# On Post-critically finite polynomials

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Alfredo Poirier

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# Abstract of the Dissertation
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We extend the work of *Bielefeld, Fisher and Hubbard* on critical portraits to the case of arbitrary postcritically finite polynomials. This determines an effective classification of postcritically finite polynomials as dynamical systems. As an application of our results, we also state and prove necessary and sufficient conditions for the realization of Hubbard Trees.

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Contents

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# Chapter I
# Basic Concepts and Main Results

This work is concerned with the classification of Postcritically finite polynomials as dynamical systems. We provide two different types of effective classification for such dynamical systems. In the first part, we extend the work of *Bielefeld, Fisher and Hubbard* on critical portraits (see [BFH] and [F]) to the case of arbitrary postcritically finite polynomials. As an application of these results we state and prove in the second part necessary and sufficient conditions for the realization of Hubbard Trees.

## First Part: Critical Portraits.

In the first three sections of this introductory chapter we define the concept of critically marked polynomials and state their main combinatorial properties. Our definition extends the concept presented in [F] and [BFH] by including the possibility of periodic critical points. This definition differs slightly from that given in the above references in the strictly preperiodic case, but our results are the same. This small modification will later be useful, because some proofs will be simplified.

Our definition is supported by a number of examples given in Section 4. We remark here that the 'hierarchic selections' in the construction, are essential only to the marking corresponding to Fatou set critical cycles. Here they are needed in order to guarantee uniqueness for the polynomial with specified critical portrait. (Compare Example 4.5, and see the remark following Lemma II.2.4). The inclusion of the 'hierarchic selection' for Julia set critical points was made to uniformize notation and is not essential (compare [BFH] where all critical points are in the Julia set).

## 1. Preliminaries.

**1.1.** Let $P$ be a polynomial of degree $d > 1$ with $\Omega(P)$ the set of critical points. For $M \subset \mathbf{C}$ denote by $\mathcal{O}(M) = \cup_{n=0}^{\infty} P^{\circ n}(M)$ the *orbit of $M$.* If the orbit $\mathcal{O}(\Omega(P))$ of the critical set is finite, we say that $P$ is *postcritically finite (PCF).* It follows that every critical point of $P$ is periodic or preperiodic. We call the orbit $\mathcal{O}(\omega)$ of a periodic critical point $\omega$ (if any) a *critical cycle.* In this postcritically finite case a criterion to decide when a preperiodic (or periodic) point is in the Fatou set is as follows.

*A preperiodic point is in the Fatou set if and only if it eventually maps to a critical cycle.*

If $P$ is postcritically finite, then the Julia set $J(P)$ and the filled in Julia set $K(P)$ of $P$ are connected and locally connected (see [M] Theorem 17.5). As there are no wandering domains for the Fatou components of this polynomial $P$, each bounded Fatou component contains exactly one point $z$ (called its *center*) which eventually maps to a critical point. If we map this component $U(z)$ onto the unit disk by an uniformizing Riemann map $\phi$ with

$\phi(z) = 0$, we can talk about *internal rays* in $U(z)$ defined as the preimages of radial segments under $\phi$. Because we are in the locally connected case those internal rays can be extended up to the boundary.

In the case of the basin of attraction of $\infty$, if the polynomial is monic and centered, the uniformizing Riemann map can be chosen tangent to the identity at $\infty$. These rays are called *external rays*, and satisfy the condition $P(R_\theta) = R_{d\theta}$.

In general, let $\omega \mapsto P(\omega) \mapsto \ldots \mapsto P^{\circ n}(\omega) = \omega$ be a critical cycle. Then $P^{\circ n} : U(\omega) \mapsto U(\omega)$ is a degree $\mathcal{D} > 1$ cover of itself ($\mathcal{D}$ is the product of the local degree of elements in the orbit $\mathcal{O}(\omega)$, and $U(\omega)$ the Fatou component with center $\omega$). It follows then that the uniformizing Riemann map $\phi_\omega$ can be chosen so that

$$\phi_\omega(z)^{\mathcal{D}} = \phi_\omega(P^{\circ n}(z)).$$

In this case the Riemann map is known as the *Böttcher coordinate* (compare [M] Theorem 6.7). This coordinate is uniquely defined up to conjugation with a $(\mathcal{D}-1)^{th}$ root of unity. In particular, it is easy to see that there are exactly $\mathcal{D}-1$ *'fixed' internal rays*, i.e, internal rays $\mathcal{R}$ satisfying $P^{\circ n}(\mathcal{R}) = \mathcal{R}$. They correspond in the Böttcher coordinate to the segments $\{re^{\frac{2\pi k i}{\mathcal{D}-1}} : r \in [0,1),\ k = 0, \ldots, \mathcal{D}-2\}$.

What is important to note here, is that the same construction is valid for all elements in the critical cycle. Note that if we choose a coordinate $\phi_\omega$ in which the internal ray $\mathcal{R}$ corresponds to the real segment $[0,1)$, then we can choose in a unique way a coordinate $\phi_{P(\omega)}$ (at $P(\omega)$) for which $P(\mathcal{R})$ corresponds to $[0,1)$. Furthermore in this case

$$\phi_{P(\omega)}(P(z)) = (\phi_\omega(z))^{deg_\omega P},$$

where $deg_\omega P$ is the local degree of $P$ at $\omega$ (for more details see [DH1, Chapter 4, Proposition 2.2]).

**1.2 Lemma.** *If a critical point $z$ belongs to a critical cycle of period $n = n_z$, then $P^{\circ n}|_{\overline{U(z)}}$ (which has degree say $\mathcal{D}_z > 1$) has exactly $\mathcal{D}_z - 1$ different fixed points in the boundary $\partial U$ of this component $U(z)$ respect to this return map. Furthermore, all external rays that land at such points have period exactly $n$.*

**Proof.** The first part is well known. For the second, we consider near this periodic point segments of all the external rays which land there, together with the internal ray joining this point to the center $z$. The cyclic order of these analytic arcs must be preserved under iteration. The result thus follows easily. #

**1.3 Supporting arguments.** Given a Fatou component $U$ and a point $p \in \partial U$, there are only a finite number of external rays $R_{\theta_1}, \ldots, R_{\theta_k}$ landing at $p$. These rays divide the plane in $k$ regions. We order the arguments of these rays in counterclockwise cyclic order $\{\theta_1, \ldots, \theta_k\}$, so that $U$ belongs to the region determined by $R_{\theta_1}$ and $R_{\theta_2}$ ($\theta_1 = \theta_2$ if there is a single ray landing at $p$). The argument $\theta_1$ (respectively the ray $R_{\theta_1}$) is by definition the *(left) supporting argument (respectively the (left) supporting ray) of the Fatou component U*. In a completely analogous way we can define right supporting rays. Note that an argument supports at most one Fatou component (compare [DH1, Chapter VII.4]). Furthermore, by definition, given a Fatou component $U$, for every boundary point $p$ there is an external ray landing at $p$, and therefore a supporting ray for $U$.

**1.4 Extended Rays.** Given an external ray $R_\theta$ supporting the Fatou component $U(z)$ with center $z$, we extend $R_\theta$ by joining its landing point with $z$ by an internal ray, and call this set an *extended ray* $\hat{R}_\theta$ *with argument* $\theta$.

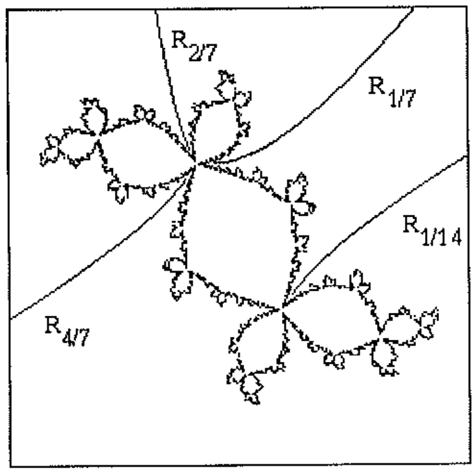



*Figure 1.1*

**1.5 Example.** Consider the postcritically finite polynomial $P_c(z) = z^2 + c$ (where the critical value $c \approx -0.12256117 + 0.74486177i$ satisfies $c^3 + 2c^2 + c + 1 = 0$). The rays with argument $1/7, 2/7, 4/7$ all land at the same fixed point. But $R_{4/7}$ is the only ray landing at this point, which supports the critical component. (Compare Figure 1.1.)

## 2. Construction of Critically Marked Polynomials.

Given a postcritically finite polynomial $P$, we associate to every critical point a finite subset of $\mathbf{Q}/\mathbf{Z}$ and construct a *critically marked polynomial* $(P, \mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_{n_F}\},\ \mathcal{J} = \{\mathcal{J}_1, \ldots, \mathcal{J}_{n_J}\})$. Here $\mathcal{F}_k$ would be the set of arguments associated with the critical point $z_k^F$ in the Fatou set, and $\mathcal{J}_k$ would be the set associated with the critical point $z_k^J$ in the Julia set. The number of elements in these finite sets would be equal to the local degree of the associated critical points. We remark that given a polynomial its critical marking is not necessarily unique. Also note that one of these two families will be empty if there are no critical points in the Fatou or Julia set. In the following definition we will always work with left supporting rays. We remark that we could equally well work with the right analogue, but there must be the same choice throughout. Also, multiplication by $d$ modulo 1 in $\mathbf{R}/\mathbf{Z}$ will be denoted by $m_d$.

**2.1 Construction of $\mathcal{F}_k$.** First we consider the case in which a given Fatou critical point $z = z_k^F$ is periodic. Let $z = z_k^F \mapsto P(z) \mapsto \ldots \mapsto P^{\circ n}(z) = z$ be a critical cycle of period $n$ and degree $\mathcal{D}_z > 1$ (compare §1.1). We construct the associated set $\mathcal{F}_\ell$ for every critical point $z_\ell^{\mathcal{F}}$ in the cycle simultaneously. Denote by $d_z$ the local degree of $P$ at $z$. We pick any periodic point $p_z \in \partial U(z)$ of period dividing $n$ (which is not critical because it is periodic and belongs to the Julia set $J(P)$) and consider the supporting ray $R_\theta$ for this component $U(z)$ at $p_z$. Note that this choice naturally determines a periodic supporting ray for every Fatou component in the cycle. The period of this ray is exactly $n$ (compare Lemma 1.2). Given this periodic supporting ray $R_\theta$, we consider the $d_z$ supporting rays for this same component $U(z)$ that are inverse images of $P(R_\theta) = R_{m_d(\theta)}$. The set of arguments of these rays is defined to be $\mathcal{F}_k$. Keeping in mind that a preferred periodic supporting ray has been already chosen, we repeat the same construction for all critical points in this cycle. Note that as the cycle has critical degree $\mathcal{D}_z$, we can produce $\mathcal{D}_z - 1$ different possible choices for $\mathcal{F}_k$. If $\mathcal{F}_k$ is the set associated with the periodic critical point $z_k$, there is only one periodic argument in $\mathcal{F}_k$ (namely $\theta$ as above), we call this angle *the preferred supporting argument associated with* $z_k^F$. Note that by definition, the period of $z_k^F$ equals the period of the associated preferred periodic argument.

Otherwise, if $z = z_k^F$ of degree $d_z > 1$, is a non periodic critical point in the Fatou set $F(P)$, there exists a minimal $n > 0$ for which $w = P^{\circ n}(z)$ is critical. If $w$ has associated a preferred supporting ray $R_\theta$ (at the beginning only periodic critical points do), then $P^{-n}(R_\theta)$ contains exactly $d_z$ rays which support this Fatou component $U(z)$. The set of arguments of these rays is defined to be $\mathcal{F}_k$. We pick any of those and call it the *preferred supporting argument associated with* $z$. We continue this process for all Fatou critical points.

**2.2 Construction of $\mathcal{J}_k$.** Given $z = z_k^J$ (a critical point in $J(P)$) of degree $d_k > 1$, we distinguish two cases. If the forward orbit of $z$ contains no other critical point, we have that for some $\theta$ (usually non unique) $R_\theta$ lands at $P(z)$. Now $P^{-1}(R_\theta)$ consists of $d$ different rays, among them exactly $d_k$ land at $z$. Define $\mathcal{J}_k$ as the set of arguments of these rays, and choose a *preferred ray.* Otherwise, $z$ will map in $n \geq 1$ iterations to a critical point, which we assume to have associated a preferred ray $R_\theta$. In the $n^{th}$ inverse image $P^{-n}(R_\theta)$ of this preferred ray, there are $d_k$ rays which land at $z$. The set of arguments of these rays is defined to be $\mathcal{J}_k$. Again we pick one of those to be preferred, and continue until every critical point has an associated set.

The critical marking itself gives information about how many iterates are needed for a given critical point to become periodic. For example we have the following lemma.

**2.3 Lemma.** *Let $\gamma$ be a preferred supporting argument in the set $\mathcal{F}_k$ (respectively in $\mathcal{J}_k$). Then the multiple $m_d^{\circ n}(\gamma)$ (with $n \geq 1$) is periodic but $m_d^{\circ n-1}(\gamma)$ is not if and only if $z_k^F$ (respectively $z_k^J$) falls in exactly $n$ iterations into a periodic orbit.*

**Proof.** This clearly follows from the construction. #

The importance of the above construction is stated in the following theorem. The proof will be given in Chapter III (compare also Theorem 3.9).

**2.4 Theorem.** *Every centered monic postcritically finite polynomial $P$ has a critical marking $(P, \mathcal{F}, \mathcal{J})$. This marking determines the polynomial $P$ in the following sense: if $(P, \mathcal{F}, \mathcal{J})$ and $(Q, \mathcal{F}, \mathcal{J})$ are critically marked polynomials, then $P = Q$. In other words, two monic centered post-critically finite polynomials with the same critical marking $(\mathcal{F}, \mathcal{J})$ must be equal.*

**Remark.** Note that the construction of associated sets was done in several steps. We first complete the choice for all critical cycles, and then proceed backwards. In both the Fatou and Julia set cases we will have to make decisions at several stages of the construction. Such decisions will affect the choice of the marking for all critical points found in the backward orbit of these starting ones. Each time that this kind of construction is made, we will informally say that it is a *hierarchic selection.* We encourage the reader to take a look at the examples in Section 4.

## 3. The Combinatorics of Critically Marked Polynomials.

In order to analyze which properties the families $(\mathcal{F}, \mathcal{J})$ satisfy, it is convenient to introduce some combinatorial notation.

**3.1 Definitions.** We say that a subset $\Lambda \subset \mathbf{R}/\mathbf{Z}$ is a *(d-)preargument set* if $m_d(\Lambda)$ is a singleton. For technical reasons we will always assume that $\Lambda$ contains at least two elements. If all elements of $\Lambda$ are rational, we say that $\Lambda$ is a *rational preargument set.* It follows by construction that whenever $(P, \mathcal{F}, \mathcal{J})$ is a marked polynomial, all the sets $\mathcal{J}_k$, and $\mathcal{F}_l$ are rational $d$-preargument sets.

Consider now a family $\mathbf{\Lambda} = \{\Lambda_1, \ldots, \Lambda_n\}$ of finite subsets of the circle $\mathbf{R}/\mathbf{Z}$. The family $\mathbf{\Lambda}$ determines the *family union* set $\mathbf{\Lambda}^{\cup} = \bigcup \Lambda_i$. We say that any $\lambda \in \mathbf{\Lambda}^{\cup}$ is an *element of the family* $\mathbf{\Lambda}$. Furthermore, we can say that it is a periodic or preperiodic element of the family if it is so with respect to $m_d$. The set of all periodic elements in the family union will be denoted by $\mathbf{\Lambda}^{\cup}_{\mathbf{per}}$.

**3.2 Hierarchic Families.** We say that a family $\mathbf{\Lambda}$ is *hierarchic* if for any elements in the family $\lambda, \lambda' \in \mathbf{\Lambda}^{\cup}$, whenever $m_d^{\circ i}(\lambda), m_d^{\circ j}(\lambda') \in \Lambda_k$ for some $i, j > 0$ then $m_d^{\circ i}(\lambda) = m_d^{\circ j}(\lambda')$. (This is useful if we think of a dynamically preferred element in each $\Lambda_k$).

**3.3 Linkage Relations.** We will say that two subsets $T$ and $T'$ of the circle $\mathbf{R}/\mathbf{Z}$ are unlinked if they are contained in disjoint connected subsets of $\mathbf{R}/\mathbf{Z}$, or equivalently, if $T'$ is contained in just one connected component of the complement $\mathbf{R}/\mathbf{Z} - T$. (In particular $T$ and $T'$ must be disjoint.) If we identify $\mathbf{R}/\mathbf{Z}$ with the boundary of the unit disk, an equivalent condition would be that the convex closures of these sets are pairwise disjoint. If $T$ and $T'$ are not unlinked then either $T \cap T' \neq \emptyset$ or there are elements $\theta_1, \theta_2 \in T$ and $\theta_1', \theta_2' \in T'$ such that the cyclic order can be written $\theta_1, \theta_1', \theta_2, \theta_2', \theta_1$. In this second case we say that $T$ and $T'$ are *linked*. More generally a family $\mathbf{\Lambda} = \{\Lambda_1, \ldots, \Lambda_n\}$ is an *unlinked family* if $\Lambda_1, \ldots, \Lambda_n$ are pairwise unlinked. Alternatively, each $\Lambda_i$ is completely contained in a component of $\mathbf{R}/\mathbf{Z} - \Lambda_j$ for all $j \neq i$.

The preceding definition has its motivation in the description of the dynamics of external rays for a polynomial map. Suppose the external rays $R_{\theta_i}, R_{\psi_i}$ land at $z_i$ for $i = 1, 2$. If $z_1 \neq z_2$ then the sets $\{\theta_1, \psi_1\}, \{\theta_2, \psi_2\}$ are unlinked, for otherwise the rays will cross each other. The same argument applies if we consider rays supporting Fatou components. But if we analyze linkage relations arising from rays supporting a Fatou component and rays that land at some point, we may get minor problems. Anyway, it is easy to see that even in this case the associated sets of arguments will be 'almost' unlinked. (Compare condition (c.2) and as well as Proposition 3.8 below.)

**3.4 Weak linkage relations.** Consider two families $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ and $\mathcal{J} = \{\mathcal{J}_1, \dots, \mathcal{J}_m\}$; we say that $\mathcal{J}$ is *weakly unlinked to* $\mathcal{F}$ *in the right* if we can chose arbitrarily small $\epsilon > 0$ so that the family $\{\mathcal{F}_1, \dots, \mathcal{F}_n, \mathcal{J}_1 - \epsilon, \dots, \mathcal{J}_m - \epsilon\}$ is unlinked. (Here $\Lambda - \epsilon = \{\lambda - \epsilon \ (mod\ 1) : \lambda \in \Lambda\}$.) In particular each family should be unlinked. Note that the definition allows empty families. To simplify notation we will simply say that *"$\mathcal{F}$ and $\mathcal{J}^-$ are unlinked"*.

**3.5 Formal Critical Portraits.** Consider families $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ and $\mathcal{J} = \{\mathcal{J}_1, \dots, \mathcal{J}_m\}$ of rational ($d$-)prearguments. We say that the pair $(\mathcal{F}, \mathcal{J})$ is a *degree d formal critical portrait* if the following conditions are satisfied.

*(c.1)* $d - 1 = \sum(\#(\mathcal{F}_k) - 1) + \sum(\#(\mathcal{J}_l) - 1)$
*(c.2)* $\mathcal{F}$ *and* $\mathcal{J}^-$ *are unlinked.*
*(c.3) Each family is hierarchic.*
*(c.4) Given* $\gamma \in \mathcal{F}^{\cup}$, *there is an* $i > 0$ *such that* $m_d^{\circ i}(\gamma) \in \mathcal{F}^{\cup}_{\mathbf{per}}$.
*(c.5) No* $\theta \in \mathcal{J}^{\cup}$ *is periodic.*

This set of conditions represent the simplest conditions satisfied by the critical marking of a postcritically finite polynomial. Condition (c.1) says that we have chosen the right number of arguments. Condition (c.2) means that the rays and extended rays determine sectors which do not cross each other, and that $\mathcal{F}$ was constructed from arguments of left supporting rays. This reflects our decision to chose the supporting arguments as the rightmost possible argument of an external ray. Condition (c.3) reflects our choice of dynamically preferred rays. Condition (c.4) indicates that arguments in $\mathcal{F}$

are related to Fatou critical points. Condition (c.5) indicates that arguments in $\mathcal{J}$ are related to Julia set critical points. Unfortunately there are formal critical portraits which do not correspond to a postcritically finite polynomial (compare Example II.2.8). In order to state necessary and sufficient conditions we need to study the dynamically defined partitions of the unit circle determined by these elements.

**3.6.** Given two families $\mathcal{F}, \mathcal{J}$ as above, we form a partition $\mathcal{P} = \{L_1, \ldots, L_d\}$ of the unit circle minus a finite number of points $\mathbf{R}/\mathbf{Z} - \mathcal{F}^{\cup} - \mathcal{J}^{\cup}$, in the following way. We consider two points $t, t' \in \mathbf{R}/\mathbf{Z} - \mathcal{F}^{\cup} - \mathcal{J}^{\cup}$. By definition, $t, t'$ are *unlink equivalent* if they belong to the same connected component of $\mathbf{R}/\mathbf{Z} - \mathcal{F}_i$ and $\mathbf{R}/\mathbf{Z} - \mathcal{J}_j$, for all possible $i, j$. Let $L_1, \ldots, L_d$ be the resulting unlink equivalence classes with union $\mathbf{R}/\mathbf{Z} - \mathcal{F}^{\cup} - \mathcal{J}^{\cup}$. It is easy to check that each $L_p$ is a finite union of open intervals with total length $1/d$.

Each element $L_i \in \mathcal{P}$ of the partition is a finite union $L_i = \cup(x_j, y_j)$ of open connected intervals. We also define the sets $L_i^+ = \cup[x_j, y_j)$ and $L_i^- = \cup(x_j, y_j]$. It is easy to see that both $\mathcal{P}^+ = \{L_1^+, \ldots, L_d^+\}$ and $\mathcal{P}^- = \{L_1^-, \ldots, L_d^-\}$ are partitions of the unit circle. As every $\theta \in \mathbf{R}/\mathbf{Z}$ belongs to exactly one set $L_k^+$, we define its *right address* $A^+(\theta) = L_k$. In an analogous way we define the *left address* $A^-(\theta)$ *of* $\theta$. We associate to every argument $\theta \in \mathbf{R}/\mathbf{Z}$ a *right symbol sequence* $S^+(\theta) = (A^+(\theta), A^+(m_d(\theta)), \ldots)$ and a *left symbol sequence* $S^-(\theta) = (A^-(\theta), A^-(m_d(\theta)), \ldots)$. Note that for all but a countable number of arguments $\theta \in \mathbf{R}/\mathbf{Z}$ (namely the arguments present in the families and their iterated inverses), the left $S^-(\theta)$ and the right $S^+(\theta)$ symbol sequences coincide. By $S(\theta)$ will be meant either (left or right) symbol sequence.

**3.7 Admissible Critical Portraits.** Let $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ and $\mathcal{J} = \{\mathcal{J}_1, \ldots, \mathcal{J}_m\}$ be two families of rational ($d$-)prearguments. We say that $(\mathcal{F}, \mathcal{J})$ is a *degree $d$ admissible critical portrait* if $(\mathcal{F}, \mathcal{J})$ is a degree $d$ formal critical portrait and the following two extra conditions are satisfied.

*(c.6) Let $\gamma \in \mathcal{F}^{\cup}_{\mathbf{per}}$ and $\lambda \in \mathbf{R}/\mathbf{Z}$, then $\lambda = \gamma$ if and only if $S^+(\gamma) = S^+(\lambda)$.*

*(c.7) Let $\theta \in \mathcal{J}_l$ and $\theta' \in \mathcal{J}_k$. If for some $i$, $S^-(m_d^{\circ i}(\theta)) = S^-(\theta')$, then $m_d^{\circ i}(\theta) \in \mathcal{J}_k$.*

**3.8 Proposition.** *If $(P, \mathcal{F}, \mathcal{J})$ is a critically marked polynomial, then $(\mathcal{F}, \mathcal{J})$ is an admissible critical portrait.*

Condition (c.6) indicates that arguments in $\mathcal{F}_l$ must support Fatou components. Condition (c.7) indicates that different elements in the family $\mathcal{J}$ are associated with different critical points. The proof of this proposition will be given in Section II.2.

Now we can state the main result for critically marked polynomials as follows (the proof of this theorem will be given in Chapter III).

**3.9 Theorem.** *Let $(\mathcal{F}, \mathcal{J})$ be a degree $d$ admissible critical portrait. Then there is a unique monic centered postcritically finite polynomial $P$, with critical marking $(P, \mathcal{F}, \mathcal{J})$.*

Now we should ask if conditions *(c.1)-(c.7)* represent a finite amount of information to be checked. This question is answered in a positive way by the following lemma. The proof would be given in Section II.1.

**3.10 Lemma.** *Suppose $\theta$ and $\theta'$ have the same periodic left (or right) symbol sequence. Then $\theta$ and $\theta'$ are both periodic and of the same period.*

In particular condition $(c.6)$ can be replaced by condition $(c.6)'$:

*(c.6)' Let $\gamma \in \mathcal{F}^{\cup}_{\mathbf{per}}$ and let $\lambda$ have the same period as $\gamma$, then $\lambda = \gamma$ if and only if $S^+(\gamma) = S^+(\lambda)$.*

**3.11.** The next question that we ask is what kind of information about the Julia set can be gained by looking carefully into the combinatorics. For example, if can we determine if two rays land at the same point by only looking at their arguments. In fact, left symbol sequences convey all the information necessary to effectively decide whether two rays land at the same point or not. This is done as follows. Suppose $\mathcal{J}_i = \{\theta_1, ..., \theta_k\} \in \mathcal{J}$ with corresponding left symbol sequences $S^-(\theta_1), \ldots, S^-(\theta_k)$. As we expect the rays with those arguments to land at the same critical point, we declare them ($i$-)equivalent; i.e, we write $S^-(\theta_\alpha) \equiv_i S^-(\theta_\beta)$. Then we set $\theta \approx \theta'$ either if $S^-(\theta) = S^-(\theta')$ or there is an $n \geq 0$ such that $A^-(m_d^{\circ j}(\theta)) = A^-(m_d^{\circ j}(\theta'))$ for all $j < n$ and $S^-(m_d^{\circ n}(\theta)) \equiv_i S^-(m_d^{\circ n}(\theta'))$ for some $i$. This relation $\approx$ is not necessarily an equivalence relation, because transitivity may fail. To make this into an equivalence relation we say that $\theta \sim_l \theta'$ if and only if there are arguments $\lambda_0 = \theta, \lambda_1, \ldots, \lambda_m = \theta'$, such that $\lambda_0 \approx \ldots \approx \lambda_m$. The importance of this equivalence relation is shown by the following proposition. The proof will be given in Chapter II.

**3.12 Proposition.** *Let $(P, \mathcal{F}, \mathcal{J})$ be a critically marked polynomial. Then $R_\theta$ and $R_{\theta'}$ land at the same point if and only if $\theta \sim_l \theta'$.*

# 4. Examples.

We will illustrate with examples the definitions of the previous sections. We will try to isolate and illustrate all possible complications. Of course, the worst possible examples will involve several of these at the same time.

**4.1 The rabbit.** (See Figure 1.1.) Once again consider the degree two polynomial $P_c(z) = z^2 + c$ with $c \approx -0.12256117 + 0.74486177i$. The Fatou critical point $z = 0$ has a period 3 orbit under iteration. Therefore $P^{\circ 3}$ restricted to the critical component is a degree 2 cover of itself. It follows that the map $P^{\circ 3}$ has a unique fixed point in the boundary of this critical Fatou component. As noticed above, among the three rays $R_{1/7}, R_{2/7}, R_{4/7}$ landing at this fixed point, only the ray $R_{4/7}$ supports the critical component. By the definition of marking, we must look for the other ray that supports this component and maps to $P(R_{4/7}) = R_{1/7}$. This ray can only be $R_{1/14}$. Thus, we have constructed a marking for $P$. In this case $\mathcal{F} = \{\mathcal{F}_1\}$ and $\mathcal{J} = \emptyset$, where $\mathcal{F}_1 = \{4/7, 1/14\}$.

It is important to note that we were looking for a fixed point of $P^{\circ 3}$ restricted to the boundary of the critical Fatou component. Such a fixed point for $P^{\circ 3}$ turned out to be a fixed point for $P$ as well, but the rays landing there have period equal 3.

**4.2 The Ulam-von Neumann map.** We consider now the strictly preperiodic case. Let $P(z) = z^2 - 2$, and note that the orbit of the critical point $z = 0$ is $0 \mapsto -2 \mapsto 2 \mapsto 2 \ldots$. Only the external ray $R_0$ lands at $z = 2$,

**4.4 Non trivial critical cycle.** (See Figure 1.3.) Consider the degree 3 polynomial $P(z) = z^3 - \frac{3}{2}z$. The critical points satisfy $z^2 = 1/2$, and is easy to see that they are interchanged by $P$ (i.e, if $a$ is a critical point then $P(a) = -a$). In each of the critical Fatou components the map $P^{\circ 2}$ is a degree 4 (the product of the degrees of the cycle!) covering of itself. In this way, there must be in the boundary of each component 3 ($= 4 - 1$) possible choices of periodic points. One of those fixed points ($z = 0$) belongs to the boundary of both components. The rays landing at $z = 0$ are $R_{1/4}$ and $R_{3/4}$, and each one supports exactly one of the Fatou critical components. The period 2 rays that support the *'rightmost'* component are $R_{3/4}, R_{7/8}, R_{1/8}$ (their respective images $R_{1/4}, R_{5/8}, R_{3/8}$ support the other). Therefore, the choice of a periodic supporting ray for one component, forces the choice of its image for the other.

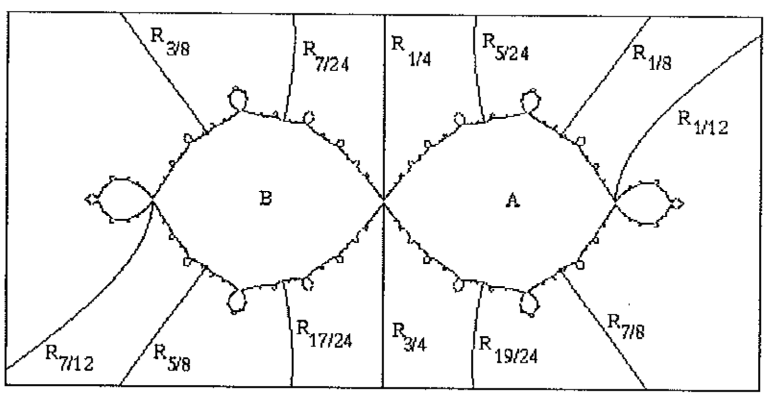



*Figure 1.3*

This polynomial has exactly three markings, all of type $\mathcal{F} = \{\mathcal{F}_A, \mathcal{F}_B\}$, $\mathcal{J} = \emptyset$. The periodic supporting rays are listed on the left.

| *Component A* | *Component B* | $\mathcal{F}_A$ | $\mathcal{F}_B$ |
|---|---|---|---|
| $R_{3/4}$ | $R_{1/4}$ | $\{3/4, 1/12\}$ | $\{1/4, 7/12\}$ |
| $R_{7/8}$ | $R_{5/8}$ | $\{7/8, 5/24\}$ | $\{5/8, 7/24\}$ |
| $R_{1/8}$ | $R_{3/8}$ | $\{1/8, 19/24\}$ | $\{3/8, 17/24\}$ |

The question is now, why can we not take $\mathcal{F}_A = \{3/4, 1/12\}$ and $\mathcal{F}_B = \{3/8, 17/24\}$ as a marking? This is forbidden by the rules of §3 since 3/4 and 3/8 do not belong to the same cycle. A good reason for this rule is given in the next example.

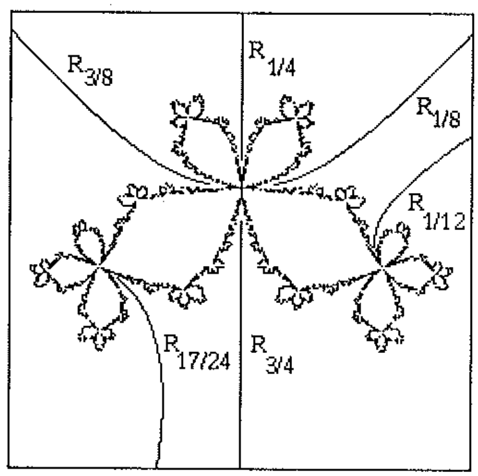



*Figure 1.4*

**4.5 Bad choice, wrong polynomial.** (See Figure 1.4.) There is a polynomial with marking $\mathcal{F} = \{\mathcal{F}_A, \mathcal{F}_B\}, \mathcal{J} = \emptyset$, where $\mathcal{F}_A = \{3/4, 1/12\}$, $\mathcal{F}_B = \{3/8, 17/24\}$. But it is not the one in Example 4.4.

For the polynomial $P(z) = z^3 + az + b$ (where $a = -0.75, b \approx 0.661438i$), the rays $R_{3/4}, R_{1/8}, R_{1/4}, R_{3/8}$, land at a fixed point which belongs to the

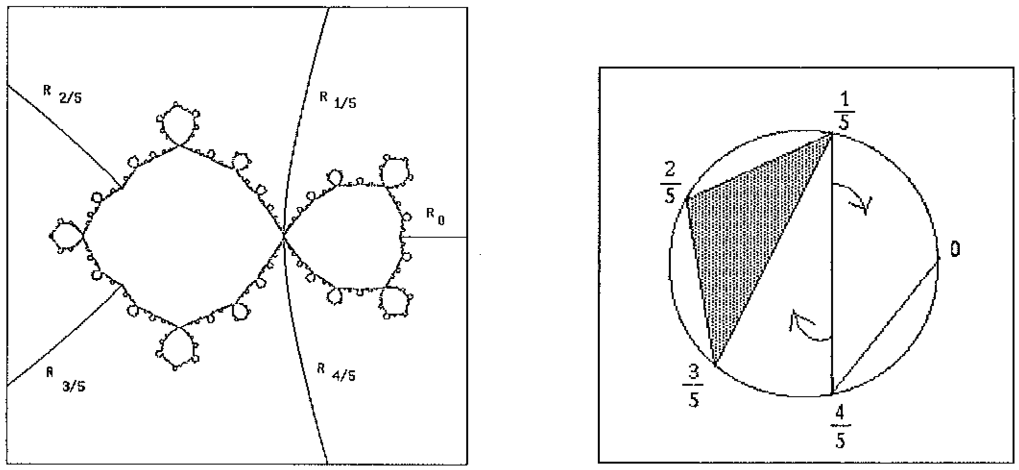



*Figure 1.6*

**4.7 Badly mixed case.** (See Figure 1.6.) Consider the degree 5 polynomial $P(z) = c(z^5 + 3z^4 + 3z^3 + z^2)$, where $c \approx 4.3582708$. It has two Fatou critical components, one (on the right) fixed of degree 2, and one (on the left) preperiodic of degree 3 (absorbed by the first in one iteration). The boundaries of these two Fatou components share a point, which happens to be critical. The image of this Julia set critical point is the only fixed point lying in the boundary of the fixed Fatou critical component. Only the ray $R_0$ lands at this fixed point. The rays $R_{1/5}, R_{4/5}$ are thus the only rays landing at the Julia set critical point. Now, one of these rays ($R_{4/5}$) supports the fixed Fatou component, while the other supports the preperiodic one. Also $R_0$ must have two inverses supporting the fixed Fatou component $(R_0, R_{4/5})$, and three supporting the preperiodic one $(R_{1/5}, R_{2/5}, R_{3/5})$. Thus, the marking is $\mathcal{F} = \{\{0, 4/5\}, \{1/5, 2/5, 3/5\}\}, \mathcal{J} = \{\{1/5, 4/5\}\}$. Note that in this case there are arguments that belong to one family and to the other. Of course, if this happens, these arguments must be strictly preperiodic.

| *Component B* | $\mathcal{F}_B$ | $\mathcal{F}_{B'}$ |
|---|---|---|
| $R_{1/4}$ | $\{1/4, 26/72, 42/72, 50/72\}$ | $\{66/72, 2/72\}$ |
| $R_{5/8}$ | $\{5/8, 53/72, 21/72, 29/72\}$ | $\{31/72, 39/72\}$ |
| $R_{3/8}$ | $\{3/8, 43/72, 51/72, 19/72\}$ | $\{5/72, 69/72\}$ |

This implies that we have 9 possible markings. Note that the marking for the components $A, B$ are independent, but they uniquely determine the marking for $A', B'$.

**4.9** (See Figure 1.7.) In our final example we show the importance of working with two separate families $\mathcal{F}, \mathcal{J}$. Consider the sets $\mathcal{A} = \{0, \frac{1}{3}\}$, $\mathcal{B} = \{\frac{5}{9}, \frac{8}{9}\}$. The polynomial $P(z) = z^3 + Az + B$ ($A = 2.25$, $B \approx -0.4330127i$) has marking $\mathcal{F} = \{\mathcal{A}, \mathcal{B}\}$, $\mathcal{J} = \emptyset$, while the polynomial $P(z) = z^3 + A'z + B'$ ($A' \approx 2.181104577$, $B' \approx -0.3871686256i$) has marking $\mathcal{F} = \{\mathcal{A}\}$, $\mathcal{J} = \{\mathcal{B}\}$.

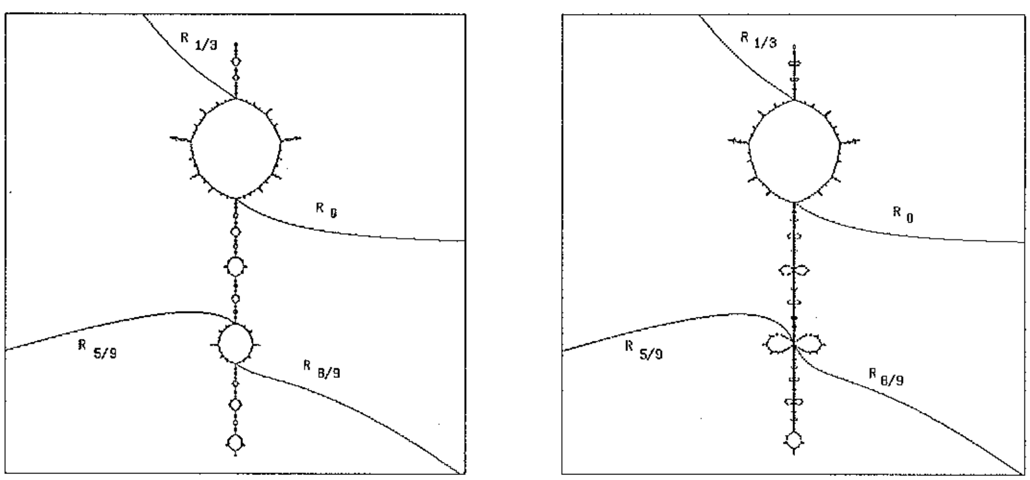



*Figure 1.7. Almost the same marking.*

# Second Part: Hubbard Trees.

We provide necessary and sufficient conditions for the realization of Hubbard Trees. This gives an effective classification of Post-critically finite polynomials as dynamical systems.

## 5. Hubbard Trees.

Given a polynomial $P$ of degree $n \geq 2$, we consider the set $K(P)$ (called the *filled Julia set*) of points whose orbit under iteration is bounded. This set is known to be compact and its complement consists of a unique unbounded component (see [M, Lemma 17.1]). The behavior under iteration of the critical points of this polynomial dramatically influences the topology of this set $K(P)$. For example, this set is connected if and only if all critical points are contained within (see [M, Theorem 17.3]). We are interested in the special case where the orbit of every critical points is finite, i.e, the case where the orbits of all critical points are periodic or eventually periodic. We call such polynomials *postcritically finite* ($PCF$ in short). For such polynomials the filled Julia set $K(P)$ is connected. Furthermore, it is also known in this case that $K(P)$ is locally connected (see [M, Theorem 17.5]).

In order to proceed further we establish some notation. The set $J(P) = \partial K(P)$ is called the *Julia set*, and its elements *Julia points*. The complement $F(P) = \mathbf{C} - J(P)$ of the Julia set is called the *Fatou set* and its elements *Fatou points*. A periodic orbit $z_0 \mapsto z_1 = P(z_0) \mapsto \ldots \mapsto z_n = z_0$

which contains a critical point is called a *critical cycle*. In the $PCF$ case a periodic orbit belongs to the Fatou set $F(P)$ if and only if it is a critical cycle (see [M, Corollary 11.6]).

In this $PCF$ case the dynamics of the polynomial admits a further decomposition. When restricted to the interior of $K(P)$ (which is not empty if and only if there exists a critical cycle), $P$ maps each component onto some other as a branched covering map. Furthermore, every component is eventually periodic (see [M, Theorem 13.4]). It is well known (see [M, Theorem 6.7]) that each component can be uniformized so that in local coordinates $P$ can be written as $z \mapsto z^n$ for some $n \geq 1$. Furthermore, if $U$ is a periodic bounded Fatou component, the first return map is conjugate in local coordinates to $z \mapsto z^n$ for some $n \geq 2$. In particular such cycles of components are in one to one correspondence with critical cycles. Also, in each component there is a unique point which eventually maps to a critical point (these points are those which correspond to 0 in local coordinates).

In the work [DH1], Douady and Hubbard suggested a combinatorial description of the dynamics of such polynomials using a tree-like structure. First we note the following (see [DH1, Corollary VII.4.2 p 64]).

**Lemma.** *Let $P$ be a $PCF$ polynomial. Then for any $z \in J(P)$, the set $J(P) - \{z\}$ consists of only a finite number of connected components.*

Thus, the filled Julia set is arranged in a tree like fashion. To simplify this tree we consider a finite invariant set $M$ (i.e, $P(M) \subset M$) containing all critical points. We join them in $K(P)$ by paths subject to the restriction that if they intersect a Fatou component, this intersection must consist of radial segments in the coordinate described above. Douady and Hubbard proved that this construction is unique and defines a finite topological tree $T_M$ in which all points in $M$ (and perhaps more) are vertices. Now, if from this tree we retain the dynamics and local degree at every vertex, the way this tree is embedded in the complex plane (up to isotopy class), and "a bit of extra information to recover the tree generated by $P^{-1}(M)$" (there are several ways to state this condition in a non-ambiguous way), they proved that different $PCF$ polynomials (i.e, not conjugated as dynamical systems) give rise to different tree-structures. No criterion for realization was given at the time. (The only previous partial results about realization are given in Lavaurs' thesis [L]).

A way to deal with this conditions is to introduce angles around vertices in the tree structure (see [DH1, p.46]). In what follows we will measure angles in turns (i.e, $360° = 1$ turn). Around a Fatou vertex $v$ (which correspond to 0 in the uniformizing coordinate), an angle between edges incident at $v$ is naturally defined by means of the local coordinate system. At Julia vertices, where $m$ components of $K(P)$ meet (compare the lemma above), the angle is defined to be a multiple of $1/m$ (this normalization is introduced here for the first time). These angles satisfy two conditions. First, they are compatible with the embedding of the tree. Second, we have that $\angle_{P(v)}(P(\ell), P(\ell')) = \delta(v)\angle_v(\ell, \ell')$ *(mod 1)*, where $\delta(v)$ is the local degree of $P$ at $v$ and $\ell, \ell'$ are edges incident at $v$ ($\angle_v$ and $\angle_{P(v)}$ measure the angles at $v$ and $P(v)$ respectively). When this further structure is given, we have a 'dynamical tree', which we denote by $\mathbf{H}_{P,M}$.

Now let us start with an abstract tree and try to reconstruct the appropriate polynomial.

**Definition.** By an *(angled) tree* $H$ will be meant a finite connected acyclic $m$-dimensional simplicial complex ($m = 0, 1$), together with a function $\ell, \ell' \mapsto \angle(\ell, \ell') = \angle_v(\ell, \ell') \in \mathbf{Q}/\mathbf{Z}$ which assigns a rational modulo 1 to each pair of edges $\ell, \ell'$ which meet at a common vertex $v$. This angle $\angle(\ell, \ell')$ should be skew-symmetric, with $\angle(\ell, \ell') = 0$ if and only if $\ell = \ell'$, and with $\angle_v(\ell, \ell'') = \angle_v(\ell, \ell') + \angle_v(\ell', \ell'')$ whenever three edges are incident at a vertex $v$. Such an angle function determines a preferred isotopy class of embeddings of $H$ into $\mathbf{C}$.

Let $V$ be the set of vertices. We specify a mapping $\tau : V \to V$ and call it the *vertex dynamics*, and require that $\tau(v) \neq \tau(v')$ whenever $v$ and $v'$ are endpoints of a common edge $\ell$. We consider also a *local degree function* $\delta : V \to \mathbf{Z}$ which assigns an integer $\delta(v) \geq 1$ to each vertex $v \in V$. We require that $deg(\delta) = 1 + \sum_{v \in V}(\delta(v) - 1)$ be greater that 1. By definition a vertex $v$ *is critical* if $\delta(v) > 1$, and *non-critical* otherwise. The *critical set* $\Omega(\delta) = \{v \in V : v \text{ is critical}\}$ is thus non empty.

The maps $\tau$ and $\delta$ must be related in the following way. Extend $\tau$ to a map $\tau : H \to H$ which carries each edge homeomorphically onto the shortest path joining the images of its endpoints. We require then that $\angle_{\tau(v)}(\tau(\ell), \tau(\ell')) = \delta(v)\angle_v(\ell, \ell')$ whenever $\ell, \ell'$ are incident at $v$ (in this case $\tau(\ell)$ and $\tau(\ell')$ are incident at the vertex $\tau(v)$ where the angle is measured).

A vertex $v$ is *periodic* if for some $n > 0$, $\tau^{\circ n}(v) = v$. The orbit of a periodic critical point is a *critical cycle*. We say that a vertex $v$ is of *Fatou* *type or a Fatou vertex* if it eventually maps into a critical cycle. Otherwise, if it eventually maps to a non critical cycle, it is of *Julia type or a Julia vertex.*

We define the distance $d_H(v, v')$ between vertices in $H$ as the number of edges in a shortest path $\gamma$ between $v$ and $v'$. We say that $(H, V, \tau, \delta)$ is *expanding* if the following condition is satisfied. For any edge $\ell$ whose end points $v, v'$ are Julia vertices, there is an $n \geq 1$ such that $d_H(\tau^{\circ n}(v), \tau^{\circ n}(v')) > 1$.

The angles at Julia vertices are rather artificial, so we normalize them as follows. If $m$ edges $\ell_1, \ldots, \ell_m$ meet at a periodic Julia vertex $v$, then we assume that the angles $\angle_v(\ell_i, \ell_j)$ are all multiples of $1/m$. (It follows that the angles at a periodic Julia vertex convey no information beyond the cyclic order of these $m$ incident edges.)

By an *abstract Hubbard Tree* we mean an angled tree $\mathbf{H} = ((H, V, \tau, \delta), \angle)$ so that the angles at any periodic Julia vertex where $m$ edges meet are multiples of $1/m$.

The basic existence and uniqueness theorem can now be stated as follows (compare Theorem V.4.7).

**Theorem A.** *Any abstract Hubbard Tree* $\mathbf{H}$ *can be realized as a tree associated with a postcritically finite polynomial* $P$ *if and only if* $\mathbf{H}$ *is expanding. Such a realization is necessarily unique up to affine conjugation.*

This abstract Hubbard Tree also gives information about external rays as the following theorem essentially due to Douady and Hubbard shows (compare [DH1, Chap VII]). This will follow in our case from Propositions V.3.3, VI.4.3 and the fact that $J(P)$ is locally connected.

**Theorem B.** *The number of rays which land at a periodic Julia vertex $v$ is equal to the number of incident edges of the tree $T$ at $v$, and in fact, there is exactly one ray landing between each pair of consecutive edges. Furthermore, the ray which lands at $v$ between $\ell$ and $\ell'$ maps to the ray which lands at $f(v)$ between $f(\ell)$ and $f(\ell')$.*

After these theorems there is no reason to distinguish between the abstract Hubbard Tree and the unique polynomial which realizes it.

**Definition.** A point $p \in J(P)$ is *terminal* if there is only one external ray landing at $p$. Otherwise $p$ is an *incidence point.* For incidence points we distinguish between *branching* (if there are more than two rays landing at $p$) and *non branching* (exactly two rays landing at $p$). For a postcritically finite polynomial $P$, every branching point must be periodic or preperiodic. Also every periodic branching point is present as vertex in any tree $\mathbf{H}_{P,M}$.

**Proposition IV.3.2.** *Let $P$ be a Postcritically Finite Polynomial and $z \in J(P)$ a branching point. Then $z$ is preperiodic (or periodic).*

**Proposition IV.3.3.** *Let $P$ be a Postcritically Finite Polynomial and $z \in J(P)$ a periodic incidence point. For any invariant finite set $M$ containing the critical points of $P$, we have $z \in T_{P,M}$. Furthermore, the number of components of $T_{P,M} - \{z\}$ is independent of $M$ and equals the number of components of $J(P) - \{z\}$.*

Now we give a brief description of Chapters IV,V and VI which are devoted to Hubbard Trees. In Chapter IV we have included the basic background of Hubbard Trees following the original exposition of Douady and Hubbard. We have done so because there is nowhere in the literature where we can find in a systematic way what was known up to now. In Chapter V, we introduce our basic abstract framework. We have carefully justified why there is the need to introduce all the abstract elements in our definition. In Chapter VI we give the proof of our main result. This proof is based in the theory of critical portraits developed in the first part of this work. In Appendix B, we study necessary and sufficient conditions under which an $n^{th}$ fold covering of a finite cyclic set to a proper subset can be given a compatible 'argument coordinate' so that it becomes multiplication by $n$.

# Chapter II
# Critical Portraits.

In this chapter we isolate the combinatorial properties of a critical portrait $(\mathcal{F}, \mathcal{J})$ as defined in Section I.3, and relate them to the dynamics of the respective critically marked polynomial. Section 1 deals with the partition in the unit circle determined by this marking. We also prove here Lemma I.3.10. Section 2 translates to the Julia set the language of Section 1. As a consequence we prove that the critical marking defines an admissible critical portrait. In Section 3 we prove Proposition I.3.12 which gives the combinatorial criterion for deciding when two external rays land at the same point. Section 4 characterizes the preimages of marked periodic rays landing at that same Fatou component from the combinatorial point of view. Almost all the material in this chapter can be found in a weaker formulation in [BFH]. The essential novelty here is Section 4, which plays a central role in the proof of the realization Theorem for Critical Portraits.

## 1. Partitions of the unit circle.

In this section we fix a formal critical portrait $(\mathcal{F}, \mathcal{J})$, and study some dynamical properties of the partition determined by these families.

Given a formal critical portrait $(\mathcal{F}, \mathcal{J})$, we defined in Chapter I the partitions $\mathcal{P} = \{L_1, \ldots, L_d\}$ and $\mathcal{P}^{\pm} = \{L_1^{\pm}, \ldots, L_d^{\pm}\}$. The first partition omits the arguments in $\mathcal{F}^{\cup} \cup \mathcal{J}^{\cup}$; while the other two cover the whole circle $\mathbf{R}/\mathbf{Z}$. We also know that each $L_p$ ($L_p^{\pm}$) is a finite union of open (semiopen) intervals with total length $1/d$ (compare Section I.3.6). From the dynamical point of view we can say even more.

**1.1 Lemma.** *Each $L_p$ is mapped bijectively by $m_d$ onto the complement of a finite set. Each $L_p^{\pm}$ is mapped bijectively by $m_d$ onto the whole unit circle. Furthermore these correspondences preserve the circular order.*

**Proof.** The proof is straightforward and is left to the reader. #

Before the next corollary, we recall briefly the standard language for manipulation of symbol sequences. Let $\mathbf{S} = (S_0, S_1, \ldots)$, where $S_i \in \mathcal{P}$. The *shift of* $\mathbf{S}$ is the sequence $\sigma(\mathbf{S}) = (S_1, S_2, \ldots)$. (Formally $\sigma$ is a map from the space of symbol sequences to itself.) The $i^{th}$ projection $\pi_i$ is the map from symbol sequences to the partition space $\mathcal{P}$ defined by $\pi_i(\mathbf{S}) = S_i$. The proof of the following corollary is an easy induction using Lemma 1.1 and is left to the reader.

**1.2 Corollary.** *Suppose $m_d^{\circ n}(\theta) = m_d^{\circ n}(\theta')$ and $\pi_j(S^+(\theta)) = \pi_j(S^+(\theta'))$ for all $j < n$, then $\theta = \theta'$. (The same is true if we consider left symbol sequences instead.)* #

**Warning.** Corollary 1.2 is not necessarily true if we compare left with right symbol sequences. From $S^+(\theta) = S^-(\theta')$ and $m_d(\theta) = m_d(\theta')$, we can not infer $\theta = \theta'$. For example, in the Ulam-von Neumann map (compare Example I.4.2), $S^+(1/4) = S^-(3/4)$, and both arguments become equal after doubling.

As our partitions are well behaved under iteration, it is natural to introduce dynamically defined refinements. The fact that these refinements are also unlinked allow us derive some basic properties of symbol sequences.

**1.3 Definition.** For $S_0, S_1, \ldots \in \mathcal{P}$, set $U_{S_0,\ldots,S_n} = \{\theta \in \mathbf{R}/\mathbf{Z} : m_d^{\circ i}\theta \in S_i, i = 0, \ldots, n\}$. The Lebesgue measure of this set is $1/d^{n+1}$ as can be easily verified by induction. Also set $U_{S_0,S_1,\ldots} = \bigcap_{n=0}^{\infty} cl(U_{S_0,\ldots,S_n})$. This last set being a nested intersection of non empty compact sets, is non empty. It is easy to see that if $S(\theta) = (S_0, S_1, S_2, \ldots)$ then $\theta \in U_{S_0,S_1,S_2,\ldots}$. It follows that given $S_0, S_1, \ldots \in \mathcal{P}$, there exists an argument which has either left or right symbol sequence $(S_0, S_1, S_2, \ldots)$.

**1.4 Lemma.** *For each $n \geq 0$ the family $\{U_{S_0,\ldots,S_n}\}$ is unlinked.*

**Proof.** This follows by construction and Lemma 1.1. #

**1.5 Lemma.** *There are only a finite number of arguments which admit a given symbol sequence.*

**Proof.** Consider the full orbit of both families $\Lambda = \mathcal{O}(\mathcal{F}^\cup) \cup \mathcal{O}(\mathcal{J}^\cup)$. It is enough to prove that the number of connected components of $U_{S_0,S_1,\ldots,S_n} - \Lambda$ is bounded by a number which depends only on $(\mathcal{F}, \mathcal{J})$. We claim that the cardinality $N = \#(\Lambda)$ of $\Lambda$ is the bound we are looking for. We prove this by induction. For $n = 0$ this is clear. Now suppose $U_{S_1,S_2,\ldots,S_n} - \Lambda = \cup_{i=1}^{k} I_1$, where each $I_\alpha$ is connected and $k \leq N$. By construction every set $S_0 \cap m_d^{-1}(I_\alpha)$ is completely contained in a component of $\mathbf{R}/\mathbf{Z} - \Lambda$ and therefore is connected. The result follows. #

**1.6 Lemma.** *Suppose $\theta, \theta'$ have the same periodic left (or right) symbol sequence. Then $\theta, \theta'$ are periodic and have the same period.*

**Proof.** First note that $\theta$ can not be strictly preperiodic. For otherwise, eventually it becomes periodic, and such periodic argument would have at least two different inverses with the same symbol sequence, in contradiction with Corollary 1.2. If $\theta, \theta'$ are periodic of different period, we assume without loss of generality that $\theta$ is fixed, but $\theta'$ is not. In this case, we have at least three points with the same symbol sequence, for which the cyclic order is not preserved under iteration, but this is a contradiction to Lemma 1.1. Finally, $\theta$ can not be irrational because of Lemma 1.5. #

**1.7 Remark.** We conclude this section with a trivial remark that will be used later several times. If we take $\theta, \theta' \in \mathcal{J}_k$ and $\lambda$ such that $A^-(\lambda) = A^-(\theta)$, then by definition $\lambda \in (\theta', \theta]$. Analogously, if $\theta, \theta' \in \mathcal{F}_k$ and $\lambda$ is such that $A^+(\lambda) = A^+(\theta)$, then $\lambda \in [\theta, \theta')$. (There is nothing special about $\mathcal{J}$ or $\mathcal{F}$ in this formulation; but this is the way in which these statements will be used.)

## 2. The induced partitions in the dynamical plane.

In this section we introduce the induced partition of the Julia set with respect to the given critical marking. The main result is that this partition is Markov. As a consequence of this, we establish that the critical marking of a postcritically finite polynomial is in fact an admissible critical portrait, establishing in this way Proposition I.3.8.

Let $(P, \mathcal{F}, \mathcal{J})$ be critically marked. In analogy with the way we constructed a partition $\mathcal{P}$ of the unit circle where only the arguments in $\mathcal{F}^{\cup} \cup \mathcal{J}^{\cup}$ were omited, we will construct a partition of the dynamical plane off the rays with argument in $\mathcal{J}^{\cup}$ and extended rays with argument in $\mathcal{F}^{\cup}$. To simplify this construction we introduce some notation. For a set $\Lambda \subset \mathbf{R}/\mathbf{Z}$ we denote by $\mathcal{R}(\Lambda)$ the set of all external rays with argument in $\Lambda$ and their landing points. Also, whenever $\Lambda \subset \mathbf{R}/\mathbf{Z}$ is a set of arguments each of them supporting a Fatou component, we denote by $\mathcal{E}(\Lambda)$ the set of all extended rays with argument in $\Lambda$ and the respective centers of Fatou components.

**Definition.** We say that two points $z_1, z_2$ in $\mathbf{C} - \mathcal{R}(\mathcal{J}^{\cup}) - \mathcal{E}(\mathcal{F}^{\cup})$ are *"unlink equivalent"*, if they belong to the same connected component of $\mathbf{C} - \mathcal{R}(\mathcal{J}_i)$ and of $\mathbf{C} - \mathcal{E}(\mathcal{F}_l)$ for all possible choices of $\mathcal{J}_i$ and $\mathcal{F}_l$ in the marking.

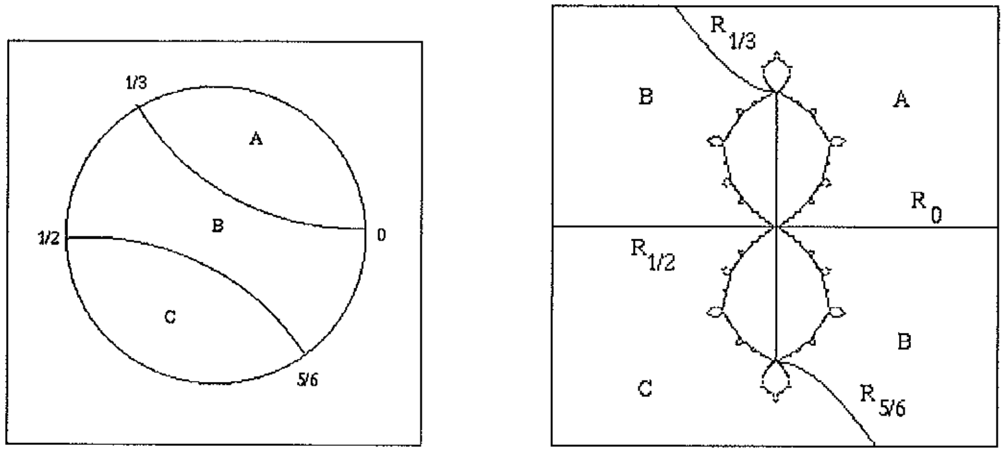



*Figure 2.1 The critically marked polynomial $P(z) = z^3 + 1.5z$ with critical portrait ($\mathcal{F} = \{\{0, 1/3\}, \{1/2, 5/6\}\}, \mathcal{J} = \emptyset$) determines a partition of the dynamical plane. However the elements of this partition are not necessarily connected open sets. Note that 0 and 1/2 share the same left symbol sequence in the circle, while the rays $R_0$ and $R_{1/2}$ land at the same point in the dynamical plane.*

Looking at the circle at infinity we immediately derive some properties. First, it is easy to see that there are exactly $d$ (=deg $P$) equivalence classes. Next, we have that either an external ray is completely contained in an equivalence class, or is disjoint from it. Furthermore, we have that two rays $R_\theta$ and $R_{\theta'}$ belong to the same equivalence class if and only if their arguments $\theta$ and $\theta'$ belong to the same element $S \in \mathcal{P}$. Thus, these equivalence classes are in canonical correspondence with the elements of the partition $\mathcal{P}$. For $S \in \mathcal{P}$ we denote by $\mathcal{U}_S$ the corresponding equivalence class in the dynamical plane. Each equivalence class is by definition a finite union of unbounded open sets. Note that if two arguments belong to the same connected component of some $S \in \mathcal{P}$, then the respective rays will be contained within the same connected open region in the dynamical plane.

**2.1 Lemma.** *Each region $\mathcal{U}_S$ is mapped bijectively by $P$ into the complement of a finite number of rays and extended rays.* #

**2.2 Lemma.** *The closure $cl(\mathcal{U}_S)$ and its restriction to the Julia set $J_S = J(P) \cap cl(\mathcal{U}_S)$ are connected.* #

Both proofs are somehow trivial and are left to the reader (compare also the proofs of Lemma 2.3 and Corollary 2.4).

We can go a step beyond, and take the regions determined by the $n$-fold inverse images of those rays and extended rays. Or alternatively we can dynamically define sets $\mathcal{U}_{S_0,\ldots,S_n}$ in analogy with §1.3. The analogy between this and the definition given in §1.3, is clear: by definition, $R_\theta \subset \mathcal{U}_{S_0,\ldots,S_n}$ if and only if $\theta \in U_{S_0,\ldots,S_n}$. Even if the sets $U_{S_0,\ldots,S_n}$ are usually disconnected we have that their closures are not.

**2.3 Lemma.** *Let $\gamma : [0,1] \to \mathbf{C}$ be an arc which crosses neither a ray with argument in $\mathcal{O}(m_d(\mathcal{J}^\cup))$ nor an extended ray with argument in $\mathcal{O}(m_d(\mathcal{F}^\cup))$. Suppose further that the image of $\gamma$ is disjoint from the forward orbit of all Fatou critical points. If $\gamma$ contains an interior point disjoint from these rays and extended rays, then $\gamma$ can be lifted in a unique way within any $cl(\mathcal{U}_S)$, for all $S \in \mathcal{P}$.*

**Proof.** Pick an $S \in \mathcal{P}$ and start the lifting of $\gamma$ at an image point not in the above rays or extended rays. Note that the hypothesis guarantees that the lifting can be chosen in such way that it never gets into any region $\mathcal{U}_{S'}$ other than $\mathcal{U}_S$. Uniqueness follows from Lemma 2.1. #

**2.4 Corollary.** *The closure* $cl(\mathcal{U}_{S_0,\ldots,S_n})$ *and its restriction to the Julia set* $J_{S_0,\ldots,S_n} = J(P) \cap cl(\mathcal{U}_{S_0,\ldots,S_n})$ *are connected.*

**Proof.** Note that if we cut open the plane along all extended rays with argument in $\mathcal{O}(m_d(\mathcal{F}^{\cup}))$ and remove the forward orbit of all Fatou critical points, we are left with a connected set. In fact, given a Fatou component $U$, there is at most one argument in $\mathcal{O}(m_d(\mathcal{F}^{\cup}))$ which supports $U$. This follows by construction of critical marking using the hierarchic selection. (This is the only place where the hierarchic selection is essentially used in this work!) Therefore we can join any two points in the Julia set with a path satisfying the hypothesis of Lemma 2.3. The result now follows by induction on $n$. #

**Remark.** That $J_{S_0,\ldots,S_n}$ is connected depends upon the fact that the definition of critical marking follows a hierarchic selection. Without hierarchic selection for extended supporting rays, the statement above is definitely not true.

At the end, we are mostly interested in the effect of this partition in the Julia set. We set $J_{S_0,S_1,\ldots} = \bigcap_{n=0}^{\infty} J_{S_0,\ldots,S_n}$ Note that because $J(P)$ is locally connected, it follows easily that the external ray $R_\theta$ lands somewhere in the set $J_{S^+(\theta)} \cap J_{S^-(\theta)}$. Therefore we should ask if $J_{S(\theta)}$ consists of exactly one point.

**2.5 Lemma.** *For any sequence* $(S_0, S_1, \ldots)$ *the set* $J_{S_0,S_1,\ldots}$ *contains exactly one point.*

**Proof** (Compare with [GM, Lemma 4.2]) We will make use of the Thurston *orbifold metric* associated with $P$. Let $M_P$ be the surface with boundary, equal to the disjoint union of all $\tilde{\mathcal{U}}_S$ defined as $cl(\mathcal{U}_S)$ cut open along all marked rays, extended rays and their forward images, and with the orbit of the Fatou critical points removed. Define the distance $\rho(z, z')$ between two points of $M_P$ to be the infimum of the lengths with respect to the orbifold metric of smooth paths joining $z$ to $z'$ within $M_P$ (or $\infty$ if they belong to different components). If $z$ and $z'$ belong to the same subset $J_{S_0,S_1} \subset J(P)$, then any path from $P(z)$ to $P(z')$ within $\tilde{\mathcal{U}}_{S_1}$ can be lifted back uniquely to a path from $z$ to $z'$ within $\tilde{\mathcal{U}}_{S_0}$ (compare Lemma 2.3). Since the orbifold metric is locally strictly expanding, a compactness argument shows that

$$\rho(P(z), P(z')) \geq c\rho(z, z')$$

for some constant $c > 1$, independent of $S_i$ for this $P$. Therefore, the inverse map

$$P_{S_0}^{-1} : J_S \mapsto J_{S_0,S}$$

contracts lengths by at least $1/c$. Hence the iterated inverse images $P_{S_0}^{-1} \circ \ldots \circ P_{S_n}^{-1}(J_{S_{n+1}})$ have diameter less than some constant divided by $1/c^n$. Taking the limit as $n \to \infty$, we obtain the required unique point. #

**2.6 Corollary.** *For any sequence* $(S_0, S_1, \ldots)$ *we have* $P(J_{S_0,S_1,\ldots}) = J_{S_1,S_2,\ldots}$.

**Proof.** For some $\theta$, either its left or right symbol sequence $S(\theta)$ equals $(S_0, S_1, \ldots)$. As the ray $R_\theta$ lands at the unique point contained in $J_{S_0,S_1,\ldots}$, the result follows. #

**2.7 Corollary.** *If $(S_0, S_1, \ldots)$ is a periodic sequence of period $m$, then the unique point in $J_{S_0,S_1,\ldots}$ is periodic of period dividing $m$.*

**Proof.** This follows from Lemma 2.5 and Corollary 2.6. In fact, the period is $m$ but this is not a priori obvious, this will follow from Proposition 3.6. #

**2.8 A formal critical portrait not coming from a polynomial.** Consider the degree 4 formal critical portrait

$$\mathcal{J} = \{\{\frac{3}{60}, \frac{18}{60}\}, \{\frac{19}{60}, \frac{34}{60}\}, \{\frac{1}{60}, \frac{46}{60}\}\},$$

which does not came from the marking of a polynomial. (Compare condition (c.7) in §I.3.7 and Corollary 2.9, here $S^-(19/60) = S^-(46/60)$).

If there is a polynomial $P$ of degree 4 which realizes this critical portrait, there should be critical points $\omega_1 \neq \omega_2$ associated with $\{\frac{19}{60}, \frac{34}{60}\}$ and $\{\frac{1}{60}, \frac{46}{60}\}$ respectively. But as $S^-(19/60) = S^-(46/60)$, then Lemma 2.5 tell us $\omega_1 = \omega_2$. Thus, the critical points associated with $\{\frac{19}{60}, \frac{34}{60}\}, \{\frac{1}{60}, \frac{46}{60}\}$ must be actually the same. Therefore we do not have three degree 2 critical points, but one of degree 3 and the other of degree 2. In this case, the rays $R_{4/60}$, and $R_{16/60}$ land at the same fixed point. This fixed point has exactly one other preimage, the degree 3 critical point. At this critical point the rays

$R_{19/60}$, $R_{34/60}$, $R_{49/60}$, $R_{1/60}$, $R_{31/60}$, and $R_{46/60}$ land. Therefore, the actual polynomial must have as critical marking either of the following,

$$\mathcal{J} = \{\{\frac{3}{60}, \frac{18}{60}\}, \{\frac{19}{60}, \frac{34}{60}, \frac{49}{60}\}\},$$

or

$$\mathcal{J} = \{\{\frac{3}{60}, \frac{18}{60}\}, \{\frac{1}{60}, \frac{31}{60}, \frac{46}{60}\}\}.$$

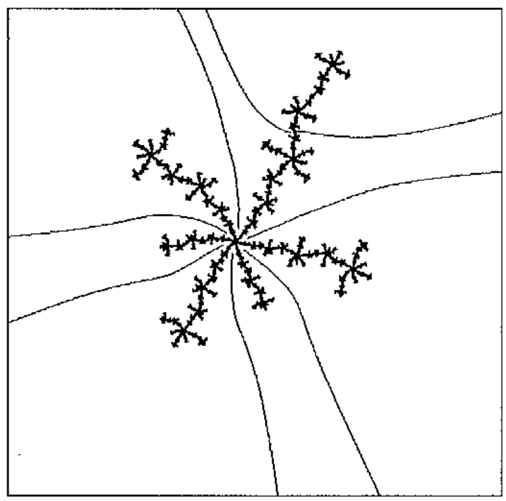


*Figure 2.2. Julia set of $P(z) = z^4 + Az^2 + Bz + C$ with the rays $\frac{1}{60}$, $\frac{3}{60}$, $\frac{18}{60}$, $\frac{19}{60}$, $\frac{31}{60}$, $\frac{34}{60}$, $\frac{46}{60}$, $\frac{49}{60}$ shown. Here*

$$A \approx 0.38437710 - 0.56951210i$$

$$B \approx 0.30830201 + 0.03253718i$$

$$C \approx 0.49119643 + 0.93292127i$$

**2.9 Corollary.** *Let $(P, \mathcal{F}, \mathcal{J})$ be a critically marked polynomial. Suppose $\theta \in \mathcal{J}_k$ and $\theta' \in \mathcal{J}_l$. If $S^-(m_d^{\circ i}(\theta)) = S^-(\theta')$ for some $i \geq 0$, then $m_d^{\circ i}(\theta) \in \mathcal{J}_l$.*

**Proof.** It follows from Lemma 2.5 that the rays with argument $m_d^{\circ i}(\theta)$ and $\theta'$ land at the same critical point. The result then follows from the hierarchic selection of rays. (Compare §I.2.) #

**2.10 Corollary.** *Let $\gamma \in \mathcal{F}^{\cup}_{\mathbf{per}}$, and $\lambda \in \mathbf{R}/\mathbf{Z}$, then $\lambda = \gamma$ if and only if $S^+(\gamma) = S^+(\lambda)$.*

**Proof.** Suppose $\mathcal{F}_k = \{\gamma = \gamma_1, \ldots, \gamma_n\}$, where the arguments $\gamma_1, \ldots, \gamma_n$ are in counterclockwise cyclic order. Suppose $\lambda \neq \gamma$ but $S^+(\gamma) = S^+(\lambda)$. By Lemma 2.5 the rays $R_\gamma, R_\lambda$ land at the same point. As $\lambda$ is periodic by Lemma 1.6, it follows that $\lambda \notin \mathcal{F}_k$. But then, by definition of the right address $A^+(\lambda)$ of $\lambda$, it follows that the cyclic order is $\gamma_1, \lambda, \gamma_2, \ldots, \gamma_n$ (compare Remark 1.7). By definition of supporting argument (see §I.1.3), the corresponding Fatou component must be in the sector determined by $R_{\gamma_1}, R_\lambda$ (in the counterclockwise sense). But this is a contradiction with the fact that $R_{\gamma_2}, \ldots, R_{\gamma_n}$ also support this component. #

The following now follows from Corollaries 2.9 and 2.10.

**2.11 Proposition.** *If $(P, \mathcal{F}, \mathcal{J})$ is a marked polynomial, then the pair $(\mathcal{F}, \mathcal{J})$ is an admissible critical portrait.* #

## 3. Which rays land at the same point?

We would like to have a combinatorial criterion to decide when two rays land at the same point. Two arguments $\theta, \theta'$ in the same $\mathcal{J}_k$ do not have equal (left or right) symbol sequences. Nevertheless, the external rays $R_\theta, R_{\theta'}$ both land at the same critical point. In general, all exceptions are a consequence of this fact. Furthermore, all the information we need is already contained in left symbol sequences.

**3.1 The landing equivalence ($\sim_l$).** We recall briefly the definition of the "landing equivalence" $\sim_l$ between angles, introduced in Chapter I (compare §I.3.11). Let $(\mathcal{F}, \mathcal{J})$ be an admissible critical portrait. For $\theta_\alpha, \theta_\beta \in \mathcal{J}_i \in \mathcal{J}$ we set $S^-(\theta_\alpha) \equiv_i S^-(\theta_\beta)$. Then we write $\theta \approx \theta'$ if either $S^-(\theta) = S^-(\theta')$ or there is an $n \geq 0$ such that $\pi_j(S^-(\theta)) = \pi_j(S^-(\theta'))$ for all $j < n$ and $\sigma^n(S^-(\theta)) \equiv_i \sigma^n(S^-(\theta'))$ for some $i$. Finally we make this into an equivalence relation by letting $\theta \sim_l \theta'$ if and only if there are arguments $\lambda_0 = \theta, \lambda_1, \ldots, \lambda_m = \theta'$, such that $\lambda_0 \approx \ldots \approx \lambda_m$. Note that condition $(c.7)$ together with $(c.3)$ guarantee that whenever $\theta_i \in \mathcal{F}_i$ $(i = 0, 1)$; then $\theta_0 \sim_l \theta_1$ if and only if $\mathcal{F}_1 = \mathcal{F}_2$.

If the family $\mathcal{J}$ is empty, two arguments are equivalent if and only if their left symbol sequences coincide. As $S^-(\theta)$ is strictly preperiodic for every argument $\theta$ in the family union $\mathcal{J}^\cup$, two periodic or irrational arguments $\theta, \theta'$ are $\sim_l$ equivalent if and only if $S^-(\theta) = S^-(\theta')$. Of course, a preperiodic argument would never be equivalent to a non preperiodic one.

By definition, if $\theta \sim_l \theta'$ there is an $m \geq 0$ such that $\sigma^m(S^-(\theta)) = \sigma^m(S^-(\theta'))$. Also note that whenever $\theta \approx \theta'$ then also $m_d(\theta) \approx m_d(\theta')$. Therefore the following lemma is trivial.

**3.2 Lemma.** *If $\theta \sim_l \theta'$ then $m_d(\theta) \sim_l m_d(\theta')$.* #

Now let $(P, \mathcal{F}, \mathcal{J})$ be a critically marked polynomial. We will show now that the $\sim_l$ equivalence classes defined from the associated admissible critical portrait effectively characterize the arguments of rays landing at a common point.

**3.3 Lemma.** *Suppose $R_\theta, R_{\theta'}$ both land at the same point $z$. If $z$ is non critical then $A^-(\theta) = A^-(\theta')$.*

**Proof.** If $z$ is not the landing point of a ray with argument in $\mathcal{F}^\cup$, then it is in the interior of some region $\mathcal{U}_S$. Otherwise, let $R_{\theta_1}, \ldots, R_{\theta_k}$ be all rays with argument in $\mathcal{F}^\cup$ landing at $z$. Around $z$ we consider locally segments of these rays together with internal rays joining this point $z$ to the center of the $k$ associated Fatou components. This configuration divides a neighborhood of $z$ into $2k$ consecutive regions. As every other region is contained in $\mathcal{U}_S$ where $S = A^-(\theta_1) = \ldots = A^-(\theta_k)$, the result follows. #

**3.4 Lemma.** *Suppose $R_\theta$ lands at a critical point $\omega$, then $A^-(\theta) = A^-(\theta')$ for some $\theta' \in \mathcal{J}_\omega$.*

**Proof.** The external ray $R_\theta$ is contained within some $\mathcal{U}_{A^-(\theta')}$. #

**3.5 Corollary.** *Suppose $\theta, \theta'$ are such that $A^-(\theta) = A^-(\theta')$. Then $R_\theta, R_{\theta'}$ land at the same point if and only if $R_{m_d(\theta)}, R_{m_d(\theta')}$ land at the same point.* #

**3.6 Proposition.** *Let $(P, \mathcal{F}, \mathcal{J})$ be a marked polynomial. Then $R_\theta$ and $R_{\theta'}$ land at the same point if and only if $\theta \sim_l \theta'$.*

**Proof.** First suppose that $\theta \sim_l \theta'$. If $S^-(\theta) = S^-(\theta')$ then the rays $R_\theta, R_{\theta'}$ land at the same point by Lemma 2.5. Otherwise, it is enough to assume $\theta \approx \theta'$. In this way, for some $n \geq 0$, $\sigma^n(S^-(\theta)) \equiv_i \sigma^n(S^-(\theta'))$ and $\pi_j(S^-(\theta)) = \pi_j(S^-(\theta'))$ for $j < n$. By definition there are arguments in $\mathcal{J}_i$ with symbol sequences $\sigma^n(S^-(\theta))$ and $\sigma^n(S^-(\theta'))$. As the rays with these arguments land at the same critical point $\omega_i$, the rays $R_{m_d^{\circ n}(\theta)}$ and $R_{m_d^{\circ n}(\theta')}$ also land at $\omega_i$. The result follows now from Corollary 3.5.

Conversely, suppose $R_\theta$ and $R_{\theta'}$ land at the same point $z$. There is a minimal $m \geq 0$ such that $P^{\circ m}(z)$ neither is critical nor contains a critical point in its forward orbit. We will prove by induction in $m$ that $\theta \sim_l \theta'$. Let $P^{\circ n}(z)$ be non critical for all $n \geq 0$ (this is the case $m = 0$). For all $n \geq 0$, $R_{d^{\circ n}(\theta)}, R_{d^{\circ n}(\theta')}$ will be rays landing at the same non critical point. In this case the result follows from Lemma 3.3. Now, let $m_d(\theta) \sim_l m_d(\theta')$ (this is the inductive hypothesis). If $z$ is not a critical point we use again Lemma 3.3; if $z$ is a critical point we use Lemma 3.4. In either case we deduce that $\theta \sim_l \theta'$. #

**3.7 Corollary.** *If $(S_0, S_1, \ldots)$ is a periodic sequence of period $m$, then the unique point in $J_{S_0,S_1,\ldots}$ has period $m$.* #

## 4. Which rays support the same Fatou component?

In general it is impossible to give a combinatorial description of when two arguments support the same Fatou component. This because the closure of two Fatou components may share a periodic point which is not the landing point of a marked ray. In this case, the arguments of all rays landing at such point will have the same left and right symbol sequences, and thus they are undistinguishable from the combinatoric point of view. However, for some cases we will study which rays support some given periodic Fatou component. We will only consider rays for which some forward image belongs to the periodic part of the family union $\mathcal{F}^{\cup}_{\mathbf{per}}$. The importance of the combinatorial construction below will become clear in the next chapter. In the meanwhile we can tell the reader that in order to apply the theory of "Levy Cycles" (compare Appendix A), we should artificially introduce some preperiodic arguments for every periodic critical point. These preperiodic arguments are what we call in this section "special arguments".

As motivation for the combinatorial construction to follow, we consider a critically marked polynomial $(P, \mathcal{F}, \mathcal{J})$. Let $\gamma \in \mathcal{O}(\mathcal{F}^{\cup}_{\mathbf{per}})$, be of period $k$. Suppose also that $m_d^{\circ nk}(\lambda) = \gamma$.

**4.1 Lemma.** *With the above hypothesis, $R_\lambda$ supports the same Fatou component as $R_\gamma$ if and only if, for each $i \geq 0$ either*

*i)* $\pi_i(S^+\gamma) = \pi_i(S^+\lambda)$, *or*

*ii)* $m_d^{\circ i}(\gamma)$ *belongs to some* $\mathcal{F}_\alpha$ *and* $A^+(m_d^{\circ i}(\lambda)) = A^+(\gamma')$ *for some* $\gamma' \in \mathcal{F}_\alpha$.

**Proof.** The proof is straightforward and is left to the reader. #

This motivates the following definition.

**4.2 Special arguments.** Let $(\mathcal{F}, \mathcal{J})$ be an admissible critical portrait. To every $\gamma \in \mathcal{O}(\mathcal{F}^{\cup}_{\mathbf{per}})$ we associate a (periodic) sequence of sets $\mathcal{T}(\gamma, i)$ as follows. First we define $\mathcal{T}(\gamma, 0)$:

$$\mathcal{T}(\gamma, 0) = \begin{cases} \{A^+(\gamma') : \gamma' \in \mathcal{F}_\alpha\} & \text{if } \gamma \in \mathcal{F}_\alpha \text{ for some } \alpha; \\ \{A^+(\gamma)\} & \text{otherwise.} \end{cases}$$

In the general case set $\mathcal{T}(\gamma, j) = \mathcal{T}(m_d^{\circ j}(\gamma), 0)$.

**Definition.** Let $\gamma \in \mathcal{O}(\mathcal{F}^{\cup}_{\mathbf{per}})$ be of period $k = k(\gamma)$. We say that $\lambda$ is a *special argument* for $\gamma$, if there is an $n \geq 0$ such that $\pi_i(S^+(\lambda)) \in \mathcal{T}(\gamma, i)$ for all $i < nk$ and $m_d^{\circ nk}(\lambda) = \gamma$. In case both $\theta$ and $\theta'$ are special arguments for $\gamma \in \mathcal{O}(\mathcal{F}^{\cup}_{\mathbf{per}})$ we write $\theta \sim_\gamma \theta'$.

The following establishes an equivalence relation between *'special arguments'*.

**4.3 Lemma.** *If $\lambda$ is a special argument for both $\gamma, \gamma'$, then $\gamma = \gamma'$.*

**Proof.** Let $n$ be a multiple of $k(\gamma)k(\gamma')$ big enough, then $S^+(\gamma) = \sigma^n(S^+(\lambda)) = S^+(\gamma')$ and the result follows from condition (c.6) in the definition of admissible critical portrait and Corollary 1.2. #

**4.4 Remark.** If $\theta \sim_\gamma \theta'$ and $S^+(\theta) = S^+(\theta')$, it follows from the definition of $\sim_\gamma$, condition (c.6) and Corollary 1.2 that $\theta = \theta'$.

These relations between special arguments are compatible with $m_d$ in the following sense.

**4.5 Lemma.** *If* $\lambda_1 \sim_\gamma \lambda_2$ *then* $m_d(\lambda_1) \sim_{m_d(\gamma)} m_d(\lambda_2)$.

**Proof.** For some high iterate $\gamma = m_d^{\circ k}(\lambda_1) = m_d^{\circ k}(\lambda_2)$. Thus $m_d(\gamma) = m_d^{\circ k}(m_d(\lambda_1)) = m_d^{\circ k}(m_d(\lambda_2))$ and the result follows from the definition of $\sim_{m_d(\gamma)}$. #

The following proposition, is a technical result needed in the proof of the main theorem (Theorem I.3.9). Its meaning when translated to the context of $PCF$ polynomials, is that inverse images of a (marked) periodic ray supporting that same Fatou component, can be found very close to the starting periodic ray (this is obvious in the context of dynamics, because we are in the subhyperbolic case).

**4.6 Proposition.** *Let* $(\mathcal{F}, \mathcal{J})$ *be an admissible critical portrait. If* $\gamma \in \mathcal{F}^{\cup}_{\mathbf{per}}$ *then there exist arbitrary small* $\epsilon > 0$ *such that* $\gamma + \epsilon \sim_\gamma \gamma$.

**Proof.** Let $\mathbf{S}_\gamma = (A^+(\gamma), \ldots, A^+(m_d^{\circ k-1}(\gamma)))$ and take any $W \in \mathcal{T}_\gamma^0 \times \ldots \times \mathcal{T}_\gamma^{k-1}$ different from $\mathbf{S}_\gamma$. We form a sequence $\gamma_n \sim_\gamma \gamma$, where $S^+(\gamma_n) = \mathbf{S}_\gamma^{\mathbf{n}} W \bar{\mathbf{S}}_\gamma$. Take a convergent subsequence to $\lambda$. As $S^+(\lambda) = \bar{\mathbf{S}}_\gamma = S^+(\gamma)$ it follows by condition (c.6) that $\lambda = \gamma$. Now, for $\epsilon > 0$ small enough, $\gamma_n$ can not be of the form $\gamma - \epsilon$ by Remark 1.7, therefore it must be of the form $\gamma + \epsilon$. #

In the language of special arguments Lemma 4.1 reads.

**4.7 Proposition.** *Let* $(P, \mathcal{F}, \mathcal{J})$ *be a marked polynomial. If* $\theta$ *is a special argument for* $\gamma \in \mathcal{F}^{\cup}_{\mathbf{per}}$ *then* $R_\theta$ *and* $R_\gamma$ *support the same Fatou component.* #

# Chapter III
# Realizing Critical Portraits

In this Chapter we give the proof of the Realization Theorem for Critical Portraits. In Section 1 we prove that the combinatorial data is 'compatible' in the sense that it allows us to construct a Topological Polynomial. The actual construction is carried out in Section 2, where we also indicate (following [BFH]) that it is essentially unique. In Section 3 we prove that every admissible critical portrait has associated a unique (up to affine conjugation) polynomial which is Thurston equivalent to the topological polynomial so far constructed. In Section 4 we show that the isotopies between the 'actual' and 'topological' polynomials can be chosen fixed not only relative to certain 'marked' points, but also relative to the whole boundary when suitably chosen neighborhoods of Fatou points are deleted. In Section 5 we complete the proof of the Theorem by assigning the expected critical marking to the associated polynomial.

## 1. Combinatorial Information of Admissible Critical Portraits.

In this Section we analyze the linkage relations that arise when we consider the full orbit of the families and special arguments together. The main result is summarized in Proposition 1.2 and is used in Section 2. This fact is easy to believe but its proof is extremely technical.

**1.1** Consider an admissible critical portrait $(\mathcal{F}, \mathcal{J})$. The orbit set $\mathcal{O}(\mathcal{F}^{\cup})$ can be partitioned in a natural way as $\mathcal{F} \cup \{\{\gamma\} : \gamma \in \mathcal{O}(\mathcal{F}^{\cup}) - \mathcal{F}^{\cup}\}$. In the context of dynamics, two elements in the orbit $\mathcal{O}(\mathcal{F}^{\cup})$ belong to the same element of this partition if and only if they support the same Fatou component (compare Proposition II.4.7). If in addition we consider a finite invariant set of special arguments $\Gamma$ (i.e, satisfying $m_d(\Gamma) \subset \Gamma \cup \mathcal{F}^{\cup}$), we can include an element $\lambda \in \Gamma$ in that same class as $\gamma$, whenever $\lambda \sim_\gamma \gamma$. In this way, we construct a family $\mathcal{F}^* = \{\mathcal{F}_1^*, \ldots, \mathcal{F}_n^*\}$ which is a partition of $\mathcal{O}(\mathcal{F}^{\cup}) \cup \Gamma$.

Next, we partition the set $\mathcal{O}(\mathcal{F}^{\cup}) \cup \mathcal{O}(\mathcal{J}^{\cup}) \cup \Gamma \cup \{0\}$ into $\sim_l$ equivalence classes to form the family $\mathcal{J}^* = \{\mathcal{J}_1^*, \ldots, \mathcal{J}_m^*\}$. In the *PCF* context we are grouping all those rays we expect to land at the same point (compare Proposition II.3.6). Here we are adding the argument $\theta = 0$ to simplify things later. This will reflect the choice of $R_0$ as a preferred fixed 'internal' ray in the basin of attraction of $\infty$. (Compare Example 3.7.)

In the way the pair $(\mathcal{F}^*, \mathcal{J}^*)$ was constructed, it is clear that if we think in terms of external rays, the proposition below must be true.

**1.2 Proposition.** *Let $(\mathcal{F}, \mathcal{J})$ be an admissible critical portrait and $\Gamma$ a finite invariant set of special arguments. With the notation above, $\mathcal{J}^*$ is weakly unlinked to $\mathcal{F}^*$ in the right.*

The reader can skip the rest of this section without any loss of continuity. The proof of the proposition follows immediately from Lemmas 1.3-1.9.

**1.3 Lemma.** *Suppose* $\theta_1 \approx \theta_2$, $\psi_1 \approx \psi_2$ *but* $\theta_1 \not\sim_l \psi_1$. *Then* $\{\theta_1, \theta_2\}$ *and* $\{\psi_1, \psi_2\}$ *are unlinked.*

**Proof.** Suppose this is not the case. We assume then that $\{\theta_1, \theta_2\}$ and $\{\psi_1, \psi_2\}$ are linked because $\theta_2 = \psi_2$ implies $\theta_1 \sim_l \psi_1$. As a preliminary remark suppose $A^-(\theta_1) = A^-(\theta_2) = A^-(\psi_1) = A^-(\psi_2)$; then as the cyclic order of these elements is preserved by $m_d$ (compare Lemma II.1.1), $\{m_d(\theta_1), m_d(\theta_2)\}$ and $\{m_d(\psi_1), m_d(\psi_2)\}$ are still linked. For the proof we distinguish several cases.

*Case 1:* $S^-(\theta_1) = S^-(\theta_2)$ and $S^-(\psi_1) = S^-(\psi_2)$. This possibility is easily ruled out using Lemma II.1.4. We can say even more. If $A^-(\theta_1) = A^-(\theta_2)$ and $A^-(\psi_1) = A^-(\psi_2)$ then by that same lemma we have also $A^-(\theta_1) = A^-(\psi_1)$. Thus, according to our preliminary remark, it is enough to consider the case when $A^-(\theta_1) \neq A^-(\theta_2)$.

*Case 2:* $\theta_1, \theta_2 \in \mathcal{J}_k$. As $\psi_1$ and $\psi_2$ belong to different components of $\mathbf{R}/\mathbf{Z} - \mathcal{J}_k$, by definition $A^-(\psi_1) \neq A^-(\psi_2)$. Thus, also by definition $S^-(\psi_1) \equiv_i S^-(\psi_2)$ for some $i$. But then, again by definition, there are $\psi_j' \in \mathcal{J}_i$ $(j = 1, 2)$, each in the same connected component of $\mathbf{R}/\mathbf{Z} - \{\theta_1, \theta_2\}$ as $\psi_j$, with $S^-(\psi_j') = S^-(\psi_j)$. But this is a contradiction with the fact that $\mathcal{J}_k, \mathcal{J}_i$ are unlinked.

*Case 3:* $S^-(\theta_1) \equiv_k S^-(\theta_2)$ and $S^-(\psi_1) = S^-(\psi_2)$. By definition, there is $\theta_1' \in \mathcal{J}_k$ such that $S^-(\theta_1') = S^-(\theta_1)$. Now, if $\theta_1$ and $\theta_1'$ belong to different components of $\mathbf{R}/\mathbf{Z} - \{\psi_1, \psi_2\}$ then $\{\theta_1, \theta_1'\}$, and $\{\psi_1, \psi_2\}$ are linked and we are in case 1. Otherwise, we repeat the same reasoning using now $\theta_2$ and we reach either case 1 or case 2.

*Case 4:* $S^-(\theta_1) \equiv_k S^-(\theta_2)$ and $S^-(\psi_1) \equiv_j S^-(\psi_2)$. We proceed as in case 3 and this is reduced to either case 2 or case 3. #

**1.4 Corollary.** *The $\sim_l$ equivalence classes are unlinked.* #

**1.5 Lemma.** *For any $\mathcal{F}_k \in \mathcal{F}$ and any $\sim_l$ equivalence class $\Lambda$, $\{\Lambda\}$ is weakly unlinked to $\{\mathcal{F}_k\}$ in the right.*

**Proof.** Let $\theta_0 \in \Lambda$ and take $\gamma_1, \gamma_2$ consecutive in $\mathcal{F}_k$ so that $\theta_0 \in (\gamma_1, \gamma_2]$. It is enough to prove that if $\theta_0 \approx \theta_1$ then also $\theta_1 \in (\gamma_1, \gamma_2]$. If $A^-(\theta_0) = A^-(\theta_1)$, this follows by definition ($\theta_0$ and $\theta_1$ by definition belong to the same connected component of $\mathbf{R}/\mathbf{Z} - \mathcal{F}_k$). So suppose that $S^-(\theta_0) \equiv_i S^-(\theta_1)$ with $\theta_1 \notin (\gamma_1, \gamma_2]$. In this case there exist $\mathcal{J}_i \in \mathcal{J}$ so that $\theta_0' \in \mathcal{J}_i \cap (\gamma_1, \gamma_2]$ and $\theta_1' \in \mathcal{J}_i \cap (\gamma_2, \gamma_1]$ with $S^-(\theta_j) = S^-(\theta_j')$. But this is a contradiction with condition (c.2) in the definition of critical portraits ($\mathcal{J}_i$ will not be weakly unlinked to $\mathcal{F}_k$ in the right). #

**1.6 Lemma.** *Let $\psi_1 \sim_\gamma \psi_2$ and $\gamma \notin \mathcal{F}_k$, then $\{\psi_1, \psi_2\}$ and $\mathcal{F}_k$ are unlinked.*

**Proof.** If $A^+(\psi_1) = A^+(\psi_2)$ this follows by definition and Remark II.1.7. Otherwise we must have that $\gamma \in \mathcal{F}_i$ for some $i \neq k$. But then a similar argument as that used in Lemma 1.5 shows that $\mathcal{F}_i$ and $\mathcal{F}_k$ are not unlinked. #

**1.7 Lemma.** *Let $\theta_i \sim_{\gamma_i} \psi_i$, $i = 1, 2$ with $\gamma_1 \neq \gamma_2$. Then $\{\theta_1, \psi_1\}$ and $\{\theta_2, \psi_2\}$ are unlinked.*

**Proof.** We will consider right symbol sequences $S^+(\theta_j)$ and $S^+(\psi_j)$. Suppose is not the case that they are unlinked. Then $\{\theta_1, \psi_1\}$ and $\{\theta_2, \psi_2\}$ are linked because $\theta_2 = \psi_2$ will imply $\gamma_1 = \gamma_2$ by Lemma II.4.3. As preliminary remarks, suppose $A^+(\theta_1) = A^+(\psi_1) = A^+(\theta_2) = A^+(\psi_2)$. Then as the cyclic order of these elements is preserved by $m_d$ (compare Lemma II.1.1), $\{m_d(\theta_1), m_d(\theta_2)\}$ and $\{m_d(\psi_1), m_d(\psi_2)\}$ are linked. Furthermore, if $A^+(\theta_1) = A^+(\theta_2)$ and $A^+(\psi_1) = A^+(\psi_2)$, by Lemma II.1.4 we must have $A^+(\theta_1) = A^+(\psi_1)$.

Now, suppose $\theta_1$ is in the same connected component of $\mathbf{R}/\mathbf{Z} - \{\theta_2, \psi_2\}$ as $\gamma_1$ (if not $\psi_1$ will be). In this case $\{\theta_1' = \gamma_1, \psi_1\}$, and $\{\theta_2, \psi_2\}$ are linked, so we assume $\theta_1 = \gamma_1$. In an analogous way we may suppose that $\theta_2 = \gamma_2$. Under this assumption we will prove that for all $j \geq 0$, $\{m_d^{\circ j}(\theta_1), m_d^{\circ j}(\psi_1)\}$ and $\{m_d^{\circ j}(\theta_2), m_d^{\circ j}(\psi_2)\}$ should be linked. Of course this is absurd because by definition, for $j$ big enough we have $m_d^{\circ j}(\theta_1) = m_d^{\circ j}(\psi_1) = m_d^{\circ j}(\gamma_1)$.

Suppose that $A^+(\theta_1) \neq A^+(\psi_1)$. Then by definition $\theta_1 \in \mathcal{F}_k$ for some $k$. Furthermore, there is $\psi_1' \in \mathcal{F}_k$ with $A^+(\psi_1') = A^+(\psi_1)$. It follows from Lemma 1.6 that $\theta_1, \psi_1' \in \mathcal{F}_k$ are in the same component of $\mathbf{R}/\mathbf{Z} - \{\theta_2, \psi_2\}$. Thus, $\{\psi_1', \psi_1\}$ and $\{\theta_2, \psi_2\}$ are still linked. Note that $m_d(\psi') = m_d(\theta_1)$. Also by symmetry we may take $A^+(\theta_2) = A^+(\psi_2)$ (note that the property $m_d(\theta_2) = m_d(\gamma_2)$ will not be lost). But then by the second preliminary remark $A^+(\theta_1) = A^+(\psi_1) = A^+(\theta_2) = A^+(\psi_2)$, and so, by the first $\{m_d(\theta_1) = m_d(\gamma_1), m_d(\psi_1)\}$ and $\{m_d(\theta_2) = m_d(\gamma_2), m_d(\psi_2)\}$ are linked. This is the desired contradiction. #

**1.8 Corollary.** *The family* $\{\{\theta : \theta \sim_\gamma \gamma\} : \gamma \in \mathcal{O}(\mathcal{F}_{\mathbf{per}}^{\cup})\}$ *is unlinked.* #

**1.9 Lemma.** *Let $\gamma \in \mathcal{O}(\mathcal{F}^{\cup}_{\mathbf{per}})$ and $\Lambda$ an $\sim_l$ equivalence class. Then $\Lambda$ is weakly unlinked in the right to any finite subset of $\{\theta : \theta \sim_\gamma \gamma\}$.*

**Proof.** Take $\gamma \in \mathcal{F}_\gamma \in \mathcal{F}$. We will prove by induction that any $\sim_l$ equivalence class $\Lambda$, is weakly unlinked to $\Psi_n(\gamma') = \{\theta \sim_{\gamma'} \gamma' : m_d^{\circ n}(\theta) \in \mathcal{F}_\gamma\}$ (here $\gamma'$ belongs to the same cycle as $\gamma$, and $m_d^{\circ n}(\gamma') = \gamma$). The result follows easily. For $n = 0$, this is Lemma 1.5. In general take $\theta_1 \approx \theta_2$ and assume that $\{\theta_1, \theta_2\}$ is not weakly unlinked in the right to $\{\psi_1, \psi_2\} \subset \Psi_n(\gamma')$.

*Case 1:* $A^+(\psi_1) \neq A^+(\psi_2)$. Then by definition $\gamma' \in \mathcal{F}_k \in \mathcal{F}$ for some $k$. Thus, there are $\psi_i' \in \mathcal{F}_k$ such that $A^+(\psi_i') = A^+(\psi_i)$, and because of Lemma 1.5, it is easy to see that $\{\theta_1, \theta_2\}$ is not weakly unlinked in the right to either $\{\psi_1, \psi_1'\}$ or to $\{\psi_2, \psi_2'\}$ (both being subsets of $\Psi_n(\gamma')$). Thus it is enough to consider case 2.

*Case 2:* $A^+(\psi_1) = A^+(\psi_2)$. In this case we can not have simultaneously $\theta_1 = \psi_1$ and $\theta_2 = \psi_2$. In fact, in this case Lemma II.1.1 would imply that $\{m_d(\theta_1), m_d(\theta_2)\}$ is not weakly unlinked in the right to $\{m_d(\psi_1), m_d(\psi_2)\}$ in contradiction with the inductive hypothesis. Thus we may suppose that $\theta_1 \in (\psi_1, \psi_2)$ (and $\theta_2 \in (\psi_2, \psi_1]$). If $A^-(\theta_1) = A^-(\theta_2)$ it follows from Lemma II.1.4 that for $\epsilon > 0$ small enough $A^+(\theta_1 - \epsilon/d) = A^+(\theta_2 - \epsilon/d) = A^+(\psi_1) = A^+(\psi_2)$. By Lemma II.1.1 we have then that $\{m_d(\theta_1) - \epsilon, m_d(\theta_2) - \epsilon\}$ and $\{m_d(\psi_1), m_d(\psi_2)\}$ are not unlinked, in contradiction with the inductive hypothesis. Therefore $A^-(\theta_1) \neq A^-(\theta_2)$, and then by definition we must have $S^-(\theta_1) \equiv_i S^-(\theta_2)$. But then, using the same reasoning as in the previous lemmas, we can assume that $\theta_1, \theta_2 \in \mathcal{J}_i$. But if this is the case, we get a contradiction because it follows by definition and Remark II.1.7 that $A^+(\psi_1) \neq A^+(\psi_2)$. #

Proposition 1.2 follows now easily from the above lemmas. #

## 2. Abstract and embedded webs.

In this section we construct from the combinatorial data a topological polynomial of degree $d$. We also study some of its basic properties. None of the material presented here is essentially new, and can be found in a slightly different formulation in [BFH].

**2.1** Let $(\mathcal{F}, \mathcal{J})$ be an admissible critical portrait. For any finite invariant set of special arguments $\Gamma$, we consider the pair $(\mathcal{F}^*, \mathcal{J}^*)$ as in Section 1. With these families, we construct first an abstract topological graph $W(\mathcal{F}^*, \mathcal{J}^*)$ as follows. We pick a vertex $v = \infty$, and take as many edges $\mathcal{R}_\theta$ incident at $\infty$ as elements $\theta \in \mathcal{J}^{*\cup}$. Let $v_\theta$ be the other adjacent vertex to $\mathcal{R}_\theta$. We identify the vertices $v_\theta, v_{\theta'}$ if and only if $\theta, \theta' \in \mathcal{J}_k^*$ for some $k$; that is, if and only if $\theta \sim_l \theta'$. (This because we are expecting the rays with arguments $\sim_l$ related to land at the same point.) We write this vertex as $v(\mathcal{J}_k^*)$. As each $\mathcal{R}_\theta$ is labeled by an argument $\theta$, we call it *the web ray of argument* $\theta$. By abuse of language we will say that $v_\theta$ ($= v(\mathcal{J}_k^*)$ whenever $\theta \in \mathcal{J}_k^*$) is the *landing point of the web ray* $\mathcal{R}_\theta$.

Next, for each subset $\mathcal{F}_k^* \in \mathcal{F}^*$ we consider a new vertex $\omega(\mathcal{F}_k^*)$. We join this vertex to the landing points of $\mathcal{R}_\gamma$ for all $\gamma \in \mathcal{F}_k^*$. (This because, all those rays are supposed to support the same Fatou component; compare Proposition II.4.7). In this case the *extended web ray* $\mathcal{E}_\gamma$ is the set formed by the web ray of argument $\gamma$, its landing point, and the edge joining this landing point with the vertex $\omega(\mathcal{F}_k^*)$. In each set $\mathcal{F}_k^* \in \mathcal{F}^{*\cup}$ there is a preferred argument $\gamma_k$. We call the edge $\ell_{\mathcal{F}_k^*}$ joining $\omega(\mathcal{F}_k^*)$ with $v_\gamma$, *the preferred internal ray associated with the "Fatou type" point* $\omega(\mathcal{F}_k^*)$.

Note that by construction (compare §1.1), the argument 0 is always present in our construction. We say that the web ray $\mathcal{R}_0$ is *the preferred internal ray associated with* $v = \infty$. The graph $W(\mathcal{F}^*, \mathcal{J}^*)$ constructed in this way, is the *abstract web* associated with $(\mathcal{F}, \mathcal{J}, \Gamma)$. We will denote by $\mathbf{V}$ the set of vertices of this graph.

**2.2 Embedded webs.** We consider embeddings in the Riemann Sphere $\hat{\mathbf{C}}$ of this abstract web $W = W(\mathcal{F}^*, \mathcal{J}^*)$. An embedding such that the cyclic order of the web rays corresponds to the cyclic order of the labeling by arguments can always be constructed because of Proposition 1.2. We can always assume that the respective points at $\infty$ correspond. Any such embedding is an *embedded web.* We still call the image of edges incident at "$\infty$" *web rays.* Unless strictly necessary we will not distinguish between an embedding and its image.

**2.3 Web maps.** The following two properties follow immediately from the construction of $(\mathcal{F}^*, \mathcal{J}^*)$ and Lemmas II.3.2 and II.4.5.

*If* $\theta, \theta' \in \mathcal{J}_k^*$, *there is a unique* $\mathcal{J}_{f(k)}^*$, *such that* $m_d(\theta), m_d(\theta') \in \mathcal{J}_{f(k)}^*$.
*If* $\gamma, \gamma' \in \mathcal{F}_k^*$, *there is a unique* $\mathcal{F}_{f(k)}^*$, *such that* $m_d(\gamma), m_d(\gamma') \in \mathcal{F}_{f(k)}^*$.

These two conditions allow us to define a map $f$ between the set vertices of the web $W(\mathcal{F}^*, \mathcal{J}^*)$ (also define $f(\infty) = \infty$). We can extend this map to *ii) The diagram*

$$\begin{array}{ccc} \hat{\mathbf{C}} & \xrightarrow{\psi_\beta} & \hat{\mathbf{C}} \\ \hat{f}_1 \downarrow & & \downarrow \hat{f}_2 \\ \hat{\mathbf{C}} & \xrightarrow{\psi_\alpha} & \hat{\mathbf{C}} \end{array}$$

*is commutative.*

**Proof.** It is not difficult and can be found in [BFH, Theorem 6.8]. #

**2.5 Lifting Webs.** Suppose $\mathcal{W} = \phi(W(\mathcal{J}^*, \mathcal{F}^*))$ is an embedded web. Given this embedding, we fix a regular extension $\hat{f} : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ of the web map. If $\mathcal{W}'$ is another embedded web isotopic to $\mathcal{W}$ relative to the set $\phi(\mathbf{V})$, then $\hat{f}$ uniquely determines an embedded web $\mathcal{W}'' \subset \hat{f}^{-1}(\mathcal{W}')$ which is also isotopic to $\mathcal{W}$ relative to $\phi(\mathbf{V})$, as the following construction shows.

It is convenient first to define "the web ray of argument 0" in $\mathcal{W}''$. For this we need the following remark.

*Let $\theta \neq 0$ belong to $\mathcal{J}^{*\cup}$. If $0 \sim_l \theta$, then the web rays $\mathcal{R}_\theta$ and $\mathcal{R}_0$ in $\mathcal{W}$ can not be isotopic relative to the set $\phi(\mathbf{V})$.*

To see this we note that these web rays determine two sectors. By construction each of these two sectors contains all web rays with arguments in $(0, \theta)$ and $(\theta, 1)$ respectively. Now, by Lemma II.1.6, $\theta$ is of the form $k/(d-1)$, so each of the sets $\mathcal{J}^{*\cup} \cap (0, \theta)$ and $\mathcal{J}^{*\cup} \cap (\theta, 1)$ is non empty. The result follows easily.

As a consequence we have that there is a unique edge $\mathcal{R}'_0$ in $\mathcal{W}'$ which can correspond to $\mathcal{R}_0$. Thus there is a unique 'edge' $\mathcal{R}''_0 \subset \hat{f}^{-1}(\mathcal{R}'_0)$ joining

$\phi(v_0)$ and $\infty$, which is isotopic to $\mathcal{R}_0$ relative $\phi(\mathbf{V})$. This is to be defined as the zero web ray in $\mathcal{W}''$.

To construct the web $\mathcal{W}''$ we consider first all edges $\ell \subset \mathcal{W}$ incident at vertices $v = \phi(v(\mathcal{J}_k^*))$ which are not critical. By definition, $\hat{f}(\ell)$ is also an edge in $\mathcal{W}$; now, there is a unique edge $\ell' \in \mathcal{W}'$ which is isotopic to $\hat{f}(\ell)$ relative to $\phi(\mathbf{V})$. As $\hat{f}$ is locally one to one near $v$, starting at $\hat{f}(v)$, $\ell'$ can be lifted back in a unique way by $\hat{f}$ to an arc $\ell''$. As $\hat{f}(\ell)$ and $\ell'$ are in particular isotopic relative to the critical values of $\hat{f}$, it follows that $\ell$ and $\ell''$ are isotopic relative to $\phi(\mathbf{V})$.

Finally, we consider all edges $\ell$ incident at critical vertices $v = \phi(v(\mathcal{J}_k^*))$. Again we repeat the same procedure but keeping in mind that the correct indexing for web rays can be found by its relative position respect to the web ray $\mathcal{R}_0$. The adequate choice of inverses can now be easily determined. This finishes the construction of $\mathcal{W}''$. By abuse of notation, we denote this embedded web $\mathcal{W}''$ by $\hat{f}^{-1}(\mathcal{W}')$.

Note that we can apply the same construction to the web $\mathcal{W}'' = \hat{f}^{-1}(\mathcal{W}')$ and so on; in this way we can form a sequence of webs

$$\mathcal{W}', \hat{f}^{-1}(\mathcal{W}'), \ldots, \hat{f}^{-n}(\mathcal{W}'), \ldots$$

all isotopic relative to $\phi(\mathbf{V})$.

## 3. There are no Levy cycles.

In this Section we will prove that any admissible critical portrait is 'naturally' associated to a unique polynomial $P$ (see Corollary 3.6). The natural way to proceed is to construct from the family $(\mathcal{F}^*, \mathcal{J}^*)$ with $\Gamma = \emptyset$ a web map $\hat{f}$. The next step can be (as in [BFH]) to prove that any regular extension has no Thurston's obstruction by proving there are no Levy cycles. This fact is by no means obvious. In fact, it is easier to prove this fact for maps $\hat{f}'$ associated to a bigger family $(\mathcal{F}'^*, \mathcal{J}'^*)$ with $\Gamma$ suitably chosen. Now, as a Levy cycle for the map $\hat{f}$ will determine a Levy cycle for the map $\hat{f}'$ we can conclude that the former map has no Levy cycles.

We start with some notation and another result borrowed from [BFH] Section 7.

**3.1 Definition.** Let $\mathcal{W}$ be an embedded web and $\ell \subset \mathcal{W}$ an edge. A Jordan curve $\mathcal{C}$ disjoint from $\phi(\mathbf{V})$ is said to *intersect* $\ell$ *essentially*, if for every $\mathcal{C}'$ homotopic to $\mathcal{C}$ in $\hat{\mathbf{C}} - \phi(\mathbf{V})$, we have that $\ell \cap \mathcal{C}'$ is non empty.

The following is together with Theorem A.5 a technical result needed for the proof of the main theorem.

**3.2 Lemma.** *Suppose $\hat{f}$ admits a Levy cycle $\Lambda = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ (see appendix A). Then any $C_i$ does not intersect a preperiodic edge $\ell$ of the web in an essential way.*

**Proof.** See [BFH] Lemma 7.7. #

**3.3 Remark.** Using Proposition II.4.6 it is easy to construct a finite set of special arguments $\Gamma$ with the following properties.

*i)* $m_d(\Gamma) \subset \Gamma \cup \mathcal{O}(\mathcal{F}^{\cup})$.

*ii) If* $\lambda \in \mathcal{F}^{\cup}_{\mathbf{per}}$, *and* $\lambda'$ *is the successor (counterclockwise) of* $\lambda$ *in* $\mathcal{J}^{*\cup}$ *then* $\lambda \sim_{\lambda} \lambda'$.

In the following lemma we assume that the web and a regular extension where constructed with this set of special arguments. Here if $\mathcal{C}$ is a Jordan curve, the *interior of* $\mathcal{C}$ is defined as the bounded component of $\hat{\mathbf{C}} - \mathcal{C}$.

**3.4 Lemma.** *Let* $\mathcal{C}$ *be a Jordan curve disjoint from* $\phi(\mathbf{V})$. *Suppose further that* $\mathcal{C}$ *has the following properties,*

*a) All vertices in* $\phi(\mathbf{V})$ *which belong to the interior of* $\mathcal{C}$ *are periodic and do not belong to a critical cycle.*

*b)* $\mathcal{C}$ *does not intersect essentially any preperiodic edge* $\ell$.

*Under theses hypothesis, if* $v_\theta, v_{\theta'} \in \phi(\mathbf{V})$ *(corresponding to the landing point of the web rays* $\mathcal{R}_\theta, \mathcal{R}_{\theta'}$ *respectively) belong to the interior of* $\mathcal{C}$, *then* $A^-(\theta) = A^-(\theta')$.

**Proof.** Suppose $v_\theta, v_{\theta'}$ are in the interior of $\mathcal{C}$ (and therefore $\theta, \theta'$ are periodic). Let $\gamma, \gamma' \in \mathcal{J}_k$ for some $k$. The rays $\mathcal{R}_\gamma$ and $\mathcal{R}_{\gamma'}$ divide the plane in two regions. If $v_\theta, v_{\theta'}$ do not belong to the same region, then $\mathcal{C}$ will cut either $\mathcal{R}_\gamma$ or $\mathcal{R}_{\gamma'}$ in an essential way. Thus, $\theta, \theta'$ belong to the same connected component of $\mathbf{R}/\mathbf{Z} - \mathcal{J}_k$. Now, let $\gamma, \gamma' \in \mathcal{F}_k$ for some $k$. The extended rays $\mathcal{E}_\gamma$ and $\mathcal{E}_{\gamma'}$ divide the plane in two regions. If both $\gamma, \gamma'$ are preperiodic the same argument as above applies, and again $\theta, \theta'$ belong to the same connected component of $\mathbf{R}/\mathbf{Z} - \{\gamma, \gamma'\}$. Otherwise, suppose that $\gamma$ is periodic (and thus, $\gamma'$ must be preperiodic). By hypothesis there is $\epsilon > 0$

such that $\gamma + \epsilon$ is a special argument for $\gamma$, and $(\gamma, \gamma + \epsilon) \cap \mathcal{J}^{*\cup} = \emptyset$. Now we apply the same reasoning with the extended rays $\mathcal{E}_{\gamma+\epsilon}$ and $\mathcal{E}_{\gamma'}$ and thus $\theta, \theta'$ belong to the same connected component of $\mathbf{R}/\mathbf{Z} - \{\gamma + \epsilon, \gamma'\}$ (compare Figure 3.1). As $\epsilon$ can be chosen arbitrarily small, it follows that for $\epsilon > 0$ small enough, $\theta - \epsilon$ and $\theta' - \epsilon$ belong to the same connected component of $\mathbf{R}/\mathbf{Z} - \mathcal{F}_k$. It follows by definition that $A^-(\theta) = A^-(\theta')$. #

**3.5 Proposition.** *Let $\hat{f} : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a regular extension of the web map over $(\mathcal{F}^*, \mathcal{J}^*)$ for some $\Gamma$. Then $\hat{f}$ admits no Levy cycles.*

**Proof.** We are going to add points to $\Gamma$ as needed (see the introduction to this section). Suppose by contradiction that $\hat{f}$ has a Levy cycle $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$.

*Step 1.* As all "Fatou points" (i.e, vertices of the form $\phi(\omega(\mathcal{F}_j^*))$) are preperiodic or belong to a critical cycle, no such points are in the interior of an element of a Levy cycle (compare Theorem A.5).

*Step 2.* If $\theta \sim_l \theta'$ but $S^-(\theta) \neq S^-(\theta')$ then $\theta$ is preperiodic, and so is $v_\theta$. Thus, $v_\theta$ is not in the interior of a curve in a Levy cycle.

*Step 3.* If $v_\theta, v_{\theta'}$ are in the interior of an element of a Levy cycle, then by Lemma 3.4 $A^-(\theta) = A^-(\theta')$.

*Step 4.* There are no Levy cycles:

If $v_\theta, v_{\theta'}$ belong to the interior of an element $\mathcal{C}_1$ of a Levy cycle, then there is another element $\mathcal{C}$ in this Levy cycle such that $v_{m_d(\theta)}$ and $v_{m_d(\theta')}$

belong to the interior of $\mathcal{C}$. This immediately implies $S^{-}(\theta) = S^{-}(\theta')$ by step 3 and the definition of Levy cycles. In this way $v_\theta = v_{\theta'}$ by construction of the Web. But this implies there is a unique point in the interior of an element of a Levy cycle, and this is a contradiction with the definition of Levy cycles. #

**3.6 Corollary.** *Let $(\mathcal{F}, \mathcal{J})$ be an admissible critical portrait. There is a unique (up to conjugation) polynomial $P(\mathcal{F}, \mathcal{J})$ which is Thurston equivalent to $\hat{f}$. Here $\hat{f}$ is any regular extension of the web map.* #

**3.7 Example.** We are left with the awkward situation of illustrating a result about the impossibility of Levy cycles. In order to do this, some hypothesis must be violated. We have chosen to violate the condition which avoids the existence of Levy cycles, namely that $\sim_l$ equivalence classes determine only one point in the Julia set.

We consider the admissible critical portrait $\mathcal{F} = \{\{\frac{1}{4}, \frac{7}{12}\}, \{\frac{3}{4}, \frac{1}{12}\}\}$ and $\mathcal{J} = \emptyset$ (compare example I.4.4). It is easy to check that $S^{-}(\frac{1}{4}) = S^{-}(\frac{3}{4})$ (thus expecting the rays $R_{\frac{1}{4}}$ and $R_{\frac{3}{4}}$ to land at the same point in the Julia set). We consider also the set of special arguments $\Gamma = \{\frac{13}{36}, \frac{31}{36}\}$ which satisfies the hypothesis stated in 3.3 (here $\frac{13}{36} \sim_{\frac{1}{4}} \frac{1}{4}$ and $\frac{31}{36} \sim_{\frac{3}{4}} \frac{3}{4}$). Thus we have formed

$$\mathcal{F}^* = \{\{\frac{1}{4}, \frac{13}{36}, \frac{7}{12}\}, \{\frac{3}{4}, \frac{31}{36}, \frac{1}{12}\}\}$$

$$\mathcal{J}^* = \{\{0\}, \{\frac{1}{12}\}, \{\frac{1}{4}, \frac{3}{4}\}, \{\frac{13}{36}\}, \{\frac{7}{12}\}, \{\frac{31}{36}\}\}$$

(recall the meaning of the elements in each family).

To illustrate Lemma 3.4 (and Proposition 3.5), we construct a web $\mathcal{W}(\mathcal{F}^*, \mathcal{J}^*)$ without identifying $v_{\frac{1}{4}}$ and $v_{\frac{3}{4}}$. We will show how this leads to a Levy cycle (compare Figure 3.1).

Lemma 3.4 claims that if there is a Levy cycle, then arguments of any two $v_\theta$, $v_{\theta'}$ in the interior of a constituent element $\mathcal{C}$ of this cycle should have the same left address. In our case this means that any such $\mathcal{C}$ can not cross any solid segment in Figure 3.1 because of Lemma 3.2. Thus, the only possibility of a cycle is as shown in Figure 3.1. Of course, with the appropriate identification of $v_{\frac{1}{4}}$ and $v_{\frac{3}{4}}$, this is impossible.

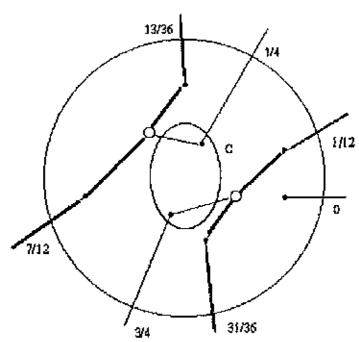



*Figure 3.1*

## 4. Untwisting the conjugacy.

Up to this point Corollary 3.6 tells us there is a polynomial (unique up to conjugation) associated with the admissible critical portrait $(\mathcal{F}, \mathcal{J})$. We must still prove that external and internal rays land at the expected places. In other words, we have to prove that such post-critically finite polynomial admits the required marking. The proof of this fact is not as obvious as it will seem. We will consider first a particular example in order to show which difficulties we can still find and describe a way to handle them.

**4.1 Example.** Consider the admissible critical portrait formed with $\mathcal{F} = \{\{0, \frac{1}{3}, \frac{2}{3}\}\}$, $\mathcal{J} = \emptyset$. We first look at the map $f(z) = z^3$ as a *'topological polynomial'* in the web $\mathcal{W}(\mathcal{F}, \mathcal{J})$ with *vertices* $\mathrm{V} = \{0, 1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ and *extended web rays* $\mathcal{E}_{k/3} = \{re^{\frac{2k\pi i}{3}} : r \in [0, \infty)\}$ for $k = 0, 1, 2$. By Corollary 3.6 this topological polynomial is equivalent to a unique polynomial, which will surely be $P(z) = z^3$.

Consider the homeomorphisms

$$\psi_0(r^3 e^{2\pi i\theta}) = \begin{cases} r^3 e^{2\theta\pi i} & \text{if } r \leq 3; \\ r^3 e^{2\pi i[\theta + \frac{3}{2}(\frac{\ln r - \ln 3}{\ln 4 - \ln 3})]} & \text{if } 3 \leq r \leq 4; \\ r^3 e^{2\pi i[\theta + \frac{3}{2}]} & \text{if } 4 \leq r. \end{cases}$$

$$\psi_1(re^{2\pi i\theta}) = \begin{cases} re^{2\theta\pi i} & \text{if } r \leq 3; \\ re^{2\pi i[\theta + \frac{1}{2}(\frac{\ln r - \ln 3}{\ln 4 - \ln 3})]} & \text{if } 3 \leq r \leq 4; \\ re^{2\pi i[\theta + \frac{1}{2}]} & \text{if } 4 \leq r. \end{cases}$$

Then clearly the following diagram is commutative

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{\psi_1} & \hat{\mathbb{C}} \\ f \downarrow & & \downarrow P. \\ \hat{\mathbb{C}} & \xrightarrow{\psi_0} & \hat{\mathbb{C}} \end{array} \qquad (1)$$

We describe what is happening in the following terms. The map $\psi_0$ makes a *'Dehn twist'* of 3/2 turns far from $\infty$. Thus the 'Web' $\psi_0(\mathcal{W}(\mathcal{F}, \mathcal{J}))$ itself is twisted 3/2 turns. By this we mean that when keeping track of the image $\psi_0(\mathcal{R}_0)$ of the web ray $\mathcal{R}_0$, we start as the actual ray $R_0$ for a while, then twist in counterclockwise direction until we have completed 3/2 turns, and finally continue our way to $\infty$ following the ray $R_{1/2}$! Similarly with all other web rays.

Now, when lifting back the web $\psi_0(\mathcal{W}(\mathcal{F},\mathcal{J}))$ by $P^{-1}$ (compare §2.6), we see that the resulting embedded web $\psi_1(\mathcal{W}(\mathcal{F},\mathcal{J}))$ has a completely different behavior (but they are isotopic). The image web ray $\psi_1(\mathcal{R}_0)$ in this case goes for a while in the direction of the actual ray $R_0$, then twists 1/2 turns, and finally continues in the direction of the actual ray $R_{1/2}$ to $\infty$.

The situation is even worse if we consider successive liftings of the web ray $\psi_0(\mathcal{R}_0)$. In these cases, near $\infty$ they will be successively identified with the rays $R_{\frac{1}{2}}, R_{\frac{1}{6}}, R_{\frac{1}{18}}, \ldots$. Of course, we will prefer to have always near $\infty$ the correct identification. In order to describe a possible solution to this dilemma, we note that $\psi_0(z) = \psi_1(z)$ for $|z|$ big enough. If we remove the set $\{z : |z| > \alpha\}$ for $\alpha$ big enough, $\psi_0$ and $\psi_1$ would not be isotopic in this new Riemann surface relative to the boundary (they will differ by exactly 'one turn' around $\{z : |z| = \alpha\}$. This is hardly a surprise because the difference in 1 turn can be easily measured by comparing the embedded web to its lift. Now, it is clear that we have not started with the best possible choice of a web. Our original web was 'twisted' by a given number of turns (3/2 in this case); when we 'lift back' the web, this twist will be divided by the degree of the polynomial (3 in this case). Thus, the 'difference in twist' (which can always be measured) allows us to state the relation

$$twist - \frac{twist}{d} = difference\ in\ twist. \qquad (2)$$

Where $d$ is the degree of the polynomial (here $d = 3$) and *difference in twist* is the *relative twist of the ray* $\psi_1(\mathcal{R}_0)$ in the lifted web respect to the original

$\psi_0(\mathcal{R}_0)$. In this way, equation (2) suggests that any possible odd behavior when lifting webs is because of a 'Dehn twist' in a neighborhood of Fatou points. This is going to be in general the case as we will show below.

**4.2.** In the general case, we have that starting from the admissible critical portrait $(\mathcal{F}, \mathcal{J})$ we can construct a unique up to conjugation polynomial $P$ of degree $d$ (which we take here to be monic and centered). Also diagram (1) holds. Furthermore, by replacing $f$ by $\psi_0 \circ f \circ \psi_0^{-1}$ and $\psi_1$ by $\psi_1 \circ \psi_0^{-1}$, we may assume without loss of generality that $\psi_0 = id$.

For notational convenience we include $\infty$ in the critical set $\Omega(P)$ of the polynomial $P$. For each periodic Fatou point $\omega \in \Omega(P)$, let $\phi_\omega$ denote a fixed Böttcher coordinate associated with $\omega$ ($\infty$ included). For $r < 1$ define $N_r(w) = \{z \in U(\omega) : |\phi_\omega(z)| < r\}$. For each strictly preperiodic Fatou point $c \in \mathcal{O}(\Omega(P))$, we inductively define $N_r(c)$ as the connected component of $P^{-1}(N_r(P(c)))$ containing $c$. For $X \subset \mathcal{O}(\Omega(P))$ set $N_r(X) = \cup_{c \in X} N_r(c)$.

Now, as there is no topological way to distinguish between the sets $\hat{\mathbf{C}} - \mathcal{O}(\Omega(P))$ and $\hat{\mathbf{C}} - N_r(\mathcal{O}(\Omega(P)))$, we can construct an embedded web in $\hat{\mathbf{C}}$ and a regular extension $f$ such that the following conditions are satisfied,

i) *$f = P$ in $N_{1/2}(\mathcal{O}(\Omega(P)))$,*

ii) *preferred internal web rays are equal to internal preferred rays in $N_{1/2}(\omega)$ if $\omega$ is in a critical cycle,* and

iii) *Web edges correspond to internal rays in $N_{1/2}(\mathcal{O}(\Omega(P)))$.*

Denote by $\mathcal{W}$ the so constructed web, and by $\mathbf{V}$ be the respective set of vertices (there is no further need to write this set as $\phi(\mathbf{V})$). Recall we are assuming that $\psi_0$ is the identity in diagram (1). Note also that the construction implies that near periodic critical points, $\psi_1$ is a rotation in the Böttcher coordinate.

**4.3 Untwisting external rays.** We consider first what happens near $\infty$ (for example, in the set $N_{1/2}(\infty)$). As diagram (1) is commutative, we have that for any positive $r \leq 1/2$, $\psi_0(\mathcal{W}) \cap N_r(\infty)$ is by construction $\infty$ and some segments of actual external rays. The portion of the web ray $\psi_1(\mathcal{R}_0) \cap N_r(\infty)$ must then be a segment of a ray of the form $R_{j/d}$. Furthermore, we can measure the relative twist of $\psi_1(\mathcal{R}_0)$ respect to $\psi_0(\mathcal{R}_0)$ in $\partial N_r(\infty)$ (which by construction is a rational number of the form $k/d$). Stating this as an equation

$$\textit{possible twist} - \frac{\textit{possible twist}}{d} = \textit{difference in twist}$$

we have necessarily a rational solution of the form $k/(d-1)$ (same $k$ as above).

To prove that this 'possible twist' is in fact a twist we proceed as follows. Take a positive $s < r$ and consider the annulus $N_{r^d}(\infty) - N_{s^d}(\infty)$. We modify $\psi_0$ in $N_{r^d}(\infty)$ by making a twist of $-\frac{k}{d-1}$ turns inside this annulus. This forces us to modify $\psi_1$ in $N_r(\infty)$ by a twist of $-\frac{k}{d(d-1)}$ turns inside the annulus $N_r(\infty) - N_s(\infty)$ in order to make diagram (1) commutative. Clearly there is no problem in doing so because $\psi_0$ is the identity in $N_{r^d}(\infty)$, and $\psi_1$ is a rotation in the set $N_r(\infty)$ respect to the Böttcher coordinate.

Formally, we have that in the set $\hat{\mathbf{C}} - \mathbf{V} - N_r(\infty)$, $\psi_0$ and $\psi_1$ are not isotopic respect to the boundary because they differ by $k/d$ turns. In the annulus $N_r(\infty) - N_{s^d}(\infty)$, the modified $\psi_0, \psi_1$ differ by $-k/d$ turns. In this way, for $x_i$ rational with denominator $\mathcal{D}-1$. But it is clear that this can be done if we rewrite the system as

$$\begin{array}{ccccccc} d_0 d_1 \dots d_{n-1} & x_0 & = & d_1 \dots d_{n-1} & x_1 & + & d_0 d_1 \dots d_{n-1} & y_0 \\ d_1 \dots d_{n-1} & x_1 & = & d_2 \dots d_{n-1} & x_2 & + & d_1 \dots d_{n-1} & y_1 \\ & \vdots & & & \vdots & & & \vdots \\ d_{n-2} d_{n-1} & x_{n-2} & = & d_{n-1} & x_{n-1} & + & d_{n-2} d_{n-1} & y_{n-2} \\ d_{n-1} & x_{n-1} & = & & x_0 & + & d_{n-1} & y_{n-1} \end{array}$$

With the given solutions $x_0, \dots, x_{n-1}$ we proceed to untwist the conjugacy in all neighborhoods of the cycle simultaneously as in §4.3.

**4.5 Untwisting non periodic Fatou critical components.** The last basins that need to be 'untwisted' are the ones that correspond to strictly preperiodic Fatou critical points. Let $\omega$ be such critical point, and $\omega' = f^{\circ n}(\omega)$ the first critical point in its forward orbit. We assume that near $\omega'$ the conjugacy has been already 'untwisted'. In this case the resulting equation is simply $x_\omega = y_\omega$ so we proceed again as in §4.3.

## 5. Proof of Theorem I.3.9.

**5.1** Now we apply successively the construction in §2.5. The webs $\mathcal{W}_n = P^{-n}(\psi_0(\mathcal{W}))$ have edges which coincide with the actual internal and external rays in a bigger set after each lifting. Given $n$, for the web $\mathcal{W}_n$ we consider for each $\bar{v}$ landing point of "web rays" and for each edge $\ell$ incident at it, the orbifold length of $\ell_n = \hat{\mathbf{C}} - N_{r^{-d^n}}(\mathcal{O}(\Omega(P))) \cap \ell$. For fixed $n$ denote by $\delta_n$ the supremum of such numbers over all possible vertices and edges. Note that, as the orbifold metric is strictly expanding for $P$ in $\hat{\mathbf{C}} - N_r(\mathcal{O}(\Omega(P)))$, and each $\ell_n$ is the inverse image of some $\ell'_{n-1}$ we have that $\delta_n \downarrow 0$. In this way we have that the respective rays (internal and external) of $P$ can be found arbitrarily close to the expected landing points. As $J(P)$ is locally connected they actually land there.

**5.2** To finish the proof of the theorem, we only have to prove that the rays $R_\gamma$ associated with a Fatou periodic critical point actually support the respective component. But this is trivial if we consider Proposition II.4.6. In this case $R_\gamma, R_{\gamma+\epsilon}$ land in the boundary of the same critical component (compare Proposition II.4.7). Thus, in the region determined by the extended rays $\hat{R}_\gamma, \hat{R}_{\gamma+\epsilon}$ there is no place for a periodic ray $R_\lambda$ of the same period as $R_\gamma$, if $\epsilon > 0$ was chosen small enough. This completes the proof of Theorem I.3.9. #

# Chapter IV
# Hubbard Trees

In this Chapter we recall the definition and survey the main properties of Hubbard Trees as defined by Douady and Hubbard in [DH1]. In Section 1 we define the main concepts and deduce some properties. We ask the reader to pay special attention to Proposition 1.21. In Section 2 we define the inverse of Hubbard Trees. In Section 3 we define and study the incidence number at every point $p$ of the tree and relate this concept with the number of connected components of $J(P) - \{p\}$.

## 1. Regulated Trees.

**1.1** Let $P$ be a Postcritically Finite Polynomial. Given two points in the closure of a bounded Fatou component, they can be joined in a unique way by a Jordan arc consisting of (at most two) segments of internal rays. We call such arcs (following Douady and Hubbard) *regulated.* The filled Julia set $K(P)$ being connected and locally connected in a compact metric space is also arcwise connected. This means that given two points $z_1, z_2 \in K(P)$ there is a continuous injective map $\gamma : I = [0,1] \mapsto K(P)$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. In general we will not distinguish between the map and its image. Such arcs (actually their images) can be chosen in a unique way so that the intersection with the closure of a Fatou component is regulated (see [DH1, Chapter 2]). We still call such arcs *regulated*, and denote them by $[z_1, z_2]_K$.

The following immediate properties hold for regulated arcs (compare also [DH1, Chapter 2]).

**1.2 Lemma.** *Let $\gamma_1, \gamma_2$ be regulated arcs, then $\gamma_1 \cap \gamma_2$ is regulated.* #

**1.3 Lemma.** *Every subarc of a regulated arc is regulated.* #

**1.4 Lemma.** *Let $z_1, z_2, z_3 \in K(P)$, then there exists $p \in K(P)$ such that $[z_1, z_2]_K \cap [z_2, z_3]_K = [p, z_2]_K$. In particular if $[z_1, z_2]_K \cap [z_2, z_3]_K = \{z_2\}$, the set $[z_1, z_2]_K \cup [z_2, z_3]_K$ is a regulated arc.* #

**1.5 Regulated Sets.** We say that a subset $X \subset K(P)$ is *regulated connected* if for every $z_1, z_2 \in X$ we have $[z_1, z_2]_K \subset X$. We define the *regulated hull* $[X]_K$ *of* $X \subset K(P)$ as the minimal closed regulated connected subset of $K(P)$ containing $X$.

**1.6 Proposition.** *If $z_1, ..., z_n$ are points in $K(P)$, the regulated hull $[z_1, ..., z_n]_K$ of $\{z_1, ..., z_n\}$ is a finite topological tree.*

**Proof** (Compare [DH1]). The proof is by induction in the number of points. This is clearly true for small $n$ $(= 1, 2)$. Suppose $[z_1, ..., z_n]_K$ is a finite topological tree, and let $z_{n+1} \in K(P)$. Let $p$ any point in $[z_1, ..., z_n]_K$ and $y$

the first point in the arc $[z_{n+1}, p]_K$ that belongs to $[z_1, ..., z_n]_K$. In this way $[z_1, ..., z_{n+1}]_K = [z_1, ..., z_n]_K \cup [y, z_{n+1}]_K$ and $[z_1, ..., z_n]_K \cap [y, z_{n+1}]_K = \{y\}$. The result follows. #

**1.7 Remark.** By definition every end of the tree $[z_1, ..., z_n]_K$ is one of $z_k$, but there may be $z_k$ which are not ends.

**1.8 Lemma.** *Let $\gamma(I) \subset K(P)$ be a regulated arc containing no critical point of $P$, except possibly for its end points. Then $P|_{\gamma(I)}$ is injective and $P(\gamma(I))$ is a regulated arc.*

**Proof.** The second part follows from the first, so let us show that $P|_\gamma$ is injective. As $P \circ \gamma$ is locally one to one, the set $\Delta = \{(t_1, t_2) : t_1 < t_2$ and $P(\gamma(t_1)) = P(\gamma(t_2))\}$ is compact. If this set is non empty we can take $(t_1, t_2) \in \Delta$ with $t_2 - t_1$ minimal. Let $t \in (t_1, t_2)$, then $P(\gamma([t_1, t]))$ and $P(\gamma([t, t_2]))$ are regulated arcs with the same end points; therefore they are equal and $t_2 - t_1$ is not minimal. #

**1.9 Definition.** For a finite invariant set $M$, containing the set $\Omega(P)$ of critical points of $P$, we denote by $T(M)$ the tree generated by $M$, i.e, the regulated hull $[M]_K$. The *minimal tree* $T(M_0)$, is the tree generated by $M_0 = \mathcal{O}(\Omega(P))$ the orbit of the critical set. This last tree is usually called in the literature *the Hubbard Tree of P.*

**1.10 Lemma.** *For a finite invariant set $M$, containing the set $\Omega(P)$ of all critical points, $P(T(M)) = [P(M)]_K$.*

**Proof.** The tree $T(M)$ is the union of regulated arcs of the form $[z_1, z_2]_K$ with $z_1, z_2 \in M$ not containing a critical point except possibly for their end points. By Lemma 1.8, $P(T(M))$ is the union of the regulated arcs $[P(z_1), P(z_2)]_K$. As this set is regulated connected and contains all of $P(M)$, by definition this set equals $[P(M)]_K$. #

**1.11 Remark.** If $X \subset K(P)$ is arbitrary, the same argument shows that $P(T(X)) \subset [P(X \cup \Omega(P))]_K$.

**1.12 Definition.** Let $T^*(M)$ be the family whose elements are the closures of components of $T(M) - \Omega(P)$.

**1.13 Lemma.** *$P$ induces a continuous map from $T(M)$ to itself, where the restriction to every element (component) of $T^*(M)$ is injective.*

**Proof.** This follows from Lemmas 1.8 and 1.10. #

**1.14 Lemma.** *Let $\gamma(I) \subset K(P)$ be a regulated arc containing no critical value of $P$ except possibly for its end points. Then any lift of $\gamma(I)$ by $P$ is a regulated arc.*

**Proof.** As $\gamma|_{(0,1)}$ contains no critical value of $P$, it can be pulled back by $P$ in $d$ different ways, each being a regulated arc. #

**1.15 Definition.** Given $z \in T(M)$ the *incidence number* $\nu_{T(M)}(z)$ *of $T(M)$ at $z$* is the number of components of $T(M) - \{z\}$. In other words,

$\nu_{T(M)}(z)$ is the number of branches of $T(M)$ that are incident at $z$. Note that this number might be different from the number of connected components of $K(P) - \{z\}$ (the incidence number at $z$ for $P$).

A point $z \in T(M)$ is called a *branching point of* $T(M)$ if $\nu_{T(M)}(z) > 2$ and an *end* if $\nu_{T(M)}(z) = 1$. The *preferred set* of $T(M)$ is $V_{T(M)} = M \cup \{z \in T(M) : \nu_{T(M)}(z) > 2\}$. Note that $V_{T(M)}$ is finite. This because there are only a finite number of vertices in this tree.

**1.16 Proposition.** *The set $V_{T(M)}$ is invariant. Furthermore, it generates the same tree as M; i.e, $T(M) = T(V_{T(M)})$.*

**Proof.** If $z$ is a branching point and $deg_z P = 1$, then $P(z)$ is also a branching point with $\nu_{T(M)}(P(z)) \geq \nu_{T(M)}(z)$ because $P$ maps $T(M)$ into itself and $P$ is a local homeomorphism in a neighborhood of $z$.

We must prove that $[M]_K = [V_{T(M)}]_K$. As $M \subset V_{T(M)}$ then $[M]_K \subset [V_{T(M)}]_K$. Also by definition $V_{T(M)} \subset [M]_K$, so $[V_{T(M)}]_K \subset [[M]_K]_K = [M]_K$. #

**1.17 Corollary.** *Let $M, M'$ be finite invariant subsets containing $\Omega(P)$. If $V_{T(M)} = V_{T(M')}$ then $T(M) = T(M')$.* #

**1.18 Proposition.** *Let $v, v' \in J(P)$ be two periodic points. If for all $n \geq 0$, $P^{\circ n}(z)$ and $P^{\circ n}(z')$ belong to the same element (component) of $T^*(M)$, then $v = v'$.*

**Proof.** Suppose $P^{\circ n}(v), P^{\circ n}(v')$ belong to the same component of $T^*(M)$ for all $n \geq 0$. By Lemma 1.8 there is no precritical point in $[v, v']_K$. It follows easily that $[v, v']_K \subset J(P)$. Next, let $m$ be the least common multiple of the periods of $v$ and $v'$. Thus, $v, v'$ are fixed by $P^{\circ m}$. As there are only a finite number of such fixed points, we may assume that there are no other in this set $[v, v']_K$. Both endpoints of this regulated arc are repelling. Also by Lemma 1.8, $P^{\circ m}$ induces an homeomorphism of $[v, v']_K$ onto itself. It follows that there must be other fixed point in the interior of the arc $[v, v']_K$, in contradiction to what was assumed. #

**1.19 Remark.** Note that the same is true if $v, v'$ are assumed only to be preperiodic. In this case, high enough iterates of both points must be periodic and therefore coincide. Lemma 1.13 will imply that $v, v'$ are identified as well.

**1.20 Definition.** We define the distance $d_{T(M)}(v, v')$ between points $v, v' \in V_{T(M)}$ as follows. Set $d_{T(M)}(v, v) = 0$. Otherwise, take a regulated arc $[v, v']_K$ and define $d_{T(M)}(v, v') = \#(V_{T(M)} \cap [v, v']_K) - 1$ (# denotes as usual cardinality). Thus, $d_{T(M)}$ measures the number of 'edges' between $v$ and $v'$. In this language Proposition 1.18 can be read as follows.

**1.21 Proposition: Expanding Property of the tree T(M).** *For all pairs $v, v' \in V_{T(M)} \cap J(P)$ satisfying $d_{T(M)}(v, v') = 1$, there is an $n \geq 1$ such that $d_{T(M)}(P^{\circ n}(v), P^{\circ n}(v')) > 1$.*

**Proof.** As $v, v'$ are eventually periodic, the result follows from Proposition 1.18. #

## 2. The Regulated Trees $T(P^{-n}M)$

In this section we study the inverse under $P$ of the tree $T(M)$.

**2.1 Proposition.** $P^{-1}T(M) = T(P^{-1}M) = T(P^{-1}V_{T(M)})$. *In this case the vertices of the tree are given by* $V_{T(P^{-1}M)} = P^{-1}V_{T(M)}$.

**Proof.** As $P^{-1}M \subset P^{-1}V_{T(M)}$ we have $T(P^{-1}M) \subset T(P^{-1}V_{T(M)})$.

From Lemma 1.10, $PT(P^{-1}V_{T(M)}) = [PP^{-1}V_{T(M)}]_K = [V_{T(M)}]_K = T(M)$. It follows that $T(P^{-1}V_{T(M)}) \subset P^{-1}T(M)$.

Now let $z \in P^{-1}T(M) - P^{-1}M$, then $P(z)$ belongs to a regulated arc $\gamma(I) \subset T(M)$, with only end points in $M$. By Lemma 1.14 any inverse of this regulated arc is also regulated with endpoints in $P^{-1}M$ and therefore belongs to $T(P^{-1}M)$; in this way $z \in T(P^{-1}M)$. If $z \in P^{-1}M$ then by definition $z \in T(P^{-1}M)$. This completes the proof of the chain of inequalities.

The second part follows from the first together with the definition of $V_{T(P^{-1}M)}$ and Proposition 1.16. #

Proposition 2.1 extends easily.

**2.2 Corollary.** $P^{-n}T(M) = T(P^{-n}M) = T(P^{-n}V_{T(M)})$. *In this case the vertices of the tree are given by* $V_{T(P^{-n}M)} = P^{-n}V_{T(M)}$. #

**2.3** As $T(M) \subset P^{-1}T(M)$ there are two incidence functions, $\nu_{0,M} = \nu_{T(M)}$ for $T(M)$ and $\nu_{-1,M} = \nu_{T(P^{-1}M)}$ for $P^{-1}T(M)$. It is immediate that

$\nu_{0,M}(z) \leq \nu_{-1,M}(z)$ at every point of $T(M)$. Furthermore, we have the following (here $deg_z P$ denotes the local degree of $P$ at $z$).

**2.4 Lemma.** $\nu_{-1,M}(z) = \nu_{0,M}(P(z))\ deg_z P$, *for every* $z \in P^{-1}T(M)$.

**Proof.** This follows from Lemma 1.10 and Proposition 2.1. #

These inequalities can be easily generalized for the incidence functions $\nu_{-n,M}$ of the trees $P^{-n}T(M)$. For example, $\nu_{-n,M}(z) \leq \nu_{-n-1,M}(z)$ at every point of $P^{-n}T(M)$.

The next proposition is a weak attempt to reconstruct the tree $P^{-1}T(M)$ starting from $T(M)$. An improved version will be given in Chapter VI (compare Proposition VI.2.5).

**2.5 Proposition.** *Let* $X$ *be a component of* $T^*(P^{-1}M)$. *Denote by* $C(X) = \Omega(P) \cap X$ *the critical points in* $X$. *Then* $P$ *induces a homeomorphism between* $X$ *and the component* $T_\alpha$ *of* $T(M)$ *cut along* $P(C(X))$ *that contains* $P(X)$.

**Proof.** By Lemma 1.13 $P|X$ is injective. Also, $P(X)$ is relatively open in $T_\alpha$. As it is also compact it must be the whole component. #

## 3. Incidence.

In this Section we take a closer look at terminal, incidence, branching and non branching points of the Postcritically Finite Polynomial $P$. A point $p \in J(P)$ is terminal if there is only one external ray landing at $p$. Otherwise $p$ is an incidence point. For incidence points we distinguish between branching (if there are more than two rays landing at $p$) and non branching (exactly two rays landing at $p$). We will show that for a postcritically finite polynomial $P$, every branching point must be periodic or preperiodic. Also we will prove that every periodic branching point is present as a preferred point (see §1.15) in the minimal tree $T(M_0)$, and thus in any tree $T(M)$.

**3.1** Let $P$ be a Postcritically Finite Polynomial, and $z$ an arbitrary point in the Julia set $J(P)$. Every component of $J(P) - \{z\}$ is eventually mapped onto the whole Julia set, and therefore contains points whose orbit contains any specified point. We will use this fact in the following two propositions.

**3.2 Proposition.** *Let $P$ be a Postcritically Finite Polynomial and $z \in J(P)$ a branching point. Then $z$ is preperiodic (or periodic).*

**Proof.** Suppose $z$ does not eventually map to $\mathcal{O}(\Omega(P))$ (otherwise $z$ is already preperiodic). Fix $w \in \Omega(P)$ and pick in every component of $J(P) - \{z\}$ a point $p_i$ which eventually maps to $w$. The orbit $\mathcal{O}(\{p_1, \ldots, p_k\})$ of this set $\{p_1, \ldots, p_k\}$ is a finite set. In this way, the set $M' = M \cup \mathcal{O}(\{p_1, \ldots, p_k\})$ is invariant and contains the critical points of $P$. As $z \in V_{T(M')}$, the result follows from Proposition 1.16. #

**3.3 Proposition.** *Let $P$ be a Postcritically Finite Polynomial and $z \in J(P)$ a periodic incidence point. Then $z \in T(M_0)$, and in this way $z \in T(M)$ for any finite invariant set $M \supset \Omega(P)$. Furthermore, $\nu_{0,M}(z)$ is independent of $M$ and equals the number of components of $J(P) - \{z\}$. In particular there are exactly $\nu_{0,M_0}(z)$ external rays landing at $z$.*

**Proof.** The number of rays landing at $z$ equals the number of components of $J(P) - \{z\}$. After this remark the proof is analogous to that of last proposition. Further details are left to the reader (compare also Lemma 1.10 and Remark 1.11). #

**3.4 Corollary.** *Let $z \in J(P) \cap T(M)$ be such that $P^{\circ n}(z)$ is periodic. Then $\nu_{-n,M_0}(z)$ equals the number of components of $J(P) - \{z\}$. In particular there are exactly $\nu_{-n,M}(z)$ external rays landing at $z$.*

**Proof.** This follows from Proposition 3.3 and Lemma 2.4. #

**3.5 Corollary.** *T(M) contains a fixed point of P.*

**Proof.** If $P$ has a fixed critical point, then such point is in $M$ and by definition in $T(M)$. Otherwise, as there are only $d-1$ fixed rays, but $d$ fixed points, one must be an incidence point. By Proposition 3.3, this fixed point is in $T(M)$. #

# Chapter V
# Abstract Hubbard Trees.

In this Chapter we set our basic abstract framework. We carefully justify the importance of all the elements in the definition of abstract Hubbard Trees given in the introduction (compare Examples 2.11-13). In Section 1, we introduce some basic notation related to finite topological trees. In Section 2, we introduce dynamics in finite topological trees, and explain why further structure should be added in order to have a characterization of postcritically finite polynomials. In Section 3, the elements needed for this characterization are defined. In Section 4, we give a normalization in order to simplify notation, and we state our main result, namely necessary and sufficient conditions for the realization of Hubbard Trees.

## 1. Cyclic Trees.

In this Section we only introduce some notation related to finite topological trees which would be used throughout the rest of this work.

**1.1 Definition.** By a *topological tree* $T$ will be meant a finite connected acyclic $m$-dimensional simplicial complex ($m = 0, 1$). Given $p \in T$ we define the *incidence number* $\nu_T(p)$ *of* $T$ *at* $p$ as the number of connected components of $T - \{p\}$. We say that $p \in T$ is a *branching point* if $\nu_T(p) > 2$, and an *end* if $\nu_T(p) = 1$.

A homeomorphism $\gamma : I = [0,1] \to T$ is called a *regulated path in* $T$. In general we will not distinguish between the map $\gamma$ and its image $\gamma(I)$. This because given two points $p, p' \in T$, any regulated path joining them will have the same image, which we denote by $[p, p']_T$. Given $X \subset T$ we denote by $[X]_T$ the smallest subtree of $T$ which contains $X$. Clearly this notation is compatible with that introduced before.

**1.2 Definition.** A *cyclic tree* is a triple $(T, V, \chi)$, where

(a) $T$ is the *underlying topological tree*;

(b) $V \subset T$ is finite set of *vertices* so that each component of $T - V$ is an open 1-cell (an *edge*);

(c) For each $v \in V$, $\chi_v$ represents a *cyclic order* in the set $E_v = \{\ell_1, \ldots, \ell_k\}$ of all edges with $v$ as a common endpoint.

The presence of these $\chi_v$ naturally determines an isotopy class of embeddings of this tree $T$ into $\mathbf{C}$.

**1.3 Pseudoaccesses.** If $\ell, \ell' \in E_v$ are consecutive in the cyclic order of $E_v$, we say that $(v, \ell, \ell')$ is a *pseudoaccess to* $v$. Take a pseudoaccess $(v, \ell, \ell')$ to $v$, and let the end points of the edge $\ell'$ be $v, v' \in V$. At $E_{v'}$ let $\ell''$ be the successor of $\ell'$ in the cyclic order. We say that $(v', \ell', \ell'')$ is the *successor* of $(v, \ell, \ell')$.

**1.4 Lemma.** *Let $(T, V, \chi)$ be a cyclic tree. The successor function in the set of pseudoaccess to the vertices in $V$ is a complete cyclic order.*

**Proof.** A trivial induction in the cardinality of $V$. #

**1.5 Remark.** A Postcritically Finite Polynomial $P$ and a finite invariant set $M$ containing the critical set $\Omega(P)$ of $P$, naturally defines a cyclic tree $(T(M), V_{T(M)}, \chi)$. Here $\chi_v$ represents the cyclic order of the components around a point $v \in V_{T(M)}$ taken counterclockwise.

**1.6 Definition.** Let $(T, V, \chi)$ be a cyclic tree, and let $M \subset V$. We define the *restriction of* $(T, V, \chi)$ *to* $M$, as the cyclic tree $([M]_T, V_M, \chi')$ where $V_M$ is the union of $M$ and the branching points in the topological tree $[M]_T$, and $\chi'_v$ is the natural restriction of the cyclic order $\chi_v$ of $E_v$ to the set $E'_v$ of all edges of $[M]_T$ incident at $v$.

## 2. Dynamical Abstract Trees.

In this Section we give our first attempt to describe the dynamics of a Postcritically Finite polynomial by means of the dynamics in a finite topological tree. Unfortunately this simple characterization proves to be weak (compare Examples 2.11-13), and further structure has to be added. This will be done in Sections 3 and 4.

**2.1 Definition.** A *dynamical abstract tree* is a triple $\mathbf{T} = ((T, V, \chi), \tau, \delta)$ where

(a) $(T, V, \chi)$ is *the underlying cyclic tree*,

(b) $\tau : V \to V$ is the *vertex dynamics*,

(c) $\delta : V \to \mathbf{Z}$ is a positive *local degree function*.

We require these elements to be related as follows,

(i) For any edge $\ell$ with endpoints $v, v' \in V$ we must have $\tau(v) \neq \tau(v')$.

This condition allows us to extend $\tau$ to the underlying tree as follows. For any edge $\ell$ with endpoints $v, v' \in V$, map $\ell$ homeomorphically to the shortest path joining $\tau(v)$ and $\tau(v')$. Any extension '$\tau$' well defines a map $\tau_v : E_v \to E_{\tau(v)}$. We require,

(ii) For any $v \in V$, there exists a cyclic ordered set $\mathcal{E}_v$ such that $E_v$ embeds in an order preserving way into $\mathcal{E}_v$. We require that $\tau_v$ can be extended to a degree $\delta(v)$ orientation preserving covering map between $\mathcal{E}_v$ and $E_{\tau(v)}$ (see appendix B). For the practical interpretation of this set $\mathcal{E}_v$ we refer to Remark 2.2 and Proposition VI.2.5.

We define the *degree* of $\mathbf{T}$ as $deg(\mathbf{T}) = 1 + \sum_{v \in V}(\delta(v) - 1)$. We require

(iii) $deg(T) > 1$.

**2.2 Remark.** A Postcritically Finite Polynomial $P$ of degree $n > 1$ and a finite invariant set $M \supset \Omega(P)$ naturally defines a dynamical abstract tree $\mathbf{T}_{P,M} = ((T(M), V_{T(M)}, \chi), P, deg_z P)$ of degree $n$. Here $\mathcal{E}_v$ represents the components around $v$ in the tree $T_M^{-1}$ (see §IV.2).

**2.3 Definitions.** Let $\mathbf{T} = ((T, V, \chi), \tau, \delta)$ be a dynamical abstract tree. We extend $\delta$ to all the tree $T$ by letting $\delta(p) = 1$ if $p \notin V$. We define the *critical set of* $\mathbf{T}$ as $\Omega(\mathbf{T}) = \{p \in T : \delta(p) > 1\}$. Condition (iii) above implies that $\Omega(\mathbf{T})$ is always non empty. A point $p \in \Omega(\mathbf{T})$ is a *critical point*; otherwise, it is *non critical.*

The orbit of $S \subset V$ is the set $\mathcal{O}(S) = \cup_{k=0}^{\infty} \tau^{\circ k}(S)$.

**2.4 Definition.** Let $\ell \in E_v$, we denote by $\mathcal{B}_{v,T}(\ell)$ the closure of the connected component of $T - \{v\}$ that contains $\ell$. This is just the *branch at $v$ determined by $\ell$ in the tree $T$.*

**2.5 Definition.** Let $\mathbf{T} = ((T, V, \chi), \tau, \delta)$ be an abstract tree, and let $M \subset V$ be an invariant set of vertices containing the critical set ($\tau(M) \cup \Omega(\mathbf{T}) \subset M$). We define the *restriction* $\mathbf{T}(M)$ *of* $\mathbf{T}$ *determined by* $M$, as the abstract tree $\mathbf{T}(M) = ((T(M), V_M, \chi), \tau', \delta')$, where $(T(M), V_M, \chi)$ is the restriction of the angled tree as defined in §1.8, and $\tau'$, $\delta'$ are restrictions of the functions $\tau, \delta$ to the set $V_M$.

**2.6 Definition.** Let $\mathbf{T}, \mathbf{T}'$ be two abstract trees of degree $n = deg(\mathbf{T}) = deg(\mathbf{T}') > 1$. We say that $\mathbf{T}'$ is an *extension of* $\mathbf{T}$ (in symbols $\mathbf{T} \preceq \mathbf{T}'$), if there is an embedding $\phi : T \to T'$ which satisfies the obvious conditions:

(i) $\phi(V) \subset V'$,

(ii) $\tau'(\phi(v)) = \phi(\tau(v))$ and

(iii) $\delta(v) = \delta'(\phi(v))$ for all $v \in V$,

(iv) $\phi$ induces a cyclic order preserving embedding of $E_v$ into $E_{\phi(v)}$. (At this point it is convenient to think of the elements of $E_v$ as 'germs of edges'.)

Clearly $\preceq$ is an order relation.

**2.7** Let $\mathbf{T}, \mathbf{T}'$ be two abstract trees of degree $n = deg(\mathbf{T}) = deg(\mathbf{T}') > 1$. We say that $\mathbf{T}'$ is *equivalent to* $\mathbf{T}$ (in symbols $\mathbf{T} \approx \mathbf{T}'$), if $\mathbf{T} \preceq \mathbf{T}'$ and $\mathbf{T}' \preceq \mathbf{T}$. This determines an equivalence relation between abstract trees. Furthermore, the order relation $\preceq$ extends to a partial order between equivalence classes of dynamical abstract trees of degree $n > 1$.

**2.8 Definition.** We say that an equivalence class $[\mathbf{T}]$ of dynamical abstract trees of degree $n > 1$ is *minimal* if given $[\mathbf{T}'] \preceq [\mathbf{T}]$ we necessarily have $[\mathbf{T}'] = [\mathbf{T}]$.

From the definition of extension tree we can deduce that if $[\mathbf{T}']$ is an extension of $[\mathbf{T}]$, then $[\mathbf{T}]$ is a restriction of $[\mathbf{T}']$ in the sense of Definition 2.5. Therefore we have the following.

**2.9 Proposition.** *Every abstract tree* $\mathbf{T}$ *contains a unique minimal tree* $min(\mathbf{T})$. *Furthermore, this unique minimal tree is the tree generated by the orbit* $\mathcal{O}(\Omega(\mathbf{T}))$ *of the critical set.* #

**2.10** The question now is if this description completely characterizes Postcritically Finite Polynomials. In other words, given a class $[\mathbf{T}]$ of dynamical abstract trees, is there a unique (up to affine conjugation) Postcritically Finite Polynomial $P$ and an invariant set $M \supset \Omega(P)$ such that $\mathbf{T}_{P,M} \in [\mathbf{T}]$?

The answer is negative as the following examples show.

**2.11 Non uniqueness.** Suppose a degree 3 polynomial has the following minimal tree $\mathbf{T}$ (where the double star stands for a double critical point, i.e, its local degree is 3).

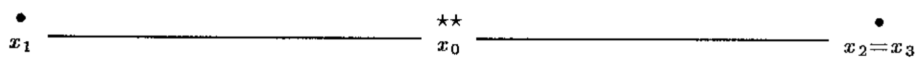



*Figure 2.1. The vertex dynamics is given by* $x_0 \mapsto x_1 \mapsto x_2 \mapsto x_3 = x_2$.

If we want a centered monic polynomial with this minimal tree we suppose that $x_0 = 0$. We have then $P(z) = z^3 + c$. (For polynomials of the form $P_c(z) = z^3 + c$, the number $c^2$ is a complete invariant up to conjugacy. In other words $P_c$ is affine conjugate to $P_{c'}$ if and only if $c^2 = c'^2$.) If $P$ has this minimal tree, then the orbit of the critical point is as follows, $0 \mapsto c \mapsto c^3 + c \mapsto c^3 + c$.

In this way, the relation $P_c^{\circ 2}(0) = P_c^{\circ 3}(0)$ determines the equation $c^3 + c = (c^3 + c)^3 + c$. Thus $c$ must satisfy $c^5(c^4 + 3c^2 + 3) = 0$. If we want $c^3 + c \neq 0$ we must have $c \neq 0$, and we have two different possible values for $c^2 = \frac{-3 \pm \sqrt{-3}}{2}$. For both values of $c^2$ the respective polynomials $P_c$ have minimal tree $T(M_0)$ as shown in Figure 2.1. In fact, by Lemma IV.1.13, $c$ and $c^3 + c$ belong to different components of $T - \{0\}$.

In this way, we have constructed two different non affine conjugate polynomials $P, P'$ which define the same class of minimal trees. Nevertheless, the trees $\mathbf{T}_{P,\mathcal{O}P^{-1}\Omega(P)}$, $\mathbf{T}_{P',\mathcal{O}P'^{-1}\Omega(P')}$ belong to different classes (see Figure 2.2 below).

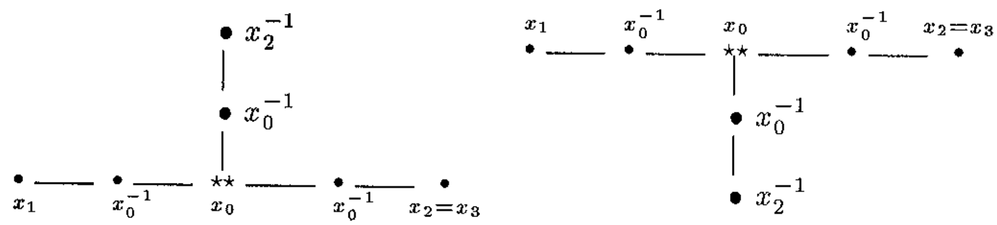



*Figure 2.2. Here $x_j^{-1}$ maps to $x_j$. Even if the trees are isomorphic, they fail to have the same cyclic order around $x_0$.*

**2.12 Non existence** (compare Figure 2.3). The class of the tree below can not be obtained from a polynomial map. It must correspond to a degree two polynomial with three fixed points, which is impossible.

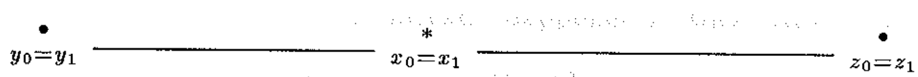



*Figure 2.3. All vertices are fixed. Here $\delta(x_0)=2$, and $\delta(y_0)=\delta(z_0)=1$.*

Here is an alternative description of the obstruction for 'realizing' this tree. If this tree is equivalent to a tree $\mathbf{T}_{P,M}$ of a degree two polynomial $P$, edges whose common vertex is the Fatou critical point must be realized near this vertex as internal rays in the uniformizing coordinate (see Section IV.1.1). Let $\alpha$ be the difference of the two arguments in this coordinate. We have then $2\alpha = \alpha + 1$ (*mod* 1). But this implies that the two segments should be identified. Note that the minimal tree corresponding to this tree has only $x_0$ as vertex. Thus, this minimal tree can be realized as $\mathbf{T}_{z\mapsto z^2,\{0\}}$.

**2.13 Non existence** (compare Figure 2.4). The class of the minimal tree below can not be obtained from a polynomial map. If there is a tree $\mathbf{T}_{P,M}$ in the class of such tree, it will not satisfy the expanding property (compare Propositions IV.1.18 and IV.1.21).

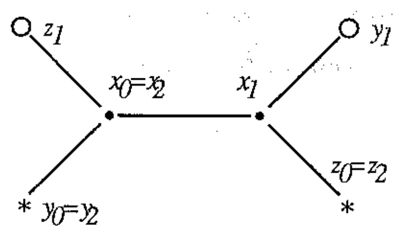



*Figure 2.4. For any $k \geq 0$ there is no vertex between $\tau^{\circ k}(x_0)$ and $\tau^{\circ k}(x_1)$.*

**2.14** All that can go wrong already happened in these three examples. Uniqueness failed because we had too little information. Here *too little information* means that we do not have enough information to recover in a unique way the tree $T^{-1}(M)$ (see Section IV.2, compare also Propositions IV.2.5 and VI.2.5). Examples 2.12 and 2.13 failed because they do not satisfied necessary conditions. Namely, the trees must have well defined angles around Fatou critical points (see Section IV.1) and should satisfy the expanding condition between Julia type vertices (see Proposition IV.1.21).

# 3. Angled Trees.

In this section we introduce the class of trees that will model our results. In this class we must be able to consider an analogue of the expanding condition, and also to define angles between edges near Fatou points.

**3.1 Definition.** An *angled tree* is a pair $\mathbf{A} = (\mathbf{T}, \angle)$, where

(a) $\mathbf{T} = ((T, V, \chi), \tau, \delta)$ is a dynamical abstract tree,

(b) together with a function $\ell, \ell' \mapsto \angle(\ell, \ell') = \angle_v(\ell, \ell') \in \mathbf{Q}/\mathbf{Z}$ which assigns a rational modulo 1 to each pair of edges $\ell, \ell'$ which meet at a common vertex. This angle $\angle(\ell, \ell')$ should be skew-symmetric, with $\angle_v(\ell, \ell') = 0$ if and only if $\ell = \ell'$, and with $\angle_v(\ell, \ell'') = \angle_v(\ell, \ell') + \angle_v(\ell', \ell'')$ whenever three edges are incident at a vertex $v$.

The maps $\angle$, $\tau$ and $\delta$ must be related as follows. Again we extend $\tau$ to a map $\tau : T \to T$ which carries each edge homeomorphically onto the shortest path joining the images of its endpoints. Any extension well defines a map between 'germs' $\tau_v : E_v \to E_{\tau(v)}$. We require then that

$$\angle_{\tau(v)}\big(\tau_v(\ell), \tau_v(\ell')\big) = \delta(v)\angle_v\big(\ell, \ell'\big), \qquad (1)$$

whenever $\ell, \ell' \in E_v$ (in this case $\tau_v(\ell), \tau_v(\ell')$ contain edges incident at $\tau(v)$ where the angle between them is measured).

Such an angle function determines a cyclic order in $E_v$ which we suppose to coincide with $\chi$. Note that in this case the angle function $\angle_v$ at $v$ can be extended to a bigger set $\mathcal{E}_v$ (see §3.3 below).

The degree $deg(\mathbf{A})$ of the angled tree $\mathbf{A} = (\mathbf{T}, \angle)$ is by definition the degree of the abstract tree $\mathbf{T}$. The critical set $\Omega(\mathbf{A})$ of $\mathbf{A}$ is by definition $\Omega(\mathbf{T})$.

**3.2** A vertex $v \in V$ is called *periodic* if for some $m > 0$ we have $\tau^{\circ m}(v) = v$. The orbit of a periodic critical vertex is a *critical cycle.* We say that a vertex $v$ is of *Fatou type (or a Fatou vertex)* if eventually maps to a critical cycle. Otherwise it is of *Julia type (or a Julia vertex).* If $v_0 \mapsto v_1 \mapsto \ldots \mapsto v_m = v_0$ is a critical cycle, we define the *degree of the cycle* as the product $\delta(v_0) \times \ldots \times \delta(v_{n-1})$ of the degrees of the elements in said cycle.

**3.3** The function $\tau$ induces a function $\tau_v$ between the set $E_v$ of edges incident at $v$ and the set $E_{\tau(v)}$ of edges incident at $\tau(v)$. Given a Fatou periodic vertex we can find embeddings $\phi_v \to \mathbf{R}/\mathbf{Z}$ called *local coordinates of the set* $E_v$ (see Appendix B) such that the diagram

$$\begin{array}{ccc} E_v & \xrightarrow{\tau_v} & E_{\tau(v)} \\ \phi_v \downarrow & & \downarrow \phi_{\tau(v)} \\ \mathbf{R}/\mathbf{Z} & \xrightarrow{m_v} & \mathbf{R}/\mathbf{Z} \end{array} \qquad (2)$$

commutes. Here $m_v$ is multiplication by $\delta(v)$ (modulo 1). Note that the number of possible embeddings for each critical cycle is the degree of the cycle minus one.

At other Fatou vertices $v$ we can still make diagram (2) hold by pulling back the local coordinate at $\tau(v)$ and using relation (1).

At periodic Julia vertices relation (1) easily implies that $\tau_v$ is a bijection. We pick an element $\ell \in E_v$ to which we assign the 0 coordinate $(\phi_v(\ell) = 0)$.

If $E_{\tau(v)}$ has not been assigned a local coordinate, we assign to each edge $\tau_v(\ell) \in E_{\tau(v)}$ the argument $\phi_v(\ell)$. In general we can not make diagram (2) commute for all vertices. (It might fail at the starting vertex $v$). In this last case the induced function $m_v$ in $\mathbf{R}/\mathbf{Z}$ becomes translation by some constant.

At non periodic Julia vertices a local coordinate $\phi_v$ can be pulled back from $E_{\tau(v)}$ in $\delta(v)$ different ways such that diagram (2) commutes.

**3.4 Definition.** Let $\mathbf{A} = (\mathbf{T} = ((T, V, \chi), \tau, \delta), \angle)$ be an angled tree. For a finite set of invariant vertices $M \supset \Omega(\mathbf{A})$, we denote by $\mathbf{A}(M) = (\mathbf{T}(M), \angle_M)$ the angled tree generated by $M$, i.e, take $\mathbf{T}(M)$ the dynamical abstract tree determined by $M$ (see section 2.5), and let $\angle_M$ be the restriction of $\angle$ to the vertices of $\mathbf{T}(M)$. Of course, $\mathbf{A} = \mathbf{A}(V)$.

**3.5 Lemma.** *For any extension '$\tau$' and invariant set of vertices $M \supset \Omega(\mathbf{A})$ we have $\tau(T(M)) = [\tau(M)]_T$.*

**Proof.** A copy of Lemma IV.1.10 with the appropriate change of notation (see also Lemma 3.7). #

**3.6 Definition.** Let $\mathbf{A}$ be an angled tree of degree $n$, and let $\Omega(\mathbf{A}) = \{v_1, \ldots, v_l\}$ be the critical set. For a fixed family of local coordinates $\{\phi\}_{v \in V}$, we construct a partition $T^* = T^*(\{\phi\})$ of $T$ consisting (counting possible repetitions) of exactly $n$ subtrees of $T$. This partition will have the property that every point $p \in T$ will belong to exactly $\delta(p)$ elements of $T^*$. Note that this will be possible only if we somehow 'unglue' the tree around every critical point. This is formally done as follows (compare the example within the proof of Proposition VI.2.5).

Let $T_0$ be $\{T\}$. We will inductively define partitions $T_i$ $(i \leq l)$ of $T$ with the following properties

(a) For $j \leq i$, $v_j$ belongs to exactly $\delta(v_j)$ elements of $T_i$,

(b) For $j > i$, $v_j$ belongs to exactly one element of the family $T_i$,

(c) $T_i$ is constructed from $T_{i-1}$ by replacing the unique element $T(\alpha)$ of $T_{i-1}$ to which $v_i$ belongs by $\delta(v_i)$ subtrees of $T(\alpha)$.

We proceed as follows. Let $T(\alpha)$ be the only element of $T_{i-1}$ to which $v_i$ belongs. We partition $T(\alpha)$ into $\delta(v_i)$ pieces as follows. First divide the set $E_i = E_{v_i}$ in $\delta(v_i)$ subsets using the local coordinate. For this we define for $k = 0, \ldots, \delta(v_i) - 1$,

$$E^k = E_i^k = \{\ell \in E_i : \phi_{v_i}(\ell) \in [\frac{k}{\delta(v_i)}, \frac{k+1}{\delta(v_i)})\}$$

Now, we take the union of all branches in a set $E^k$, i.e, define

$$T^k(\alpha) = T(\alpha) \cap \big(v_i \cup \bigcup_{\ell \in E^k} \mathcal{B}_{v_i,T}(\ell)\big).$$

Define now $T_i$ by removing $T(\alpha)$ of the family and including all such $T^k(\alpha)$. By definition $T^*$ is the last partition $T_l$.

**3.7 Lemma.** *Let* $\mathbf{A}$ *be an angled tree. Then the vertex dynamics* $\tau$ *induces a continuous map of* $T$ *into itself, where the restriction to every element (component) of* $T^*$ *is injective.*

**Proof.** (Compare Lemma IV.1.8.) Let $T_\alpha$ be an element of $T^*$. Suppose there are different $p_1, p_2 \in T_\alpha$ so that $\tau(p_1) = \tau(p_2)$. Take a path $\gamma : I \to [p_1, p_2]_T \subset T_\alpha$ joining $p_1, p_2$. As $\tau|_{T_\alpha}$ is locally one to one, the set $\Delta = \{(t_1, t_2) : t_1 < t_2 \text{ and } \tau(\gamma(t_1)) = \tau(\gamma(t_2))\}$ is compact. As we have assumed that this set is not empty we can take $(t_1, t_2) \in \Delta$ with $t_2 - t_1$ minimal. Let $t \in (t_1, t_2)$, then $\tau(\gamma([t_1, t]))$ and $\tau(\gamma([t, t_2]))$, are regulated arcs with the same end points. Therefore they are equal and thus $t_2 - t_1$ is not minimal. #

**3.8 Remark.** As $T^*$ consists of $n = deg(\mathbf{A})$ elements (counting possible repetitions), it follows from the last lemma that for any $p \in \tau(T)$ and any possible extension '$\tau$'

$$\sum_{\{q \in T : \tau(q) = p\}} \delta(q) \leq n.$$

**3.9 Lemma.** *Let $v$ be a periodic Fatou vertex, and $\ell_1, \ell_2 \in E_v$ be different edges. There is an $n \geq 0$ so that $\tau_v^{\circ n}(\ell_1)$ and $\tau_v^{\circ n}(\ell_2)$ belong to different components of $T^*$.*

**Proof.** Let $d > 1$ be the degree of the cycle $v_0 = v \mapsto v_1 \mapsto \ldots \mapsto v_m = v_0$. We write $\phi_v(\ell_1)$ and $\phi_v(\ell_2)$ in base $d$ expansion. If for all $n$, $\tau_v^{\circ n}(\ell_1)$ and $\tau_v^{\circ n}(\ell_2)$ belong to the same component of $T^*$, by construction for all $k > 0$ the integer parts of $m_{\delta(v_k)} \phi_{v_k}(\tau^{\circ k}(\ell_1))$ and $m_{\delta(v_k)} \phi_{v_k}(\tau^{\circ k}(\ell_2))$ are equal. But this implies that $\ell_1 = \ell_2$. #

**3.10 Definition.** (Compare §IV.1.20.) We define the distance $d_T(v, v')$ between vertices as follows. Set $d_T(v, v) = 0$. Otherwise let $d_T(v, v')$ be the number of edges between $v$ and $v'$.

We say that the angled tree $\mathbf{A} = (\mathbf{T}, \angle)$ is *expanding* if the following property is satisfied (see also Propositions IV.1.18 and IV.1.21).

*For any edge $\ell$ whose end points $v, v'$ are Julia vertices there is an $m \geq 1$ such that $d_T(\tau^{\circ m}(v), \tau^{\circ m}(v') > 1$.*

Equivalently, $\mathbf{A}$ is not expanding if and only if *there exists periodic Julia vertices $v, v'$ such that $d_T(\tau^{\circ m}(v), \tau^{\circ m}(v')) = 1$ for all $m \geq 0$.*

**3.11 Lemma.** *An angled tree $\mathbf{A}$ is expanding if and only if for any two periodic Julia vertices $v, v'$ there is an $m \geq 0$ such that $\tau^{\circ m}(v)$ and $\tau^{\circ m}(v')$ belong to different components of $T^*$.*

**Proof.** Suppose $\mathbf{A}$ is not expanding. By definition there are periodic Julia vertices $v, v'$ with $d_T(\tau^{\circ m}(v), \tau^{\circ m}(v')) = 1$ for all $m \geq 0$. As there are no critical points in the orbit of periodic Julia vertices, by construction $\tau^{\circ m}(v)$ and $\tau^{\circ m}(v')$ will be in the same element of $T^*$ for any possible choice of the family $\{\phi_v\}$.

Let now $\mathbf{A}$ be expanding. Suppose there are different Julia vertices $v, v'$ such that $\tau^{\circ m}(v)$,$\tau^{\circ m}(v')$ belong to the same component of $T^*$ for all $m \geq 0$. Among such pairs we can take $v, v'$ periodic and with the property that $d_T(v, v')$ is minimal. By assumption the regulated path $[\tau^{\circ m}(v)\tau^{\circ m}(v')]_T$ is completely contained within a component of $T^*$ for all $m \geq 0$. It follows from Lemma 3.7 that all $[\tau^{\circ m}(v)\tau^{\circ m}(v')]_T$ are homeomorphic. We take $v'' \in [v, v']_T \cap V$ such that $d_T(v, v'') = 1$. As $\mathbf{A}$ is expanding it follows that $v''$ is a periodic Fatou vertex. In this way $E_{v''} \cap [v, v']_T = \{\ell_1, \ell_2\}$ with $\ell_1 \neq \ell_2$. We get a contradiction in applying Lemma 3.9. #

**3.12 Corollary.** *Let* **A** *be an expanding angled tree. The induced angled tree* $\mathbf{A}(M)$ *is expanding for every invariant set of vertices* $M \supset \Omega(\mathbf{A})$. #

**3.13 Lemma.** *Let* **A** *be an expanding angled tree. Given a periodic Julia vertex* $v$*, every component of* $T - \{v\}$ *contains a vertex which belongs to* $\mathcal{O}(\Omega(\mathbf{A}))$.

**Proof.** Suppose that $\mathcal{B}_{v,T}(\ell)$ does not contain a vertex in $\mathcal{O}(\Omega(\mathbf{A}))$ different from $v$ for some $\ell \in E_v$. The relation $\tau_v(\ell) \in E_{\tau(v)}$ determines a cyclic sequence of edges $\ell = \ell_0 \in E_v, \ell_1 \in E_{\tau(v)}, \ldots, \ell_n = \ell_0 \in E_{\tau^{\circ m}(v)} = E_v$. If for some $k < m$ the branch $\mathcal{B}_{\tau^{\circ k}(v),T}(\ell_k)$ contains a critical point, we may assume that $k$ is as big as possible and derive a contradiction by using Lemma 3.7. We assume though that $\mathcal{B}_{\tau^{\circ k}(v),T}(\ell_k)$ does not contains a critical point for all $k$. This implies using again Lemma 3.7 that all $\mathcal{B}_{\tau^{\circ k}(v),T}(\ell_k)$ are homeomorphic with only periodic Julia vertices. Thus, **A** is not expanding. #

## 4. Abstract Hubbard Trees.

The angles at Julia vertices are rather artificial, so we normalize them as follows. If $m$ edges $\ell_1, \ldots, \ell_m$, meet at a periodic Julia vertex $v$, then we assume that the angles $\angle_v(\ell_l, \ell_k)$ are all multiples of $1/m$ (it follows that the angles at periodic Julia vertices convey no information beyond the cyclic order of these $m$ incident edges). Fortunately, this number is preserved under restrictions which contain the orbit of the critical set. This will allow us to give a coherent description.

**4.1 Definition.** By an *abstract Hubbard Tree* we mean an expanding angled tree $\mathbf{H} = (\mathbf{T}, \angle)$ such that the angles at any periodic Julia vertex where $m$ edges meet are multiples of $1/m$.

**4.2** Let $\mathbf{H}, \mathbf{H}'$ be two abstract Hubbard Trees of degree $n = deg(\mathbf{H}) = deg(\mathbf{H}') > 1$. We say that $\mathbf{H}'$ is an *extension* of $\mathbf{H}$ (in symbols $\mathbf{H} \preceq \mathbf{H}'$), if there is an embedding $\phi : T \to T'$ which satisfies the obvious conditions:

(i) $\phi(V) \subset V'$,
(ii) $\tau'(\phi(v)) = \phi(\tau(v))$ and
(iii) $\delta(v) = \delta'(\phi(v))$ for all $v \in V$,
(iv) $\angle_v(\ell, \ell') = \angle'_{\phi(v)}(\phi(\ell), \phi(\ell'))$ for all $\ell, \ell' \in E_v$.

Clearly $\preceq$ is an order relation.

**4.3** Let $\mathbf{H}, \mathbf{H}'$ be two abstract Hubbard Trees of degree $n = deg(\mathbf{H}) = deg(\mathbf{H}') > 1$. We say that $\mathbf{H}'$ is *equivalent to* $\mathbf{H}$ (in symbols $\mathbf{H} \cong \mathbf{H}'$), if $\mathbf{H} \preceq \mathbf{H}'$ and $\mathbf{H}' \preceq \mathbf{H}$.

This determines an equivalence relation between abstract Hubbard Trees. Furthermore, the order relation $\preceq$ well defines a partial order between equivalence classes of abstract Hubbard Trees of degree $n > 1$.

**4.4 Lemma.** *Let* $\mathbf{H}$ *be an abstract Hubbard Tree, and* $M \supset \Omega(\mathbf{H})$ *a finite invariant set of vertices. Then* $\mathbf{H}(M)$ *is an abstract Hubbard Tree and* $\mathbf{H}(M) \preceq \mathbf{H}$.

**Proof.** This follows from Corollary 3.12 and Lemma 3.13. #

**4.5 Proposition.** *Every abstract Hubbard Tree* $\mathbf{H}$ *contains a unique minimal tree* $min([\mathbf{H}])$. *Furthermore, this unique minimal tree is the tree generated by the orbit* $\mathcal{O}(\Omega(\mathbf{H}))$ *of the critical set.*

**Proof.** This follows from Proposition 2.8 and Lemma 4.4. #

**4.6 Remark.** A Postcritically Finite Polynomial $P$ and a finite invariant set $M \supset \Omega(P)$ naturally defines an abstract Hubbard Tree $\mathbf{H}_{P,M} = (\mathbf{T}_{P,M}, \angle)$. To define the angle function we note the following. At Fatou periodic vertices the edges of the tree are by definition segments of constant argument in the Böttcher coordinate (see IV.1.1), we define the angle between two such edges as the difference of their coordinates. For other Fatou points the coordinate can be defined such that the diagram (2) commutes, and we proceed as above. For a Julia set point $v$, $J(P) - \{v\}$ consists of a finite number (say $m$) of components. We define the 'angle' between these components to be a multiple of $1/m$. As edges in the tree correspond locally to some of these components we have an angle function between them. (This procedure is well defined and compatible with the definition above, see Proposition IV.3.3). It is easy to see that the minimal tree $T_{M_0}$ (see IV.1.9) corresponds to the minimal tree $\mathbf{H}_{P,M_0} = min(\mathbf{H}_{P,M})$ of any bigger invariant set $M$.

The main result of this work is the following.

**4.7 Theorem.** *Let* $\mathbf{H}$ *be an abstract Hubbard Tree. Then there is a unique (up to affine conjugation) Postcritically Finite polynomial* $P$*, and an invariant set* $M \supset \Omega(P)$ *such that* $\mathbf{H}_{P,M} \in [\mathbf{H}]$.

**4.8 Theorem.** *Equivalence classes of minimal abstract Hubbard Trees of degree* $n > 1$ *are in one to one correspondence with affine conjugate Postcritically finite polynomials.*

We prove Theorem 4.7 in the next chapter. Theorem 4.8 is an easy consequence of this result and Proposition 4.5.

# Chapter VI
# Realizing Abstract Hubbard Trees.

In this chapter we give the proof of the realization Theorem for Abstract Hubbard Trees (Theorem V.4.7). Our proof depends in the theory of Critical Portraits developed in the first part of this work. In Section 1 we define the class of extensions which do not add any essential information to the tree. We will prove later that every extension belongs to this class (compare Corollary 4.6). Section 2 gives the abstract analogue of §IV.2, where we show that a Hubbard Tree contains all the information required to reconstruct its 'inverse'. Section 3 gives the abstract analogue of §IV.3. In Section 4 we relate the 'accesses to Julia points' with the argument of a possible 'external ray' (compare Theorem B in the introduction). As a consequence of this, we prove that every extension of a Hubbard Tree is canonical in the sense described in Section 1. In Section 5 we associate a Formal Critical Portrait to our Tree. This Critical Portrait is also admissible as shown in Section 6. Finally we prove that the Hubbard Tree associated with this critical portrait is equivalent to the starting one, thus establishing the result. From now on, we omit the trivial case in which $T$ is a single critical vertex.

## 1. Canonical Extensions.

In this Section we define what we call 'canonical extensions'. We will prove in Section 4 that every extension which itself is a Hubbard Tree, is canonical in the sense described here. This fact will allow us later to associate in a natural way a critical portrait to every Hubbard Tree.

**1.1 Definition.** Let $\mathbf{H}_0 \preceq \mathbf{H}_1$ be abstract Hubbard Trees. We say that $\mathbf{H}_1$ is a *canonical extension* of $\mathbf{H}_0$ if for every extension $\mathbf{H} \succeq \mathbf{H}_0$, there is a common extension of $\mathbf{H}$ and $\mathbf{H}_1$. Canonical extensions always exist. By definition every Hubbard Tree is a canonical extension of itself. Our final goal in this direction will be to prove that every extension is canonical (compare Corollary 4.6).

**1.2 Proposition.** *Let* $\mathbf{H}$ *be an abstract Hubbard Tree and* $\omega$ *a periodic Fatou vertex. There is a canonical extension* $\mathbf{H}'$ *of* $\mathbf{H}$ *such that*

*(a)* $E_v = E'_v$ *at all vertices of the original Hubbard Tree* $\mathbf{H}$.

*(b) For every periodic* $\ell \in E'_\omega$ *with end points* $\omega, v$ *in* $\mathbf{H}'$*, the vertex* $v$ *is of Julia type and* $d_{\mathbf{H}'}(\tau^{\circ m}(\omega), \tau^{\circ m}(v)) = 1$ *for all* $m \geq 0$.

*In fact, the underlying topological trees can be chosen to be the same, with only new Julia vertices to be added.*

**Proof.** Suppose the edge $\ell$ has end points $\omega, v$, and its germ is of period $k$. In other words suppose the induced maps $\tau_v$ determine a periodic sequence of edges $\ell_0 = \ell \in E_\omega, \ell_1 \in E_{\tau(\omega)}, \ldots, \ell_k = \ell_0 \in E_{\tau^{\circ k}(\omega)} = E_\omega$. We distinguish two cases.

Suppose $d_{\mathbf{H}}(\tau^{\circ m}(\omega), \tau^{\circ m}(v)) = 1$ for all $m$. If $v$ is of Julia type, condition (b) is already satisfied. If $v$ is of Fatou type then by Lemma V.3.7 all

$\ell_k = [\tau^{\circ m}(\omega), \tau^{\circ m}(v)]_T$ are homeomorphic. In this case we insert a vertex $v_m$ in each $\ell_m$ (if $\ell_m = \ell_l$ then $v_m = v_l$) and define $\tau(v_m) = v_{m+1}$. Then clearly $v_1$ is periodic of period $k$ or $k/2$. The angles at $v_k$ are $1/2$ because two edges will meet now. Note that in this case this is the only possible extension that involves the segments $[\tau^{\circ m}(\omega), \tau^{\circ m}(v)]_T$ and gives an expanding tree.

Otherwise, suppose $d_{\mathbf{H}}(\tau^{\circ m}(\omega), \tau^{\circ m}(v)) > 1$ for some $m \geq 1$. In this case we insert a vertex $v_m$ in each $\ell_m$ as close as possible to $\tau^{\circ m}(\omega)$ (note here that if $\ell_m = \ell_l$ then we must have $v_m \neq v_l$) and define $\tau(v_m) = v_{m+1}$. Clearly $v_1$ is periodic of period $k$. The angles at $v_k$ are $1/2$ because two edges meet now.

The only obstruction to this construction is if condition (b) is already satisfied. Therefore the extension is canonical. #

**1.3 Corollary.** *Every abstract Hubbard Tree has a canonical extension with at least one Julia vertex.* #

## 2. Inverse Hubbard Trees.

We now describe an important type of canonical extension. In the case of the Hubbard Tree $\mathbf{H}_{P,M}$ generated by a polynomial $P$ and an invariant set $M$, the interpretation is simple. We will reconstruct the equivalence class of the abstract Hubbard Tree generated by $P^{-1}M$ starting from $\mathbf{H}_{P,M}$. Thus, this section is the abstract analogue of Section IV.2.

**2.1 Definition.** An abstract Hubbard Tree $\mathbf{H}$ of degree $n > 1$ is *homogeneous* if

(a) $\forall v \in \tau(V), n = \sum_{\{v' \in V: v = \tau(v')\}} \delta(v')$, and

(b) $\Omega(\mathbf{H}) \subset \tau(V)$.

In other words, every vertex with at least one inverse must have a maximal number counting multiplicity (compare Remark V.3.8). Furthermore, all critical vertices must have a preimage. The terminology is justified by the fact (proved below) that the underlying topological tree can be 'chopped' into $n$ pieces; each piece being homeomorphic as a graph to the abstract tree generated by restriction to $\tau(V)$. More formally, $\tau$ establishes a homeomorphism between each of the $n$ elements of $T^*$ (compare Section V.3.6) and the abstract Hubbard Tree $\mathbf{H}(\tau(V))$.

**2.2 Lemma.** *For any election of local coordinate system $\{\phi_v\}_{v \in V}$, each $T_\alpha \in T^*(\{\phi_v\})$ is homeomorphic to $\mathbf{H}(\tau(V))$.*

**Proof.** By Lemma V.3.5 we have $\tau(T) = [\tau(V)]_T$. Also, every $v \in [\tau(V)]_T$ has at most one inverse in $T_\alpha$ by Lemma V.3.7. It follows easily from condition a) that every $v \in [\tau(V)]_T$ should have a unique inverse in $T_\alpha$. The result follows. #

**2.3 Corollary.** *Let $\mathbf{H}$ be an abstract tree of degree $n > 1$, such that $\Omega(\mathbf{H}) \subset \tau(V)$. Then $\mathbf{H}$ is homogeneous if and only if $\#(V) - 1 = n(\#(\tau(V)) - 1)$.*

**Proof.** This follows from Lemma 2.2 and Remark V.3.8. #

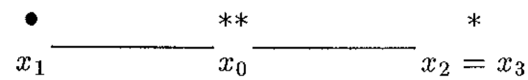
**2.4 Definition.** Let $\mathbf{H}' \preceq \mathbf{H}$ be abstract Hubbard Trees with $\mathbf{H}$ homogeneous. We say that $\mathbf{H}'$ is the image of $\mathbf{H}$ if the embedding which defines the order $\mathbf{H}' \preceq \mathbf{H}$ is such that also $\mathrm{H}(\tau(V)) \cong \mathbf{H}'$.

This definition clearly extends to equivalence classes of abstract Hubbard Trees.

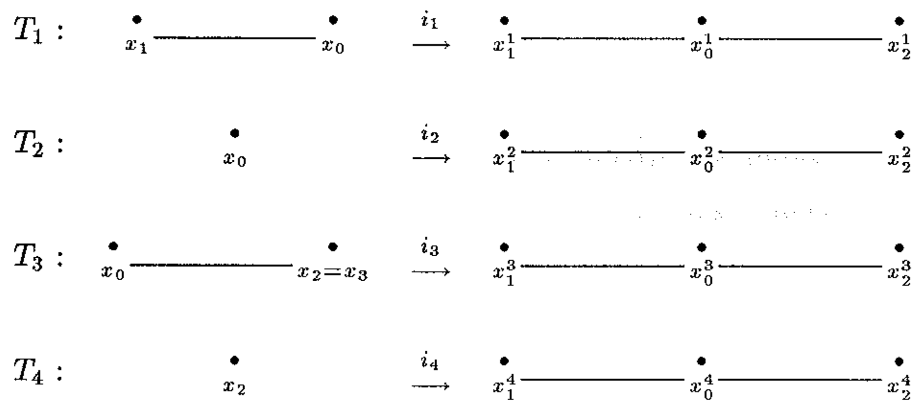
**2.5 Proposition.** *Every equivalence class* $[\mathbf{H}]$ *of abstract Hubbard Trees is the image of a unique class of homogeneous abstract Hubbard Trees.*

**Proof.** The proof of existence is constructive using only necessary conditions, uniqueness follows. Let $\{\phi_v\}_{v \in V}$ be a family of local coordinates for $V \subset T$. We will work with the family $T^* = T^*(\{\phi_v\})$ (compare §V.3.6). We construct a new simplicial complex by gluing a different copy of $T$ to each component $T_\alpha \in T^*$ following $\tau$ (compare Lemma 2.2). In other words we consider $n$ disjoint copies $H^\alpha$ of $T$ ($\alpha = 1 \dots n$), with a suitable identification at "critical points" described below. By Lemma V.3.7, the dynamics $\tau$ restricted to each subtree $T_\alpha$ of the family $T^* = T^*(\{\phi_v\})$ is one to one. We denote this restriction by $i_\alpha$ ("$i$" stands for identification). Thus we have a family of maps $i_\alpha : T_\alpha \to H^\alpha$. We establish an equivalence relation $\sim$ between points in the disjoint union $\coprod H^\alpha$ as follows. Whenever $\omega \in T_\alpha \cap T_\beta$ (and this can only happen if $\omega$ is critical), we write $i_\alpha(\omega) \sim i_\beta(\omega)$. Thus the new underlying topological tree is $X = \coprod H^\alpha / \sim$. There is a 'natural inclusion $T \subset X$' induced by the maps $i_\alpha$. The new set of vertices is the disjoint union of vertices of $H_\alpha$ modulo $\sim$.

In order to avoid confusion in the above notation, we will interrupt the proof in order to exemplify our construction.




The abstract tree in the figure above can be chopped into 4 pieces according to the construction in §V.3.6. We think of these pieces as mapping onto different copies $H_\alpha$ of $T$ (this is emphasized below by the superscripts in the right).




In this way the new tree is given by identifying $x_1^1 = i_1(x_0), x_1^2 = i_2(x_0)$ and $x_1^3 = i_3(x_0)$ (because $x_0 \in T_1 \cap T_2 \cap T_3$) and by identifying $x_2^3 = i_3(x_2)$ with $x_2^4 = i_4(x_2)$ (because $x_2 \in T_3 \cap T_4$). Note that the original tree is canonically embedded in this new one by using $i_\alpha$.

**Proof of 2.5 (Continue).** We continue the proof by defining the dynamics and angle functions. What we have done so far is simply to replace each piece $T_\alpha$ by the copy $H^\alpha$. In this way, if we think of the $H^\alpha$ as the corresponding pieces for the new tree, the final structure is induced by the old one by gluing the $H^\alpha$ following that same pattern of the $T_\alpha$.

The vertex dynamics $\bar{\tau}$ maps each new vertex to the actual point in $V$ from which it was constructed. More formally, take $v \in H^\alpha$ a vertex of $X$; as $H^\alpha$ is also partitioned by the family $\mathbf{T}^*$, it follows that $v \in T_\beta$ for some $\beta$. We define $\bar{\tau}(v) = i_\beta(v) \in H_\beta \subset X$. (Clearly this is well defined and two consecutive vertices have different image). The degree is one at each vertex not present in the original tree. In other words, if $v \in T_\alpha$ (that is if $v$ belongs to the original tree $T$), we define the degree at $i_\alpha(v)$ (which is the point in $X$ to which $v$ is identified) as $\bar{\delta}(i_\alpha(v)) = \delta(v)$. If $v \in X$ is not of the form $i_\beta(\omega)$ for some $\omega$, we set $\bar{\delta}(v) = 1$.

The angle function at non critical points is pulled back from the identification: if $\bar{\delta}(w) = 1$, we have a natural homeomorphism between a neighborhood of $w \in X$ and a neighborhood of $w \in T$. The angle function is then copied from the original Hubbard Tree $\mathbf{H}$. At critical points, it is enough to extend the coordinate functions $\phi_v$ in a compatible way; the angle between edges can be read from this. We proceed as follows. Let $v \in T_\alpha$ be critical. We will define the coordinate $\phi_{i_\alpha(v)}$ at $i_\alpha(v) \in X$ as follows. By definition (compare §V.3.6) there is a $k$ such that $\ell \in E_v$ belongs to $T_\alpha$ if and only if $\phi_v(\ell) \in [\frac{k}{\delta(v)}, \frac{k+1}{\delta(v)})$. Now, an edge $\ell$ incident at $i_\alpha(v)$ must belong to a unique $H_\alpha$ and therefore corresponds to a unique edge $\ell' \in E_{\bar{\tau}(v)}$ in the original tree $T$. Define $\phi_{i_\alpha(v)}(\ell) = \frac{k+\phi_{\tau(v)}(\ell')}{\delta(v)}$.

As no new periodic vertices are added the tree is still expanding. At periodic Julia vertices no new edges are added (compare §V.3.3). Therefore, we have a Hubbard Tree which is homogeneous by Corollary 2.3 and satisfies the required properties.

To prove uniqueness, we note that any other local coordinate system

$\{\phi_v\}_{v\in V}$ is also canonically present in the new tree constructed. It follows from Lemma 2.2 that the corresponding partition with respect to this coordinate is independent of the starting local coordinate system. #

**2.6 Definition.** Let $[\mathbf{H}], [\mathbf{H}']$ be equivalence classes of abstract Hubbard Trees. We say that the equivalence class $[\mathbf{H}']$ of homogeneous abstract Hubbard Trees is the *inverse* of $[\mathbf{H}]$ (in symbols $inv(\mathbf{H}) = \mathbf{H}'$), if $[\mathbf{H}]$ is the image of $[\mathbf{H}']$.

Thus, by Propositions 2.3 and 2.5, *inv* determines a one to one mapping from equivalence classes of abstract Hubbard Trees of degree $n > 1$ to itself. Furthermore, in this new language Proposition 2.5 reads as follows.

**2.7 Proposition.** *Let* $\mathbf{H}$ *be an abstract Hubbard Tree, then* $inv(\mathbf{H})$ *is a canonical extension of* $\mathbf{H}$. #

**2.8 Corollary.** *Let* $\mathbf{H}$ *be an abstract Hubbard Tree and* $\omega$ *a Fatou vertex. There is a canonical extension* $\mathbf{H}'$ *of* $\mathbf{H}$ *such that*

*(a)* $E_v = E'_v$ *at all vertices of* $\mathbf{H}$.

*(b) For every* $\ell \in E'_\omega$ *with end points* $\omega, v$, *we have that* $v$ *is of Julia type, and* $d_{\mathbf{H}'}(\tau^{\circ k}(\omega), \tau^{\circ k}(v)) = 1$ *for all* $k \geq 0$.

**Proof.** We apply first Proposition 1.2 and then take a finite number of 'inverses' (Proposition 2.5). Finally we restrict to the tree generated by the original vertices. #

**2.9 Corollary.** *Let* **H** *be an abstract Hubbard Tree. Then* **H** *has a canonical extension in which all ends are of Julia type.*

**Proof.** We apply first Proposition 1.2 and then take a finite number of 'inverses' (Proposition 2.5). Finally we restrict to the tree generated by the required vertices. #

## 3. Incidence.

In this section we study from the dynamical point of view, how the number of edges incident at a Julia vertex can grow as we take inverses. This section is the abstract analogue of Section IV.3.

**3.1 Definition.** Let [**H**] be an equivalence class of abstract Hubbard Trees. We define the incidence number $\nu_{\mathbf{H}}(v)$ at a vertex $v \in V$ as the number of connected components of $T - \{v\}$ in any underlying topological tree $T$. In the inverse trees $inv^{\circ m}([\mathbf{H}])$ we have also incidence functions $\nu_{\mathbf{H},-m} = \nu_{inv^{\circ m}([\mathbf{H}])}$ at the vertices of $inv^{\circ m}(\mathbf{H})$. By definition $\nu_{\mathbf{H},0}(v) \leq \nu_{\mathbf{H},-1}(v)$ for $v \in V$. Also by construction of $inv(\mathrm{H})$, it follows that $\nu_{\mathbf{H},-1}(v) = \delta(v)\nu_{\mathbf{H},0}(\tau(v))$ for all vertices in $inv(\mathbf{H})$.

**3.2 Proposition.** *Let* [**H**] *be an equivalence class of abstract Hubbard Trees. For every periodic Julia vertex* $v \in V$ *and* $m \geq 0$ *we have* $\nu_{H,0}(v) = \nu_{H,-m}(v)$.

**Proof.** As $\delta(v')=1$, for every point $v' \in \mathcal{O}(v)$, no new edges are added around $v$ in the construction of $inv^{\circ m}([\mathbf{H}])$. (See also Lemma V.3.13.) #

**3.3 Corollary.** *Let* $[\mathbf{H}]$ *be an equivalence class of abstract Hubbard Trees. Let* $v \in V$ *be a Julia vertex such that* $\tau^{\circ k}(v)$ *is periodic. Then for every* $m \geq k$ *we have* $\nu_{H,-k}(v) = \nu_{H,-m}(v)$. #

**3.4 Corollary.** *Let* $[\mathbf{H}]$ *be an equivalence class of abstract Hubbard Trees. There is a* $k \geq 0$ *such that for all* $m \geq k$ *we have* $\nu_{H,-k}(v) = \nu_{H,-m}(v)$ *at every Julia vertex* $v \in V_{\mathbf{H}}$. #

We denote such numbers by $\nu_{\mathbf{H},-\infty}(v)$.

## 4. Accesses and External Coordinates.

In this section we associate to every 'access' at a Julia vertex an argument. This coordinate system will allow us to define extensions with 'reasonable' properties. Combining these two results we prove that every extension of a Hubbard Tree is canonical.

**4.1 Definition.** (Compare Definition V.1.3.) Let $\mathbf{H}$ be an abstract Hubbard Tree. Given $\ell, \ell' \in E_v$ consecutive in the cyclic order, we say that $(v, \ell, \ell')$ is an *access to* $v$ if $\nu_{\mathbf{H},0}(v) = \nu_{\mathbf{H},-\infty}(v)$. If $\nu_{\mathbf{H},0}(v) < \nu_{\mathbf{H},-\infty}(v)$ we say that $(v, \ell, \ell')$ is a *strict pseudoaccess to* $v$ in $\mathbf{H}$. Note that at Fatou vertices there are no possible accesses. Clearly an access at $v$ is periodic if and only if $v$ is periodic. These concepts extend to equivalence classes.

**4.2 Lemma.** *Let* $\mathbf{H}$ *be an abstract Hubbard Tree of degree* $n$*. Then* $\tau$ *induces a degree* $n$ *orientation preserving covering mapping between the pseudoaccesses of the trees* $inv(\mathbf{H})$ *and* $\mathbf{H}$*. Furthermore, accesses in* $inv(\mathbf{H})$ *map to accesses in* $\mathbf{H}$.

**Proof.** If $(v, \ell, \ell')$ is a pseudoaccess in $inv(\mathbf{H})$, by construction $(\tau(v), \tau_v(\ell), \tau_v(\ell'))$ is a pseudoaccess in $\mathbf{H}$. Clearly this is $n$ to 1, and order preserving by construction. The second part is obvious. #

**4.3 Proposition.** *Let* $\mathbf{H}$ *be a homogeneous abstract Hubbard Tree of degree* $n > 1$ *with at least one Julia vertex. There exist an embedding* $\phi_{\mathbf{H}}$ *of the accesses of* $\mathbf{H}$ *into* $\mathbf{R}/\mathbf{Z}$ *such that the induced map between accesses becomes multiplication by* $n$ *(modulo 1). Furthermore* $\phi_{\mathbf{H}}$ *is uniquely defined up to a global addition of a multiple of* $1/(n-1)$.

**Proof.** Instead of proving directly that we can assign an argument to each access of $\mathbf{H}$, we will prove this fact in a larger tree $inv^{\circ m}(\mathbf{H})$, where $m$ is big enough. The result follows then by restriction (compare Lemma B.1.7 and Corollary B.2.8 in Appendix B).

By Lemma 4.2 the induced map between accesses is an orientation preserving covering of degree $n$. In order to be able to assign an argument to each access we must prove that this map is expanding (compare Appendix B). Take two consecutive periodic accesses $\mathcal{A}_i = (v_i, \ell_i, \ell'_i)$ in $\mathbf{H}$ $(i = 0, 1)$.

The idea is to show that for some $m$ big enough, these accesses are not consecutive in $inv^{\circ m}(\mathbf{H})$. As no new periodic vertices are added in the construction of $inv^{\circ m}(\mathbf{H})$, we have no new periodic accesses and the conditions of Lemma B.1.7 are trivially satisfied; this will establish the result. We distinguish between $v_0 = v_1$ and $v_0 \neq v_1$.

If $v_0 = v_1$ then $\ell_0 \prec \ell'_0 \preceq \ell_1 \prec \ell'_1 \preceq \ell_0$ at $E_{v_0}$. It is enough to find an $m \geq 0$ such that $inv^{\circ m}(\mathbf{H})$ has an access in the 'branch' $\mathcal{B}_{v_0, inv^{\circ m}(\mathbf{H})}(\ell'_0)$. If there is a Julia vertex in $\mathcal{B}_{v_0,\mathbf{H}}(\ell'_0)$ this is obvious by Corollary 3.4. If not, $\ell'_0$ has end points $v_0, \omega$ where $\omega$ is a Fatou point. Now the edge $\ell'_0$ corresponds to an argument in the coordinate $\phi_\omega$ at $\omega$; as $\omega$ eventually maps to a critical point, we can find an argument $\theta \neq \phi_\omega(\ell'_0)$ which eventually maps to the same argument as $\phi_\omega(\ell)$ under successive multiplication by $deg_{\tau^{\circ i}(\omega)}$ modulo 1 (compare diagram (2) in §V.3.3). It follows that for some $m$ big enough, there is an $\ell' \in E'_\omega$ such that $\phi_\omega(\ell') = \theta$. The result then follows easily from Corollary 2.8. (Alternatively, we can use Corollary 2.9.)

Now let $v_0, v_1$ be different periodic Julia points. By Lemma V.3.7, for some $m > 0$ there is a vertex $v'$ of $inv^{\circ m}(\mathbf{H})$ in $[v_0, v_1]_T$ for otherwise $\mathbf{H}$ will not be expanding. If $v'$ is a Julia vertex we proceed as above. Otherwise, we let $(v', \ell, \ell')$ be the pseudoaccess (in $inv^{\circ m}(\mathbf{H})$ at the Fatou vertex $v'$) between $\mathcal{A}_0$, $\mathcal{A}_1$ in the cyclic order. We take an argument $\theta$ between $\phi_\omega(\ell)$ and $\phi_\omega(\ell')$ which eventually maps to the same argument as $\phi_\omega(\ell)$ and proceed as in the last paragraph. #

**4.4.** As every abstract Hubbard Tree $\mathbf{H}$ of degree $n > 1$ has a canonical extension satisfying the conditions of Proposition 4.3, we can associate to every access a coordinate compatible with the dynamics. Such map $\phi_{\mathbf{H}}$ is called an *external coordinate.* In practice, this will correspond to the argument of the external ray landing throughout this access.

Now let $\theta \mapsto m_n(\theta) \mapsto \ldots \mapsto m_n^{\circ k}(\theta) = \theta$, be a periodic orbit under the standard $n$-fold multiplication in $\mathbf{R}/\mathbf{Z}$. The question is whether there is a canonical extension of $\mathbf{H}$ at which accesses corresponding to the arguments $\{\theta, m_n(\theta), \ldots, m_n^{\circ k-1}(\theta)\}$ are present. For this we have the following proposition.

**4.5 Proposition.** *Let* $\mathbf{H}$ *be a homogeneous abstract Hubbard Tree with at least one Julia vertex. For any election of external coordinate* $\phi_{\mathbf{H}}$ *and periodic orbit* $\theta \mapsto m_n(\theta) \mapsto \ldots \mapsto m_n^{\circ k}(\theta) = \theta$ *under n-fold multiplication in* $\mathbf{R}/\mathbf{Z}$*, there is a canonical extension of* $\mathbf{H}$ *in which accesses corresponding to* $\{\theta, m_n(\theta), \ldots, m_n^{\circ k-1}(\theta)\}$ *are present.*

**Proof.** Using Corollary 2.8 we may assume that the distance between two Fatou vertices in never equal to 1; and furthermore, whenever the distance between a Fatou and a Julia vertices is one, so is the distance between all their iterates. Also, because of Corollary 2.9 we may assume without loss of generality that no Fatou vertex is an end. We assume that there are no accesses to which we can associate the referred periodic orbit and construct a canonical extension of this tree.

**Case 1.** The easiest way to construct extensions with periodic orbits of period $k$ is whenever there is a Fatou periodic orbit of period dividing $k$. Suppose the total degree of such critical cycle is $d$. In this case, for all arguments of period $k$ under $m_d$ we can include an edge which correspond in local coordinates to this argument and a periodic vertex (if they are not already present). When this is done simultaneously at all Fatou vertices of the cycle we clearly get a new expanding Hubbard Tree. Clearly this construction is canonical. If the required accesses are present in this canonical extension, we are done; otherwise we have to work harder.

To continue the general case, first note that Corollary 3.4 guarantees that for $m$ big enough $\nu_{-m,\mathbf{H}}(v) = \nu_{-\infty,\mathbf{H}}(v)$ at every original vertex $v \in V_{\mathbf{H}}$. We will only keep track of the following information: the original tree $\mathbf{H}$ and all these accesses of $inv^{\circ m}(\mathbf{H})$ at vertices $v \in V_{\mathbf{H}}$ in the original tree (we have 'pruned' the tree $inv^{\circ m}(\mathbf{H})$). In this case if $\ell \in E_v^m$ but $\ell \notin E_v$ (i.e, if the germ $\ell$ at $v$ in the tree $inv^{\circ m}(\mathbf{H})$ is not present in $\mathbf{H}$) we say that the tree $inv^{\circ m}(\mathbf{H})$ was *pruned* at $\ell$.

Let $\{\gamma_1, \ldots, \gamma_\alpha\}$ be the arguments of all such accesses ordered counterclockwise. Working if necessary in a canonical extension, we may further suppose that the Lebesgue measure of $(\gamma_i, \gamma_{i+1})$ is at most $1/n^{2k+2}$. (In fact, we may work in an inverse $inv^{\circ l}(\mathbf{H})$ with $l$ big enough thanks to the expansiveness of $m_n$ in $\mathbf{R}/\mathbf{Z}$.) It follows that $(\gamma_i, \gamma_{i+1})$ contains at most one periodic orbit of period diving $2k$ in its closure. In particular, each $m_n^{\circ i}(\theta)$ belongs to an interval $(\theta_i^+, \theta_i^-)$ with $\theta_i^{\pm}$ strictly preperiodic. It follows that the vertices $v_{\theta_i}^+, v_{\theta_i}^-$ at the respective accesses are not periodic.

Suppose first that $v_{\theta_0}^+ \neq v_{\theta_0}^-$. By further subdividing the tree (for example by taking an extra $k$ inverses and restricting to the vertices in the original underlying topological tree), we may suppose that for any edge $\ell$, the iterated maps $\tau^{\circ i}|_\ell$ are one to one ($i = 1, \ldots k$).

**Case 2.** Suppose $[v^+_{\theta_0}, v^-_{\theta_0}] \subset \tau^{\circ k}([v^+_{\theta_0}, v^-_{\theta_0}])$. It follows from standard techniques for subshifts of finite type that we can canonically extend the vertices of the tree so that it includes an orbit of period $k$ or $k/2$ with $v_{m_n^{\circ i}(\theta)} \in [v^+_{\theta_i}, v^-_{\theta_i}]$. Because $v^\pm_{\theta_i}$ are strictly preperiodic, the expansive condition for the new set of vertices is trivially satisfied . Therefore any access at $v_{m_n^{\circ i}(\theta)}$ belonging to the set $(\theta_i^+, \theta_i^-)$ should have an associated argument of period dividing $2k$. By construction this argument can only be $m_n^{\circ i}(\theta)$.

**Case 3.** Suppose $[v^+_{\theta_0}, v^-_{\theta_0}] \cap \tau^{\circ k}([v^+_{\theta_0}, v^-_{\theta_0}]) = [v_1, v_2]$. Then the vertices $v_1, v_2$ belong to the interval $[v^+_{\theta_0}, v^-_{\theta_0}]$. Now, by hypothesis this last interval contains no vertex of Julia type (for otherwise after completing the accesses at such vertex, we will have that $\theta_0^+$ and $\theta_0^-$ are not consecutive in the cyclic order) and at most one vertex $w$ of Fatou type. It follows that $v_1 = w$ and that $v_2$ equals either $v^+_{\theta_0}$ or $v^-_{\theta_0}$. In either case we get $d(\tau^{\circ k}(v^+_{\theta_0}), \tau^{\circ k}(v^-_{\theta_0})) \geq 3$. However, by assumption this is impossible since $d(\tau^{\circ k}(v^+_{\theta_0}), \tau^{\circ k}(w)) = d(\tau^{\circ k}(v^-_{\theta_0}), \tau^{\circ k}(w)) = 1$ implies $d(\tau^{\circ k}(v^+_{\theta_0}), \tau^{\circ k}(v^-_{\theta_0})) \leq 2$.

**Case 4.** Suppose $[v^+_{\theta_0}, v^-_{\theta_0}]$ intersects $\tau^{\circ k}[v^+_{\theta_0}, v^-_{\theta_0}] = [\tau^{\circ k}(v^+_{\theta_0}), \tau^{\circ k}(v^-_{\theta_0})]$ at an interior vertex $w \in [v^+_{\theta_0}, v^-_{\theta_0}]$. It follows from the preliminary discussion in case 3 that $w$ is a Fatou vertex. This Fatou vertex should be periodic of period dividing $k$ because otherwise $\tau^{\circ k}(w) \neq w$ belongs to $[\tau^{\circ k}(v^+_{\theta_0}), \tau^{\circ k}(v^-_{\theta_0})]$ and therefore $d(\tau^{\circ k}(v^+_{\theta_0}), \tau^{\circ k}(v^-_{\theta_0})) \geq 3$, which can be shown to be impossible as in case 3.

Denote by $\ell_k^\pm$ the edges $[w, \tau^{\circ k}(v^\pm_{\theta_0})]$ with local coordinates $\alpha_k^\pm$ at $w$, and by $\ell_0^\pm$ the edges $[w, v^\pm_{\theta_0}]$ with local coordinates $\alpha_0^\pm$. Clearly $(\alpha_0^+, \alpha_0^-) \subset (\alpha_k^+, \alpha_k^-)$ because this is the only ordering compatible with the order of the accesses. Denote by $d$ the local degree of $\tau^k$ at $w$.

**Claim.** *$m_d$ maps $(\alpha_0^+, \alpha_0^-)$ homeomorphicaly onto $(\alpha_k^+, \alpha_k^-)$.*

In fact, if this is not the case, in some inverse tree there is an edge $\ell' = [w, v']$ with corresponding argument $\phi_w(\ell') \in (\alpha_0^+, \alpha_0^-)$ and with $\tau^{\circ k}(\ell') = \tau^{\circ k}(\ell_0^+)$. It follows that after completing the access at the vertex $v'$ there is an access with corresponding argument $\beta \in (\theta_0^+, \theta_0^-)$ such that $m_n(\beta) = m_n(\theta_{\theta_0^+})$. But this implies that the interval $(\theta_0^+, \theta_0^-)$ has Lebesgue measure at least $1/n^k$, which is a contradiction.

To finish the proof of case 4, we notice that the claim implies that $m_d$ has a fixed point inside the interval $(\alpha_0^+, \alpha_0^-)$. Therefore we are in case 1.

**Case 5.** Suppose the intervals $[v_{\theta_0}^+, v_{\theta_0}^-]$ and $[\tau^{\circ k}(v_{\theta_0}^+), \tau^{\circ k}(v_{\theta_0}^-)]$ have disjoint interiors. In this case we consider the subtree generated by the vertices $v_0^\pm$ and $\tau^{\circ k}(v_0^\pm)$ to notice that there is vertex $v$ strictly contained in the interior of $[\tau^{\circ k}(v_{\theta_0}^+), \tau^{\circ k}(v_{\theta_0}^-)]$. Also there is an edge $\ell$ at this vertex such that $v_{\theta_0}^\pm \in \mathcal{B}(\ell)$ the branch at $\ell$. In fact, this follows from the ordering of accesses. This implies in particular that for some inverse of the tree there is a vertex $v' \in [v_{\theta_0}^+, v_{\theta_0}^-]$ with $\tau^{\circ k}(v') = v$. Also, we can find an edge $\ell'$ at $v'$ which maps locally to $\ell$ under $\tau^k$. If $v'$ is of Julia type, there are consecutive accesses (after completing the accesses) at $v'$ with associated arguments $\theta_{\mathcal{A}}$ and $\theta_{\mathcal{B}}$ such that $\theta \in (\theta_{\mathcal{A}}, \theta_{\mathcal{B}}) \subset (\theta_0^+, \theta_0^-)$. If $v'$ is of Fatou type, there is a Julia vertex $v_1$ in the branch $\mathcal{B}(\ell')$ such that (after restriction to the a tree which only includes this vertex in such branch) there are two consecutive accesses with that property described above. In fact, these two properties follow immediately from the fact that accesses at $v'$ (respectively at $v_1$) map to accesses at $\tau^{\circ k}(v')$ (respectively at $\tau^{\circ k}(v_1)$), and that $(\theta_0^+, \theta_0^-)$ has Lebesgue measure at most $1/n^{2k+2}$.

In either case we have reduced the problem to case 6.

**Case 6.** Suppose now that $v_{\theta_0}^+ = v_{\theta_0}^-$. After taking inverses and restricting if necessary we may suppose that $\tau^{\circ i}(v_{\theta_0}^\pm) = v_{\theta_i}^+$ for $i = 0, \ldots k-1$. Thus, the accesses $\mathcal{A}_i^+$ and $\mathcal{A}_i^-$ with external arguments $\theta_i^+, \theta_i^-$ share an edge $\ell_i$. As there is no further access with argument in $(\theta_i^+, \theta_i^-)$ it follows that some tree $inv^{\circ m}(\mathbf{H})$ was "pruned" at $\ell_i$. In this way, the required extension is achieved by adding the vertices $v_{m_n^{\circ i}}(\theta)$ at the other end of $\ell_i$. Note that the extension is canonical because for any extension including the vertex $v_{\theta_i}^+$, the vertex $v_{m_n^{\circ i}}(\theta)$ should belong to the branch $\ell_i$, and thus, according to Lemma V.3.13 these periodic vertices should be ends. #

**4.6 Corollary.** *Every extension of an abstract Hubbard Tree is canonical.*

**Proof.** Given any extension we assign to every periodic access its canonical argument (compare Proposition 4.3). Then starting with the minimal tree we add all these periodic orbits according to Proposition 4.5. Finally, we take a finite number of inverses and restrict if necessary. #

## 5. From Hubbard Trees to Formal Critical Portraits.

Using canonical extensions we will mimic the constructions from the first part of this work.

**5.1 Extending the tree.** Let $\mathbf{H}$ be an abstract Hubbard Tree of degree $n > 1$. We start with a canonical extension $\mathbf{H}'$ of $\mathbf{H}$ as in Corollary 2.8; i.e, we require from this extension that if $\omega$ is a Fatou point and $\ell \in E_\omega$, then for the endpoints $\omega, v$ of $\ell$ we must have that $v$ is a Julia vertex, and $d_{\mathbf{H}'}(\tau^{\circ k}(v), \tau^{\circ k}(\omega)) = 1$ for all $k \geq 0$.

We fix local coordinates $\{\phi_v\}_{v \in V}$. For any critical cycle we extend the tree by adding an edge and a vertex at every 0 argument (if they are not present). Next, for any Fatou vertex $\omega$ we proceed as follows. Inductively suppose that the 0 edge is present in the local coordinate of $\tau(\omega)$. We insert a new vertex and edge (if they are not present) at every argument of $\phi^{-1}_{\tau(\omega)}(0)$. Then we use Corollary 3.4 to guarantee that pseudoaccesses defined at such points are indeed accesses. We call any extension satisfying the above conditions *supporting* (compare §I.2).

Let $\omega$ be a Fatou vertex, an access $(v, \ell', \ell)$ is said to support $\omega$ if $\ell$ has endpoints $\omega, v$ and $d_{\mathbf{H}}(\tau^{\circ k}(\omega), \tau^{\circ k}(v)) = 1$ for all $k \geq 0$. Clearly $\tau(v, \ell', \ell) = (\tau(v), \tau_v(\ell'), \tau_v(\ell))$ supports $\tau(\omega)$. An access $(v, \ell', \ell)$ which supports the Fatou critical point $\omega$ will be denoted by $\mathcal{D}(\omega, \ell)$

**5.2 Constructing marked accesses.** Let $\mathbf{H}$ be a supporting abstract Hubbard Tree. Using Corollary 3.4 we pick an inverse $inv^{\circ m}(\mathbf{H})$ such that at every $v \in V$ we have $\nu_{\mathbf{H},-m}(v) = \nu_{\mathbf{H},-\infty}(v)$. From this it is easy to chose hierarchic accesses as in §I.2:

For each critical vertex $\omega \in \Omega(\mathbf{H})$ set

$$\Lambda_\omega = \{\ell \in E_\omega : \delta(\omega)\phi_v(\ell) = 0\}$$

(in this case the hierarchic selection is reflected in the choice of a 0 argument in the local coordinate). Let $\Omega(\mathcal{F}) = \{\omega_1^{\mathcal{F}}, \ldots, \omega_l^{\mathcal{F}}\}$ be the set of Fatou critical vertices, and $\Omega(\mathcal{J}) = \{\omega_1^{\mathcal{J}}, \ldots, \omega_k^{\mathcal{J}}\}$ the set of Julia critical vertices. For each $\omega \in \Omega(\mathcal{F})$ we construct $\delta(\omega)$ *marked* supporting accesses to $\omega$ in the following way. Take $\ell \in \Lambda_\omega$ with end points $v_\ell, \omega$; then there is a supporting access to $\omega$ at $v_\ell$ of the form $\mathcal{D}(\omega, \ell) = (v_\ell, \ell', \ell)$. The set of such accesses for all possible $\ell \in \Lambda_\omega$ is by definition $\mathcal{F}_\omega$.

For each $\omega \in \Omega(\mathcal{J})$ we construct $\delta(\omega)$ *marked* accesses in the following way. Take $\ell \in \Lambda_\omega$, then there is an accesses at $\omega$ of the form $\mathcal{E}(\omega, \ell) = (\omega, \ell, \ell')$. The set of such accesses for all possible $\ell \in \Lambda_\omega$ is by definition $\mathcal{J}_\omega$.

Note the slight difference in the construction, at a Julia critical vertex $v$, the marked accesses are at $v$. While for Fatou critical vertices the accesses are taken at the other end of each edge.

In this way we have constructed two families

$$\begin{aligned} \mathcal{F} &= \{\mathcal{F}_{\omega_1}, \ldots, \mathcal{F}_{\omega_l}\} \\ \mathcal{J} &= \{\mathcal{J}_{\omega_1}, \ldots, \mathcal{J}_{\omega_k}\} \end{aligned}$$

of accesses. As these accesses correspond in the external coordinate $\phi_{\mathbf{H}}$ to arguments, we will not distinguish between the accesses and their corresponding argument. In this way we have the following (see §I.3).

**5.3 Proposition.** *The marking $(\mathcal{F}, \mathcal{J})$ is a formal critical portrait.*

**Proof.** This follows directly from the construction. #

There are several trivial consequences of this construction that we want to point out. To simplify notation, the vertex at which an access $\mathcal{C}$ is defined will be denoted by $v_{\mathcal{C}}$. The proof in all cases is the same: by removing the edge $\ell$ we are left with two connected pieces.

**5.4 Lemma.** *Let $\omega$ be a Fatou critical vertex. If $v_{\mathcal{C}} \in \mathcal{B}_{\mathbf{H},\omega}(\ell)$, then for all $\ell' \in \Lambda_\omega - \{\ell\}$ we have $\mathcal{D}(\omega, \ell') \prec \mathcal{C} \preceq \mathcal{D}(\omega, \ell)$.* #

**5.5 Lemma.** *Let $\omega$ be a Julia critical vertex, and $\mathcal{C}$ an access at $v_{\mathcal{C}} \in \mathcal{B}_{\mathbf{H},\omega}(\ell) - \{\omega\}$. Then for any accesses $\mathcal{A}, \mathcal{A}'$ at $\omega$ we have either $\mathcal{A} \prec \mathcal{C} \prec \mathcal{A}'$ or $\mathcal{A}' \prec \mathcal{C} \prec \mathcal{A}$.* #

**5.6 Lemma.** *Suppose $\omega$ is a Fatou critical vertex and let $\ell \notin \Lambda_\omega$. If $\mathcal{C}$ an access at $v_{\mathcal{C}} \in \mathcal{B}_{\mathbf{H},\omega}(\ell)$, then for any $\ell', \ell'' \in \Lambda_\omega$ we have either $\mathcal{D}(\omega, \ell') \prec \mathcal{C} \prec \mathcal{D}(\omega, \ell'')$ or $\mathcal{D}(\omega, \ell'') \prec \mathcal{C} \prec \mathcal{D}(\omega, \ell')$.* #

## 6. From Hubbard Trees to Admissible Critical Portraits.

In this section we prove that the formal critical portrait constructed above is also admissible. For this we must verify conditions $(c.6), (c.7)$ in §I.3.7. We first verify condition $(c.6)$. The verification of condition $(c.7)$, will also show that any polynomial with critical marking $(\mathcal{F}, \mathcal{J})$ has Hubbard Tree equivalent to this starting one. In this way the main Theorem **A** will follow.

**6.1 Proposition.** *The formal critical portrait $(\mathcal{F}, \mathcal{J})$ is an admissible critical portrait.*

**Proof.** This follows from Corollaries 6.4 and 6.9 below. #

**6.2 Lemma.** *Let $\mathcal{A}_i, \mathcal{B}_i$ be accesses at $v_i$ for $i = 1, 2$ with $v_1 \neq v_2$. Then $\{\mathcal{A}_1, \mathcal{B}_1\}$, and $\{\mathcal{A}_2, \mathcal{B}_2\}$ are unlinked.*

**Proof.** This follows from the fact that $\{\mathcal{A}_2, \mathcal{B}_2\}$ are defined in the same connected component of $T - \{v_1\}$. #

**6.3 Lemma.** *Let $\mathcal{A}, \mathcal{A}'$ be periodic accesses. If either $S^+(\mathcal{A}) = S^+(\mathcal{A}')$ or $S^-(\mathcal{A}) = S^-(\mathcal{A}')$, then $v_\mathcal{A} = v_{\mathcal{A}'}$.*

**Proof.** By contradiction suppose $v_\mathcal{A} \neq v_{\mathcal{A}'}$. We distinguish two cases.

Suppose $\tau^{\circ k}|_{[v_\mathcal{A}, v_{\mathcal{A}'}]_T}$ is injective for all $k \geq 1$. In this case there is a periodic Fatou vertex $v \in [v_\mathcal{A}, v_{\mathcal{A}'}]_T$, because otherwise the tree will not be expanding. Let $d > 1$ be the degree of the critical cycle $v_0 = v \to v_1 \ldots \to v_n = v_0$. There are exactly two different edges $\ell, \ell' \in E_v$ contained in $[v_\mathcal{A}, v_{\mathcal{A}'}]_T$. The dynamics of these edges must be periodic by Lemma V.3.7. We write $\phi_v(\ell), \phi_v(\ell')$ in base $d$ expansion. As they are not equal by hypothesis, we may suppose that the first coefficient in the expansions are different. As $d$ is the product of the degrees of the vertices in the cycle, we may suppose then that when multiplying by $\delta(v_0)$ they have different integer part. But in this way by Lemma 5.6 we will have $\pi_0(S^+(\mathcal{A})) \neq \pi_0(S^+(\mathcal{A}'))$. (In fact, for $\epsilon > 0$ small enough, the arguments $\phi_{\mathrm{H}}(\mathcal{A})$ and $\phi_{\mathrm{H}}(\mathcal{A}')$ belong to different connected components of $\mathbf{R}/\mathbf{Z} - \{\phi_{\mathbf{H}}(\mathcal{D}(v,\ell)) : \ell \in \Lambda_v\} = \mathbf{R}/\mathbf{Z} - \mathcal{F}_v$.) But implies that $S^+(\mathcal{A}) \neq S^+(\mathcal{A}')$. If we consider instead of $\phi_v$ the 'coordinate' $1 - \phi_v$ the same reasoning give us $S^-(\mathcal{A}) \neq S^-(\mathcal{A}')$.

Suppose now that $\tau|_{[v_{\mathcal{A}}, v_{\mathcal{A}'}]_T}$ is not locally one to one near $\omega$. If $\omega$ is a Julia critical vertex the result follows from Lemma 5.5. If $\omega$ is a Fatou critical vertex, by Lemma 5.6 we always have $\pi_0(S^-(\mathcal{A})) \neq \pi_0(S^-(\mathcal{A}'))$ and thus $S^-(\mathcal{A}) \neq S^-(\mathcal{A}')$.

If neither $\mathcal{A}$ nor $\mathcal{A}'$ support $\omega$, again by Lemma 5.6 $\pi_0(S^+(\mathcal{A})) \neq \pi_0(S^+(\mathcal{A}'))$. We start though by assuming that $\mathcal{A}$ is a marked access associated with $\omega$. By Hypothesis there is a preperiodic marked access $\mathcal{C} \in \mathcal{F}_\omega$ (and therefore such that $\tau(\mathcal{C}) = \tau(\mathcal{A})$) with $v_{\mathcal{C}} \in [v_{\mathcal{A}}, v_{\mathcal{A}'}]_T$. Thus $\tau^{\circ k}|_{[v_{\mathcal{C}}, v_{\mathcal{A}'}]_T}$ eventually maps into $[v_{\mathcal{A}}, v_{\mathcal{A}'}]_T$. It follows there is a point $\omega' \in [v_{\mathcal{C}}, v_{\mathcal{A}'}]_T$ that eventually maps to $\omega$. Working if necessary in a canonical extension $inv^{\circ k}(\mathbf{H})$ we may assume without loss of generality that $\omega' \in V$. But then by Lemma V.3.7 for some $i \geq k$, $\tau^{\circ i}|_{[\omega, \omega']_T}$ is not locally one to one near some point $\omega''$. If $i$ is minimal, neither of the periodic accesses $\tau^{\circ i}(\mathcal{A}) = \tau^{\circ i}(\mathcal{C})$ nor $\tau^{\circ i}(\mathcal{A}')$ can support the critical point $\tau^{\circ i-1}(\omega'')$ if it is of Fatou type. It follows from the previous reasoning that $S^+(\tau^{\circ i-1}(\mathcal{A})) \neq S^+(\tau^{\circ i-1}(\mathcal{A}'))$, and therefore $S^+(\mathcal{A}) \neq S^+(\mathcal{A}')$. #

**6.4 Corollary.** *The formal critical portrait $(\mathcal{F}, \mathcal{J})$ satisfies condition (c.6).*

**Proof.** Let $\mathcal{A}$ be a periodic marked access. Suppose there is a periodic argument $\lambda$ such that $S^+(\lambda) = S^+(\mathcal{A})$. By Proposition 4.5 we can assume that there is an accesses corresponding to $\lambda$. By Lemma 6.3 this access is supported at $v_{\mathcal{A}}$. By Lemma 5.4 this access can only be $\mathcal{A}$. #

**6.5 Lemma.** *Let $v_{\mathcal{A}} = v_{\mathcal{A}'}$ be a non critical Julia vertex. Then $\mathcal{A}$ and $\mathcal{A}'$ have the same left address, i.e, $\pi_0(S^-(\mathcal{A})) = \pi_0(S^-(\mathcal{A}'))$.*

**Proof.** If $\mathcal{E}, \mathcal{E}'$ are marked accesses associated to the same Julia critical vertex, Lemma 6.2 implies that $\{\mathcal{A}, \mathcal{A}'\}$,$\{\mathcal{E}, \mathcal{E}'\}$ are unlinked.

If $\mathcal{D}, \mathcal{D}'$ are marked accesses associated to the same Fatou critical vertex, we distinguish if $v_{\mathcal{A}}$ equals $v_{\mathcal{D}}$ or not. If $v_{\mathcal{A}} \neq v_{\mathcal{D}}, v_{\mathcal{D}'}$ then clearly $\{\mathcal{A}, \mathcal{A}'\}$,$\{\mathcal{D}, \mathcal{D}'\}$ are unlinked because the regulated path $[v_{\mathcal{D}}, v_{\mathcal{D}'}]_T$ does not contain $v_{\mathcal{A}}$. If $v_{\mathcal{A}} = v_{\mathcal{D}}$ then by Lemma 5.5 $\mathcal{D}' \prec \mathcal{A} \prec \mathcal{A}' \preceq \mathcal{D}$.

All these facts together mean by definition that the accesses $\mathcal{A}$ and $\mathcal{A}'$ have the same left address, i.e, $\pi_0(S^-(\mathcal{A})) = \pi_0(S^-(\mathcal{A}'))$. #

**6.6 Lemma.** *Let $\mathcal{B}$ be an access at a Julia critical vertex $v$. Then there is a marked access $\mathcal{E}$ at $v$, such that $\pi_0(S^-(\mathcal{E})) = \pi_0(S^-(\mathcal{B}))$.*

**Proof.** Take consecutive $\mathcal{E}, \mathcal{E}'$ marked accesses at $v$, such that $\mathcal{A}' \prec \mathcal{E} \preceq \mathcal{A}$. Using Lemma 6.2 and the same reasoning as in Lemma 6.5 we get $\pi_0(S^-(\mathcal{A})) = \pi_0(S^-(\mathcal{E}))$. #

**6.7 Corollary.** *Suppose $\pi_0(S^-(\mathcal{A})) = \pi_0(S^-(\mathcal{A}'))$. Then $v_{\mathcal{A}} = v_{\mathcal{A}'}$ if and only if $v_{\tau(\mathcal{A})} = v_{\tau(\mathcal{A}')}$.*

**Proof.** One direction is obvious. On the other hand, we may assume that $v_{\tau(\mathcal{A})}$ has $n$ inverses in the tree counting multiplicity. As there are only

$n$ possible choices of addresses, the result follows combining Lemmas 6.3, 6.5, 6.6. #

**6.8 Proposition.** *$v_{\mathcal{A}} = v_{\mathcal{A}'}$ if and only if $S^-(\mathcal{A}) \sim_l S^-(\mathcal{A}')$.*

**Proof.** First suppose $S^-(\mathcal{A}) \sim_l S^-(\mathcal{A}')$. It is enough to prove that if $S^-(\mathcal{A}) \approx S^-(\mathcal{A}')$ then $v_{\mathcal{A}} = v_{\mathcal{A}'}$. If $S^-(\mathcal{A}) = S^-(\mathcal{A}')$ this follows from Lemma 6.3 and Corollary 6.7. In the other case the result follows from this fact, Lemma 6.6 and again Corollary 6.7.

Suppose now $v_{\mathcal{A}} = v_{\mathcal{A}'}$. Let $m \geq 0$ be the smallest integer such that $\tau^{\circ m}(v_{\mathcal{A}})$ does not contain in its forward orbit a critical vertex. The proof will be in induction in $m$. For $m = 0$ this is Lemma 6.5. Suppose now that the result holds for $m-1$. This implies that all accesses at $\tau(v_{\mathcal{A}})$ have equivalent symbol sequences. If $v$ is not critical we use again Lemma 6.5. If $v$ is critical we use Lemma 6.6. #

**6.9 Corollary.** *The formal critical portrait $(\mathcal{F}, \mathcal{J})$ satisfies condition (c.7).* #

## 7. Proof of the Theorem A.

The admissible critical portrait $(\mathcal{F}, \mathcal{J})$ determines a unique (up to affine conjugation) polynomial $P$ with marking $(P, \mathcal{F}, \mathcal{J})$ by Theorem I.3.9. By Propositions 6.8 and I.3.12 its Hubbard tree is the starting one. The angle function at Fatou vertices are the starting ones because of Proposition 2.7, and Corollaries 2.8 and B.2.5 #

# Appendix A
# Thurston's Topological Characterization of Rational Maps.

Let $f : S^2 \mapsto S^2$ be an orientation preserving branched covering map of the topological sphere. The set $\Omega(f)$ of all critical points of $f$ is called the *critical set of f*. The *postcritical set of f* is the set $P(\Omega(f)) = \bigcup_{n=1}^{\infty} f^{\circ n}\Omega(f)$. Whenever the set $P(\Omega(f))$ is finite, we say that $f$ is *postcritically finite.*

In what follows, we assume always that $f$ is postcritically finite. A finite invariant set $M$; i.e, $f(M) \subset M$, containing all critical points of $f$ is called a *marked set.* In analogy with the previous notation, we set $P(M) = \bigcup_{n=1}^{\infty} f^{\circ n}M$, and call it a *postmarked set.* The elements of $M$ (respectively $P(M)$) are called *marked points* (respectively *postmarked points*). We say that $(f, M)$ is a *marked branched map.*

Two marked branched maps $(f, M(f))$ and $(g, M(g))$ are *Thurston equivalent* if there are homeomorphisms $\phi_1, \phi_2 : S^2 \to S^2$, isotopic relative to the set $P(M(f))$ such that $g \circ \phi_1 = \phi_2 \circ f$, and $\phi_1(P(M(f))) = \phi_2(P(M(f))) = P(M(g))$.

We say that a simple closed curve $\gamma \subset S^2 - P(M)$ is *non-peripheral* (for the marked branched map $(f, M)$), if each component of $S^2 - \gamma$ contains at least two points of $P(M)$. A *multicurve* $\Gamma = \{\gamma_1, ..., \gamma_n\}$ is a set of simple, closed, disjoint, non-homotopic, non-peripheral curves in $S^2 - P(M)$. A multicurve $\Gamma$ is *stable*, if for every $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic (relative to $P(M)$) to a curve in $\Gamma$.

Let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to $\gamma_i$ relative to $P(M)$, and $d_{i,j,\alpha}$ be the degree of the map $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \mapsto \gamma_j$. We define the $(i,j)$ entry of the *Thurston Matrix* $f_\Gamma$ as

$$(f_\Gamma)_{i,j} = \sum 1/d_{i,j,\alpha}.$$

Note that by the Perron-Frobenious theorem there is a largest positive eigenvalue $\lambda(f_\Gamma)$.

There is a smaller function $\nu : P_M \mapsto \{1, 2, ..., \infty\}$, such that for all $x \in f^{-1}(y)$, $\nu(y)$ is a multiple of $\nu(x)deg_x f$. We have that the *orbifold* $(S^2, P_M, \nu_f)$ is *hyperbolic* if its *"Euler characteristic"* satisfies

$$2 - \sum_{x \in P(M)} (1 - 1/\nu_f(x)) < 0.$$

Note that we are allowing extensions of the critical and postcritical sets. This is because we what to use Thurston's theorem in more generality than presented in [DH2] and used in [F] or [BFH]. Our marked set is the usual one and maybe a finite number of additional periodic or preperiodic orbits. Note that at these additional points, the orbifold function has value 1, so that the orbifold structure is only determined by the original postcritical set $P(\Omega(f))$.

**A.1 Theorem (Thurston's Characterization of Rational Maps).** *A marked branched map, with hyperbolic orbifold is equivalent to a rational function if and only if for any stable multicurve* $\Gamma$, *we have* $\lambda(f_\Gamma) < 1$. *In this case the rational function is unique up to conjugation by a Mobius transformation.*

**Proof.** The proof in [DH2] applies without modification. #

**A.2 Topological Polynomials.** A branched map $f : S^2 \mapsto S^2$ is said to be a *topological polynomial* if $f^{-1}(\infty) = \infty$.

If we are interested only in topological polynomials Thurston's theorem is equivalent to the following (see [BFH Theorem 3.2]).

**A.3 Theorem.** *A marked topological polynomial (f,M) is equivalent to a polynomial if and only if for any stable multicurve* $\Gamma$ *we have* $\lambda(f_\Gamma) < 1$. *In this case, the polynomial is unique up to conjugation by an affine transformation.*

**Definition.** A stable multicurve $\Gamma$, with $\lambda(f_\Gamma) \geq 1$ is called a *Thurston Obstruction (for (f,M)).*

## Levi Cycles

Everything here is taken from [BFH] section 4.

Let $(f, M)$ be a marked topological polynomial. Let $\Gamma$ be a stable multicurve. Suppose there exists $\{\gamma_0, ..., \gamma_k = \gamma_0\} = \Lambda \subset \Gamma$ such that for each $i = 0, ..., k-1$, $\gamma_i$ is homotopic relative to $P(M)$ to exactly one component $\gamma'$ of $f^{-1}(\gamma_{i+1})$. Suppose also that $f : \gamma' \mapsto \gamma_{i+1}$ has degree 1. Then $\Lambda$ is called a *Levy cycle.*

**A.4 Theorem.** *If a marked topological polynomial (f,M) has a Thurston obstruction $\Gamma$, then (f,M) has a Levy cycle.*

**A.5 Theorem.** *The disks of the elements of $\Lambda = \{\gamma_0, ..., \gamma_k = \gamma_0\}$ (i.e, the bounded components of $S^2 - \gamma_i$), contain only cycles of periodic non-critical points of $P(M)$.*

The last two Theorems together have an interesting interpretation.

*For Post-critically finite topological Polynomials, only misidentification of periodic points can lead to an obstruction.*

# Appendix B
# Finite Cyclic Expanding Maps.

## 1. Expanding Maps.

We consider a finite cyclic set $X$, and a degree $n \geq 2$ orientation preserving map $f : X \mapsto f(X) \subset X$. We will study under which conditions we can assign an *argument* $\phi(p)$ to every point $p \in X$ such that the induced map becomes multiplication by $n$.

**1.1.** Let $k \geq 1$ and $n \geq 2$. Consider a finite cyclicly ordered set $X = \{p_1, \ldots, p_{kn}\}$ with $kn$ elements. The *cyclic order* can be realized as a *successor* function $Suc_X(p_i) = p_{i+1}$ with the convention $p_{kn} = p_0$. Given $Y \subset X$ there is an induced order in $Y$, and therefore a successor function $Suc_Y : Y \mapsto Y$. We consider a degree $n \geq 2$ orientation preserving map $f : X \mapsto f(X) \subset X$. By this we mean a function $f$ with the property that $f(p_i) = f(p_j)$ if and only if $i \equiv j$ *(mod k)*; and such that $f(Suc_X(p)) = Suc_{f(X)}(f(p))$. It follows that $f$ is an $n^{th}$-fold cover of its image. Note that because $f$ is a degree $n$ cover and order preserving, for every $p \in X$, the restriction of $f$ to the set $\{p, Suc_X(p), \ldots, Suc_X^{\circ k-1}(p)\}$ is one to one and onto $f(X)$.

Given a cyclicly ordered set $X$ as above, we define the *ordered distance* $d_X(p_1, p_2)$ *between two points* $p_1, p_2 \in X$, as the minimal $m$ for which $p_2 = Suc^{\circ m}(p_1)$. Thus, the ordered distance between two points is always less than $kn$. It follows easily that $f(p_1) = f(p_2)$ if and only if $d_X(p_1, p_2)$ is a multiple of $k$.

Given three points $p_1, p_2, p_3$ and numbers $0 \leq m \leq m' < kn$, with $m = d_X(p_1, p_2)$, $m' = d_X(p_1, p_3)$, we write $p_1 \leq p_2 \leq p_3$. If in addition $m < m'$ we write $p_1 \leq p_2 < p_3$.

**1.2 Lemma.** *Suppose* $p_1 \leq p_2 \leq p_3 < p_1$. *Then* $d_X(p_1, p_3) = d_X(p_1, p_2) + d_X(p_2, p_3)$.

**Proof.** Completely trivial. #

**1.3 Remark.** Even if we are considering two orders (one in $X$ and that induced in $f(X)$), we will only be considering the ordered distance of $X$. In other words if $p_1, p_2 \in f(X)$, the ordered distance $d_X(p_1, p_2)$ is always measured in $X$.

**1.4 Definition.** We say that $f : X \to X$ as above is *expanding*, if given $p_1, p_2$ periodic

($\star$) *there exists* $l \geq 0$ *such that* $d_X(f^{\circ l}(p_1), f^{\circ l}(p_2)) \neq 1$.

In other words, if two periodic points are consecutive, the distance between them eventually increases. From the facts that $d_X(p_1, p_2) < k$ implies

$f(p_1) \neq f(p_2)$, and every point is eventually periodic, we can easily deduce that for an expanding map, condition *($\star$)* is also satisfied for every pair of different points.

**1.5** Given a finite cyclicly ordered set $X$ and a degree $n \geq 2$ orientation preserving map $f$, we say that $f : X \to X$ can be *angled* if there is an order preserving embedding $\phi : X \mapsto \mathbf{R}/\mathbf{Z}$, such that $n\phi(p) \equiv \phi(f(p))(mod\ 1)$. Of course, an angled function is expanding.

**1.6 Remark.** If we reverse the order in all the definitions above (i.e, if we replace the successor function by a predecessor function $Pre_X$), all the definitions above make sense. In particular if $\phi_S$,$\phi_P$ are the angle functions for these two orders then clearly $\phi_S + \phi_P \equiv 1$.

**1.7 Proposition.** *Let $X$ be a finite cyclic set and $f$ an orientation preserving degree $n \geq 2$ map. Then $f : X \to X$ is angled if and only if is expanding.*

**Proof.** Being angled implies being expanding as remarked above. We prove the converse in several steps.

*Step 1:* We can assume without loss of generality that $f$ has a fixed point. In fact, if there is no fixed point, then $f(X)$ has at least two elements. We define a function $g : X \to \{1, \ldots, kn-1\}$ by the formula $g(x) = d_X(x, f(x))$. It follows easily from Lemma 1.2 that whenever $i \equiv j$ *(mod k)* then $g(x_i) \equiv g(x_j) + d_X(x_i, x_j)$ *(mod kn)*. Therefore for $y \in f(X)$, there is a unique

$x_i \in f^{-1}(y)$ for which $k(n-1) < g(x_i) < kn$ (in fact, $g(x_i) = k(n-1)$ would imply that $x_{i+k}$ is a fixed point). Let $d$ be the maximum of $g$. Among all $x$ with $g(x) = d$ take one for which $g(Suc_X(x)) < d$. It follows easily that the cyclic order can be written as

$$f(x) < x < Suc_X(x) < f(Suc_X(x)) = Suc_{f(X)}(f(x)) < f(x).$$

To simplify notation, we rewrite $X$ as $\{p_0 = x, p_1, \ldots, p_{kn-1}\}$. We insert a new point $q_i$ between every pair $p_{ki}$ and $p_{ki+1}$. All of this new points will be mapped to $q_0$. In this way, we have a degree $n \geq 2$ orientation preserving map which is an extension of the original one.

We must verify that this map is expanding. The only new periodic point included is $q_0$. The expanding property is obviously verifies if $Suc_X(q_0)$ is periodic: if $d_X(f(q_0), f(Suc_X(q_0))) = 1$ then $Suc_X(q_0)$ is a fixed point, in contradiction to what was assumed. If $Pre_X(q_0)$ is periodic the result follows analogously.

*Step 2:* We assign an argument to each point in $X$ as follows. Let $q_0 < q_1 < q_{n-1} < q_0$ be all points which map to the fixed point $q_0$. We assign to $q_i$ the argument $i/n$ for $i = 0, \ldots, n-1$. For an arbitrary point $x \in X$, we dynamically find its numerical expansion in base $n$.

*Step 3: The assignment is order preserving.* Because the function is $n$ to one order preserving, we may introduce inverse iterates of the fixed point. Thus, we may assume that given $m$ there are in the cyclic order different values $\{q_0 = q, \ldots, q_{m^n-1}\}$ with the property that $f^{\circ m}(q_i) = q_0$. Taking $m$ big enough the result follows.

*Step 4: Different points are assigned different arguments.* Consider a set $\{x_1, \ldots, x_l\}$ of maximal cardinality to which equal periodic base $n$ expansion is associated. Clearly all $x_i$ are periodic. Furthermore, if $l > 1$ we have for all $m \geq 0$ $d_X(f^{\circ m}(x_1), f^{\circ m}(x_2)) = 1$ because of maximality. But this contradicts the expanding condition. There is a case in which this argument does not apply. Suppose that in applying step 2, there is an argument to which the decimal expansion $0.n-1, n-1, \ldots$ is assigned. In this case we reverse the order, and apply the same argument to derive a contradiction. #

## 2. Finding the Coordinates.

Consider an integer $n > 1$, and denote by $m_n$ multiplication by $n$ modulo 1. From the dynamical point of view the election of 0 as the origin is arbitrary in the sense that any dynamically property present at a point $x \in \mathbf{R}/\mathbf{Q}$, is also present at $x + j/(n-1)$. In this way, with the knowledge of the dynamical behavior of a point $x$, the natural question is not which is the value of $x$, but that of $m_{n-1}(x)$.

**2.1** For $n > 1$ define $\delta_n : \mathbf{R}/\mathbf{Z} \to \{0, \ldots, n-1\}$ by

$$\delta_n(x) = i \qquad \text{if} \qquad m_{n-1}(x) \in [\frac{i}{n}, \frac{i+1}{n}).$$

In other words, if we take $m_{n-1}(x)$, we define $\delta_n(x)$ as the integer part of $n(m_{n-1}(x))$.

**2.2 Remark.** It follows that $\delta_n(x)$ is the number of inverses of $m_n(x)$ (other than $x$) in the cyclicly counterclockwise oriented interval $(x, m_d(x))$ (if $x$ is fixed this interval is interpreted to be empty). To see this, we rewrite $m_{n-1}(x)$ as $m_n(x) - x$ *(mod n)*. In this way, $\delta_n(x)$ counts the number of intervals of size $1/n$ to be found in $(x, m_d(x))$. The claim follows easily.

**2.3 Example.** Consider with $n = 3$ the point $x = 1/5$. We have $m_2(x) = 2/5$ and $1 \leq 3(2/5) < 2$; so by definition $\delta_3(1/5) = 1$. Note also that $\delta_3(1/5 + 1/2) = 1$, which is not a surprise because $m_2(x) = m_2(x + 1/2)$ for all $x \in \mathbf{R}/\mathbf{Z}$.

**2.4 Lemma.** *Let* $n > 1$ *and* $x \in \mathbf{R}/\mathbf{Z}$, *then*

$$m_{n-1}(x) = \frac{1}{n} \sum_{i=0}^{\infty} \frac{\delta_n(m_n^{\circ i}(x))}{n^i}.$$

**Proof.** We successively subdivide the interval $I_j = [\frac{j}{n-1}, \frac{j+1}{n-1})$ in $n$ semiopen intervals. This determines a parametrization of the interval $I_j$ by symbol sequences in the symbol space $\{0, \ldots, n-1\}$ (not allowing any symbol sequence with tail $(n-1, n-1, \ldots)$). A point $x \in I_j$ has symbol sequence $S_0, S_1, \ldots$ if and only if $x = \frac{j}{n-1} + \frac{1}{(n-1)n} \sum_{i=0} \frac{S_i}{n^i}$. Therefore, $m_{n-1}(x) = \frac{1}{n} \sum_{i=0}^{\infty} \frac{S_i}{n^i}$ (and all reference to the initial interval $I_j$ is lost). The result follows as $S_i = \delta_n(m_n^{\circ i}(x))$ by construction. #

**2.5 Coordinates for expanding maps.** We return to the case described in §1. It follows by definition of covering map and Remark 2.2 that $\delta_n(x)$ equals the integer part of $\frac{d_X(x,f(x))}{k}$. Thus, according to Lemma 2.4, $m_{n-1}(x)$ is independent of the coordinate assigned in Proposition 1.7. Furthermore, we have proved the following.

**2.6 Theorem.** *Let $X$ be a finite cyclic ordered set and $f : X \mapsto X$ be a degree $n$ orientation preserving expanding map. Then $f$ can angled in exactly $n-1$ ways.*

**2.7 Example.** (Compare Figure B.1.) Let $X$ be the cyclic set shown in Figure $B$.1. (The notation is justified by the dynamics.) We consider a map $f : X \to X$ for which

$$f(A) = f(A') = f(A'') = B$$
$$f(B) = f(B') = f(B'') = C$$
$$f(C) = f(C') = f(C'') = D$$
$$f(D) = f(D') = f(D'') = A$$

The unique periodic orbit is given by $A \mapsto B \mapsto C \mapsto D \mapsto A$. This map is clearly expanding. According to Remark 2.5, we have that $\delta(A) = 1$, $\delta(B) = 0$, $\delta(C) = 1$, and $\delta(D) = 2$. Using Lemma 2.4, we can easily find the base 3 expansions of $m_2(A) = 0.\overline{1012}$. It follows that $m_2(A) = 2/5$, and therefore A takes value either $1/5$ or $7/10$.

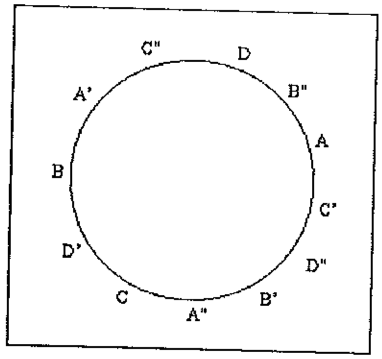



*Figure B.1*

**2.8 Corollary.** *Let $x$ be periodic under $m_n$, and denote by $\mathcal{O}(x)$ its orbit. Then $m_{n-1}(x)$ is uniquely determined by the cyclic order of $m_n^{-1}(\mathcal{O}(x))$.*

**Proof.** This follows directly from Remark 2.2 and Lemma 2.4. (Compare also Example 2.7).

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