{ (\max_{x \in D^2} |\phi|) \left(\int_{\gamma^L_{r_i} \cup \gamma^R_{r_i}} |\xi^1_\infty| \right) \right\} \\ &= 0\end{aligned}$$

by Lemma 4.0.11. This finishes the proof of (2), and therefore the proof of the removable singularity lemma. □

Lemma 4.0.12 *The conditions (i) and (ii) in Lemma 4.0.8 also imply that*

$$\lim_{\alpha\to\infty} \|D_\alpha \xi_\alpha^0\|_{0,p;\varepsilon} = 0.$$

Proof. Use exp_x to identify a neighbourhood $\tilde{U}$ of 0 in T_xV with a neighbourhood U of x in V. Under this correspondence we have

$$\tilde{f}_\alpha \leftrightarrow f_\alpha, \quad \varphi\tilde{\xi}_\alpha^0 \leftrightarrow \xi_\alpha^0, \quad \tilde{E}_\alpha \leftrightarrow E_\alpha,$$

where E_α is the contraction of D_α along the direction $\frac{\partial}{\partial\tau}$.

Here $\tilde{E}_\alpha$ is of the form

$$\tilde{E}_\alpha(\tilde{\xi}) = \tilde{\nabla}_\tau\tilde{\xi} + \tilde{J}(\tilde{f}_\alpha)\tilde{\nabla}_t\tilde{\xi} + \tilde{N}(\tilde{\partial}_J\tilde{f}_\alpha, \tilde{\xi}),$$

where $\tilde{\nabla}$, $\tilde{N}$, $\tilde{J}$ and $\tilde{\partial}_J$ are the liftings of ∇, N, J and ∂_J respectively. Note that

$$|(f_\alpha)_*\left(\frac{\partial}{\partial\tau}\right)| \sim e^{-\tau}, \quad |(f_\alpha)_*\left(\frac{\partial}{\partial t}\right)| \sim e^{-\tau}$$

when α and τ are large enough. This implies that

$$\begin{aligned}
\tilde{\nabla}_\tau\tilde{\xi} &= \frac{\partial}{\partial\tau}\tilde{\xi} + O(e^{-\tau})\tilde{\xi}, \\
\tilde{\nabla}_t\tilde{\xi} &= \frac{\partial}{\partial t}\tilde{\xi} + O(e^{-\tau})\tilde{\xi}, \\
\tilde{N}(\tilde{\partial}_J\tilde{f}_\alpha, \tilde{\xi}) &= O(e^{-\tau})\tilde{\xi}.
\end{aligned}$$

Therefore

$$|\tilde{E}_\alpha(\varphi\tilde{\xi}_\alpha^0)| \le C\left(|\frac{\partial\varphi}{\partial\tau}\tilde{\xi}_\alpha^0| + e^{-\tau}|\tilde{\xi}_\alpha^0|\right).$$

Now $\frac{\partial\varphi}{\partial\tau}$ is only supported on $0 \le \tau \le 1$. Therefore we have

$$\|\tilde{E}_\alpha(\varphi\tilde{\xi}_\alpha^0)\|_{0,p;\varepsilon} \;\le\; C\left\{\int_{[0,1]\times S^1} e^{p\varepsilon\tau}|\frac{\partial\varphi}{\partial\tau}|^p|\tilde{\xi}_\alpha^0|^p\right\}^{\frac{1}{p}} +$$

$$
\begin{aligned}
& C\left\{\int_{\mathbb{R}^+\times S^1} e^{p(\varepsilon-1)\tau}|\tilde{\xi}^0_\alpha|^p\right\}^{\frac{1}{p}} \\
\leq\ & C\left(e\left\{\int_{[0,1]\times S^1}|\frac{\partial\varphi}{\partial\tau}|^p d\tau dt\right\}^{\frac{1}{p}} + \right. \\
& \left.\left\{\int_{\mathbb{R}^+\times S^1} e^{p(\varepsilon-1)\tau} d\tau dt\right\}^{\frac{1}{p}}\right)\cdot|\tilde{\xi}^0_\alpha| \\
\leq\ & C_1|\tilde{\xi}^0_\alpha|
\end{aligned}
$$

which tends to zero when $\alpha\to\infty$ by Lemma 4.0.8. This implies that

$$\lim_{\alpha\to\infty}\|D_\alpha\xi^0_\alpha\|_{0,p;\varepsilon}=0.$$

□

From Lemmas 4.0.8 and 4.0.12, we conclude that

(I) $\|\xi^1_\alpha\|_{1,p;\varepsilon}=1$,

(II) $\|D_\alpha\xi^1_\alpha\|_{0,p;\varepsilon}\to 0$ when $\alpha\to\infty$.

We will prove that (I) and (II) contradict each other. To do this, we need to have an estimate on the middle part of ξ^1_α.

Let β_2 be a 'bump' function on S^2_α which is supported in $-2<s<+2$ and equal to 1 on $-\log 2<s<+\log 2$ where (s,t) or $w_\alpha(s,t)$ is the cylindrical coordinate of S^2_α starting from the middle.

Lemma 4.0.13 $\lim\limits_{\alpha\to\infty}\|\beta_2\xi^1_\alpha\|_{1,p;\varepsilon}=0.$

Proof. Let $\rho_\alpha=-\log\tilde{z}_\alpha+\log 2$, which is the length of the cylindrical coordinate $w_\alpha(s,t)$ along the t-direction. Define

$$\zeta_\alpha:[-\rho_\alpha,+\rho_\alpha]\times S^1\to T_xV$$

by

$$Dexp_x(\tilde{f}_\alpha \circ w_\alpha(s,t))(\zeta_\alpha(s,t)) = e^{|\rho_\alpha|\varepsilon} \cdot \xi^1_\alpha(w_\alpha(s,t)).$$

Extend ζ_α trivially over the whole cylinder. Then from (I) there exists a constant C such that

$$\|e^{-\varepsilon|s|} \cdot \zeta_\alpha\|_p \leq C \tag{3}$$

for all α. Let $\zeta_{\alpha;R}$ be the restriction of ζ_α to the domain $Z_R = [-R,+R] \times S^1$, then from (I) again there exists a constant $C(R)$ depending on R such that $\|\zeta_{\alpha;R}\|_{1,p} \leq C(R)$ for all α. Therefore as $\alpha \to \infty$,

$$\zeta_{\alpha;R} \to \zeta_{\infty,R} \tag{4}$$

weakly in $L^p_1(Z_R, T_xV)$ for some $\zeta_{\infty;R} \in L^p_1(Z_R, T_xV)$.

The same argument as in the proof of Lemma 4.0.8 will prove that when $R \to \infty$, all these $\zeta_{\alpha;R}$'s agree with each other on their overlaps to form a single element

$$\zeta_\infty \in L^p_{1,\mathrm{loc}}(\mathbf{R} \times S^1, T_xV)$$

such that $\zeta_\infty|_{Z_R} = \zeta_{\infty;R}$. Now (4) implies that when $\alpha \to \infty$,

$$\overline{\partial}_{J_0}\zeta_{\alpha;R} \to \overline{\partial}_{J_0}\zeta_{\infty;R} \tag{5}$$

weakly in $L^p(Z_R, T_xV)$ where $\overline{\partial}_{J_0}$ is the standard Cauchy-Riemann operator.

Let $\tilde{E}_\alpha$ be the lifting of E_α under exp_x as before. Then it is of the form

$$\tilde{E}_\alpha\zeta_{\alpha;R} = \frac{\partial\zeta_{\alpha;R}}{\partial s} + J_0\frac{\partial\zeta_{\alpha;R}}{\partial t} + (\tilde{J} - J_0)(\tilde{f}_\alpha)\frac{\partial\zeta_{\alpha;R}}{\partial t} + A_{\alpha;R} \cdot \zeta_{\alpha;R},$$

where $A_{\alpha;R}$ is the restriction to Z_R of some zero order operator A_α. It is easy to see that when R is fixed,

$$\lim_{\alpha\to\infty} |A_{\alpha;R}| = 0, \quad \text{and} \quad \lim_{\alpha\to\infty} |(\tilde{J} - J_0)|_{\tilde{f}_{\alpha;R}}| = 0.$$

From (II) we have

$$\lim_{\alpha\to\infty} \|\tilde{E}_\alpha \zeta_{\alpha;R}\|_p = 0.$$

Therefore

$$\lim_{\alpha\to\infty} \|\overline{\partial}_{J_0}\zeta_{\alpha;R}\|_p \leq \lim_{\alpha\to\infty} \left\{\|\tilde{E}_\alpha\zeta_{\alpha;R}\|_p + |A_{\alpha;R}|\|\zeta_{\alpha;R}\|_p + |(\tilde{J} - J_0)|_{\tilde{f}_{\alpha;R}}|\|\zeta_{\alpha;R}\|_{1,p}\right\} = 0. \tag{6}$$

From this, (5) and Lemma 4.0.7, we have

$$\|\overline{\partial}_{J_0}\zeta_{\infty;R}\|_p \leq \liminf_{\alpha\to\infty} \|\overline{\partial}_{J_0}\zeta_{\alpha;R}\|_p = 0.$$

This implies that $\overline{\partial}_{J_0}\zeta_{\infty;R} = 0$ for any $R > 0$. Therefore

$$\overline{\partial}_{J_0}\zeta_\infty = 0 \tag{7}$$

From (3) and Lemma 4.0.7, we conclude that

$$\|\zeta_{\infty;R}\|_{0,p;(-\varepsilon)} \leq \liminf_{\alpha\to\infty} \|e^{-\varepsilon|s|}\zeta_\alpha\|_p$$

is bounded independently of R. This implies that $\|\zeta_\infty\|_{0,p;(-\varepsilon)} < \infty$. This together with (7) and the fact that the constant Fourier component of $\zeta_\infty|_{\{0\}\times S^1}$ is zero implies that $\zeta_\infty = 0$.

By a Sobolev embedding argument we conclude that for any fixed $R > 0$, $\zeta_{\alpha;R}$ is uniformly C^0-convergent to zero. Therefore, when $\alpha \to \infty$,

$$\|\beta_2\zeta_\alpha\|_{1,p} \leq C\|\overline{\partial}_{J_0}(\beta_2\zeta_\alpha)\|_p \leq C(\|\beta_2'\zeta_\alpha\|_p + \|\beta_2\overline{\partial}_{J_0}\zeta_\alpha\|_p) \to 0.$$

This implies that $\lim_{\alpha\to\infty} e^{\varepsilon\rho_\alpha}\|\beta_2\xi_\alpha^1\|_{1,p} = 0$. Hence

$$\lim_{\alpha\to\infty} \|\beta_2\xi_\alpha^1\|_{1,p;\varepsilon} = 0.$$

□

Finishing the proof of Proposition 4.0.2:

Let $W^p_{1;\varepsilon}(f, a_1, a_2, a_3, a_4)$ be the weighted Sobolev space of sections ξ of f^*TV over $S^2 \vee S^2 - \{y\}$ with cylindrical coordinates, which consists of all ξ with $\|\xi\|_{1,p;\varepsilon} < \infty$ which satisfy the obvious constraints at the four marked points.

It is proved in [F1] and [LM] that when $\varepsilon < 1$,

$$D_f : W^p_{1;\varepsilon}(f, a_1, a_2, a_3, a_4) \to L^p_{0,\varepsilon;}(\Omega^{0,1}(f^*TV))$$

is Fredholm. The same argument as in the proof of the removable singularity lemma will show that any element ξ in the kernel of the above operator will be in $W^p_1(f, a_1, a_2, a_3, a_4)$. But by assumption this is impossible unless $\xi = 0$.

Now $(1-\beta_2)\xi_\alpha^1$ is in $W^p_{1,\varepsilon}(f, a_1, a_2, a_3, a_4)$. Therefore there exists a constant C independent of α such that

$$\begin{aligned} \|(1-\beta_2)\xi_\alpha^1\|_{1,p;\varepsilon} &\leq C\|D_f\{(1-\beta_2)\xi_\alpha^1\}\|_{0,p;\varepsilon} \\ &= C\|D_\alpha\{(1-\beta_2)\xi_\alpha^1\}\|_{0,p;\varepsilon} \end{aligned}$$

$$\begin{aligned} &\leq C\left\{\|D_\alpha\xi_\alpha^1\|_{0,p;\varepsilon} + \|D_\alpha(\beta_2\xi_\alpha^1)\|_{0,p;\varepsilon}\right\} \\ &\leq C\left\{2\|D_\alpha\xi_\alpha^1\|_{0,p;\varepsilon} + \|\beta_2'\xi_\alpha^1\|_{0,p;\varepsilon}\right\} \\ &\to 0 \end{aligned}$$

when $\alpha \to \infty$. Therefore

$$\|\xi_\alpha^1\|_{1,p;\varepsilon} \leq \|(1-\beta_2)\xi_\alpha^1\|_{1,p;\varepsilon} + \|\beta_2\xi_\alpha^1\|_{1,p;\varepsilon} \to 0$$

when $\alpha \to 0$. This contradicts (I). □

Lemma 4.0.14 $\lim_{\alpha\to\infty} \|\overline{\partial}_J f_\alpha\|_{\chi,0} = 0.$

Proof. If we use $\sim$ to denote the corresponding lifted maps and operators under exp_x, then

$$\tilde{f}_\alpha = \beta_1\tilde{f}_1 + \beta_2\tilde{f}_2$$

where β_1' and β_2' are supported in the middle part of S_α^2 of length $\log 2$. Since $\tilde{\overline{\partial}}_J\tilde{f}_i = 0$, $|\tilde{f}_i(\tau,t)| \sim e^{-\tau}$ when τ is large, we have

$$\|\overline{\partial}_J f_\alpha\|_{\chi,0} \leq C\|\tilde{\overline{\partial}}_J\tilde{f}_\alpha\|_{\chi,0} \leq C(\|\beta_1'\tilde{f}_1\|_{\chi,0} + \|\beta_2'\tilde{f}_2\|_{\chi,0}) \sim e^{(\varepsilon-1)\rho_\alpha} \to 0$$

when $\alpha \to \infty$. □

Lemma 4.0.15 *There exists a constant $C_1(f)$ only depending on f such that for any $\xi_\alpha, \zeta_\alpha \in W_1^p(f_\alpha, a_1, a_2, a_3, a_4)$*

(i) $\|N(\xi_\alpha)\|_{\chi,0} \leq C_1(f)\|\xi_\alpha\|_\infty\|\xi_\alpha\|_{\chi,1}$

(ii) $\|N(\xi_\alpha) - N(\zeta_\alpha)\|_{\chi,0} \leq C_1(f)(\|\xi_\alpha\|_{\chi,1} + \|\zeta_\alpha\|_{\chi,1})\|\xi_\alpha - \zeta_\alpha\|_{\chi,1}$.

Proof. By Lemma 3.0.3 we have

$$N(\xi)(\frac{\partial}{\partial\tau}) = L_1(\xi)\nabla_\tau\xi + L_2(\xi)\nabla_t\xi + Q_1(\xi)\frac{\partial f_\alpha}{\partial\tau} + Q_2(\xi)\frac{\partial f_\alpha}{\partial t}$$

with

$$\|L_i(\xi)\|_\infty \le C(f_\alpha)\|\xi\|_\infty, \quad \|Q_i(\xi)\|_\infty \le C(f_\alpha)\|\xi\|^2_\infty,$$

for $i = 1, 2$ and some constant $C(f_\alpha)$, which depends on α but can be uniformly controlled by a constant $C(f)$ depending only on f. Now

$$\nabla_\tau\xi = \frac{\partial}{\partial\tau}\xi + A_{\alpha,\tau}\xi, \quad \nabla\xi = \frac{\partial}{\partial t}\xi + A_{\alpha,t}\xi$$

where $A_{\alpha,\tau}$ and $A_{\alpha,t}$ are the contractions of $\frac{\partial f_\alpha}{\partial\tau}$ and $\frac{\partial f_\alpha}{\partial t}$ with the connection matrix A of ∇ under the trivialization $Dexp_x$ of TV near x. But $|\frac{\partial f_\alpha}{\partial\tau}|$ and $|\frac{\partial f_\alpha}{\partial t}|$, and therefore $A_{\alpha,\tau}$ and $A_{\alpha,t}$, are of order $e^{-\tau}$. Therefore we have

$$\begin{aligned}
\|N(\xi)\|_{\chi,0} \;&\le\; C(f)\|\xi_\infty\|^2(\|e^{\varepsilon\tau}\frac{\partial f_\alpha}{\partial\tau}\|_p + \|e^{\varepsilon\tau}\frac{\partial f_\alpha}{\partial t}\|_p) + \\
&\qquad C(f)\|\xi\|_\infty(C_0\|\xi^1\|_{\chi,1} + \|\varphi' e^{\varepsilon\tau}\tilde{\xi^0}\|_p + \|\varphi e^{\varepsilon\tau}A_\alpha\tilde{\xi^0}\|_p) \\
&\le\; C_0(f)\|\xi\|_\infty(\|\xi\|_{\chi,1} + \|\xi\|_\infty) \\
&\le\; C_1(f)\|\xi\|_\infty\|\xi\|_{\chi,1}.
\end{aligned}$$

Here the last two inequalities follow from the fact that $|\tilde{\xi^0}|$ can be controlled by $\|\xi\|_\infty$ which itself, in turn, can be uniformly controlled by $\|\xi\|_{\chi,1}$. The proof for the second estimate is similar. $\square$

Lemma 4.0.16 (Picard method) *Assume that a smooth map $f : E \to F$ from Banach spaces $(E, \|\cdot\|)$ to F has a Taylor expansion*

$$f(\xi) = f(0) + Df(0)\xi + N(\xi)$$

such that $Df(0)$ has a finite dimensional kernel and a right inverse G satisfying

$$\|GN(\xi) - GN(\zeta)\| \leq C(\|\xi\| + \|\zeta\|)\|\xi - \zeta\|$$

for some constant C. Let $\delta = \frac{1}{8C}$. If $\|G \circ f(0)\| \leq \frac{\delta}{2}$, then the zero set of f in $B_\delta = \{\xi | \|\xi\| < \delta\}$ is a smooth manifold of of dimension equal to the dimension of $\ker Df(0)$. In fact, if

$$K_\delta = \{\xi | \xi \in \ker Df(0), \|\xi\| < \delta\}$$

and $K^\perp = G(F)$, then there exists a smooth function

$$\phi : K_\delta \to K^\perp$$

such that $f(\xi + \phi(\xi)) = 0$ and all zeros of f in B_δ are of the form $\xi + \phi(\xi)$.

The proof of this lemma is an elementary application of Banach's fixed point theorem. Applying this to our case we have

Proposition 4.0.3 *If $A, B \in H_2(V)$ with $A + B = P$, $c_1(P) + n = \sum \alpha_i$, and*

$$f = (f_1, f_2) \in \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4),$$

then for generic J and a parameter $z = \frac{2}{\bar{z}^2} \in \mathbf{C}^*$ with $|z|$ large enough, there exists a gluing map

$$\#_z : \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4) \to \mathcal{M}^{\bar{z}}(P, J, a_1, a_2, a_3, a_4)$$

$$f = (f_1, f_2) \mapsto f_1 \#_z f_2.$$

Moreover, if g_z *is another element in* $\mathcal{M}^{\bar{z}}(P, a_1, a_2, a_3, a_4)$ *'close' to the pre-gluing* $f_1\,\chi_z\,f_2$ *in the sense that* $\|\tilde{g}_z\|_{\chi,1} \leq \delta = \frac{1}{8CC_1}$*, then*

$$g_z = f_1 \#_z f_2.$$

Here $\tilde{g}_z$ *is a vector field along* $f_z = f_1\,\chi_z\,f_2$ *defined by*

$$Dexp(f_z(\tau,t))(\tilde{g}_z(\tau,t)) = g_z(\tau,t),$$

and C *and* C_1 *are the maxima of the constants* $C(f)$ *and* $C_1(f)$ *appeared in Proposition 4.0.2 and Lemma 4.0.15 respectively when* f *varies in* $\mathcal{M}(A, B, J; a_1, a_2, a_3, a_4)$*, which is a finite set.*

Proof. By Proposition 4.0.2 and Lemma 4.0.15 we have

$$\begin{aligned} \|GN(\xi) - GN(\zeta)\|_{\chi,1} &\leq C\|N(\xi) - N(\zeta)\|_{\chi,0} \\ &\leq CC_1\|\xi - \zeta\|_{\chi,1}(\|\xi\|_{\chi,1} + \|\zeta\|_{\chi,1}), \end{aligned}$$

for any ξ and ζ over f_z. Let $\delta = \frac{1}{8CC_1}$. Then by Lemma 4.0.14 we have

$$\|G(\overline{\partial}_J f_z)\|_{\chi,1} \leq C\|\overline{\partial}_J f_z\|_{\chi,0} < \frac{\delta}{2}$$

when $|z|$ is large enough. The conclusion of the lemma follows by applying the Picard method to the above situation. □

Corollary 4.0.3 *If the* g_z *in Proposition 4.0.3 is* C^0*-close to* f_z *in the sense that*

$$\|\tilde{g}_z\|_\infty \leq \frac{1}{2CC_1},$$

then $g_z = f_1 \#_z f_2$*.*

Proof. Since $0 = \overline{\partial}_{J,f_z}\tilde{g}_z = \overline{\partial}_J(f_z) + D_z(\tilde{g}_z) + N(\tilde{g}_z)$, we have

$$\tilde{g}_z = -G(\overline{\partial}_J(f_z)) - GN(\tilde{g}_z).$$

Therefore

$$\begin{aligned} \|\tilde{g}_z\|_{X,1} &\leq C(\|\overline{\partial}_J(f_z)\|_{X,0} + \|N(\tilde{g}_z)\|_{X,0}) \\ &\leq C\|\overline{\partial}_J(f_z)\|_{X,0} + CC_1\|\tilde{g}_z\|_\infty\|\tilde{g}_z\|_{X,1} \\ &\leq C\|\overline{\partial}_J(f_z)\|_{X,0} + \frac{1}{2}\|\tilde{g}_z\|_{X,1}. \end{aligned}$$

This implies that $\|\tilde{g}_z\|_{X,1} \leq 2C\|\overline{\partial}_J(f_z)\|_{X,0} < \delta$ when $|z|$ is large enough. □

Remark.(i) If f we give the orientation to the moduli spaces involved as in [M] and [F2], then the gluing map $\#_z$ becomes an orientation preserving map.

(ii) The above construction of gluing is also applicable to the case that A or B is zero. We observe that in that case we can do 'gluing' directly. Let B be zero, for example. Then $a_3 \cap a_4$ is non-empty, and for generic J, the 4−fold evaluation map $e_{A,J} : \mathcal{M}(A, J) \times S^2 \to V^4$ given by

$$(\phi, z) \mapsto (\phi(0), \phi(1), \phi(\infty), \phi(z))$$

and its restriction $e^1_{A,J}$ to $\mathcal{M}(A, J) \times \{\infty\}$ are transversal to $a_1 \times a_2 \times a_3 \times a_4$. Let

$$\mathcal{M}^+(A, J; a_1, a_2, a_3, a_4) = e^{-1}_{A,J}(a_1 \times a_2 \times a_3 \times a_4),$$

and consider the restriction π of

$$\pi_2 : \mathcal{M}^+(A, J; a_1, a_2, a_3, a_4) \to S^2$$

where $\pi_2 : \mathcal{M}(A, J) \times S^2 \to S^2$ is the projection. Then the transversality of $e^1_{A,J}$ just means that $\infty \in S^2$ is a regular value of π. From this and finiteness of $\#(\pi^{-1}(\infty))$, we can easily construct $\#_z$ for big $|z|$ by making use of the local covering structure of π over some neighbourhood of ∞.

Chapter 5

Compactness

In this chapter, we use Wolfson's version of Gromov compactness theorem to analyze the convergence of sequences of parametrized J-curves. As a consequence of this analysis, we will see how to ensure that the conditions of Corollary 4.0.3 are satisfied.

We assume throughout this chapter that (V, ω) is monotone with $c_1(A) > 1$ for every effective class $A \in H_2(V)$ for generic J. Here 'generic' means that all transversality results in Chapter 3 hold.

Fix a generic J and a class $P \in H_2(V)$ with $c_1(P) + n = \sum \alpha_i$. Since the only J-holomorphic spheres of class zero are the constant maps, we may assume that P is not zero. Assume that the given four cycles a_1, a_2, a_3, a_4, which have been assumed to be submanifolds of V as we remarked in chapter 1, are put in general position in V so that all possible intersections among them are still submanifolds of V.

Proposition 5.0.4 *Let J be generic and P and the a_i be as above. Consider* *a sequence* $\{f_n\} \in \mathcal{M}^{\tilde{z}_n}(P, J; a_1, a_2, a_3, a_4)$ *with*

$$|z| = \frac{2}{|\tilde{z}|^2} \to \infty.$$

Each such f_n *gives rise to the two* J*-holomorphic spheres* $f_{L,n}$ *and* $f_{R,n}$ *under the 'left' and 'right' coordinates of* $S^2 \#_{z_n} S^2$ *respectively, both mapping the 'standard' sphere* S^2 *to* V*. Here we have identified the coordinate chart* C *with* $S^2 - \{\infty\}$ *and used the fact that any* J*-holomorphic map form* $S^2 - \{\infty\}$ *to* V *with finite aera can be extended over* ∞ *smoothly .*

Then there are three possibilities:

(I) $a_1 \cap a_2$ *is not empty and* $\{f_{R,n}\}$ *is* C^∞*-convergent to some*

$$f_R \in \mathcal{M}(P, J, a_1 \cap a_2, a_3, a_4);$$

(II) $a_3 \cap a_4$ *is not empty and* $\{f_{L,n}\}$ *is* C^∞*-convergent to some*

$$f_L \in \mathcal{M}(P, J, a_1, a_2, a_3 \cap a_4).$$

(III) *There exists a parametrized cusp-curve*

$$(f_L, f_R) \in \mathcal{M}(A, B, J; a_1, a_2; a_3, a_4)$$

for some $A+B = P$ *with* f_L *not equal to* f_R *as unparametrized curves and such that* $\{(f_{L,n}, f_{R,n})\}$ *is locally* C^∞*-convergent to* (f_L, f_R) *as parametrized curves.*

Proof. We will use $\hat{}$ to denote the corresponding unparametrized curve and moduli space.

By Gromov's compactness theorem, we have that $\{\hat{f}_n\}$ converges to a cusp-curve

$$\hat{f}_\infty = \cup_{i=1}^m \hat{f}_{i,\infty}$$

with $P = [\hat{f}_n] = \sum_{i=1}^m [\hat{f}_{i,\infty}]$. Moreover for any C^0-neighbourhood U of the image of $\hat{f}_\infty$, $\hat{f}_n$ is contained in U when n is large enough. The last statement implies that $\hat{f}_\infty$ has a non-empty intersection with a_i', for $i = 1, 2, 3, 4$. By a detailed combinatorial analysis of the intersection pattern of $\hat{f}_{i,\infty}$'s and a dimension counting argument, we conclude that $m = 1$ or 2 . Details of this kind of argument may be found in [MS1], Chapter 6 and [R], Sec.3.

The cases that $m = 2$, $\hat{f}_{1,\infty} = \hat{f}_{2,\infty}$ and that $m = 1$ but the curve is multiply-covered do not occur. To rule out the first possibility, let $[\hat{f}_{i,\infty}] = A$ for $i = 1, 2$. Then

$$\hat{f}_{i,\infty} \in \hat{\mathcal{M}}(A, J; a_1, a_2, a_3, a_4),$$

and by our genericity assumptions, the dimension of this space is

$$2(c_1(A) + n + 1 - \sum \alpha_i) = 2(c_1(P) + n + 1 - \sum \alpha_i) - 2(c_1(A))$$

which is less than zero by our assumption. Similarly for the latter case.

Now we can use Wolfson's version of Gromov's compactness theorem to analyze the limit behavior of the sequence $\{f_n\}$, as parametrized curves. Let

$$\lim_n f_n(0_L) = l_1 \in a_1, \quad \lim_n f_n(1_L) = l_2 \in a_2,$$

$$\lim_n f_n(0_R) = l_3 \in a_3, \quad \lim_n f_n(1_R) = l_4 \in a_4.$$

Proof of cases (I) and (II) If $l_1 = l_2 \in a_1 \cap a_2$ or $l_3 = l_4 \in a_3 \cap a_4$, then we have $m = 1$. Otherwise, for example, in the case $l_1 = l_2$, $\hat{f}_\infty = (\hat{f}_{1,\infty}, \hat{f}_{2,\infty})$ is in the moduli space of $(A-B)$-cusp-curves with $A+B=P$, which intersects with $a_1 \cap a_2$,a_3 and a_4. A dimension counting argument shows that the dimension of this moduli space is less than zero.

In the case $l_1 = l_2$, we claim that $\{f_{R,n}\}$ is C^∞- convergent to f_R. If this is not true, then we have only one bubble at some point x_1 in S^2, and $\{f_{R,n}\}|_{S^2-\{x_1\}}$ is locally convergent to a constant map. The point x_1 must be 0_R, or 1_R. Otherwise $l_3 = \lim_n f_n(0_R) = \lim_n f_n(1_R) = l_4$. This implies that

$$\hat{f}_\infty \in \hat{\mathcal{M}}(P, J; a_1 \cap a_2, a_3 \cap a_4),$$

which is empty for reasons of dimension. If x_1, for example,is 0_R, then a similar argument show that $l_1 = l_2 = l_4$. Therefore $\hat{f}_\infty$ is in $\hat{\mathcal{M}}(P, J; a_1 \cap a_2 \cap a_4, a_3)$, which is empty again.

Similarly, in the case $l_3 = l_4$, we have that $\{f_{L,n}\}$ is C^∞-convergent to f_L.

This gives the possibilities (I) and (II) of the lemma.

Proof of case (III) Now we can assume that $l_1 \neq l_2$, and $l_3 \neq l_4$.

Consider the sequence $\{f_{L,n}\}$. When n tends to ∞, z_n tends to the point ∞ in S^2. But

$$\lim_{n\to\infty} f_{L,n}(z_n) = l_3 \neq l_4 = \lim_{n\to\infty} f_{L,n}(\infty).$$

This implies that the derivative of $f_{L,n}$ at ∞ blows up when n tends to ∞. Therefore we have one bubble at ∞.

We claim that this is the only bubble $\{f_{L,n}\}$ can have, in other words that $\{f_{L,n}\}$ is locally C^∞-convergent. Suppose that this is not true. Let x_1 be another bubble point. Then x_1 must be 0_L or 1_L. Otherwise since *when $|z_n|$ is large enough. Here*

$$d(f_n, f_L \chi_{z_n} f_R) = \max_{x \in S_1^2 \#_{z_n} S_2^2} d(f_n(x), f_L \chi_{z_n} f_R(x))$$

measured by some metric on V.

Proof. We first prove that $f_L(\infty) = f_R(\infty)$. Note that if we view f_L as the 'base' curve of the cusp-curve (f_L, f_R), then f_R becomes the bubble. But its parametrization may differ from that coming from the bubbling procedure described in [PW]. Therefore the statement does not immediately follow from that part of the compactness theorem concerning bubble intersections, although its proof is based on the same idea there, namely to use monotonicity for minimal surfaces.

Assume that $d(f_L(\infty), f_R(\infty)) = 5\delta > 0$. Let $B_1(R)$ and $B_2(R)$ be the open balls of radius R in the 'left' and 'right' coordinates of $S_1^2 \#_{z_n} S_2^2$ respectively. Note that for a fixed R, $B_1(R)$ does not intersect with $B_2(R)$ when $|z_n|$ is big enough. Denote $B_1(r) \cup B_2(R)$ by $B(R)$ and $\partial B_i(R)$ by $C_i(R)$, $i = 1, 2$.

By using the fact that f_n is locally convergent to (f_L, f_R) and that the area $A(f_n) = A(f_L) + A(f_R)$, it is easy to see that when R and $|z_n|$ are large enough,

(a) $d(f_n(C_1(R)), f_L(\infty)) < \delta, \quad d(f_n(C_2(R)), f_R(\infty)) < \delta,$

(b) $A(f_n|_{S^2 - B(R)}) < C\delta^2$ for some fixed constant C which we will specify soon.

Now $f_n : (S^2 - B(R)) \to V$ is a minimal surface with respect to the metric g_J. Its two boundaries lie on the two disjoint balls $B_{f_L(\infty)}(\delta)$ and $B_{f_R(\infty)}(\delta)$ respectively.

Since $B_{f_L(\infty)}(2\delta) \cap f_n(S^2 - B(R))$ and $B_{f_R(\infty)}(2\delta) \cap f_n(S^2 - B(R))$ are two disjoint open subsets of the connected surface $f_n(S^2 - B(R)$, there exists a point x_1 in $f_n(S^2 - B(R))$ such that the distance between x_1 and $f_L(\infty)$ or $f_R(\infty)$ is larger than 2δ. Therefore $B_{x_1}(\delta)$ does not intersect the two boundary components of $f_n(S^2 - B(R))$. Now we can apply monotonicity for minimal surfaces to conclude that

$$A(f_n|_{S^2-B(R)}) > A(B_{x_1}(\delta) \cap f_n(S^2 - B(R))) > C\delta^2$$

for some constant C ,which only depends on the geometry of (V, J, g_J). If we choose the constant C appeared in (b) above same as the one here, then we get a contradiction.

This proves that $f_L(\infty) = f_R(\infty) = x$, the cuspidal point of f.

A similar argument, again using monotonicity for minimal surfaces, will show that when R and $|z_n|$ are large enough, $f_n(S^2 - B(R))$ is contained in $B_x(\frac{\delta}{4})$. From this the conclusion of the proposition follows immediately. □

Proof of theorem 2.6:

Let $P \in H_2(V)$ with $c_1(P) + n = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. By Proposition 4.0.3, when $|z_n|$ is big enough, there exists a gluing map

$$\#_{z_n} : \coprod_{A+B=P} \mathcal{M}(A, B, J, a_1, a_2; a_3, a_4) \to \mathcal{M}^{\tilde{z}_n}(P, J; a_1, a_2, a_3, a_4).$$

Note that for any cusp-curve $f = (f_L, f_R)$, $f_L \#_{z_n} f_R$ is locally convergent to f. This plus the fact that the domain of $\#_{z_n}$ is a finite set implies that $\#_{z_n}$ is injective. The surjectivity of $\#_{z_n}$ is a consequence of Proposition 5.0.5 and Corollary 4.0.3. As we remarked before, $\#_{z_n}$ also preserves the orientation.

This proves the special decomposition rule, and therefore the associativity of quantum multiplication. □

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