K-Theory Index of Dirac Extensions with Periodic Multipliers on a Universal Cover

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Peter Armand Miegom

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Abstract of the Dissertation

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After defining an adjoined Dirac extension with periodic multipliers on the universal cover of a compact riemannian spin manifold, we show that its index maps in K-theory are 1-1 in the case where M is odd-dimensional and has nonpositive curvature. Moreover, a connection is established between these index maps and the Thom isomorphisms in K-theory. A rough index theorem is also established.

In the special case where the base manifold is the flat odd-dimensional m-torus, the index maps are shown to be isomorphisms.

Dedicated to the survival of the land, the waters, the animals, the plants, and the people.

Contents

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Chapter 1

Introduction

In [**BmDg3**], a Toeplitz extension is constructed for each self adjoint, elliptic, pseudodifferential operator A on a compact riemannian manifold M. The index map for this extension is the Fredholm index of certain Fredholm operators and in [**BmDg3**], a formula is given for this analytic index in terms of topological data related to M and to the symbols of A and T_φ.

We study in this thesis, a certain Toeplitz-like extension on the universal cover of a compact riemannian spin manifold M. The multipliers chosen for this Toeplitz-like extension, are those continuous functions on the universal cover which lift from continuous functions on M. The index maps of this extension are not ordinary Fredholm index since the ideal in the extension is not the algebra of compact operators.

In the case where M is the flat odd-dimensional torus $\mathbf{T}^m$, the ideal $\mathcal{L}$ in this extension is isomorphic to the algebra of $k \times k$ matrices over $C(\mathbf{T}^m) \times_\alpha \mathbf{R}^m$ where α is the action of $\mathbf{R}^m$ on $C(\mathbf{T}^m)$ given by translations. The index maps in this case are isomorphisms.

The apparent similarity between these index maps and the Connes-Thom Isomorphism in the special case $M = \mathbf{T}^m$, suggested to the author an approach to studying the general case involving the construction of a "Wiener-Hopf" extension analogous to the Wiener-Hopf construction used in Rieffel's proof of the Connes-Thom Isomorphism.

In order to carry out this construction, assumptions of nonpositive curvature on M are used. With these assumptions, it is shown that the index maps obtained are $1-1$. Also, a rough index theorem is obtained connecting these index maps with the topological Thom isomorphism.

In Chapter 2, we look at index maps of extensions involving multipliers vanishing at ∞. This is to be used later in Chapter 7.

In Chapter 3, 4, and 5, the "Wiener-Hopf" extension is constructed. In Chapter 6, the index maps of the slanted-cone Thom extension is shown to be the same as the topological Thom isomorphisms. In Chapter 7, the main theorem is proved and the special case of $M = \mathbf{T}^m$ is treated.

Chapter 2

Dirac Extensions

2.1 Flip Functions and the Fourier Transform

Notation 1.1 *Let $\mathcal{S}$ denote the Schwartz space on* $\mathbf{R}$, $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ *the Fourier transform. If $f \in \mathcal{S}$ $\hat{f}$ will denote the Fourier transform of f. If M is a differentiable manifold, $C_c^\infty(M)$ will denote the algebra of C^∞ functions on M with compact support, and $C_0^\infty(M)$ the algebra of C^∞ functions on M which vanish at infinity.*

Also, let $f^-(x) = f(-x)$ for every function f on $\mathbf{R}$ *and every $x \in \mathbf{R}$.*

Proposition 1.2 *The set $\mathcal{F}(C_c^\infty(\mathbf{R})) \subset C_0^\infty(\mathbf{R})$ is a dense $*$-subalgebra of $C_0^\infty(\mathbf{R})$ closed under the operation $f \mapsto f^-$.*

Proof: The Fourier transform is an algebra homomorphism from the algebra $\mathcal{S}$ with convolution as multiplication, to the algebra $\mathcal{S}$ with ordinary multiplication of functions. Since $C_c^\infty(\mathbf{R})$ is an algebra with convolution as multiplication, it follows that $\mathcal{F}(C_c^\infty(\mathbf{R}))$ is a subalgebra of $C_0^\infty(\mathbf{R})$ with respect to ordinary multiplication.

Now, it is not hard to see that $\hat{f}(-x) = (\widehat{f^-})(x)$ for all x in $\mathbf{R}$. That is,

$$(\hat{f})^- = (\widehat{f^-}).$$

Suppose that g belongs to $\mathcal{F}(C_c^\infty(\mathbf{R}))$. Say, $g = \hat{f}$ where $f \in C_c^\infty(\mathbf{R})$. Then, of course, f^- also belongs to $C_c^\infty(\mathbf{R})$, which implies that $g^- = (\hat{f})^- = (\widehat{f^-})$ belongs to $\mathcal{F}(C_c^\infty(\mathbf{R}))$. Thus, $\mathcal{F}(C_c^\infty(\mathbf{R}))$ is closed under the operation $g \mapsto g^-$.

Now, for $f \in \mathcal{S}$, it is also easy to see that $\overline{\hat{f}(x)} = \hat{\bar{f}}(-x)$ for all x in $\mathbf{R}$. That is, $\bar{\hat{f}} = (\hat{\bar{f}})^-$ which means that

$$\bar{\hat{f}} = (\widehat{\bar{f}^-})$$

by using the previous formula. Suppose that g belongs to $\mathcal{F}(C_c^\infty(\mathbf{R}))$ and $g = \hat{f}$ where $f \in C_c^\infty(\mathbf{R})$, as before. Of course, $\bar{f}^-$ also belongs to $C_c^\infty(\mathbf{R})$. So, from the above formula, we have that $\bar{g} = \bar{\hat{f}} = (\widehat{\bar{f}^-})$ belongs to $\mathcal{F}(C_c^\infty(\mathbf{R}))$. Thus, $\mathcal{F}(C_c^\infty(\mathbf{R}))$ is closed under the operation of taking the adjoint, or the conjugate, of a function. That is, $\mathcal{F}(C_c^\infty(\mathbf{R}))$ is a $*$-algebra.

To show that $\mathcal{F}(C_c^\infty(\mathbf{R}))$ is dense in $C_0^\infty(\mathbf{R})$, it therefore suffices to show that $\mathcal{F}(C_c^\infty(\mathbf{R}))$ separates points in $\mathbf{R}$ (by the Stone-Weierstrass Theorem).

So, take $x, y \in \mathbf{R}$, $x \neq y$, and suppose $f(x) = f(y)$ for all $f \in \mathcal{F}(C_c^\infty(\mathbf{R}))$. Take an $f \in \mathcal{F}(C_c^\infty(\mathbf{R}))$. Since $C_c^\infty(\mathbf{R})$ is closed under multiplication by the function $u \mapsto e^{iau}$ for all $a \in \mathbf{R}$, then $\mathcal{F}(C_c^\infty(\mathbf{R}))$ is closed under translations. Thus, $f(x-t) = f(y-t)$ for all $t \in \mathbf{R}$. This implies $f(t) = f(t + (y-x))$ for all $t \in \mathbf{R}$. Hence f is periodic with period $y - x$. But, $f \in C_0^\infty(\mathbf{R})$, and the only periodic function in $C_0^\infty(\mathbf{R})$ is the 0-function. Thus, $f \equiv 0$. This implies that $\mathcal{F}(C_c^\infty(\mathbf{R})) = \{0\}$, which is certainly not true. For instance, $\mathcal{F}$ is a one to one mapping. So, it is impossible for $\mathcal{F}(C_c^\infty(\mathbf{R}))$ to be $\{0\}$. Therefore, there exists $f \in \mathcal{F}(C_c^\infty(\mathbf{R}))$ such that $f(x) \neq f(y)$. That is, $\mathcal{F}(C_c^\infty(\mathbf{R}))$ separates points in $\mathbf{R}$. ♠

Corollary 1.3 *If $f \in C_0^\infty(\mathbf{R})$ and $\epsilon > 0$, then there exists $g \in \mathcal{S}$ such that $\hat{g}$ has compact support, and $\|f - g\|_\infty < \epsilon$.*

Proof: By Proposition 1.2, there exists $g \in \mathcal{F}(C_c^\infty(\mathbf{R}))$ such that $\|f - g\|_\infty < \epsilon$. Of course, $g \in \mathcal{S}$. Suppose $g = \hat{h}$ where $h \in C_c^\infty(\mathbf{R})$. Then, h^- also belongs to $C_c^\infty(\mathbf{R})$. It is not hard to show that $\hat{\hat{u}} = u^-$ for every $u \in \mathcal{S}$. Therefore, $\hat{g} = \hat{\hat{h}} = h^-$ belongs to $C_c^\infty(\mathbf{R})$. ♠

Definition 1.4 *Let* ***Flip*** *denote the algebra of all continuous functions f on $\mathbf{R}$ such that $f(x)$ approaches finite limits as x goes to $+\infty$, and as x goes to $-\infty$.*

If $f \in$ Flip, let $\boldsymbol{f}(-\infty)$ equal the limit of $f(x)$ as $x \to -\infty$, and let $\boldsymbol{f}(\infty)$ equal the limit of $f(x)$ as $x \to \infty$.

Also, let ***Flip***$_c$ *denote the set of all C^∞ functions f in Flip such that f' has compact support. This is the set of all C^∞ Flip functions which are constant outside some compact subset of $\mathbf{R}$.*

Recall the definition of a tempered distribution. (See, for example, page 15 of [Ter].)

Proposition 1.5 *Every Flip$_c$ function f can be written as the sum $f = f_1 + f_2$ of a C^∞-Flip function f_1 and a Schwartz function f_2 in such a way that (when viewed as a tempered distribution) f_1 has a compactly supported Fourier transform $\widehat{f_1}$. Moreover, if $\epsilon > 0$, we can choose f_1, f_2 so that the support of $\widehat{f_1}$ is a subset of $[-\epsilon, \epsilon]$.*

Proof: All bounded measurable functions on $\mathbf{R}$ are tempered distributions (Exercise 10(a) of Section 1.2, Chapter I, of [**Ter**]). Hence, f is a tempered distribution. Since $f \in Flip_c$, then f' has compact support and is C^∞, which implies $\widehat{(f')} \in \mathcal{S}$. By Theorem 3(2), Section I.1.2 of [**Ter**], we have that

$$2\pi i x \hat{f} = \widehat{(f')} \in \mathcal{S}.$$

Now, choose any interval $(-\epsilon, \epsilon)$ about 0 and any C^∞ bump function ρ such that $\rho \equiv 1$ on $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ and $\rho \equiv 0$ on $\mathbf{R}\backslash(-\epsilon, \epsilon)$. Let

$$g_2 = (1-\rho) \cdot \frac{1}{2\pi i x}\widehat{(f')}.$$

(At $x = 0, g_2(x) = 0$.) Since $\widehat{(f')} \in \mathcal{S}$, then $g_2 \in \mathcal{S}$. Let $f_2 \in \mathcal{S}$ be the Schwartz function such that $\widehat{f_2} = g_2$ and let $f_1 = f - f_2$. Since $f_2 \in \mathcal{S}$ and $f \in Flip_c$, it follows that $f_1 \in Flip$ and f_1 is C^∞. Also, $\widehat{f_1} = \hat{f} - \widehat{f_2}$, which implies $2\pi i x\widehat{f_1} = 2\pi i x \hat{f} - 2\pi i x \widehat{f_2} = \widehat{(f')} - 2\pi i x g_2 = \widehat{(f')} - (1-\rho)\widehat{(f')} = \rho \cdot \widehat{(f')}$. That is, $2\pi i x \widehat{f_1} = \rho \cdot \widehat{(f')}$. Since ρ has support in $[-\epsilon, \epsilon]$, then $2\pi i x \widehat{f_1}$ has support in $[-\epsilon, \epsilon]$, which gives us that $\widehat{f_1}$ has support in $[-\epsilon, \epsilon]$. ♠

Corollary 1.6 *Given a $Flip_c$ function f and an $\epsilon > 0$, there exists a C^∞ Flip function g such that $\|f - g\|_\infty < \epsilon$, $\hat{g}$ has compact support, $g(-\infty) = f(-\infty)$, and $g(\infty) = f(\infty)$.*

Proof: By Proposition 1.5, $f = f_1 + f_2$ for some C^∞-$Flip$ function f_1 such that $\widehat{f_1}$ has compact support, and some $f_2 \in \mathcal{S}$. By Corollary 1.3, there exists $g_2 \in \mathcal{S}$ such that $\widehat{g_2}$ has compact support, and $\|f_2 - g_2\|_\infty < \epsilon$. Let $g = f_1 + g_2$. Since both $\widehat{f_1}$ and $\widehat{g_2}$ have compact support, then $\hat{g} = \widehat{f_1} + \widehat{g_2}$ also has compact support. In addition, $\|f-g\|_\infty = \|f_1+f_2-(f_1+g_2)\|_\infty = \|f_2-g_2\|_\infty < \epsilon$, and $g \in C^\infty\text{-}Flip$ since $f_1 \in C^\infty\text{-}Flip$ and $g_2 \in \mathcal{S} \subset C^\infty\text{-}Flip$. Finally, since $f = f_1+f_2$ and $f_2 \in \mathcal{S}$, then $f(-\infty) = f_1(-\infty)$ and $f(\infty) = f_1(\infty)$. Similarly, since $g = f_1 + g_2$ and $g_2 \in \mathcal{S}$, then $g(-\infty) = f_1(-\infty)$ and $g(\infty) = f_1(\infty)$. Hence, $g(-\infty) = f(-\infty)$ and $g(\infty) = f(\infty)$. The corollary is therefore true. ♠

Proposition 1.7 *Given $f \in Flip$ and $\epsilon > 0$, there exists $g \in Flip_c$ such that $g(\infty) = f(\infty)$, $g(-\infty) = f(-\infty)$, and $\|g-f\| < \epsilon$.*

Proof: This follows from the Stone-Weierstrass Theorem. ♠

Corollary 1.8 *Given $f \in Flip$ and $\epsilon > 0$, there exists $g \in C^\infty\text{-}Flip$ such that $g(\infty) = f(\infty)$, $g(-\infty) = f(-\infty)$, $\hat{g}$ has compact support, and $\|f-g\|_\infty < \epsilon$.*

Proof: This is a consequence of Proposition 1.7 and Corollary 1.6. ♠

Corollary 1.9 *Given $f \in Flip$ and $\epsilon > 0$, there exists a C^∞-Flip function f_1 and a Schwartz function f_2 such that $f_1(-\infty) = f(-\infty)$, $f_1(\infty) = f(\infty)$, $\|f-(f_1+f_2)\|_\infty < \epsilon$, and f_1, viewed as a tempered distribution, has Fourier transform $\widehat{f_1}$ with compact support in $[-\epsilon, \epsilon]$.*

Proof: This follows from Propositions 1.7 and 1.5. ♠

Corollary 1.10 *Given $a, b \in \mathbf{R}$ and $r > 0$, there exists $f \in C^\infty\text{-}Flip$ such that $f(-\infty) = a, f(\infty) = b$ and $\hat{f}$ has support in $(-r, r)$.*

Proof: Follows immediately from Corollary 1.9. ♠

Remark 1.11 *By Equation 2.16 of [**BDT**], Corollary 1.9 is also true for those functions f in the space S_1^0 defined in (2.14) of [**BDT**].*

Definition 1.12 *Let*

$$\boldsymbol{Flip}_d = \{f \in Flip : \hat{f} \text{ has compact support}\}.$$

Proposition 1.13 *The algebra of C^∞-$Flip_d$ functions, is dense in Flip in the $\|\cdot\|_\infty$ norm. Hence $Flip_d \subset Flip$ is dense in Flip in the $\|\cdot\|_\infty$ norm.*

Proof: This is just a rephrasing of Corollary 1.8. ♠

Corollary 1.10 can be made a little stronger as we now show.

Lemma 1.14 *For every $\epsilon > 0$ there exists $h \in \mathcal{S}$ such that $h \neq 0, h \geq 0$, and $\hat{h}$ has support in $(-\epsilon, \epsilon)$. We may choose such an h so that $\int h = 1$.*

Proof Take $\epsilon > 0$. Start with any $g \in \mathcal{S}, g \neq 0$, such that $\hat{g}$ has support in $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. It is easy to show that $\hat{\bar{g}}(x) = \overline{\hat{g}(-x)}$. Hence, $\hat{\bar{g}}$, like $\hat{g}$, also has support in $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. It follows that $\hat{g} * \hat{\bar{g}}$ has support in $(-\epsilon, \epsilon)$. Since $g \neq 0$ then $h = |g|^2 = g \cdot \bar{g}$ is a nonzero, nonnegative, Schwartz function whose Fourier transform $\hat{h} = \hat{g} * \hat{\bar{g}}$ has support in $(-\epsilon, \epsilon)$. The function $f = (\frac{1}{\int h}) \cdot h$ has these same properties with the additional property that $\int f = 1$. This completes the proof. ♠

Lemma 1.15 *For every $\epsilon > 0$, there exists $f \in C^\infty$-Flip such that $-1 \leq f(x) \leq 1$ for all $x \in \mathbf{R}, f(-\infty) = -1, f(+\infty) = 1$, and $\hat{f}$ has support in $[-\epsilon, \epsilon]$.*

Proof Suppose $\epsilon > 0$. By Lemma 1.14, there exists a nonzero Schwartz function h such that $h \geq 0$, $\hat{h}$ has support in $(-\epsilon, \epsilon)$ and $\int h = 1$. Take any $C^\infty Flip_c$ function g such that $g(x) = -1$ for $x \in (-\infty, -1], g(x) = 1$ for

$x \in [1,\infty)$ and $-1 \le g(x) \le 1$ for all x. Let $f = h * g$. Then f is C^∞, real-valued, and, since $\hat{h}$ has support in $(-\epsilon,\epsilon), \hat{f} = \hat{h}\cdot\hat{g}$ also has support in $(-\epsilon,\epsilon)$. Also, for every $x \in \mathbf{R}, |f(x)| = |(h*g)(x) \le \int h(t)\cdot|g(x-t)|dt \le \|g\|_\infty \cdot \int h(t)dt = 1\cdot 1 = 1$. Which shows that $\|f\|_\infty \le 1$. Hence, for every $x \in \mathbf{R}$,

$$
\begin{aligned}
0 \le 1 - f(x) &= 1 - \int_{-\infty}^{\infty} h(x-t)g(t)dt \\
&= \int_{-\infty}^{\infty} h(x-t)dt - \int_{-\infty}^{\infty} h(x-t)g(t)dt \\
&= \int_{-\infty}^{\infty} h(x-t)(1-g(t))dt \\
&= \int_{-\infty}^{1} h(x-t)(1-g(t))dt \\
&\quad \text{(since } g(t) = 1 \text{ if } t \ge 1) \\
&\le 2\int_{-\infty}^{1} h(x-t)dt \\
&\quad \text{(since } 0 \le 1-g(t) \le 2 \text{ for all } t) \\
&= 2\int_{x-1}^{\infty} h(t)dt \to 0 \quad \text{as} \quad x \to \infty.
\end{aligned}
$$

Therefore $f(x) \to 1$ as $x \to \infty$. Similarly,

$$
\begin{aligned}
0 \le f(x) - (-1) &= f(x) + 1 \\
&= \int_{-\infty}^{\infty} h(x-t)g(t)dt + \int_{-\infty}^{\infty} h(x-t)dt \\
&= \int_{-\infty}^{\infty} h(x-t)[g(t)+1]dt \\
&= \int_{-1}^{\infty} h(x-t)(g(t)+1)dt \\
&\quad \text{(since } g(t) = -1 \text{ for } t \le -1) \\
&\le 2\int_{-1}^{\infty} h(x-t)dt \\
&\quad \text{(since } 0 \le g(t)+1 \le 2 \text{ for all } t)
\end{aligned}
$$

$$= \int_{-\infty}^{x+1} h(t)dt \to 0 \text{ as } x \to -\infty.$$

Thus $f(x) \to -1$ as $x \to -\infty$. This completes the proof. ♠

Now, we present the stronger version of Corollary 1.10.

Corollary 1.16 *Given $a, b \in \mathbf{C}$ and $r > 0$, there exists $f \in C^\infty$-Flip such that $f(-\infty) = a, f(\infty) = b, \hat{f}$ has support in $[-r, r]$ and $f(x)$ lies on the closed line segment $[a, b] \subseteq \mathbf{C}$ with endpoints a and b for all $x \in \mathbf{R}$.*

Proof Suppose $a, b \in \mathbf{C}$ and $r > 0$. By Lemma 1.15, there exists a C^∞-*Flip* function g such that $g(-\infty) = -1, g(\infty) = 1, -1 \leq g(x) \leq 1$ for all x, and $\hat{g}$ has support in $[-r, r]$. By multiplying g by an appropriate constant $\lambda \in \mathbf{C}$ and adding to this another constant $c \in \mathbf{C}$, we obtain another function $f = \lambda g + c$ with all the required properties of the function we are seeking including the property that $\hat{f}$ has support in $[-r, r]$ (which is due to the fact that the Fourier transform of a constant function has support concentrated at 0). The corollary is therefore true. ♠

Corollary 1.17 *Given a Flip function f and $r > 0$, there exists a C^∞-Flip function g such that $f - g \in C_0(\mathbf{R}), \hat{g}$ has support in $(-r, r)$, and*

$$\|g\|_\infty \leq \max\{|f(-\infty)|, |f(\infty)|\} \leq \|f\|_\infty.$$

Proof Let f be a *Flip* function with $f(-\infty) = a, f(+\infty) = b$. By Corollary 1.16, there exists a $g \in C^\infty$-*Flip* such that $g(-\infty) = a, g(+\infty) = b, \hat{g}$ has support in $(-r, r)$, and $g(x)$ lies on $[a, b] \subseteq \mathbf{C}$ for all x. Thus, we also have $f - g \in C_0(\mathbf{R})$ and $\min\{|a|, |b|\} \leq$ the infimum of $|g| \leq$ the supremum of $|g|$ $\leq \max\{|a|, |b|\}$. The corollary is therefore true. ♠

2.2 Symbols

Let M be a complete riemannian manifold and $E \to M$ a finite dimensional hermitian vector bundle over M. Let $\pi : T^*M \to M$ be the projection map, and let $\pi^*E \to T^*M$ be the pullback of the bundle $E \to M$ by the projection map $\pi : T^*M \to M$. This determines another bundle $Hom(\pi^*E) \to T^*M$. If $x \in M$ and $\xi \in (T^*M)_x$, then the fiber $Hom(\pi^*E)_\xi$ may be regarded as the algebra $B(E_x)$ of bounded operators on the finite dimensional Hilbert space E_x. That is, each element $T \in Hom(\pi^*E)_\xi$ is a bounded operator $T : E_x \to E_x$ on E_x. If $A : C^\infty(E) \to C^\infty(E)$ is a differential operator on M, there is associated to A a smooth section $\sigma_L A$ of the bundle $Hom(\pi^*E) \to T^*M$ called the **principal symbol** of the differential operator A. (See Section III.1 of [**L&M**].) Let d be any real number. In the case where M is a compact riemannian manifold, recall the definition of the space $S^d(E)$ of symbols of order d with respect to the bundle E. (See Definition III.3.18 of [**L&M**].) Elements of $S^d(E)$ are smooth sections of the bundle $Hom(\pi^*E) \to T^*M$. Now, every pseudodifferential operator $P : C^\infty(E) \to C^\infty(E)$ of order d on a compact riemannian manifold has an associated **principal symbol** $[\sigma_L(P)] \in S^d(E)/S^{d-1}(E)$ where $\sigma_L(P) \in S^d(E)$. ($\sigma_L P$ is well-defined up to $S^{d-1}(E)$.) (See Thereom III.3.19 of [**L&M**].) If $P : C^\infty(E) \to C^\infty(E)$ happens to be a differential operator of order m on a compact riemannian manifold the two notions of principal symbol $\sigma_L P$ of P given, namely the principal symbol of P where P is considered a differential operator, and the principal symbol of P where P is considered a pseudodifferential operator, are actually the same. That is, if $\sigma_L P$ is the principal symbol of the *differential* operator P, then $\sigma_L P$ is an element of $S^m(E)$, and the principal symbol of the *pseudodifferential* operator P is the element $[\sigma_L P]$ of $S^m(E)/S^{m-1}(E)$. So, there is no ambiguity in the definitions of the principal symbol of P.

Proposition 2.1 *Let $P : C^\infty(E) \to C^\infty(E)$ be a self-adjoint pseudodifferential operator of order d on a compact riemannian manifold. Then we can find a self-adjoint $\sigma \in S^d(E)$ whose symbol class $[\sigma] \in S^d(E)/S^{d-1}(E)$ is the principal symbol of P.*

Proof: Suppose the principal symbol of P is $[\tau]$ where $\tau \in S^d(E)$. Then the pseudodifferential operator P^* of order d has principal symbol $[\tau^*] \in S^d(E)/S^{d-1}(E)$. But P is self-adjoint. So $P = P^*$, and therefore $[\tau] = [\tau^*]$ in $S^d(E)/S^{d-1}(E)$. Thus $\tau - \tau^* \in S^{d-1}(E)$. Let $\sigma = \frac{1}{2}(\tau + \tau^*)$. Then $\sigma \in S^d(E)$ is self-adjoint, and

$$\begin{aligned} [\tau] &= [\frac{1}{2}(\tau + \tau^*)] + [\frac{1}{2}(\tau - \tau^*)] \\ &= [\frac{1}{2}(\tau + \tau^*)] = [\sigma] \end{aligned}$$

in $S^d(E)/S^{d-1}(E)$, since $\frac{1}{2}(\tau - \tau^*) \in S^{d-1}(E)$. Therefore the principal symbol of P is represented by a self-adjoint $\sigma \in S^d(E)$. ♠

The following theorem is due to Seeley [**See3**] and Kohn-Nirenberg [**K&N**] for "classical" pseudodifferential operators.

Theorem 2.2 : *Let $P : C^\infty(E) \to C^\infty(E)$ be a pseudodifferential operator or order 0 on a compact riemannian manifold M. Let K $(L^2(E))$ denote the compact operators on $L^2(E)$. Since P has order 0 it extends to a bounded operator $P : L^2(E) \to L^2(E)$. Suppose the principal symbol of P is equal to*

$[\sigma_L(P)] \in S^0(E)\backslash S^{-1}(E)$ where $\sigma_L(P)$ belongs to $S^0(E)$. Then the distance of P from $\mathcal{K}(L^2(E))$ is equal to

$$\lim_{r\to\infty} \sup\{\|(\sigma_L P)(w)\| : w \in T^*M, \|w\| \geq r\}.$$

(Note: Each $(\sigma_L P)(w)$ is a bounded operator $(\sigma_L P)(w) : E_{\pi(w)} \to E_{\pi(w)}$ on the finite dimensional Hilbert space $E_{\pi(w)}$ (where $\pi : T^*M \to M$ is the projection map) and so has an operator norm $\|(\sigma_L P)(w)\|$.)

Proof: This is given by the remark following the proof of Theorem 3.3 in Hormander's paper **[Hor1]**. ♠

2.3 The Dirac Operator

In this section, M is a **complete, riemannian, spin manifold of dimension m, k is a positive integer**, and

$$\Delta_M \to M$$

is the bundle of spinors over M. Wherever possible, I will suppress the subscript M and use Δ instead of Δ_M.

Let

$$D_M : C_c^\infty(\Delta) \to C_c^\infty(\Delta)$$

denote the Dirac operator acting on the algebra $C_c^\infty(\Delta)$ of smooth sections of Δ with compact support, and let $L^2(\Delta)$ be the L^2-sections of Δ with respect to the natural measure on M induced by the given riemannian metric on M. I will use D_M and D interchangably unless there is a need to be precise, in which case I will only use D_M.

It is well known that $C_c^\infty(\Delta)$ is dense in the Hilbert space $L^2(\Delta)$, and that the Dirac operator $D_M : C_c^\infty(\Delta) \to C_c^\infty(\Delta)$ is formally self-adjoint. That is, D has the property

$$(D\xi, \eta) = (\xi, D\eta)$$

for all ξ, η in $C_c^\infty(\Delta)$, where (,) denotes the $L^2(\Delta)$ inner product.

Thus, considered as an unbounded operator on $L^2(\Delta)$ with dense domain $C_c^\infty(\Delta)$, D is closable, and we let $\bar{D}$ denote the closure of D on $L^2(\Delta)$.

We have the following theorem.

Theorem 3.1 *The unbounded operator D on $L^2(\Delta)$ is essentially self-adjoint. That is, $\bar{D}$ is a self-adjoint operator on $L^2(\Delta)$.*

Proof: This is Theorem 5.7 of the book, **[L&M]**, by Lawson and Michelsohn. ♠

From now on, we will use $\bar{D}$ and D interchangably unless some confusion arises.

Since D $(= \bar{D})$ is self-adjoint, then, for any bounded, continuous (complex-valued), function f on $\mathbf{R}$, we can define the bounded operator $f(D)$ on $L^2(\Delta)$ using the functional calculus for self-adjoint operators.

Notation 3.2 *If X is a metric space, $A \subset X$, and $r > 0$, let*

$$\boldsymbol{B(A, r)} = \{x \in X : d(x, A) < r\}.$$

If f is any function on any space X, we let $\mathbf{supp}(\boldsymbol{f}) \subset X$ stand for the closed support of f.

Definition 3.3 *Let $r > 0$, let X be a complete riemannian manifold, E a finite dimensional hermitian vector bundle over X, and $A : L^2(E) \to L^2(E)$ a possibly unbounded operator on $L^2(E)$. Then A is said to be* **r-local** *if, for every $u \in L^2(E)$, the support of Au lies inside $B(\mathrm{supp}(u), r)$.*

Theorem 3.4 (Unit Propagation Speed Property) *If $t \in \mathbf{R}$, then the operator $e^{itD} \in B(L^2(\Delta))$ is $|t|$-local, and, for every $u \in C_c^\infty(\Delta)$, $e^{itD}u$ also belongs to $C_c^\infty(\Delta)$.*

Proof: This is Theorem 1.3 of [**Roe2**]. ♠

Corollary 3.5 *If f is a bounded continuous function on $\mathbf{R}$ and if $\hat{f}$ has support in the interval $[-r, r]$ for some $r > 0$, then $f(D) \in B(L^2(\Delta))$ is an r-local operator. ($\hat{f}$ is regarded as a tempered distribution.)*

Proof: We can write

$$\begin{aligned} f(D) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{itD}dt \\ &= \frac{1}{2\pi}\int_{-r}^{r} \hat{f}(t)e^{itD}dt. \end{aligned}$$

Since e^{itD} is r-local for every $t \in [-r, r]$ (by Theorem 3.4), it follows that

$$f(D) = \frac{1}{2\pi}\int_{-r}^{r} \hat{f}(t)e^{itD}dt$$

is also r-local. ♠

We now turn our attention to the symbol of D.

Let m **be any positive integer**, and let $\boldsymbol{Cl_m}$ and $\boldsymbol{\Delta_m}$ denote the complex Clifford algebra, and complex spinors respectively, of the euclidean space

$\mathbf{R}^m$. Recall that Δ_m is a hermitian vector space over $\mathbf{C}$ and a module over the algebra Cl_m. Let S^{m-1} denote the unit $(m-1)$-sphere sitting inside $\mathbf{R}^m$. (Note: S^0 is the two-point set $\{-1, 1\}$.)

Remarks 3.6 *1. It is well known that if v is in S^{m-1}, then the operator $s \mapsto v \cdot s$ for s in Δ_m (where $\cdot$ denotes Clifford multiplication) is a unitary operator on Δ_m.*

2. If v in $\mathbf{R}^m$, define the operator

$$\sigma_v : \Delta_m \to \Delta_m$$

by letting

$$\sigma_v(s) = iv \cdot s$$

for every s in Δ_m. From (1), it follows that σ_v is unitary for every $v \in S^{m-1}$.

3. For every $v \in \mathbf{R}^m, (iv)^2 = \|v\|^2$, which implies that

$${\sigma_v}^2 = \|v\|^2 \cdot Id_{\Delta_m}.$$

Thus, the spectrum of σ_v is a subset of $\{-\|v\|, +\|v\|\}$ for every $v \in \mathbf{R}^m$. It follows that σ_v is self-adjoint and $\|\sigma_v\| = \|v\|$ for every v in $\mathbf{R}^m$.

4. Thus, for v in S^{m-1}, $\sigma_v : \Delta_m \to \Delta_m$ is a self-adjoint unitary, and the operator

$$\frac{1}{2}(1 + \sigma_v) : \Delta_m \to \Delta_m,$$

is the projection onto the +1 eigenspace of σ_v.

5. *It is clear from the definition of* σ_v *that*

$$\sigma_{-v} = -\sigma_v$$

for every $v \in \mathbf{R}^m$.

6. *From 5, it follows that, for every* $v \in \mathbf{R}^m$,

$$spec(\sigma_v) \cup spec(\sigma_{-v}) = \{-\|v\|, \|v\|\}.$$

Proposition 3.7 *For each* $x \in M$, *identify* T_x^*M *with* T_xM *in the canonical way, so that Clifford multiplication* $\xi \cdot \omega$ *makes sense for every* $\xi \in T_x^*M$ *and* $\omega \in \Delta_x$. *Then the principal symbol* $\boldsymbol{\sigma_L(D)}$ *of* D *is such that*

$$\sigma_L(D)(\xi)(\omega) = i\xi \cdot \omega \tag{3.8}$$

for all $x \in M$, $\xi \in T_x^*M$ *and* $\omega \in \Delta_x$, *and the map* $\sigma_L(D)(\xi) : \Delta_x \to \Delta_x$, *on the inner product space* Δ_x, *is self-adjoint. Moreover,* D *is elliptic over* M.

Proof: The equality (3.8) is Lemma II.5.1 of **[L&M]**. It immediately gives us that D is elliptic over M. By Remarks 3.6, it also implies that the map $\sigma_L(D)(\xi) : \Delta_x \to \Delta_x$ on the inner product space Δ_x, is self-adjoint. ♠

2.4 Multiplication Operators

If φ is a function on a manifold M and u is a section of a bundle E over M, then $\boldsymbol{M_\varphi u}$ will stand for the section

$$\boldsymbol{M_\varphi u} = \varphi \cdot u$$

of E. That is, $(M_\varphi u)(x) = \varphi(x) \cdot u(x)$ for every $x \in M$. If M is a riemannian manifold, E is a smooth finite-dimensional vector bundle over M, and φ is a bounded measurable function on M, then M_φ gives a bounded operator M_φ on $L^2(E)$. When the symbol M_φ is used we will usually mean this operator on $L^2(E)$. If $k > 0$ is a positive integer, φ is a bounded measurable $k \times k$ matrix-valued function on a riemannian manifold M and E is again a smooth finite dimensional hermitian bundle over M, then

$$M_\varphi : L^2(E)^k \to L^2(E)^k$$

is the bounded operator satisfying $M_\varphi u = \varphi u$ for every $u \in L^2(E)^k$. That is, $(M_\varphi u)(x) = \varphi(x) \cdot u(x)$. More precisely, each $u \in L^2(E)^k$ can be regarded as an element of $L^2(E^k)$. So, $\varphi(x) \cdot u(x)$ is the obvious multiplication of the vector $u(x) \in (E_x)^k$ by a $k \times k$ matrix $\varphi(x)$.

Remark 4.1 *If E and M are as above, φ is a bounded measurable $k \times k$ matrix-valued function on M, and $A : L^2(E) \to L^2(E)$ is a bounded operator, then, in the future, when we write AM_φ and $M_\varphi A$, what we will really mean are the operators*

$$\begin{pmatrix} A & & & \\ & A & & 0 \\ & & \ddots & \\ & 0 & & A \end{pmatrix} \cdot M_\varphi \text{ and } M_\varphi \cdot \begin{pmatrix} A & & & \\ & A & & 0 \\ & & \ddots & \\ & 0 & & A \end{pmatrix}$$

on $L^2(E)^k$. *In fact, if* H *is any Hilbert space and* $A \in B(H)$, *when we write* $A \in B(H^k)$ *we will mean the operator*

$$\begin{pmatrix} A & & & \\ & A & & 0 \\ & & \ddots & \\ & 0 & & A \end{pmatrix}$$

in $B(H^k)$ *with respect to the orthogonal decomposition* $H^k = \underbrace{H \oplus \ldots \oplus H}_{k}$

Finally, operators M_φ will be called **multiplication operators with symbol** φ, and the symbol φ will sometimes be referred to as a **multiplier**.

2.5 Coincidences

Definition 5.1 *Let* $E \to M$, $F \to N$ *be smooth hermitian bundles over isometric riemannian manifolds* M *and* N *respectively. A* **bundle isometry from** E **to** F *is a smooth bundle isomorphism from* E *to* F *which preserves both the hermitian structure on the fibers of* E *and the riemannian structure on* M.

Definition 5.2 *Let* $E \to M$ *and* $F \to N$ *be smooth hermitian vector bundles over complete riemannian manifolds* M *and* N *respectively. Let* $U \subseteq M$, $V \subseteq N$ *be open balls of the same radius, and suppose there is a bundle isometry* $h : E|_U \to F|_V$. *Let* A *be a linear operator from smooth sections of* E *with compact support to* L^2*-sections of* E, *and let* B *be a linear operator from* *smooth sections of F with compact support to L^2- sections of F. We say that* $(A, B; h)$ *is a* **coincidence**, *or that* $(A, B; U, V; h)$ *is a coincidence, or that* $(A, B; U, V; E, F; h)$ *is a coincidence, or that h is a* **coincidence between A and B**, *or that* **A coincides locally with B** *(over U), if*

1. *For every $u \in C_c^\infty(E)$ with compact support in U, Au is an L^2-section of E with compact support also in U;*

2. *For every $v \in C_c^\infty(F)$ with compact support in V, Bv is an L^2-section of F with compact support in V; and*

3. *The linear operator induced by A from $C_c^\infty(U, E)$ to L^2- sections of E over U, is identified via h with the linear operator induced by B from $C_c^\infty(V, F)$ to L^2-sections of F over V. That is, if $h_* : L^2(U, E) \to L^2(V, F)$ is the unitary operator induced by h, then h_* gives a unitary equivalence between the operator $A|_{C_c^\infty(U,E)} : C_c^\infty(U, E) \to L^2(U, E)$ and the operator $B|_{C_c^\infty(V,F)} : C_c^\infty(V, F) \to L^2(V, F)$. In other words,*

$$h_* \cdot A|_{C_c^\infty(U,E)} = B|_{C_c^\infty(V,F)} \cdot h_*.$$

We say $(A, B; h)$ is an **exact coincidence**, *or that h is* **an exact coincidence between A and B** *if (A, B, h) is a coincidence, and if $Au = 0$ whenever $u \in C_c^\infty(E)$ has support outside U, and $Bv = 0$ whenever $v \in C_c^\infty(F)$ has support outside V.*

Proposition 5.3 *Let M be a complete riemannian manifold in which U is an open ball, and suppose φ is a continuous function on M with compact support in U. Then, for every coincidence $(A, B; U; V, h)$,*

$$(AM_\varphi, BM_{\varphi \circ h^{-1}}; U, V; h)$$

is an exact coincidence.

Proof: Let $(A, B; U, V; E, F; h)$ be a coincidence. Suppose $u \in C^\infty(E)$ has support in U and $v \in C^\infty(F)$ has support in V. Then φu has support in U and $(\varphi \circ h^{-1}) \cdot v$ has support in V. Since $(A, B; U, V; h)$ is a coincidence, it follows that $A(\varphi u)$ has support in U and $B((\varphi \circ h^{-1}) \cdot v)$ has support in V. That is $(AM_\varphi)(u)$ has support in U and $(BM_{\varphi o h^{-1}})(V)$ has support in V. Moreover, $h_*((AM_\varphi)u) = h_*(A(\varphi \cdot u)) = B(h_*(\varphi \cdot u))$ since $(A, B; U, V; h)$ is a coincidence. Hence, $h_*((AM_\varphi)u) = B((\varphi \circ h^{-1}) \cdot (h_* u)) = (BM_{\varphi o h^{-1}})(h_* u)$. Thus, $(AM_\varphi, BM_{\varphi o h^{-1}}, U, V, h)$ is a coincidence.

To show it is an exact coincidence, take u with support outside U and v with support outside V. Since φ has support in U, then $\varphi \circ h^{-1}$ has support in V. Therefore $\phi \cdot u = 0$ and $(\varphi o h^{-1}) \cdot v = 0$, which implies that $(AM_\varphi)u = 0$ and $(BM_{\varphi o h^{-1}})(v) = 0$. Thus $(AM_\varphi, BM_{\varphi o h^{-1}}; U, V; h)$ is an exact coincidence. ♠

Proposition 5.4 *Let $(A, B; U, V; E, F; h)$ be an exact coincidence. Then the following are true.*

1. *A extends to a bounded operator on $L^2(E)$ if and only if B extends to a bounded operator on $L^2(F)$.*

2. *A extends to a compact operator on $L^2(E)$ if and only if B extends to a compact operator on $L^2(F)$.*

3. *A extends to a Fredholm operator on $L^2(E)$ if and only if B extends to a Fredholm operator on $L^2(F)$, and, in this case, $ind(A) = ind(B)$.*

4. If A extends to a bounded operator on $L^2(E)$ (and so B also extends to a bounded operator on $L^2(F)$), then $(A^*, B^*; U, V; E, F; h)$ is also an exact coincidence.

Proof: The operator

$$A|_{C_c^\infty(U,E)} : C_c^\infty(U,E) \to L^2(U,E)$$

is unitarily equivalent to the operator

$$B|_{C_c^\infty(V,F)} : C_c^\infty(V,F) \to L^2(V,F)$$

(via h_*) and both $A|_{C_c^\infty(\setminus \bar{U},E)}$ and $B|_{C_c^\infty(\setminus \bar{V},F)}$ are equal to 0. Thus, A extends to a bounded operator A on $L^2(E) \Leftrightarrow A|_{C_c^\infty(V,E)}$ extends to a bounded operator A_0 on $L^2(V,E) \Leftrightarrow B|_{C_c^\infty(V,F)}$ extends to a bounded operator B_0 on $L^2(V,F) \Leftrightarrow$ B extends to a bounded operator B on $L^2(F)$. (This proves 1.) When this happens, A_0 will be unitarily equivalent to B_0 via h_*, and, with respect to the decompositions

$$L^2(E) = L^2(U,E) \oplus L^2(\setminus U, E),$$

and

$$L^2(F) = L^2(V,F) \oplus L^2(\setminus V, F),$$

we may write

$$A = A_0 \oplus 0$$

and

$$B = B_0 \oplus 0.$$

Statements 2, 3, and 4 now follow immediately from the fact that A_0 is unitarily equivalent to B_0 via the unitary h_*. ♠

Theorem 5.5 *Let D_M and D_N be the Dirac operators on complete spin riemannian manifolds M and N respectively. Suppose (D_M, D_N, U, V, h) is a coincidence. Suppose $r > 0$, and that $\varphi \in C_c^\infty(U)$ is such that the distance of the support of φ from the complement of U is larger than r. Let f be a C^∞ Flip function whose Fourier transform $\hat{f}$ has support in the interval $[-r, r]$. Then*

$$(f(D_M)M_\varphi, f(D_N)M_{\varphi\circ h^{-1}}; U, V; h),$$

$$(M_\varphi f(D_M), M_{\varphi\circ h^{-1}} f(D_N); U, V; h),$$

and

$$([f(D_M), M_\varphi], [f(D_N), M_{\varphi\circ h^{-1}}]; U, V; h)$$

are all exact coincidences, and if $|t| \leq r$, then $(e^{itD_M} M_\varphi, e^{itD_N} M_{\varphi\circ h^{-1}}; U, V; h)$ and $(M_\varphi e^{itD_M}, M_{\varphi\circ h^{-1}} e^{itD_N}; U, V; h)$ are also exact coincidences.

Proof: Suppose $u \in C_c^\infty(U)$, then

$$u_t = e^{itD_M} M_\varphi u = e^{itD_M}(\varphi u), \quad t \in \mathbf{R}$$

is a solution of the initial value problem

$$\left.\begin{array}{rcl} \frac{d}{dt} u_t & = & i D_M u_t \\ u_0 & = & \varphi \cdot u \end{array}\right\}. \qquad (*)$$

Also,

$$\begin{aligned} \frac{d}{dt}(e^{itD_N} M_{\varphi\circ h^{-1}}(h_* u)) &= \frac{d}{dt}(e^{itD_N}(h_*(\varphi u)) \\ &= i D_N \cdot e^{itD_N}(h_*(\varphi u)). \end{aligned}$$

Now, if $|t| \leq r$, then by Theorem 3.4, the operator e^{itD_N} is r-local. Together with the fact that the distance from the support of $h_*(\varphi \cdot u)$ from the complement of V is larger than r, this gives that the support of $e^{itD_N}(h_*(\varphi \cdot u))$ remains inside V if $|t| \leq r$. The differential operator D_N does not enlarge support. Thus $iD_N e^{itD_N}(h_*(\varphi \cdot u))$ also has support in V when $|t| \leq r$. Therefore, for $t \in [-r, r]$, the expression $h^*(e^{itD_N}(h_*(\varphi \cdot u)))$ makes sense and we have

$$\begin{aligned}
\frac{d}{dt} h^*(e^{itD_N}(h_*(\varphi u))) &= h^*(\frac{d}{dt} e^{itD_N}(h_*(\varphi u))) \\
&= h^*(iD_N \cdot e^{itD_N}(h_*(\varphi u))) \\
&= i(h^* D_N) \cdot h^*(e^{itD_N}(h_*(\varphi u))) \\
&= iD_M \cdot h^*(e^{itD_N}(h_*(\varphi u))).
\end{aligned}$$

Also, if $t = 0$ then $h^*(e^{itD_N}(h_*(\varphi u))) = h^*(h_*(\varphi u)) = \varphi \cdot u$. Thus, $h^*(e^{itD_N}$ $(h_*(\varphi u)))$ is a solution of the initial value problem $(*)$ for $|t| \leq r$. By uniqueness of the solution of $(*)$, it follows that

$$\begin{aligned}
e^{itD_M} M_\varphi(u) &= h^*(e^{itD_N}(h_*(\varphi u))) \\
&= h^*(e^{itD_N} M_{\varphi o h^{-1}}(h_* u))
\end{aligned}$$

for $t \in [-r, r]$. This means that $(e^{itD_M} M_\varphi, e^{itD_N} M_{\varphi o h^{-1}}, U, V, h)$ is a coincidence for all $t \in [-r, r]$.

It is an exact coincidence, for if u is a compactly supported section, with compact support outside U,then, because φ has support in $U, M_\varphi u = 0$, and therefore $e^{itD_M} M_\varphi u = 0$. Similarly, $e^{itD_N} M_{\varphi o h^{-1}} v = 0$ if v has support outside V. This proves that

$$(e^{itD_M} M_\varphi, e^{itD_N} M_{\varphi o h^{-1}}; U, V; h)$$

is an exact coincidence when $|t| \leq r$. By replacing φ with $\bar{\varphi}$ and t with $-t$, we get that

$$(e^{-itD_M} M_{\bar{\varphi}}, e^{-itD_N} M_{\bar{\varphi} o h^{-1}}; U, V; h)$$

is an exact coincidence when $|t| \leq r$. By Property 4 of Proposition 5.4, the adjoint

$$(M_{\varphi} e^{itD_M}, M_{\varphi o h^{-1}} e^{itD_N}; U, V; h)$$

is also an exact coincidence when $|t| \leq r$. For the rest of the theroem, note that, since $\hat{f}$ has support in $[-r, r]$, then

$$\begin{aligned} f(D_M) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{f}(t) e^{itD_M} dt \\ &= \frac{1}{2\pi i} \int_{-r}^{r} \hat{f}(t) e^{itD_M} dt. \end{aligned}$$

and

$$f(D_N) = \frac{1}{2\pi i} \int_{-r}^{r} \hat{f}(t) e^{itD_N} dt.$$

We just showed that

$$(e^{itD_M} M_{\varphi}, e^{itD_N} M_{\varphi o h^{-1}}, U, V, h)$$

and

$$(M_{\varphi} e^{itD_M}, M_{\varphi o h^{-1}} e^{itD_N}, U, V, h)$$

are exact coincidences for $|t| \leq r$. Thus,

$$h_*((e^{itD_M} M_{\varphi})(u)) = e^{itD_N} M_{(\varphi o h^{-1})}(h_* u)$$

and

$$h_*(M_{\varphi} e^{itD_M} u) = (M_{\varphi o h^{-1}} e^{itD_N})(h_* u)$$

whenever $|t| \leq r$ and $u \in C_c^\infty(U, E)$. It follows that

$$\begin{aligned} h_*(f(D_M)M_\varphi(u) &= \frac{1}{2\pi i}\int_{-r}^{r} \hat{f}(t)h_*(e^{itD_M}M_\varphi u)dt \\ &= \frac{1}{2\pi i}\int_{-r}^{r} \hat{f}(t)e^{itD_N}M_{\varphi o h^{-1}}(h_* u)dt \\ &= f(D_N)M_{\varphi o h^{-1}}(h_* u) \end{aligned}$$

for all $u \in C_c^\infty(U, E)$, which implies that $(f(D_M)M_\varphi, f(D_N)M_{\varphi o h^{-1}}; U, V; h)$ is a coincidence. It is exact for if $u \in C_c^\infty(E)$ has support outside the support of φ, then $f(D_M)M_\varphi u = 0$ and if $v \in C_c^\infty(F)$ has support outside the support of $\varphi \circ h^{-1}$ then $f(D_N)M_{\varphi o h^{-1}} v = 0$. By a similar argument, $(M_\varphi f(D_M), M_{\varphi o h^{-1}} f(D_N); U, V; h)$ is also an exact coincidence.

Finally, the property that

$$([f(D_M), M_\varphi], [f(D_N), M_{\varphi o h^{-1}}], U, V, h)$$

is an exact coincidence follows from the fact that

$$(f(D_M)M_\varphi, f(D_N)M_{\varphi o h^{-1}}; U, V; h)$$

and

$$(M_\varphi f(D_M), M_{\varphi o h^{-1}} f(D_N); U, V; h)$$

are exact coincidences. ♠

Proposition 5.6 *Let M be a complete, riemannian, spin manifold, D_M the Dirac operator on M, $U, V \subseteq M$ open coordinate balls centered at the same point such that $U \subseteq \bar{U} \subseteq V$, and N a compact spin manifold with the same dimension as M. Then there is a riemannian metric on N whose corresponding*

Dirac operator D_N *(with respect to any chosen spin structure on* N*) coincides locally with* D_M *over* U*.*

Moreover, if (D_M, D_N, h) *is a coincidence over* U*, then*

$$((D_M{}^2+1)^{-1}, (D_N{}^2+1)^{-1}, h)$$

is a coincidence over U*. If, in addition,* $\varphi \in C_c^\infty(U), r > 0$ *is smaller than the distance between the support of* φ *and the complement of* U*, and* f *is a* C^∞*-Flip function whose Fourier transform has compact support in* $[-r, r]$*, then*

$$(f(D_M)M_\varphi, f(D_N)M_{\varphi \circ h^{-1}}; h),$$

$$(M_\varphi f(D_M), M_{\varphi \circ h^{-1}} f(D_N); h),$$

and

$$([f(D_M), M_\varphi], [f(D_N), M_{\varphi \circ h^{-1}}]; h)$$

are exact acoincidences over U*. And if* $|t| \leq r$*, then*

$$(e^{itD_M} M_\varphi, e^{itD_N} M_{\varphi \circ h^{-1}}; h)$$

and

$$(M_\varphi e^{itD_M}, M_{\varphi \circ h^{-1}} e^{itD_N}; h)$$

are also exact coincidences over U*.*

Proof : Let $\varphi : B \to V$ be a normal coordinate mapping where $B \subseteq \mathbf{R}^m$ is a ball centered at 0. Take any point y in N and any coordinate mapping $\eta : B \to W$ where W is an open neighbourhood of y and $\eta(0) = y$. Transfer the riemannian metric on V to a riemannian metric on W by using the map

$\eta \circ \varphi^{-1}$. This gives a riemannian metric on $\mathcal{O} \stackrel{\text{def}}{=} (\eta \circ \varphi^{-1})(U)$. By a simple partition of unity argument, we may extend this riemannian metric on $\mathcal{O}$ to a riemannian metric ρ on the whole of N. Thus, ρ is a riemannian metric on N whose restriction to $\mathcal{O}$ agrees with the metric on U once we identify $\mathcal{O}$ with U by the map $\eta \circ \varphi^{-1}$. Now, let D_N be the Dirac operator on N determined by the riemannian metric ρ and any chosen spin structure on N. Let Δ_M and Δ_N be the bundle of spinors over M and N respectively. Let $h : U \to \mathcal{O}$ be the restriction of the map $\eta \circ \varphi^{-1}$ to the open set U. The map $h : U \to \mathcal{O}$ is an isometry. It induces a bundle isometry which we also denote by h from $\Delta_M|_{\mathcal{O}}$ to $\Delta_N|_{\mathcal{O}}$. That is, $h : \Delta_M|_U \to \Delta_N|_{\mathcal{O}}$ is a bundle isometry. It is then clear from the definition of the Dirac operator, that $(D_M, D_N; h)$ is a coincidence over U. Thus the first part of the proposition is true.

Now, let (D_M, D_N, h) be any coincidence over U, and let $\mathcal{O} = h(U)$. It is clear that $({D_M}^2 + 1, {D_N}^2 + 1, h)$ is also a coincidence over U. Since the operators ${D_M}^2 + 1$ and ${D_N}^2 + 1$ are differential operators, they are 0-local. Hence, the operators $({D_M}^2 + 1)^{-1}$ and $({D_N}^2 + 1)^{-1}$ are also 0-local. Thus, if $v \in C_c^\infty(\mathcal{O}, \Delta_N)$, then $({D_N}^2 + 1)^{-1}(v) = w$ for some $w \in L^2(\mathcal{O}, \Delta_N)$, in the domain of ${D_N}^2 + 1$ and $v = ({D_N}^2 + 1)(w)$. Since $({D_M}^2 + 1, {D_N}^2 + 1, h)$ is a coincidence, it follows that $h^* w \in L^2(U, \Delta_M)$ belongs to the domain of ${D_M}^2 + 1$ and that $h^* v = ({D_M}^2 + 1)(h^* w)$. Thus, $({D_M}^2 + 1)^{-1}(h^* v) = h^* w$, which implies that $w = h_*(({D_M}^2 + 1)^{-1}(h^* v))$. Hence,

$$({D_N}^2 + 1)^{-1}(v) = h_*(({D_M}^2 + 1)^{-1}(h^* v))$$

which shows that $(({D_M}^2 + 1)^{-1}, ({D_N}^2 + 1)^{-1}, h)$ is a coincidence over U. The rest of the proposition now follows from Theorem 5.5. ♠

2.6 Elliptic Operators

Theorem 6.1 *Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic pseudodifferential operator of order d on a compact manifold. Suppose the principal symbol of P is $[\sigma_L P] \in S^d(E)/S^{d-1}(E)$. Then there exists a pseudodifferential operator $Q : C^\infty(F) \to C^\infty(E)$ of order $-d$, called* **a parametrix for *P***, *such that $PQ = Id + S$ and $QP = Id + S'$ where S and S' are infinitely smoothing operators. The principal symbol of Q is $[(\sigma_L P)^{-1}] \in S^{-d}(E)/S^{-d-1}(E)$.*

Remark 6.2 *If $[\sigma_L P]$ is the principal symbol of an elliptic pseudodifferential operator P, then $(\sigma_L P)(\xi)^{-1}$ may make sense only for large enough ξ. But the principal symbol of a pseudodifferential operator Q only depends on what $(\sigma_L Q)(\xi)$ does for large ξ. So $[(\sigma_L P)(\xi)^{-1}]$ is well-defined for elliptic P.*

Proof: This is part of Lemma 1.3.5 of [**Gil**]. ♠

Proposition 6.3 *If E and F are smooth hermitian vector bundles over a compact riemannian manifold M, and if $P : C^\infty(E) \to C^\infty(F)$ is an elliptic pseudodifferential operator of order d, and if $H_s(E)$ is the s-th Sobolev space of E for every $s \in \mathbf{R}$, then the kernel of the operator*

$$P : H_s(E) \to H_{s-d}(F)$$

is a finite dimensional subspace of $C^\infty(E)$, for every $S \in \mathbf{R}$.

Proof: This is Lemma 1.4.5 of [**Gil**]. ♠

Definition 6.4 *Let $P : C^\infty(E) \to C^\infty(E)$ be a pseudodifferential operator on a compact riemannian manifold. Then $P : C^\infty(E) \to C^\infty(E)$ is said to be*

invertible *if it is invertible as a map from* $C^\infty(E)$ *to* $C^\infty(E)$. *In this case, the inverse map is denoted by* $P^{-1} : C^\infty(E) \to C^\infty(E)$.

Lemma 6.5 *Let* $P : C^\infty(E) \to C^\infty(E)$ *be a pseudodifferential of order* d *on a compact riemannian manifold. If it so happens that the extended operator* $P : H_s(E) \to H_{s-d}(E)$ *is invertible for every* $s \in \mathbf{R}$, *then* $P : C^\infty(E) \to C^\infty(E)$ *is invertible and the inverse* $P^{-1} : C^\infty(E) \to C^\infty(E)$ *extends to the inverse* $P^{-1} : H_{s-d}(E) \to H_s(E)$ *of the operator* $P : H_s(E) \to H_{s-d}(E)$.

Proof: Assume $P : H_s(E) \to H_{s-d}(E)$ is invertible for every $s \in \mathbf{R}$. Then for every $s \in \mathbf{R}$, there is a bounded operator $P^{-1} : H_{s-d}(E) \to H_s(E)$ inverse to $P : H_s(E) \to H_{s-d}(E)$. This gives a well defined operator

$$P^{-1} : \bigcup_{s\in\mathbf{R}} H_s(E) \to \bigcup_{s\in\mathbf{R}} H_s(E)$$

If $u \in C^\infty(E)$, then $u \in H_{s-d}(E)$ for every $s \in \mathbf{R}$, which implies that $P^{-1}u \in H_s(E)$ for every $s \in \mathbf{R}$. Therefore $P^{-1}u \in C^\infty(E)$ for every $u \in C^\infty(E)$. So

$$P^{-1} : \bigcup_{s\in\mathbf{R}} H_s(E) \to \bigcup_{s\in\mathbf{R}} H_s(E)$$

restricts to a map

$$P^{-1} : C^\infty(E) \to C^\infty(E).$$

It is now easy to show that $PP^{-1}u = u$ and $P^{-1}Pu = u$ for every $u \in C^\infty(E)$. Thus $P : C^\infty(E) \to C^\infty(E)$ is invertible with inverse $P^{-1} : C^\infty(E) \to C^\infty(E)$. ♠

Proposition 6.6 *Let* $P : C^\infty(E) \to C^\infty(E)$ *be an elliptic pseudodifferential operator on a compact riemannian manifold. Then the following are equivalent.*

1. *$P : C^\infty(E) \to C^\infty(E)$ is 1-1 and the index of P is* 0.

2. *P is invertible.*

3. *For some $s \in \mathbf{R}$, $P : H_s(E) \to H_{s-d}(E)$ is invertible.*

Proof: Suppose $P : C^\infty(E) \to C^\infty(E)$ is 1-1 and that the index of P is 0. Take $s \in \mathbf{R}$. Since $P : C^\infty(E) \to C^\infty(E)$ is 1-1, then so is $P : H_s(E) \to H_{s-d}(E)$. Hence the kernel of $P : H_s(E) \to H_{s-d}(E)$ is 0. Since the Fredholm index of $P : H_s(E) \to H_{s-d}(E)$ is 0 and the dimension of its kernel is 0, then the image of $P : H_s(E) \to H_{s-d}(E)$ must be all of $H_{s-d}(E)$. Thus, $P : H_s(E) \to H_{s-d}(E)$ is both 1-1 and onto which means it is invertible. This is true for all $s \in \mathbf{R}$. Hence, from Lemma 6.5, $P : C^\infty(E) \to C^\infty(E)$ is invertible. Thus (1) $\Rightarrow$ (2). Now suppose $P : C^\infty(E) \to C^\infty(E)$ is invertible. Then it is 1-1 and its range is all of $C^\infty(E)$. It follows that, for every $s \in \mathbf{R}$, the extended Fredholm operator $P : H_s(E) \to H_{s-d}(E)$ is 1-1, onto (since its closed range contains the dense subspace $C^\infty(E)$ of $H_{s-d}(E)$) and therefore invertible. Thus (2) $\Rightarrow$ (3). Now suppose that there exists $s \in \mathbf{R}$ such that $P : H_s(E) \to H_{s-d}(E)$ is invertible. Then the index of P would be 0, and $P : C^\infty(E) \to C^\infty(E)$ would be 1-1 since it is the restriction to $C^\infty(E)$ of the map $P : H_s(E) \to H_{s-d}(E)$. Thus (3) $\Rightarrow$ (1). ♠

Proposition 6.7 *If $A : C^\infty(E) \to C^\infty(E)$ is a self-adjoint elliptic pseudodifferential operator on a compact riemannian manifold, and if the spectrum of A does not contain* 0*, then $A : C^\infty(E) \to C^\infty(E)$ is invertible.*

Proof: Suppose the spectrum of A does not contain 0. Let us use $\bar{A}$ to denote the closure $\bar{A} : Dom(\bar{A}) \to L^2(E)$ of $A : C^\infty(E) \to C^\infty(E)$. Since 0 is not in the spectrum of A, then $\bar{A} : Dom(\bar{A}) \to L^2(E)$ is 1-1. This implies that $A : C^\infty(E) \to C^\infty(E)$ is 1-1. Since A is self-adjoint, we also have that the index of A is 0. Thus, by Proposition 6.6, $A : C^\infty(E) \to C^\infty(E)$ is invertible. ♠

Proposition 6.8 *Let d be any real number, and $P : C^\infty(E) \to C^\infty(E)$ an invertible elliptic pseudodifferential operator of order d on a compact riemannian manifold, (with principal symbol $[\sigma_L P] \in S^d(E)/S^{d-1}(E)$ where $\sigma_L P \in S^d(E)$.) Then the inverse $P^{-1} : C^\infty(E) \to C^\infty(E)$ is an elliptic pseudodifferential operator of order $-d$ with principal symbol $[(\sigma_L P)^{-1}] \in S^{-d}(E)/S^{-d-1}(E)$. Moreover, for each $s \in \mathbf{R}$, the extended operators*

$$P : H_s(E) \to H_{s-d}(E)$$

and

$$P^{-1} : H_{s-d}(E) \to H_s(E)$$

are inverse bounded operators.

Proof: By Theorem 6.1, there exists an elliptic pseudodifferential operator Q of order $-d$ with principal symbol $[(\sigma_L P)^{-1}] \in S^{-d}(E) \backslash S^{-d-1}(E)$ and an infinitely smoothing S such that

$$PQ = 1 + S.$$

Multiplying this equation on the left by $P^{-1} : C^\infty(E) \to C^\infty(E)$ gives us that $Q = P^{-1} + P^{-1}S$, or that

$$P^{-1} = Q - P^{-1}S.$$

But S is infinitely smoothing and so maps each $H_s(E)$ to $C^\infty(E)$. Since P^{-1} maps $C^\infty(E)$ to $C^\infty(E)$, it follows that $P^{-1}S$ is an infinitely smoothing operator. Putting this together with the fact that Q is an elliptic pseudodifferential operator of order $-d$ gives us that $P^{-1} = Q - P^{-1}S$ is an elliptic pseudodifferential operator of order $-d$ with the same principal symbol $[(\sigma_L P)^{-1}]$ as Q. The rest follows from Lemma 6.5. ♠

Lemma 6.9 *Let $P : C^\infty(E) \to C^\infty(E)$ be an invertible, self-adjoint, elliptic pseudodifferential operator of positive order $d > 0$ on a compact riemannian manifold. Then there exists a complete orthonormal basis $\{v_k\}_{k=1}^\infty$ for $L^2(E)$ consisting of $C^\infty(E)$ eigenvectors v_k of P, and, if λ_k is the eigenvalue of P corresponding to v_k, then $\lambda_k \in \mathbf{R}$ and $|\lambda_k| \to \infty$ as $k \to \infty$.*

Proof: The inverse operator $P^{-1} : C^\infty(E) \to C^\infty(E)$ of $P : C^\infty(E) \to C^\infty(E)$ is an elliptic pseudodifferential operator of order $-d$ (by Proposition 6.8). Moreover, for each $s \in \mathbf{R}$, the extended bounded operators

$$P : H_s(E) \to H_{s-d}(E)$$

and

$$P^{-1} : H_{s-d}(E) \to H_s(E)$$

are inverse to each other. Now P^{-1} maps $H_0(E) = L^2(E)$ onto $H_d(E)$. Since $d > 0$, $H_d(E)$ is included in $L^2(E)$ and the inclusion map is compact. Thus, P^{-1} maps $L^2(E)$ into $L^2(E)$ and this operator $P^{-1} : L^2(E) \to L^2(E)$ is compact. Now, since P is self-adjoint, then $P^{-1} : L^2(E) \to L^2(E)$ is also self-adjoint. By a well-known thereom on self-adjoint, compact operators (see

Thereoms 12.29 and 12.30 of [**Rud**] for example) there is a complete orthonormal basis $\{v_1, v_2, \ldots\}$ for $L^2(E)$ consisting of eigenvectors $v_1, v_2, v_3, \ldots$ of P^{-1}. Moreover, if μ_k is the eigenvalue of P^{-1} corresponding to v_k, then each μ_k belongs to $\mathbf{R}$, and $\mu_k \to 0$ as $k \to \infty$. Of course, no μ_k is equal to 0. Otherwise, for such a μ_k, we would have $P^{-1}v_k = 0$ and thus $P(0) = v_k$ which is not true. So $\mu_k \neq 0$ for all k. Let $\lambda_k = \frac{1}{\mu_k}$. Then $v_k = \lambda_k P^{-1}(v_k)$. Since the image of $P^{-1} : L^2(E) \to L^2(E)$ is actually $H_d(E)$, it follows that $v_k \in H_d(E)$ and that $Pv_k = \lambda_k v_k$, or that $(P - \lambda_k)v_k = 0$. That is, v_k belongs to the kernel of the operator

$$P - \lambda_k : H_d(E) \to L^2(E).$$

Now $P - \lambda_k : C^\infty(E) \to C^\infty(E)$ is also an elliptic pseudodifferential operator of order d. By Proposition 6.3, it follows that the kernel of $P - \lambda_k : H_d(E) \to L^2(E)$ is a subspace of $C^\infty(E)$. Thus, $v_k \in C^\infty(E)$ for each k. Also, from above, $\{v_k\}_{k=1}^\infty$ is a complete orthonormal basis for $L^2(E)$ and each v_k is an eigenvector of P with eigenvalue λ_k. Finally, since $\mu_k \to 0$ as $k \to \infty$, then $|\lambda_k| = |\frac{1}{\mu_k}| \to \infty$ as $k \to \infty$, which completes the proof. ♠

Theorem 6.10 *Let $P : C^\infty(E) \to C^\infty(E)$ be a self-adjoint, elliptic pseudodifferential operator of positive order $d > 0$ on a compact riemannian manifold. Then there exists a complete orthonormal basis $\{u_k\}_{k=1}^\infty$ for $L^2(E)$ consisting of $C^\infty(E)$ eigenvectors v_k of P, and, if λ_k is the eigenvalue of P corresponding to v_k, then $\lambda_k \in \mathbf{R}$ and $|\lambda_k| \to \infty$ as $k \to \infty$.*

Proof: By Proposition 6.3, ellipticity of P implies that the kernel $\ker(P)$ of $P : C^\infty(E) \to C^\infty(E)$ is a finite dimensional subspace of $C^\infty(E)$. Since P is self-adjoint then we also have that $\ker(P)$ is orthonormal to the image $Im(P)$ of $P : C^\infty(E) \to C^\infty(E)$. Let $Q : L^2(E) \to L^2(E)$ be the orthogonal projection onto $\ker(P)$. Then $P + Q : C^\infty(E) \to C^\infty(E)$ is 1-1 and self-adjoint (since both P and Q are self-adjoint). Moreover, since $\ker P$ is a finite dimensional subspace of $C^\infty(E)$, it is not difficult to show that $Q : L^2(E) \to L^2(E)$ is an integral operator with a C^∞ kernel. Thus Q is infinitely smoothing, which, together with the fact that P is an elliptic pseudodifferential operator of order d, implies that $P + Q : C^\infty(E) \to C^\infty(E)$ is also an elliptic pseudodifferential operator of order $d > 0$. Since the operator is self-adjoint then its index is 0. Thus $P + Q : C^\infty(E) \to C^\infty(E)$ is both 1-1 and has index 0. By Proposition 6.6, it is invertible. We can therefore apply Lemma 6.9 to the operator $P + Q$ to obtain a complete orthonormal basis $\{w_k\}_{k=1}^\infty$ for $L^2(E)$ consisting of $C^\infty(E)$ eigenvectors w_k of $P+Q$ such that, if μ_k is the eigenvalue of $P + Q$ corresponding to w_k, then $\mu_k \in \mathbf{R}$ and $|\mu_k| \to \infty$ as $k \to \infty$. If $\ker(P) = \{0\}$, then $Q = 0$ and we are done. Suppose $\ker(P) \neq \{0\}$. Note that $\ker(P)$ is a subspace of the eigenspace $V_1 \subseteq C^\infty(E)$ of $P + Q$ corresponding to the eigenvalue 1. Suppose V_1 has dimension n and that the dimension of $\ker P$ is $m < n$. We may assume that $w_1, w_2, \ldots, w_n$ is a basis for V_1. Change this to another orthonormal basis $v_1, \ldots, v_n$ of V_1 so that the first m vectors $v_1, v_2, \ldots, v_m$ span $\ker(P)$. Then let $v_k = w_k$ for $k \geq n+1$, and let $\lambda_k = 0$ for $1 \leq k \leq n$, and $\lambda_k = \mu_k$ for $k \geq n+1$. Then it is clear that $Pv_k = \lambda_k v_k$ for all k, that $\{v_k\}_{k=1}^\infty$ spans $L^2(E)$, and that $|\lambda_k| \to \infty$ as $k \to \infty$. Thus Theorem 6.10 is true. ♠

Recall: If $d \in \mathbf{R}$, then a Calderón-Zygmund operator (or CZO) $P : C^\infty(E) \to C^\infty(E)$ of order d on a compact riemannian manifold M is a special case of a pseudodifferential operator of order d on M. (Please see [**Hor2**], [**K&N**], [**Pal**], or [**See2**] for its definition.) For this thesis, we only need to know that differential operators of order n, are special cases of $CZ0's$ of order n, and that $CZ0's$ of order d are special cases of pseudodifferential operators of order d.

Theorem 6.11 *Let $A : C^\infty(E) \to C^\infty(E)$ be a self-adjoint, invertible, elliptic Calderón-Zygmund operator $(CZ0)$ of positive order $d > 0$ and with principal symbol $[\sigma_L A]$ in $S^d(E)/S^{d-1}(E)$ on a compact riemannian manifold. We will assume that $\sigma_L A$ is self adjoint. This assumption is justified by Proposition 2.1. Then, for every $s \in \mathbf{C}$, A^s is an invertible elliptic $CZ0$ (and hence a pseudodifferential operator) of order $d \cdot \mathrm{Re}(S)$, with principal symbol $[(\sigma_L A)^s]$ in $S^{d \cdot \mathrm{Re}(s)}(E)/S^{d \cdot \mathrm{Re}(s)-1}(E)$.*

Proof: This is a special case of Theorem 3 of [**See1**]. ♠

Theorem 6.12 *Let M be a compact riemannian manifold, E a smooth, finite dimensional, hermitian vector bundle over M, and $A : C^\infty(E) \to C^\infty(E)$ an elliptic self adjoint Calderón-Zygmund operator (CZO) of positive order $d > 0$. If $I \subseteq \mathbf{R}$ is an interval, let χ_I denote the characteristic function on I. Then, for every $a \in \mathbf{R}$, the operators $\chi_{[a,\infty)}(A), \chi_{(-\infty,a]}(A), \chi_{(a,\infty)}(A)$ and $\chi_{(-\infty,a)}(A)$ are Calderón-Zygmund operators (and hence pseudodifferential operators) of order 0. Moreover, suppose $\sigma_L A \in S^d(E)$ is self-adjoint and represents the* *principal symbol* $[\sigma_L(A)] \in S^d(E)\backslash S^{d-1}(E)$ *of* A. *Such a* $\sigma_L A$ *exists by Proposition 2.1. Then, for* $a \in \mathbf{R}$ *and for* $I = (-\infty, a)$, $(-\infty, a]$, (a, ∞), *or* $[a, \infty)$, *the principal symbol of* $\chi_I(A)$ *in* $S^0(E)/S^{-1}(E)$ *is* $[\chi_I(\sigma_L(A))]$.

Proof: Suppose $\sigma_L(A) \in S^d(E)$ is self-adjoint and represents the principal symbol of A. By ellipticity of A and Proposition 6.3, the kernel $\ker(A)$ is a finite dimensional subspace of $C^\infty(E)$. Let $Q : L^2(E) \to L^2(E)$ be the orthogonal projection onto $\ker(A)$. Then, as mentioned before, Q is an integral operator with C^∞ kernel, and therefore an infinitely smoothing projection. Thus, the operator $B = A + Q$ is an elliptic CZO of the same order d as A, and with the same principal symbol $[\sigma_L(A)]$. It is also self-adjoint (since both A and Q are self-adjoint) and therefore has index 0. Moreover, it is clearly one to one. By Proposition 6.6, it follows that B is an invertible elliptic self-adjoint CZO of order $d > 0$. Therefore B^2 is also an invertible elliptic self-adjoint CZO of order $2d > 0$, and with principal symbol $[(\sigma_L A)^2] \in S^{2d}(E)/S^{2d-1}(E)$. From Theorem 6.11, it follows that $|B|^{-1} = (B^2)^{-\frac{1}{2}}$ is an invertible elliptic self-adjoint CZO of order $-d$ and with principal symbol $[((\sigma_L A)^2)^{-\frac{1}{2}}] = [|\sigma_L A|^{-1}] \in S^{-d}(E)/S^{-d-1}(E)$. Therefore $B \cdot |B|^{-1}$ is an invertible elliptic CZO of order $d + (-d) = 0$ with principal symbol $\sigma_L(B \cdot |B|^{-1}) = [(\sigma_L A) \cdot |\sigma_L A|^{-1}] = [g(\sigma_L A)]$ in $S^0(E)/S^{-1}(E)$ where

$$g(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}$$

Note that, with respect to the decomposition

$$L^2(E) = \ker(A)^\perp \oplus \ker(A),$$

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where A_0 is the operator $A|_{\ker(A)^\perp} : \ker(A)^\perp \to \ker(A)^\perp$. Thus,

$$B = A + Q = \begin{pmatrix} A_0 & 0 \\ 0 & 1 \end{pmatrix}, \ |B|^{-1} = \begin{pmatrix} |A_0|^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} B \cdot |B|^{-1} &= \begin{pmatrix} A_0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} |A_0|^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} A_0 \cdot |A_0|^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot= \begin{pmatrix} g(A_0) & 0 \\ 0 & g(0) \end{pmatrix}. \end{aligned}$$

Therefore,

$$B \cdot |B|^{-1} = g\left(\begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\right) = g(A).$$

So, we have shown that $g(A)$ is an invertible elliptic CZO of order 0 and with principal symbol $[g(\sigma_L A)]$. Since $\chi_{[0,\infty)} = \frac{1}{2}(1+g)$, then $\chi_{[0,\infty)}(A) = \frac{1}{2}(1+g(A))$ is a CZO of order 0 with principal symbol $[\frac{1}{2}(1+g(\sigma_L A))] = [\chi_{[0,\infty)}(\sigma_L A)]$. From this it follows that $\chi_{(-\infty,0)}(A) = 1 - \chi_{[0,\infty)}(A)$ is also a CZO of order 0 with principal symbol $[1 - \chi_{[0,\infty,)}(\sigma_L A)] = [\chi_{(-\infty,0)}(\sigma_L A)]$. Thus we have shown that the principal symbol of $\chi_I(A)$ is $[\chi_I(\sigma_L A)]$ for $I = [0,\infty)$ and $I = (-\infty, 0)$.

Now let I be any finite interval $\subseteq \mathbf{R}$. Since A is a self-adjoint, elliptic pseudodifferential operator of positive order, then, by Theorem 6.10, there is a complete orthonormal basis $\{v_k\}_{k=1}^{\infty}$ for $L^2(E)$ consisting of $C^\infty(E)$ eigenvectors v_k of P, and if we let λ_k be the eigenvalue of A corresponding to v_k, then $\lambda_k \in \mathbf{R}$ at $|\lambda_k| \to \infty$ as $k \to \infty$. From this we see that $\chi_I(A)$ is an orthogonal projection onto a finite dimensional subspace of $C^\infty(E)$. As we have seen before, such a projection is an infinitely smoothing operator. Thus $\chi_I(A)$ is infinitley smoothing. Let $r > 0$ be such that $I \subseteq [-r, r]$. By the fact that A is an elliptic operator of positive order $d > 0$, there exists an $R > 0$ such that whenever $\xi \in T^*M, \|\xi\| > R$, and λ is in the spectrum of $(\sigma_L A)(\xi)$, then $|\lambda| > r$. Thus

$$\|\xi\| > R \Rightarrow \chi_I((\sigma_L A)(\xi)) = 0.$$

Therefore, the symbol class $[\chi_I(\sigma_L A)]$ of $\chi_I(\sigma_L A)$ in $S^0(E)/S^{-1}(E)$ is 0. That is, $[\chi_I(\sigma_L A)] = 0$ for every finite interval $I \subseteq \mathbf{R}$.

Now, if $a \in \mathbf{R}$, then $\chi_{[a,\infty)} = \chi_{[0,\infty)} + \chi_I$ for some finite interval I. Thus $\chi_{[a,\infty)}(A) = \chi_{[0,\infty)}(A) + \chi_I(A)$. We just showed that $\chi_I(A)$ is infinitely smoothing and that $\chi_{[0,\infty)}(A)$ is a CZO of order 0. Thus, $\chi_{[a,\infty)}(A)$ is a pseudodifferential operator of order 0 with principal symbol equal to the principal symbol $[\chi_{[0,\infty)}\,(\sigma_L(A))] \in S^0(E)/S^{-1}(E)$ of $\chi_{[0,\infty)}\,(\sigma_L A)$. Since $[\chi_I(\sigma_L A)] = 0$ in $S^0(E)/S^{-1}(E)$, then $[\chi_{[0,\infty)}\,(\sigma_L A)] = [\chi_{[0,\infty)}\,(\sigma_L A)] + [\chi_I(\sigma_L A)] = [\chi_{[a,\infty)}\,(\sigma_L A)]$ in $S^0(E)/S^{-1}(E)$. We can therefore say that the principal symbol of $\chi_J(A)$ is $[\chi_J(\sigma_L A)]$ whenever $J = [a,\infty), a \in \mathbf{R}$. The rest of the theorem can now be proved by a similar argument. ♠

2.7 Toeplitz Extensions

Lemma 7.1 *If $D : C^\infty(\Delta) \to C^\infty(\Delta)$ is the Dirac operator on a compact riemannian spin manifold. Then $(D^2+1)^{-1}$ is a compact operator on $L^2(\Delta)$.*

Proof: The operator $D^2+1 : C^\infty(\Delta) \to C^\infty(\Delta)$ is an elliptic self-adjoint differential operator of order 2 on a compact manifold. Its spectrum which lies in $[1,\infty)$ does not contain 0. It is therefore invertible and, by Proposition 6.8, $(D^2+1)^{-1} : C^\infty(\Delta) \to C^\infty(\Delta)$ is a pseudodifferential operator of order -2 on a compact manifold. It therefore extends to a compact operator on $L^2(\Delta)$. ♠

Proposition 7.2 *If $D : C^\infty(\Delta) \to C^\infty(\Delta)$ is the Dirac operator on a compact riemannian spin manifold, and if $f \in C_0(\mathbf{R})$, then $f(D)$ is a compact operator on $L^2(\Delta)$.*

Proof: Since $\mathcal{S}$ is dense in $C_0(\mathbf{R})$, it suffices to show that $f(D)$ is compact for $f \in \mathcal{S}$. So take $f \in \mathcal{S}$. Then $f(D) = g(D)\cdot(D^2+1)^{-1}$ where g is the function on $\mathbf{R}$ sending $x \in \mathbf{R}$ to $f(x)\cdot(x^2+1)$. Since f is Schwartz, it follows that g is Schwartz and therefore $g(D)$ is a bounded operator on

$L^2(\Delta)$. By Lemma 7.1, $(D^2+1)^{-1}$ is a compact operator on $L^2(\Delta)$. Thus, $f(D) = g(D)\cdot(D^2+1)^{-1}$ is compact. ♠

Proposition 7.3 *If $D : C^\infty(\Delta) \to C^\infty(\Delta)$ is the Dirac operator on a compact riemannian spin manifold $M, f \in Flip$, and $\varphi \in C(M)$, then the commutator $[f(D), M_\varphi]$ is compact on $L^2(\Delta)$.*

Proof: Take $f \in Flip$ and $\varphi \in C(M)$. Let $a = f(-\infty)$ and $b = f(+\infty)$. Since M is compact, D has discrete spectrum. Hence, there exists $g \in Flip$ such that $g(\infty) = b, g(-\infty) = a, g(\lambda) = b$ for all $\lambda \geq 0$ in the spectrum of D, and $g(\lambda) = a$ for all $\lambda < 0$ in the spectrum of D. So, for all $\lambda \in spec(D)$, we have

$$g(\lambda) = a\cdot\chi_{(-\infty,0)}(\lambda) + b\cdot\chi_{[0,\infty)}(\lambda),$$

and therefore

$$g(D) = a\cdot\chi_{(-\infty,0)}(D) + b\cdot\chi_{[0,\infty)}(D).$$

Therefore,

$$[g(D), M_\varphi] = a\cdot[\chi_{(-\infty,0)}(D), M_\varphi] + b\cdot[\chi_{[0,\infty)}(D), M_\varphi].$$

Now, since D is an elliptic self-adjoint elliptic differential operator on compact M, it follows from Proposition 6.12 that $\chi_{(-\infty,0)}(D)$ and $\chi_{[0,\infty)}(D)$ are pseudodifferential operators of order 0 on M. M_φ is also a pseudodifferential operator of order 0 on M. Thus $[\chi_{(-\infty,0)}(D), M_\varphi]$ and $[\chi_{[0,\infty)}(D), M_\varphi]$ are pseudodifferential operators of negative order -1 on the compact manifold M. They are therefore compact on $L^2(\Delta)$. Hence $[g(D), M_\varphi]$ is compact on $L^2(\Delta)$. ♠

Lemma 7.4 *Let M be a complete riemannian spin manifold and D the Dirac operator on M. Then, for every $\varphi \in C_0^\infty(M)$, the operators $(D^2+1)^{-1}M_\varphi$ and $M_\varphi(D^2+1)^{-1}$ are compact.*

Proof: Since $C_c^\infty(M)$ is dense in $C_0^\infty(M)$, it suffices to prove this for the case where φ has compact support. By a partition of unity argument, we can then reduce to the case where φ has compact support in an open coordinate ball $V \subseteq M$. So take such a φ. We only need to show that $(D^2+1)^{-1}M_\varphi$ is compact since $M_\varphi(D^2+1)^{-1} = [(D^2+1)^{-1}M_{\bar{\varphi}}]^*$. Since φ has compact support in V, we can find an open ball $U \subseteq M$ with the same center as V such that $U \subseteq \bar{U} \subseteq V$ and such that U contains the closed support of φ. Now, let S denote the sphere of the same dimension as M. By Proposition 5.6, there is a riemannian metric on S whose corresponding Dirac operator D_S (acting on $C^\infty(\Delta_S)$) coincides locally with D over U. Moreover, by the same proposition, $(D^2+1)^{-1}$ coincides locally with $({D_S}^2+1)^{-1}$ over U. By Lemma 5.3, $(D^2+1)^{-1}M_\varphi$ coincides exactly with $({D_S}^2+1)^{-1}M_{\varphi o h^{-1}}$ over U. By Lemma 7.1, we have that $({D_S}^2+1)^{-1}$ is a compact operator on $L^2(\Delta_S)$. It follows that $({D_S}^2+1)^{-1}M_{\varphi o h^{-1}}$ is compact on $L^2(\Delta_S)$. Since $({D_S}^2+1)^{-1}M_\varphi$ coincides exactly with $({D_S}^2+1)^{-1}M_{\varphi o h^{-1}}$, it follows from Property 2 of Proposition 5.4 that $(D^2+1)^{-1}M_\varphi$ is compact. This completes the proof. ♠

Proposition 7.5 *Let M be a complete riemannian spin manifold and D the Dirac operator on M. Then, for every $f \in C_0(\mathbf{R})$, for every $\varphi \in C_0(M)$, the operators $f(D)M_\varphi$ and $M_\varphi f(D)$ (acting on L^2-sections of the bundle of spinors over M) are compact.*

Proof: Since $\mathcal{S}$ is dense in $C_0(\mathbf{R})$, it suffices to prove this for the case where $f \in \mathcal{S}$. So, take $f \in \mathcal{S}$ and $\varphi \in C_0(M)$. Then the operator $f(D)M_\varphi$ is equal to the operator $g(D)\cdot(D^2+1)^{-1}M_\varphi$ where g is the function $x \mapsto f(x)(x^2+1)$ on $\mathbf{R}$. Since f is Schwartz, then so is g. Thus $g(D)$ is a bounded operator. By Lemma 7.4, the operator $(D^2+1)^{-1}M_\varphi$ is compact. Hence $f(D)M_\varphi$ is the product of a bounded operator $g(D)$ and a compact operator $(D^2+1)^{-1}M_\varphi$. We have therefore that $f(D)M_\varphi$ is compact. Now since $\bar{f} \in \mathcal{S}$ and $\bar{\varphi} \in C_0(M)$, then $\bar{f}(D)M_{\bar{\varphi}}$ is also compact. Therefore, the adjoint $M_\varphi \cdot f(D)$ is compact. This completes the proof. ♠

Proposition 7.6 *Let M be a complete riemannian spin manifold, D the Dirac operator on M, f an element of $Flip$, and φ an element of $C_0(M)$. Then the commutator $[f(D), M_\varphi]$ is compact.*

Proof: Since $C_c^\infty(M)$ is dense in $C_0(M)$, and $Flip_c$ is dense in $Flip$, it suffices to prove this for the case where $\varphi \in C_c^\infty(M)$ and $f \in Flip_c$. By using a partition of unity, we can then further reduce to the case where φ has support in an open coordinate ball. Assume this to be the case and that $f \in Flip_c$. Let $U \subseteq V$ be an open ball with the same center as V such that U contains the closed support of φ and $U \subseteq \bar{U} \subseteq V$. Pick a number $r > 0$ smaller than the distance between the support of φ and the complement of U. Since f is in $Flip_c$, then by Proposition 1.5, f can be written as the sum $f = f_1 + f_2$ where f_2 is a Schwartz function, and f_1 is a C^∞-$Flip$ function whose Fourier transform $\widehat{f_1}$ has support in $[-r, r]$. Now $[f_2(D), M_\varphi]$ is compact by Proposition 7.5. Since $[f(D), M_\varphi] = [f_1(D), M_\varphi] + [f_2(D), M_\varphi]$, the problem reduces to showing that $[f_1(D), M_\varphi]$ is compact.

Let S be a sphere of the same dimension as M. By Proposition 5.6, there is a riemannian metric on S whose corresponding Dirac operator D_S coincides locally with D over U. Let $h : U \to V$ be the coincidence map. By Theorem 5.5, it follows that $[f_1(D), M_\varphi]$ coincides exactly with $[f_1(D_S), M_{\varphi o h^{-1}}]$ over U. But S is compact. Hence $[f_1(D_s), M_{\varphi o h^{-1}}]$ is compact by Proposition 7.3. Since $[f_1(D), M_\varphi]$ coincides exactly with $[f_1(D_s), M_{\phi o h^{-1}}]$ then, by Proposition 5.4 (part 2), $[f_1(D), M_\varphi]$ is also compact. This completes the proof of Proposition 7.6. ♠

Definition 7.7 *: Let*

$$\boldsymbol{Flip}_l = \{f \in Flip : f(-\infty) = 1 \text{ and } f(\infty) = 0\}$$

$$\boldsymbol{Flip}_r = \{f \in Flip : f(-\infty) = 0 \text{ and } f(\infty) = 1\}$$

$$\boldsymbol{Flip}_{lc} = Flip_l \bigcap Flip_c$$

$$\boldsymbol{Flip}_{rc} = Flip_r \bigcap Flip_c.$$

Definition 7.8 *Let M be a complete riemannian spin manifold, Δ the bundle of spinors over M and $D : C^\infty(\Delta) \to C^\infty(\Delta)$ the Dirac operator on M. Let $\mathcal{K}$ denote the algebra of all compact operators on $L^2(\Delta)$. Define the* **(double) Toeplitz algebra $\mathcal{T}$ (of D)** *as the C^*-subalgebra of $B(L^2(\Delta))$ generated by $\mathcal{K}$, and by the set of all $f(D) \cdot M_\varphi$ and $M_\varphi \cdot f(D)$ in $B(L^2(\Delta))$ such that $\varphi \in C_0(M)$ and $f \in Flip$. If $T \in \mathcal{T}$ let $[T]$ denote the class $T + \mathcal{K}$ in the quotient C^*-algebra $\mathcal{T}/\mathcal{K}$. Define the* **Toeplitz map**

$$\tau : C_0(M) \oplus C_0(M) \to \mathcal{T}/\mathcal{K}$$

by letting

$$\tau(\varphi \oplus \eta) = [f(D)M_\varphi + g(D)M_\eta]$$

where f is any $Flip_l$ function and g is any $Flip_r$ function.

Lemma 7.9 *The Toeplitz map*

$$\tau : C_0(M) \oplus C_0(M) \to \mathcal{T}/\mathcal{K}$$

of Definition 7.8 is a well-defined surjective $*$-homomorphism.

Proof: If $f_1, f_2 \in Flip_l, g_1, g_2 \in Flip_r$, and $\varphi, \eta \in C_0(M)$, then

$$(f_1(D)M_\varphi + g_1(D)M_\eta) - (f_2(D)M_\varphi + g_2(D) \cdot M_\eta)$$

$$= (f_1 - f_2)(D) \cdot M_\varphi + (g_1 - g_2)(D) \cdot M_\eta$$

is compact by Proposition 7.5, since $f_1 - f_2$ and $g_1 - g_2$ belong to $C_0(\mathbf{R})$ and $\varphi, \eta \in C_0(M)$. So τ is well-defined. It is clear that τ is linear. To show that τ is a C^*-algebra homomorphism, we use Proposition 7.6 which says that

$$f(D)M_\varphi \equiv M_\varphi f(D) \bmod \mathcal{K}$$

whenever $f \in Flip$ and $\varphi \in C_0(M)$, and Proposition 7.5 which says that

$$f(D)M_\varphi \equiv M_\varphi f(D) \equiv 0 \bmod \mathcal{K}$$

whenever $f \in C_0(\mathbf{R})$ and $\varphi \in C_0(M)$. Now, take $\varphi, \eta, \mu, \nu \in C_0(M), f \in Flip_l$, and $g \in Flip_r$. Then

$$(f(D)M_\varphi + g(D)M_\mu)(f(D)M_\eta + g(D)M_\nu)$$

$$
\begin{aligned}
&= f(D)M_\varphi f(D)M_\eta + f(D)M_\varphi g(D)M_\nu \\
&\quad + g(D)M_\mu f(D)M_\eta + g(D)M_\mu g(D)M_\nu \\
&\equiv f(D)f(D)M_\varphi M_\eta + f(D)g(D)M_\varphi M_\nu \\
&\quad + g(D)f(D)M_\mu M_\eta + g(D)g(D)M_\mu M_\nu \\
&= f^2(D)M_{\varphi\cdot\eta} + (fg)(D)M_{\varphi\nu} \\
&\quad + (g\cdot f)(D)M_{\mu\cdot\eta} + g^2(D)M_{\mu\cdot\nu} \\
&\equiv f^2(D)M_{\varphi\cdot\eta} + g^2(D)\cdot M_{\mu\nu} \qquad \text{mod } \mathcal{K}
\end{aligned}
$$

for $fg \in C_0(\mathbf{R})$ and so $(fg)(D)M_{\varphi\nu} \equiv 0 \text{ mod } \mathcal{K}$ and $(fg)(D)M_{\mu\eta} \equiv 0 \text{ mod } \mathcal{K}$. Thus

$$
\begin{aligned}
&\tau(\varphi \oplus \mu)\tau(\eta \oplus \nu) \\
&\quad = [f(D)M_\varphi + g(D)M_\mu][f(D)M_\eta + g(D)M_\nu] \\
&\quad = [f^2(D)\cdot M_{\varphi\cdot\eta} + g^2(D)M_{\mu\cdot\nu}] \\
&\quad = \tau(\varphi\eta, \mu\nu)
\end{aligned}
$$

since $f^2 \in Flip_l$ and $g^2 \in Flip_r$. This shows that τ is an algebra homomorphism.

Next we note that if we take the same f, g, φ, μ as before, then

$$(f(D)M_\varphi + g(D)M_\mu)^* = M_{\bar\varphi}\cdot \bar f(D) + M_{\bar\mu}\cdot \bar g(D)$$

$$\equiv \bar f(D)M_{\bar\varphi} + \bar g \cdot M_{\bar\mu} \text{ mod } \mathcal{K}$$

so that

$$\tau(\varphi \oplus \mu)^* \;=\; [(f(D)M_\varphi + g(D)M_\mu)^*]$$

$$= [\bar{f}(D)M_{\bar{\varphi}} + \bar{g}(D)M_{\bar{\mu}}]$$
$$= \tau(\bar{\varphi} \oplus \bar{\mu})$$

since $\bar{f} \in Flip_l$ and $\bar{g} \in Flip_r$. Therefore τ is a $*$-homomoprphism.

We now show that τ is onto. Note that, since $[f(D)M_\varphi] = [M_\varphi \cdot f(D)]$ in $\mathcal{T}/\mathcal{K}$, then $\mathcal{T}/\mathcal{K}$ is generated by all the $[f(D)M_\varphi]$ such that $f \in Flip$ and $\varphi \in C_0(M)$. Also note that $Flip$ is the algebra generated by $Flip_l$ and the constant functions. Therefore $\mathcal{T}/\mathcal{K}$ is generated by the set A of all the $[M_\varphi]$ such that $\varphi \in C_0(M)$ and the set B of all $[f(D)M_\varphi]$ such that $f \in Flip_l$ and $\varphi \in C_0(M)$. B is certainly in the image of τ since each $[f(D)M_\varphi]$, where $f \in Flip_l$ and $\varphi \in C_0(M)$, is equal to $\tau(\varphi \oplus 0)$. The set A is also in the image of τ since each $[M_\varphi]$ in A is equal to $[f(D)M_\varphi + (1-f)(D)M_\varphi] = \tau(\varphi \oplus \varphi)$ where f is any $Flip_l$ function. This shows that all of $\mathcal{T}/\mathcal{K}$ lies in the image of τ, or that τ is onto. This completes the proof. ♠

Theorem 7.10 *Let M be a compact riemannian spin manifold, and let Δ, D, $\mathcal{T}$, $\mathcal{K}$, and*

$$\tau : C(M) \oplus C(M) \to \mathcal{T}/\mathcal{K}$$

be as in Definition 7.8. (When M is compact, $C_0(M)$ is the same as $C(M)$.) Then τ is a C^*-algebra isomorphism. This implies that if $f \in Flip_l, g \in Flip_r, \varphi, \eta \in C(M)$, then $f(D)M_\varphi + g(D)M_\eta$ is compact if and only if $\varphi = \eta = 0$.

Proof: We already know that τ is a surjective C*-algebra homomorphism (by Lemma 7.9). So we only need to show that it is $1-1$, or that $\ker(\tau) = \{0\}$. Suppose $\tau(\varphi, \eta) = 0$. Since M is compact, the spectrum of D is discrete. We

$$= [\bar{f}(D)M_{\bar{\varphi}} + \bar{g}(D)M_{\bar{\mu}}]$$
$$= \tau(\bar{\varphi} \oplus \bar{\mu})$$

since $\bar{f} \in Flip_l$ and $\bar{g} \in Flip_r$. Therefore τ is a $*$-homomoprphism.

We now show that τ is onto. Note that, since $[f(D)M_\varphi] = [M_\varphi \cdot f(D)]$ in $\mathcal{T}/\mathcal{K}$, then $\mathcal{T}/\mathcal{K}$ is generated by all the $[f(D)M_\varphi]$ such that $f \in Flip$ and $\varphi \in C_0(M)$. Also note that $Flip$ is the algebra generated by $Flip_l$ and the constant functions. Therefore $\mathcal{T}/\mathcal{K}$ is generated by the set A of all the $[M_\varphi]$ such that $\varphi \in C_0(M)$ and the set B of all $[f(D)M_\varphi]$ such that $f \in Flip_l$ and $\varphi \in C_0(M)$. B is certainly in the image of τ since each $[f(D)M_\varphi]$, where $f \in Flip_l$ and $\varphi \in C_0(M)$, is equal to $\tau(\varphi \oplus 0)$. The set A is also in the image of τ since each $[M_\varphi]$ in A is equal to $[f(D)M_\varphi + (1-f)(D)M_\varphi] = \tau(\varphi \oplus \varphi)$ where f is any $Flip_l$ function. This shows that all of $\mathcal{T}/\mathcal{K}$ lies in the image of τ, or that τ is onto. This completes the proof. ♠

Theorem 7.10 *Let M be a compact riemannian spin manifold, and let Δ, D, $\mathcal{T}$, $\mathcal{K}$, and*

$$\tau : C(M) \oplus C(M) \to \mathcal{T}/\mathcal{K}$$

be as in Definition 7.8. (When M is compact, $C_0(M)$ is the same as $C(M)$.) Then τ is a C^*-algebra isomorphism. This implies that if $f \in Flip_l, g \in Flip_r, \varphi, \eta \in C(M)$, then $f(D)M_\varphi + g(D)M_\eta$ is compact if and only if $\varphi = \eta = 0$.

Proof: We already know that τ is a surjective C*-algebra homomorphism (by Lemma 7.9). So we only need to show that it is $1-1$, or that $\ker(\tau) = \{0\}$. Suppose $\tau(\varphi, \eta) = 0$. Since M is compact, the spectrum of D is discrete. We can therefore find $f \in Flip_r$ such that $f(\lambda) = \chi_{[0,\infty)}(\lambda)$ for all λ in the spectrum of D. Therefore $f(D) = \chi_{[0,\infty)}(D)$ and $(1-f)(D) = \chi_{(-\infty,0)}(D)$. So, $0 = \tau(\varphi \oplus \eta) = [(1-f)(D) \cdot M_\varphi + f(D)M_\eta] = [\chi_{(-\infty,0)}(D) \cdot M_\varphi + \chi_{[0,\infty)}(D)M_\eta]$ which means that $\chi_{(-\infty,0)}(D) \cdot M_\varphi + \chi_{[0,\infty)}(D)M_\eta$ is compact. Multiplying on the left by $\chi_{[0,\infty)}(D)$ gives us that $\chi_{[0,\infty)}(D)M_\eta$ is compact, and multiplying instead by $\chi_{(-\infty,0)}(D)$ gives us that $\chi_{(-\infty,0)}(D) \cdot M_\varphi$ is compact. By Theorem 6.12, $P = \chi_{[0,\infty)}(D)$ and $Q = \chi_{(-\infty,0)}(D)$ are both pseudodifferential operators of order 0 with principal symbols $[\sigma_L P]$ and $[\sigma_L Q]$ respectively in $S^0(\Delta)/S^{-1}Del)$, where

$$\begin{aligned} \sigma_L P &= \chi_{[0,\infty)}(\sigma_L D), \\ \sigma_L Q &= \chi_{(-\infty,0)}(\sigma_L D), \end{aligned}$$

and $\sigma_L D$ is the principal symbol of D. Thus PM_η and QM_φ are pseudodifferential operators of order 0 with principal symbols $[\sigma_L(PM_\eta)], [\sigma_L(QM_\varphi)] \in S^0(\Delta)/S^{-1}(\Delta)$, where $\sigma_L(PM_\eta)$ is the section

$$\xi \mapsto \eta(x) \cdot (\sigma_L P)(\xi), \ \forall \xi \in (T^*M)_x$$

and $\sigma_L(QM_\varphi)$ is the section

$$\xi \mapsto \varphi(x) \cdot (\sigma_L Q)(\xi), \forall \xi \in (T^*M)_x.$$

By Proposition 3.7, $\sigma_L D \in S^1(\Delta)$ satisfies

$$(\sigma_L D)(\xi)(w) = i\xi \cdot w$$

whenever $x \in M, \xi \in (T^*M)_x$ and $w \in \Delta_x$. So, for every $\xi \in (T^*M)_x$, $(\sigma_L D)(\xi) : \Delta_x \to \Delta_x$ is the map $\sigma_\xi : \Delta_x \to \Delta_x$ given by Clifford multiplication on the left by $i\xi$. From Remarks 3.6, Point 5, we see that, for every $\xi \in T^*M$,

$$\text{spec}\,((\sigma_L D)\xi) \cup \text{spec}\,((\sigma_L D)(-\xi)) = \{-\|\xi\|, \|\xi\|\}. \tag{7.11}$$

Hence, for all $\xi \neq 0$ in $(T^*M)_x$

$$\begin{aligned} &\text{spec}(\sigma_L(PM_\eta)(\xi)) \bigcup \text{spec}\,(\sigma_L(PM_\eta)(-\xi)) \\ &\quad = \{\eta(x) \cdot \chi_{[0,\infty)}(-\|\xi\|), \eta(x) \cdot \chi_{[0,\infty)}(+\|\xi\|)\} \\ &\quad = \{0, \eta(x)\} \end{aligned}$$

and

$$\begin{aligned} &\text{spec}\,(\sigma_L(QM_\varphi)(\xi)) \cup \text{spec}(\sigma_L(QM_\varphi)(-\xi)) \\ &\quad = \{\varphi(x) \cdot \chi_{(-\infty,0)}(-\|\xi\|), \varphi(x) \cdot \chi_{(-\infty,0)}(\|\xi\|)\} \\ &\quad = \{\varphi(x), 0\}. \end{aligned}$$

Hence, for every $r > 0$, for every $x \in M$,

$$\sup\{\|\sigma_L(PM_\eta)(\xi)\| : \xi \in (T^*M)_x, \|\xi\| \geq r\} = |\eta(x)|$$

and

$$\sup\{\|\sigma_L(QM_\varphi)(\xi)\| : \xi \in (T^*M)_x, \|\xi\| \geq r\} = |\varphi(x)|.$$

It follows that, for every $r > 0$, $a_r \overset{\text{def}}{=} \sup\{\|\sigma_L(PM_\eta)(\xi)\| : \xi \in T^*M, \|\xi\| \geq r\}$ $= \|\eta\|_\infty$, and $b_r \overset{\text{def}}{=} \sup\{\|\sigma_L(QM_\varphi)(\xi)\| : \xi \in T^*M, \|\xi\| \geq r\} = \|\varphi\|_\infty$. Now, since PM_η and QM_φ are compact pseudodifferential operators of order 0, then, by Theorem 2.2, $a_r \to 0$ as $r \to \infty$, and $b_r \to 0$ as $r \to \infty$. Since $a_r = \|\eta\|_\infty, b_r = \|\varphi\|_\infty$ for all $r > 0$, it follows that $\|\eta\|_\infty = \|\varphi\|_\infty = 0$, which implies that $\eta = 0$ and $\varphi = 0$, and thus $(\varphi, \eta) = 0$. This proves that τ is 1-1. ♠

Theorem 7.12 *Let M be a complete riemannian spin manifold, and let Δ, D, $\mathcal{T}$, $\mathcal{K}$ and*

$$\tau : C_0(M) \oplus C_0(M) \to \mathcal{T}/\mathcal{K}$$

be as in Definition 7.8. Then τ is a C^*-algebra isomorphism. This implies that if $f \in Flip_l, g \in Flip_r, \varphi, \eta \in C_0(M)$, then $f(D)M_\varphi + g(D)M_\eta$ is compact if and only if $\varphi = \eta = 0$.

Proof : We already know that τ is a surjective C^*-algebra homomorphism (Lemma 7.9). So we only have to show that τ is 1-1, or that ker $\tau = \{0\}$. Assume $\tau(\varphi, \eta) = 0$ where $\varphi, \eta \in C_0(M)$. Suppose $f \in Flip_{lc}$. Then there is a $g \in Flip_{lc}$ such that $g \cdot f = f$. Note that $1 - g \in Flip_{rc}$. Since $\tau(\varphi, \eta) = 0$, it follows that

$$f(D)M_\varphi + (1-g)(D) \cdot M_\eta \in \mathcal{K}.$$

Multiplying on the left by $g(D)$ gives us that

$$f(D)M_\varphi + (g-g^2)(D) \cdot M_\eta \in \mathcal{K}.$$

But $g - g^2 \in C_0(\mathbf{R})$ which implies that $(g-g^2)(D) \cdot M_\eta \in \mathcal{K}$ (by Proposition 7.5). Therefore $f(D)M_\varphi \in \mathcal{K}$ for every $f \in Flip_{lc}$. Since $Flip_{lc}$ is dense in $Flip_l$, then $f(D)M_\varphi \in \mathcal{K}$ for every $f \in Flip_l$. Similarly, we can show that $g(D)M_\eta \in \mathcal{K}$ for every $g \in Flip_r$.

Now take any $x \in M$ and let $U, V \subseteq M$ be coordinate balls centered at x such that $\bar{U} \subseteq V$. Let $\rho \in C_c^\infty(U)$ be a bump function such that $\rho(x) = 1$. Let $r > 0$ be smaller than the distance from the support of ρ to the complement of U. Let f be a C^∞-$Flip_l$ function and g be a C^∞-$Flip_r$ function such that $\hat{f}$ and $\hat{g}$ has support in $[-r, r]$. (This is possible by Corollary 1.10.) Let S be the sphere of the same dimension as M. By Proposition 5.6, there exists a riemannian metric on S with corresponding Dirac operator $D_S : C^\infty(\Delta_S) \to C^\infty(\Delta_S)$, and exact coincidences

$$(f(D)M_{\varphi\cdot\rho}, f(D_S)\cdot M_{\varphi\rho\circ h^{-1}}; h)$$

and

$$(g(D)M_{\eta\cdot\rho}, g(D_S)\cdot M_{\eta\cdot\rho\circ h^{-1}}; h)$$

over U. Since $f(D)M_{\varphi\cdot\rho}$ and $g(D)M_{\eta\cdot\rho}$ are both compact (for $f(D)M_\varphi$ and $g(D)M_\eta$ are compact from above), it follows from Proposition 5.4 (part 2), that $f(D_S)\cdot M_{\varphi\rho\circ h^{-1}}$ and $g(D_S)M_{\eta\cdot\rho\circ h^{-1}}$ are compact. This means that

$$\tau_S(\varphi\cdot\rho\circ h^{-1}, \eta\cdot\rho\circ h^{-1}) = 0$$

where

$$\tau_S : C(S)\oplus C(S) \to \mathcal{T}_S/\mathcal{K}(L^2(\Delta_S))$$

is the Toeplitz map corresponding to D_s and $\mathcal{T}_S$ is the (double) Toeplitz algebra for D_S. But since S is compact, we know that τ_S is 1-1 (by Theorem 7.10). Hence, $\varphi\cdot\rho\circ h^{-1} = 0$, and $\eta\cdot\rho\circ h^{-1} = 0$. This implies that $\varphi\cdot\rho = 0$ and $\eta\cdot\rho = 0$. Since $\rho(x) = 1$, it follows that $\varphi(x) = \eta(x) = 0$. This is true for all $x \in M$. Hence $\varphi = 0$ and $\eta = 0$, which proves that τ is 1-1. ♠

If M is a complete riemannian spin manifold and $\Delta, D, \mathcal{T}, \mathcal{K}$ and τ are as in Definition 7.8, then, since

$$\tau : C_0(M)\oplus C_0(M) \to \mathcal{T}/\mathcal{K}$$

is an isomorphism (Theorem 7.12), we get a C*-algebra extension

$$0 \to \mathcal{K} \xrightarrow{i} \mathcal{T} \to C_0(M)\oplus C_0(M) \to 0 \qquad (*)$$

where $i : \mathcal{K} \to \mathcal{T}$ is the inclusion map and the map

$$\mathcal{T} \to C_0(M) \oplus C_0(M)$$

is the composition

$$\mathcal{T} \xrightarrow{q} \mathcal{T}/\mathcal{K} \xrightarrow{\tau^{-1}} C_0(M) \oplus C_0(M)$$

where $q : \mathcal{T} \to \mathcal{T}/\mathcal{K}$ is the quotient map.

Definition 7.13 *The C^*-algebra extension* $(*)$ *above will be called the* **(double) Toeplitz extension of $C_o(M)$**.

The single Toeplitz extensions for compact M. Now, we introduce two related extensions, for the case where M is compact. Let M be a compact riemannian spin manifold and let $D : C^\infty(\Delta) \to C^\infty(\Delta)$ be the Dirac operator on M. Let

$$P\chi_{[0,\infty)}(D)$$

$$Q = \chi_{(-\infty,0]}(D)$$

$$H_r = P(L^2(\Delta)),$$

and

$$H_l = Q(L^2(\Delta)).$$

Since M is compact then D has discrete spectrum. Hence, $P = f(D)$ for some $f \in Flip_r$, and $Q = g(D)$ for some $g \in Flip_l$. Thus, for every $\varphi \in C(M)$, $[P, M_\varphi] = [f(D), M_\varphi]$ is compact, and $[Q, M_\varphi] = [g(D), M_\varphi]$ is compact. Define the **positive, single Toeplitz algebra $\mathcal{T}_r$** to be the C^*-subalgebra of $B(H_r)$ generated by $\mathcal{K}(H_r)$ and the set of all the compressions

$T_\varphi = PM_\varphi P \in B(H_r)$ such that $\varphi \in C(M)$. Also, define the **negative single Toeplitz algebra** $\mathcal{T}_l$ to be the C^*-subalgebra of $B(H_l)$ generated by $\mathcal{K}(H_l)$ and the set of all $S_\varphi = QM_\varphi Q \in B(H_l)$ such that $\varphi \in C(M)$. Since $[P, M_\varphi]$ and $[Q, M_\varphi]$ are compact, we get $*$-homomorphisms

$$\tau_r : C(M) \to \mathcal{T}_r/\mathcal{K}(H_r)$$

$$\tau_l : C(M) \to \mathcal{T}_l/\mathcal{K}(H_l)$$

defined by

$$\tau_r(\varphi) = [PM_\varphi P]$$

$$\tau_l(\varphi) = [QM_\varphi Q].$$

The map τ_r is 1-1 for the following reason. Suppose $\tau_r(\varphi) = 0$. Then $PM_\varphi P$ would be compact. But $PM_\varphi P = f(D)M_\varphi f(D)$ for some $f \in Flip_r$. Hence, $f(D)M_\varphi f(D)$ would be compact. This would imply that $\tau(0, \varphi) = 0$ where $\tau : C(M) \oplus C(M) \to \mathcal{T}/\mathcal{K}$ is the Toeplitz map for D. Since τ is 1-1, we get that $\varphi = 0$. Therefore τ_r is 1-1. Similarly τ_l is 1-1. (Note that this also shows that H_r and H_l are infinite dimensional, or that P and Q are not compact.) These maps τ_r and τ_l are clearly onto. So they are $*$-isomorphisms. Hence we have corresponding extensions

$$0 \to \mathcal{K}(H_r) \to \mathcal{T}_r \to C(M) \to 0$$

$$0 \to \mathcal{K}(H_l) \to \mathcal{T}_l \to C(M) \to 0$$

which we call the **right Toeplitz extension of D** and the **left Toeplitz extension of D**. They generated index maps (Fredholm index maps)

$$\partial_r : K_1(C(M)) \to \mathbf{Z}$$

$$\partial_l : K_1(C(M)) \to \mathbf{Z}$$

since K_0 of the algebra of compacts is $\mathbf{Z}$. Now identify $K_1(C(M) \oplus C(M))$ with $K_1(C(M)) \oplus K_1(C(M))$. The Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \to C(M) \oplus C(M) \to 0$$

also generates an index map

$$\partial : K_1(C(M)) \oplus K_1(C(M)) \to \mathbf{Z}.$$

It is clear from the definition of the index maps of C^*-algebra extensions that the following is true.

Proposition 7.14

$$\partial(0, a) = \partial_r(a)$$

and

$$\partial(a, 0) = \partial_l(a)$$

for every $a \in K_1(C(M))$.

2.8 Odd-Dimensional Spheres

Let S be an odd-dimensional sphere with a given riemannian metric, and with Dirac operator $D : C^\infty(\Delta) \to C^\infty(\Delta)$. We know that $K^0(S) \cong \mathbf{Z}$ and that $K^1(S) \cong \mathbf{Z}$. Let

$$K^0(S) \otimes K_1(S) \xrightarrow{\cap} K_1(S)$$

denote the cap product (see [**BmDg3**]). The positive Toeplitz extension of D defines an element $[D]$ of $K_1(S)$. Poincaré duality for K-theory ([**BmDg3**]) tells us that cap product with $[D]$ gives an isomorphism

$$K^0(S) \xrightarrow{\cong} K_1(S).$$

From this, we see that $K_1(S) \cong \mathbf{Z}$ and that $[D] \in K_1(S) \cong \mathbf{Z}$ is a generator of $K_1(S)$. Now, there is an isomorphism

$$\gamma : K_1(S) \xrightarrow{\cong} \text{Hom}\ (K^1(S), K^0(\mathcal{K})) \cong \text{Hom}\ (\mathbf{Z}, \mathbf{Z}).$$

For every $[\tau] \in K_1(S)$ represented by the extension τ,

$$0 \to \mathcal{K} \to \mathcal{A} \to C(S) \to 0,$$

$\gamma([\tau]) : K^1(S) \to K^0(\mathcal{K})$ is the index map determined by τ. Thus,

$$\gamma([D]) : K^1(S) \to \mathbf{Z}$$

is the index map

$$\partial_r : K_1(C(S)) \to \mathbf{Z}$$

determined by the positive Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T}_r \to C(S) \to 0.$$

($K_1(C(S))$ is the same as $K^1(S)$.) Since $[D]$ is a generator for $K_1(S)$ and since γ is an isomorphism, then $\gamma([D]) = \partial_r$ must be an isomorphism from $K_1(C(S)) \cong \mathbf{Z}$ to $\mathbf{Z}$. So, the following is true.

Theorem 8.1 *If S is an odd-dimensional sphere with a given riemannian metric and with a corresponding Dirac operator $D : C^\infty(\Delta) \to C^\infty(\Delta)$ (with respect to a chosen spin structure), and if τ_r is the positive Toeplitz extension*

$$0 \to \mathcal{K} \to \mathcal{T}_r \to C(S) \to 0$$

for D, then the induced index map

$$\partial_r : K_1(C(S)) \to K_0(\mathcal{K})$$

is an isomorphism from $K_1(C(S)) \cong \mathbf{Z}$ to $K_0(\mathcal{K}) \cong \mathbf{Z}$. Thus, if $a \in K_1(C(S))$ is a generator of $K_1(C(S))$, then $\partial_r(a) = +1$ or -1.

If

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S) \oplus C(S) \to 0$$

is the double Toeplitz extension of D, and if

$$\partial : K_1(C(S)) \oplus K_1(C(S)) \to \mathbf{Z}$$

is the induced index map, then from Theorem 8.1 and Proposition 7.14, we see that the map $a \mapsto \partial(0, a) \in \mathbf{Z}$ is an isomorphism from $K_1(C(S))$ to $\mathbf{Z}$.

Now, let $a \in K_1(C(S))$ and suppose $a = [\varphi]$ where $\varphi \in M_k(C(S))$ is a unitary. Then $M_\varphi \in B(L^2(\Delta)^k)$ is Fredholm with Fredholm index 0, and so the element $[M_\varphi] \in K_1(C(S) \oplus C(S))$ represented by M_φ is such that $\partial([M_\varphi]) = 0$. But if $f \in Flip_r$, then

$$M_\varphi = (1 - f)(D) \cdot M_\varphi + f(D) \cdot M_\varphi$$

and therefore $[M_\varphi] \in K_1(C(S) \oplus C(S))$ is the same as the element $([\varphi],[\varphi]) = (a,a)$ in $K_1(C(S)) \oplus K_1(C(S))$. So $\partial(a,a) = \partial([M_\varphi]) = 0$ which implies that

$$\partial(a,0) = -\partial(0,a)$$

for all $a \in K_1(C(S))$. Since the map $a \mapsto \partial(0,a)$ from $K_1(C(S))$ to $\mathbf{Z}$ is an isomorphism, it follows that the map $a \mapsto \partial(a,0)$ is also an isomorphism. Also, if $a \in K_1(C(S))$ is a generator of $K_1(C(S))$, then

$$\partial(a,0) = -\partial(0,a) = +1 \text{ or } -1.$$

Moreover, since $\partial(a,0) = \partial_l(a)$ for all $a \in K_1(C(S))$ (Proposition 7.14), then $\partial_l : K_1(C(S)) \to \mathbf{Z}$, like ∂_r, is also an isomorphism, $\partial_l(a) = \partial(a,0) = -\partial(0,a) = -\partial_r(a)$ for all $a \in K_1(C(S))$, and if a is a generator of $K_1(C(S))$ then $\partial_l(a) = -\partial_r(a) = +1$ or -1. So, we have proved the following two theorems.

Theorem 8.2 *Both $\partial_r : K_1(C(S)) \to \mathbf{Z}$ and $\partial_l : K_1(C(S)) \to \mathbf{Z}$ are isomorphisms. Moreover, $\partial_l(a) = -\partial_r(a)$ for every $a \in K_1(C(S))$, and if $a \in K_1(C(S))$ is a generator, then $\partial_l(a) = -\partial_r(a) = +1$ or -1.*

Theorem 8.3 *The index map*

$$\partial : K_1(C(S)) \oplus K_1(C(S)) \to \mathbf{Z}$$

is an isomorphism on each component. That is the maps $a \mapsto \partial(0,a)$ and $a \mapsto \partial(a,0)$ are isomorphisms from $K_1(C(S))$ to $\mathbf{Z}$. Also $\partial(a,a) = 0$ or $\partial(a,0) = -\partial(0,a)$ for every $a \in K_1(C(S))$. If a is a generator of $K_1(C(S))$, then $\partial(a,0) = -\partial(0,a) = +1$ or -1.

2.9 Odd Dimensional Manifolds Homeomorphic to $\mathbf{R}^m$

Here we consider an odd-dimensional, complete riemannian manifold M, homeomorphic to $\mathbf{R}^m$, with Dirac operator $D : C_c^\infty(\Delta) \to C_c^\infty(\Delta)$, and Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{q} C_0(M) \oplus C_0(M) \to 0 \tag{9.1}$$

which we call τ. This extension induces an index map

$$\partial : K_1(C_0(M)) \oplus K_1(C_0(M)) \to \mathbf{Z}.$$

Since M is homeomorphic to $\mathbf{R}^m$ and m is odd, then

$$K_1(C_0(M)) \cong K_1(C_0(\mathbf{R}^m)) \cong \mathbf{Z}.$$

Thus ∂ is a map

$$\partial : \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}.$$

We use $C_0(M)^+$ to denote the C^*-algebra $C_0(M)$ with identity adjoined. (See Definition 3.2.1 of [**Bla**].) Elements of $C_0(M)^+$ will be regarded as functions φ on M for which there exists a constant $\lambda_\varphi \in \mathbf{C}$ such that for every $\epsilon > 0$, there exists compact $K \subseteq M$ such that $|\varphi(x) - \lambda_\varphi| < \epsilon$ for all $x \notin K$. Roughly these are functions with a limit at infinity, or continuous functions on the one point compactification of M. We have the following theorem.

Theorem 9.2 *The index map*

$$\partial : K_1(C_0(M)) \oplus K_1(C_0(M)) \to \mathbf{Z}$$

is an isomorphism on each component. That is, the maps $a \mapsto \partial(0,a)$ *and* $a \mapsto \partial(a,0)$ *are isomorphisms from* $K_1(C_0(M))$ *to* $\mathbf{Z}$. *also,* $\partial(a,a) = 0$, *or* $\partial(a,0) = -\partial(0,a)$ *for every* $a \in K_1(C_0(M))$. *So, if* $a \in K_1(C_0(M))$ *is a generator of* $K_1(C_0(M))$ *then* $\partial(a,0) = -\partial(0,a) = +1$ *or* -1.

Proof : Let $a \in K_1(C_0(M))$. Then $a = [\varphi]$ for some unitary $\varphi \in M_k(C_0(M)^+)$, for some $k \geq 1$. Let f be any element of $Flip_r$. Then $1 - f \in Flip_l$ and $\partial(a,a) = \partial([(1-f)(D)\cdot M_\varphi + f(D)\cdot M_\varphi]) = \partial([M_\varphi])$ is the Fredholm index of the Fredholm operator $M_\varphi \in B(L^2(\Delta)^k)$. Since φ is unitary, so is M_φ, and hence the index of M_φ is 0. Thus $\partial(a,a) = 0$, which gives us $\partial(a,0) = -\partial(0,a)$.

Thus, to complete the proof, we only have to show that $\partial(0,a) = +1$ or -1 when a is a generator of $K_1(C_0(M))$. For this would also imply that $\partial(a,0) = -\partial(0,a) = +1$ or -1 for a generator a. For this, we take a generator $a \in K_1(C_0(M))$. If $a = [\varphi]$ and $f \in Flip_r$, then $(0,a) = [(1-f)(D) + f(D)\cdot M_\varphi]$, which means that $(1-f)(D) + f(D)\cdot M_\varphi$ is Fredholm, and $\partial(0,a)$ is the Fredholm index of $(1-f)(D) + f(D)\cdot M_\varphi$. So it suffices to find appropriate f and φ as above, and then show that the Fredholm operator $(1-f)(D) + f(D)\cdot M_\varphi$ has index +1 or −1. To do this, we take any two coordinate balls $U, V \subseteq M$ with the same center such that $\bar{U} \subseteq V$. Then take a unitary $\varphi \in M_\infty(C_0(M)^+)$ (say $\varphi \in M_k(C_0(M)^+)$ for some $k \geq 1$) such that $a = [\varphi]$ and $\varphi - 1$ has compact support in U. Let $r > 0$ be a positive number smaller than the distance between the support of $\varphi - 1$ and the complement of U. Let f be a $Flip_r$ function whose Fourier transform has support in $[-r, r]$. (This is possible by Corollary 1.10.) Let S be the sphere of dimension m.

By Proposition 5.6, there is a riemannian metric on S, a corresponding Dirac operator $D_S : C^\infty(\Delta_S) \to C^\infty(\Delta_S)$, and an exact coincidence

$$(f(D) \cdot M_{\varphi-1}, f(D_S) \cdot M_{(\varphi-1)\circ h^{-1}}, U, V, h).$$

Since the coincidence is exact, $L^2(U,\Delta)^k$ and $L^2(V,\Delta_S)^k$ are reducing for $f(D)M_{\varphi-1}$ and $f(D_S) \cdot M_{(\varphi-1)\circ h^{-1}}$ respectively, and, with respect to the decompositions

$$L^2(\Delta)^k = L^2(U,\Delta)^k \oplus L^2(M/U,\Delta)^k,$$

$$L^2(\Delta_S)^k = L^2(V,\Delta_S)^k \oplus L^2(S/S,\Delta_S)^k,$$

we may write

$$f(D) \cdot M_{\varphi-1} = A \oplus 0,$$

$$f(D_S) \cdot M_{(\varphi-1)\circ h^{-1}} = A_S \oplus 0,$$

where A and A_S are the compressions of $f(D)M_{\varphi-1}$ and $f(D_S).M_{(\varphi-1)\circ h^{-1}}$ to the spaces $L^2(U,\Delta)^k$ and $L^2(V,\Delta_S)^k$ respectively. Now, consider the operators

$$C = 1 + f(D) \cdot M_{\varphi-1}$$

and

$$C_S = 1 + f(D_S) \cdot M_{(\varphi-1)\circ h^{-1}}.$$

With respect to the decompositions of $L^2(\Delta)^k$ and $L^2(\Delta_S)^k$ given above, we note that

$$C = (1 + A) \oplus 1,$$

and

$$C_S = (1 + A_S) \oplus 1.$$

Also, we may write

$$C = (1 - f)(D) + f(D) \cdot M_{\varphi}.$$

So, from the discussion above, we know that C is Fredholm, and, to complete the proof, it suffices to show that the index of C is $+1$ or -1.

Well, since $C = (1 + A) \oplus 1$ and C is Fredholm, then $1 + A \in B(L^2(\Delta)^k)$ must also be Fredholm and $\text{ind}(C) = \text{ind}(1 + A) + \text{ind}(1) = \text{ind}(1 + \text{A}) + 0$ $= \text{ind}(1 + A)$. That is,

$$\text{ind } (C) = \text{ ind } (1 + A).$$

From the definition of coincidences, we have that $A \in B(L^2(U, \Delta)^k)$ is unitarily equivalent to $A_S \in B(L^2(V, \Delta_S)^k)$. Therefore $1 + A \in B(L^2(U, \Delta)^k)$ is unitarily equivalent to $1 + A_S \in B(L^2(V, \Delta_S)^k)$. Thus, $1 + A_S$ is also Fredholm and

$$\text{ind } (1 + A) = \text{ ind } (1 + A_S).$$

Since $C_S = (1 + A_S) \oplus 1$, it follows that C_S is Fredholm and $\text{ind } (C_S) = \text{ind}(1 + A_S) + 0 = \text{ind}(1 + A_S)$. That is,

$$\text{ind } (1 + A_S) = \text{ ind } (C_S).$$

Thus we have shown that $\text{ind } (C) = \text{ind } (1 + A) = \text{ind } (1 + A_S) = \text{ind } (C_S)$. That is,

$$\text{ind } (C) = \text{ind } (C_S).$$

So, to finish the proof, we only have to show that $\text{ind } (C_S) = +1$ or -1.

Now let

$$\eta(x) = \begin{cases} (\varphi \circ h^{-1})(x), & x \in V \\ \\ 1, & x \notin V \end{cases}$$

Then $\eta : S \to M_k(\mathbf{C})$ has value 1 outside V and coincides with φ via $h : U \to V$ over V. Since φ is a unitary in $M_k(C_0(M)^+)$, it follows that $\eta \in M_k(C(S))$ is a unitary. More important, since $[\varphi] = K_1(C_0(M))$ is a generator, it is clear that $[\eta] \in K_1(C(S))$ is a generator. Now, we observe that $(\varphi - 1) \circ h^{-1} = \eta - 1$ and therefore

$$C_S = 1 + f(D_S) \cdot M_{\eta - 1} = (1 - f)(D_S) + f(D_S) \cdot M_\eta.$$

Since $f \in Flip_r$ and η is a unitary, then $[C_S] = (0, [\eta])$ in $K_1(C(S)) \oplus K_1(C(S))$, and

$$\text{ind } (C_S) = \partial(0, [\eta]).$$

But we just showed that $[\eta]$ is a generator of $K_1(C(S))$. Hence, by Theorem 8.3, $\partial(0, [\eta]) = +1$ or -1. It follows that ind $(C_S) = +1$ or -1, which completes the proof. ♠

2.10 r-strings

Definition 10.1 *Let X be a metric space and let $r > 0$. An* **r-string (of beads)** *in X is any disjoint union $R_1 \cup R_2 \cup R_3 \cup \ldots$ of a sequence $R_1, R_2, R_3, \ldots$ of Borel subsets $R_i \subseteq X$ such that each R_i has diameter $\leq 2r$, and such that $B(R_i, r) \cap B(R_j, r) = \emptyset$ whenever $i \neq j$. An r-string will also be called simply a* **string**, *and the R_i will be called* **r-beads** *(of the string).*

If $E \to X$ is a hermitian vector bundle over a complete riemannian manifold X, $A : L^2(E) \to L^2(E)$ a bounded operator on $L^2(E)$, and $S \subseteq X$ a Borel subset of X, we will use $\boldsymbol{A|_S}$ to denote the operator

$$A|_S = A|_{L^2(S,E)} : L^2(S, E) \to L^2(E).$$

Proposition 10.2 *Let $r > 0$, $E \to X$ a hermitian vector bundle over a complete riemannian manifold X, $A : L^2(E) \to L^2(E)$ an r-local bounded operator on $L^2(E)$, and $S = \bigcup R_i$ an r-string with beads R_i. Then the operator $A|_S$ has norm*

$$\|A|_S\| = \sup_i \|A|_{R_i}\|.$$

Proof: Let $u \in L^2(E)$. Then we can write $u = \sum u_i$ where each $u_i \in L^2(R_i, E)$. Since A is r-local, then each Au_i has support in $B(R_i, r)$. Since S is an r-string, the $B(R_i, r)$ are mutually disjoint. Hence, the Au_i have mutually disjoint supports. This implies that

$$\begin{aligned} \|Au\|^2 &= \sum_i \|Au_i\|^2 \\ &\leq \sum_i \|A|_{R_i}\|^2 \cdot \|u_i\|^2 \end{aligned}$$

$$\leq \left(\sup_i \|A|_{R_i}\|^2\right) \cdot \sum_i \|u_i\|^2$$
$$= \left(\sup_i \|A|_{R_i}\|\right)^2 \cdot \|u\|^2,$$

which gives us that $\|A|_S\| \leq \sup_i \|A|_{R_i}\|$. Of course, each $\|A|_{R_i}\| \leq \|A|_S\|$. Therefore, $\|A|_S\| = \sup_i \|A|_{R_i}\|$. ♠

Proposition 10.3 *Let $r > 0$, $E \to M$ a hermitian vector bundle over a complete riemannian manifold M, and $A : L^2(E) \to L^2(E)$ an r-local bounded operator on $L^2(E)$ with the property that, for some $c > 0$, $\|A|_B\| \leq c$ for all balls B of radius r. Assume further that M can be covered by a finite number q of r-strings. Then $\|A\| \leq c \cdot q$.*

Proof: Let S be an r-string and R one of its beads. then R has diameter $\leq 2r$ and is therefore contained in a closed ball B of radius r. Since the operator $A|_B$ has norm $\leq c$, then $A|_R$ also has norm $\leq c$. This is true for all beads of S. So, by Proposition 10.2, the norm of $A|_S$ is $\leq c$, and this is true for every r-string S in M.

Now, let $S_1, \ldots, S_q$ be q r-strings which cover M. Let

$$\hat{S}_i = S_i \backslash (S_1 \cup \cdots \cup S_{i-1}).$$

From above, $\|A|_{S_i}\| \leq c$ for all i. Since $\hat{S}_i \subset S_i$, we also have $\|A|_{\hat{S}_i}\| \leq c$ for all i. Take $u \in L^2(E)$. Note that the $\hat{S}_i$ are mutually disjoint. Hence, $u = \sum_{i=1}^q u_i$ where each u_i has support in $\hat{S}_i$. It follows that $\|Au\| \leq \sum_{i=1}^q \|Au_i\| = \sum_{i=1}^q \|(A|_{\hat{S}_i})u_i\| \leq \sum_{i=1}^q \|A|_{\hat{S}_i}\| \cdot \|u_i\| \leq \sum_{i=1}^q c \cdot \|u_i\| \leq \sum_{i=1}^q c \cdot \|u\| = q \cdot c \cdot \|u\|$. Thus $\|A_u\| \leq qc \cdot \|u\|$ for all $u \in L^2(E)$. Therefore $\|A\| \leq q \cdot c$. ♠

Definition 10.4 *Let M be a complete riemannian manifold. Then M is said to have* **bounded geometry** *if M has positive injectivity radius, and if the curvature tensor of M is uniformly bounded as are all its covariant derivatives.*

Proposition 10.5 *If M is a compact riemannian manifold, then M has bounded geometry. If M is the universal covering space of a compact, riemannian manifold, and has the geometry induced by the lifting, then M has bounded geometry.*

Proof: Easy. ♠

Proposition 10.6 *If M has bounded geometry and $r > 0$, then M can be covered by a finite number of r-strings.*

Proof: This is Lemma 7.3 of [**Roe2**]. ♠

Corollary 10.7 *If M is a compact riemannian manifold and $r > 0$, then its universal cover $\tilde{M}$ can be covered by a finite number of r-strings.*

Proof: By Proposition 10.5, $\tilde{M}$ has bounded geometry. Hence (by Proposition 10.6), $\tilde{M}$ can be covered by a finite number of r-strings. ♠

We will need something slightly stronger than Corollary 10.7 for small r.

If $\alpha : [a, b] \to M$ is a smooth curve in a riemannian manifold M, we will use $|\alpha|$ to denote the length of α. If S is a subset of a metric space, we will use $|S|$ to denote the diameter of S. If S, T are subsets of the same metric space, we will use $d(S, T)$ to denote the distance from S to T.

Lemma 10.8 *For every* $t > 0$ *let* $B(t) \subseteq \mathbf{R}^m$ *denote the ball of radius* t *centered at* 0. *Assume* $B(t)$ *has the usual euclidean metric coming from* $\mathbf{R}^m$. *Let* p *be a point in a compact riemannian manifold* M, *let* $r > 0$ *be a number smaller than the injectivity radius of* M *and small enough so that for every* $x, y \in B(p,r)$, *there is only one shortest geodesic between* x *and* y, *and it lies completely in* $B(p,r)$. *Let*

$$\exp_p : B(r) \to B(p,r)$$

be a normal coordinate map at p. *Then there exist* $\alpha, \beta > 0$ *such that, for every smooth curve* γ *in* $B(r)$,

$$\alpha \cdot |\gamma| \leq |(exp_p)_* \gamma| \leq \beta \cdot |\gamma|, \qquad (1)$$

for every subset $S \subseteq B(r)$,

$$\alpha |S| \leq |exp_p(S)| \leq \beta \cdot |S| \qquad (2)$$

(where $|exp_p(S)|$ *stands for the diameter of* $exp_p(S)$ *in* M*), and, for every two sets* $S, T \subseteq B(r)$,

$$\alpha \cdot d(S,T) \leq d(exp_p(S), exp_p(T)) \leq \beta \cdot d(S,T) \qquad (3)$$

(where $d(exp_p(S), exp_p(T))$ *stands for the distance between* $exp_p(S)$ *and* $exp_p(T)$ *in* M*).*

Proof First we note that, by the assumption made on r, the distance in M between two points in $B(p,r)$ is the length of a geodesic lying completely in $B(p,r)$ and is therefore the distance in $B(p,r)$ between the two points.

From this, it follows that if $S, T \subseteq B(r)$, then $|\exp_p(S)|$, which is the diameter of $\exp_p(S)$ in M, is the same as the diameter of $\exp_p(S)$ in $B(p,r)$, and $d(\exp_p(S), \exp_p(T))$, which is the distance between $\exp_p(S)$ and $\exp_p(T)$ in M, is the same as the distance between $\exp_p(S)$ and $\exp_p(T)$ in $B(p,r)$. We see therefore that (2) and (3) will follow from (1). So it is enough to prove (1).

To do this, it suffices to show that there exists $\alpha, \beta > 0$ such that, for every $x \in B(r)$ and $v \in T_x(B(r))$, $\alpha\|v\| \leq \|D_x(\exp_p)v\| \leq \beta\|v\|$ where

$$D_x(\exp_p) : T_x(B(r)) \to T_{\exp_p x}M$$

is the derivative of $\exp_p$ at the point x. Since r is smaller than the injectivity radius at p, we can find $s > r$, still smaller than the injectivity radius at p, and a normal coordinate map

$$\exp_p : B(s) \to B(p,s)$$

which extends $\exp_p : B(r) \to B(p,r)$. Since s is smaller than the injectivity radius of M, then the derivative

$$D_x(\exp_p) : T_x B(s) \to T_{\exp_p x}M$$

at x is invertible with inverse A_x say, for every x in $B(s)$. Let $\|D_x(\exp_p)\|$ and $\|A_x\|$ denote the operator norms of these operators, for every $x \in B(s)$. Since $\overline{B(r)}$ is compact, there exist $\delta, \beta > 0$ such that $\|A_x\| \leq \delta$ and $\|D_x(\exp_p)\| \leq \beta$ for every $x \in \overline{B(r)}$. This implies that $\|D_x(\exp_p)v\| \leq \beta \cdot \|v\|$ for every $x \in \overline{B(r)}$ and for every $v \in T_x(B(r))$. Since $D_x(\exp_p)$ is the inverse of A_x, it also implies that $\|D_x(\exp_p)v\| \geq \frac{1}{\delta}\|v\|$ for every $x \in \bar{B(r)}$ and every $v \in T_x(B(r))$. Thus, the lemma is true. ♠

Proposition 10.9 *Let M be a compact riemannian manifold of dimension M and let $\tilde{M}$ be its universal cover. Assume $\tilde{M}$ has the riemannian metric induced by the lifting. Then there is an $R > 0$ and an integer $L > 0$ such that for every $r > 0$ with $0 < r < R$, there are L r-strings which cover $\tilde{M}$.*

Proof Let a be the injectivity radius of M and let $\rho : \tilde{M} \to M$ be the covering map. If $x, y \in \tilde{M}$ are two different lifts of the same point in M then the distance from x to y is $\geq 2a$. Take $R < \frac{1}{2}a$ small enough so that if $B \subseteq M$ is a ball of radius R and $x, y \in B$ there is only one shortest geodesic joining x to y and it lies completely in B. Since M is compact, it is covered by a finite number, say k, of balls of radius R. Let $B(q_1, R), B(q_2, R), \ldots, B(q_R, R)$ be such a cover for M, where $q_1, \ldots, q_k \in M$. For every q_i, we can write

$$\rho^{-1}(B(q_i, R)) = \dot{\bigcup_{g \in \pi_1(M)}} B(g \cdot p_i, R)$$

where $p_i \in \tilde{M}$ is a lift of q_i and $\pi_1(M)$ is the fundamental group of M. This is a disjoint union and, in fact, if $g, h \in \pi_1(M), g \neq h$, and $i \in \{1, \ldots, k\}$, the distance between $g \cdot p_i$ and $h \cdot p_i$ is $\geq 2a > 4R$ (as mentioned above), and thus the distance from $B(g \cdot p_i, R)$ to $B(h \cdot p_i, R)$ is greater than $2R$.

Let $\exp_{qi} : B(R) \to B(q_i, R)$ be a normal coordinate map for each i. By Lemma 10.8, for each $i \in \{1, \ldots, k\}$, there exist $\alpha_i, \beta_i > 0$ such that

$$\alpha_i |S| \leq |\exp_{q_i}(S)| \leq \beta_i |S|$$

for every $S \subseteq B(R)$, and

$$\alpha_i \cdot d(S, T) \leq d(\exp_{q_i}(S), \exp_{q_i}(T)) \leq \beta_i \cdot d(S, T)$$

for every $S, T \subseteq B(R)$. Let $\alpha = \min \alpha_i$, and $\beta = \max \beta_i$. Then $\alpha, \beta > 0$,

$$\alpha|S| \leq |\exp_{q_i}(S)| \leq \beta \cdot |S|,$$

and

$$\alpha \cdot d(S,T) \leq d(\exp_{q_i}(S), \exp_{q_i}(T) \leq \beta \cdot d(S,T)$$

for every $i \in \{1, 2, \ldots, k\}$ and every $S, T \subseteq B(R)$.

Now, for every $g \in \pi_1(M)$, lift the normal coordinate map

$$\exp_{q_i} : B(R) \to B(q_i, R) \subseteq M$$

up to a normal coordinate map

$$\exp_{g \cdot p_i} : B(R) \to B(g \cdot p_i, R) \subseteq \tilde{M}.$$

Then, for every $S, T \in B(R), g \in \pi_1(M)$, we have

$$\alpha \cdot |S| \leq |\exp_{g \cdot p_i}(S)| \leq \beta \cdot |S|$$

and

$$\alpha \cdot d(S,T) \leq d(\exp_{g \cdot p_i}(S), \exp_{g \cdot p_i}(T)) \leq \beta \cdot d(S,T).$$

Let l be the smallest integer such that

$$l > \frac{\beta\sqrt{m}}{\alpha} + 1$$

and let

$$L = k \cdot l^m.$$

Take any $r > 0$ less than R, and let

$$\lambda = \frac{2r}{\beta\sqrt{m}}.$$

Let

$$I = [0,1)^m, \quad I_x = I + x$$

for every $x \in \mathbf{R}^m$, and, for every $n \in \mathbf{Z}^m$, let

$$\begin{aligned} T_n &= \dot{\bigcup_{j \in \mathbf{Z}^m}} (I_n + l \cdot j) \\ &= \dot{\bigcup_{j \in l \cdot \mathbf{Z}^m}} I_{n+j} \quad . \end{aligned}$$

Then

$$\begin{aligned} \mathbf{R}^m &= \dot{\bigcup_{j \in \mathbf{Z}^m}} I_j \\ &= \dot{\bigcup_{n \in l \cdot I \cap \mathbf{Z}^m}} \dot{\bigcup_{j \in l \cdot \mathbf{Z}^m}} I_{n+j}, \\ \Rightarrow \mathbf{R}^m &= \dot{\bigcup_{n \in lI \cap \mathbf{Z}^m}} T_n. \\ \Rightarrow \mathbf{R}^m &= \lambda \mathbf{R}^m \\ &= \lambda \dot{\bigcup_{n \in (lI) \cap \mathbf{Z}^m}} T_n \\ &= \dot{\bigcup_{n \in (lI) \cap \mathbf{Z}^m}} \lambda \cdot T_n \quad . \end{aligned}$$

Thus

$$B(R) = \dot{\bigcup_{n \in (lI) \cap \mathbf{Z}^m}} ((\lambda \cdot T_n) \cap B(R))$$

So, for each $g \cdot p_i$ where $g \in \pi_1(M), i \in \{1, \ldots, k\}$, we have that

$$\begin{aligned} B(g \cdot p_i, R) &= \exp_{g \cdot p_i} (B(R)) \\ &= \dot{\bigcup_{n \in (lI) \cap \mathbf{Z}^m}} \exp_{g \cdot p_i} ((\lambda \cdot T_n) \cap B(R)) \\ &= \dot{\bigcup_{n \in (lI) \cap \mathbf{Z}^m}} S_{n,i,g} \end{aligned}$$

Where

$$S_{n,i,g} \stackrel{\text{def}}{=} \exp_{g \cdot p_i}((\lambda \cdot T_n) \cap B(R))$$

for $n \in (lI) \cap \mathbf{Z}^m, 1 \leq i \leq k$, and $g \in \pi_1(M)$. Since, for each $i \in \{1, \ldots, k\}, \{B(g \cdot p_i, R) : g \in \pi_1(M)\}$ is a disjoint collection of balls, each pair of which is separated by a distance greater than $2R > 2r$, then, for each $n \in (lI) \cap \mathbf{Z}^m$, and $i \in \{1, \ldots, k\}$, the collection $\{S_{n,i,g} : g \in \pi_1, (M)\}$ is disjoint and each pair of this collection is separated by a distance greater than $2r$. We now let

$$S_{n,i} = \dot{\bigcup_{g \in \pi_1(M)}} S_{n,i,g}$$

for each $n \in (lI) \cap \mathbf{Z}^m, i \in \{1, \ldots, k\}$.

Claim 10.10 *$S_{n,i}$ is an r-string for each $n \in (lI)\mathbf{Z}^m, i \in \{1, \ldots, k\}$.*

Proof of Claim 10.10: Let $n \in (lI) \cap \mathbf{Z}^m, i \in \{1, \ldots, k\}$. Since each pair of elements in $\{S_{n,i,g} : g \in \pi_1(M)\}$ is separated by a distance greater than $2r$, it suffices to show that $S_{n,i,g}$ is an r-string for each $g \in \pi_1(M)$. Now

$$T_n = \dot{\bigcup_{j \in l \cdot \mathbf{Z}^m}} I_{n+j}.$$

So

$$(\lambda T_n) \cap B(R) = \dot{\bigcup_{j \in l \cdot \mathbf{Z}^m}} (\lambda I_{n+j}) \cap B(R),$$

and thus

$$\begin{aligned} S_{n,i,g} &= \exp_{g \cdot p_i} ((\lambda T_n) \cap B(R)) \\ &= \dot{\bigcup_{j \in l \cdot \mathbf{Z}^m}} \exp_{g \cdot p_i} ((\lambda \cdot I_{n+j} \cap B(R)) \\ &= \dot{\bigcup_{j \in l \cdot \mathbf{Z}^m}} R_{n,i,g,j} \end{aligned}$$

where

$$R_{n,i,g,j} \stackrel{\text{def}}{=} \exp_{g\cdot p_i}((\lambda \cdot I_{n+j}) \cap B(R)).$$

We will now show that $S_{n,i,g}$ is an r-string with r-beads $R_{n,i,g,j}$. For this, it suffices to show that $|R_{n,i,g,j}| \leq 2r$ for each $R_{n,i,g,j} \neq \emptyset$ and that

$$d(R_{n,i,g,j_1}, R_{n,i,g,j_2}) > 2r$$

whenever $j_1, j_2 \in l \cdot \mathbf{Z}^m, j_1, \neq j_2$, and $R_{n,i,g,j_1}, R_{n,i,g,j_2}$ are both nonempty.

First we show that $|R_{n,i,g,j}| \leq 2r$. Well,

$$\begin{aligned} |R_{n,i,g,j}| &= |\exp_{g\cdot p_i}((\lambda \cdot I_{n+j}) \cap B(R))| \\ &\leq \beta \cdot |(\lambda \cdot I_{n+j}) \bigcap B(R))| \\ &\leq \beta \cdot |\lambda \cdot I_{n+j}| \\ &= \beta \cdot \lambda \cdot |I_{n+j}| \\ &= \beta \cdot \lambda\sqrt{m} \\ &= \beta \cdot \frac{2r}{\beta\sqrt{m}} \cdot \sqrt{m} \\ &= 2r. \end{aligned}$$

This shows that $|R_{n,i,g,j}| \leq 2r$.

Now, we take $j_1, j_2 \in l \cdot \mathbf{Z}^m, j_1 \neq j_2$. The distance from R_{n,i,g,j_1} to R_{n,i,g,j_2}, (assuming both are nonempty) which by definition of $R_{n,i,g,j}$, is the distance from $\exp_{g\cdot p_i}((\lambda \cdot I_{n+j_1}) \bigcap B(R))$ to $\exp_{g\cdot p_i}((\lambda \cdot I_{n+j_2}) \bigcap B(R))$, is

$$\begin{aligned} &\geq \alpha \cdot d((\lambda \cdot I_{n+j_1}) \cap B(R), (\lambda \cdot I_{n+j_2}) \cap B(R)) \\ &\geq \alpha \cdot d(\lambda I_{n+j_1}, \lambda I_{n+j_2}) \\ &= \alpha\lambda d(I_{n+j_1}, I_{n+j_2}) \end{aligned}$$

Since $j_1 - j_2 \in l \cdot \mathbf{Z}^m$ and is not 0, it is clear that the distance from I_{n+j_1} to I_{n+j_2} is $\geq l-1$. Thus the distance from R_{n,i,g,j_1} to R_{n,i,g,j_2} is $\geq \alpha \cdot \lambda \cdot (l-1)$

$$> \alpha \cdot \frac{2r}{\beta \cdot \sqrt{m}} \cdot \frac{\beta\sqrt{m}}{\alpha} = 2r.$$

This completes the proof of Claim 10.10. ♠

Proof of Proposition 10.9 (cont'd): We showed that, for each $i \in \{1, \ldots, k\}, g \in \pi_1(M)$,

$$B(g \cdot p_i, R) = \dot{\bigcup_{n \in (lI) \cap \mathbf{Z}^m}} S_{n,i,g}.$$

Since M is covered by the set of all $B(q_i, R)$, then $\tilde{M}$ is covered by the collection of all $B(g \cdot p_i, R)$ such that $i \in \{1, 2, \ldots, k\}$ and $g \in \pi_1(M)$. Hence,

$$\begin{aligned}
\tilde{M} &= \bigcup B(g \cdot p_i, R) \\
&= \bigcup_{i=1}^{k} \bigcup_{g \in \pi_1(M)} \bigcup_{n \in (lI) \cap \mathbf{Z}^m} S_{n,i,g} \\
&= \bigcup_{n \in (lI) \cap \mathbf{Z}^m} \bigcup_{i=1}^{k} \bigcup_{g \in \pi_1(M)} S_{n,i,g} \\
&= \bigcup_{n \in (lI) \cap \mathbf{Z}^m} \bigcup_{i=1}^{k} S_{n,i}
\end{aligned}$$

Now the set $(lI) \cap \mathbf{Z}^m$ is equal to the cross product of $\{0, 1, 2, \ldots, l-1\}$ with itself m times, and therefore has cardinality l^m. Since each $S_{n,i}$ is an r-string in $\tilde{M}$ (Claim 10.10), it follows that $\tilde{M}$ is covered by L ($= kl^m$) r-strings, which completes the proof. ♠

Lemma 10.11 *Let M be a complete riemannian spin manifold with Dirac operator $D : C_c^\infty(\Delta) \to C_c^\infty(\Delta)$. Let f be a C^∞ Flip function and let φ be a bounded continuous function on M. Suppose $r > 0$, $\hat{f}$ has support in $[-r, r]$,* *and that, for some $\mu > 0$, $|\varphi(x) - \varphi(y)| \leq \mu$ whenever $d(x,y) \leq 3r$. Then, for every r-string S in M, we have*

$$\|[f(D), M_\varphi]|_S\| \leq 2\|f\|_\infty \mu.$$

Proof: Let S be an r-string of r-beads $R_1, R_2, R_3, \ldots$. Since $\hat{f}$ has support in $[-r, r]$, then $f(D)$ is r-local (by Corollary 3.5). Of course, M_φ is 0-local. So, $[f(D), M_\varphi]$ is r-local. By Proposition 10.2, it follows that

$$\|[f(D), M_\varphi]|_S\| = \sup_i \|[f(D), M_\varphi]|_{R_i}\|.$$

So, it suffices to show that $\|[f(D), M_\varphi]|_{R_i}\| \leq 2\|f\|_\infty \cdot \mu$ for every $i \geq 1$.

Take $i \geq 1$, pick $p \in R_i$ and let $\lambda = \varphi(p)$. Since R_i is an r-bead for each i, then $d(x,p) < 3r$ for all $x \in B(R_i, r)$ and so by the assumption on φ, $\|\varphi(x) - \lambda\| \leq \mu$ for all $x \in B(R_i, r)$. Therefore $\|M_{\varphi-\lambda} v\| \leq \mu\|v\|$ for all $v \in L^2(\Delta)$ with support in $B(R_i, r)$.

Now take $u \in L^2(\Delta)$ with support in R_i. Since $f(D)$ is r-local, then $f(D)u$ has support in $B(R_i, r)$. Thus

$$\begin{aligned}
\|[f(D), M_\varphi]u\| &= \|f(D)M_\varphi u - M_\varphi f(D)u\| \\
&= \|f(D)M_\varphi u - f(D)(\lambda u) + f(D)(\lambda u) - M_\varphi f(D)u\| \\
&\leq \|f(D)M_{\varphi-\lambda} u\| + \|M_{\lambda-\varphi} f(D)u\| \\
&\leq \|f(D)\| \cdot \mu\|u\| + \mu \cdot \|f(D)u\| \\
&\qquad \text{(since both } u \text{ and } f(D)u \text{ have support in } B(R_i, r)) \\
&\leq \|f\|_\infty \cdot \mu\|u\| + \mu \cdot \|f\|_\infty \|u\| \\
&= 2\|f\|_\infty \cdot \mu\|u\|.
\end{aligned}$$

Therefore $\|[f(D), M_\varphi]|_{R_i}\| \leq 2\|f\|_\infty \mu$ which completes the proof. ♠

2.11 Dirac Extensions on a Cover

Let M be the universal cover of a compact riemannian spin manifold. Assume M has the riemannian metric and bundle Δ of spinors induced by the lifting. Since M is the universal cover of a complete riemannian manifold, then M itself is complete. Let $D : C_c^\infty(\Delta) \to C_c^\infty(\Delta)$ be the resulting Dirac operator. Let $UC(M)$ denote the C^*-algebra of all bounded uniformly continuous functions on M. Let C be any C^* subalgebra of $M_k(UC(M))$. Each element of C is a bounded, uniformly continuous $k \times k$ matrix-valued function on M. Let $\mathcal{D}'$ be the C^*-subalgebra of $B(L^2(\Delta)^k)$ generated by the set of all $f(D)$ and $M_\varphi \in B(L^2(\Delta)^k)$ such that $f \in Flip$ and $\varphi \in C$. Define **the Dirac algebra of C** to be the ideal $\mathcal{D}$ of $\mathcal{D}'$ generated by the set of all M_φ such that $\varphi \in C$.

Remark 11.1 *In the case where the constant function 1 belongs to C, we actually have $\mathcal{D} = \mathcal{D}'$. So, in this case, $\mathcal{D}$ is the C^*-subalgebra of $B(L^2(\Delta)^k)$ generated by the set of all $f(D)$ and M_φ such that $f \in Flip$ and $\varphi \in C$. Also, since the set A of all $f(D)$ such that $f \in Flip$, and the set B of all M_φ such that $\varphi \in C$, are both C^*-algebras, then when $1 \in C$, $\mathcal{D}$ is actually the closed algebra generated by these two sets A and B.*

Next, we let $\mathcal{L}' \subseteq B(L^2(\Delta)^k)$ denote the C^*-subalgebra of $B(L^2(\Delta^k)$ generated by the set of all $f(D)$ and M_φ such that $f \in C_0(\mathbf{R})$ and $\varphi \in C$. Then we let $\mathcal{L}$ be the ideal of $\mathcal{L}'$ generated by the set of all $M_\varphi f(D)$ and $f(D) \cdot M_\varphi$ such that $f \in C_0(\mathbf{R})$ and $\varphi \in C$. Of course, $\mathcal{L}$ is a C^*-subalgebra of $\mathcal{A}$.

Lemma 11.2 *Let A be a family of functions in $UC(M, M_k(\mathbf{C}))$ with the property that, for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in M, d(x,y) < \delta$, and $\varphi \in A$, then $\|\varphi(x) - \varphi(y)\| < \epsilon$. Then, for every $\epsilon > 0$, there is an $r > 0$ such that if $g \in C^\infty$-Flip, $\hat{g}$ has support in $(-r, r), \|g\|_\infty \leq 1$, and $\varphi \in A$, then $\|[g(D), M_\varphi]\| < \epsilon$.*

Proof: Take any $\epsilon > 0$. By Proposition 10.9, there is an $R > 0$ and an integer $l > 0$ such that, for every $r > 0$ with $0 < r < R$, there are l r-strings which cover M. By the assumption on the family A, there exists $\delta > 0$ such that $\|\varphi(x) - \varphi(y)\| < \frac{\epsilon}{2l}$ whenever $\varphi \in A$ and $d(x,y) < \delta$. We can, of course, choose δ so that $\delta < 3R$. Let us do this. Let $r = \frac{\delta}{3}$. So $0 < r < R$. Since $r < R$, then, from above, we can find l r-strings $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_l$ which cover M.

Now, suppose $\varphi \in A, g \in C^\infty$-*Flip*, $\hat{g}$ has support in $(-r, r)$ and $\|g\|_\infty \leq 1$. Since $\hat{g}$ has support in $(-r, r)$ and since $\|\varphi(x) - \varphi(y)\| < \frac{\epsilon}{2l}$ whenever $d(x,y) < 3r$, it follows from Lemma 10.11 that $\|[g(D), M_\varphi]|_{S_i}\| \leq 2\|g\|_\infty \cdot \frac{\epsilon}{2l} \leq \frac{\epsilon}{l}$ (since $\|g\|_\infty \leq 1$) for all $i \in \{1, \ldots k\}$. This implies that $\|[g(D), M_\varphi]\| = \|\sum_{i=1}^k [g(D), M_\varphi]|_{S_i}\| \leq \sum_{i=1}^k \frac{\epsilon}{l} = k \cdot \frac{\epsilon}{l} \leq \epsilon$(since $k \leq l$). That is $\|[g(D), M_\varphi]\| \leq \epsilon$. Thus, the lemma is true. ♠

Lemma 11.3 *If $\varphi \in UC(M, M_k(\mathbf{C}))$, then for every $\epsilon > 0$, there exists $r > 0$ such that if g is a C^∞-Flip function, $\hat{g}$ has support in $(-r, r)$, and $\|g\|_\infty \leq 1$, then $\|[g(D), M_\varphi]\| < \epsilon$.*

Proof : This is a special case of Lemma 11.2. ♠

Lemma 11.4 *If $\varphi \in C$ and $f \in Flip$, then $[f(D), M_\varphi] \in \mathcal{L}$.*

Proof : Suppose $\varphi \in C$ and $f \in Flip$. If $f = 0$ then we are done. If $f \neq 0$, then, by taking $\frac{f}{\|f\|_\infty} = 1$.

Take $\epsilon > 0$. By Lemma 11.3, there exists $r > 0$ such that if $g \in C^\infty\text{-}Flip, \hat{g}$ has support in $(-r, r)$, and $\|g\|_\infty \leq 1$ then $\|[g(D), M_\varphi]\| < \epsilon$. By Corollary 1.17, there exists $g \in C^\infty\text{-}Flip$ such that $\hat{g}$ has support in $(-r, r), g - f \in C_0(\mathbf{R})$, and $\|g\|_\infty \leq \|f\|_\infty = 1$. Since $h = f - g \in C_0(\mathbf{R})$ then $[h(D), M_\varphi] \in \mathcal{L}$. From this and the equation $[f(D), M_\varphi] = [g(D), M_\varphi] + [h(D), M_\varphi]$, it follows that the distance from $[f(D), M_\varphi]$ to $\mathcal{L}$ is $\leq \|[g(D), M_\varphi\| \leq \epsilon$. This is true for every $\epsilon > 0$. Therefore $[f(D), M_\varphi] \in \mathcal{L}$. ♠

Lemma 11.5 *Suppose $\varphi \in C, g \in C_0(\mathbf{R})$, and $f \in Flip$. Then multiplying $M_\varphi \cdot g(D)$ and $g(D) \cdot M_\varphi$ on the left or on the right by $f(D)$ gives an element of $\mathcal{L}$.*

Proof : Multiplying $M_\varphi \cdot g(D)$ on the right and $g(D) \cdot M_\varphi$ on the left by $f(D)$ gives $M_\varphi \cdot (g \cdot f)(D)$ and $(f \cdot g)(D) \cdot M_\varphi$ which are clearly in $\mathcal{L}$ (since $f \cdot g \in C_0(\mathbf{R})$). Also, if we show that $f(D) \cdot M_\varphi g(D) \in \mathcal{L}$ then we can conclude that $g(D) \cdot M_\varphi \cdot f(D) = (\bar{f}(D) \cdot M_{\bar{\varphi}} \cdot \bar{g}(D))^* \in \mathcal{L}$. So it suffices to show that $f(D) M_\varphi g(D) \in \mathcal{L}$. But

$$f(D) M_\varphi g(D) = [f(D), M_\varphi] g(D) + M_\varphi \cdot (fg)(D).$$

Since $fg \in C_0(\mathbf{R})$, the second term $M_\varphi (f \cdot g)(D)$ on the right belongs to $\mathcal{L}$. By Lemma 11.4, $[f(D), M_\varphi] \in \mathcal{L}$. Since $g(D) \in \mathcal{L}'$ and $\mathcal{L}$ is an ideal of $\mathcal{L}'$, it follows that the first term $[f(D), M_\varphi] \cdot g(D)$ also belongs to $\mathcal{L}$. Thus $f(D) M_\varphi g(D)$ belongs to $\mathcal{L}$. ♠

Lemma 11.6 *If $f \in Flip, a \in \mathcal{L}'$, then $[f(D), a] \in \mathcal{L}$.*

Proof : $\mathcal{L}'$ is generated by the set of all M_φ and $g(D)$ such that $\varphi \in C$ and $g \in C_0(\mathbf{R})$. Using this and the fact that $\mathcal{L}$ is an ideal of $\mathcal{L}'$, we can reduce the proof of the lemma to showing that $[f(D), g(D)]$ and $[f(D), M_\varphi]$ both belong to $\mathcal{L}$ when $f \in Flip, g \in C_0(\mathbf{R})$, and $\varphi \in C$. But for such f, g and φ, we have $[f(D), g(D)] = 0$ and, by Lemma 11.4, $[f(D), M_\varphi] \in \mathcal{L}$. Thus Lemma 11.6 is true. ♠

Corollary 11.7 *$\mathcal{L}$ is an ideal of both $\mathcal{D}'$ and $\mathcal{D}$.*

Proof: Since $\mathcal{D} \subseteq \mathcal{D}'$ it suffices to show that $\mathcal{L}$ is an ideal of $\mathcal{D}'$. Since $\mathcal{D}'$ is generated by the set of all M_φ and $f(D)$ such that $\varphi \in C$ and $f \in Flip$, then we only have to show that, given $L \in \mathcal{L}, f \in Flip$, and $\varphi \in C$, that $M_\varphi L, LM_\varphi, f(D)L$ and $Lf(D)$ belong to $\mathcal{L}$. By taking adjoints, it is only necessary to show that $M_\varphi L$ and $f(D)L \in \mathcal{L}$ for such φ, f, and L.

So take such φ, f, and L. It is easy to see that $M_\varphi L \in \mathcal{L}$. For $M_\varphi \in \mathcal{L}'$ (by definition of $\mathcal{L}'$), $L \in \mathcal{L}$, and $\mathcal{L}$ is an ideal of $\mathcal{L}'$. Therefore $M_\varphi L \in \mathcal{L}$. To show that $f(D)L \in \mathcal{L}$, we observe first that L can be approximated by finite sums of terms of the form $a \cdot g(D) \cdot M_\eta \cdot b$ or $a \cdot M_\eta \cdot g(D) \cdot b$ where $g \in C_0(\mathbf{R}), \eta \in C$, and $a, b \in \mathcal{L}' \bigcup \{1\}$ (where 1 is the identity operator on $L^2(\Delta)^k$). So, it suffices to show that $f(D) \cdot a \cdot g(D) \cdot M_\eta$ and $f(D) \cdot aM_\eta \cdot g(D) \cdot b$ belong to $\mathcal{L}$ when $g \in C_0(\mathbf{R})$, $a, b \in \mathcal{L}' \bigcup \{1\}$, and $\eta \in C$. We take such g, a, b, and η. By Lemma 11.6, $[f(D), a] \in \mathcal{L}$. (If $a = 1$ then $[f(D), a] = 0$.) Similarly, $[f(D), b] \in \mathcal{L}$. Now,

$$f(D) \cdot a \cdot g(D) M_\eta b = [f(D), a] \cdot g(D) M_\eta \cdot b + a \cdot (f \cdot g)(D) \cdot M_\eta \cdot b.$$

Note that $g(D) \cdot M_\eta \cdot b \in \mathcal{L}'$. From above, $[f(D), a] \in \mathcal{L}$. Therefore $[f(D), a] \cdot g(D) \cdot M_\eta \cdot b$ belongs to $\mathcal{L}$. Also, since $f \cdot g \in C_0(\mathbf{R})$, then $(f \cdot g)(D)M_\eta \in \mathcal{L}$. Since $a, b \in \mathcal{L}' \bigcup \{1\}$, it follows that $a(f \cdot g)(D)M_\eta \cdot b \in \mathcal{L}$. Therefore $f(D) \cdot a \cdot g(D) \cdot M_\eta b$ belongs to$\mathcal{L}$. Similarly, we have that

$$f(D) \cdot a \cdot M_\eta g(D) \cdot b = [f(D), a \cdot M_\eta] \cdot g(D) \cdot b + a \cdot M_\eta \cdot (f \cdot g)(D) \cdot b$$

belongs to $\mathcal{L}$. ♠

Proposition 11.8 *If $a, b \in \mathcal{D}'$, then $[a, b] \in \mathcal{L}$. Consequently if $a, b \in \mathcal{D}$, then $[a, b] \in \mathcal{L}$.*

Proof: This follows immediately from Lemma 11.4 and Corollary 11.7. ♠

If $a \in \mathcal{D}$, we will let $[a] \in \mathcal{D}/\mathcal{L}$ denote the class $a + \mathcal{L}$ in $\mathcal{D}/\mathcal{L}$.

Proposition 11.9 *$\mathcal{D}$ is the C^*-subalgebra of $B(L^2(\Delta)^k)$ generated by $\mathcal{L}$ and the set of all $f(D) \cdot M_\varphi$ such that $f \in Flip$ and $\varphi \in C$. Therefore $\mathcal{D}/\mathcal{L}$ is generated by elements $[f(D) \cdot M_\varphi]$ where $f \in Flip$ and $\varphi \in C$.*

Proof: Since $[f(D), M_\varphi] \in \mathcal{L}$ for every $f \in Flip, \varphi \in C$, then, from its definition, $\mathcal{D}$ contains as a dense subset the set of all finite sums of terms of the form $f(D)M_\varphi + L$ such that $f \in Flip, \varphi \in C$, and $L \in \mathcal{L}$. Thus, $\mathcal{D}$ is generated by $\mathcal{L}$ and the set of all $f(D) \cdot M_\varphi$ such that $f \in Flip$ and $\varphi \in C$. ♠

Lemma 11.10 *If $L \in \mathcal{L}$ and $\mu \in C_0(M)$, then $M_\mu \cdot L$ and $L \cdot M_\mu$ are compact.*

Proof: Take $\mu \in C_0(M)$. From its definition, $\mathcal{L}$ contains as a dense subset a vector space spanned by operators L of the form $f(D)M_\phi T$ or $M_\varphi f(D)T$

where $T \in B(L^2(\Delta)^k), \varphi \in C$, and $f \in C_0(\mathbf{R})$. So it suffices to prove that $M_\mu L$ and LM_μ are compact for such operators L.

By taking adjoints, we only have to show that $M_\mu \cdot L$ is compact for such L. So take T, φ, and f as above. Let $L = f(D) \cdot M_\varphi T$. Then $M_\mu L = M_\mu f(D) \cdot M_\varphi \cdot T$. But $M_\mu \cdot f(D)$ is compact since $\mu \in C_0(M)$ and $f \in C_0(\mathbf{R})$ (Proposition 7.5). Therefore $M_\mu L$ is compact. Next, let $L = M_\varphi f(D) \cdot T$. In this case $M_\mu L = M_{\mu\varphi} f(D) \cdot T$. The operator $M_{\mu\varphi} \cdot f(D)$ is compact since $\mu\varphi \in C_0(M)$ and $f \in C_0(\mathbf{R})$. Thus $M_\mu L$ is compact. ♠

Corollary 11.11 *Suppose $f \in Flip_l, g \in Flip_r$, and $\varphi, \eta \in C$. Then $f(D)M_\varphi + g(D)M_\eta$ belongs to $\mathcal{L} + K(L^2(\Delta)^k)$ if and only if $\varphi = \eta = 0$.*

Proof: Suppose $f(D)M_\varphi + g(D)M_\eta \in \mathcal{L} + \mathcal{K}(L^2(\Delta)^k)$. Then, by Lemma 11.10, $(f(D) \cdot M_\varphi + g(D)M_\eta)M_\mu$ is compact for every $\mu \in C_0(M)$. This means that $f(D)M_{\varphi\mu} + g(D)M_{\eta\mu}$ is compact for every $\mu \in C_0(M)$. Since $\varphi\mu, \eta\mu \in C_0(M)$ for $\mu \in C_0(M)$, it follows from Theorem 7.12 that $\varphi\mu = \eta\mu = 0$ for every $\mu \in C_0(M)$. Hence $\varphi = \eta = 0$. ♠

Definition 11.12 *Define the* **Dirac map**

$$\tau : C \oplus C \to \mathcal{D}/\mathcal{L}$$

by letting

$$\tau(\varphi, \eta) = [f(D) \cdot M_\varphi + g(D) \cdot M_\eta]$$

where f is any $Flip_l$ function and g is any $Flip_r$ function.

Proposition 11.13 *The Dirac map $\tau : C \oplus C \to \mathcal{D}/\mathcal{L}$ is a well-defined surjective $*$-homorphism.*

Proof: The proof of this is identical to the proof of Lemma 7.9. The only difference is that here we use Proposition 11.8, which says that the commutator of two elements of $\mathcal{D}$ belongs to $\mathcal{L}$, and Proposition 11.9, which says that $\mathcal{D}$ is the C^*-subalgebra of $B(L^2(\Delta)^k)$ generated by $\mathcal{L}$ and the set of all $f(D)M_\varphi$ such that $f \in Flip$ and $\varphi \in C$. ♠

Theorem 11.14 *The Dirac map $\tau : C \oplus C \to \mathcal{D}/\mathcal{L}$ is a $*$-isomorphism. This implies that if $f \in Flip_l$, $g \in Flip_r$, $\varphi, \eta \in C$, then $f(D)M_\varphi + g(D)M_\eta$ belongs to $\mathcal{L}$ if and only if $\varphi = \eta = 0$.*

Proof: We already know that τ is a surjective $*$-homomorphism (Proposition 11.13). We only have to show that τ is 1-1, or that $\ker(\tau) = \{0\}$. So, take $(\varphi, \eta) \in C \oplus C$ and suppose $\tau(\varphi, \eta) = 0$. Take any $f \in Flip_l$ and $g \in Flip_r$. Then $[f(D) \cdot M_\varphi + g(D)M_\eta] = \tau(\varphi, \eta) = 0$, Which means that $f(D) \cdot M_\varphi + g(D) \cdot M_\eta \in \mathcal{L}$. Hence $\varphi = \eta = 0$ by Corollary 11.11. Therefore τ is 1-1. ♠

Theorem 11.14 gives us a C^*-algebra extension

$$0 \to \mathcal{L}_C \xrightarrow{i} \mathcal{D}_C \to C \oplus C \to 0 \tag{11.15}$$

where we now put subscripts on $\mathcal{L}$ and $\mathcal{D}$ to show their dependence on C. Also, i is the inclusion of $\mathcal{L}_C$ into $\mathcal{D}_C$, and the map from $\mathcal{D}_C$ to $C \oplus C$ is the composition of $\tau^{-1} : \mathcal{D}_C/\mathcal{L}_C \to C \oplus C$ with the quotient map $\mathcal{D}_C \to \mathcal{D}_C/\mathcal{L}_C$. This extension (11.15) will be called **the Dirac extension of C**. By Corollary 11.11 we also have an extension

$$0 \to \mathcal{L}_C + \mathcal{K} \xrightarrow{i} \mathcal{D}_C + \mathcal{K} \to C \oplus C \to 0$$

where $\mathcal{K} \stackrel{\text{def}}{=} K(L^2(\Delta)^k)$ and i is the inclusion map. This extension will be called **the Dirac extension of C with compacts adjoined.** If $C = C_0(M)$ (which is allowed since every function in $C_0(M)$ is uniformly continuous) then $\mathcal{L}_{C_0(M)}$ is an ideal generated by operators of the form $M_\varphi f(D)$ and $f(D)M_\varphi$ where $\varphi \in C_0(M)$ and $f \in C_0(\mathbf{R})$. By Proposition 7.5, every such operator is compact. Hence $\mathcal{L}_{C_0(M)} \subseteq \mathcal{K}$, which implies the following.

Remark 11.16 *The Dirac extension of $C_0(M)$ with compacts adjoined is the same as the (double) Toeplitz extension of $C_0(M)$ of Definition 2.7.13.*

Now, we say a function is **periodic on M** if it is the lift of a continuous function on the base manifold. Let Per denote the C^*-algebra of periodic functions on M. Of course every periodic function on M is uniformly continuous. Thus we can let $C = Per$ in (11.15) to get the Dirac extension of Per. This extension

$$0 \to \mathcal{L}_{Per} \to \mathcal{D}_{Per} \to Per \oplus Per \to 0 \tag{11.17}$$

will also be called **the Dirac extension with periodic multipliers.** In this thesis, we are mainly interested in the K-theory index maps (or connective maps) induced by this extension.

Chapter 3

Tangential Cones

3.1 Tangential Cones

If A is any algebra, we will use the term **matrix over A** to mean any $k \times k$ matrix over A. The symbol $\boldsymbol{M_\infty(A)}$ will stand for the algebra of all matrices over A. ($M_k(A)$, as usual, will stand for the algebra of $k \times k$ matrices over A.)

If A is a C^*-algebra, we will use $\boldsymbol{Proj(A)}$ for the set of all projections in A and let $\boldsymbol{Proj_k(A)} = Proj(M_k(A))$. An element of $Proj_k(A)$ will be called a **$k \times k$ projection over A** or simply a **projection over A**. We will use $\boldsymbol{Proj_\infty(A)}$ for the set of all projections over A. So, a projection over A is a matrix over A which happens to be a projection.

If X is a topological space, a $(k \times k)$ matrix-valued ($M_k(\mathbf{C})$-valued) function (not necessarily continuous) on X will be called a **$(k \times k)$ matrix on X**. If a $(k \times k)$ matrix on X, as a function on X, is continuous, it will be called a **continuous $(k \times k)$ matrix on X**. A $(k \times k)$ matrix on X which is projection-valued (has values in $Proj_k(\mathbf{C})$) will be called a **$(k \times k)$ projection on X**. A **continuous $(k \times k)$ projection on X** has the obvious definition.

If M is a complete riemannian manifold, $\pi : TM \to M$ will denote the tangent bundle over M, $\pi : S \to M$, the (unit) sphere bundle over M, and $\exp : TM \to M$, the exponential map.

Lemma 1.1 *Let M be a complete riemannian manifold of dimension m, and, for $r > 0$, let V_r be the set of all v in TM such that $\|v\| < r$. Then, for every compact subset K of M, and, for every $r > 0$, the set $\pi^{-1}(K) \cap \overline{V_r}$ is compact in TM.*

Proof : For every $r > 0$, we let $B(r)$ stand for the ball in $\mathbf{R}^m$ centered at 0 and with radius r.

Let $r > 0$. Since every compact subset of M can be covered by a finite number of balls $B(x, a)$ where a is smaller than the injectivity radius at x, it suffices to show that $\pi^{-1}\overline{(B(x,a))} \cap \overline{V_r}$ is compact in TM for such balls $B(x, a)$. So take $B(x, a) \subseteq M$ where a is smaller than the injectivity radius at x. Let s be a number $> a$ but still smaller than the injectivity radius at x. Identify T_xM with $\mathbf{R}^m$. The exponential map $\exp_x : T_xM \to M$ can then be regarded as a map $\exp_x : \mathbf{R}^m \to M$ and the derivative $\exp_x'$ is a map $\exp_x' : \mathbf{R}^m \times \mathbf{R}^m \to TM$. The ball $B(0, s) \subseteq T_xM$ is identified with the ball $B(s) \subseteq \mathbf{R}^m$ of radius s centered at 0. $TB(0, s)$ is identified with $B(s) \times \mathbf{R}^m$. So we have

$$\exp_x : B(s) \to B(x, s)$$

and

$$\exp_x' : B(s) \times \mathbf{R}^m \to TB(x, s).$$

Both of these maps are diffeomorphisms since s is smaller than the injectivity radius at x. Thus, for each $u \in B(s)$, the derivative

$$\exp_x{}'(u) : T_u B(s) \to T_{\exp_x(u)} B(x,s)$$

or

$$\exp_x{}'(u) : B(s) \times \mathbf{R}^m \to T_{\exp_x(u)} B(x,s),$$

is invertible, and, since $B(a)$ is compact, there exists $C > 0$ such that $\| (\exp_x{}'(u))^{-1} \| \leq C$ for all $u \in \overline{B(a)}$. So, if $v \in T\overline{B(x,a)}$, and $(\exp')^{-1}(v) = (u, w) \in B(s) \times \mathbf{R}^m$, then $\|w\| \leq C \cdot \|v\|$. Hence, $(\exp')^{-1}(\pi^{-1}(\overline{B(x,a)}) \cap \overline{V_r})$ is a closed subset of the compact set $\overline{B(a)} \times \overline{B(Ca)}$ and is therefore compact. Since

$$\exp_x{}' : B(s) \times \mathbf{R}^m \to TB(x,s)$$

is a diffeomorphism, it follows that $\pi^{-1}(\overline{B(x,a)}) \cap \overline{V_r}$ is compact. ♠

If M is any complete riemannian manifold the symbol $\boldsymbol{TM \backslash M}$ will denote the tangent bundle minus the zero section of TM, and

$$\boldsymbol{r} : TM \backslash M \to S,$$

will denote the retraction map given by

$$r(v) = \frac{v}{\|v\|}.$$

The map r will be called **the retraction onto $\boldsymbol{S}$**, or **the tangent bundle retraction map**. If φ and ψ are matrices on $TM \backslash M$, we will say that φ **equals $\boldsymbol{\psi}$ at infinity** if for every $\epsilon > 0$, there exists $L > 0$ such that

$$\|\varphi(v) - \psi(v)\| < \varepsilon$$

whenever $\|v\| > L$.

Definition 1.2 *Let M be a compact, riemannian manifold. Suppose p is a nonzero continuous $k \times k$ projection on the sphere bundle S where k is some positive integer. Then* **the (upright) tangential p-cone of M**, *or simply* **the (upright) p-cone of M**, *denoted by $C(p)$, is the C^*-algebra of all continuous $k \times k$ matrices φ on TM, which, for some function f_φ in $C(M)$, equals the matrix $(\pi^* f_\varphi) \cdot (r^* p)$ at infinity. (Note that r^*p is a projection on $TM \backslash M$.)* **The slanted tangential p-cone**, *or* **the slanted p-cone**, *denoted by $SC(p)$, is the C^*-algebra of all continuous $k \times k$ matrices φ on TM which, for some function f_φ in $C(M)$, equals the matrix $exp^*(f_\varphi) \cdot (r^*p)$ at infinity.*

Remark 1.3 *Let $C(p)$ and $SC(p)$ be upright and slanted p-cones of a compact riemannian manifold M. Let φ be an element of either $C(p)$ or $SC(p)$. Since φ is equal to a bounded matrix at infinity, and since M is compact, it is easy to see that φ is bounded on TM. That is $\|\varphi\|_\infty$ is $< \infty$ and is the C^*-algebra norm of φ in the C^*-algebra $C(p)$ or $SC(p)$.*

Proposition 1.4 *Let $C(p)$ and $SC(p)$ be upright and slanted p-cones of a compact riemannian manifold M. Let φ be an element of either $C(p)$ or $SC(p)$. Then f_φ is unique.*

Proof: Let $p_x = p|_{S_x}$ for every x in M. Let $\|p_x\|$ equal the supremum of $\|p_x(v)\|$ as v ranges over S_x. Since p is a projection on S then, for every x in $M, \|p_x\|$ is either 0 or 1. By continuity of $p, \|p_x\|$ varies continuously with x. Since M is connected it follows that the function $\|p_x\|$ of x is either the constant function 1 or the constant function 0 on M. But p is a nonzero projection by assumption. Hence $\|p_x\| = 1$ for all x in M.

Therefore, for every x in M, there exists $v \in S_x$ such that $p(v) \neq 0$ (i.e. $\|p(v)\| = 1$). From this, it follows that, for every $N > 0$, there exists $v \in T_xM$ such that $\|v\| > N$ and such that $\|(r^*p)(v)\| = 1$.

Suppose $\varphi \in C(p)$, $f, g \in C(M)$, and that φ is equal to both $(\pi^* f)(r^*p)$ and $(\pi^* g)(r^*p)$ at infinity. We want to show that $f = g$. From the assumption on f and g, it follows that $\pi^*(f-g)(r^*p)$ is equal to 0 at infinity. Let $\epsilon > 0$. Then it follows that there is an $N > 0$ such that $\|\pi^*(f-g)(v)(r^*p)(v)\| < \epsilon$ whenever $v \in TM$ and $\|v\| > N$. Take $x \in m$. From above there exists $v \in T_xM$ such that $\|v\| > N$ and $\|(r^*p)(v)\| = 1$. Since $\|v\| > N$, we have $|f(x) - g(x)| = |\pi^*(f-g)(v)| = |\pi^*(f-g)(v)| \cdot \|r^*(p)(v)\| = \|\pi^*(f-g)(v)(r^*p)(v)\| < \epsilon$. That is, $|f(x) - g(x)| < \epsilon$ for every $\epsilon > 0$. Thus $f(x) - g(x) = 0$ for every x in M, which implies that $f = g$.

So the proposition is true for $\varphi \in C(p)$.

Now, suppose $\varphi \in SC(p)$, $f, g \in C(M)$, and that φ is equal to both $(\exp^* f)(r^*p)$ and $(\exp^* g)(r^*p)$ at infinity. We want to show that $f = g$. This implies that $\exp^*(f-g)(r^*p)$ is equal to 0 at infinity. Thus, if $t \mapsto v(t)$ is a curve in TM such that $\|v(t)\| \to \infty$ as $t \to \infty$, then $\|exp^*(f-g)(v(t)) \cdot (r^*p)(v(t))\| \to 0$ as $t \to \infty$.

Take now x in M. From above there exists v in S_x such that $\|p(v)\| = \|(r^*p)(v)\| = 1$. Let $\alpha : (-\infty, \infty) \to M$ be the geodesic with the property that $\alpha(0) = x$ and $\alpha'(0) = -v$. Let

$$w(t) = -t \cdot \alpha'(t) \in T_{\alpha(t)}M.$$

Then $t \mapsto w(t)$ is a curve in TM such that $\|w(t)\| = |t| \to \infty$ as $t \to \infty$. From above, it follows that

$$\|exp^*(f-g)(w(t)) \cdot (r^*p)(w(t))\| \to 0 \quad \text{as} \quad t \to \infty. \tag{1.5}$$

Now,

$$\begin{aligned}
\exp^*(f-g)(w(t)) &= (f-g)(\exp(w(t))) \\
&= (f-g)(exp(-t\alpha'(t))) \\
&= (f-g)(\alpha(0)) \\
&= (f-g)(x)
\end{aligned}$$

since α is a geodesic in M with $\alpha(0) = x$. Therefore, from (1.5), we have that

$$|(f-g)(x)|\|(r^*p)(w(t))\| \to 0 \quad \text{as} \quad t \to \infty. \tag{1.6}$$

Also, $(r^*p)(w(t)) = p(r(w(t))) = p(r(-t\alpha'(t))) = p(-\alpha'(t))$ for all $t > 0$. This is because $\|\alpha'(t)\| = 1$ (since α is a geodesic) for all $t > 0$, and, by definition, $r(-t\alpha'(t)) = \frac{-t\alpha'(t)}{\|-t\alpha'(t)\|} = \frac{-t\alpha'(t)}{t\|\alpha'(t)\|} = -\alpha'(t)$ (since $t > 0$). So, $(r^*p)(w(t))$ $= p(-\alpha'(t))$ for all $t > 0$. Thus $\lim_{t\to 0^+} (r^*p)(w(t)) = \lim_{t\to 0^+} p(-\alpha'(t))=$ $p(-\alpha'(0))= p(-(-v))= p(v) \neq 0$ by our pick of v. So, $\|(r^*p)(w(t))\| \to$ $\|p(v)\| = 1$ as $t \to 0^+$. Thus $\|(r^*p)(w(t))\| = 1$ for t close to 0. Since $t \mapsto$ $(r^*p)(w(t))$ is a continuous curve of projections, it follows that $\|(r^*p)(w(t))\| =$ 1 for all $t > 0$. This together with (1.6) implies that $|f(x) - g(x)| = |(f - g)(x)| \cdot \|(r^*p)(w(t))\| \to 0$ as $t \to \infty$. Hence $f(x) = g(x)$ for every $x \in M$, which implies $f = g$. This completes the proof. ♠

Definition 1.7 *Let $C(p)$ and $SC(p)$ be upright and slanted p-cones of a compact riemannian manifold M. Let*

$$\boldsymbol{l} : C(p) \to C(M)$$

and

$$\boldsymbol{l} : SC(p) \to C(M)$$

denote the maps which sends φ to $l(\varphi) = f_\varphi$ in both cases. (By Proposition 1.4, these maps are well defined.)

It is clear that both of these maps are surjective $*$-homomorphisms. In both cases, $l(\varphi)$ will be called **the limit of φ at ∞**.

Remark 1.8 *Even though the same symbol l is used for two different maps, it should be clear from the context what $l(\varphi)$ means.*

If M is a compact riemannian manifold, it is not difficult to see that $C_0(TM)$, the C^*-algebra of continuous functions on TM which vanish at ∞, is the same as the set of all continuous functions φ on TM such that for every $\epsilon > 0$, there is an $N > 0$ such that $|\varphi(v)| < \epsilon$ whenever $\|v\| \geq N$. So $C_0(TM)$ is the set of all continuous functions on TM which equal 0 at infinity. Hence, for each integer $k > 0, M_k(C_0(TM))$ is the set of all continuous $k \times k$ matrices on TM which equal $0 \cdot 1_k$ at ∞ (where 1_k is the $k \times k$ identity matrix). Thus, if $C(p)$ and $SC(p)$ are upright and slanted p-cones of M, where p is a nonzero, continuous $k \times k$ matrix on the sphere bundle of M, then $M_k(C_0(TM))$ is the set of all continuous $k \times k$ matrices on TM which are equal to $0 \cdot (r^*p)$ at infinity. Therefore, $M_k(C_0(TM))$ is a subalgebra of both $C(p)$ and $SC(p)$ and is the set of all $\varphi \in C(p)$ such that $l(\varphi) = 0$, or the set of all φ in $SC(p)$ such that $l(\varphi) = 0$. In other words, $M_k(C_0(TM))$ is the kernel of both maps l, whether l is from $C(p)$ to $C(M)$ or from $SC(p)$ to $C(M)$.

We therefore have two C^*-algebra extensions given in the next proposition.

Proposition 1.9 *Let M be a compact riemannian manifold, p a nonzero continuous $k \times k$ projection on the sphere bundle over M. Then we have two C^*-algebra extensions*

$$0 \to M_k(C_0(TM)) \xrightarrow{i} C(p) \xrightarrow{l} C(M) \to 0 \qquad (*)$$

and

$$0 \to M_k(C_0(TM)) \xrightarrow{i} SC(p) \xrightarrow{l} C(M) \to 0, \qquad (**)$$

where i is the inclusion map in both cases.

Proof: Follows from the previous remarks. ♠

Definition 1.10 *The two extensions $(*)$ and $(**)$ given in Proposition 1.9 will be referred to as* **Thom extensions**. *The first extension $(*)$ will be called the* **$C(p)$ Thom extension of $C(M)$**, *and $(**)$ will be called the* **$SC(p)$ Thom extension of $C(M)$**.

Now, suppose M is a compact riemannian manifold and that

$$\rho : \tilde{M} \to M$$

is its universal cover. Give $\tilde{M}$ the riemannian metric that comes from lifting the riemannian metric on M up to $\tilde{M}$.

We will use $\boldsymbol{Per(M)}$ to denote the C*-algebra of periodic functions on $\tilde{M}$. Recall that periodic functions on $\tilde{M}$ are those functions which are lifts of continuous functions on M. Clearly, $Per(M)$ is isomorphic to $C(M)$ as C*-algebras.

If φ is a matrix on M, then $\tilde{\varphi}$ will denote the lift of φ up to a matrix on $\tilde{M}$. That is,

$$\tilde{\varphi} = \varphi \circ \rho.$$

Also, if φ is a matrix on TM, then $\tilde{\varphi}$ will denote the lift of φ to a matrix on $T\tilde{M}$. That is,

$$\tilde{\varphi} = \varphi \circ \rho_*$$

where

$$\rho_* : T\tilde{M} \to TM$$

is the map induced by ρ. The symbol $\tilde{S}$ will stand for the unit sphere bundle over $\tilde{M}$, while, as before, S will stand for the unit sphere bundle over M. If A is a matrix on the sphere bundle S over M, then we let $\tilde{A}$ denote the lift of A up to a matrix on the sphere bundle $\tilde{S}$ over $\tilde{M}$. That is, we let

$$\tilde{A} = A \circ \rho_*|_{\tilde{S}}.$$

Definition 1.11 *If p is a nonzero continuous projection on the sphere bundle S over M, we define* $\boldsymbol{\widetilde{SC}(p)}$, **the lift of the slanted cone of p**, *as the C*-algebra of all matrices $\tilde{\varphi}$ on $T\tilde{M}$ such that φ belongs to $SC(p)$.*

Note 1.12 *The map $\varphi \mapsto \tilde{\varphi}, \varphi \in SC(p)$, clearly gives a $*$-isomorphism between the C*-algebras $SC(p)$ and $\widetilde{SC}(p)$. Hence, $\|\tilde{\varphi}\|_\infty = \|\varphi\|_\infty < \infty$ for every $\tilde{\varphi}$ in $\widetilde{SC}(p)$.*

Remark 1.13 *Note that, for each* φ *in* $\widetilde{SC}(p)$, *there exists* $\boldsymbol{f_\varphi}$ in $\boldsymbol{Per(M)}$ *such that* φ *is equal to*

$$\exp^*(f_\varphi)\cdot(r^*\tilde{p}) = (f_\varphi \circ \exp)\cdot(\tilde{p}\circ r)$$

at infinity. (The projection $\tilde{p}$ *is the lift of* p *to a projection on* $\tilde{S}$.*) This gives a definition of* f_φ *for each* $\varphi \in \widetilde{SC}(p)$. *It should be clear that the map* l *from* $\widetilde{SC}(p)$ *to* $Per(M)$ *which sends* φ *to* f_φ *is essentially the same as the map* $l : SC(p) \to C(M)$, *once* $\widetilde{SC}(p)$ *is identified with* $SC(p)$ *and* $Per(M)$ *is identified with* $C(M)$. *We can express this by saying that the following diagram*

$$\begin{array}{ccc} \widetilde{SC}(p) & \xrightarrow{\ l\ } & Per(M) \\ \cong\uparrow\rho^* & & \cong\uparrow\rho^* \\ SC(p) & \xrightarrow{\ l\ } & C(M) \end{array}$$

commutes. In other words, if η *belongs to* $SC(p)$, *then* $f_{\tilde{\eta}}$ *in* $Per(M)$ *is equal to* $\widetilde{f_\eta}$.

Now define $\boldsymbol{\tilde{C}_0(TM)}$ as the lift of $C_0(TM)$ to functions on $T\tilde{M}$. That is, $\tilde{C}_0(TM)$ is the set of $\tilde{\varphi}$ such that φ belongs to $C_0(TM)$. Clearly $\tilde{C}_0(TM)$ is a C^*-algebra and is isomorphic to $C_0(TM)$ via the lifting map ρ^*.

Proposition 1.14 *Let* M *be a compact riemannian manifold with universal cover* $\rho : \tilde{M} \to M$. *Suppose* $\tilde{M}$ *is given the riemannian structure induced by the lifting, and that* p *is a nonzero, continuous* $k \times k$ *matrix on the sphere bundle over* M. *Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_k(\tilde{C}_0(TM)) & \xrightarrow{\ i\ } & \widetilde{SC}(p) & \xrightarrow{\ l\ } & Per(M) & \longrightarrow & 0 \\ & & \cong\uparrow\rho^* & & \cong\uparrow\rho^* & & \cong\uparrow\rho^* & & \\ 0 & \longrightarrow & M_k(C_0(TM)) & \xrightarrow{\ i\ } & SC(p) & \xrightarrow{\ l\ } & C(M) & \longrightarrow & 0 \end{array}$$

commutes, and the two rows are C^*-algebra extensions.

Proof : Follows immediately from Proposition 1.9 and Remark 1.13. ♠

The top extension of Proposition 1.14 and Remark 1.13. will be called **the $\widetilde{SC}(p)$ Thom extension of Per(M).**

3.2 Nonpositive Curvature

In this section, we assume that M is a **simply connected, complete, riemannian manifold of nonpositive sectional curvature.** It is well known that, in this case, the exponential maps

$$exp_x : T_xM \to M, \quad x \in M,$$

are C^∞ diffeomorphisms. For this reason, we let

$$M_x = T_xM$$

for every $x \in M$.

Proposition 2.1 *The map*

$$\pi \times exp : TM \to M \times M$$

is a homeomorphism.

Proof: The exponential map exp : $TM \to M$ is a C^∞ map (by Theorem 7.1 of [**Boo**] for example). The projection $\pi : TM \to M$ is also C^∞ by the definition of the C^∞ structure on TM. Hence $\pi \times \exp : TM \to M \times M$ is C^∞

and therefore continuous. It is bijective for it has the map $(x,y) \mapsto (\exp_x)^{-1}(y)$ from $M \times M$ to TM as its inverse.

To show that $\pi \times \exp$ is a homeomorphism, we need to show that $(\pi \times \exp)(U)$ is open in $M \times M$ for every open U in TM. If $r > 0$, let $V_r = \{v \in TM : \|v\| < r\}$. It suffices to show that $(\pi \times \exp)(U)$ is open in $M \times M$ for those U which are open subsets of the open set $(\pi \times \exp)^{-1}(B(x,r) \times B(y,s))$ for some $x, y \in M, r > 0, s > 0$. For these U's certainly form a basis for the topology of TM.

So choose $x, y \in M$ and $r, s > 0$, and let

$$C = r + d(x,y) + s.$$

We note that if $v \in (\pi \times \exp)^{-1}(B(x,r) \times B(y,s))$, then $\pi v \in B(x,r)$ and $\exp(v) \in B(y,s)$ so that the distance $d(\pi v, \exp v)$ from πv to $\exp v$ is $\leq$ the diameter of $B(x,r) \cup B(y,s)$ which is $\leq r + d(x,y) + s = C$. But $d(\pi v, \exp v) = \|v\|$. Hence $\|v\| \leq C$ or $v \in \bar{V_C}$ for every v in $(\pi \times \exp)^{-1}(B(x,r) \times B(y,s))$. It follows that

$$(\pi \times \exp)^{-1}(B(x,r) \times B(y,s)) \subseteq \pi^{-1}(\overline{B(x,r)}) \cap \bar{V_C}.$$

By Lemma 1.1, $\pi^{-1}(\overline{B(x,r)}) \cap \bar{V_C}$ is compact in TM. Hence, $(\pi \times \exp)^{-1}(\overline{B(x,r)} \times \overline{B(y,s)})$ is a closed subset of a compact set and is therefore compact. We know that $\pi \times \exp$ is a continuous bijection from $(\pi \times \exp)^{-1}(\overline{B(x,r)} \times \overline{B(y,s)})$ onto $\overline{B(x,r)} \times \overline{B(y,s)}$. Since $(\pi \times \exp)^{-1}(\overline{B(x,r)} \times \overline{B(y,s)})$ is compact, it is also a homeomorphism. Thus the map $\pi \times \exp$ from $(\pi \times \exp)^{-1}(B(x,r) \times B(y,s))$ onto $B(x,r) \times B(y,s)$ is a homeomorphism.

It follows that $(\pi\times\exp)(U)$ is an open subset of $B(x,r)\times B(y,s)$ and therefore an open subset of $M\times M$ for every open subset U of $(\pi\times\exp)^{-1}(B(x,r)\times B(y,s))$. ♠

Definition 2.2 (**Metric space structure on TM**.) *We give TM the metric space structure that comes from pulling the metric space structure of $M\times M$ back to TM via the homeomorphism*

$$\pi\times\exp : TM \to M\times M$$

of Proposition 2.1. In this metric, we have

$$d(v,w)^2 = d(\pi v,\pi w)^2 + d(\exp(v),\exp(w))^2$$

for every v, w in TM.

Proposition 2.3 *Every closed bounded subset of TM is compact.*

Proof: Since M is a complete riemannian manifold, every closed, bounded subset of M is compact. It follows that every closed bounded subset of $M\times M$ is compact. Since TM is isometric to $M\times M$ as a metric space, every closed bounded subset of TM is therefore compact. ♠

Definition 2.4 *If X is a metric space, $\boldsymbol{UC(X)}$ will denote the C^*-algebra of all bounded, uniformly continuous functions on X, and $\boldsymbol{UC(X, M_k(\mathbf{C}))}$ will denote the C^*-algebra of all bounded, uniformly continuous maps from X to $M_k(\mathbf{C})$. If $\varphi\in UC(X, M_k(\mathbf{C}))$ then the C^*-algebra norm of φ is the same as the infinity norm $\|\varphi\|_\infty$. We will often identify $UC(X, M_k(\mathbf{C}))$ with $M_k(UC(X))$.*

Remark 2.5 *Note that $UC(X, M_k(\mathbf{C}))$ is a C^*-subalgebra of the C^*-algebra $BC(X, M_k(\mathbf{C}))$ of all bounded continuous maps from X to $M_k(\mathbf{C})$.*

Since TM and M are both metric spaces then $UC\ (M)$, $UC(TM)$, $UC(M, M_k(\mathbf{C}))$ and $UC(TM, M_k(\mathbf{C}))$ all make sense.

Definition 2.6 *If φ is a matrix on TM, and if x belongs to M, then* **the x-component of φ** *is, by definition, the matrix*

$$\varphi_x = \exp_{x*}(\varphi|_{M_x})$$

on M.

Proposition 2.7 *If φ belongs to $UC(TM, M_k(\mathbf{C}))$ then φ_x belongs to UC $(M, M_k(\mathbf{C}))$ for every x in M, and the map $x \mapsto \varphi_x$ is a bounded continuous map from M to $UC(M, M_k(\mathbf{C}))$.*

Proof: Follows immediately from boundedness and uniform continuity on TM of every φ in $UC(TM, M_k(\mathbf{C}))$. ♠

Notation 2.8 *If A, B are points in M, we will sometimes use $|AB|$ to denote the distance $d(A, B)$ from A to B which is the same as the length of the unique geodesic segment from A to B.*

For each x in M, let Sph_x denote the set of all z in M whose distance from x is equal to 1. Let

$$r_x : M \backslash \{x\} \to M$$

denote the retraction which sends each z in $M \backslash \{x\}$ to the unique element on Sph_x which lies on the geodesic ray starting at x and passing through z.

Lemma 2.9 *let A, B, C be three points in M such that $|CA| > 1$ and $|CB| > 1$. Let $E = r_A(C)$ and $D = r_B(C)$. Then $|DE| \leq |AB|$.*

Proof: Note that from the definition of the retractions, we have that E belongs to Sph_A and D belongs to Sph_B, which means

$$|AE| = 1, \text{ and } |BD| = 1. \tag{2.10}$$

So, by the assumptions $|AC| > 1, |BC| > 1$, the point E lies inside the segment AC, and D lies inside BC.

Let us assume that

$$|AC| \leq |BC|.$$

Make the definition,

$$\lambda = \frac{|CD|}{|BC|}. \tag{2.11}$$

Let F be the point on the geodesic segment AC satisfying

$$\frac{|CF|}{|AC|} = \frac{|CD|}{|BC|} = \lambda. \tag{2.12}$$

Of course,

$$|DE| \leq |DF| + |EF|. \tag{2.13}$$

We now get upper bounds for $|DF|$ and $|EF|$. Since $\frac{|CF|}{|AC|}$ is equal to $\frac{|CD|}{|BC|}$, then $1 - \frac{|CF|}{|AC|}$ is equal to $1 - \frac{|CD|}{|BC|}$, which implies $\frac{|AF|}{|AC|}$ equals $\frac{|BD|}{|BC|}$. That is, $\frac{|AF|}{|AC|}$ is the same as $\frac{1}{|BC|}$, from which we get

$$|AF| = \frac{|AC|}{|BC|}. \tag{2.14}$$

Since $|AC|$ is less than or equal to $|BC|$, by assumption, it follows that $|AF|$ is less than or equal to 1. But, $|AE|$ equals 1, by (2.10), and F lies on the geodesic line segment AEC. Therefore, F must lie on the segment AE. From this, it follows that $|EF|$ equals $1-|AF|$. From 2.14, it then follows that $|EF|$ is equal to $1-\frac{|AC|}{|BC|}$, which is the same as $\frac{|BC|-|AC|}{|BC|}$. But, $|BC|-|AC|$ is less than or equal to $|AB|$, by the triangle inequality. Thus,

$$|EF| \leq \frac{|AB|}{|BC|}. \tag{2.15}$$

Now, from p.2, Section 1A, of ([**BGS**]), we have that

$$|DF| \leq \lambda |AB|. \tag{2.16}$$

Thus, from (2.11), (2.13), (2.15), and (2.16), it follows that

$$\begin{aligned} |DE| &\leq \lambda|AB| + \frac{|AB|}{|BC|} = \frac{|CD|}{|BC|}|AB| + \frac{|AB|}{|BC|} \\ &= (|CD|+1)\frac{|AB|}{|BC|} = (|CD|+|BD|)\frac{|AB|}{|BC|} \\ &= |BC|\frac{|AB|}{|BC|} = |AB|. \end{aligned}$$

Thus, the lemma is true. ♠

Lemma 2.17 *Let A, B, C be three points in M such that $|CA| \geq 2$ and $|CB| \geq 2$. Let $D = r_C(A)$ and $E = r_C(B)$. Then $|DE| \leq |AB|$.*

Proof: Of course, D lies on the segment CA, and E lies on the segment CB. Also

$$|DC| = 1 \quad \text{and} \quad |CE| = 1.$$

Assume $|AC| \leq |BC|$. Let F be the unique point on the geodesic segment CB satisfying

$$\frac{|CF|}{|CB|} = \frac{|CD|}{|CA|} = \frac{1}{|CA|} \stackrel{\text{def}}{=} \lambda.$$

Then, as in the proof of Lemma 2.9,

$$|DF| \leq \lambda \cdot |AB| = \frac{1}{|AC|} \cdot |AB|.$$

Also, since $|CF| = \frac{|BC|}{|AC|} \geq 1$, and since $|CE| = 1$, then E lies on the geodesic segment CF and $|CF| = |CE| + |EF| = 1 + |EF|$. Hence

$$\begin{aligned} |EF| &= |CF| - 1 \\ &= \frac{|BC|}{|AC|} - 1. \end{aligned}$$

By the triangle inequality, $|BC| \leq |AB| + |AC|$. So

$$\begin{aligned} |EF| &\leq \frac{|AB| + |AC|}{|AC|} - 1 \\ &= \frac{|AB|}{|AC|}. \end{aligned}$$

Thus,

$$\begin{aligned} |DE| &\leq |DF| + |EF| \\ &\leq \frac{|AB|}{|AC|} + \frac{|AB|}{|AC|} \\ &= 2\frac{|AB|}{|AC|}. \end{aligned}$$

So to show that $|DE| \leq |AB|$, it suffices to show that $\frac{2}{|AC|} \leq 1$ or that $|AC| \geq 2$. But this is true by assumption. Therefore Lemma 2.17 is true. ♠

Corollary 2.18 *Let* $V_N = \{v \in TM : \|v\| < N\}$*, and let* S *be the sphere bundle over* M*. Then the retraction map*

$$r : TM \backslash V_2 \to S$$

is uniformly continuous on $TM \backslash V_2$*.*

Proof: It suffices to show that

$$d(r(u), r(v)) \leq d(u, v)$$

when $u, v \in TM \backslash V_2$ and $\pi u = \pi v$, and that

$$d(ru, rv) \leq \sqrt{2}\ d(u, v)$$

when $u, v \in TM \backslash V_2$ and $\exp u = \exp v$.

Look first at the case where $u, v \in TM \backslash V_2$ and $\pi u = \pi v = x$ say. Then both u and v belong to $T_x M$, $\|u\| \geq 2$, and $\|v\| \geq 2$. Let $a = \exp(u)$, $b = \exp(v)$. Then $d(x, a) \geq 2$, $d(x, b) \geq 2$, and

$$d(u, v) = \sqrt{d(\pi u, \pi v)^2 + d(\exp(u), \exp(v))^2} = d(a, b)$$

since $\pi u = \pi v$, $\exp(u) = a$ and $\exp(v) = b$. By Lemma 2.17, it follows that

$$d(r_x(a), r_x(b)) \leq d(a, b).$$

Notice that

$$\exp(r(w)) = r_x(\exp(w))$$

for every w in $T_x M \backslash \{0\}$. Thus

$$\begin{aligned}
d(r(u), r(v)) &= d((\exp_x)(ru), (\exp_x)(rv)) \\
&= d(r_x(\exp_x u), r_x(\exp_x v)) \\
&= d(r_x a, r_x b) \\
&\leq d(a, b) \\
&= d(\exp_x u, \exp_x v) \\
&= d(u, v).
\end{aligned}$$

So $d(ru, rv) \leq d(u, v)$ when $u, v \in TM\backslash V_2$ and $\pi u = \pi v$.

Now look at the second case where $u, v \in TM\backslash V_2$ and $\exp u = \exp v = x$ say. Let $\pi u = a, \pi v = b$. That is, $u \in T_a M$ and $v \in T_b M$. Since $u, v \in TM\backslash V_2$, then $\|u\| \geq 2$ and $\|v\| \geq 2$, and therefore $d(\exp(u), a) \geq 2$, and $d(\exp(v), b) \geq 2$. That is, $d(x, a) \geq 2$ and $d(x, b) \geq 2$. By Lemma 2.9, it follows that

$$d(r_a x, r_b x) \leq d(a, b).$$

Also, we have that

$$\begin{aligned} d(u, v) &= \sqrt{d(\pi u, \pi v)^2 + d(\exp(u), \exp(v))^2} \\ &= \sqrt{d(a, b)^2 + d(x, x)^2} \\ &= d(a, b) \end{aligned}$$

Thus,

$$\begin{aligned} d(ru, rv) &= \sqrt{d(\pi(ru), \pi(rv))^2 + d(\exp(ru), \exp(rv))^2} \\ &= \sqrt{d(a, b)^2 + d(r_a(\exp(u)), r_b(\exp(v)))^2} \\ &= \sqrt{d(a, b)^2 + d(r_a x, r_b x)^2} \\ &\leq \sqrt{d(a, b)^2 + d(a, b)^2} \\ &= \sqrt{2}\, d(a, b) \\ &= \sqrt{2}\, d(u, v). \end{aligned}$$

The corollary is therefore true. ♠

3.3 Components of a Slanted Cone Element

In this section, M is a **compact, riemannian manifold of nonpositive curvature**, and

$$\rho : \tilde{M} \to M$$

is its universal cover. We give $\tilde{M}$ the riemannian structure that comes from lifting the riemannian structure on M up to the cover $\tilde{M}$. So, $\tilde{M}$ also has nonpositive curvature.

Proposition 3.1 *If p is a continuous $k \times k$ projection on the sphere bundle over M, then $\widetilde{SC}(p) \subseteq UC(T\tilde{M}, M_k(\mathbf{C}))$.*

Proof: Let p be a continuous $k \times k$ projection on the sphere bundle S over M, let φ be an element of $\widetilde{SC}(p)$, and let η be the unique element of $SC(p)$ such that $\varphi = \tilde{\eta}$. If $N > 0$, let

$$V_N = \{v \in TM : \|v\| < N\}$$

and let

$$V = V_2.$$

Since M is compact, there exists $x \in \tilde{M}$ and $r > 0$ such that $B(x, r)$ covers M. That is, $\rho(B(x, r)) = M$ where $\rho : \tilde{M} \to M$ is the covering map. Let

$$B = B(x, 2r).$$

We want to show that φ is a bounded, uniformly continuous matrix on $T\tilde{M}$. By Note 1.12, it is bounded. To show it is uniformly continuous on $T\tilde{M}$, it suffices to show that it is uniformly continuous on $\pi^{-1}(\bar{B})$. For if so, then for every $\epsilon > 0$ there would exist $\delta > 0$ and $< r$ such that $\|\varphi v - \varphi w\| < \epsilon$ whenever $v, w \in \pi^{-1}(\bar{B})$ and $d(v, w) < \delta$. Suppose now that $v, w \in T\tilde{M}$ and $d(v, w) < \delta$. Since B covers M, there exists $\hat{v}$ in $TB = \pi^{-1}(B)$ such that $\rho(\hat{v}) = \rho(v)$ in TM, where $\rho : T\tilde{M} \to TM$ is the map induced by $\rho : \tilde{M} \to M$. The action of the fundamental group $\Gamma = \pi_1(M)$ on $\tilde{M}$ induces an action of Γ on $T\tilde{M}$. So $\hat{v} = g \cdot v$ for a unique $g \in \Gamma$. This action preserves distance in $T\tilde{M}$ (easy to check). Therefore, if we let $\hat{w} = g \cdot w$ then $d(\hat{v}, \hat{w}) = d(v, w) < \delta$ and $\|\varphi(\hat{v}) - \varphi(\hat{w})\|$ would be smaller than ϵ. But φ is the lift of a matrix on TM and so is invariant under the action of Γ. Thus, $\|\varphi v - \varphi w\| = \|\varphi(\hat{v}) - \varphi(\hat{w})\|$ would be $< \epsilon$. That is, $\|\varphi v - \varphi w\|$ would be $< \epsilon$ whenever $v, w \in T\tilde{M}$ and $d(v, w) < \delta$. Hence φ would be uniformly continuous on $T\tilde{M}$ if it were uniformly continuous on $TB = \pi^{-1}(\bar{B})$.

Let us now show uniform continuity of φ on $\pi^{-1}(\bar{B})$. By Lemma 2.11.11, $\pi^{-1}(\bar{B}) \cap \bar{V}_N$ is compact for every $N > 0$. Thus φ is uniformly continuous on $\pi^{-1}(\bar{B}) \cap \bar{V}_N$ for every $N > 0$. Using this and the fact (Remark 1.13) that φ is equal to $\exp^*(f_\varphi) \cdot (r^*\tilde{p})$ at infinity, we can reduce the proof to showing that $\exp^*(f_\varphi) \cdot (r^*\tilde{p})$ is uniformly continuous on $\pi^{-1}(\bar{B}) \backslash V$. (Recall $V = V_2$.)

Now, f_φ is a periodic function. That is, it is the lift of a continuous function on M. f_φ is therefore uniformly continuous on $\tilde{M}$. It follows that $\exp^*(f_\varphi)$ is uniformly continuous on all of $T\tilde{M}$. For

$$|\exp^*(f_\varphi)(u) - \exp^*(f_\varphi)(v)| = |f_\varphi(\exp(u)) - f_\varphi(\exp(v))|.$$

and, $d(\exp(u), \exp(v)) \leq d(u, v)$ for every $u, v \in T\tilde{M}$. Hence $\exp^*(f_\varphi)$ is uniformly continuous on all of $T\tilde{M}$.

So, to complete the proof, we only have to show that $r^*\tilde{p} = \tilde{p} \circ r$ is uniformly continuous on $\pi^{-1}(\bar{B})\backslash V$.

Let $\tilde{S}$ be the sphere bundle over $\tilde{M}$. The map $\tilde{p} \circ r$ on $\pi^{-1}(\bar{B})\backslash V$ is the composition of the function

$$\tilde{p} : \tilde{S} \cap \pi^{-1}(\bar{B}) \to M_k(\mathbf{C})$$

with the function

$$r : \pi^{-1}(\bar{B})\backslash V \to \tilde{S} \cap \pi^{-1}(\bar{B}).$$

To show

$$\tilde{p} \circ r : \pi^{-1}(\bar{B})\backslash V \to M_k(\mathbf{C})$$

is uniformly continuous, it suffices to show that both maps above are uniformly continuous.

Now, it is not hard to see that $\tilde{S} \cap \pi^{-1}(\bar{B})$ is a bounded subset of $T\tilde{M}$. For $\pi(\tilde{S} \cap \pi^{-1}(\bar{B})) \subseteq \bar{B}$ and $\exp(\tilde{S} \cap \pi^{-1}(\bar{B})) \subseteq B(\bar{B}, 1) = B(x, 2r+1)$. Since $\tilde{S}$ and $\pi^{-1}(\bar{B})$ are closed in $T\tilde{M}$, then $\tilde{S} \cap \pi^{-1}(\bar{B})$ is a closed bounded subset of $T\tilde{M}$. It follows, by Proposition 2.3, that $\tilde{S} \cap \pi^{-1}(\bar{B})$ is compact in $T\tilde{M}$. So $\tilde{p} : \tilde{S} \cap \pi^{-1}(\bar{B}) \to M_k(\mathbf{C})$ is a continuous function on a compact metric space, and is therefore uniformly continuous.

Thus, to complete the proof we only have to show that $r : \pi^{-1}(\bar{B})\backslash V \to \tilde{S} \cap \pi^{-1}(\bar{B})$ is uniformly continuous. But Corollary 2.18 tells us that, in fact, r is uniformly continuous on all of $TM\backslash V$. So it certainly is uniformly continuous on $\pi^{-1}(\bar{B})\backslash V$. Proposition 3.1 is therefore true. ♠

Proposition 3.2 *If p is a continuous $k \times k$ projection on the sphere bundle over M, and φ belongs to $\widetilde{SC}(p)$, then the map $x \mapsto \varphi_x$, $x \in \tilde{M}$, is a bounded continuous map from $\tilde{M}$ to $UC(\tilde{M}, M_k(\mathbf{C}))$.*

Proof: Suppose $\varphi \in \widetilde{SC}(p)$. By Proposition 3.1, φ is a bounded, uniformly continuous function from $T\tilde{M}$ to $M_k(\mathbf{C})$. By Proposition 2.7, it follows that the map $x \mapsto \varphi_x$ is a bounded continuous map from $\tilde{M}$ to $UC(\tilde{M}, M_k(\mathbf{C}))$. Hence, Proposition 3.2 is true. ♠

Chapter 4

Hilbert and C*-algebra Bundles

4.1 Hilbert and C*-algebra Bundles over $\tilde{M}$

In this section, M is a **compact, riemannian spin manifold of non-positive curvature**, and $\tilde{M}$ is its universal cover with covering map $\rho : \tilde{M} \to M$. Also, throughout the section, $\boldsymbol{k}$ will denote a **positive integer**.

If X and Y are riemannian manifolds, the term **isometry between $\boldsymbol{X}$ and $\boldsymbol{Y}$** will be reserved for any diffeomorphism f from X to Y such that, for x in X, and v, w in T_xX, $(f_*v, f_*w) = (v, w)$. If this condition is satisfied by some f which is not necessarily bijective, then we say only that f is a **locally isometric map from $\boldsymbol{X}$ to $\boldsymbol{Y}$**. The group of diffeomorphisms on X will be denoted by *Diff*$(\boldsymbol{X})$, while the group of isometries on X will be denoted by $\boldsymbol{Isom}(\boldsymbol{X})$.

Similarly, if $V \to X$ and $W \to Y$ are two hermitian bundles over X and Y, then we will use the term, **bundle isometry from $\boldsymbol{V}$ to $\boldsymbol{W}$**, for a bundle isomorphism from V to W which preserves the hermitian structure, and whose restriction to a map from the zero elements of V (which we identify with X) to the zero elements of W (which we identify with Y) is an isometry from X to Y. If a bundle homomorphism f from V to W preserves the hermitian structure, and is a local isometry from X to Y, when restricted to a map from the zero elements of V to those of W, but is not necessarily bijective, then we will call f only a **locally isometric bundle homorphism** from V to W. We will use $\boldsymbol{Isom(V)}$ to denote the group of bundle isometries on V.

If h is any Hilbert space, then $\boldsymbol{U(h)}$ will stand for the group of unitaries on h, and we will use the symbol $\boldsymbol{h^k}$ to denote the direct sum of h with itself k times. That is, we let

$$h^k = \underbrace{h \oplus \cdots \oplus h}_{k\ times}.$$

Of course, h^k is also a Hilbert space. Also, we have an obvious $*$-isomorphism between $B(h^k)$ and $M_k(B(h))$.

Similarly, if $h \to X$ is a Hilbert bundle, we let $\boldsymbol{h^k \to X}$ denote the Hilbert bundle

$$h^k = \underbrace{h \oplus \cdots \oplus h}_{k\ times}$$

over X. We also let $\boldsymbol{B(h) \to X}$ denote the C*-algebra bundle over X associated to the Hilbert bundle h. So, $B(h)$ is the C*-algebra bundle over X whose fiber $B(h)_x$, at each point x in X, is the C*-algebra $B(h_x)$. The bundle structure on $B(h_x)$ is the obvious one coming from the bundle structure on h. Note that if a in $B(h)$ and ξ in h lie over the same point x in X, that is, if a belongs to $B(h)_x$ and ξ belongs to h_x, then we can multiply ξ by a to get the element $a \cdot \xi$ of h_x.

If $A \to X$ is any C*-algebra bundle over X then we have, associated to A, the C*-algebra bundle $\boldsymbol{M_k(A)} \to \boldsymbol{X}$ whose fiber at each x in X is the algebra $M_k(A_x)$ of matrices over the C*-algebra A_x. The bundle structure on $M_k(A)$ is the one induced by the bundle structure on A.

It is clear from the definitions that the C*-algebra bundle $B(h^k)$ associated to the bundle h^k is canonically isomorphic to the C*-algebra bundle $M_k(B(h))$. So, every element of $B(h^k)_x$ can be represented as a matrix (a_{ij}) over the C*-algebra $B(h_x)$, where each a_{ij} is an element of $B(h_x)$. If A is in $B(h^k)_x$, and $A = (a_{ij})$ where each a_{ij} is an element of $B(h)_x$, and if ξ belongs to $(h^k)_x = (h_x)^k$, and $\xi = (\xi_1, \ldots, \xi_k)$ where each ξ_i belongs to h_x, then the multiplication of ξ by A is given by matrix multiplication of the k-column $(\xi_1, \ldots, \xi_k)$ by the $k \times k$ matrix (a_{ij}). That is, $A \cdot \xi = (a_{ij}) \cdot (\xi_1, \ldots, \xi_k)$.

Now, for each x in $\tilde{M}$, we use the diffeomorphism

$$exp_x : \tilde{M}_x \to \tilde{M},$$

to pull the riemannian and spin structures on $\tilde{M}$ back to the manifold $\tilde{M}_x$. This makes $\tilde{M}_x$ into a riemannian manifold of nonpositive curvature, and the map

$$exp_x : \tilde{M}_x \to \tilde{M}$$

becomes an isometry of riemannian manifolds.

The spin structure on $\tilde{M}_x$ thus obtained gives us a bundle

$$\tilde{\Delta}(x) \to \tilde{M}_x$$

of spinors over $\tilde{M}_x$, and the isometry

$$exp_x : \tilde{M}_x \to \tilde{M}$$

extends to a bundle isometry

$$exp_x : \tilde{\Delta}(x) \to \tilde{\Delta}. \tag{1.1}$$

This map then induces the unitary operator

$$exp_x : L^2(\tilde{\Delta}(x)) \to L^2(\tilde{\Delta}),$$

which, in turn, induces the unitary operator

$$\pi_x : L^2(\tilde{\Delta}(x))^k \to L^2(\tilde{\Delta})^k.$$

Note that if $(\xi_1, \ldots, \xi_k)$ is an element of $L^2(\tilde{\Delta}(x))^k$, then

$$exp_x(\xi_1, \ldots, \xi_k) = (exp_x(\xi_1), \ldots, exp_x(\xi_k)). \tag{1.2}$$

We now let $\tilde{\boldsymbol{H}}_x$ denote the Hilbert space

$$\tilde{H}_x = L^2(\tilde{\Delta}(x)).$$

So, we have the unitaries

$$exp_x : \tilde{H}_x \to L^2(\tilde{\Delta}),$$

and

$$exp_x : (\tilde{H}_x)^k \to L^2(\tilde{\Delta})^k,$$

and if $(\xi_1, \ldots, \xi_k)$ is an element of $(\tilde{H}_x)^k$ then we have that

$$exp_x(\xi_1, \ldots, \xi_k) = (exp_x(\xi_1), \ldots, exp_x(\xi_k)). \tag{1.3}$$

Now, define $\tilde{\boldsymbol{H}}$ as the set

$$\tilde{H} = \bigcup_{x \in \tilde{M}} \tilde{H}_x.$$

We have the map

$$exp : \tilde{H} \to L^2(\tilde{\Delta})$$

defined by setting

$$exp|_{\tilde{H}_x} = exp_x$$

for all x in $\tilde{M}$.

Since each $\tilde{H}_x$ is a Hilbert space of L^2 sections living on the manifold $\tilde{M}_x$, it makes sense to denote the map from $\tilde{H}$ to $\tilde{M}$, which sends everything in each $\tilde{H}_x$ to x, by the symbol π. That is,

$$\pi : \tilde{H} \to \tilde{M}$$

is the map that satisfies

$$\pi(\xi) = x$$

for all ξ in $\tilde{H}_x$.

We have the map

$$\pi \times exp : \tilde{H} \to \tilde{M} \times L^2(\tilde{\Delta})$$

given by

$$(\pi \times exp)(\xi) = (\pi(\xi), exp(\xi))$$

for every ξ in $\tilde{H}$. This is a bijective map since $exp_x : \tilde{H}_x \to L^2(\tilde{\Delta})$ is a unitary for every x. Also, when restricted to each $\tilde{H}_x, \pi \times exp$ is a unitary operator from $\tilde{H}_x$ onto $\{x\} \times L^2(\tilde{\Delta})$.

Now, $\tilde{M} \times L^2(\tilde{\Delta})$, with the product topology, is a trivial Hilbert bundle over $\tilde{M}$ with bundle projection map

$$\tilde{M} \times L^2(\tilde{\Delta}) \to \tilde{M}$$

defined by

$$(x, \xi) \mapsto x$$

and with Hilbert space fiber $\{x\} \times L^2(\tilde{\Delta})$ at each point x in $\tilde{M}$. If we now give $\tilde{H}$ the topology which makes the map

$$\pi \times exp : \tilde{H} \to \tilde{M} \times L^2(\tilde{\Delta})$$

into a homeomorphism, then, from the fact that $\pi \times exp$ restricted to each $\tilde{H}_x$ is a unitary operator from $\tilde{H}_x$ to $\{x\} \times L^2(\tilde{\Delta})$, we see that $\tilde{H}$ becomes a trivial Hilbert bundle over $\tilde{M}$, with bundle projection map

$$\pi : \tilde{H} \to \tilde{M},$$

defined above, with fiber $\tilde{H}_x$ at each point x in $\tilde{M}$, and with trivializing map

$$\pi \times exp : \tilde{H} \to \tilde{M} \times L^2(\tilde{\Delta}).$$

Similarly, we also have the Hilbert bundle

$$\tilde{H}^k \to \tilde{M}$$

with fiber $(\tilde{H}_x)^k$ at each x in $\tilde{M}$, and with bundle projection map

$$\pi : \tilde{H}^k \to \tilde{M}$$

which sends everything in the fiber $(\tilde{H}_x)^k$ to x. We have the map

$$\exp : \tilde{H}^k \to L^2(\tilde{\Delta})^k$$

whose restriction to the fiber $(\tilde{H}^k)_x$ is the unitary operator

$$\exp_x : (\tilde{H}_x)^k \to L^2(\tilde{\Delta})^k.$$

Moreover, the Hilbert bundle $\tilde{H}^k$ over $\tilde{M}$ is trivial, with trivializing map

$$\pi \times exp : \tilde{H}^k \to \tilde{M} \times L^2(\tilde{\Delta})^k.$$

Definition 1.4 *The Hilbert bundle*

$$\pi : \tilde{H}^k \to \tilde{M}$$

will be called **the kth Hilbert bundle over $\tilde{M}$ associated to the bundle Δ over M.**

Now let $\tilde{\mathcal{B}}$ be the C*-algebra bundle over $\tilde{M}$ associated to the Hilbert bundle $\tilde{H}$, and let $\tilde{\mathcal{B}}_k$ be the C*-algebra bundle over $\tilde{M}$ associated to the Hilbert bundle $\tilde{H}^k$. That is,

$$\tilde{\mathcal{B}} = B(\tilde{H})$$

and

$$\tilde{\mathcal{B}}_k = B(\tilde{H}^k).$$

Of course, we have that

$$\tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}},$$

since

$$\tilde{H}^1 = \tilde{H}.$$

The fiber of $\tilde{\mathcal{B}}_k$ at each x in $\tilde{M}$ will be denoted by $\tilde{\mathcal{B}}_{k,x}$.

From remarks made earlier, we have a canonical C*-algebra bundle isomorphism between $\tilde{\mathcal{B}}_k = B(\tilde{H}^k)$ and $M_k(B(\tilde{H})) = M_k(\tilde{\mathcal{B}})$. That is,

$$\tilde{\mathcal{B}}_k \cong M_k(\tilde{\mathcal{B}})$$

as C*-algebra bundles. So,

$$\tilde{\mathcal{B}}_{k,x} \cong B((\tilde{H}_x)^k)$$

for each x in $\tilde{M}$, and each element A of $\tilde{\mathcal{B}}_{k,x}$ can be expressed as a $k \times k$ matrix (a_{ij}) over $\tilde{\mathcal{B}}_x$. We write, in this case,

$$A = (a_{ij}).$$

Multiplication of an element $(\xi_1, \ldots, \xi_k)$ of $(\tilde{H}_x)^k$ by an element (a_{ij}) of $\tilde{\mathcal{B}}_{k,x}$ is given by ordinary multiplication of a k-column by a $k \times k$ matrix.

We let π represent the bundle projection map for this bundle. So,

$$\pi : \tilde{\mathcal{B}}_k \to \tilde{M}$$

maps every element in each $\tilde{\mathcal{B}}_{k,x}$ to x.

For every x in $\tilde{M}$, the unitary

$$\exp_x : (\tilde{H}_x)^k \to L^2(\tilde{\Delta})^k$$

induces a C*-algebra isomorphism

$$\exp_x : \tilde{\mathcal{B}}_{k,x} = B((\tilde{H}_x)^k) \to B(L^2(\tilde{\Delta})^k).$$

Using the canonical identification of $B(L^2(\tilde{\Delta})^k)$ with $M_k(B(L^2(\tilde{\Delta})))$, we can write each element A of $B(L^2(\tilde{\Delta})^k)$ as a $k \times k$ matrix (a_{ij}) over $B(L^2(\tilde{\Delta}))$. By Equation 1.3 on page 109, we then have that

$$\exp_x(a_{ij}) = (\pi_x(a_{ij})) \tag{1.5}$$

for all (a_{ij}) in $\tilde{\mathcal{B}}_{k,x}$.

Define

$$\exp : \tilde{\mathcal{B}}_k \to B(L^2(\tilde{\Delta})^k)$$

as the map whose restriction to each $\tilde{\mathcal{B}}_{k,x}$ gives the $*$-isomorphism $\exp_x$. Then, as before, we have that $\tilde{\mathcal{B}}_k$ is a trivial C*-algebra bundle over the manifold $\tilde{M}$ with the map

$$\pi \times exp : \tilde{\mathcal{B}}_k \to \tilde{M} \times B(L^2(\tilde{\Delta})^k)$$

being a C*-algebra bundle trivialization.

Definition 1.6 *The C*-algebra bundle*

$$\pi : \tilde{\mathcal{B}}_k \to \tilde{M}$$

will be called **the k-th C*-algebra bundle over $\tilde{M}$ associated to the bundle $\tilde{\Delta}$.**

Definition 1.7 *We have the identification map*

$$\pi \times exp : T\tilde{M} \to \tilde{M} \times \tilde{M}$$

of $T\tilde{M}$ with $\tilde{M} \times \tilde{M}$, and trivialization maps

$$\pi \times exp : \tilde{H}^k \to \tilde{M} \times L^2(\tilde{\Delta})^k,$$

$$\textit{and} \qquad \pi \times exp : \tilde{\mathcal{B}}_k \to \tilde{M} \times B(L^2(\tilde{\Delta})^k).$$

of the bundles $\tilde{H}^k$, and $\tilde{\mathcal{B}}_k$. If a belongs to one of $T\tilde{M}$, $\tilde{H}^k$, or $\tilde{\mathcal{B}}_k$, we will call $\pi(a)$ and $exp(a)$ the **π- and exp-coordinates**, *respectively,* **of a**, *and I will say*

$$a = (\pi(a), exp(a))$$

in π-exp coordinates.

Remark 1.8 *Note that a map*

$$f : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_k$$

is a C^*-algebra bundle isomorphism if and only if the corresponding map from the bundle $\tilde{M} \times B(L^2(\tilde{\Delta})^k)$ to itself, that is, the map f, **in π-exp coordinates,** *is a C^*-algebra bundle isomorphism.*

Similarly, a map

$$A : \tilde{H}^k \to \tilde{H}^k$$

is a Hilbert bundle isomorphism if and only if the corresponding map from the bundle $\tilde{M} \times L^2(\tilde{\Delta})^k$ to itself, that is, the map A, **in π-exp coordinates,** *is a Hilbert bundle isomorphism.*

Definition 1.9 *If A is a section of $\tilde{\mathcal{B}}_k$, and x is in $\tilde{M}$, we define* **the x-component of A**, *denoted by $\boldsymbol{A_x}$, as the element of $B(L^2(\tilde{\Delta})^k)$ given by the equation*

$$A_x = exp(A(x)).$$

(Note that $A(x)$ is an element of $\tilde{\mathcal{B}}_{k,x}$.) So A_x can also be described as the exp-coordinate of $A(x)$.

Definition 1.10 *Suppose A is a section of $\tilde{\mathcal{B}}_k$. For each x in $\tilde{M}$, let $A(x) = (a_{ij}(x))$ where each $a_{ij}(x)$ is an element of $\tilde{\mathcal{B}}_x$. The a_{ij} are then sections of the bundle $\tilde{\mathcal{B}}$. We write, in this case, $A = (a_{ij})$.*

Proposition 1.11 *If A is a section of $\tilde{\mathcal{B}}_k$, and $A = (a_{ij})$, then*

$$A_x = ((a_{ij})_x)$$

for all x in $\tilde{M}$.

Proof:

$$\begin{aligned} A_x &= exp_x(A(x)) \\ &= exp_x\left((a_{ij}(x))\right) \\ &= (exp_x(a_{ij}(x))) \quad \text{(by Eq. 1.5)} \\ &= ((a_{ij})_x). \spadesuit \end{aligned}$$

Proposition 1.12 *A section A of $\tilde{\mathcal{B}}_k$ is a continuous section of $\tilde{\mathcal{B}}_k$ if and only if the map*

$$x \mapsto exp(A(x)), \qquad x \in \tilde{M}$$

is a continuous map from $\tilde{M}$ into $B(L^2(\tilde{\Delta})^k)$. That is, if and only if the x component, A_x, of A varies continuously with x.

Proof: This follows immediately from the fact that

$$\pi \times exp : \tilde{\mathcal{B}}_k \to \tilde{M} \times B(L^2(\tilde{\Delta})^k)$$

is a trivialization of $\tilde{\mathcal{B}}_k$. ♠

Definition 1.13 *A section A of $\tilde{\mathcal{B}}_k$ is said to be* **bounded** *if there exists an $N > 0$ such that $\|A(x)\|$, which is the same as $\|A_x\|$, is less than N for all x in $\tilde{M}$. The C^*-algebra of all bounded, continuous sections of $\tilde{\mathcal{B}}_k$ will be denoted by* $\boldsymbol{BC}(\tilde{\mathcal{B}}_k)$. *Of course, the norm on each element A of $BC(\tilde{\mathcal{B}}_k)$ is given by*

$$\|A\| = \sup_{x \in \tilde{M}} \|A(x)\|.$$

Definition 1.14 *Let $b \to X$ be a C^*-algebra bundle. Any section a of $M_k(b)$ can be written as*

$$a = (a_{ij})$$

where the a_{ij} are sections of b. If $\mathcal{A}$ is a C^*-algebra of sections of b, define $\boldsymbol{M_k(\mathcal{A})}$ *as the algebra of all sections $a = (a_{ij})$ of $M_k(b)$ such that, each a_{ij} belongs to $\mathcal{A}$.*

Proposition 1.15 *If A is a section of $\tilde{\mathcal{B}}_k$ and $A = (a_{ij})$, then A is a bounded, continuous section of $\tilde{\mathcal{B}}_k$ — that is, A is in $BC(\tilde{\mathcal{B}}_k)$ — if and only if each a_{ij} belongs to $BC(\tilde{\mathcal{B}})$. Hence, we can write that*

$$BC(\tilde{\mathcal{B}}_k) \cong M_k(BC(\tilde{\mathcal{B}}))$$

as $C*$-algebras.

Proof: By Proposition 1.11, we have $A_x = ((a_{ij})_x)$. Therefore A_x varies continuously in x, and is bounded in x, if and only if each entry $(a_{ij})_x$ varies continuously with x, and is bounded in x. That is, A belongs to $BC(\tilde{\mathcal{B}}_k)$ if and only if each a_{ij} belongs to $BC(\tilde{\mathcal{B}})$. ♠

4.2 Actions by the Fundamental Group

In this section, we assume that M is a **compact, spin manifold**, and that k is a **positive integer**.

Definition 2.1 *If $E \to X$ and $F \to Y$ are hermitian vector bundles over the riemannian manifolds X and Y respectively, if $f : E \to F$ is a locally isometric bundle homomorphism from E to F, and if $g : X \to Y$ is a local isometry from X to Y, then we say that* $\boldsymbol{f}$ **is an extension of** $\boldsymbol{g}$ *or* $\boldsymbol{f}$ **extends** $\boldsymbol{g}$ *if, after identifying the zero elements of the bundles E and F with X and Y*

respectively, we have that f restricted to X equals g. Another way of saying this is that $f(E_x) = F_{g(x)}$ for all x in X.

If G is a group, and

$$\alpha : G \to Isom(E)$$

is an action of G on E by bundle isometries, and

$$\beta : G \to Isom(X)$$

is an action of G on X by isometries, then we say that **$\boldsymbol{\alpha}$ is an extension of $\boldsymbol{\beta}$**, *or* **$\boldsymbol{\alpha}$ extends $\boldsymbol{\beta}$** *if, $\alpha(g)$ extends $\beta(g)$ for every $g \in G$.*

Now, let $\mathbf{\Gamma}_M$ equal the fundamental group of M. Wherever possible, I will drop the subscript M and use only $\mathbf{\Gamma}$ to represent this group. Also, let

$$\widetilde{Cl} \to \tilde{M}$$

and

$$\widetilde{Cl} \to M$$

denote the complex Clifford bundles over $\tilde{M}$ and M respectively, and let

$$\tilde{\nabla} : T^*M \otimes C^\infty(\tilde{\Delta}) \to C^\infty(\tilde{\Delta})$$

be the covariant derivative on $C^\infty(\tilde{\Delta})$. The group Γ acts on $\tilde{M}$ by deck transformations. These deck transformations are actually isometries on $\tilde{M}$. I will use

$$\alpha : \Gamma \to Isom(\tilde{M})$$

to denote this action. Recall that

$$\rho : \tilde{M} \to M$$

is the covering map from $\tilde{M}$ to M. The action

$$\alpha : \Gamma \to Isom(\tilde{M})$$

has the property that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\alpha(g)} & \tilde{M} \\ & {\scriptstyle \rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & M & \end{array}$$

commutes for every g in Γ.

Since the riemannian and spin structures on $\tilde{M}$ were defined as the lifts of the corresponding structures on M by the covering map $\rho : \tilde{M} \to M$, it follows that $\rho : \tilde{M} \to M$ extends to locally isometric bundle homomorphisms

$$\rho : \tilde{\Delta} \to \Delta, \ \rho : T\tilde{M} \to TM, \text{ and } \rho : \widetilde{Cl} \to Cl, \tag{2.2}$$

and the action

$$\alpha : \Gamma \to Isom(\tilde{M})$$

of Γ on $\tilde{M}$, extends to actions

$$\alpha : \Gamma \to Isom(\widetilde{Cl}), \alpha : \Gamma \to Isom(\tilde{\Delta}) \text{ and } \alpha : \Gamma \to Isom(T\tilde{M}) \tag{2.3}$$

of Γ on $\widetilde{Cl}, \tilde{\Delta}$, and $T\tilde{M}$ in such a way that the diagrams

$$\begin{array}{ccc} \widetilde{Cl} & \xrightarrow{\alpha(g)} & \widetilde{Cl} \\ & {\scriptstyle \rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & Cl & \end{array}, \quad \begin{array}{ccc} \tilde{\Delta} & \xrightarrow{\alpha(g)} & \tilde{\Delta} \\ & {\scriptstyle \rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & \Delta & \end{array}, \quad \text{and} \quad \begin{array}{ccc} T\tilde{M} & \xrightarrow{\alpha(g)} & T\tilde{M} \\ & {\scriptstyle \rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & TM & \end{array} \tag{2.4}$$

commute for all g in Γ.

Also, the two actions of Γ on $\tilde{\Delta}$ and $\widetilde{Cl}$ respect Clifford multiplication. In other words, if we use $\boldsymbol{g \cdot a}$ to denote the action of g on a, for g in Γ and a in either $\tilde{\Delta}$ or $\widetilde{Cl}$, then, for every x in $\tilde{M}$, the following equation,

$$g \cdot (a \cdot s) = (g \cdot a) \cdot (g \cdot s), \tag{2.5}$$

is true for all g in Γ, a in $\widetilde{Cl}_x$, and s in $\tilde{\Delta}_x$.

Moreover, since $\alpha(g) : \tilde{\Delta} \to \tilde{\Delta}$ is a bundle isometry preserving the spin structure on $\tilde{M}$, then

$$\nabla_{g \cdot v}(g \cdot \xi) = g \cdot \tilde{\nabla}_v \xi \tag{2.6}$$

for every v in $T\tilde{M}$ and ξ in $C^\infty(\tilde{\Delta})$.

Now, each bundle isometry $\alpha(g)$ in $Isom(\tilde{\Delta})$ induces a unitary operator, which I also call $\alpha(g)$, on the Hilbert space $L^2(\tilde{\Delta})^k$. So, the action α of Γ on $\tilde{\Delta}$ induces an action

$$\alpha : \Gamma \to U(L^2(\tilde{\Delta})^k)$$

of Γ on $L^2(\tilde{\Delta})^k$ by unitary operators. This, in turn, gives an action

$$\alpha : \Gamma \to InnerAut(B(L^2(\tilde{\Delta})^k))$$

of Γ on $B(L^2(\tilde{\Delta})^k)$ by inner $*$-automorphisms. (An inner $*$-automorphism of a C*-algebra A, or an element of $InnerAut(A)$ is an automorphism of A given by conjugation by a unitary.)

For g in Γ and a in one of $\tilde{M}$, $L^2(\tilde{\Delta})^k$, or $B(L^2(\tilde{\Delta})^k)$, let $g \cdot a$ represent the action of g on a. Note that, if $(\xi_1, \ldots, \xi_k)$ is an element of $L^2(\tilde{\Delta})^k$, where the ξ_i are elements of $L^2(\tilde{\Delta})$, then

$$g \cdot (\xi_1, \ldots, \xi_k) = (g \cdot \xi_1, \ldots, g \cdot \xi_k). \tag{2.7}$$

Hence, since the action of Γ on $B(L^2(\tilde{\Delta})^k)$ is induced by the action of Γ on $L^2(\tilde{\Delta})^k$, it follows that if A is an element of $B(L^2(\tilde{\Delta})^k)$, and $A = (a_{ij})$ where the entries a_{ij} are elements of $B(L^2(\tilde{\Delta}))$, then

$$g \cdot A = (g \cdot a_{ij}) \tag{2.8}$$

for all g in Γ.

Now, Γ also acts naturally by bundle isometries on the bundle

$$T\tilde{M} \to \tilde{M}.$$

Each bundle isometry $\alpha(g)$ on $T\tilde{M}$ is the one induced naturally by the isometry $\alpha(g)$ on $\tilde{M}$. We let

$$\alpha : \Gamma \to Isom(T\tilde{M})$$

denote this action, and we let $g \cdot v$ denote the action of g on v for g in Γ and v in $T\tilde{M}$. When I need to distinguish the Γ action α on one space from the Γ action α on another, I will use the symbols for the different spaces as subscripts on the symbol α. For example, $\alpha_{\tilde{M}}$ may be used for the action of Γ on the manifold $\tilde{M}$, whereas I may use $\alpha_{T\tilde{M}}$ for the action of Γ on the space $T\tilde{M}$.

Proposition 2.9 *If v belongs to $T\tilde{M}$, g belongs to Γ, and x belongs to $\tilde{M}$, then*

$$\pi(g \cdot v) = g \cdot \pi(v), \tag{2.10}$$

$$exp(g \cdot v) = g \cdot exp(v), \tag{2.11}$$

$$g \cdot exp^{-1}(x) = exp^{-1}(g \cdot x), \tag{2.12}$$

$$\text{and} \quad g \cdot T_x\tilde{M} = T_{g \cdot x}\tilde{M}. \tag{2.13}$$

If M has nonpositive curvature, then we can write 2.13 as

$$g \cdot \tilde{M}_x = \tilde{M}_{g \cdot x}. \tag{2.14}$$

Proof: Here, I will use $\alpha_{\tilde{M}}$ for the action of Γ on $\tilde{M}$. Note that

$$\begin{aligned} \pi(g \cdot v) &= \pi(\alpha_{\tilde{M}}(g)_*(v)) \\ &= \alpha_{\tilde{M}}(g)(\pi(v)) \\ &= g \cdot \pi(v), \end{aligned}$$

which proves 2.10.

To prove 2.11, we take the unit-speed, geodesic segment

$$\beta : t \mapsto exp(tv), \qquad 0 \leq t \leq 1$$

and move it by α to the segment

$$\alpha(g) \circ \beta : [0,1] \to \tilde{M}.$$

Since $\alpha(g)$ is an isometry, then the segment $\alpha(g) \circ \beta$ is also a geodesic segment of unit speed. Therefore,

$$(\alpha(g) \circ \beta)(1) = exp\left((\alpha(g) \circ \beta)'(0)\right). \tag{2.15}$$

But, $\beta(1)$ is equal to $exp(v)$, and $\beta'(0)$ is equal to v. Thus,

$$\begin{aligned} (\alpha(g) \circ \beta)(1) &= \alpha(g)(exp(v)) \\ &= g \cdot exp(v), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned}(\alpha(g) \circ \beta)'(0) &= \alpha(g)'(\beta'(0)) \\ &= \alpha(g)'(v) \\ &= \alpha(g)_*(v) \\ &= g \cdot v \end{aligned} \tag{2.17}$$

Thus, by Equations 2.15, 2.16, and 2.17, we get

$$g \cdot exp(v) = exp(g \cdot v),$$

which proves 2.11.

Now, note that 2.12 follows from 2.11, and that 2.13 follows from 2.10. Thus, the proposition is true. ♠

4.3 Hilbert and C*-algebra Bundles over M

In this section, M is a **compact riemannian spin manifold of non-positive curvature**, as in Section 4.1, and k is a **positive integer**.

From Equations 2.10 and 2.14, it follows that the diagram

$$\begin{array}{ccc} \tilde{M}_x & \xrightarrow{\alpha(g)} & \tilde{M}_{g\cdot x} \\ {\scriptstyle \exp_x}\downarrow & & \downarrow{\scriptstyle \exp_{g\cdot x}} \\ \tilde{M} & \xrightarrow[\alpha(g)]{} & \tilde{M} \end{array} \tag{3.1}$$

commutes for all x in $\tilde{M}$ and g in Γ. The maps $\exp_x$, $\exp_{g\cdot x}$, and

$$\alpha(g) : \tilde{M} \to \tilde{M}$$

are all isometries. Therefore, from 3.1, the map

$$\alpha(g) : \tilde{M}_x \to \tilde{M}_{g\cdot x}$$

is also an isometry.

We now define a bundle isometry

$$\alpha(g) : \tilde{\Delta}(x) \to \tilde{\Delta}(g \cdot x)$$

as the map which makes the diagram

$$\begin{array}{ccc} \tilde{\Delta}(x) & \xrightarrow{\alpha(g)} & \tilde{\Delta}(g\cdot x) \\ {\scriptstyle \exp_x}\downarrow & & \downarrow{\scriptstyle \exp_{g\cdot x}} \\ \tilde{\Delta} & \xrightarrow[\alpha(g)]{} & \tilde{\Delta} \end{array} \tag{3.2}$$

commute, where the vertical maps, $\exp_x$ and $\exp_{g\cdot x}$, are given in 1.1 on page 109, and the map

$$\alpha(g) : \tilde{\Delta} \to \tilde{\Delta}$$

is the map given by 2.3 and 2.4 on page 119. In other words,

$$\alpha(g) : \tilde{\Delta}(x) \to \tilde{\Delta}(g \cdot x)$$

is defined by setting

$$\alpha(g) = {\exp_{g\cdot x}}^{-1} \circ \alpha(g) \circ \exp_x,$$

where $\exp_{g\cdot x}$, $\alpha(g)$, and $\exp_x$, are the three other maps given in Diagram 3.2. Since the three maps in the composition are all bundle isometries, which are extensions of the corresponding isometries in Diagram 3.1, it is clear that the map

$$\alpha(g) : \tilde{\Delta}(x) \to \tilde{\Delta}(g \cdot x)$$

is a bundle isometry which extends the isometry

$$\alpha(g) : \tilde{M}_x \to \tilde{M}_{g\cdot x}.$$

The commutative diagram, 3.2, of bundle isometries, induces a commutative diagram

$$\begin{array}{ccc} \tilde{H}_x & \xrightarrow{\alpha(g)} & \tilde{H}_{g\cdot x} \\ {\scriptstyle \exp_x}\downarrow & & \downarrow{\scriptstyle \exp_{g\cdot x}} \\ L^2(\tilde{\Delta}) & \xrightarrow[\alpha(g)]{} & L^2(\tilde{\Delta}) \end{array} \tag{3.3}$$

of unitary operators, which, in turn, induces a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{B}}_x & \xrightarrow{\alpha(g)} & \tilde{\mathcal{B}}_{g\cdot x} \\ {\scriptstyle \exp_x}\downarrow & & \downarrow{\scriptstyle \exp_{g\cdot x}} \\ B(L^2(\tilde{\Delta})) & \xrightarrow[\alpha(g)]{} & B(L^2(\tilde{\Delta})) \end{array} \tag{3.4}$$

of $*$-isomorphisms of C*-algebras.

The diagram, 3.3, also induces a commutative diagram

$$\begin{array}{ccc} (\tilde{H}_x)^k & \xrightarrow{\alpha(g)} & (\tilde{H}_{g\cdot x})^k \\ \exp_x \downarrow & & \downarrow \exp_{g\cdot x} \\ L^2(\tilde{\Delta})^k & \xrightarrow[\alpha(g)]{} & L^2(\tilde{\Delta})^k \end{array} \tag{3.5}$$

of unitary operators, which induces a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{B}}_{k,x} & \xrightarrow{\alpha(g)} & \tilde{\mathcal{B}}_{k,g\cdot x} \\ \exp_x \downarrow & & \downarrow \exp_{g\cdot x} \\ B(L^2(\tilde{\Delta})^k) & \xrightarrow[\alpha(g)]{} & B(L^2(\tilde{\Delta})^k) \end{array} \tag{3.6}$$

of $*$-isomorphisms.

Note that Diagrams 3.2, 3.3, 3.4, 3.5, and 3.6, are all extensions of Diagram 3.1.

Let

$$\boldsymbol{g \cdot a} = \alpha(g)(a)$$

for g in Γ and a in $(\tilde{H}_x)^k$ or $\tilde{\mathcal{B}}_{k,x}$. Since the action of Γ on $\tilde{H}^k$ is induced by the action of Γ on $\tilde{H}$, then we have

$$g \cdot (\xi_1, \ldots, \xi_k) = (g \cdot \xi_1, \ldots, g \cdot \xi_k) \tag{3.7}$$

for all g in Γ and $(\xi_1, \ldots, \xi_k)$ in $(\tilde{H}_x)^k$.

Also, if a belongs to $\tilde{\mathcal{B}}_{k,x} = B((\tilde{H}_x)^k)$, and $a = (a_{ij})$ where each a_{ij} belongs to $\tilde{\mathcal{B}}_x = B(\tilde{H}_x)$, then

$$g \cdot (a_{ij}) = (g \cdot a_{ij}) \tag{3.8}$$

for all g in Γ.

We now put the maps

$$\alpha(g) : \tilde{\mathcal{B}}_{k,x} \to \tilde{\mathcal{B}}_{k,g\cdot x},$$

together to form the map

$$\alpha(g) : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_k,$$

and we do the same with the maps

$$\alpha(g) : (\tilde{H}^k)_x \to (\tilde{H}^k)_{g\cdot x},$$

to form the map

$$\alpha(g) : \tilde{H}^k \to \tilde{H}^k.$$

These are the maps which satisfy

$$\alpha(g)(a) = g \cdot a$$

for a either in $\tilde{H}^k$ or in $\tilde{\mathcal{B}}_k$.

Proposition 3.9 *In* π-exp *coordinates, the maps*

$$\alpha(g) : \tilde{H}^k \to \tilde{H}^k$$

$$\textit{and} \qquad \alpha(g) : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_k$$

can be written as

$$\alpha(g)(a, x) = (g \cdot x, g \cdot a)$$

for all a *in* $L^2(\tilde{\Delta})^k$ *or* $B(L^2(\tilde{\Delta})^k)$*, and all* x *in* $\tilde{M}$*.*

Proof: This follows from Diagrams 3.5 and 3.6. ♠

Since the map

$$\alpha(g) : L^2(\tilde{\Delta})^k \to L^2(\tilde{\Delta})^k$$

is a unitary operator, and the map

$$\alpha(g) : B(L^2(\tilde{\Delta})^k) \to B(L^2(\tilde{\Delta})^k)$$

is a $*$-isomorphism, it follows from Proposition 3.9 that the map

$$\alpha(g) : \tilde{H}^k \to \tilde{H}^k,$$

in π-exp coordinates, is a Hilbert bundle isomorphism on $\tilde{M} \times L^2(\tilde{\Delta})^k$, and the map

$$\alpha(g) : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_k,$$

in π-exp coordinates, is a C*-algebra bundle isomorphism on $\tilde{M} \times B(L^2(\tilde{\Delta})^k)$.

Thus, by Remark 1.8, the map

$$\alpha(g) : \tilde{H}^k \to \tilde{H}^k$$

is a Hilbert bundle isomorphism on $\tilde{H}^k$, and the map

$$\alpha(g) : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_k$$

is a C*-algebra bundle isomorphism on $\tilde{\mathcal{B}}_k$.

For any C*-algebra bundle $\mathcal{A}$ and any Hilbert bundle h, we let $\boldsymbol{Isom(\mathcal{A})}$ and $\boldsymbol{Isom(h)}$ denote the group of all C*-algebra bundle isomorphisms on $\mathcal{A}$, and the group of all Hilbert bundle isomorphisms on h, respectively.

From Equation 3.9, it is clear that the maps

$$\alpha : \Gamma \to Isom(\tilde{H}^k)$$

and

$$\alpha : \Gamma \to Isom(\tilde{\mathcal{B}}_k)$$

which sends g to $\alpha(g)$ are actions of Γ on $\tilde{H}^k$ and $\tilde{\mathcal{B}}_k$ respectively.

Now, recall that an action of a group G on a topological space X is said to be **properly discontinuous** if, for every x in X, there is an open neighbourhood U of x such that $(g \cdot U) \cap U$ is empty whenever g is not the identity element of the group.

Definition 3.10 *Let F be a fiber bundle over X. If U is a subset of X, we let*

$$\boldsymbol{F}_U = \bigcup_{x \in U} F_x.$$

If a group G acts on F by bundle isomorphisms, then the action is said to be **properly discontinuous** *if, for every x in X, there is an open neighbourhood U of x such that $(g \cdot F_U) \cap F_U$ is empty whenever g is not the identity element of G.*

From its very definition, the action of the fundamental group Γ on $\tilde{M}$ is properly discontinuous. Since $g \cdot (\tilde{H}_x)^k = (\tilde{H}_{g \cdot x})^k$ and $g \cdot \tilde{\mathcal{B}}_{k,x} = \tilde{\mathcal{B}}_{k,g \cdot x}$, for all g in Γ and x in $\tilde{M}$, we have that the diagrams

$$\begin{array}{ccc} \tilde{H}^k & \xrightarrow{\alpha(g)} & \tilde{H}^k \\ \pi \downarrow & & \downarrow \pi \\ \tilde{M} & \xrightarrow[\alpha(g)]{} & \tilde{M} \end{array} \tag{3.11}$$

and

$$\begin{array}{ccc} \tilde{\mathcal{B}}_k & \xrightarrow{\alpha(g)} & \tilde{\mathcal{B}}_k \\ \pi\big\downarrow & & \big\downarrow\pi \\ \tilde{M} & \xrightarrow[\alpha(g)]{} & \tilde{M} \end{array} \tag{3.12}$$

commute for every g in Γ. It follows, therefore, from the fact that the action of Γ on $\tilde{M}$ is properly discontinuous, that the actions of Γ on $\tilde{H}^k$ and $\tilde{\mathcal{B}}_k$ are both properly discontinuous actions.

This implies that we can take quotients to get a Hilbert bundle

$$\tilde{H}^k/\Gamma \to \tilde{M}/\Gamma = M$$

over M, and a C*-algebra bundle

$$\tilde{\mathcal{B}}_k/\Gamma \to \tilde{M}/\Gamma = M$$

over M, with bundle projection maps

$$\pi : \tilde{H}^k/\Gamma \to M(= \tilde{M}/\Gamma),$$

and

$$\pi : \tilde{\mathcal{B}}_k/\Gamma \to M(= \tilde{M}/\Gamma),$$

defined by setting

$$\pi(\Gamma \cdot a) = \Gamma \cdot \pi(a) \tag{3.13}$$

for a in either $\tilde{H}^k$ or $\tilde{\mathcal{B}}_k$.

It also implies that the quotient maps

$$\rho : \tilde{H}^k \to \tilde{H}^k/\Gamma$$

and

$$\rho : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_k/\Gamma$$

defined by the equation

$$\rho(a) = \Gamma \cdot a, \tag{3.14}$$

for a in either $\tilde{H}^k$ or $\tilde{\mathcal{B}}_k$, are covering bundle maps.

Definition 3.15 *Let*

$$\boldsymbol{H} \to M$$

denote the hilbert bundle

$$\tilde{H}/\Gamma \to M$$

and let

$$\mathcal{B}_k \to M$$

denote the C^*-algebra bundle

$$\tilde{\mathcal{B}}_k/\Gamma \to M.$$

By Equation 3.7 on page 126, we have that

$$g \cdot (\xi_1, \ldots, \xi_k) = (g \cdot \xi_1, \ldots, g \cdot \xi_k)$$

for all g in Γ and all $(\xi_1, \ldots, \xi_k)$ in $\tilde{H}^k$. This implies that

$$\tilde{H}^k/\Gamma \cong (\tilde{H}/\Gamma)^k = H^k \tag{3.16}$$

as Hilbert bundles over M, with Hilbert bundle isomorphism, the map

$$\Psi : \Gamma \cdot (\xi_1, \ldots, \xi_k) \mapsto (\Gamma \cdot \xi_1, \ldots, \Gamma \cdot \xi_k). \tag{3.17}$$

The isomorphism Ψ maps each fiber $(\tilde{H}^k/\Gamma)_x$ onto the fiber $\left((\tilde{H}/\Gamma)^k\right)_x$. That is, it is the kind of bundle isomorphism that does not change the base point on which the fiber sits.

So, from now on, the symbols H^k and $\tilde{H}^k/\Gamma$ will be used interchangeably, and if ξ is an element of $\tilde{H}^k/\Gamma$, then we can write

$$\xi = (\xi_1, \ldots, \xi_k)$$

for some $(\xi_1, \ldots, \xi_k)$ in H^k.

Definition 3.18 *The bundle $\pi : H^k \to M$ will be called the* **the k-th Hilbert bundle associated to the bundle Δ over M.**

We have the bundle projection map

$$\pi : H^k \to M$$

and the bundle covering map

$$\rho : \tilde{H}^k \to H^k.$$

Also, from Equation 3.17, we have that

$$\rho(\xi_1, \ldots, \xi_k) = (\rho(\xi_1), \ldots, \rho(\xi_k)) \tag{3.19}$$

for all $(\xi_1, \ldots, \xi_k)$ in $\tilde{H}^k$.

Note that the covering map

$$\rho : \tilde{M} \to M$$

can be described by the equation

$$\rho(x) = \Gamma \cdot x$$

for every x in $\tilde{M}$. From Equations 3.13 and 3.14, we see that

$$\pi(\rho(a)) = \rho(\pi(a)) \tag{3.20}$$

for a in either $\tilde{H}^k$ or $\tilde{\mathcal{B}}_k$. This means that if x belongs to $\tilde{M}$, and a belongs to either $(\tilde{H}_x)^k$ or $\tilde{\mathcal{B}}_{k,x}$, then $\rho(a)$ belongs to $(H^k)_{\rho(x)}$ or $\mathcal{B}_{k,\rho(x)}$, respectively. So, the bundle covering map $\rho : \tilde{H}^k \to H^k$ restricts to unitaries

$$\rho : (\tilde{H}_x)^k \to (H_{\rho(x)})^k$$

and the bundle covering map $\rho : \tilde{\mathcal{B}}_k \to \mathcal{B}_k$ restricts to $*$-isomorphisms

$$\rho : \tilde{\mathcal{B}}_{k,x} \to \mathcal{B}_{k,\rho(x)}.$$

Now, since the action

$$\alpha : \Gamma \to Isom(\tilde{\mathcal{B}}_k)$$

is the action induced by the action

$$\alpha : \Gamma \to Isom(\tilde{H}^k),$$

and since the bundle

$$\pi : \tilde{\mathcal{B}}_k \to \tilde{M}$$

was defined as the C*-algebra bundle over $\tilde{M}$ associated to the Hilbert bundle

$$\pi : \tilde{H}^k \to \tilde{M},$$

then the bundle

$$\pi : \mathcal{B}_k = \tilde{\mathcal{B}}_k/\Gamma \to M$$

is the C*-algebra bundle over M associated to the Hilbert bundle

$$\pi : H^k = \tilde{H}^k/\Gamma \to M.$$

That is,

$$\mathcal{B}_k \cong B(H^k)$$

as C*-algebra bundles. If a belongs to $\tilde{\mathcal{B}}_{k,x}$ and ξ belongs to $(\tilde{H}^k)_x$ for some x in $\tilde{M}$, then we have

$$\rho(a)(\rho(\xi)) = \rho(a(\xi)). \tag{3.21}$$

In particular,

$$\mathcal{B} \cong B(H).$$

By remarks made earlier, it follows that

$$\mathcal{B}_k \cong M_k(\mathcal{B}).$$

So, every element a of $\mathcal{B}_k$ can be represented by a matrix (a_{ij}) over $\mathcal{B}$, and we express this by writing

$$a = (a_{ij}).$$

If $\xi = (\xi_1, \ldots, \xi_k)$ belongs to $(H_x)^k$, for some x in M, and $a = (a_{ij})$ belongs to $\mathcal{B}_{k,x}$, where each a_{ij} is in $B(H_x)$, then $a \cdot \xi$ is given by ordinary multiplication of the k-column $(\xi_1, \ldots, \xi_k)$ by the $k \times k$ matrix (a_{ij}).

Proposition 3.22 *If $a = (a_{ij})$ belongs to $\tilde{\mathcal{B}}_k$, where each a_{ij} belongs to $\tilde{\mathcal{B}}$, then*

$$\rho(a) = (\rho(a_{ij})). \tag{3.23}$$

Proof: Let

$$\rho(a) = (b_{ij})$$

where each b_{ij} belongs to $\mathcal{B}$. We want to show that $b_{ij} = \rho(a_{ij})$ for every (i, j). Assume that a belongs to $\tilde{\mathcal{B}}_{k,x}$, where x is in $\tilde{M}$. Take ξ in $\tilde{H}_x$. Let

$$j(\xi) = (0, \ldots, 0, \xi, 0, \ldots, 0)$$

be the element in $(\tilde{H}_x)^k$ with 0 in all the entries except in the jth position where we have ξ as an entry. By Equation 3.19, we have that

$$\rho(j(\xi)) = j(\rho(\xi)) = (0, \ldots, 0, \rho(\xi), 0, \ldots, 0), \tag{3.24}$$

the element in $(H_{\rho(x)})^k$ with $\rho(\xi)$ in the jth position, and 0 everywhere else. Also,

$$\begin{aligned} \rho(a(j(\xi))) &= \rho(a_{1j}\xi, \ldots, a_{kj}\xi) \\ &= (\rho(a_{1j}\xi), \ldots, \rho(a_{kj}\xi)) \quad \text{(by Eq. 3.19)} \\ &= (\rho(a_{1j})\rho(\xi), \ldots, \rho(a_{kj})\rho(\xi)) \quad \text{(by Eq. 3.21)} \end{aligned}$$

At the same time, we have that

$$\begin{aligned} \rho(a(j(\xi))) &= \rho(a)\rho(j(\xi)) \quad \text{(by Eq. 3.21)} \\ &= \rho(a)(j(\rho(\xi))) \quad \text{(by 3.24)} \\ &= (b_{1j}\rho(\xi), \ldots, b_{kj}\rho(\xi)). \end{aligned}$$

Thus,

$$(\rho(a_{1j})\rho(\xi), \ldots, \rho(a_{kj})\rho(\xi)) = (b_{1j}\rho(\xi), \ldots, b_{kj}\rho(\xi))$$

for every ξ in $\tilde{H}_x$. Since

$$\rho : \tilde{H}_x \to H_{\rho(x)}$$

is a unitary, it follows that

$$(\rho(a_{1j})\eta, \ldots, \rho(a_{kj})\eta) = (b_{1j}\eta, \ldots, b_{kj}\eta)$$

for all η in $H_{\rho(x)}$. This implies $\rho(a_{ij}) = b_{ij}$ for all (i, j). Thus, the proposition is true. ♠

Chapter 5

A "Wiener-Hopf" Extension on the Tangent Bundle

5.1 $Per(\tilde{\mathcal{B}}_k)$

In this section, I carry over all the assumptions of the previous section. So, in particular, M is a **compact riemannian spin manifold of nonpositive curvature.**

Definition 1.1 *Let* $\boldsymbol{C(\mathcal{B}_k)}$ *denote the algebra of all continuous sections of the C^*-algebra bundle $\mathcal{B}_k$ over M.*

Let $\boldsymbol{BS(\tilde{\mathcal{B}})_k}$ denote the C*-algebra of all bounded sections of $\tilde{\mathcal{B}}_k$. If A belongs to $BS(\tilde{\mathcal{B}}_k)$, then

$$\|A\| = \sup_{x\in\tilde{M}} \|A_x\|$$

where A_x is the x-component of A. Elements of $BS(\tilde{\mathcal{B}}_k)$ need not be continuous. If A is any C*-algebra, we let $\boldsymbol{Isom(A)}$ denote the group of *-

isomorphisms of A. Note that the action α of Γ on $\tilde{\mathcal{B}}_k$ induces an action

$$\alpha : \Gamma \to Isom(BS(\tilde{\mathcal{B}}_k))$$

of Γ on $BS(\tilde{\mathcal{B}}_k)$ by $*$-isomorphisms. If g is in Γ and A belongs to $BS(\tilde{\mathcal{B}}_k)$, I will write $\boldsymbol{g \cdot A}$ for $\alpha(g)(A)$.

Recall that $BC(\tilde{\mathcal{B}}_k)$ is the C*-algebra of all bounded, continuous sections of $\tilde{\mathcal{B}}_k$. Note that the action α of Γ on $BS(\tilde{\mathcal{B}}_k)$ restricts to an action

$$\alpha : \Gamma \to Isom(BC(\tilde{\mathcal{B}}_k))$$

of Γ on $BC(\tilde{\mathcal{B}}_k)$ by $*$-isomorphisms.

Proposition 1.2 *If $A = (a_{ij})$ belongs to $BC(\tilde{\mathcal{B}}_k)$, then*

$$g \cdot A = (g \cdot a_{ij}).$$

Proof: For each x in $\tilde{M}$,

$$\begin{aligned}
(g \cdot A)(x) &= g \cdot A(g^{-1} \cdot x) \\
&= g \cdot (a_{ij}(g^{-1} \cdot x)) \\
&= (g \cdot a_{ij}(g^{-1} \cdot x)) \quad \text{(by Eq. 3.8 on p.126)} \\
&= ((g \cdot a_{ij})(x))
\end{aligned}$$

for all g in Γ. Thus,

$$g \cdot A = (g \cdot a_{ij})$$

for all g in Γ. ♠

Definition 1.3 *An element A of $BC(\tilde{\mathcal{B}}_k)$ is said to be* **periodic** *if $g \cdot A$ equals A for every g in Γ. We will use* $\boldsymbol{Per}(\tilde{\mathcal{B}}_k)$ *to denote the C^*-algebra of all periodic, bounded, continuous sections of $\tilde{\mathcal{B}}_k$.*

Proposition 1.4 *If $A = (a_{ij})$ is an element of $BC(\tilde{\mathcal{B}}_k)$, then A belongs to $Per(\tilde{\mathcal{B}}_k)$ if and only if each a_{ij} belongs to $Per(\tilde{\mathcal{B}})$. Hence,*

$$Per(\tilde{\mathcal{B}}_k) \cong M_k(Per(\tilde{\mathcal{B}})).$$

Proof: By Proposition 4.1.2, $g \cdot A = (g \cdot a_{ij})$. Thus, $g \cdot A = A$ if and only if $g \cdot a_{ij} = a_{ij}$ for each (i, j). Therefore, the proposition is true. ♠

Proposition 1.5 *If a and b are elements of $BC(\tilde{\mathcal{B}}_k)$, and g belongs to Γ, then $b = g \cdot a$ if and only if*

$$b_{g \cdot x} = g \cdot a_x$$

for every x in $\tilde{M}$.

Proof: By definition,

$$(g \cdot a)(g \cdot x) = g \cdot a(x).$$

Also, we have

$$a(x) = (x, a_x)$$

in π-exp coordinates. By Proposition 4.3.9, we have that

$$g \cdot a(x) = (gx, g \cdot a_x)$$

in π-exp coordinates. So,

$$(g \cdot a)(g \cdot x) = (g \cdot x, g \cdot a_x)$$

in π-exp coordinates. This means that the $(g \cdot x)$-component, $(g \cdot a)_{g \cdot x}$, of $g \cdot a$ is given by the equation

$$(g \cdot a)_{g \cdot x} = g \cdot a_x. \tag{1.6}$$

This proves the "only if" part.

Now, if $b_{g \cdot x} = g \cdot a_x$ for every x in $\tilde{M}$, then (1.6) implies that

$$b_{g \cdot x} = (g \cdot a)_{g \cdot x}$$

for all x in $\tilde{M}$. Thus,

$$b_x = (g \cdot a)_x$$

for all x in $\tilde{M}$. Hence, $b = g \cdot a$. ♠

Since $\mathcal{B}_k$ was defined as the quotient $\tilde{\mathcal{B}}_k/\Gamma$ of $\tilde{\mathcal{B}}_k$ by the properly discontinuous action of Γ on $\tilde{\mathcal{B}}_k$, then periodic continuous sections of $\tilde{\mathcal{B}}_k$ correspond exactly to continuous sections of $\mathcal{B}_k$, and the correspondence is given by the map

$$\Phi : \mathrm{Per}(\tilde{\mathcal{B}}_k) \to C(\mathcal{B}_k)$$

defined by the equation

$$(\Phi(A))(x) = \rho(A(\tilde{x}))$$

for every A in Per $(\tilde{\mathcal{B}}_k)$ and $x \in M$, where $\tilde{x}$ is any lift of x up to the manifold $\tilde{M}$. We state this formally.

Proposition 1.7 *Φ, defined above, is a C^*-algebra isomorphism between $Per(\tilde{\mathcal{B}}_k)$ and $C(\mathcal{B}_k)$.*

Note that, since $\mathcal{B}_k = B(H^k)$, then $C(\mathcal{B}_k) \cong M_k(C(\mathcal{B}))$ in the obvious way. So, every a in $C(\mathcal{B}_k)$ can be written uniquely as $a = (a_{ij})$ where the a_{ij} belong to $C(\mathcal{B})$.

Proposition 1.8 *If*

$$\Phi : Per(\tilde{\mathcal{B}}_k) \to C(\mathcal{B}_k)$$

is the $*$-isomorphism given in Proposition 1.7, and $a = (a_{ij})$ is an element of $Per(\tilde{\mathcal{B}}_k)$, then

$$\Phi(a) = (\Phi(a_{ij})).$$

Proof: Take x in M. Choose any $\tilde{x}$ in $\tilde{M}$ such that $\rho(\tilde{x}) = x$. Then

$$\begin{aligned}
(\Phi(a))(x) &= \rho(a(\tilde{x})) \\
&= \rho((a_{ij}(\tilde{x}))) \\
&= (\rho(a_{ij}(\tilde{x}))) \quad \text{(by Prop. 3.22)} \\
&= ((\Phi(a_{ij}))(x)).
\end{aligned}$$

Since this is true for all x in M, then $\Phi(a) = (\Phi(a_{ij}))$. ♠

Definition 1.9 *If $h \to X$ is a Hilbert bundle, let $K(h)$ denote the subbundle of $B(h)$ consisting of all A in $B(h)$ such that A is compact. Thus, for each x in X, the fiber $K(h)_x$ of $K(h)$ is the algebra $K(h_x)$ of all compact operators on h_x.*

Definition 1.10 *Let $\tilde{\mathcal{K}}_k$ denote the C^*-algebra bundle $K(\tilde{H}^k)$, and let $\mathcal{K}_k$ denote the C^*-algebra bundle $K(H^k)$.*

Also, let $\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_1$, and $\mathcal{K} = \mathcal{K}_1$. Note that $\tilde{\mathcal{K}}_k \cong M_k(\tilde{\mathcal{K}}) = M_k(K(\tilde{H}))$ and $\mathcal{K}_k \cong M_k(\mathcal{K}) = M_k(K(H))$.

As usual, $C(\tilde{\mathcal{K}}_k)$ will denote the continuous sections of $\tilde{\mathcal{K}}_k$, $C(\mathcal{K}_k)$, the continuous sections of $\mathcal{K}_k$, and $BC(\tilde{\mathcal{K}}_k)$, the bounded, continuous sections of

$\tilde{\mathcal{K}}_k$. By compactness of M, $C(\mathcal{K}_k)$ is the same as the algebra of all bounded, continuous sections of $\mathcal{K}_k$. Of course, $C(\tilde{\mathcal{K}}_k) \cong M_k(C(\tilde{\mathcal{K}}))$ and $C(\mathcal{K}_k) \cong M_k(C(\mathcal{K}))$.

Also, let $\boldsymbol{Per}(\tilde{\mathcal{K}}_k)$ denote the algebra of all periodic elements of $BC(\tilde{\mathcal{K}}_k)$. That is,

$$Per(\tilde{\mathcal{K}}_k) = BC(\tilde{\mathcal{K}}_k) \cap Per(\tilde{\mathcal{B}}_k).$$

Then $Per(\tilde{\mathcal{K}}_k) \cong M_k(Per(\tilde{\mathcal{K}}))$.

Remark 1.11 *Note that the bundle $\tilde{\mathcal{K}}_k$ is the lift of the bundle $\mathcal{K}_k$ by the map $\rho : \tilde{M} \to M$. Also, note that the $*$-isomorphism $\Phi : Per(\tilde{\mathcal{B}}_k) \to C(\mathcal{B}_k)$ of Proposition 1.7 restricts to a $*$-isomorphism from $Per(\tilde{\mathcal{K}}_k)$ onto $C(\mathcal{K}_k)$. So,*

$$Per(\tilde{\mathcal{K}}_k) \cong C(\mathcal{K}_k).$$

5.2 Invariance of $f(\tilde{D})$ under the Γ Action

In this section, we assume that M is a **compact, spin riemannian manifold of dimension** $\boldsymbol{m}$ and that k is a **positive integer.**

There is an obvious action α of Γ on $C^\infty(\tilde{\Delta})$. It is given by the equation

$$\alpha(g)(\xi)(x) = g \cdot \xi(g^{-1} \cdot x)$$

for g in Γ and ξ in $C^\infty(\tilde{\Delta})$. If we let $g \cdot \xi = \alpha(g)(\xi)$, then we can write this as

$$(g \cdot \xi)(x) = g \cdot \xi(g^{-1} \cdot x).$$

Proposition 2.1 *The Dirac operator*

$$\tilde{D} : C^\infty(\tilde{\Delta}) \to C^\infty(\tilde{\Delta})$$

is invariant under the action of the fundamental group Γ *of* M. *That is, if* g *belongs to* Γ *and* ξ *belongs to* $C^\infty(\tilde{\Delta})$, *then*

$$\tilde{D}(g \cdot \xi) = g \cdot \tilde{D}(\xi).$$

Or, we can say that

$$g \cdot \tilde{D} = \tilde{D}$$

where $g \cdot \tilde{D}$ *is, by definition, the operator* $\alpha(g) \circ \tilde{D} \circ \alpha(g)^{-1}$ *on* $C^\infty(\tilde{\Delta})$. *So, we can write*

$$\tilde{D} = \alpha(g) \circ \tilde{D} \circ \alpha(g)^{-1} \tag{2.2}$$

for all g *in* Γ.

Proof: By Equation 2.6,

$$\tilde{\nabla}_{g\cdot v}(g\cdot\xi) = g\cdot\tilde{\nabla}_v\xi$$

for every v in $T\tilde{M}$, and ξ in $C^\infty(\tilde{\Delta})$. Let $e_1,\ldots,e_m$ be a local orthonormal frame, Since $\alpha(g)$ is an isometry on $\tilde{M}$, $g\cdot e_1,\ldots g\cdot e_m$ is also a local orthonormal frame. Hence, locally, we can write

$$\begin{aligned}
\tilde{D}(g\cdot\xi) &= \sum_{i=1}^{m}(g\cdot e_i)\cdot\tilde{\nabla}_{g\cdot e_i}(g\cdot\xi) \\
&= \sum_{i=1}^{m}(g\cdot e_i)\cdot(g\cdot\tilde{\nabla}_{e_i}\xi) \quad \text{(from above)} \\
&= \sum_{i=1}^{m} g\cdot(e_i\cdot\tilde{\nabla}_{e_i}\xi) \qquad \text{(by Eq. 2.5 on p. 120)} \\
&= g\cdot\sum_{i=1}^{m} e_i\cdot\tilde{\nabla}_{e_i}\xi \\
&= g\cdot\tilde{D}\xi,
\end{aligned}$$

which proves the proposition. ♠

Proposition 2.3 *If f is a bounded, continuous function on* $\mathbf{R}$, *and if g is an element of* Γ, *then*

$$f(\tilde{D}) = \alpha(g)\circ f(\tilde{D})\circ\alpha(g^{-1}).$$

That is,

$$f(\tilde{D}) = g\cdot f(\tilde{D}).$$

Proof: This follows from Equation 2.2 of Proposition 2.1 and the fact that the operator

$$\alpha(g): L^2(\tilde{\Delta})\to L^2(\tilde{\Delta})$$

is unitary for every g in Γ. ♠

Now, if A is any C*-algebra, and $a_1, \ldots, a_k$ are elements of A, define $\boldsymbol{diag(a_1, \ldots, a_k)}$ in $M_k(A)$ as the diagonal matrix

$$\begin{pmatrix} a_1 & & & & \\ & a_2 & & 0 & \\ & & \ddots & & \\ & 0 & & a_{k-1} & \\ & & & & a_k \end{pmatrix}.$$

Also, if a is in A, let

$$\boldsymbol{a_k} = diag(\underbrace{a, \ldots, a}_{k\ times}).$$

Proposition 2.4 *If f is a bounded, continuous function on* $\mathbf{R}$*, then the operator $f(\tilde{D})_k$ in $B(L^2(\tilde{\Delta})^k)$ satisfies*

$$g \cdot f(\tilde{D})_k = f(\tilde{D})_k$$

for all g in Γ.

Proof: Since

$$f(\tilde{D})_k = \begin{pmatrix} f(\tilde{D}) & & & & \\ & f(\tilde{D}) & & 0 & \\ & & \ddots & & \\ & 0 & & f(\tilde{D}) & \\ & & & & f(\tilde{D}) \end{pmatrix},$$

then by Equation 4.2.8 on page 121, we have

$$g \cdot f(\tilde{D})_k = \begin{pmatrix} g \cdot f(\tilde{D}) & & & & \\ & g \cdot f(\tilde{D}) & & 0 & \\ & & \ddots & & \\ & 0 & & g \cdot f(\tilde{D}) & \\ & & & & g \cdot f(\tilde{D}) \end{pmatrix}.$$

But $g \cdot f(\tilde{D}) = f(\tilde{D})$ by Proposition 2.3. Thus

$$g \cdot f(\tilde{D})_k = f(\tilde{D})_k \; \spadesuit$$

Remark 2.5 *As mentioned earlier in Remark 2.4.1, when we write* $f(\tilde{D}) \in B(L^2(\tilde{\Delta})^k)$*, what we really mean is the operator* $f(\tilde{D})_k$*. Also, if* φ *is a bounded measurable function on* $\tilde{M}$*, then we may write* $f(\tilde{D})M_\varphi$ *in place of* $f(\tilde{D})_k M_\varphi$*, and* $M_\varphi f(\tilde{D})$ *in place of* $M_\varphi f(\tilde{D})_k$*.*

5.3 Dirac Extensions on the Tangent Bundle

Here, M is assumed to be a **compact riemannian, spin manifold of nonpositive curvature.**

Definition 3.1 *If* a *is an element of* $B(L^2(\tilde{\Delta}))$*, we define* $\boldsymbol{\Upsilon}_{\mathbf{k}}(\boldsymbol{a})$ *as that section of* $\tilde{\mathcal{B}}_k$ *whose* x*-component* $\Upsilon_k(a)_x$ *equals* a_k *for all* x *in* $\tilde{M}$*. That is,*

$$\Upsilon_k(a)_x = a_k = \underbrace{\begin{pmatrix} a & & & & \\ & a & & 0 & \\ & & \ddots & & \\ & 0 & & a & \\ & & & & a \end{pmatrix}}_{k}$$

for all x in $\tilde{M}$. Whenever possible, I will suppress the subscript k and use Υ instead of Υ_k. Of course, Υ will always be used instead of Υ_1.

If A is a section of $\tilde{\mathcal{B}}_k$, and $A = \Upsilon_k(a)$ for some a in $B(L^2(\tilde{\Delta}))$, then the x-component of A, being a constant a_k for all x in $\tilde{M}$, clearly varies continuously with x, and is bounded in x. Thus, by Proposition 4.1.12, such an A is bounded and continuous. So, Υ_k is a map,

$$\Upsilon_k : B(L^2(\tilde{\Delta})) \to BC(\tilde{\mathcal{B}}_k),$$

from $B(L^2(\tilde{\Delta}))$ to $BC(\tilde{\mathcal{B}}_k)$. It is not hard to see that Υ_k is an *injective* $*$-homomorphism, or an embedding of $B(L^2(\tilde{\Delta}))$ into $BC(\tilde{\mathcal{B}}_k)$.

Remark 3.2 *If $a \in B(L^2(\tilde{\Delta}))$ and we write $a \in BC(\tilde{\mathcal{B}}_k)$, what we really mean is the operator $\Upsilon_k(a)$ in $BC(\tilde{\mathcal{B}}_k)$.*

Proposition 3.3 *If f is bounded and continuous on $\mathbf{R}$, then $\Upsilon_k(f(\tilde{D}))$ is a periodic, bounded, continuous section of $\tilde{\mathcal{B}}_k$. That is, $\Upsilon_k(f(\tilde{D}))$ is an element of $Per(\tilde{\mathcal{B}}_k)$.*

Proof: In this proof, I will use Υ in place of Υ_k. We already know that the section $\Upsilon(f(\tilde{D}))$ is bounded and continuous. So, we only need to show that it is periodic. That is, we need to show that $g \cdot \Upsilon(f(\tilde{D})) = \Upsilon(f(\tilde{D}))$ for all g in Γ. By Proposition 1.5, it suffices to show that

$$(\Upsilon(f(\tilde{D})))_{g\cdot x} = g \cdot (\Upsilon(f(\tilde{D})))_x \tag{3.4}$$

for every g in Γ and x in $\tilde{M}$. But, from the definition of Υ, we have that

$$\Upsilon(a)_y = a_k$$

for all a in $B(L^2(\tilde{\Delta}))$ and all y in $\tilde{M}$. Thus, showing 3.4 is the same as showing that

$$f(\tilde{D})_k = g \cdot f(\tilde{D})_k$$

for all g in Γ. But, this is given by Proposition 2.4, which says that $f(\tilde{D})_k$ is invariant under the action of Γ. Thus, Proposition 3.3 is true. ♠

Proposition 3.5 *Let φ be a bounded, continuous function on $\tilde{M}$. Then the operator $M_\varphi \in B(L^2(\tilde{\Delta}))$ is such that*

$$g \cdot M_\varphi = M_{g\cdot\varphi}$$

for all g in Γ.

Proof: Suppose $u \in L^2(\tilde{\Delta})$, and $x \in \tilde{M}$. Then,

$$\begin{aligned}
(g \cdot M_\varphi)(u)(x) &= \left(g \cdot M_\varphi(g^{-1} \cdot u)\right)(x) \\
&= \left(g \cdot (\varphi \cdot (g^{-1} \cdot u))\right)(x) \\
&= \left(\varphi \cdot (g^{-1} \cdot u)\right)(g^{-1}x) \\
&= \varphi(g^{-1}x) \cdot (g^{-1} \cdot u)(g^{-1}x) \\
&= (g \cdot \varphi)(x) \cdot u(g \cdot g^{-1}x) \\
&= (g \cdot \varphi)(x) \cdot u(x) \\
&= (M_{g\cdot\varphi}u)(x),
\end{aligned}$$

which implies that $g \cdot M_\varphi = M_{g\cdot\varphi}$. ♠

Proposition 3.6 *Let $\varphi \in Per(M)$. Then $\Upsilon_k(M_\varphi)$ is a periodic, bounded, continuous section of $\tilde{\mathcal{B}}_k$. That is, $\Upsilon_k(M_\varphi)$ belongs to $Per(\tilde{\mathcal{B}}_k)$.*

Proof: As in the proof of Proposition 3.3, it suffices to show that $(M_\varphi)_k = g \cdot (M_\varphi)_k$ for all $g \in \Gamma$. This is equivalent to showing that $g \cdot M_\varphi = M_\varphi$ for all $g \in \Gamma$.

So, take $g \in \Gamma$. Since $\varphi \in Per(M)$, then $g \cdot \varphi = \varphi$. Thus, by Proposition 3.5, it follows that $g \cdot M_\varphi = M_{g \cdot \varphi} = M_\varphi$, which proves the proposition. ♠

Notation 3.7 *Let*

$$0 \to \mathcal{L}_{Per(M)} \to \mathcal{D}_{Per(M)} \to Per(M) \oplus Per(M) \to 0 \qquad (*)$$

be the Dirac extension of the algebra $Per(M)$. For reasons that will be apparent later we will also use $\mathcal{A}_{21}(M)$ to denote the Dirac algebra $\mathcal{D}_{Per(M)}$, $\mathcal{A}_{20}(M)$ to denote $\mathcal{L}_{Per(M)}$, and $\mathcal{A}_{22}(M)$ to denote $Per(M) \oplus Per(M)$.

Proposition 3.8 *For all a in $\mathcal{D}_{Per(M)}$, $\Upsilon_k(a)$ belongs to $Per(\tilde{\mathcal{B}}_k)$. So, we have an injective $*$-homomorphism*

$$\Upsilon_k : \mathcal{D}_{Per(M)} \to Per(\tilde{\mathcal{B}}_k)$$

or

$$\Upsilon_k : \mathcal{A}_{21}(M) \to Per(\tilde{\mathcal{B}}_k).$$

Proof: By Propositions 3.3 and 3.6, $\Upsilon_k(f(\tilde{D}))$ and $\Upsilon_k(M_\varphi)$ belong to $Per(\tilde{\mathcal{B}}_k)$ for all f in *Flip*, and φ in $Per(M)$. Since $\mathcal{D}_{Per(M)}$ is the C*-algebra generated by all such $f(\tilde{D})$ and M_φ, and since $Per(\tilde{\mathcal{B}}_k)$ is a C*-algebra, it follows that $\Upsilon_k(a) \in Per(\tilde{\mathcal{B}}_k)$ for all $a \in \mathcal{D}_{Per(M)}$. ♠

Definition 3.9 *Suppose $a \in \mathcal{A}_{21}(M)$. Then, whenever we refer to the element a of $Per(\tilde{\mathcal{B}}_k)$, we will actually mean the element $\Upsilon_k(a)$ of $Per(\tilde{\mathcal{B}}_k)$. This makes sense since the map $\Upsilon_k : \mathcal{A}_{21}(M) \to Per(\tilde{\mathcal{B}}_k)$ is an injective $*$-homomorphism.*

So, in particular, if $\varphi \in Per(M)$ *and* $f \in Flip$, *then the expressions* $M_\varphi \in Per(\tilde{\mathcal{B}}_k)$ *and* $f(\tilde{D}) \in Per(\tilde{\mathcal{B}}_k)$ *make sense.*

Definition 3.10 *If* φ *is a* $k \times k$ *matrix on* $T\tilde{M}$, *we let* M_φ *denote that section of* $\tilde{\mathcal{B}}_k$ *with* x*-component*

$$(M_\varphi)_x = M_{\varphi_x}$$

in $B(L^2(\tilde{\Delta})^k)$.

So, M_φ *is the section of* $\tilde{\mathcal{B}}_k$ *whose value* $(M_\varphi)(x) \in B((\tilde{H}_x)^k)$ *at each* x *in* $\tilde{M}$ *is the operator given by*

$$(M_\varphi)(x) = M_{(\varphi|_{\tilde{M}_x})}.$$

Thus, for every x *in* $\tilde{M}$, ξ *in* $(\tilde{H}^k)_x$, *and* v *in* $\tilde{M}_x$, *we have that*

$$(M_\varphi \xi)(v) = \varphi(v) \cdot \xi(v). \tag{3.11}$$

Proposition 3.12 *If* $\varphi \in M_k(UC(T\tilde{M}))$, *then the section* M_φ *of* $\tilde{\mathcal{B}}_k$ *is bounded and continuous. That is,* $M_\varphi \in BC(\tilde{\mathcal{B}}_k)$ *for every* φ *in* $M_k(UC(T\tilde{M}))$. *Moreover, the map* $\varphi \mapsto M_\varphi$ *is an injective* $*$*-homomorphism from* $M_k(UC(T\tilde{M}))$ *to* $BC(\tilde{\mathcal{B}}_k)$. *Hence* $\|M_\varphi\|_\infty = \|\varphi\|_\infty$ *for every* φ *in* $M_k(UC(T\tilde{M}))$.

Proof: Let φ be an element of $M_k(UC(T\tilde{M}))$. By Proposition 3.3.2, the map By Proposition 3.2.7, the map $x \mapsto \varphi_x$ is a bounded continuous map from $\tilde{M}$ to the C*-algebra $UC(\tilde{M}, M_k(\mathbf{C})) \subseteq BC(\tilde{M}, M_k(\mathbf{C}))$. Since $\|M_\psi\| = \|\psi\|_\infty$ for every ψ in $BC(\tilde{M}, M_k(\mathbf{C}))$, it follows that the map $x \mapsto M_{\varphi_x}$ is a bounded continuous map from $\tilde{M}$ to $B(L^2(\tilde{\Delta})^k)$. By definition of M_φ, the x-component $(M_\varphi)_x$ of M_φ is equal to M_{φ_x} for every x in $\tilde{M}$. Thus, the map $x \mapsto (M_\varphi)_x$

is a bounded continuous map from $\tilde{M}$ to $B(L^2(\tilde{\Delta})^k)$. In other words, the x-component of M_φ varies continuously with x in $\tilde{M}$, and is uniformly bounded in x. By Proposition 4.1.12 and the definition of a bounded section of $\tilde{\mathcal{B}}_k$, it follows that M_φ is a bounded continuous section of $\tilde{\mathcal{B}}_k$.

Moreover,

$$\begin{aligned}\|M_\varphi\|_\infty &= \sup_{x\in\tilde{M}} \|(M_\varphi)_x\| \\ &= \sup_{x\in\tilde{M}} \|M_{(\varphi x)}\| \\ &= \sup_{x\in\tilde{M}} \|\varphi_x\|_\infty \\ &= \|\varphi\|_\infty.\end{aligned}$$

This proves that the map $\varphi \mapsto M_\varphi$ from $M_k(UC(T\tilde{M}))$ to $BC(\tilde{\mathcal{B}}_k)$ is an injective $*$-homomorphism, and completes the proof of Proposition 3.12. ♠

Definition 3.13 *If C is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, define $\mathcal{D}'_C$ to be the C^*-subalgebra of $BC(\tilde{\mathcal{B}}_k)$ generated by the set of all M_φ and $f(\tilde{D})$ in $BC(\tilde{\mathcal{B}}_k)$ such that φ belongs to C and f belongs to $Flip$.*

Then define **the Dirac algebra of C** *to be the ideal $\mathcal{D}_C$ of $\mathcal{D}'_C$ generated by the set of all M_φ such that φ is in C.*

Next, we let $\mathcal{L}'_C \subseteq BC(\tilde{\mathcal{B}}_k)$ denote the C^*-subalgebra of $BC(\tilde{\mathcal{B}}_k)$ generated by the set of all $f(\tilde{D})$ and M_φ in $BC(\tilde{\mathcal{B}}_k)$ such that $f \in C_0(\mathbf{R})$ and $\varphi \in C$. Then we let $\mathcal{L}_C$ be the ideal of $\mathcal{L}'_C$ generated by the set of all $M_\varphi f(\tilde{D})$ and $f(\tilde{D})M_\varphi$ such that $f \in C_0(\mathbf{R})$ and $\varphi \in C$.

Remark 3.14 *Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$. Since the set A of all M_φ in $BC(\tilde{\mathcal{B}}_k)$ such that φ belongs to C, and the set B of all $f(\tilde{D})$*

in $BC(\tilde{\mathcal{B}}_k)$ such that $f \in Flip$, are both C^*-algebras, then $\mathcal{D}'_C$ is actually the closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$ generated by these two sets A and B.

Proposition 3.15 *Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$. Let A be a C^*-subalgebra of C. Then $\mathcal{D}'_A \subseteq \mathcal{D}'_C, \mathcal{D}_A \subseteq \mathcal{D}_C, \mathcal{L}'_A \subseteq \mathcal{L}'_C$, and $\mathcal{L}_A \subseteq \mathcal{L}_C$.*

Proof: Obvious from the definitions. ♠

Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$. For every $x \in \tilde{M}$, the map $\varphi \mapsto \varphi_x$ is a $*$-homomrphism from $M_k(UC(T\tilde{M}))$ onto $M_k(UC(\tilde{M}))$. We let C_x be the image of C under this $*$-homomorphism. That is

$$C_x = \{\varphi_x : \varphi \in C\}.$$

Then C_x is a C^*-subalgebra of $M_k(UC(\tilde{M}))$.

Proposition 3.16 *Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, and let x be an element of $\tilde{M}$. Then the $*$-homomorphism $r_x : a \mapsto a_x$ maps $\mathcal{D}_C$ onto $\mathcal{D}_{C_x}$, $\mathcal{L}_C$ onto $\mathcal{L}_{C_x}$, $C(\tilde{\mathcal{K}}_k)$ onto $K(L^2(\tilde{\Delta})^k)$, and $\mathcal{L}_C + C(\tilde{\mathcal{K}}_k)$ onto $\mathcal{L}_{C_x} + K(L^2(\tilde{\Delta})^k)$. Moreover, the diagrams*

$$\begin{array}{ccc} \mathcal{L}_C & \xrightarrow{i} & \mathcal{D}_C \\ \downarrow r_x & & \downarrow r_x \\ \mathcal{L}_{C_x} & \xrightarrow{i} & \mathcal{D}_{C_x} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{L}_C + C(\tilde{\mathcal{K}}_k) & \longrightarrow & \mathcal{D}_C + C(\tilde{\mathcal{K}}_k) \\ \downarrow r_x & & \downarrow r_x \\ \mathcal{L}_{C_x} + K(L^2(\tilde{\Delta})^k) & \longrightarrow & \mathcal{D}_{C_x} + K(L^2(\tilde{\Delta})^k) \end{array}$$

commute.

Proof: This follows from the definitions of the C^*-algebras

$$\begin{aligned}\mathcal{D}_C, \mathcal{L}_C &\subseteq BC(\tilde{\mathcal{B}}_k) \text{ and} \\ \mathcal{D}_{C_x}, \mathcal{L}_{C_x} &\subseteq B(L^2(\tilde{\Delta})^k)). \spadesuit\end{aligned}$$

Note 3.17 *If A and B are C^*-subalgebras of some larger C^*-algebra, we let* $\boldsymbol{C^*(A, B)}$ *denote the C^*-algebra generated by A and B and we let* $\boldsymbol{\mathcal{I}(A, B)}$ *denote the ideal of $C^*(A, B)$ generated by the set of all $a \cdot b$ and $b \cdot a$ such that $a \in A$ and $b \in B$.*

Proposition 3.18 *Suppose A and B are C^*-subalgebras of some larger C^*-algebra. Let 1_k denote the identity matrix in $M_k(\mathbf{C})$. Then*

$$\mathcal{I}(A \otimes M_k(\mathbf{C}), B \otimes \{1_k\}) = \mathcal{I}(A, B) \otimes M_k(\mathbf{C}).$$

Proof: For convenience, let

$$\mathcal{I}(A, B)_k = \mathcal{I}(A \otimes M_k(\mathbf{C}), B \otimes \{1_k\}).$$

For $1 \leq i \leq k$, $1 \leq j \leq k$, let E_{ij} denote the $k \times k$ matrix in $M_k(\mathbf{C})$ whose entries are all 0 except for the ij entry which has value 1. Note that

$$\mathcal{I}(A, B)_k \subseteq C^*(A, B) \otimes M_k(\mathbf{C}).$$

Let C_{ij} equal the set of all $x \in C^*(A, B)$ such that $x \otimes E_{ij} \in \mathcal{I}(A, B)_k$.

Suppose $a \in A$, $b \in B$. Then $(a \otimes E_{ij})(b \otimes 1_k) = (ab) \otimes E_{ij}$ and $(b \otimes 1_k)(a \otimes E_{ij}) = (ba) \otimes E_{ij}$ belong to $\mathcal{I}(A, B)_k$ by definition of $\mathcal{I}(A, B)_k$. Thus ab and $ba \in C_{ij}$ for every $a \in A$, $b \in B$.

Suppose now $x \in A, y \in B$ and $z \in C_{ij}$. Then both $x \otimes 1_k$ and $y \otimes 1_k$ belong to $C^*(A \otimes M_k(\mathbf{C}), B \otimes \{1_k\})$, and $z \otimes E_{ij} \in \mathcal{I}(A, B)_k$. So by definition of $\mathcal{I}(A,B)_k$, $(x\otimes 1_k)(z\otimes 1_k)(z\otimes E_{ij})$ and $(z\otimes E_{ij})(x\otimes 1_k)\in\mathcal{I}(A,B)_k$. That is, $(xz)\otimes E_{ij}$ and $(zx)\otimes E_{ij}$ belong to $\mathcal{I}(A,B)_k$ which means that both xz and zx belong to C_{ij}. Therefore C_{ij} is an ideal of $C^*(A,B)$ containing all the ab and ba such that $a\in A$ and $b\in B$. Thus $\mathcal{I}(A,B)\subseteq C_{ij}$. This is the same as saying that $\mathcal{I}(A,B)\otimes E_{ij}\subseteq\mathcal{I}(A,B)_k$ for every i,j with $1\le i\le k$ and $1\le j\le k$. From this we get that

$$\mathcal{I}(A,B)\otimes M_k(\mathbf{C})\subseteq\mathcal{I}(A,B)_k. \tag{3.19}$$

Next we show that $\mathcal{I}(A,B)\otimes M_k(\mathbf{C})$ is an ideal of $C^*(A\otimes M_k(\mathbf{C}),B\otimes\{1_k\})$. But $C^*(A\otimes M_k(\mathbf{C}),\, B\otimes\{1_k\})$ is a subalgebra of $C^*(A,B)\otimes M_k(\mathbf{C})$, and since $\mathcal{I}(A,B)$ is an ideal of $C^*(A,B)$, then $\mathcal{I}(A,B)\otimes M_k(\mathbf{C})$ is an ideal of $C^*(A,B)\otimes M_k(\mathbf{C})$. Thus, $\mathcal{I}(A,B)\otimes M_k(\mathbf{C})$ is an ideal of $C^*(A\otimes M_k(\mathbf{C}),B\otimes\{1_k\})$.

Of course, if $a\in A$, $b\in B, x\in M_k(\mathbf{C})$, and ab and $ba\in\mathcal{I}(A,B)$, $\Rightarrow$ $(a\otimes x)(b\otimes 1_k)=(ab)\otimes x\in\mathcal{I}(A,B)\otimes M_k(\mathbf{C})$ and $(b\otimes 1_k)(a\otimes x)=(ba)\otimes x\in\mathcal{I}(A,B)\otimes M_k(\mathbf{C})$.

So $\mathcal{I}(A,B)\otimes M_k(\mathbf{C})$ is a closed ideal of $C^*(A\otimes M_k(\mathbf{C}),\, B\otimes\{1_k\})$ containing $(a\otimes x)(b\otimes 1_k)$ and $(b\otimes 1_k)(a\otimes x)$ for every $a\in A, b\in B$. Since $\mathcal{I}(A,B)_k$ is the smallest closed ideal with these properties, it follows that

$$\mathcal{I}(A,B)_k\subseteq\mathcal{I}(A,B)\otimes M_k(\mathbf{C}).$$

From (3.19) we then get that

$$\mathcal{I}(A,B)_k=\mathcal{I}(A,B)\otimes M_k(\mathbf{C}).\ \spadesuit$$

Corollary 3.20 *Let C be a C^*-subalgebra of $M_l(UC(T\tilde{M}))$. Consider $M_k(C)$ as a C^*-subalgebra of $M_{kl}(UC(T\tilde{M}))$. Look at the C^*-subalgebras $\mathcal{D}_{M_k(C)}$ and $\mathcal{L}_{M_k(C)}$ of $BC(\tilde{\mathcal{B}}_{kl})$. Then the isomorphism $BC(\tilde{\mathcal{B}}_{kl}) \cong M_k(BC(\tilde{\mathcal{B}}_l))$ restricts to isomorphisms*

$$\mathcal{D}_{M_k(C)} \cong M_k(\mathcal{D}_C)$$

and

$$\mathcal{L}_{M_k(C)} \cong M_k(\mathcal{L}_C).$$

Proof: Let B be the C*-algebra of all $f(\tilde{D}) \in Per(\tilde{\mathcal{B}})$ such that f is in Flip and let B_0 =the C*-algebra of all $f(\tilde{D}) \in Per(\tilde{\mathcal{B}})$ such that $f \in C_0(\mathbf{R})$.

Note that if A is a C*-subalgebra of $M_n(UC(T\tilde{M}))$, then

$$\mathcal{D}_A = \mathcal{I}(M_A, B \otimes \{1_n\})$$

and

$$\mathcal{L}_A = \mathcal{I}(M_A, B_0 \otimes \{1_n\})$$

where M_A is the algebra of all M_φ in $BC(\tilde{\mathcal{B}}_n)$ such that φ is in A. So

$$\begin{aligned}
\mathcal{D}_{M_k(C)} &= \mathcal{I}(M_{M_k(C)}, B \otimes 1_{kl}) \\
&= \mathcal{I}(M_C \otimes M_k(\mathbf{C}), (B \otimes 1_l) \otimes 1_k) \\
&= \mathcal{I}(M_C, B \otimes 1_l) \otimes M_k(\mathbf{C}) \\
&\qquad \text{(by Proposition 3.18)} \\
&= \mathcal{D}_C \otimes M_k(\mathbf{C})
\end{aligned}$$

and similarly $\mathcal{L}_{M_k(C)} = \mathcal{L}_C \otimes M_k(\mathbf{C})$. It is a simple matter to show that the isomorphisms correspond to the isomorphism $BC(\tilde{\mathcal{B}}_{kl}) \cong M_k(BC(\tilde{\mathcal{B}}_l))$. ♠

Lemma 3.21 *The isomorphism between* $Per(\tilde{\mathcal{B}}_k)$ *and* $M_k(Per(\tilde{\mathcal{B}}))$ *restricts to an isomorphism between* $\mathcal{D}_{M_k(\tilde{C}_0(TM))}$ *and* $M_k(\mathcal{D}_{\tilde{C}_0(TM)})$*. So*

$$\mathcal{D}_{M_k(\tilde{C}_0(TM))} \cong M_k(\mathcal{D}_{\tilde{C}_0(TM)}).$$

Proof: This is a special case of Corollary 3.20. ♠

Lemma 3.22 *If* $\varphi \in UC(T\tilde{M}, M_k(\mathbf{C}))$*, then, for every* $\epsilon > 0$*, there exists* $r > 0$ *such that if* g *is a* C^∞*-Flip function,* $\hat{g}$ *has support in* $(-r, r)$*, and* $\|g\|_\infty \leq 1$*, then the section* $[g(D), M_\varphi]$ *in* $BC(\tilde{\mathcal{B}}_k)$ *has norm* $\|[g(D), M_\varphi]\|_\infty < \epsilon$*.*

Proof: Suppose $\varphi \in UC(T\tilde{M}, M_k(\mathbf{C}))$. Consider the family $A = \{\varphi_x\}_{x\in\tilde{M}}$ of functions $\varphi_x \in UC(\tilde{M}, M_k(\mathbf{C}))$. Let $\epsilon > 0$. Then, by uniform continuity of φ, there exists $\delta > 0$ such that $\|\varphi(v) - \varphi(w)\| < \epsilon$ whenever $d(v, w) < \delta$. Thus, if $x \in \tilde{M}, a, b \in \tilde{M}$, and $d(a, b) < \delta$, then $\|\varphi_x(a) - \varphi_x(b)\| = \|\varphi(\exp_x^{-1}(a)) - \varphi(\exp_x^{-1}(b))\| < \epsilon$, since $d(\exp_x^{-1}(a), \exp_x^{-1}(b))$ is equal to $d(a, b)$ which is $< \delta$. Thus, the family $\{\varphi_x\}_{x\in\tilde{M}}$ satisfies the conditions of Lemma 2.11.2. By that lemma, there exists $r > 0$ such that if $g \in C^\infty$*-Flip*, $\hat{g}$ has support in $(-r, r)$, $\|g\|_\infty \leq 1$, then, for every $x \in \tilde{M}$, $\|[g(D), M_{\varphi_x}]\| < \epsilon$. Since $[g(D), M_{\varphi_x}]$ is equal to $[g(D), M_\varphi]_x$, it follows that, for such g, $\|[g(D), M_\varphi]_x\| < \epsilon$ for all $x \in \tilde{M}$. Therefore $\|[g(D), M_\varphi]\|_\infty \leq \epsilon$ for such g. ♠

The following three propositions follow from Lemma 3.22 in exactly the same way that Corollary 2.11.7, Proposition 2.11.8, and Proposition 2.11.9 follow from Lemma 2.11.3.

Proposition 3.23 *If* C *is a* C^**-subalgebra of* $M_k(UC(T\tilde{M}))$*, then* $\mathcal{L}_C$ *is an ideal of both* $\mathcal{D}'_C$ *and* $\mathcal{D}_C$*.*

Proposition 3.24 *Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$. Then $[a,b] \in \mathcal{L}_C$ for every $a, b \in \mathcal{D}'_C$. Consequently, $[a,b] \in \mathcal{L}_C$ for every $a, b \in \mathcal{D}_C$.*

If C is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$ and $a \in \mathcal{D}_C$, we will use $[\boldsymbol{a}]$ to denote the class $a + \mathcal{L}_C$ in $\mathcal{D}_C/\mathcal{L}_C$.

Proposition 3.25 *Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$. Then $\mathcal{D}_C$ is the C^*-subalgebra of $BC(\tilde{\mathcal{B}}_k)$ generated by $\mathcal{L}_C$ and the set of all $f(\tilde{D})M_\varphi$ such that $f \in Flip$ and $\varphi \in C$. Therefore $\mathcal{D}_C/\mathcal{L}_C$ is generated by elements $[f(D)M_\varphi]$ where $f \in Flip$ and $\varphi \in C$.*

Definition 3.26 *If C is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, define the* **Dirac map**

$$\tau_C : C \oplus C \to \mathcal{D}_C/\mathcal{L}_C,$$

by letting

$$\tau_C(\varphi, \eta) = [f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta]$$

where f is any $Flip_l$ function and g is any $Flip_r$ function.

Proposition 3.27 *If C is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, the Dirac map $\tau_C : C \oplus C \to \mathcal{D}_C/\mathcal{L}_C$ is a well-defined surjective $*$-homomorphism.*

Proof: The proof is identical to the proof of Proposition 2.11.13 except here we use Proposition 3.24 and 3.25 instead of Proposition 2.11.8 and 2.11.9. ♠

Proposition 3.28 *Let C be a C^*-subalgebra of $M_k(UC(T\tilde{M}))$. Suppose $f \in Flip_l$, $g \in Flip_r$ and $\varphi, \eta \in C$. Then $f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta$ belongs to $\mathcal{L}_C + BC(\tilde{\mathcal{K}}_k)$ if and only if $\varphi = \eta = 0$.*

Proof: Suppose $f \in Flip_l$, $g \in Flip_r$ $\varphi, \eta \in C$, and $f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta$ belongs to $\mathcal{L}_C + BC(\tilde{\mathcal{K}}_k)$. Then, for every $x \in \tilde{M}$, $f(\tilde{D})M_{\varphi_x} + g(\tilde{D})M_{\eta_x}$ $= (f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta)_x$ belongs to $\mathcal{L}_{C_x} + K(L^2(\tilde{\Delta})^k))$ (by Propositon 3.16). By Corollary 2.11.11, it follows that $\varphi_x = \eta_x = 0$ for every $x \in \tilde{M}$. Therefore $\varphi = \eta = 0$. ♠

Theorem 3.29 *If C is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, then the Dirac map*

$$\tau_C : C \oplus C \to \mathcal{D}_C / \mathcal{L}_C$$

is a $*$-isomorphism.

Proof: This is a corollary of Proposition 3.27 and 3.28. ♠

For every C^*-subalgebra C of $M_k(UC(T\tilde{M}))$, Theorem 3.29 gives us a C^*-algebra extension.

$$0 \to \mathcal{L}_C \xrightarrow{i} \mathcal{D}_C \xrightarrow{q} C \oplus C \to 0 \quad (*)$$

where i is the inclusion of $\mathcal{L}_C$ into $\mathcal{D}_C$ and the map q from $\mathcal{D}_C$ to $C \oplus C$ is the composition of $\tau_C^{-1} : \mathcal{D}_C/\mathcal{L}_C \to C \oplus C$ with the quotient map $\mathcal{D}_C \to \mathcal{D}_C/\mathcal{L}_C$. This extension will be called **the Dirac extension of C.**

Proposition 3.30 *If C is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, and A is a C^*-subalgebra of C, then the following diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_C & \xrightarrow{i} & \mathcal{D}_C & \xrightarrow{q_C} & C \oplus C & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i \oplus i} & & \\
0 & \longrightarrow & \mathcal{L}_A & \xrightarrow{i} & \mathcal{D}_A & \xrightarrow{q_A} & A \oplus A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof: Suppose g is in Γ and ξ is in $(\tilde{H}^k)_x$, where x is in $\tilde{M}$. Then, for every v in $\tilde{M}_x$,

$$\begin{aligned}(g \cdot M_\varphi)(\xi)(v) &= [g \cdot (M_\varphi(g^{-1} \cdot \xi))(v) \\ &= g \cdot (M_\varphi(g^{-1} \cdot \xi)(g^{-1} \cdot v)) \\ &= g \cdot (\varphi(g^{-1} \cdot v) \cdot ((g^{-1} \cdot \xi)(g^{-1} \cdot v))) \\ &\quad \text{(by Equation 3.11)} \\ &= g \cdot (\varphi(g^{-1} \cdot v) \cdot (g^{-1} \cdot \xi(v))) \\ &= \varphi(g^{-1} \cdot v) \cdot (g \cdot (g^{-1} \cdot \xi(v))) \\ &= \varphi(g^{-1} \cdot v) \cdot \xi(v) \\ &= (g \cdot \varphi)(v) \cdot \xi(v) \\ &= (M_{g \cdot \varphi}\xi)(v),\end{aligned}$$

by Equation 3.11. Therefore, $g \cdot M_\varphi = M_{g \cdot \varphi}$ as asserted. ♠

Proposition 3.32 *If p is a continuous $k \times k$ projection on the sphere bundle S over M, and if φ is an element of $\widetilde{SC}(p)$, then M_φ is a periodic, bounded, continuous section of $\tilde{\mathcal{B}}_k$. That is, M_φ belongs to $Per(\tilde{\mathcal{B}}_k)$.*

Proof: Suppose $\varphi \in \widetilde{SC}(p)$. By Proposition 3.3.1, φ belongs to $UC(T\tilde{M}, M_k(\mathbf{C}))$. It follows (from Proposition 3.12) that M_φ is a bounded continuous section of $\tilde{\mathcal{B}}_k$.

Now, since φ belongs to $\widetilde{SC}(p)$, it is the lift of an element of $SC(p)$ and is therefore periodic. That is, $g \cdot \varphi = \varphi$ for all g in Γ. By Proposition 3.31, it follows that $g \cdot M_\varphi = M_{g \cdot \varphi} = M_\varphi$ for all g in Γ. This means that M_φ is periodic. Hence M_φ is a periodic, bounded, continuous section of $\tilde{\mathcal{B}}_k$. ♠

Now fix a nonzero,continuous $k \times k$ projection p on the sphere bundle over M, and consider the Dirac extension

$$0 \to \mathcal{L}_{\widetilde{SC}(p)} \to \mathcal{D}_{\widetilde{SC}(p)} \to \widetilde{SC}(p) \oplus \widetilde{SC}(p) \to 0$$

of $\widetilde{SC}(p)$, and the Dirac extension

$$0 \to \mathcal{L}_{M_k(\tilde{C}_0(TM))} \to \mathcal{D}_{M_k(\tilde{C}_o(TM))} \to M_k(\tilde{C}_o(TM)) \oplus M_k(\tilde{C}_o(TM)) \to 0$$

of $M_k(\tilde{C}_0(TM))$. Both of these extensions are well defined since, by Proposition 3.3.1, $\widetilde{SC}(p)$ is a C^*-subalgebra of $M_k(UC(T\tilde{M}))$, and, by Proposition 3.1.14, $M_k(\tilde{C}_o(TM))$ is a C^*-subalgebra of $\widetilde{SC}(p)$. Since $M_k(\tilde{C}_0(TM)) \subseteq \widetilde{SC}(p)$, we have, by Proposition 3.30, a commutative diagram

$$\begin{array}{ccccccc}
0 \to & \mathcal{L}_{\widetilde{SC}(p)} & \xrightarrow{i} & \mathcal{D}_{\widetilde{SC}(p)} & \xrightarrow{q} & \widetilde{SC}(p) \oplus \widetilde{SC}(p) & \to 0 \\
& \uparrow i & & \uparrow i & & \uparrow i \oplus i & \\
0 \to & \mathcal{L}_{M_k(\tilde{C}_0(TM))} & \xrightarrow{i} & \mathcal{D}_{M_k(\tilde{C}_0(TM))} & \xrightarrow{q} & M_k(\tilde{C}_0(TM)) \oplus M_k(\tilde{C}_0(TM)) & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

exact at every point.

We add to this diagram the double of the $\widetilde{SC}(p)$ Thom extension

$$0 \to M_k(\tilde{C}_o(TM)) \xrightarrow{i} \widetilde{SC}(p) \xrightarrow{l} Per(M) \to 0$$

of Proposition 3.1.14, and the Dirac extension

$$0 \to \mathcal{L}_{Per(M)} \to \mathcal{D}_{Per(M)} \to Per(M) \oplus Per(M) \to 0$$

of $Per(M)$, to get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 \to & \mathcal{L}_{Per(M)} & \xrightarrow{i} & \mathcal{D}_{Per(M)} & \xrightarrow{q} & & Per(M) \oplus Per(M) & \to 0 \\
 & & & & & & \uparrow{\scriptstyle l\oplus l} & & \\
0 \to & \mathcal{L}_{\widetilde{SC}(p)} & \xrightarrow{i} & \mathcal{D}_{\widetilde{SC}(p)} & \xrightarrow{q} & & \widetilde{SC}(p) \oplus \widetilde{SC}(p) & \to 0 \\
 & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & & \uparrow{\scriptstyle i\oplus i} & & \\
0 \to & \mathcal{L}_{M_k(\tilde{C}_0(TM))} & \xrightarrow{i} & \mathcal{D}_{M_k(\tilde{C}_0(TM))} & \xrightarrow{q} & & M_k(\tilde{C}_0(TM)) \oplus M_k(\tilde{C}_0(TM)) & \to 0 \\
 & \uparrow & & \uparrow & & & \uparrow & & \\
 & 0 & & 0 & & & 0 & &
\end{array} \qquad (3.33)$$

also exact at every point.

Proposition 3.34 *The algebras* $\mathcal{L}_{\widetilde{SC}(p)}, \mathcal{D}_{\widetilde{SC}(p)}, \mathcal{L}_{M_k(\tilde{C}_0(TM))}$, *and* $\mathcal{D}_{M_k(\tilde{C}_0(TM))}$ *are all* C^**-subalgebras of* $Per(\tilde{\mathcal{B}}_k)$.

Proof: It suffices to show that $\mathcal{D}_{\widetilde{SC}(p)}$ is a C^*-subalgebra of $Per(\tilde{\mathcal{B}}_k)$ since all of the other algebras are C^*-subalgebras of $\mathcal{D}_{\widetilde{SC}(p)}$. We already know, of course, that $\mathcal{D}_{\widetilde{SC}(p)}$ is a C^*-algebra.

By Proposition 3.3, $f(\tilde{D})$ in $BC(\tilde{\mathcal{B}}_k)$ belongs to $Per(\tilde{\mathcal{B}}_k)$ for every $f \in Flip$, and, by Proposition 3.32, M_φ belongs to $Per(\tilde{\mathcal{B}}_k)$ for every φ in $\widetilde{SC}(p)$. Since $\mathcal{D}'_{\widetilde{SC}(p)}$ is generated by all such $f(\tilde{D})$ and M_φ in $BC(\tilde{\mathcal{B}}_k)$, it follows that $\mathcal{D}'_{\widetilde{SC}(p)}$ is a C^*-subalgebra of $Per(\tilde{\mathcal{B}}_k)$. Since $\mathcal{D}_{\widetilde{SC}(p)}$ is a subalgebra of $\mathcal{D}'_{\widetilde{SC}(p)}$, then $\mathcal{D}_{\widetilde{SC}(p)}$ is a C^*-subalgebra of $Per(\tilde{\mathcal{B}}_k)$. ♠

Thus, $\mathcal{D}_{\widetilde{SC}(p)} + Per(\tilde{\mathcal{K}}_k)$, $\mathcal{L}_{\widetilde{SC}(p)} + Per(\tilde{\mathcal{K}}_k)$, $\mathcal{D}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k)$, and $\mathcal{L}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k)$, are all C^*-subalgebras of $Per(\tilde{\mathcal{B}}_k)$, and by Proposition

3.28 and an argument similar to the proof of Theorem 3.29, we obtain C^*-algebra extensions

$$0 \to \mathcal{L}_{\widetilde{SC}(p)} + Per(\tilde{\mathcal{K}}_k) \xrightarrow{i} \mathcal{D}_{\widetilde{SC}(p)} + Per(\tilde{\mathcal{K}}_k) \xrightarrow{q} \widetilde{SC}(p) \oplus \widetilde{SC}(p) \to 0$$

and

$$\begin{array}{l} 0 \longrightarrow \mathcal{L}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k) \\ \qquad\qquad\qquad \big\downarrow i \\ \qquad \mathcal{D}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k) \xrightarrow[q]{} M_k(\tilde{C}(TM)) \oplus M_k(\tilde{C}_0(TM)) \longrightarrow 0 \end{array}$$

which we will call **the adjoined Dirac extensions** of $\widetilde{SC}(p)$ and of $M_k(\tilde{C}_0(TM))$, respectively. The map q in both cases is the map which sends $f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta$ to (φ, η) for every f in $Flip_l$, g in $Flip_r$, and φ, η in $\widetilde{SC}(p)$ (or $M_k(\tilde{C}_0(TM))$).

For convenience, the adjoined Dirac extension of $\widetilde{SC}(p)$ will also be denoted by

$$0 \to \mathcal{A}_{10}(p) \xrightarrow{i} \mathcal{A}_{11}(p) \xrightarrow{q} \mathcal{A}_{12}(p) \to 0,$$

the adjoined Dirac extension of $M_k(\tilde{C}_0(TM))$ will be denoted by

$$0 \to \mathcal{A}_{00}(M)_k \xrightarrow{i} \mathcal{A}_{01}(M)_k \xrightarrow{q} \mathcal{A}_{02}(M)_k \to 0,$$

and the Dirac extension

$$0 \to \mathcal{A}_{00}(M)_1 \xrightarrow{i} \mathcal{A}_{01}(M)_1 \xrightarrow{q} \mathcal{A}_{02}(M)_1 \to 0$$

of $\tilde{C}_0(TM) = M_1(\tilde{C}_0(TM))$ will also be denoted simply by

$$0 \to \mathcal{A}_{00}(M) \xrightarrow{i} \mathcal{A}_{01}(M) \xrightarrow{q} \mathcal{A}_{02}(M) \to 0.$$

The commutative diagram (3.33) can then be written as the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{A}_{20}(M) & \xrightarrow{i} & \mathcal{A}_{21}(M) & \xrightarrow{q} & \mathcal{A}_{22}(M) & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle l\oplus l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \xrightarrow{i} & \mathcal{A}_{11}(p) & \xrightarrow{q} & \mathcal{A}_{12}(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\oplus i} & & \\
0 & \longrightarrow & \mathcal{A}_{00}(M)_k & \xrightarrow{i} & \mathcal{A}_{01}(M)_k & \xrightarrow{q} & \mathcal{A}_{02}(M)_k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \quad (3.35)$$

which is exact at every point. We state this formally.

Proposition 3.36 *Diagram 3.35, which is the same as the diagram*

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{Per(M)} & \xrightarrow{i} & \mathcal{D}_{Per(M)} & \xrightarrow{q} & Per(M) \oplus Per(M) & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle l\oplus l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \xrightarrow{i} & \mathcal{A}_{11}(p) & \xrightarrow{q} & \widetilde{SC}(p) \oplus \widetilde{SC}(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\oplus i} & & \\
0 & \longrightarrow & \mathcal{A}_{00}(M)_k & \xrightarrow{i} & \mathcal{A}_{01}(M)_k & \xrightarrow{q} & M_k(\tilde{C}_0(TM)) \oplus M_k(\tilde{C}_0(TM)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

commutes, and is exact at every point.

In the next section, we fill in the gaps in this diagram by defining a limiting map $l : \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M)$, thereby obtaining "Wiener-Hopf" extensions

$$0 \to \mathcal{A}_{01}(M)_k \xrightarrow{i} \mathcal{A}_{11}(p) \xrightarrow{l} \mathcal{A}_{21}(M) \to 0$$

and

$$0 \to \mathcal{A}_{00}(M)_k \xrightarrow{i} \mathcal{A}_{10}(p) \xrightarrow{l} \mathcal{A}_{20}(M) \to 0.$$

Before doing that however, we make the following simple observations.

Proposition 3.37

$$\mathcal{A}_{00}(M)_k \stackrel{\text{def}}{=} \mathcal{L}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k)$$

$$= Per(\tilde{\mathcal{K}}_k) \cong C(\mathcal{K}_k) \cong M_k(C(\mathcal{K})) \cong M_k(\mathcal{A}_{00}(M)).$$

Proof: It suffices to show that

$$\mathcal{L}_{M_k(\tilde{C}_0(TM))} \subseteq Per(\tilde{\mathcal{K}}_k).$$

We already know that

$$\mathcal{L}_{M_k(\tilde{C}_0(TM))} \subseteq Per(\tilde{\mathcal{B}}_k).$$

So we only have to show that for every $A \in \mathcal{L}_{M_k(\tilde{C}_0(TM))}$ and $x \in \tilde{M}$,$A_x \in K(L^2(\tilde{\Delta}))$.

Let $C = M_k(\tilde{C}_0(TM))$ and fix x in $\tilde{M}$. Then, of course,

$$C_x \stackrel{\text{def}}{=} \{\varphi_x : \varphi \in M_k(\tilde{C}_0(TM))\}$$

is the same as $M_k(C_0(\tilde{M}))$. By Proposition 3.16, if $A \in \mathcal{L}_C$, then $A_x \in \mathcal{L}_{C_x}$. Thus to complete the proof it suffices to show that $\mathcal{L}_{M_k(C_0(\tilde{M}))} \subseteq K(L^2(\tilde{\Delta}^k)$. But $\mathcal{L}_{M_k(C_0(\tilde{M}))}$ is a certain ideal generated by operators of the form $M_\varphi \cdot f(\tilde{D})$ and $f(\tilde{D}) \cdot M_\varphi$ where $\varphi \in M_k(C_0(\tilde{M}))$ and $f \in C_0(\mathbf{R})$. These operators are compact by Proposition 2.7.5. Hence $\mathcal{L}_{M_k(C_0(\tilde{M}))} \subseteq K(L^2(\tilde{\Delta})^k)$. ♠

Proposition 3.38 *The isomorphism between* $Per(\tilde{\mathcal{B}}_k)$ *and* $M_k(Per(\tilde{\mathcal{B}}))$ *restricts to an isomorphism between* $\mathcal{A}_{01}(M)_k$ *and* $M_k(\mathcal{A}_{01}(M))$. *So*

$$\mathcal{A}_{01}(M)_k \cong M_k(\mathcal{A}_{01}(M)).$$

Proof: Let Φ be the isomorphism from $Per(\tilde{\mathcal{B}}_k)$ onto $M_k(Per(\tilde{\mathcal{B}}))$. By definition, $\mathcal{A}_{01}(M)_k$ is the C*-algebra $\mathcal{D}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k)$. By Lemma 3.21, Φ restricts to an isomorphism from $\mathcal{D}_{M_k(\tilde{C}_0(TM))}$ onto $M_k(\tilde{C}_0(TM)$, and, of course, Φ also restricts to an isomorphism from $Per(\tilde{\mathcal{K}}_k)$ onto $M_k(Per(\tilde{\mathcal{K}}))$. Thus Φ restricts to an isomorphism from $\mathcal{D}_{M_k(\tilde{C}_0(TM))}+Per(\tilde{\mathcal{K}}_k)$ onto $M_k(\mathcal{D}_{\tilde{C}_0(TM)}+ Per(\tilde{\mathcal{K}})) = M_k(\mathcal{A}_{01}(M))$. ♠

For the next proposition, $\mathcal{A}_{01}(M)_k$ should be regarded in two ways: as a C*-subalgebra of $Per(\tilde{\mathcal{B}}_k)$, and as a C*-subalgebra of $C(\mathcal{B}_k)$.

Proposition 3.39 *The adjoined Dirac extension*

$$0 \to \mathcal{A}_{00}(M) \to \mathcal{A}_{01}(M) \to \mathcal{A}_{02}(M) \to 0$$

of $\tilde{C}_0(TM)$ *is the same as the extension*

$$0 \to Per(\tilde{\mathcal{K}}) \to \mathcal{A}_{01}(M) \to \tilde{C}_0(TM) \oplus \tilde{C}_0(TM) \to 0$$

which is isomorphic to the extension.

$$0 \to C(\mathcal{K}) \xrightarrow{i} \mathcal{A}_{01}(M) \xrightarrow{q} C_0(TM) \oplus C_0(TM) \to 0.$$

Also, the adjoined Dirac extension

$$0 \to \mathcal{A}_{00}(M)_k \to \mathcal{A}_{01}(M)_k \to \mathcal{A}_{02}(M)_k \to 0$$

of $M_k(\tilde{C}_0(TM))$ *is the same as the extension*

$$0 \to M_k(\mathcal{A}_{00}(M)) \to M_k(\mathcal{A}_{01}(M)) \to M_k(\mathcal{A}_{02}(M)) \to 0,$$

which is the same as the extension

$$0 \to M_k(Per(\tilde{\mathcal{K}})) \to M_k(\mathcal{A}_{01}(M)) \to M_k(\tilde{C}_0(TM))) \oplus M_k(\tilde{C}_0(TM)) \to 0,$$

which is isomorphic to the extension

$$0 \to M_k(C(\mathcal{K})) \to M_k(\mathcal{A}_{01}(M)) \to M_k(C_0(TM)) \oplus M_k(C_0(TM)) \to 0. \tag{3.40}$$

Proof: Follows from Proposition 3.38 and 3.37. ♠

Thus, Proposition 3.36 can be rephrased as follows.

Definition 3.41 *The extension (3.40) will be called* **the adjoined Dirac extension** *of* $M_k(C_0(TM))$*. By Proposition 3.39, this is isomorphic to the adjoined Dirac extension of* $M_k(\tilde{C}_0(TM))$*.*

Proposition 3.42 *Diagram 3.35 is the same as the diagram*

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{Per(M)} & \xrightarrow{i} & \mathcal{D}_{Per(M)} & \xrightarrow{q} & Per(M) \oplus PerM & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle l \otimes l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \xrightarrow{i} & \mathcal{A}_{11}(p) & \xrightarrow{q} & \widetilde{SC}(p) \oplus \widetilde{SC}(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i \oplus i} & & \\
0 & \longrightarrow & M_k(Per(\tilde{\mathcal{K}})) & \xrightarrow{i} & M_k(\mathcal{A}_{01}(M)) & \xrightarrow{q} & M_k(\tilde{C}_0(TM)) \oplus M_k(\tilde{C}_0(TM)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which is the same as the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{Per(M)} & \xrightarrow{i} & \mathcal{D}_{Per(M)} & \xrightarrow{q} & C(M) \oplus C(M) & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle l\otimes l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \xrightarrow{i} & \mathcal{A}_{11}(p) & \longrightarrow & SC(p) \oplus SC(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\oplus i} & & \\
0 & \longrightarrow & M_k(C(\mathcal{K})) & \xrightarrow{i} & M_k(\mathcal{A}_{01}(M)) & \longrightarrow & M_k(C_0(TM)) \oplus M_k(C_0(TM)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(by identifying $Per(\tilde{\mathcal{B}}_k)$ with $C(\mathcal{B}_k)$) and commutes and is exact at every point.

5.4 "Wiener-Hopf" Extensions

In this section, M is again a **compact, riemannian spin manifold of nonpositive curvature**. Moreover, we assume $\boldsymbol{k}$ **is a positive integer**, and that $\boldsymbol{p}$ is **a nonzero continuous $\boldsymbol{k} \times \boldsymbol{k}$ projection on the sphere bundle $\boldsymbol{S}$ over $\boldsymbol{M}$**. Let us also assume the notation of Section 3.3.3. So, in particular, $\tilde{S}$ will denote the sphere bundle over $\tilde{M}$.

Recall from Section 3.3.3 that $\tilde{p}$ stands for the lift of p up to a projection on $\tilde{S}$. Let $r : T\tilde{M}\backslash\tilde{M} \to \tilde{S}$ be the retraction map. Note that $\tilde{p} \circ r$ is the extension of $\tilde{p}$ to a projection on all of $T\tilde{M}\backslash\tilde{M}$ along the radial lines extending from 0 on each tangent space.

Notation 4.1 *For convenience, we will sometimes use the symbol $\hat{\mathbf{p}}$ in place of $\tilde{p} \circ r$. That is, we let $\hat{p} = \tilde{p} \circ r$.*

Definition 4.2 *If $r > 0$ let*

$$\boldsymbol{V_r} = \{v \in T\tilde{M} : \|v\| < r\}.$$

Also, let $\boldsymbol{\chi_r}$ be the characteristic function on $T\tilde{M}$ with value 1 on V_r and value 0 outside V_r. That is, $\chi_r = \chi_{V_r}$. Finally, let $\boldsymbol{P_r}$ denote the bounded section of $\tilde{\mathcal{B}}_k$ defined by setting

$$P_r = M_{\chi_r \cdot 1_k}$$

where 1_k is the diagonal $k \times k$ matrix with 1's along the diagonal.

Definition 4.3 *If $a \in BC(\tilde{\mathcal{B}}_k)$, then we say that $\boldsymbol{a}$* **has property $*$** *if there exists $b \in \mathcal{A}_{21}(M)$ which, when regarded as an element $b \in Per(\tilde{\mathcal{B}}_k)$, is such* *that* $\|(a-b)\cdot M_{\hat{p}}\cdot(1-P_N)\| \to 0$ *as* $N\to\infty$. *Any such b is said* **to have property** $*a$.

Definition 4.4 *If n is a positive integer,* $\lambda=(\lambda_1,\ldots,\lambda_n)$ *belongs to* $\mathbf{C}^n$, *h is a Hilbert space, and v belongs to h, then define* $\boldsymbol{\lambda}\otimes\boldsymbol{v}$ *in* h^n *as the vector*

$$\lambda\otimes v=(\lambda_1 v,\ldots,\lambda_n v).$$

Lemma 4.5 *If n is a positive integer, h is a Hilbert space,* λ *belongs to* $\mathbf{C}^n$, *and v belongs to h, then*

$$\|\lambda\otimes v\|=\|\lambda\|\cdot\|v\|.$$

Proof: We have

$$\begin{aligned}\|\lambda\otimes v\|^2 &= \sum_{i=1}^{n}\|\lambda_i\|^2\|v\|^2\\ &= \|\lambda\|^2\cdot\|v\|^2,\end{aligned}$$

which implies that $\|\lambda\otimes v\|=\|\lambda\|\cdot\|v\|$. ♠

Lemma 4.6 *If* $K\subset\mathbf{R}^m$ *is such that its diameter* $diam(K)$ *is finite and* $d(K,0)>0$ *(where* $d(K,0)$ *represents the distance between K and* 0*), then, for every* $x,y\in K$, *we have* $\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\|\le\frac{2diam(K)}{d(K,0)}$.

Proof: If $x,y\in K$, then

$$\begin{aligned}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &\le \left\|\frac{1}{\|x\|}(x-y)+\frac{y}{\|y\|}\left(\frac{\|y\|-\|x\|}{\|x\|}\right)\right\|\\ &\le \frac{\|x-y\|}{\|x\|}+\frac{|\,\|y\|-\|x\|\,|}{\|x\|}\\ &\le \frac{\|x-y\|}{\|x\|}+\frac{\|y-x\|}{\|x\|}\\ &= \frac{2\|x-y\|}{\|x\|}\le\frac{2diam(K)}{d(K,0)}.\ ♠\end{aligned}$$

Definition 4.7 *Let Y be a metric space, and A a subset of Y. Then we will use* ***diam(Y)*** *to denote the diameter of Y. Let $f : X \to Y$ be a map from a set X to Y, and suppose $K \subset X$. Then* **the variation of f over K**, *denoted by* **$\mathbf{Var}_K f$**, *is, by definition, the diameter of $f(K)$. That is,*

$$Var_K f = diam(f(K)).$$

If $x \in \tilde{M}$, the tangent space $T_x\tilde{M}$ has two different metrics: one coming from the riemannian structure on $\tilde{M}$, and the other coming from identifying $T\tilde{M}$ with the product space $\tilde{M} \times \tilde{M}$ via the map $\pi \times exp$.

Definition 4.8 *If $x \in \tilde{M}$ and $K \subset T_x\tilde{M}$, then the notation* ***diam(K)*** *will stand for the diameter of K in the metric on $T_x\tilde{M}$ coming from the riemannian structure on $\tilde{M}$.*

Lemma 4.9 *For every $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in M$, $v, w \in S_x$, and $\|v - w\| < \delta$, then $\|p(v) - p(w)\| < \epsilon$.*

Proof: Otherwise, we can find $\epsilon > 0$ and a sequence (v_n, w_n) in $S \times S$, such that each (v_n, w_n) belongs to $S_{x_n} \times S_{x_n}$ for some x_n in M, and such that $\|v_n - w_n\| < \frac{1}{n}$, and $\|p(v_n) - p(w_n)\| \geq \epsilon$ for all n. Since M is compact, it follows that S is compact, and therefore $S \times S$ is compact. Hence, by passing to a subsequence, we can assume that (v_n, w_n) converges to some (v, w) in $S \times S$. Thus, $x_n = \pi(v_n)$ converges to $\pi(v)$. Since x_n also equals $\pi(w_n)$, it also converges to $\pi(w)$. Let $x = \lim x_n$. Then $x = \pi(v) = \pi(w)$. That is, both v and w belong to T_xM. Now, by continuity of the riemannian metric, we have that $\|v - w\| = \lim \|v_n - w_n\| \leq \lim \frac{1}{n} = 0$, which implies that $v = w$. By continuity of p, it follows that $0 = \|p(v) - p(w)\| = \lim \|p(v_n) - p(w_n)\|$. But $\|p(v_n) - p(w_n)\| \geq \epsilon$ for every n. Thus, $0 \geq \epsilon > 0$, a contradiction. Therefore, the lemma is true. ♠

Proposition 4.10 *Suppose* $0 < s < +\infty$ *and* $\epsilon > 0$. *Then there exists* $N > 0$ *such that, for every* $x \in \tilde{M}$ *and for every* $K \subset T_x\tilde{M}$ *with* $diam(K) \leq s$ *and* $d(K, 0) > N$ *(where* $0 \in T_x\tilde{M}$*), we have that* $Var_K(\tilde{p} \circ r) \leq \epsilon$.

Proof: Recall that if $x \in M$ and $y \in \tilde{M}$, then S_x is the fiber of S at x, and $\tilde{S}_y$ is the fiber of $\tilde{S}$ at y. By Lemma 4.9, there exists $\delta > 0$ such that, if $x \in M, v, w \in S_x$ and $\|v - w\| < \delta$, then $\|p(v) - p(w)\| < \epsilon$. Since $\tilde{p}$ is the lift of p to $\tilde{S}$, the same is true for $\tilde{p}$. That is, if $x \in \tilde{M}, v, w \in \tilde{S}_x$, and $\|v - w\| \leq \delta$, then $\|\tilde{p}(v) - \tilde{p}(w)\| < \epsilon$.

Now, take any $s > 0$ such that $s < \infty$. Let $N = \frac{2s}{\delta}$. Suppose $x \in \tilde{M}, K \subset T_x\tilde{M}, diam(K) \leq s$, and $d(K, 0) > N$. Now, take any $y, z \in K$. Then, by Lemma 4.6, we have that

$$\left\| \frac{y}{\|y\|} - \frac{z}{\|z\|} \right\| \leq \frac{2diam(K)}{d(K,0)} \leq \frac{2s}{N} = \delta.$$

That is, $\|r(y) - r(z)\| \leq \delta$, which implies that $\|\tilde{p}(r(y)) - \tilde{p}(r(z))\| < \epsilon$. Therefore $Var_K(\tilde{p} \circ r) \leq \epsilon$. ♠

Proposition 4.11 *Suppose* $0 < s < \infty$. *Then, for every* $\epsilon > 0$, *there exists an* $N > 0$ *such that, if* $K \subset \tilde{M}$, $diam(K) \leq s$, $x \in \tilde{M}$, *and* $d(x, K) > N$, *then* $Var_{\exp_x^{-1}(K)}(\tilde{p} \circ r) \leq \epsilon$.

Proof: Suppose $0 < s < \infty$, and $\epsilon > 0$. By Proposition 4.10, there exists $N > 0$ such that if $x \in \tilde{M}$, $A \subset T_x\tilde{M}$, $diam(A) \leq s$, and $d(A, 0) > N$, then

$\mathrm{Var}_A(\tilde{p} \circ r) \leq \epsilon$. Now, take $x \in \tilde{M}$ and $K \subset \tilde{M}$ such that $diam(K) \leq s$ and $d(x, K) > N$. To complete the proof, it suffices to show that $diam(exp_x{}^{-1}(K)) \leq s$ and that $d(exp_x{}^{-1}(K), 0) > N$. But, $\tilde{M}$ has nonpositive curvature. Hence, the exponential map $exp_x : T_x\tilde{M} \to \tilde{M}$ increases distance. Therefore,

$$diam(exp_x{}^{-1}(K)) \leq diam(K) \leq s.$$

Also, the exponential map has the property that, for all $u \in T_x\tilde{M}$, $d(u, 0) = d(exp_x u, x)$, which implies that $d(exp_x{}^{-1}(K), 0) = d(K, x) > N$. ♠

Lemma 4.12 *If $a \in BC(\tilde{\mathcal{B}}_k)$ has property $*$ and $b \in \mathcal{A}_{21}(M)$ has property $*a$, then $\|b\| \leq \|a\|$.*

Proof: Take any $\xi \in L^2(\tilde{\Delta})$ *with compact support* K, and take any $y \in K$. Since K is compact, then, by Proposition 4.11, there exists $N_1 > 0$ such that, if $x \in \tilde{M}$ and $d(x, K) > N_1$, then

$$\mathrm{Var}_{\mathrm{exp}_x{}^{-1}(K)}\hat{p} < \frac{\epsilon}{2(1 + \|b\| \cdot \|\xi\|)}.$$

Since b has property $*a$, there exists $N_2 > 0$ such that, if $N > N_2$, then

$$\|(b - a) \cdot M_{\hat{p}} \cdot (1 - P_N)\| < \frac{\epsilon}{2(1 + \|\xi\|)}.$$

Let $N = \max\{N_1, N_2\}$.

Now, take any $x \in \tilde{M}$ such that $d(x, K) > N$ and take any λ in the range of the projection $\hat{p}(\mathrm{exp}_x{}^{-1}(y))$ such that $\|\lambda\| = 1$. Then,

$$\mathrm{Var}_{\mathrm{exp}_x{}^{-1}(K)}\hat{p} < \frac{\epsilon}{2(1 + \|b\| \cdot \|\xi\|)}. \tag{4.13}$$

Note that the support of $\exp_x{}^*(\xi)$ is $\exp_x{}^{-1}(K)$. So, if $v \in \tilde{M}_x$ lies in the support of $\exp_x{}^*(\xi)$, then, $\exp(v) = \exp_x(v) \in K$, which implies that $\|v\| = d(\exp(v), \pi(v)) = d(\exp(v), x) \geq d(K, x) > N$. Hence, $supp(\exp_x{}^*(\xi)) \subset T\tilde{M}\backslash V_N$, which implies that $(1 - P_N) \cdot \exp_x{}^*(\xi) = \exp_x{}^*(\xi)$, and therefore

$$(1 - P_N)(x) \cdot (\lambda \otimes \exp_x{}^*(\xi)) = \lambda \otimes \exp_x{}^*(\xi).$$

Furthermore, since the support of $\lambda \otimes \exp_x{}^*(\xi)$ is $\exp_x{}^{-1}(K)$, and since $\exp_x{}^{-1}(y) \in \exp_x{}^{-1}(K)$, then we have that

$$\|M_{\hat{p}(\exp_x{}^{-1}(y)) - \hat{p}}(x) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \leq \mathrm{Var}_{\exp_x{}^{-1}(K)} \hat{p} \cdot \|\lambda \otimes \exp_x{}^*(\xi)\|.$$

Hence,

$$\begin{aligned}
\|b\xi\| &= \|\lambda \otimes \exp_x{}^*(b\xi)\| \\
&= \|b(x) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \\
&\quad \text{(where we regard } b \text{ here as an element of } Per(\tilde{\mathcal{B}}_k)) \\
&= \|b(x) \cdot \hat{p}(\exp_x{}^{-1}(y)) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \\
&\quad \text{(since } \lambda \text{ is in the range of } \hat{p}(\exp_x{}^{-1}(y))) \\
&= \|b(x) \cdot M_{\hat{p}(\exp_x{}^{-1}(y)) - \hat{p}}(x) \cdot (\lambda \otimes \exp_x{}^*(\xi)) \\
&\quad + ((b - a) \cdot M_{\hat{p}})(x) \cdot (\lambda \otimes \exp_x{}^*(\xi)) \\
&\quad + (a \cdot M_{\hat{p}})(x) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \\
&\leq \|(b \cdot M_{\hat{p}(\exp_x{}^{-1}(y)) - \hat{p}})(x) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \\
&\quad + \|((b - a) \cdot M_{\hat{p}} \cdot (1 - P_N))(x) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \\
&\quad + \|(a \cdot M_{\hat{p}})(x) \cdot (\lambda \otimes \exp_x{}^*(\xi))\| \\
&\leq \|b\| \cdot \mathrm{Var}_{\exp_x{}^{-1}(K)} \hat{p} \cdot \|\lambda \otimes \exp_x{}^*(\xi)\|
\end{aligned}$$

$$
\begin{aligned}
&+\|(b-a)\cdot M_{\hat{p}}\cdot(1-P_N)\|\cdot\|\lambda\otimes\exp_x{}^*(\xi)\| \\
&+\|a\|\cdot\|\lambda\otimes\exp_x{}^*(\xi)\| \\
\leq\ & \|b\|\cdot\frac{\epsilon}{2(1+\|b\|\cdot\|\xi\|)}\cdot\|\xi\|+\frac{\epsilon}{2(1+\|\xi\|)}+\|a\|\cdot\|\xi\| \\
\leq\ & \frac{\epsilon}{2}+\frac{\epsilon}{2}+\|a\|\cdot\|\xi\| \\
=\ & \|a\|\cdot\|\xi\|+\epsilon.
\end{aligned}
$$

This is true for every ϵ. Thus, $\|b\xi\| \leq \|a\|\cdot\|\xi\|$ for every $\xi \in L^2(\tilde{\Delta})$ with compact support. Since the compactly supported $\xi \in L^2(\tilde{\Delta})$ are dense in $L^2(\tilde{\Delta})$, it follows that $\|b\| \leq \|a\|$. ♠

Corollary 4.14 *If $a \in BC(\tilde{\mathcal{B}}_k)$ has property $*$, then there is a unique $b \in \mathcal{A}_{21}(M)$ such that b has property $*a$.*

Proof: Suppose $a \in BC(\tilde{\mathcal{B}}_k)$ and that $b_1, b_2 \in \mathcal{A}_{21}(M)$ both have property $*a$. Then, $\|(a-b_1)\cdot M_{\hat{p}}\cdot(1-P_N)\| \to 0$ as $N \to \infty$ and $\|(a-b_2)\cdot M_{\hat{p}}\cdot(1-P_N)\| \to 0$ as $N \to \infty$. From this, it follows that $\|(b_1-b_2)\cdot M_{\hat{p}}\cdot(1-P_N)\| \to 0$ as $N \to \infty$. This means that $b_1 - b_2$ has property $*0$. By Lemma 4.12, it follows that $\|b_1 - b_2\| \leq \|0\| = 0$. Thus, $b_1 = b_2$, which proves the corollary. ♠

Definition 4.15 *If $a \in BC(\tilde{\mathcal{B}}_k)$ has property $*$, we define $\boldsymbol{l(a)}$ to be the unique element in $\mathcal{A}_{21}(M)$ with property $*a$. This makes sense by Corollary 4.14.*

Proposition 4.16 *We have $\|l(a)\| \leq \|a\|$ for every $a \in BC(\tilde{\mathcal{B}}_k)$.*

Proof: This follows from Lemma 4.12. ♠

Definition 4.17 *If X is a Banach space, $x, y \in X$, $\epsilon > 0$, and $\|x-y\| \leq \epsilon$, we will say that $\boldsymbol{x}$* **is ϵ-close to** *$\boldsymbol{y}$, and we will write $\boldsymbol{x \sim_\epsilon y}$.*

Proposition 4.18 *If X is a Banach space, $x, y, z \in X$, x is ϵ-close to y, and y is δ-close to z, then x is $(\epsilon + \delta)$-close to z.*

If, in addition, X is a Banach algebra, and a belongs to X, then ax is $(\|a\|\epsilon)$-close to ay.

Proof: Obvious. ♠

Lemma 4.19 *Suppose $\phi \in Per(M)$ and $\varphi \in \widetilde{SC}(p)$. If we regard $M_\phi \in B(L^2(\tilde{\Delta}))$ as an element of $Per(\tilde{\mathcal{B}}_k)$, then, for every $N > 0$, the sections M_ϕ, P_N, M_φ, and $M_{\hat{p}}$ of $\tilde{\mathcal{B}}_k$, are such that M_ϕ and P_N commute with themselves and with the other two sections, M_φ and $M_{\hat{p}}$.*

Proof: For every $x \in \tilde{M}$, the operators $M_\phi(x)$ and $P_N(x)$ in $B(L^2(\tilde{\Delta}_x)^k)$ are multiplication operators with complex-valued functions as multipliers. The operators $M_\varphi(x)$ and $M_{\hat{p}}(x)$ in $B(L^2(\tilde{\Delta}_x)^k)$ are also multiplication operators, but with complex matrix-valued functions as multipliers. Thus, $M_\phi(x)$ and $P_N(x)$ commute with themselves and with the other two operators, $M_\varphi(x)$ and $M_{\hat{p}}(x)$. This is true for every $x \in \tilde{M}$. The lemma, therefore, follows. ♠

Proposition 4.20 *Let $\varphi \in \widetilde{SC}(p)$ and $\epsilon > 0$. Then there exists $N > 0$ such that if we regard $M_{f_\varphi} \in \mathcal{A}_{21}(M)$ as an element of $Per(\tilde{\mathcal{B}}_k)$, then*

$$\|(M_\varphi - M_{f_\varphi} \cdot M_{\hat{p}}) \cdot (1 - P_N)\| < \epsilon.$$

Moreover, $M_\varphi \in Per(\tilde{\mathcal{B}}_k)$ has property $*$ and $l(M_\varphi) = M_{f_\varphi}$ in $\mathcal{A}_{21}(M)$.

Proof: By Remark 3.1.13, φ is equal to $(f_\varphi \circ \exp) \cdot \hat{p}$ at infinity. It follows that, for every $\epsilon > 0$, there exists $N > 0$ such that

$$\begin{aligned} M_\varphi \cdot (1 - P_N) &\sim_\epsilon M_{(f_\varphi \circ \exp)\cdot \hat{p}} \cdot (1 - P_N) \\ &= M_{f_\varphi} \cdot M_{\hat{p}} \cdot (1 - P_N), \end{aligned}$$

where, in the last line, we regard $M_{f_\varphi} \in \mathcal{A}_{21}(M)$ as an element M_{f_φ} of $Per(\tilde{\mathcal{B}}_k)$. This proves the first part of the lemma.

To prove the second part, we note that, for the same $N > 0$, we have that

$$\begin{aligned} M_\varphi \cdot M_{\hat{p}} \cdot (1 - P_N) &= M_\varphi \cdot (1 - P_N) \cdot M_{\hat{p}} \\ &\sim_\epsilon M_{f_\varphi} \cdot M_{\hat{p}} \cdot (1 - P_N) \cdot M_{\hat{p}} \\ &= M_{f_\varphi} \cdot M_{\hat{p}} \cdot (1 - P_N). \end{aligned}$$

Therefore, $M_\varphi \in Per(\tilde{\mathcal{B}}_k)$ has property $*$, and $l(M_\varphi) = M_{f_\varphi}$. ♠

Lemma 4.21 *If $\varphi \in \widetilde{SC}(p)$ and $\epsilon > 0$, there exists $N > 0$ such that*

$$M_\varphi \cdot M_{\hat{p}} \cdot (1 - P_N) \sim_\epsilon M_{\hat{p}} \cdot M_\varphi \cdot (1 - P_N).$$

Proof: Suppose $\varphi \in \widetilde{SC}(p)$ and $\epsilon > 0$. By Proposition 4.20, there exists $N > 0$ such that

$$M_\varphi \cdot (1 - P_N) \sim_{\frac{\epsilon}{2}} M_{f_\varphi} \cdot M_{\hat{p}} \cdot (1 - P_N).$$

It follows that

$$\begin{aligned} M_\varphi \cdot M_{\hat{p}} \cdot (1 - P_N) &= M_\varphi \cdot (1 - P_N) \cdot M_{\hat{p}} \\ &\sim_{\frac{\epsilon}{2}} M_{f_\varphi} \cdot M_{\hat{p}} \cdot (1 - P_N) \cdot M_{\hat{p}} \end{aligned}$$

$$
\begin{aligned}
&= M_{f_\varphi} \cdot M_{\hat{p}} \cdot M_{\hat{p}} \cdot (1 - P_N) \\
&= M_{\hat{p}} \cdot M_{f_\varphi} \cdot M_{\hat{p}} \cdot (1 - P_N) \\
&\sim_{\frac{\epsilon}{2}} M_{\hat{p}} \cdot M_\varphi \cdot (1 - P_N).
\end{aligned}
$$

Thus, the lemma is true. ♠

Lemma 4.22 *Let h be a Hilbert space, I the identity operator on h, K a compact operator on h, and $0 \lneq q_1 \lneq q_2 \lneq \ldots$ be a sequence of orthogonal projections on h such that $q_n \to I$ in the strong operator topology as $n \to \infty$. Then $\|K \cdot (I - q_n)\| \to 0$ and $\|(I - q_n) \cdot K\| \to 0$ as $n \to \infty$.*

Proof: This is clearly true if K is a finite rank operator. Every compact operator can be approximated in norm by a finite rank operator. Hence, by a simple approximation argument, we can prove that the lemma is true for any compact K. ♠

Proposition 4.23 *Let N be a complete, riemannian, spin manifold, Δ_N the bundle of spinors over N, x a point in N, and K a compact operator on the Hilbert space $L^2(\Delta_N)^k$. Then $\|(1 - M_{\chi_{B(x,r)}}) \cdot K\| \to 0$ as $r \to \infty$, and $\|K \cdot (1 - M_{\chi_{B(x,r)}})\| \to 0$ as $r \to \infty$.*

Proof: By completeness of N, we have that $N = \cup_{n=1}^{\infty} B(x,n)$. By Lemma 4.22, it follows that $\|(1 - M_{\chi_{B(x,n)}}) \cdot K\| \to 0$ as $n \to \infty$, for positive integers n. If $r > 1$, then, of course, $\|(1 - M_{\chi_{B(x,r)}}) \cdot K\| = \|(1 - M_{\chi_{B(x,r)}})(1 - M_{\chi_{B(x,n)}}) \cdot K\| \leq \|(1 - M_{\chi_{B(x,n)}}) \cdot K\|$ for any integer n between $r-1$ and r. So, $\|(1 - M_{\chi_{B(x,r)}}) \cdot K\|$ also goes to 0 as $r \to \infty$. Taking the adjoint then proves the second part of the proposition. ♠

Proposition 4.24 *If $K \in Per(\tilde{\mathcal{K}}_k)$, then $\|K \cdot (1 - P_r)\| \to 0$ as $r \to \infty$, and $\|(1 - P_r) \cdot K\| \to 0$ as $r \to \infty$.*

Proof Suppose $K \in Per(\tilde{\mathcal{K}}_k)$. If $x \in \tilde{M}$ and $r > 0$, let

$$\omega_r(x) = \|(1 - P_r)(x) \cdot K(x)\|.$$

By Proposition 4.23, we have that $\omega_r(x)$ decreases to 0 as $r \to \infty$ for every x in $\tilde{M}$. Since K is periodic, then ω_r is also a periodic function on $\tilde{M}$. (That is, it is periodic with respect to the Γ action.) The function ω_r is also clearly continuous on $\tilde{M}$. Since M is compact, it follows that $\omega_r(x)$ decreases to 0 uniformly in x as $r \to \infty$. Hence, $\omega_r \to 0$ in the ∞ norm, as $r \to \infty$. But, of course,

$$\|(1 - P_r) \cdot K\| = \|\omega_r\|_\infty.$$

Thus, $\|(1 - P_r) \cdot K\| \to 0$ as $r \to \infty$. By taking adjoints, we also get that $\|K \cdot (1 - P_r)\| \to 0$ as $r \to \infty$. ♠

Corollary 4.25 *If $K \in Per(\tilde{\mathcal{K}}_k)$, then K has property $*$ and $l(K) = 0$.*

Proof: Suppose $K \in Per(\tilde{\mathcal{K}}_k)$. Then, for every $N > 0$, $\|K \cdot M_{\hat{p}} \cdot (1 - P_N)\| = \|K \cdot (1 - P_N) \cdot M_{\hat{p}}\| \leq \|K \cdot (1 - P_N)\|$, which by Proposition 4.24, goes to 0 as $N \to \infty$. Hence, $\|(K - 0) \cdot M_{\hat{p}} \cdot (1 - P_N)\| \to 0$ as $N \to \infty$. Thus, K has property $*$ and $l(K) = 0$. ♠

Corollary 4.26 *If $K \in Per(\tilde{\mathcal{K}}_k)$, then $\| \, [K, M_{\hat{p}}] \cdot (1 - P_N) \, \| \to 0$ as $N \to \infty$.*

Proof: Suppose $K \in Per(\tilde{\mathcal{K}}_k)$. Then

$$\begin{aligned}\| \, [K, M_{\hat{p}}] \cdot (1 - P_N) \, \| &\leq \|K \cdot M_{\hat{p}} \cdot (1 - P_N)\| + \|M_{\hat{p}} \cdot K \cdot (1 - P_N)\| \\ &= \|K \cdot (1 - P_N) \cdot M_{\hat{p}}\| + \|M_{\hat{p}} \cdot K \cdot (1 - P_N)\| \\ &\leq 2\|K \cdot (1 - P_N)\|\end{aligned}$$

which goes to 0 as $N \to \infty$, by Proposition 4.24. Thus, $\| \, [K, M_{\hat{p}}] \cdot (1 - P_N) \, \| \to 0$ as $N \to \infty$. ♠

Proposition 4.27 *If $a \in \mathcal{A}_{21}(M)$, then the operator a, considered as an element of $Per(\tilde{\mathcal{B}}_k)$, has property $*$, and $l(a)$ is the operator $a \in \mathcal{A}_{21}(M)$. We can therefore write $l(a) = a$. More precisely, the element $\Upsilon_k(a) \in Per(\tilde{\mathcal{B}}_k)$ has property $*$, and $l(\Upsilon_k(a)) = a$.*

Proof: We have $(a - a) \cdot M_{\hat{p}} \cdot (1 - P_N) = 0$ for all $N > 0$. Therefore, a has property $*$ and $l(a) = a$. ♠

Lemma 4.28 *If $f \in Flip$, then $\| \, [f(\tilde{D}), M_{\hat{p}}] \cdot (1 - P_N) \, \| \to 0$ as $N \to \infty$, where $f(\tilde{D})$, here, is considered as an element of $Per(\tilde{\mathcal{B}}_k)$.*

Proof: By Proposition 2.1.13, the collection $Flip_d$ of $Flip$ functions with compactly supported Fourier transforms forms a dense subspace of $Flip$ in the $\| \cdot \|_\infty$ norm. Thus, $\{f(\tilde{D}) \in Per(\tilde{\mathcal{B}}_k) : f \in Flip_d\}$ is a dense subspace of $\{f(\tilde{D}) \in Per(\tilde{\mathcal{B}}_k) : f \in Flip\}$. So, it suffices to prove Lemma 4.28 only for the case where $f \in Flip_d$.

So, take such an $f \in Flip_d$ whose Fourier transform has support in $[-r, r]$ for some $r > 0$, and let $\epsilon > 0$. By Corollary 2.10.7, $\tilde{M}$ can be covered by a finite number, say q, of r-strings. Let

$$\epsilon_1 = \frac{\epsilon}{2q(1+\|f\|_\infty)}.$$

By Proposition 4.11, there exists $N_1 > 0$ such that if $K \subset \tilde{M}$ has $diam(K) \leq 4r$ and if $x \in \tilde{M}$ and $d(x, K) > N_1$, then

$$\mathrm{Var}_{\exp_x^{-1}(K)}(\hat{p}) \leq \epsilon_1. \tag{4.29}$$

Now, let

$$N_0 = N_1 + 3r$$

and take any $N > N_0$. Let

$$A = [f(\tilde{D}), M_{\hat{p}}] \cdot (1 - P_N).$$

To prove Lemma 4.28, it suffices to prove that $\|A\| \leq \epsilon$, and to prove this, we have to show that

$$\|A_x\| \leq \epsilon, \quad \forall x \in \tilde{M}. \tag{4.30}$$

So, fix $x \in \tilde{M}$. Note that $A_x \in B(L^2(\tilde{\Delta})^k)$ is the operator

$$\begin{aligned} A_x &= \exp_{x*}(A(x)) \\ &= [\exp_{x*}(f(\tilde{D})(x)), \exp_{x*}(M_{\hat{p}}(x))] \cdot (1 - \exp_{x*}(P_N(x))) \\ &= [f(\tilde{D})_k, M_{\hat{p}\circ\exp_x^{-1}}] \cdot (1 - M_{\chi_{B(x,N)}}). \end{aligned} \tag{4.31}$$

Since the Fourier transform of f has support in $[-r, r]$, then, by Corollary 2.3.5, $f(\tilde{D}) \in B(L^2(\tilde{\Delta}))$ is an r-local operator. Thus, $f(\tilde{D})_k \in B(L^2(\tilde{\Delta})^k)$ is also an r-local operator. Thus, from the expression (4.31) for A_x, we see that A_x is the sum of terms involving the product of the r-local operator $f(\tilde{D})_k$ with (0-local) multiplication operators. Hence, $A_x \in B(L^2(\tilde{\Delta})^k)$ is an r-local operator.

Now, suppose we were able to show that

$$\|A_x|_B\| \leq \frac{\epsilon}{q}, \quad \forall \text{ ball } B \subset \tilde{M} \text{ of radius } r. \tag{4.32}$$

Then, since A_x is r-local, and since $\tilde{M}$ can be covered by q r-strings, we would have, by Proposition 2.10.3, that

$$\|A_x\| \leq \frac{\epsilon}{q} \cdot q = \epsilon,$$

which would give (4.30). So, the proof of Lemma 4.28 reduces to showing (4.32).

So, take any ball $B = B(y, r) \subset \tilde{M}$ of radius r and centered at $y \in \tilde{M}$, and take any $u \in L^2(\tilde{\Delta})^k$ with support in B. Let

$$P_{x,N} = M_{\chi_{B(x,N)}} \cdot 1_k$$

and let

$$u_1 = (1 - P_{x,N})u.$$

Then

$$\begin{aligned}\|A_x u\| &= \| \, [f(\tilde{D})_k, M_{\hat{p}\circ\exp_x^{-1}}] \cdot (1 - P_{x,N})u \, \| \\ &= \| \, [f(\tilde{D})_k, M_{\hat{p}\circ\exp_x^{-1}}]u_1 \, \|.\end{aligned}$$

Note that $f(\tilde{D})_k \cdot M_{\hat{p}(\exp_x^{-1}(y))} = M_{\hat{p}(\exp_x^{-1}(y))} \cdot f(\tilde{D})_k$ since the matrix $M_{\hat{p}(\exp_x^{-1}(y))}$ is constant, and the operator $f(\tilde{D})_k$ is diagonal with $f(\tilde{D})$ on the diagonal entries. Thus, we have that

$$\begin{aligned}\|A_x u\| &= \| \Big(f(\tilde{D})_k \cdot M_{\hat{p}\circ\exp_x^{-1}}(u_1) - f(\tilde{D})_k \cdot M_{\hat{p}(\exp_x^{-1}(y))}(u_1) \Big) \\ &\quad + \Big(M_{\hat{p}(\exp_x^{-1}(y))} \cdot f(\tilde{D})_k(u_1) - M_{\hat{p}\circ\exp_x^{-1}} \cdot f(\tilde{D})_k(u_1) \Big) \| \\ &\leq \|f(\tilde{D})_k \cdot M_{(\hat{p}\circ\exp_x^{-1}) - \hat{p}(\exp_x^{-1}(y))}(u_1)\| \\ &\quad + \|M_{\hat{p}(\exp_x^{-1}(y)) - \hat{p}\circ\exp_x^{-1}} \cdot f(\tilde{D})_k(u_1)\|.\end{aligned}$$

Now, since u_1 has support in $B = B(y,r) \subset B(y,2r)$, and, since $f(\tilde{D})_k$ is r-local, then $f(\tilde{D})_k(u_1)$ has support in $B(\text{supp}(u_1),r) \subset B(B(y,r),r) = B(y,2r)$. Thus,

$$\begin{aligned}
\|A_x u\| &\leq \|f(\tilde{D})_k\| \cdot \text{Var}_{B(y,2r)}(\hat{p} \circ \exp_x{}^{-1}) \cdot \|u_1\| \\
&\quad + \text{Var}_{B(y,2r)}(\hat{p} \circ \exp_x{}^{-1}) \cdot \|f(\tilde{D})_k\| \cdot \|u_1\| \\
&\leq 2\|f\|_\infty \cdot \text{Var}_{B(y,2r)}(\hat{p} \circ \exp_x{}^{-1}) \cdot \|u_1\| \\
&= 2\text{Var}_{\exp_x{}^{-1}(B(y,2r))}(\hat{p}) \cdot \|f\|_\infty \cdot \|u_1\|.
\end{aligned}$$

That is,

$$\|A_x u\| \leq 2\text{Var}_{\exp_x{}^{-1}(B(y,2r))}(\hat{p}) \cdot \|f\|_\infty \cdot \|u_1\|. \tag{4.33}$$

Now, if B is such that $B \subset B(x,N)$, then u would have support in $B(x,N)$, which would imply that $P_{x,N}u = u$, and, therefore, that $u_1 = (1 - P_{x,N})u = 0$, which would then give us that $A_x u = 0$. So, in this case, we would clearly have that $\|A_x u\| \leq \frac{\epsilon}{q}\|u\|$.

In the other case, where $B \backslash B(x,N) \neq \phi$, there would exist a point z outside $B(x,N)$ such that z belongs to $B = B(y,r)$. For any $a \in B(y,2r)$, we would then have $d(z,a) \leq d(z,y) + d(y,a) < r + 2r = 3r$, which would imply that $d(x,a) \geq d(x,z) - d(z,a) > N - 3r > N_0 - 3r = N_1 + 3r - 3r = N_1$. So, $B(y,2r)$ would be a set of diameter $4r$ such that $d(B(y,2r),x) > N_1$. By (4.29), we would then have that

$$\text{Var}_{\exp_x{}^{-1}(B(y,2r))}(\hat{p}) \leq \epsilon_1,$$

which, together with (4.33), implies that

$$\|A_x u\| \leq 2\epsilon_1 \|f\|_\infty \cdot \|u_1\|$$

$$\leq 2\frac{\epsilon}{2q(1+\|f\|_\infty)} \cdot \|f\|_\infty \cdot \|u\|$$
$$\leq \frac{\epsilon}{q} \cdot \|u\|.$$

Thus, $\|A_x u\| \leq \frac{\epsilon}{q} \cdot \|u\|$ whether or not $B \subset B(x, N)$. Hence, $\|A_x|_B\| \leq \frac{\epsilon}{q}$. which proves (4.32) and completes the proof of Lemma 4.28. ♠

Lemma 4.34 *If $a \in \mathcal{A}_{21}(M)$ is r-local, and if we regard a as an element of $Per(\tilde{\mathcal{B}}_k)$, then, for every $N > r$, we have that $P_{N-r} \cdot a \cdot (1 - P_N) = 0$.*

Proof: Take $x \in \tilde{M}$. We need to show that $(P_{N-r} \cdot a \cdot (1 - P_N))_x = 0$. That is, we need to show that $M_{\chi_{B(x,N-r)}} \cdot a_k \cdot (1 - M_{\chi_{B(x,N)}}) = 0$. So, take $u \in L^2(\tilde{\Delta})^k$. Then $(1 - M_{\chi_{B(x,N)}})(u)$ has support in $\tilde{M} \backslash B(x, N)$. Since a is r-local, then so is a_k. Hence, $a_k((1-M_{\chi_{B(x,N)}})(u))$ has support in $\tilde{M}\backslash B(x, N-r)$, which implies that $M_{\chi_{B(x,N-r)}}(a_k((1 - M_{\chi_{B(x,N)}})(u))) = 0$. That is, $M_{\chi_{B(x,N-r)}} \cdot a_k \cdot (1 - M_{\chi_{B(x,N)}}) = 0$. ♠

Proposition 4.35 *The set of all $a \in BC(\tilde{\mathcal{B}}_k)$ such that, for every $N, \epsilon > 0$, there exists $R > N$ such that*

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon,$$

is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$.

Proof: Let $\mathcal{A}$ denote this set. It is not difficult to show that $\mathcal{A}$ is a closed vector subspace of $BC(\tilde{\mathcal{B}}_k)$. Suppose, now, that a and b are two elements of $\mathcal{A}$. We want to show that ab also belongs to $\mathcal{A}$.

Take $N > 0$ and $\epsilon > 0$. Since a belongs to $\mathcal{A}$, there exists $R > 0$ such that

$$\|P_N \cdot a \cdot (1 - P_R)\| < \frac{\epsilon}{2(1 + \|b\|)},$$

and, since b belongs to $\mathcal{A}$, there exists $S > R$ such that

$$\|P_R \cdot b \cdot (1 - P_S)\| < \frac{\epsilon}{2(1 + \|a\|)}.$$

It follows that

$$\begin{aligned}
\|P_N \cdot ab \cdot (1 - P_S)\| &= \|\,(P_N \cdot a \cdot (1 - P_R)) \cdot b \cdot (1 - P_S) \\
&\quad + (P_N \cdot a) \cdot (P_R \cdot b \cdot (1 - P_S))\,\| \\
&\leq \|P_N \cdot a \cdot (1 - P_R)\| \cdot \|b(1 - P_S)\| \\
&\quad + \|P_N a\| \cdot \|P_R \cdot b \cdot (1 - P_S)\| \\
&\leq \frac{\epsilon}{2(1 + \|b\|)}\|b\| + \|a\|\frac{\epsilon}{2(1 + \|a\|)} \\
&\leq \epsilon.
\end{aligned}$$

Hence, ab belongs to $\mathcal{A}$. Therefore, $\mathcal{A}$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$. ♠

Lemma 4.36 *Suppose $a \in \mathcal{A}_{21}(M)$, $N > 0$, and $\epsilon > 0$. Then, there exists a number $R > N$ such that, if we regard a as an element of $Per(\tilde{\mathcal{B}}_k)$, then*

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon.$$

Note that this is the same as saying that $a \cdot (1 - P_R)$ is ϵ-close to $(1 - P_N) \cdot a \cdot (1 - P_R)$.

Proof: Let $\mathcal{A}$ denote the collection of all $a \in BC(\tilde{\mathcal{B}}_k)$ such that, for every $N, \epsilon > 0$, there exists $R > N$ such that

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon.$$

We want to show that $\Upsilon_k(\mathcal{A}_{21}(M)) \subset \mathcal{A}$.

By Proposition 4.35, $\mathcal{A}$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$. Now, suppose $\varphi \in Per(M)$ and $N > 0$. By Lemma 4.19, $\Upsilon_k(M_\varphi) \in Per(\tilde{\mathcal{B}}_k)$ commutes with P_N. Thus, for every $R > N$, we have that $P_N \cdot \Upsilon_k(M_\varphi) \cdot (1 - P_R) = \Upsilon_k(M_\varphi) \cdot P_N(1 - P_R) = 0$, since $R > N$. Therefore, $\Upsilon_k(M_\varphi)$ belongs to $\mathcal{A}$ for every $\varphi \in Per(M)$.

Now, let $f \in Flip_d$ and let $r > 0$ be such that $\hat{f}$ has support in the interval $[-r, r]$. Take any $N > r$. By Corollary 2.3.5, $f(\tilde{D}) \in B(L^2(\tilde{\Delta}))$ is an r-local operator. Thus, by Lemma 4.34, $\Upsilon_k(f(\tilde{D})) \in Per(\tilde{\mathcal{B}}_k)$ has the property that $P_{N-r} \cdot \Upsilon_k(f(\tilde{D})) \cdot (1 - P_N) = 0$. Therefore, $\Upsilon_k(f(\tilde{D}))$ belongs to $\mathcal{A}$ for every $f \in Flip_d$. By Proposition 2.1.13, $Flip_d$ is dense in $Flip$. Together with the fact that $\mathcal{A}$ is closed, this implies that $\Upsilon_k(f(\tilde{D}))$ belongs to $\mathcal{A}$ for all f in $Flip$.

Therefore, $\mathcal{A}$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$ containing all the $\Upsilon_k(M_\varphi)$ and all the $\Upsilon_k(f(\tilde{D}))$ such that $\varphi \in Per(M)$ and $f \in Flip$. By Remark 2.11.1, $\mathcal{A}_{21}(M)$ is the closed subalgebra of $B(L^2(\tilde{\Delta}))$ generated by all such M_φ and $f(\tilde{D})$. Hence, $\Upsilon_k(\mathcal{A}_{21}(M)) \subset \mathcal{A}$. ♠

Definition 4.37 *For convenience, we let*

$$\mathcal{A}_{11}(p)' = \mathcal{D}_{\widetilde{SC}(p)}{}' + Per(\tilde{\mathcal{K}}_k)$$

and

$$\mathcal{A}_{01}(M)'_k = \mathcal{D}_{M_k(\tilde{C}_0(TM))}{}' + Per(\tilde{\mathcal{K}}_k).$$

Remark 4.38 *Since $\mathcal{D}_{M_k(\tilde{C}_0(TM))}{}'$ is a C^*-subalgebra of $\mathcal{D}_{\widetilde{SC}(p)}{}'$, then $\mathcal{A}_{01}(M)'_k$ is a C^*-subalgebra of $\mathcal{A}_{11}(p)'$.*

Remark 4.39 *Recall (Remark 3.14) that* $\mathcal{D}_{\widetilde{SC}(p)}{}'$ *is a closed subalgebra of* $Per(\tilde{\mathcal{B}}_k)$ *generated by* A*, the set of all* M_φ *in* $Per(\tilde{\mathcal{B}}_k)$ *such that* $\varphi \in \widetilde{SC}(p)$*, and* B*, the set of all* $f(\tilde{D}) \in Per(\tilde{\mathcal{B}}_k)$ *such that* $f \in Flip$*.*

It follows that $\mathcal{A}_{11}(p)' = \mathcal{D}_{\widetilde{SC}(p)}{}' + Per(\tilde{\mathcal{K}}_k)$ *is the closed subalgebra of* $Per(\tilde{\mathcal{B}}_k)$ *generated by the sets* A, B*, and* $Per(\tilde{\mathcal{K}}_k)$*.*

Remark 4.40 *By definition (Definition 3.13), the Dirac algebra* $\mathcal{D}_{\widetilde{SC}(p)}$ *is the closed ideal of* $\mathcal{D}_{\widetilde{SC}(p)}{}'$ *generated by the set of all* M_φ *in* $Per(\tilde{\mathcal{B}}_k)$ *such that* φ *belongs to* $\widetilde{SC}(p)$*.*

Hence, since $Per(\tilde{\mathcal{K}}_k)$ *is an ideal of* $Per(\tilde{\mathcal{B}}_k)$*,* $\mathcal{A}_{11}(p) \stackrel{\text{def}}{=} \mathcal{D}_{\widetilde{SC}(p)} + Per(\tilde{\mathcal{K}}_k)$ *is the closed ideal of* $\mathcal{A}_{11}(p)' \stackrel{\text{def}}{=} \mathcal{D}_{\widetilde{SC}(p)}{}' + Per(\tilde{\mathcal{K}}_k)$ *generated by* $Per(\tilde{\mathcal{K}}_k)$ *and the set of all* M_φ *in* $Per(\tilde{\mathcal{B}}_k)$ *such that* $\varphi \in \widetilde{SC}(p)$*.*

Similarly, $\mathcal{D}_{M_k(\tilde{C}_0(TM))}$ *is by definition the closed ideal of* $\mathcal{D}_{M_k(\tilde{C}_0(TM))}{}'$ *generated by the set of all* M_φ *in* $Per(\tilde{\mathcal{B}}_k)$ *such that* φ *is in* $M_k(\tilde{C}_0(TM))$*, and* $\mathcal{A}_{01}(M)_k$ *is the closed ideal of* $\mathcal{A}_{01}(M)_k'$ *generated by* $Per(\tilde{\mathcal{K}}_k)$ *and the set of all* M_φ *in* $Per(\tilde{\mathcal{B}}_k)$ *such that* $\varphi \in M_k(\tilde{C}_0(TM))$*.*

Lemma 4.41 *Suppose* $a \in \mathcal{A}_{11}(p)'$*,* $N > 0$*, and* $\epsilon > 0$*. Then there exists a number* $R > N$ *such that*

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon.$$

This is the same as saying that $a \cdot (1 - P_R)$ *is* ϵ*-close to* $(1 - P_N) \cdot a \cdot (1 - P_R)$*.*

Proof: Let $\mathcal{A}$ denote the collection of all $a \in BC(\tilde{\mathcal{B}}_k)$ such that, for every $N, \epsilon > 0$, there exists $R > N$ such that

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon.$$

We want to show that $\mathcal{A}_{11}(p)' \subset \mathcal{A}$.

By Proposition 4.35, $\mathcal{A}$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$. Suppose $\varphi \in \widetilde{SC}(p), f \in Flip, K \in Per(\tilde{\mathcal{K}}_k)$, and $N > 0$. By Lemma 4.19, $M_\varphi \in \mathcal{A}_{11}(p)'$ commutes with P_N, and so, for all $R > N$, $P_N \cdot M_\varphi \cdot (1 - P_R) = M_\varphi \cdot P_N \cdot (1 - P_R) = 0$. Therefore, M_φ belongs to $\mathcal{A}$. By Lemma 4.36, $f(\tilde{D}) \in \mathcal{A}_{11}(p)'$ belongs to $\mathcal{A}$. By Proposition 4.24, $\|K \cdot (1 - P_R)\| \to 0$ as $R \to \infty$. Since $\|P_N \cdot K \cdot (1-P_R)\| \leq \|K \cdot (1-P_R)\|$, it follows that $\|P_N \cdot K \cdot (1-P_R)\|$ also goes to 0 as $R \to \infty$. Thus, K also belongs to $\mathcal{A}$. Therefore, $\mathcal{A}$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$ containing all the $M_\varphi, f(\tilde{D})$, and K such that $\varphi \in \widetilde{SC}(p)$, $f \in Flip$, and $K \in Per(\tilde{\mathcal{K}}_k)$. Thus, $\mathcal{A}_{11}(p)'$, which, by Remark 4.39, is the closed subalgebra of $Per(\tilde{\mathcal{B}}_k)$ generated by such elements, is contained in $\mathcal{A}$. ♠

Corollary 4.42 *Let $\mathcal{A}' \subset Per(\tilde{\mathcal{B}}_k)$ denote the C^*-algebra generated by the C^*-subalgebras $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$ of $Per(\tilde{\mathcal{B}}_k)$. Then, every $a \in \mathcal{A}'$ has the property that, for every $N > 0$, there exists a number $R > N$ such that*

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon.$$

This is the same as saying that $a \cdot (1 - P_R)$ is ϵ-close to $(1 - P_N) \cdot a \cdot (1 - P_R)$.

Proof: Since both $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$ are C*-algebras, then $\mathcal{A}'$ is the same as the closed algebra generated by $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$.

Now, let $\mathcal{A}$ denote the collection of all $a \in BC(\tilde{\mathcal{B}}_k)$ such that, for every $N, \epsilon > 0$, there exists $R > N$ such that

$$\|P_N \cdot a \cdot (1 - P_R)\| < \epsilon.$$

We want to show that $\mathcal{A}' \subset \mathcal{A}$. By Proposition 4.35, the algebra $\mathcal{A}$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$. By Lemmas 4.36 and 4.41, both $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$ are contained in $\mathcal{A}$. But $\mathcal{A}'$ is the closed algebra generated by $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$. So, $\mathcal{A}'$ is also contained in $\mathcal{A}$. ♠

Lemma 4.43 *Suppose a belongs to the C^*-algebra generated by $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$. Then $\| \, [a, M_{\hat{p}}] \cdot (1 - P_N) \, \| \to 0$ as $N \to \infty$.*

Proof: Let $\mathcal{A}' \subset Per(\tilde{\mathcal{B}}_k)$ denote the C*-algebra generated by the subalgebras $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$ of $Per(\tilde{\mathcal{B}}_k)$. Let $\mathcal{A}$ denote the set of all $a \in \mathcal{A}'$ such that $\| \, [a, M_{\hat{p}}] \cdot (1 - P_N) \, \| \to 0$ as $N \to \infty$. We want to show that $\mathcal{A} = \mathcal{A}'$.

It is not hard to see that $\mathcal{A}$ is a closed vector subspace of $\mathcal{A}'$. Suppose, now, that a and b belong to $\mathcal{A}$, and that $\epsilon > 0$. Since $a, b \in \mathcal{A}$, there exists $R > 0$ such that

$$a \cdot M_{\hat{p}} \cdot (1 - P_R) \sim_{\frac{\epsilon}{3(1+\|b\|)}} M_{\hat{p}} \cdot a \cdot (1 - P_R)$$

and

$$b \cdot M_{\hat{p}} \cdot (1 - P_R) \sim_{\frac{\epsilon}{3(1+\|a\|)}} M_{\hat{p}} \cdot b \cdot (1 - P_R).$$

Since $b \in \mathcal{A}'$, then by Corollary 4.42, there exists $N > R$ such that

$$b \cdot (1 - P_N) \sim_{\frac{\epsilon}{6(1+\|a\|)}} (1 - P_R) \cdot b \cdot (1 - P_N).$$

It follows that

$$\begin{aligned} ab \cdot M_{\hat{p}} \cdot (1 - P_N) &= ab \cdot M_{\hat{p}} \cdot (1 - P_R)(1 - P_N) \\ &\sim_{\|a\| \frac{\epsilon}{3(1+\|a\|)}} a \cdot M_{\hat{p}} \cdot b \cdot (1 - P_R)(1 - P_N) \\ &= a \cdot M_{\hat{p}} \cdot b \cdot (1 - P_N) \end{aligned}$$

$$\begin{aligned}
&\sim_{\|a\|\frac{\epsilon}{6(1+\|a\|)}} \quad a \cdot M_{\hat{p}} \cdot (1-P_R) \cdot b \cdot (1-P_N) \\
&\sim_{\|b\|\frac{\epsilon}{3(1+\|b\|)}} \quad M_{\hat{p}} \cdot a \cdot (1-P_R) \cdot b \cdot (1-P_N) \\
&\sim_{\|a\|\frac{\epsilon}{6(1+\|a\|)}} \quad M_{\hat{p}} \cdot ab \cdot (1-P_N).
\end{aligned}$$

Since

$$\begin{aligned}
&\|a\|\frac{\epsilon}{3(1+\|a\|)} + \|a\|\frac{\epsilon}{6(1+\|a\|)} + \|b\|\frac{\epsilon}{3(1+\|b\|)} + \|a\|\frac{\epsilon}{6(1+\|a\|)} \\
&\quad \leq \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon,
\end{aligned}$$

it follows that $ab \cdot M_{\hat{p}} \cdot (1-P_N)$ is ϵ-close to $M_{\hat{p}} \cdot ab \cdot (1-P_N)$. That is, $\| \, [ab, M_{\hat{p}}] \cdot (1-P_N) \, \| < \epsilon$. Therefore, $ab \in \mathcal{A}$. Hence, $\mathcal{A}$ is a closed subalgebra of $\mathcal{A}'$.

Now, $\mathcal{A}'$ is, by definition, the C*-algebra generated by $\Upsilon_k(\mathcal{A}_{21}(M))$ and $\mathcal{A}_{11}(p)'$. Thus, $\mathcal{A}'$ is the C*-algebra generated by the C*-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and $Per(\tilde{\mathcal{K}}_k)$, where $\mathcal{A}_1$ is the set of all $f(\tilde{D}) \in Per(\tilde{\mathcal{B}}_k)$ such that $f \in Flip$, $\mathcal{A}_2$ is the set of all $M_\phi \in Per(\tilde{\mathcal{B}}_k)$ such that $\phi \in Per(M)$, and $\mathcal{A}_3$ is the set of all $M_\varphi \in Per(\tilde{\mathcal{B}}_k)$ such that $\varphi \in \widetilde{SC}(p)$. Since these four generating subsets of $\mathcal{A}'$ are all C*-algebras, then $\mathcal{A}'$ is actually the closed subalgebra of $Per(\tilde{\mathcal{B}}_k)$ generated by these four sets. So, to show that $\mathcal{A} = \mathcal{A}'$, it is enough to show that the four elements, $M_\phi, f(\tilde{D}), K$, and $M_\varphi \in Per(\tilde{\mathcal{B}}_k)$ all belong to $\mathcal{A}$ for $\phi \in Per(M), f \in Flip, K \in Per(\tilde{\mathcal{K}}_k)$, and $\varphi \in \widetilde{SC}(p)$.

So, take such ϕ, f, K, and φ. By Lemma 4.19, $M_\phi \in Per(\tilde{\mathcal{B}}_k)$ commutes with $M_{\hat{p}}$. Hence, $M_\phi \in \mathcal{A}$. By Lemmas 4.21 and 4.28, and Corollary 4.26, we have that M_φ, $f(\tilde{D})$, and K in $Per(\tilde{\mathcal{B}}_k)$ also belong to $\mathcal{A}$. Therefore, Proposition 4.43 is true. ♠

Proposition 4.44 *The set of all elements of $BC(\tilde{\mathcal{B}}_k)$ with property $*$ is a closed subalgebra of $BC(\tilde{\mathcal{B}}_k)$, and the map $a \mapsto l(a)$ is a bounded algebra homomorphism from this algebra to $\mathcal{A}_{21}(M)$ with $\|l(a)\| \leq \|a\|$ for every a in this algebra.*

Proof: Let $\mathcal{A}$ denote the set of all elements of $BC(\tilde{\mathcal{B}}_k)$ with property $*$.

Suppose $a, b \in \mathcal{A}$ and $\lambda \in \mathbf{C}$. Then, $\|((a+b)-(l(a)+l(b)))\cdot M_{\hat{p}}\cdot(1-P_N)\| \leq \|(a-l(a))\cdot M_{\hat{p}}\cdot(1-P_N)\| + \|(b-l(b))\cdot M_{\hat{p}}\cdot(1-P_N)\| \to 0$ as $N \to \infty$. Hence, $a+b$ belongs to $\mathcal{A}$ and $l(a+b) = l(a)+l(b)$.

Similarly, $\|(\lambda a - \lambda l(a))\cdot M_{\hat{p}}\cdot(1-P_N)\| = |\lambda|\cdot\|(a-l(a))\cdot M_{\hat{p}}\cdot(1-P_N)\| \to 0$ as $N \to \infty$. Thus, $\lambda a \in \mathcal{A}$ and $l(\lambda a) = \lambda l(a)$.

Now, suppose $\epsilon > 0$. Let

$$\epsilon' = \frac{\epsilon}{6(1+\|a\|+\|b\|)}.$$

Since $a \in \mathcal{A}$, there exists $R > 0$ such that

$$a \cdot M_{\hat{p}} \cdot (1-P_R) \sim_{\epsilon'} l(a) \cdot M_{\hat{p}} \cdot (1-P_R).$$

By Lemmas 4.36 and 4.43, and the fact that b belongs to $\mathcal{A}$, there exists $N > R$ such that

$$b \cdot M_{\hat{p}} \cdot (1-P_N) \sim_{\epsilon'} l(b) \cdot M_{\hat{p}} \cdot (1-P_N),$$

$$l(b) \cdot (1-P_N) \sim_{\epsilon'} (1-P_R) \cdot l(b) \cdot (1-P_N),$$

and

$$M_{\hat{p}} \cdot l(b) \cdot (1-P_N) \sim_{\epsilon'} l(b) \cdot M_{\hat{p}} \cdot (1-P_N).$$

It follows that

$$\begin{aligned}
ab \cdot M_{\hat{p}} \cdot (1-P_N) &\sim_{\|a\|\epsilon'} a \cdot l(b) \cdot M_{\hat{p}} \cdot (1-P_N) \\
&\sim_{\|a\|\epsilon'} a \cdot M_{\hat{p}} \cdot l(b) \cdot (1-P_N) \\
&\sim_{\|a\|\epsilon'} a \cdot M_{\hat{p}} \cdot (1-P_R) \cdot l(b) \cdot (1-P_N) \\
&\sim_{\|l(b)\|\epsilon'} l(a) \cdot M_{\hat{p}} \cdot (1-P_R) \cdot l(b) \cdot (1-P_N) \\
&\sim_{\|l(a)\|\epsilon'} l(a) \cdot M_{\hat{p}} \cdot l(b) \cdot (1-P_N) \\
&\sim_{\|l(a)\|\epsilon'} l(a) \cdot l(b) \cdot M_{\hat{p}} \cdot (1-P_N).
\end{aligned}$$

Now, both $\|a\|\epsilon'$ and $\|b\|\epsilon'$ are $\leq \frac{\epsilon}{6}$. From Lemma 4.16, we have that $\|l(a)\|\epsilon' \leq \|a\|\epsilon' \leq \frac{\epsilon}{6}$. Similarly, we have that $\|l(b)\|\epsilon' \leq \frac{\epsilon}{6}$. Therefore, $ab \cdot M_{\hat{p}} \cdot (1-P_N) \sim_\epsilon l(a) \cdot l(b) \cdot M_{\hat{p}} \cdot (1-P_N)$. This shows that $ab \in \mathcal{A}$ and $l(ab) = l(a)l(b)$.

Thus, we have shown that $\mathcal{A}$ is a subalgebra of $BC(\tilde{\mathcal{B}}_k)$ and that the map $l : \mathcal{A} \to \mathcal{A}_{21}(M)$ is an algebra homomorphism. We already know from Lemma 4.16 that $\|l(a)\| \leq \|a\|$ for every $a \in \mathcal{A}$. It remains to show that $\mathcal{A}$ is closed. So, take a in $\bar{\mathcal{A}}$. Let a_n be a sequence of elements of $\mathcal{A}$ such that $a_n \to a$ as $n \to \infty$. By Lemma 4.16, the map l from $BC(\tilde{\mathcal{B}}_k)$ to $\mathcal{A}_{21}(M)$ is norm reducing. It follows that the sequence $l(a_n)$ is Cauchy, and therefore converges to some $b \in \mathcal{A}_{21}(M)$. Thus,

$$\begin{aligned}
\|(a-b) \cdot M_{\hat{p}} \cdot (1-P_N)\| &\leq \|(a-a_n) \cdot M_{\hat{p}} \cdot (1-P_N)\| \\
&\quad + \|((a_n - l(a_n)) \cdot M_{\hat{p}} \cdot (1-P_N)\| \\
&\quad + \|(l(a_n) - b) \cdot M_{\hat{p}} \cdot (1-P_N)\| \\
&\leq \|a - a_n\| \\
&\quad + \|((a_n - l(a_n)) \cdot M_{\hat{p}} \cdot (1-P_N)\|
\end{aligned}$$

$$+\|l(a_n) - b\|.$$

Since the term $\|((a_n - l(a_n)) \cdot M_{\hat{p}} \cdot (1 - P_N)\|$ goes to 0 as $N \to \infty$ for every n, it follows that

$$\limsup_{N\to\infty} \|(a - b) \cdot M_{\hat{p}} \cdot (1 - P_N)\| \leq \|a - a_n\| + \|b - l(a_n)\|$$

for every n. But, the right hand term goes to 0 as $n \to \infty$. Hence,

$$\lim_{N\to\infty} \|(a - b) \cdot M_{\hat{p}} \cdot (1 - P_N)\| = 0.$$

Therefore, $a \in \mathcal{A}$ and $l(a) = b$. That is, $l(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} l(a_n)$. ♠

Proposition 4.45 *All elements of $\mathcal{A}_{11}(p)'$ have property $*$. Moreover, l is defined on all of $\mathcal{A}_{11}(p)'$, and $l : \mathcal{A}_{11}(p)' \to B(L^2(\tilde{\Delta}))$ is a $*$-homomorphism.*

Proof: Let $\mathcal{A}$ denote the set of all a in $\mathcal{A}_{11}(p)'$ with property $*$. We want to show that $\mathcal{A} = \mathcal{A}_{11}(p)'$. Let $\mathcal{A}_1 = Per(\tilde{\mathcal{K}}_k)$, be the set $\mathcal{A}_2$ of all $f(\tilde{D}) \in \mathcal{A}_{11}(p)'$ such that f is $Flip$, and $\mathcal{A}_3$ be the set of all $M_\varphi \in \mathcal{A}_{11}(p)'$ such that $\varphi \in \widetilde{SC}(p)$. By Remark 4.39, $\mathcal{A}_{11}(p)'$ is the closed algebra generated by the sets $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$. But, by Propositions 4.44, 4.25, 4.27, and 4.20, all three algebras $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$, are contained in $\mathcal{A}$, and $\mathcal{A}$ is a closed algebra. Therefore, $\mathcal{A}_{11}(p)'$, is a subalgebra of $\mathcal{A}$. On the other hand, $\mathcal{A}$ is a subset of $\mathcal{A}_{11}(p)'$. Thus, we have $\mathcal{A} = \mathcal{A}_{11}(p)'$. That is, every element of $\mathcal{A}_{11}(p)'$ has property $*$. From this and Lemma 4.44, we get that the map l is defined on all of $\mathcal{A}_{11}(p)'$ and that $l : \mathcal{A}_{11}(p)' \to B(L^2(\tilde{\Delta}))$ is an algebra homomorphism.

To show that l is a $*$-homomorphism, it suffices to prove

$$l(a^*) = l(a)^*, \tag{4.46}$$

for a equal to either $K, f(\tilde{D})$, or M_φ, where $K \in Per(\tilde{\mathcal{K}}_k)$, $f \in Flip$, and $\varphi \in \widetilde{SC}(p)$. So, take such elements K, $f(\tilde{D})$ and M_φ. Note that $l(K^*) = 0 = 0^* = l(K)^*$, which proves 4.46 for $a = K$. Also, $l(\Upsilon_k(f(\tilde{D}))^*) = l(\Upsilon_k(\bar{f}(\tilde{D}))) = \bar{f}(\tilde{D}) = f(\tilde{D})^* = l(\Upsilon_k(f(\tilde{D})))^*$. Hence, 4.46 is true for $a = \Upsilon_k(f(\tilde{D}))$. Next, we see that $l((M_\varphi)^*) = l(M_{\varphi^*}) = M_{f_{\varphi^*}}$. By Remark 3.1.13 on page 92, we have that $f_{\varphi^*} = \bar{f}_\varphi$. Therefore, $l(M_{\varphi^*}) = M_{\bar{f}_\varphi} = (M_{f_\varphi})^* = l(M_\varphi)^*$. Thus, 4.46 is also true for $a = M_\varphi$. Therefore, l is a $*$-homomorphism. ♠

Proposition 4.47 *The image of the $*$-homomorphism*

$$l : \mathcal{A}_{11}(p)' \to B(L^2(\tilde{\Delta}))$$

is the C^*-algebra $\mathcal{A}_{21}(M)$. So, we have a surjective $*$-homomorphism

$$l : \mathcal{A}_{11}(p)' \to \mathcal{A}_{21}(M).$$

Its restriction

$$l : \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M)$$

is also surjective.

Proof: By Proposition 4.45,

$$l : \mathcal{A}_{11}(p)' \to B(L^2(\tilde{\Delta}))$$

is a $*$-homomorphism. Since $\mathcal{D}_{\widetilde{SC}(p)}{}'$ is the C*-algebra generated by the set of all M_φ and $f(\tilde{D}) \in Per(\tilde{\mathcal{B}}_k)$ such that $\varphi \in \widetilde{SC}(p)$, and $f \in Flip$, then $l(\mathcal{D}_{\widetilde{SC}(p)}{}')$ is the C*-subalgebra of $B(L^2(\tilde{\Delta}))$ generated by the set of all $l(M_\varphi)$, and $l(f(\tilde{D}))$, with M_φ, and $f(\tilde{D})$ as above. By Propositions 4.20, and 4.27, it follows that $l(\mathcal{D}_{\widetilde{SC}(p)}{}')$ is the C*-algebra generated by the set of all $f(\tilde{D})$ and M_{f_φ} in $B(L^2(\tilde{\Delta}))$ such that $f \in Flip$ and $\varphi \in Per(M)$. By Proposition 3.1.14 the map $\varphi \mapsto l(\varphi) = f(\varphi)$ from $\widetilde{SC}(p)$ to Per(M) is onto. Thus, $l(\mathcal{D}_{\widetilde{SC}(p)}{}')$ is the C^*-subalgebra of $B(L^2(\tilde{\Delta})$ generated by the set of all $f(\tilde{D})$ and M_φ in $B(L^2(\tilde{\Delta}))$ such that $f \in Flip$ and $\varphi \in Per(M)$. But, this is exactly the algebra $\mathcal{A}_{21}(M)$. Hence,

$$l(\mathcal{D}_{\widetilde{SC}(p)}{}') = \mathcal{A}_{21}(M).$$

Now, it follows from this, and the fact that $\mathcal{D}_{\widetilde{SC}(p)}$ is the closed ideal of $\mathcal{D}_{\widetilde{SC}(p)}{}'$ generated by the algebra of all M_φ such that φ is in $\widetilde{SC}(p)$, that $l(\mathcal{D}_{\widetilde{SC}(p)})$ is the closed ideal of $\mathcal{A}_{21}(M)$ generated by the algebra of all $l(M_\varphi) = M_{f_\varphi}$ in $B(L^2(\tilde{\Delta}))$ such that $\varphi \in \widetilde{SC}(p)$. But, as just mentioned, the algebra of all M_{f_φ} in $B(L^2(\tilde{\Delta}))$ such that $\varphi \in \widetilde{SC}(p)$, is the same as the set of all of all $M_\varphi \in B(L^2(\tilde{\Delta}))$ such that $\varphi \in Per(M)$. Since $Per(M)$ contains the constant function 1, then this algebra contains the identity operator on $L^2(\tilde{\Delta})$. Thus, $l(\mathcal{D}_{\widetilde{SC}(p)})$ is a closed ideal of $\mathcal{A}_{21}(M)$ containing the identity operator, and therefore equals the whole algebra $\mathcal{A}_{21}(M)$.

That is,

$$l(\mathcal{D}_{\widetilde{SC}(p)}) = \mathcal{A}_{21}(M).$$

Now, since $\mathcal{A}_{11}(p)$ is the C*-algebra generated by the algebras $\mathcal{D}_{\widetilde{SC}(p)}$ and $Per(\tilde{\mathcal{K}}_k)$, and since, by Proposition 4.25, $l(K) = \{0\}$ for all $K \in Per(\tilde{\mathcal{K}}_k)$, then we have $l(\mathcal{A}_{11}(p)) = l(\mathcal{D}_{\widetilde{SC}(p)}) = \mathcal{A}_{21}(M)$. ♠

Proposition 4.48 *We have that*

$$l(\mathcal{A}_{10}(p)') = l(\mathcal{A}_{10}(p)) = \mathcal{A}_{20}(M),$$

which gives a surjective $*$-homomorphism

$$l : \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M).$$

Moreover, the diagram

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ \mathcal{A}_{20}(M) & \xrightarrow{i} & \mathcal{A}_{21}(M) \\ \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l} \\ \mathcal{A}_{10}(p) & \xrightarrow[i]{} & \mathcal{A}_{11}(p) \end{array}$$

commutes, and is exact at every point.

Proof: The proof that $l(\mathcal{A}_{10}(p))' = l(\mathcal{A}_{10}(p)) = \mathcal{A}_{20}(M)$ is similar to the proof of Proposition 4.47. That the diagram commutes is obvious since $l : \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M)$ is the restriction of $l : \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M)$ to $\mathcal{A}_{10}(p)$. That it is exact at every point is the same as saying that the two maps $l : \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M)$ and $l : \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M)$ are onto. We just showed that $l : \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M)$ is onto, and Proposition 4.47 gives us that $l : \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M)$ is onto. The proposition is therefore true. ♠

Proposition 4.49 *For every a in $\mathcal{A}_{01}(M)_k, l(a) = 0$.*

Proof: By Proposition 4.45, l is defined on all of $\mathcal{A}_{01}(M)'_k$ and

$$l : \mathcal{A}_{01}(M)'_k \to \mathcal{A}_{21}(M)$$

is a $*$-homomorphism. Let $\mathcal{I}$ denote the kernel of this map. Then $\mathcal{I}$ is a closed ideal of $\mathcal{A}_{01}(M)'_k$. By Remark 4.40, $\mathcal{A}_{01}(M)_k$ is the closed ideal of $\mathcal{A}_{01}(M)'_k$

generated by $Per(\tilde{\mathcal{K}}_k)$ and the set of all M_φ such that $\varphi \in M_k(\tilde{C}_0(TM))$. By Proposition 4.25, $l(K) = 0$ (*i.e.* $K \in \mathcal{I}$) for every $K \in Per(\tilde{\mathcal{K}}_k)$, and, by Proposition 4.20, for every $\varphi \in M_k(\tilde{C}_0(TM))$, $l(M_\varphi) = M_{f_\varphi} = M_0 = 0$ (i.e. $M_\varphi \in \mathcal{I}$). Hence, $\mathcal{A}_{01}(M)_k$ is a closed ideal of $\mathcal{A}_{01}(M)'_k$ generated by certain elements of the ideal $\mathcal{I}$ of $\mathcal{A}_{01}(M)'_k$. It follows that $\mathcal{A}_{01}(M)_k \subseteq \mathcal{I}$.♠

Proposition 4.50 *The following diagram*

$$\begin{array}{ccc} \mathcal{D}_{Per(M)} & \xrightarrow{q} & Per(M) \oplus Per(M) \\ \uparrow l & & \uparrow l\oplus l \\ \mathcal{A}_{11}(p) & \xrightarrow{q} & \widetilde{SC}(p) \oplus \widetilde{SC}(p) \end{array},$$

which is the same as

$$\begin{array}{ccc} \mathcal{A}_{21}(M) & \xrightarrow{q} & \mathcal{A}_{22}(M) \\ l\uparrow & & \uparrow l\oplus l \\ \mathcal{A}_{11}(p) & \xrightarrow[q]{} & \mathcal{A}_{12}(p) \end{array},$$

commutes.

Proof: Take $a \in \mathcal{A}_{11}(p)$. Suppose

$$q(a) = (\varphi, \eta)$$

where $\varphi, \eta \in \widetilde{SC}(p)$. Take $f \in Flip_l$, $g \in Flip_r$. Then $q(f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta)$ is also equal to (φ, η). By exactness of the extension

$$0 \to \mathcal{A}_{10}(p) \xrightarrow{i} \mathcal{A}_{11}(p) \xrightarrow{q} \widetilde{SC}(p) \oplus \widetilde{SC}(p) \to 0$$

there exists $b \in \mathcal{A}_{10}(p)$ such that $a = (f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta) + b$. By Proposition 4.48, $l(b)$ belongs to $\mathcal{A}_{20}(M)$. By exactness of $0 \to \mathcal{A}_{20}(M) \xrightarrow{i} \mathcal{A}_{21}(M) \xrightarrow{q} \mathcal{A}_{22}(M) \to 0$ it follows that

$$q(l(b)) = 0.$$

Also, by Propositions 4.20 and 4.27, $l(M_\mu) = M_{l(\mu)}$ in $B(L^2(\tilde{\Delta}))$ for every $\mu \in \widetilde{SC}(p)$, and $l(h(\tilde{D})) = h(\tilde{D})$ in $B(L^2(\tilde{\Delta}))$ for every $h \in Flip$. Thus,

$$l(a) = (f(\tilde{D})M_{l(\varphi)} + g(\tilde{D})M_{l(\eta)}) + l(b)$$

in $\mathcal{A}_{21}(M)$ and

$$\begin{aligned} q(l(a)) &= q(f(\tilde{D})M_{l(\varphi)} + g(\tilde{D})M_{l(\eta)}) + q(l(b)) \\ &= (l(\varphi), l(\eta)) + q(l(b)) \\ & \quad (l(\varphi), l(\eta))\text{since } q(l(b)) = 0 \\ &= (l \oplus l)(\varphi, \eta) \\ &= (l \oplus l)(q(a)). \end{aligned}$$

So the above diagram commutes. ♠

Definition 4.51 *If $a \in BC(\tilde{\mathcal{B}}_k)$, we say that a* **has property **** *if there exists b in $\mathcal{A}_{21}(M)$ such that $\|(a - b \cdot M_{\hat{p}}) \cdot (1 - P_N)\| \to 0$ as $N \to \infty$. When this happens, we say that* ***b* has property * * *a***.

Proposition 4.52 *The set of all $a \in BC(\tilde{\mathcal{B}}_k)$ with property **, is a closed, left ideal of the closed algebra of all $a \in BC(\tilde{\mathcal{B}}_k)$ with property *. If a has property **, there is a unique $b \in \mathcal{A}_{21}(M)$ with property * * a, and we have that $l(a) = b$.*

Proof: Let $\mathcal{A}$ be the closed algebra of all $a \in BC(\tilde{\mathcal{B}}_k)$ with property *, and let $\mathcal{I}$ denote the set of all $a \in BC(\tilde{\mathcal{B}}_k)$ with property **. If $a \in \mathcal{I}$, and b has property * * a, then $\|(a - b \cdot M_{\hat{p}}) \cdot (1 - P_N)\| \to 0$ as $N \to \infty$. Therefore, by multiplying on the right by $M_{\hat{p}}$, we have that $\|(a \cdot M_{\hat{p}} - b \cdot M_{\hat{p}}) \cdot (1 - P_N)\| \to 0$ as

$N \to \infty$, which implies that a has property $*$, b has property $*a$, and $l(a) = b$. That is, $a \in \mathcal{A}$ and $l(a) = b$. This, of course, gives uniqueness of b, and shows that $\mathcal{I} \subset \mathcal{A}$.

The proof that $\mathcal{I}$ is a closed subvector space of $\mathcal{A}$ is similar to the proof that $\mathcal{A}$ is a closed vector space in the proof of Proposition 4.44. So, to complete the proof of the proposition, we only need to show that, if $b \in \mathcal{I}$ and $a \in \mathcal{A}$, then ab belongs to $\mathcal{I}$. Take such a and b, and $\epsilon > 0$. Let

$$\epsilon' = \frac{\epsilon}{6(1 + \|a\| \cdot \|b\|)}.$$

Since $a \in \mathcal{A}$, there exists $R > 0$ such that

$$a \cdot M_{\hat{p}} \cdot (1 - P_R) \sim_{\epsilon'} l(a) \cdot M_{\hat{p}} \cdot (1 - P_R).$$

By Lemmas 4.36 and 4.43, and the fact that b belongs to $\mathcal{I}$, there exists $N > R$ such that

$$b \cdot (1 - P_N) \sim_{\epsilon'} l(b) \cdot M_{\hat{p}} \cdot (1 - P_N),$$

$$l(b) \cdot (1 - P_N) \sim_{\epsilon'} (1 - P_R) \cdot l(b) \cdot (1 - P_N),$$

and

$$M_{\hat{p}} \cdot l(b) \cdot (1 - P_N) \sim_{\epsilon'} l(b) \cdot M_{\hat{p}} \cdot (1 - P_N).$$

It follows that

$$\begin{aligned} ab \cdot (1 - P_N) &\sim_{\|a\|\epsilon'} a \cdot l(b) \cdot M_{\hat{p}} \cdot (1 - P_N) \\ &\sim_{\|a\|\epsilon'} a \cdot M_{\hat{p}} \cdot l(b) \cdot (1 - P_N) \\ &\sim_{\|a\|\epsilon'} a \cdot M_{\hat{p}} \cdot (1 - P_R) \cdot l(b) \cdot (1 - P_N) \\ &\sim_{\|l(b)\|\epsilon'} l(a) \cdot M_{\hat{p}} \cdot (1 - P_R) \cdot l(b) \cdot (1 - P_N) \end{aligned}$$

$$\sim_{\|l(a)\|\epsilon'} \quad l(a) \cdot M_{\hat{p}} \cdot l(b) \cdot (1 - P_N)$$

$$\sim_{\|l(a)\|\epsilon'} \quad l(a) \cdot l(b) \cdot M_{\hat{p}} \cdot (1 - P_N).$$

Now, both $\|a\|\epsilon'$ and $\|b\|\epsilon'$ are less than $\frac{\epsilon}{6}$. From Lemma 4.16, we have that $\|l(a)\|\epsilon' \leq \|a\|\epsilon' \leq \frac{\epsilon}{6}$. Similarly, we have that $\|l(b)\|\epsilon' \leq \frac{\epsilon}{6}$. Therefore, $ab \cdot (1 - P_N) \sim_\epsilon l(a) \cdot l(b) \cdot M_{\hat{p}} \cdot (1 - P_N)$. This shows that $ab \in \mathcal{A}$ and $l(ab) = l(a)l(b)$. ♠

Proposition 4.53 *The set of all $a \in \mathcal{A}_{11}(p)'$ with property **, is a closed ideal of $\mathcal{A}_{11}(p)'$.*

Proof: Let $\mathcal{A}$ denote the set of all $a \in \mathcal{A}_{11}(p)'$ with property $**$. By Proposition 4.45, every element of $\mathcal{A}_{11}(p)'$ has property $*$. By Proposition 4.52, it follows that $\mathcal{A}$ is a closed, left ideal of $\mathcal{A}_{11}(p)'$. So, it suffices to show that $\mathcal{A}$ is a right ideal of $\mathcal{A}_{11}(p)'$.

To do this, take $a \in \mathcal{A}_{11}(p)'$, $b \in \mathcal{A}$, and $\epsilon > 0$. Again, note that a has property $*$ by Proposition 4.45. Now, let

$$\epsilon' = \frac{\epsilon}{5(1 + \|a\| + \|b\|)}.$$

Since $b \in \mathcal{A}$, then there exists $R > 0$ such that

$$b \cdot (1 - P_R) \sim_{\epsilon'} l(b) \cdot M_{\hat{p}} \cdot (1 - P_R).$$

By Corollary 4.42, Lemma 4.43, and the fact that a has property $*$, there exists $N > R$ such that

$$a \cdot (1 - P_N) \sim_{\epsilon'} (1 - P_R) \cdot a \cdot (1 - P_N),$$

$$M_{\hat{p}} \cdot a \cdot (1 - P_N) \sim_{\epsilon'} a \cdot M_{\hat{p}} \cdot (1 - P_N),$$

$$a \cdot M_{\hat{p}} \cdot (1 - P_N) \sim_{\epsilon'} l(a) \cdot M_{\hat{p}} \cdot (1 - P_N),$$

and

$$(1 - P_R) \cdot l(a) \cdot (1 - P_N) \sim_{\epsilon'} l(a) \cdot (1 - P_N).$$

It follows that

$$\begin{aligned}
ba \cdot (1 - P_N) &\sim_{\|b\|\epsilon'} b \cdot (1 - P_R) \cdot a \cdot (1 - P_N) \\
&\sim_{\|a\|\epsilon'} l(b) \cdot M_{\hat{p}} \cdot (1 - P_R) \cdot a \cdot (1 - P_N) \\
&= l(b) \cdot (1 - P_R) \cdot M_{\hat{p}} \cdot a \cdot (1 - P_N) \\
&\sim_{\|l(b)\|\epsilon'} l(b) \cdot (1 - P_R) \cdot a \cdot M_{\hat{p}} \cdot (1 - P_N) \\
&\sim_{\|l(b)\|\epsilon'} l(b) \cdot (1 - P_R) \cdot l(a) \cdot M_{\hat{p}} \cdot (1 - P_N) \\
&= l(b) \cdot (1 - P_R) \cdot l(a) \cdot (1 - P_N) \cdot M_{\hat{p}} \\
&\sim_{\|l(b)\|\epsilon'} l(b) \cdot l(a) \cdot (1 - P_N) \cdot M_{\hat{p}} \\
&= l(b) \cdot l(a) \cdot M_{\hat{p}} \cdot (1 - P_N).
\end{aligned}$$

Now, $\|a\|\epsilon'$ and $\|b\|\epsilon'$ are both $\leq \frac{\epsilon}{5}$, and, since $\|l(b)\| \leq \|b\|$, then $\|l(b)\|\epsilon' \leq \|b\|\epsilon' \leq \frac{\epsilon}{5}$. Therefore, $ba \cdot (1 - P_N) \sim_\epsilon l(b) \cdot l(a) \cdot M_{\hat{p}} \cdot (1 - P_N)$, which implies that ba belongs to $\mathcal{A}$ and that $\mathcal{A}$ is a right ideal of $\mathcal{A}_{11}(p)'$. ♠

Proposition 4.54 *All elements of $\mathcal{A}_{11}(p)$ have property $**$. That is, if $a \in \mathcal{A}_{11}(p)$, then $\|(a - l(a) \cdot M_{\hat{p}}) \cdot (1 - P_N)\| \to 0$ as $N \to \infty$.*

Proof: Let $\mathcal{A}$ denote the set of all $a \in \mathcal{A}_{11}(p)'$ such that a has property $**$. We want to show that $\mathcal{A}_{11}(p) \subset \mathcal{A}$. By Proposition 4.53, $\mathcal{A}$ is a closed ideal of $\mathcal{A}_{11}(p)'$. By Remark 4.40, $\mathcal{A}_{11}(p)$ is the closed ideal of $\mathcal{A}_{11}(p)'$ generated by $Per(\tilde{\mathcal{K}}_k)$ and the set of all M_φ such that $\varphi \in \widetilde{SC}(p)$. So, to show that $\mathcal{A}_{11}(p) \subset \mathcal{A}$, it suffices to show that M_φ and K belong to $\mathcal{A}$ for every $\varphi \in$

$\widetilde{SC}(p)$ and $K \in Per(\tilde{\mathcal{K}}_k)$. So, take such φ and K. By Proposition 4.20, M_φ has property $**$. Since $l(K) = 0$, then showing that K has property $**$ is equivalent to showing that $\|K \cdot (1 - P_N)\| \to 0$ as $N \to \infty$. But, this is given by Proposition 4.24. Therefore, $\mathcal{A}_{11}(p) \subset \mathcal{A}$. ♠.

Corollary 4.55 *If $a \in \mathcal{A}_{11}(p)$ and $l(a) = 0$, then $\|a \cdot (1 - P_N)\| \to 0$ as $N \to \infty$.*

Proof: This follows directly from Proposition 4.54. ♠

Lemma 4.56 *Suppose $a \in \mathcal{A}_{11}(p)$ and $l(a) = 0$. Let $\{\varphi_\eta\}$ be a sequence of functions $\varphi_e ta$ in $\tilde{C}_0(TM)$ such that $P_N \leq \varphi_N \leq 1$ for every integer $N > 0$. Then $aM_{\varphi_N} \to a$ as $N \to \infty$ in $\mathcal{A}_{11}(p)$.*

Proof: By Corollary 4.55, $\|a \cdot (1 - P_N)\| \to 0$ as $N \to \infty$. Note that $(1 - \chi_N) \cdot (1 - \varphi_N) = 1 - \varphi_N$ where χ_N is the characteristic function on V_N. Thus, $\|a(1 - M_{\varphi_N})\| = \|a(1 - P_N)(1 - M_{\varphi_N})\| \leq \|a(1 - P_N)\| \to 0$ as $N \to \infty$. Therefore $\|a \cdot (1 - M_{\varphi_N})\| \to 0$ as $N \to \infty$. That is $\|a - aM_{\varphi_N}\| \to 0$ as $N \to \infty$, which means that $aM_{\varphi_N} \to a$ as $N \to \infty$. ♠

Proposition 4.57 *$\mathcal{D}_{M_k(\tilde{C}_0(TM))}$ is a closed ideal of both $\mathcal{D}_{\widetilde{SC}(p)}{}'$ and $\mathcal{D}_{\widetilde{SC}(p)}$.*

Proof: For convenience, let $\mathcal{D}_0 = \mathcal{D}_{M_k\tilde{C}_0(TM))}$ and let $\mathcal{D}_0{}' = \mathcal{D}_{M_k(\tilde{C}_0(TM))}$. We already know that $\mathcal{D}_0$ is closed. Since $\mathcal{D}_{\widetilde{SC}(p)} \subseteq \mathcal{D}_{\widetilde{SC}(p)}{}'$ we only have to show that $\mathcal{D}_0$ is an ideal of $\mathcal{D}_{\widetilde{SC}(p)}{}'$. Since $\mathcal{D}_{\widetilde{SC}(p)}{}'$ is generated by the set of all M_φ and $f(\tilde{D})$ in $Per(\tilde{\mathcal{B}}_k)$ such that $f \in Flip$ and $\varphi \in \widetilde{SC}(p)$, then we only have to show that, for such φ and f, $M_\varphi \cdot a, f(\tilde{D}) \cdot a, a \cdot M_\varphi$, and $a \cdot f(\tilde{D})$

belong to $\mathcal{D}_0$ for every $a \in \mathcal{D}_0$. By taking adjoints, we only need to show that $a \cdot M_\varphi$ and $a \cdot f(\tilde{D})$ belong to $\mathcal{D}_0$ for such a, φ, and f.

Now, $f(\tilde{D})$ in $Per(\tilde{\mathcal{B}}_k)$ belongs to $\mathcal{D}_0'$ (by definition of $\mathcal{D}_0'$) for every $f \in Flip$, and $\mathcal{D}_0$ is an ideal of $\mathcal{D}_0'$. Therefore, $a \cdot f(\tilde{D})$ belongs to $\mathcal{D}_0$ for every $a \in \mathcal{D}_0$ and every $f \in Flip$.

So, it remains to show only that $a \cdot M_\varphi$ belongs to $\mathcal{D}_0$ for every $a \in \mathcal{D}_0$ and every φ in $\widetilde{SC}(p)$. Now, let $g \in Flip, \eta \in M_k(\tilde{C}_0(TM))$, and $c \in \mathcal{D}_0$. Then $\mathcal{D}_0$ has, as a dense subset, the vector space generated by elements a of the form $a = g(\tilde{D})M_\eta, a = c\ M_\eta$, and $a = c \cdot g(\tilde{D})$. It suffices therefore to show that $g(\tilde{D})M_\eta M_\varphi$, $c\tilde{M}_\eta M_\varphi$, and $cg(\tilde{D})M_\varphi$ belong to $\mathcal{D}_0$ when $\varphi \in \widetilde{SC}(p)$. But $\varphi\eta \in M_k(\tilde{C}_0(TM))$ and therefore $M_{\varphi \cdot \eta} \in \mathcal{D}_0$. Since c is also in $\mathcal{D}_0$, then of course, $cM_\eta M_\varphi = cM_{\varphi\eta}$ belongs to $\mathcal{D}_0$. Also since $g(\tilde{D})$ in $Per(\tilde{\mathcal{B}}_k)$ belongs to $\mathcal{D}_0'$, $M_{\varphi\eta} \in \mathcal{D}_0$ and $\mathcal{D}_0$ is an ideal of $\mathcal{D}_0'$ then $g(\tilde{D})M_\eta M_\varphi = g(\tilde{D})M_{\varphi\eta}$ belongs to $\mathcal{D}_0$.

Finally we have to show that $c \cdot g(\tilde{D})M_\varphi \in \mathcal{D}_0$. For this we note that $cg(\tilde{D})M_\varphi \in \mathcal{A}_{11}(p)$ and that $l(cg(\tilde{D})M_\varphi) = l(c)l(g(\tilde{D}M_\varphi). = 0$ since $l(c) = 0$ by Proposition 4.49. By Lemma 4.56, it follows that there is a sequence μ_N of functions $\mu_N \in \tilde{C}_0(TM)$ such that $c \cdot g(\tilde{D})M_{\varphi\mu_N} = cg(\tilde{D})\ M_\varphi M_{\mu_N} \to cg(\tilde{D})M_\varphi$ in $\mathcal{A}_{11}(p)$ as $N \to \infty$. Since $\varphi \cdot \mu_N \in M_k(\tilde{C}_0(TM))$ for every $N > 0$, then $M_{\varphi\mu_N} \in \mathcal{D}_0$ for every $N > 0$. But $c \cdot g(\tilde{D})$ also belongs to $\mathcal{D}_0$. Therefore $cg(\tilde{D})M_{\varphi\mu_N} \in \mathcal{D}_0$ for every N. Thus $cg(\tilde{D}) \cdot M_\varphi$ is the limit of elements in $\mathcal{D}_0$ and is therefore in $\mathcal{D}_0$. ♠

Corollary 4.58 *$\mathcal{A}_{01}(M)_k$ is a closed ideal of both $\mathcal{A}_{11}(p)'$ and $\mathcal{A}_{11}(p)$.*

Proof: Since $\mathcal{A}_{11}(p) \subseteq \mathcal{A}_{11}(p)'$, we only have to show that $\mathcal{A}_{01}(M)_k$ is an ideal of $\mathcal{A}_{11}(p)'$.

From the definitions, $\mathcal{A}_{01}(M)_k = \mathcal{D}_{M_k(\tilde{C}_0(TM))} + Per(\tilde{\mathcal{K}}_k)$, and $\mathcal{A}_{11}(p)' = \mathcal{D}_{\widetilde{SC}(p)}{}' + Per(\tilde{\mathcal{K}}_k)$. By Proposition 4.57, $\mathcal{D}_{M_k(\tilde{C}_0(TM))}$ is a closed ideal of $\mathcal{D}_{\widetilde{SC}(p)}{}'$. This, together with the fact that $Per(\tilde{\mathcal{K}}_k)$ is a closed ideal of $Per(\tilde{\mathcal{B}}_k)$ gives us that $\mathcal{A}_{01}(M)_k$ is a closed ideal of $\mathcal{A}_{11}(p)'$. ♠

Theorem 4.59 *The surjective $*$-homomorphism $l : \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M)$ has kernel $ker(l) = \mathcal{A}_{01}(M)_k$. Thus, we have a $C*$-algebra extension,*

$$0 \longrightarrow \mathcal{A}_{01}(M)_k \xrightarrow{i} \mathcal{A}_{11}(p) \xrightarrow{l} \mathcal{A}_{21}(M) \longrightarrow 0$$

where $i : \mathcal{A}_{01}(M)_k \to \mathcal{A}_{11}(p)$ is the inclusion map.

Proof: By Propositions 4.49 and Corollary 4.58, $\mathcal{A}_{01}(M)_k$ is a closed ideal of $ker(l)$. We have to show that $ker(l) \subseteq \mathcal{A}_{01}(M)_k$.

Let $a \in ker(l)$. By Lemma 4.56, there is a sequence $\{\varphi_N\}$ of functions $\varphi_N \in \tilde{C}_0(TM)$ such that $aM_{\varphi_N} \to a$ in $\mathcal{A}_{11}(p)$ as $N \to \infty$. But $M_{\varphi_N} \in \mathcal{A}_{01}(M)_k$,$a \in \mathcal{A}_{11}(p)$, and (by Corollary 4.58) $\mathcal{A}_{01}(M)_k$ is a closed ideal of $\mathcal{A}_{11}(p)$. Therefore aM_{φ_N} belongs to $\mathcal{A}_{01}(M)_k$ for every $N > 0$. Since $aM_{\varphi_N} \to a$ in $\mathcal{A}_{11}(p)$, and since $\mathcal{A}_{01}(M)_k$ is closed, it follows that a is an element of $\mathcal{A}_{01}(M)_k$. This proves that $ker(l) \subseteq \mathcal{A}_{01}(M)_k$. ♠

Definition 4.60 *The extension of Theorem 4.59 will be called* **the Wiener-Hopf p-extension of $\mathcal{D}_{Per(M)}$**. *(Recall $\mathcal{A}_{21}(M) = \mathcal{D}_{Per(M)}$.)*

For the next proposition, we consider the $*$-homomorphim $l : \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M)$ of Proposition 4.48.

Proposition 4.61 *The sequence*

$$0 \to \mathcal{A}_{00}(M)_k \xrightarrow{i} \mathcal{A}_{10}(p) \xrightarrow{l} \mathcal{A}_{20}(M) \to 0$$

is a C∗-algebra extension which will be called **the Wiener-Hopf p-extension of $\mathcal{L}_{Per(M)}$.**

Proof: By Proposition 4.48, we already know that $l : \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M)$ is surjective. So we only need to show that this map has kernel $ker(l)$ equal to $\mathcal{A}_{00}(M)_k$.

By Theorem 4.59, $l(a) = 0$ for every a in $\mathcal{A}_{01}(M)_k$. Since $\mathcal{A}_{00}(M)_k \subseteq \mathcal{A}_{01}(M)_k$, it follows that $\mathcal{A}_{00}(M)_k \subseteq ker(l)$.

Now take an element a in $ker(l) \subseteq \mathcal{A}_{10}(p)$. Then $a \in \mathcal{A}_{11}(p)$ and $l(a) = 0$. By exactness of $0 \to \mathcal{A}_{01}(M)_k \to \mathcal{A}_{11}(p) \to \mathcal{A}_{21}(M) \to 0$ (Theorem 4.59), it follows that a is an element of $\mathcal{A}_{01}(M)_k$. Also, by exactness of the adjoined Dirac extension $0 \to \mathcal{A}_{10}(p) \xrightarrow{i} \mathcal{A}_{11}(p) \xrightarrow{q} \mathcal{A}_{12}(p) \to 0$ (Proposition 3.36), $q(a) = q(i(a)) = 0$ in $\mathcal{A}_{12}(p)$. By commutativity of

$$\begin{array}{ccc} \mathcal{A}_{11}(p) & \xrightarrow{q} & \mathcal{A}_{12}(p) \\ \uparrow i & & \uparrow i\oplus i \\ \mathcal{A}_{01}(M)_k & \xrightarrow{q} & \mathcal{A}_{02}(M)_k \end{array}$$

(Proposition 3.36), it follows that a in $\mathcal{A}_{01}(M)_k$ is sent to 0 in $\mathcal{A}_{02}(M)_k$ by the map $q : \mathcal{A}_{01}(M)_k \to \mathcal{A}_{02}(M)_k$. By exactness of the adjoined Dirac extension $0 \to \mathcal{A}_{00}(M)_k \xrightarrow{i} \mathcal{A}_{01}(M)_k \xrightarrow{q} \mathcal{A}_{02}(M)_k \to 0$, (Proposition 3.36) it follows that a is an element of $\mathcal{A}_{00}(M)_k$. Thus, $ker(l) \subseteq \mathcal{A}_{00}(M)_k$, which implies $ker(l) = \mathcal{A}_{00}(M)_k$. ♠

We now add the two Wiener-Hopf extensions of Theorem 4.59 and Proposition 4.61 to the commutative diagram (3.35) of Proposition 3.36 to get the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{A}_{20}(M) & \stackrel{i}{\longrightarrow} & \mathcal{A}_{21}(M) & \stackrel{q}{\longrightarrow} & \mathcal{A}_{22}(M) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle L} & & \uparrow{\scriptstyle l\oplus l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \stackrel{i}{\longrightarrow} & \mathcal{A}_{11}(p) & \stackrel{q}{\longrightarrow} & \mathcal{A}_{12}(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\oplus i} & & \\
0 & \longrightarrow & \mathcal{A}_{00}(M)_k & \stackrel{i}{\longrightarrow} & \mathcal{A}_{01}(M)_k & \stackrel{q}{\longrightarrow} & \mathcal{A}_{02}(M)_k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\quad , \qquad (4.62)
$$

which is the same as

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{Per(M)} & \stackrel{i}{\longrightarrow} & \mathcal{D}_{Per(M)} & \stackrel{q}{\longrightarrow} & Per(M) \oplus Per(M) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l\oplus l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \stackrel{i}{\longrightarrow} & \mathcal{A}_{11}(p) & \stackrel{q}{\longrightarrow} & \widetilde{SC}(p) \oplus \widetilde{SC}(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\oplus i} & & \\
0 & \longrightarrow & Per(\tilde{\mathcal{K}}_k) & \stackrel{i}{\longrightarrow} & \mathcal{A}_{01}(M)_k & \stackrel{q}{\longrightarrow} & M_k(\tilde{C}_0(TM)) \oplus M_k(\tilde{C}_0(TM)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

which is isomorphic to

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{Per(M)} & \xrightarrow{i} & \mathcal{D}_{Per(M)} & \xrightarrow{q} & C(M) \oplus C(M) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l \oplus l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \longrightarrow & \mathcal{A}_{11}(p) & \xrightarrow{q} & SC(p) \oplus SC(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i \oplus i} & & \\
0 & \longrightarrow & M_k(C(\mathcal{K})) & \longrightarrow & M_k(\mathcal{A}_{01}(M)) & \xrightarrow{q} & M_k(C_0(TM)) \oplus M_k(C_0(TM)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{4.63}
$$

Theorem 4.64 *Diagram 4.62 which is isomorphic to Diagram 4.63, is a commutative diagram exact at every point.*

Proof: This follows from Proposition 3.36, Theorem 4.59, Proposition 4.61, and Propositions 4.48 and 4.50. ♠

Chapter 6

The Thom Isomorphism in K-theory

6.1 A Fundamental Projection on an Even Sphere

Suppose l is an even positive integer and S^l is the even l-sphere. Then $K_0(C(S^l))$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ where $\mathbf{Z}$ is the group of integers, and if p is a projection over $C(S^l)$, and $[p]$, the class in $K_0(C(S^l))$ determined by p, and if $[p]$ is equal to (m, n) for two integers m and n, then one of the integers measures the rank of p, whereas the other measures the amount of twisting of the vector bundle over S^l determined by p. Let us assume that m is the rank of p and that n measures the twisting. Then the projection p over $C(S^l)$ will be called a **fundamental projection on S^l** if $[p]$ in $K_0(C(S^l))$ is equal to $(m, 1)$ or $(m, -1)$ for some positive integer m, and a **trivial projection on S^l** if $[p] = (m, 0)$ for some positive integer m. Also, a complex vector bundle V over S^l will be called a **fundamental vector bundle over S^l** if the $K_0(C(S^l))$ class $[V]$ determined by V equals $(m, 1)$ or $(m, -1)$ for some positive integer m.

We also need to look at the "0-sphere" S^0, which is the two-point disconnected set $\{-1, 1\}$. We identify $C(S^0)$ with $\mathbf{C} \oplus \mathbf{C}$. If $f \in C(S^0)$, then f is identified with $(f(-1), f(1)) \in \mathbf{C} \oplus \mathbf{C}$.

The elements $(0,1), (1,0)$, and $(1,1)$ in $C(S^0)$ are all projections on S^0 and determine elements $[(0,1)], [(1,0)]$, and $[(1,1)]$ in $K_0(C(S^0))$. Since $C(S^0) \cong \mathbf{C} \oplus \mathbf{C}$ under the identification given above, then

$$K_0(C(S^0)) \cong \mathbf{Z}[(1,0)] \oplus \mathbf{Z}[(0,1)] \tag{1.1}$$

where $\mathbf{Z}[(1,0)]$ is the subgroup of $K_0(C(S^0))$ generated by $[(1,0)]$, and $\mathbf{Z}[(0,1)]$ the subgroup generated by $[(0,1)]$.

Lemma 1.2 *Define* $\varphi : \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ *by letting*

$$\varphi(m,n) = m(1,1) + (0,1).$$

Then φ *is an isomorphism of groups. So, each* $(k,l) \in \mathbf{Z} \oplus \mathbf{Z}$ *can be written uniquely as* $m(1,1) + n(0,1)$ *for some* $(m,n) \in \mathbf{Z} \oplus \mathbf{Z}$.

Proof: Easy. ♠

Using Lemma 1.2, we see that $K_0(C(S^0))$ is also isomorphic to $\mathbf{Z} \cdot [(1,1)] \oplus \mathbf{Z}[(0,1)]$. So, each element $a \in K_0(C(S^0))$ can be written uniquely as $a = m[(1,1)] + n[(0,1)]$ for two integers m and n. If $a = m \cdot [(1,1)] + n[(0,1)]$, we will also write $\boldsymbol{a = (m,n)}$, and when we talk about an element (m,n) in $K_0(C(S^0))$, we mean the element $m[(1,1)] + n[(0,1)]$.

When we write $K_0(C(S^0)) \cong \mathbf{Z} \oplus \mathbf{Z}$, we mean

$$K_0(C(S^0)) \cong \mathbf{Z}[(1,1)] \oplus \mathbf{Z}[(0,1)],$$

not the isomorphism given in (1.1).

We say a projection p on S^0 is **fundamental on S^o** if $[p]$ in $K_0(C(S^0))$ is equal to $(m,1)$ or $(m,-1)$ for some integer m. We say it is **trivial** if $[p] = (m,0)$ in $K_0(C(S^0))$ for some $m \geq 0$. Also, a complex vector bundle V over S^0 is a **fundamental vector bundle over S^o** if the $K_0(C(S^0))$ class $[V]$ determined by V equals $(m,1)$ or $(m,-1)$ for some integer m.

Remark 1.3 *The projection* $(0,1) \in C(S^0)$ *is fundamental by definition. (It determines the element* $(0,1)$ *in* $K_0(C(S^0))$.

Let us assume from now on in this section that **$m \geq 3$ is an odd integer**, and let $\boldsymbol{Cl_m}$ and $\boldsymbol{\Delta_m}$ denote the complex Clifford algebra, and complex spinors respectively, of the euclidean space $\mathbf{R}^m$. Recall that Δ_m is a hermitian vector space which is also a module over the algebra Cl_m. Also, let S^{m-1} denote the unit $(m-1)$-sphere which sits inside $\mathbf{R}^m$.

Recall the map

$$\sigma_v : \Delta_m \to \Delta_m$$

defined for each $v \in \mathbf{R}^m$ by the formula

$$\sigma_v(s) = iv \cdot s$$

for every $s \in \Delta_m$. (See Remarks 3.6.) By Part 4 of Remarks 2.3.6, if $v \in S^{m-1}$ then $\sigma_v : \Delta_m \to \Delta_m$ is a self-adjoint unitary and the operator

$$\frac{1}{2}(1 + \sigma_v) : \Delta_m \to \Delta_m$$

is the projection onto the $+1$ eigenspace of σ_v. Now, after choosing an orthonormal basis for Δ_m, each $\frac{1}{2}(1+\sigma_v)$, for v in S^{m-1}, can be regarded as a projection over the complex numbers C. Let

$$p_F : S^{m-1} \to Proj_\infty(\mathbf{C})$$

denote the map given by

$$p_F(v) = \frac{1}{2}(1 + \sigma_v).$$

Then, of course, we can regard p_F as a projection over $C(S^{m-1})$.

Theorem 1.4 *p_F, defined above, is a fundamental projection on S^{m-1}. (Note we are assuming $m \geq 3$.)*

Proof: Of course, it does not matter which orthonormal basis of Δ_m is chosen, since projections over $C(S^{m-1})$ obtained by two different choices of orthonormal bases will clearly be unitarily equivalent.

Let $(e_1, e_2, \ldots, e_m)$ denote the standard orthonormal basis of $\mathbf{R}^m$, and set

$$\beta_2 = 1 + ie_1e_2, \quad \beta_4 = 1 + ie_3e_4, \quad \ldots, \quad \beta_{m-1} = 1 + ie_{m-2}e_{m-1},$$

and

$$\beta_{m+1} = 1 + ie_m.$$

By definition,

$$\Delta_m = Cl_m \cdot \beta_2\beta_4 \cdots \beta_{m+1}.$$

Let

$$E \to S^{m-1}$$

be the vector bundle over S^{m-1} determined by p_F. That is, E is the vector bundle with fiber

$$E_v = (1 + iv) \cdot \Delta_m$$

at each v in S^{m-1}. Then, proving Theorem 1.4 is equivalent to showing that E is a fundamental vector bundle over S^{m-1}.

Now, from the proof of Proposition 9.25 in Chapter I of **[L&M]**, one can construct a fundamental vector bundle over S^{m-1} as follows.

A construction of a fundamental vector bundle over an even sphere. Define the two hemispheres,

$$S_{+}^{m-1} = \{v \in S^{m-1} : (v, e_m) \geq 0\},$$

and

$$S_{-}^{m-1} = \{v \in S^{m-1} : (v, e_m) \leq 0\}.$$

Let us call S_{+}^{m-1} the top hemisphere, and S_{-}^{m-1}, the bottom hemisphere, of the sphere S^{m-1}. Note that the intersection of S_{+}^{m-1} and S_{-}^{m-1} is the sphere

$$S^{m-2} = \{v \in S^{m-1} : (v, e_m) = 0\}.$$

Now, regard $\mathbf{R}^{m-1}$ specifically as the subspace of $\mathbf{R}^m$ generated by the vectors $e_1, \ldots, e_{m-1}$, and let $Cl(\mathbf{R}^{m-1})$ denote the complex Clifford algebra of $\mathbf{R}^{m-1}$ generated by the vectors $e_1, \ldots, e_{m-1}$. Define $\Delta(\mathbf{R}^{m-1})$ specifically by setting

$$\Delta(\mathbf{R}^{m-1}) = Cl(\mathbf{R}^{m-1}) \cdot \beta_2 \cdot \beta_4 \cdots \beta_{m-1}.$$

Next, let $\Delta^{+}(\mathbf{R}^{m-1})$ and $\Delta^{-}(\mathbf{R}^{m-1})$ be the spaces

$$\Delta^{+}(\mathbf{R}^{m-1}) = (1 + \tau_{m-1}) \cdot \Delta(\mathbf{R}^{m-1}),$$

and

$$\Delta^{-}(\mathbf{R}^{m-1}) = (1 - \tau_{m-1}) \cdot \Delta(\mathbf{R}^{m-1}),$$

where τ_{m-1} is the volume element of $Cl(\mathbf{R}^{m-1})$. These are the positive and negative complex spinors respectively of the space $\mathbf{R}^{m-1}$. Also, note that S^{m-2} is contained in the space $\mathbf{R}^{m-1}$.

Take $S_{+}^{m-1} \times \Delta^{+}(\mathbf{R}^{m-1})$ and $S_{-}^{m-1} \times \Delta^{-}(\mathbf{R}^{m-1})$ regarded as trivial vector bundles over S_{+}^{m-1} and S_{-}^{m-1} respectively. For each v in S^{m-2}, let

$$\rho_v : \Delta^{+}(\mathbf{R}^{m-1}) \to \Delta^{-}(\mathbf{R}^{m-1})$$

be the linear isomorphism which sends each s in $\Delta^{+}(\mathbf{R}^{m-1})$ to

$$\rho_v(s) = v \cdot s$$

in $\Delta^{-}(\mathbf{R}^{m-1})$. This makes sense since each v in S^{m-2} is actually a member of $\mathbf{R}^{m-1}$. Let us use $Isom(V, W)$ to denote the collection of all linear isomorphisms between any two vector spaces V and W. Now, use the map

$$\rho : S^{m-2} \to Isom(\Delta^{+}(\mathbf{R}^{m-1}), \Delta^{-}(\mathbf{R}^{m-1}))$$

which sends v in S^{m-2} to ρ_v in $Isom(\Delta^{+}(\mathbf{R}^{m-1}), \Delta^{-}(\mathbf{R}^{m-1}))$, to glue the vector bundle

$$\Delta^{+}(\mathbf{R}^{m-1}) \to S_{+}^{m-1}$$

to the bundle

$$\Delta^{-}(\mathbf{R}^{m-1}) \to S_{-}^{m-1}$$

along the intersection, S^{m-2}, in order to get a new vector bundle

$$V^F \to S^{m-1}$$

over S^{m-1}. According to the proof of Proposition I.9.25 of [L&M], V^F is a fundamental vector bundle over S^{m-1}.

Thus, in order to prove Theorem 1.4, it suffices to construct vector bundle isomorphisms

$$\eta_+ : E|_{S_+^{m-1}} \to S_+^{m-1} \times \Delta^+(\mathbf{R}^{m-1}),$$

and

$$\eta_- : E|_{S_-^{m-1}} \to S_-^{m-1} \times \Delta^-(\mathbf{R}^{m-1}),$$

such that, for (v, s) in $S^{m-2} \times \Delta^+(\mathbf{R}^{m-1})$,

$$\eta_- \left((\eta_+)^{-1}(v, s) \right) = (v, v \cdot s).$$

This, we will do after a few lemmas. First, let $\boldsymbol{Pin}_m$ and $\boldsymbol{Spin}_m$ represent the Pin and Spin groups respectively of $\mathbf{R}^m$, and let

$$\alpha : Pin_m \to Gl(Cl_m)$$

denote the action of Pin_m on Cl_m given by

$$(\alpha(a))(b) = aba^t$$

for every a and b in Pin_m, where a^t stands for the transpose of a. For convenience, we let

$$\alpha_a = \alpha(a)$$

for every a in Pin_m. If ω is a $Spin_m$ element, and v a unit vector in $\mathbf{R}^m$, then we know that α_ω and α_v act as a rotation and a reflection respectively on $\mathbf{R}^m$. We will call α_ω **the rotation on $\mathbf{R}^m$ induced by $\boldsymbol{\omega}$**, and we will call α_v **the reflection on $\mathbf{R}^m$ induced by $\boldsymbol{v}$.**

Lemma 1.5 *If v, u, belong to S^{m-1}, let*

$$\omega_{v,u} = u \cdot \frac{v + u}{\|v + u\|}.$$

Then $\omega_{v,u}$ is a $Spin_m$ element, and the rotation of $\mathbf{R}^m$ induced by $\omega_{v,u}$ is the one which sends v to u, and which leaves the orthogonal complement of $\{v,u\}$ fixed. Furthermore,

$$\omega_{v,u}{}^t = \omega_{u,v}.$$

Proof of Lemma: The Clifford algebra element $\omega_{v,u}$ is the product of two vectors of norm 1. Hence, it is a $Spin_m$ element.

Now,

$$\alpha_{\omega_{v,u}} = \alpha\left(u \cdot \frac{v+u}{\|v+u\|}\right) = \alpha(u) \circ \alpha\left(\frac{v+u}{\|v+u\|}\right).$$

Thus, to obtain $\alpha_{\omega_{v,u}}(v)$, we first reflect v about the orthogonal complement of $\frac{v+u}{\|v+u\|}$. If we note that $\frac{v+u}{\|v+u\|}$ is the unit vector lying exactly between v and u, then it is not difficult to see that the reflection of v about the orthogonal complement of $\frac{v+u}{\|v+u\|}$ is just $-u$. Or, one can show this algebraically by showing that

$$\frac{v+u}{\|v+u\|} \cdot v \cdot \frac{v+u}{\|v+u\|} = -u.$$

Next, we take $-u$ and reflect it about the orthogonal complement of u itself. Clearly, what we end up with is u. Hence, $\alpha_{\omega_{v,u}}(v)$ equals u.

To show that everything in the orthogonal complement of $\{v,u\}$ is left fixed by $\alpha_{\omega_{v,u}}$ is easy, because anything in the orthogonal complement of $\{v,u\}$ must lie in the orthogonal complements of both $\frac{v+u}{\|v+u\|}$ and u, and therefore is reflected onto itself by the two reflections which compose $\alpha_{\omega_{v,u}}$.

Now,

$$\omega_{v,u}{}^t = \frac{v+u}{\|v+u\|} \cdot u = \frac{v \cdot u - 1}{\|u+v\|},$$

and

$$\omega_{u,v} = \frac{v \cdot (u+v)}{\|u+v\|} = \frac{v \cdot u - 1}{\|u+v\|}.$$

Hence,

$$\omega_{v,u}{}^{t} = \omega_{u,v}$$

as claimed. Therefore the lemma is true. ♠ **(End Proof of Lemma 1.5)**

Lemma 1.6 *If u and v belong to $\mathbf{R}^m$, then Clifford multiplication by $\omega_{u,v}$ on the left gives a linear isomorphism from the space $(1+iu)\cdot\Delta_m$ to the space $(1+iv)\cdot\Delta_m$, and if s belongs to Δ_m, then*

$$\omega_{u,v}\cdot(1+iu)\cdot s = (1+iv)\cdot(-1)\frac{1+iu}{\|u+v\|}\cdot s.$$

Proof: If a belongs to Cl_m, let

$$\varphi_{u,v}(a) = \omega_{u,v}\cdot a.$$

Then

$$\begin{aligned}
\varphi_{u,v}\left((1+iu)\cdot\Delta_m\right) &= \omega_{u,v}\cdot(1+iu)\cdot\Delta_m \\
&= \omega_{u,v}\cdot(1+iu)\cdot\omega_{u,v}{}^{t}\cdot\omega_{u,v}\cdot\Delta_m \\
&= (1+iv)\cdot\omega_{u,v}\cdot\Delta_m \\
&\subseteq (1+iv)\cdot\Delta_m,
\end{aligned}$$

since Δ_m is a module over Cl_m.

So, Clifford multiplication on the left by $\omega_{u,v}$ maps $(1+iu)\cdot\Delta_m$ into the space $(1+iv)\cdot\Delta_m$. Similarly, Clifford multiplication on the left by $\omega_{v,u}$ maps

$(1+iv)\cdot\Delta_m$ to the space $(1+iu)\cdot\Delta_m$. Since the inverse of $\omega_{u,v}$ is $\omega_{u,v}{}^t$ which equals $\omega_{v,u}$, these two maps must be inverse maps of each other.

Now, if s belongs to Δ_m, then, from above,

$$\begin{aligned}
\varphi_{u,v}((1+iu)\cdot s) &= \omega_{u,v}\cdot(1+iu)\cdot s \\
&= (1+iv)\cdot\omega_{u,v}\cdot s \\
&= (1+iv)\cdot v\cdot\frac{v+u}{\|v+u\|}\cdot s \\
&= (1+iv)\cdot\frac{(-1+vu)}{\|v+u\|}\cdot s \\
&= (1+iv)\cdot\frac{(-iv+vu)}{\|v+u\|}\cdot s \\
&= (1+iv)\cdot iv\cdot(-1)\frac{1+iu}{\|v+u\|}\cdot s \\
&= (1+iv)\cdot(-1)\frac{1+iu}{\|v+u\|}\cdot s.
\end{aligned}$$

Note that we used twice the fact that $(1+iv)\cdot iv$ equals $1+iv$. Thus, Lemma 1.6 is true. ♠ **(End Proof of Lemma 1.6)**

Now, define the linear isomorphism

$$\varphi_{u,v}:(1+iu)\cdot\Delta_m\to(1+iv)\cdot\Delta_m$$

as in the previous lemma. For each v in the top hemisphere S_+^{m-1} and a in the fiber $E_v=(1+iv)\cdot\Delta_m$ of the bundle E, we let

$$\boxed{\varphi_v(a)=\varphi_{v,e_m}(a)=\omega_{v,e_m}\cdot a,}$$

which lies in $(1+ie_m)\cdot\Delta_m$. Also, for v in the bottom hemisphere S_-^{m-1} and a in the fiber E_v, we let

$$\boxed{\hat{\varphi}_v(a)=\varphi_{v,-e_m}(a)=\omega_{v,-e_m}\cdot a,}$$

which lies in $(1 - ie_m) \cdot \Delta_m$. Now, define bundle maps

$$\varphi : E|_{S_+^{m-1}} \to S_+^{m-1} \times (1 + ie_m) \cdot \Delta_m$$

and

$$\hat{\varphi} : E|_{S_-^{m-1}} \to S_-^{m-1} \times (1 - ie_m) \cdot \Delta_m$$

by letting φ equal the map φ_v on the fiber E_v, for v in the top hemisphere S_+^{m-1}, and by letting $\hat{\varphi}$ equal the map $\hat{\varphi}_v$ on the fiber E_v for v in the bottom hemisphere S_-^{m-1}. That is, for each a in E_v, v in S_+^{m-1},

$$\varphi(a) = (v, \varphi_v(a)),$$

and, for each a in E_v, v in S_-^{m-1},

$$\hat{\varphi}(a) = (v, \hat{\varphi}_v(a)).$$

Since φ equals φ_{v,e_m} on the fiber E_v, for v in S_+^{m-1}, it follows from Lemma 1.6 that φ maps the fiber, E_v, of E isomorphically onto the fiber $(1 + ie_m) \cdot \Delta_m$ of $S_+^{m-1} \times (1 + ie_m) \cdot \Delta_m$. Similarly, for v in S_-^{m-1}, $\hat{\varphi}$ maps the fiber E_v isomorphically onto the space $(1 - ie_m) \cdot \Delta_m$. Hence, the following is true.

Lemma 1.7 *The maps*

$$\varphi : E|_{S_+^{m-1}} \to S_+^{m-1} \times (1 + ie_m) \cdot \Delta_m$$

and

$$\hat{\varphi} : E|_{S_-^{m-1}} \to S_-^{m-1} \times (1 - ie_m) \cdot \Delta_m$$

are vector bundle isomorphisms.

Proof: The lemma follows from the remarks just made. ♠ **(End Proof of Lemma 1.7)**

Next, we define linear isomorphisms

$$\mu : (1 + ie_m) \cdot \Delta_m \to \Delta^+(\mathbf{R}^{m-1}),$$

and

$$\hat{\mu} : (1 - ie_m) \cdot \Delta_m \to \Delta^-(\mathbf{R}^{m-1})$$

as follows. Every element a of Cl_m has an algebraic expression in terms of the elements $e_1, \ldots, e_m$. We will write

$$a = f(e_1, \ldots, e_m)$$

to denote that expression. We may think of $f(x_1, \ldots, x_m)$ as a polynomial of $x_1, \ldots, x_m$ with complex coefficients. We will write $a = f(e_1, e_2)$, say, if a can be expressed in terms of e_1 and e_2 only. Note that any element s of Δ_m can be written as

$$s = f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1}.$$

For, by definition,

$$\Delta_m = Cl_m \cdot \beta_2\beta_4 \cdots \beta_{m+1},$$

and so s can be written as

$$s = a \cdot \beta_2\beta_4 \cdots \beta_{m+1}$$

for some a in Cl_m. Of course, we can write a as

$$a = g(e_1, \ldots, e_{m-1}) + h(e_1, \ldots, e_{m-1}) \cdot ie_m$$

for some g and h, and so we can write s as

$$s = g(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1} + h(e_1, \ldots, e_{m-1}) \cdot ie_m \cdot \beta_2\beta_4 \cdots \beta_{m+1}.$$

But, note that ie_m commutes with all of the elements $\beta_2, \ldots, \beta_{m-1}$, and that $ie_m \cdot \beta_{m+1}$ is the same as β_{m+1}. It thus follows that

$$ie_m \cdot \beta_2\beta_4 \cdots \beta_{m+1} = \beta_2\beta_4 \cdots \beta_{m+1}, \tag{1.8}$$

and therefore s can be written as

$$s = [g(e_1, \ldots, e_{m-1}) + h(e_1, \ldots, e_{m-1})] \cdot \beta_2\beta_4 \cdots \beta_{m+1}.$$

So, the claim is true, and we can write

$$\Delta_m = Cl(\mathbf{R}^{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1}. \tag{1.9}$$

Now, if a belongs to $(1 + ie_m) \cdot \Delta_m$, then, by Equation 1.9, we can write

$$a = (1 + ie_m) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1}.$$

For such an a, we make the definition

$$\boxed{\mu(a) = (1 + \tau_{m-1}) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1}},$$

which is an element of $\Delta^+(\mathbf{R}^{m-1})$.

Similarly, if a belongs to $(1 - ie_m) \cdot \Delta_m$, then we can write

$$a = (1 - ie_m) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1},$$

and we make the definition

$$\boxed{\hat{\mu}(a) = (1 - \tau_{m-1}) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1}},$$

which is an element of $\Delta^-(\mathbf{R}^{m-1})$.

Lemma 1.10 *The maps*

$$\mu : (1 + ie_m) \cdot \Delta_m \to \Delta^+(\mathbf{R}^{m-1}),$$

and

$$\hat{\mu} : (1 - ie_m) \cdot \Delta_m \to \Delta^-(\mathbf{R}^{m-1})$$

are well-defined linear isomorphisms.

Proof: I first show that they are well-defined. First, we note that the volume element τ_{m-1} relates to the elements of $\Delta(\mathbf{R}^{m-1})$ in the same way that ie_m relates to the elements of $(1 + ie_m) \cdot \Delta_m$. That is, we have

$$ie_m \cdot f(e_1, \dots, e_{m-1}) = f(-e_1, \dots, -e_{m-1}) \cdot ie_m, \tag{1.11}$$

$$\tau_{m-1} \cdot f(e_1, \dots, e_{m-1}) = f(-e_1, \dots, -e_{m-1}) \cdot \tau_{m-1}, \tag{1.12}$$

$$ie_m \cdot \beta_2\beta_4 \cdots \beta_{m+1} = \beta_2\beta_4 \cdots \beta_{m+1}, \tag{1.13}$$

and
$$\tau_{m-1} \cdot \beta_2\beta_4 \cdots \beta_{m-1} = \beta_2\beta_4 \cdots \beta_{m-1}, \tag{1.14}$$

Equations 1.11 and 1.12 simply say that ie_m and τ_{m-1} anticommutes with each $e_1, \dots, e_{m-1}$. Equation 1.13 is the same as Equation 1.8 above, and Equation 1.14 is true because τ_{m-1} actually equals $\beta_2 \cdots \beta_{m-1}$, because the elements $\beta_2, \dots, \beta_{m-1}$ all commute with each other, and because each ${\beta_j}^2$ is equal to 1.

Now, notice that it follows from Equations 1.11 and 1.13, that, for any f,

$$\begin{aligned} &(1 + ie_m) \cdot f(e_1, \dots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1} \\ = \;& [f(e_1, \dots, e_{m-1}) + f(-e_1, \dots, -e_{m-1})] \cdot \beta_2\beta_4 \cdots \beta_{m+1}. \end{aligned}$$

Similarly, from Equations 1.12 and 1.14, it follows that

$$\begin{aligned} &(1 + \tau_{m-1}) \cdot f(e_1, \dots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1} \\ = \;& [f(e_1, \dots, e_{m-1}) + f(-e_1, \dots, -e_{m-1})] \cdot \beta_2\beta_4 \cdots \beta_{m-1}. \end{aligned}$$

Hence, we have that

$$\begin{aligned}&(1+\tau_{m-1})\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\\ =\;&(1+ie_m)\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\end{aligned}\tag{1.15}$$

By a similar argument, we also have

$$\begin{aligned}&(1-\tau_{m-1})\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\\ =\;&(1-ie_m)\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\end{aligned}\tag{1.16}$$

Now, take a in $(1+ie_m)\cdot\Delta_m$, and suppose a can be written in two ways,

$$a=(1+ie_m)\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1},$$

and

$$a=(1+ie_m)\cdot g(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}.$$

Then,

$$\begin{aligned}&(1+ie_m)\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\\ =\;&(1+ie_m)\cdot g(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}.\end{aligned}$$

Hence, by Equation 1.15, we have

$$\begin{aligned}&(1+\tau_{m-1})\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m-1}\cdot(1+ie_m)\\ =\;&(1+\tau_{m-1})\cdot g(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m-1}\cdot(1+ie_m).\end{aligned}$$

If, in the above equation, we keep only the terms not involving e_m, we are then left with the equation

$$\begin{aligned}&(1+\tau_{m-1})\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m-1}\\ =\;&(1+\tau_{m-1})\cdot g(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m-1}.\end{aligned}$$

which implies $\mu(a)$ is well defined.

Moreover, by Equation 1.15, we have that

$$a = \mu(a) \cdot (1 + ie_m) \tag{1.17}$$

for every a in $(1 + ie_m) \cdot \Delta_m$.

Similarly, if a is in $(1 - ie_m) \cdot \Delta_m$, and a can be written in two ways

$$a = (1 - ie_m) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1},$$

and

$$a = (1 - ie_m) \cdot g(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m+1},$$

then, by Equation 1.16, we have

$$\begin{aligned} & (1 - \tau_{m-1}) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1} \cdot (1 + ie_m) \\ = \ & (1 - \tau_{m-1}) \cdot g(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1} \cdot (1 + ie_m), \end{aligned}$$

and by keeping only those terms not involving e_m, we obtain

$$\begin{aligned} & (1 - \tau_{m-1}) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1} \\ = \ & (1 - \tau_{m-1}) \cdot g(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1}, \end{aligned}$$

from which follows that $\hat{\mu}(a)$ is well defined. Moreover, by Equation 1.16, we have that

$$a = \hat{\mu}(a) \cdot (1 + ie_m) \tag{1.18}$$

for every a in $(1 - ie_m) \cdot \Delta_m$.

Now, it is clear that both maps μ and $\hat{\mu}$ are linear. From Equation 1.17, we have that, if $\mu(a)$ is 0, then a, which equals $\mu(a) \cdot (1 + ie_m)$, is also 0, and therefore μ is 1-1.

Similarly, from Equation 1.18, we have that the kernel of $\hat{\mu}$ is 0, and therefore $\hat{\mu}$ is 1-1.

Also, since $\Delta^+(\mathbf{R}^{m-1})$ is generated by elements of the form $(1+\tau_{m-1})\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m-1}$, it is clear that the map

$$\mu : (1+ie_m)\cdot\Delta_m \to \Delta^+(\mathbf{R}^{m-1})$$

is onto.

Similarly, $\hat{\mu}$ is onto, which completes the proof of the lemma. ♠ **(End Proof of Lemma 1.10)**

Now, the isomorphisms

$$\mu : (1+ie_m)\cdot\Delta_m \to \Delta^+(\mathbf{R}^{m-1}),$$

and

$$\hat{\mu} : (1-ie_m)\cdot\Delta_m \to \Delta^-(\mathbf{R}^{m-1})$$

determine vector bundle isomorphisms

$$\mu : S_+^{m-1}\times(1+ie_m)\cdot\Delta_m \to S_+^{m-1}\times\Delta^+(\mathbf{R}^{m-1}),$$

and

$$\hat{\mu} : S_-^{m-1}\times(1-ie_m)\cdot\Delta_m \to S_-^{m-1}\times\Delta^-(\mathbf{R}^{m-1}).$$

Now, take the vector bundle isomorphisms

$$\varphi : E|_{S_+^{m-1}} \to S_+^{m-1}\times(1+ie_m)\cdot\Delta_m$$

and

$$\hat{\varphi} : E|_{S_-^{m-1}} \to S_-^{m-1}\times(1-ie_m)\cdot\Delta_m$$

of Lemma 1.7, and let

$$\eta = \mu \circ \varphi,$$

and

$$\hat{\eta} = \hat{\mu} \circ \hat{\varphi}.$$

This gives vector bundle isomorphisms

$$\eta : E|_{S_+^{m-1}} \to S_+^{m-1} \times \Delta^+(\mathbf{R}^{m-1}),$$

and

$$\hat{\eta} : E|_{S_-^{m-1}} \to S_-^{m-1} \times \Delta^-(\mathbf{R}^{m-1}).$$

We define

$$\eta_v : E_v \to \Delta^+(\mathbf{R}^{m-1}), \quad v \in S_+^{m-1},$$

$$\hat{\eta}_v : E_v \to \Delta^-(\mathbf{R}^{m-1}), \quad v \in S_-^{m-1},$$

by the equations

$$\eta(a) = (v, \eta_v(a)), \quad v \in S_+^{m-1},$$

$$\textit{and} \quad \hat{\eta}_v(a) = (v, \hat{\eta}_v(a)), \quad v \in S_-^{m-1}.$$

We note that

$$\eta_v = \mu \circ \varphi_v, \quad v \in S_+^{m-1},$$

$$\textit{and} \quad \hat{\eta}_v = \hat{\mu} \circ \hat{\varphi}_v, \quad v \in S_-^{m-1}.$$

Now, suppose v belongs to S^{m-2} and a belongs to

$$E_v = (1 + iv) \cdot \Delta_m.$$

Say, a is equal to $(1+iv)\cdot s$ for some s in Δ_m. Then, by Lemma 1.6,

$$\begin{aligned}\varphi_v(a) &= \omega_{v,e_m}\cdot(1+iv)\cdot s\\ &= (1+ie_m)\cdot(-1)\frac{1+iv}{\|v+e_m\|}\cdot s\end{aligned}$$

By Equation 1.9,

$$\Delta_m = Cl(\mathbf{R}^{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1},$$

and so we can write

$$s = f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}$$

for some f. Hence,

$$\begin{aligned}\eta_v(a) &= \mu(\varphi_v(a))\\ &= \mu\left((1+ie_m)\cdot(-1)\frac{1+iv}{\|v+e_m\|}\cdot s\right)\\ &= \mu\left((1+ie_m)\cdot(-1)\frac{1+iv}{\|v+e_m\|}\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\right).\end{aligned}$$

Now, use the fact that v is in S^{m-2}. Since v is in S^{m-2}, then $\|v+e_m\|$ equals $\sqrt{2}$. Also, S^{m-2} is contained in $\mathbf{R}^{m-1}$. So, v can be written in terms of $e_1,\ldots,e_{m-1}$ only. Therefore,

$$\eta_v(a) = \mu\left((1+ie_m)\cdot\left(\frac{-1}{\sqrt{2}}\right)(1+iv)\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m+1}\right),$$

which yields

$$\eta_v(a) = (1+\tau_{m-1})\cdot\left(\frac{-1}{\sqrt{2}}\right)(1+iv)\cdot f(e_1,\ldots,e_{m-1})\cdot\beta_2\beta_4\cdots\beta_{m-1}. \quad (1.19)$$

Similarly,

$$\hat{\varphi}_v(a) = \omega_{v,-e_m}\cdot(1+iv)\cdot s$$

$$
\begin{aligned}
&= (1 - ie_m) \cdot (-1) \frac{1 + iv}{\|v - e_m\|} \cdot s \\
&= (1 - ie_m) \cdot \left(\frac{-1}{\sqrt{2}} \right) (1 + iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2 \beta_4 \cdots \beta_{m+1},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\hat{\eta}_v(a) &= \hat{\mu}(\hat{\varphi}_v(a)) \\
&= \hat{\mu} \left((1 - ie_m) \cdot \left(\frac{-1}{\sqrt{2}} \right) (1 + iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2 \beta_4 \cdots \beta_{m+1} \right),
\end{aligned}
$$

and therefore

$$
\hat{\eta}_v(a) = (1 - \tau_m) \cdot \left(\frac{-1}{\sqrt{2}} \right) (1 + iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2 \beta_4 \cdots \beta_{m-1}. \qquad (1.20)
$$

Now, let

$$
\eta_+ = \eta, \qquad \eta_- = -i\hat{\eta}.
$$

Then,

$$
\eta_+ : E|_{S_+^{m-1}} \to S_+^{m-1} \times \Delta^+(\mathbf{R}^{m-1}),
$$

and

$$
\eta_- : E|_{S_-^{m-1}} \to S_-^{m-1} \times \Delta^-(\mathbf{R}^{m-1})
$$

are vector bundle isomorphisms, since η and $\hat{\eta}$ are vector bundle isomorphisms. Hence, from Equations 1.19 and 1.20, we have that, for a given as above,

$$
\eta_+(a) = \left(v, (1 + \tau_{m-1}) \cdot \left(\frac{-1}{\sqrt{2}} \right) (1 + iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2 \beta_4 \cdots \beta_{m-1} \right)
$$

and

$$
\eta_-(a) = \left(v, -i(1 - \tau_m) \cdot \left(\frac{-1}{\sqrt{2}} \right) (1 + iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2 \beta_4 \cdots \beta_{m-1} \right).
$$

It follows that

$$\begin{aligned}
& \eta_- \left((\eta_+)^{-1} (v, (1+\tau_{m-1}) \cdot (1+iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1}) \right) \\
= \quad & (v, -i(1-\tau_m) \cdot (1+iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1}) \\
= \quad & (v, v \cdot (1+\tau_{m-1}) \cdot (1+iv) \cdot f(e_1, \ldots, e_{m-1}) \cdot \beta_2\beta_4 \cdots \beta_{m-1}),
\end{aligned}$$

for every f, since

$$\begin{aligned}
v \cdot (1+\tau_{m-1}) \cdot (1+iv) &= (-i)iv \cdot (1+\tau_{m-1}) \cdot (1+iv) \\
&= (-i)(1-\tau_{m-1}) \cdot iv \cdot (1+iv) \\
&= (-i)(1-\tau_{m-1}) \cdot (1+iv).
\end{aligned}$$

Since η_+ and η_- are vector bundle isomorphisms, it follows that

$$\eta_- \left((\eta_+)^{-1}(v, s) = (v, v \cdot s) \right)$$

for every (v, s) in $S_+^{m-1} \times \Delta^+(\mathbf{R}^{m-1})$.

Therefore, Theorem 1.4 is true. ♠ **(End Proof of Theorem 1.4)**

6.2 Projection Orientations

If $n \geq 0$ is an even integer and $S \to X$ is a an S^n-bundle over a compact space X, then a projection p over $C(S)$ will be called a **projection orientation for S** if, for every x in X, the restriction of p to the fiber S_x of S is a fundamental projection on the even sphere S_x. If this is the case, then we will also say that **S has a projection orientation p**, **S is projection oriented**, or that **p is a fundamental projection on S**. Also, a complex vector bundle V over the sphere bundle S is said to be a **fundamental vector bundle over S** if V restricted to each fiber, S_x, of S, is a fundamental vector bundle over the even sphere S_x.

In the case where X is an odd-dimensional compact riemannian manifold M, and $S \to M$ is the sphere bundle over M, then a projection orientation p for S will also be called **a projection orientation for M**, and we will also say that M **has a projection orientation p**, or that M is **projection oriented.**

Theorem 2.1 *If M is an odd-dimensional, compact, riemannian, spin manifold, then the unit sphere bundle of M is projection oriented.*

Proof: Let m be the dimension of M, and let

$$\pi : S \to M$$

denote the sphere bundle over M.

Look first at the case where $m \geq 3$. By Swan's theorem (1.2.2 of [**Bla**]), every complex vector bundle over S is a summand of a trivial complex vector bundle over S. Thus every complex vector bundle over S corresponds to a continuous projection on S, and fundamental vector bundles over S correspond to fundamental projections on S.

Thus, to prove the proposition in this case, it suffices to show that there exists a fundamental vector bundle over S.

To do this, we let $\pi^{-1}(\Delta)$ denote the lift of the bundle Δ of spinors over M, up to S. The lift is via the projection map π. If x is in M, then $\pi^{-1}(\Delta)$

restricted to S_x is really the same as the trivial bundle $S_x \times \Delta_x$ over the sphere S_x. For each v in S, we let

$$E_v = (1 + iv) \cdot \Delta_x.$$

We define $E \to S$ to be the subbundle of $\pi^{-1}(\Delta)$ whose fiber at each v in S is

$$E_v = (1 + iv) \cdot \Delta_x.$$

By the proof of Theorem 1.4, E restricted to each fiber S_x is a fundamental vector bundle over the sphere S_x. Therefore, E is a fundamental vector bundle over S. So the proposition is true when $m \geq 3$.

Now, look at the case $m = 1$. In this case, M is a compact one dimensional manifold. M is therefore diffeomorphic to the one dimensional unit sphere S^1. So, $TM \to M$ is isomorphic to $TS^1 \to S^1$ which is isomorphic to the trivial bundle $S^1 \times \mathbf{R} \to S^1$. The sphere bundle of M is therefore isomorphic to the sphere bundle

$$S^1 \times \{-1, 1\} \to S^1.$$

Define p on $S^1 \times \{-1, 1\}$ by letting $p(x, 1) = 1$ and $p(x, -1) = 0$ for every x in S^1. Then p is a projection on $S^1 \times \{-1, 1\}$ which, when restricted to each fiber $\{x\} \times \{-1, 1\} \cong S^0$ is the fundamental projection $(0, 1)$ on S^0. Hence p is a fundamental projection on $S^1 \times \{-1, 1\}$. So, the sphere bundle over M is projection oriented in the case $m = 1$. ♠

6.3 A Fundamental Unitary on Odd Euclidean Space

If X is a topological space, a $\boldsymbol{k \times k}$ **unitary on** $\boldsymbol{X}$ is any map from X to the group $U_k(\mathbf{C})$ of $k \times k$ unitaries over $\mathbf{C}$. A $k \times k$ unitary on X will also be called simply a **unitary on** $\boldsymbol{X}$. If a $k \times k$ unitary on X is continuous, as a function on X, we call it a continuous $k \times k$ unitary on X.

If A is any C*-algebra, we let $\boldsymbol{A^+}$ denote the C^*-algebra A with identity adjoined. Even if A already has an identity, this definition still makes sense and A^+ will be a strictly larger C^*-algebra than A. We define $\boldsymbol{U_k(A)}$ to be the group of all unitaries in $M_k(A^+)$ which are equivalent to 1_k modulo $M_k(A)$. (1_k is the $k \times k$ identity matrix over $\mathbf{C}$.) In the case where A already has a unit, $U_k(A)$ can be identified with the group of all unitaries in $M_k(A)$.

In the commutative case where $A = C_0(X)$ for some locally compact Hausdorff space X, $U_k(C_0(X))$ is the same as the algebra of all continuous $k \times k$ unitaries on X whose limit at infinity is 1_k.

If A is a C*-algebra, we let $\boldsymbol{\Omega(A)}$ denote **the suspension of** $\boldsymbol{A}$ which may be regarded as the C^*-algebra of all continuous functions f from the unit circle S^1 to A such that $f(1) = 0$. That is

$$\Omega(A) = \{f \in C(S^1, A) : f(1) = 0\}.$$

Alternatively, $\Omega(A)$ may also be regarded as the C^*-algebra $C_0(I, A)$ where I is any open subinterval of $\mathbf{R}$. Both views of $\Omega(A)$ will be used.

Definition 3.1 *If $f : A \to B$ is a C^*-algebra homomorphism, the* **suspension of f** *denoted* $\boldsymbol{\Omega(f)}$ *is the map*

$$\Omega(f) : \Omega(A) \to \Omega(B)$$

defined by the formula

$$\Omega(f)(g)(t) = f(g(t)), \ \forall t \in (0,1)$$

for every $g \in \Omega(A)$. (Elements of $\Omega(A)$ and $\Omega(B)$ are considered here as maps on the interval $(0,1)$).

Remark 3.2 *Note that if we make the identifications*

$$\Omega(A) \cong C_0(0,1) \otimes A,$$

$$\Omega(B) \cong C_0(0,1) \otimes B,$$

and $f : A \to B$ is a C^*-algebra homomorphism, then the suspension map

$$\Omega(f) : \Omega(A) \to \Omega(B)$$

is the same as the map

$$Id \otimes f : C_0(0,1) \otimes A \to C_0(0,1) \otimes B$$

where Id is the identity map on $C_0(0,1)$. That is,

$$\Omega(f) = Id \otimes f.$$

Let m be a positive integer and let $S^{m-1} \subseteq \mathbf{R}^m$ denote the unit sphere. We can imbed the suspension $\Omega(C(S^{m-1}))$ of $C(S^{m-1})$ into $C_0(\mathbf{R}^m)$ by first identifying $\Omega(C(S^{m-1}))$ with $C_0((0,\infty), C(S^{m-1}))$ and then letting

$$i : \Omega(C(S^{m-1})) \to C_0(\mathbf{R}^m) \tag{3.3}$$

be the embedding

$$(if)(v) = \begin{cases} f(\|v\|)(\frac{v}{\|v\|}), & \text{if } \|v\| \neq 0 \\ 0, & \text{if } \|v\| = 0 \end{cases} \tag{3.4}$$

for every $f \in \Omega(C(S^{m-1})), v \in \mathbf{R}^m$. This induces an embedding

$$i : U_k(\Omega(C(S^{m-1}))) \to U_k(C_0(\mathbf{R}^m)) \tag{3.5}$$

and a map

$$i_* : K_1(\Omega(C(S^{m-1}))) \to K_1(C_0(\mathbf{R}^m)). \tag{3.6}$$

If A is any C^*-algebra, let

$$\beta : K_0(A) \overset{\cong}{\to} K_1(\Omega(A))$$

denote the Bott map (See Section 9.1 of [**Bla**]). Note that depending on how we view $\Omega(A)$, if A has a unit, $U_k(\Omega(A))$ may be described by

$$U_k(\Omega(A)) \cong \{f \in C(S^1, U_k(A)) : f(1) = 1_k\},$$

or by

$$U_k(\Omega(A)) \cong \{f \in C([0,1], U_k(A)) : f(0) = f(1) = 1_k\}.$$

If A has a unit, the Bott map $\beta : K_0(A) \to K_1(\Omega(A))$ is the map in K-theory determined by maps

$$\beta : Proj_k(A) \to U_k(\Omega(A)), \tag{3.7}$$

also denoted by β, which, depending on how the $U_k(\Omega(A))$ are viewed, can be described as the map

$$\beta(p)(z) = zp + (1_k - p), \quad \forall z \in S^1,$$

or as the map

$$\beta(p)(t) = e^{2\pi i t} p + (1_k - p),\ \forall t \in [0,1].$$

These maps $\beta : Proj_k(A) \to U_k(\Omega(A))$ will also be referred to as Bott maps.

In the case where $A = C(S^{m-1})$, we have Bott maps

$$\beta : Proj_k(C(S^{m-1})) \to U_k(\Omega(C(S^{m-1}))), \tag{3.8}$$

and

$$\beta : K_0(C(S^{m-1})) \to K_1(\Omega(C(S^{m-1}))). \tag{3.9}$$

Composing with the maps $i : U_k(\Omega(C(S))) \to U_k(C_0(\mathbf{R}^m))$ and $i_* : K_1(\Omega(C(S))) \to K_1(C_0(\mathbf{R}^m))$, we get maps

$$i \circ \beta : Proj_k(C(S^{m-1})) \to U_k(C_0(\mathbf{R}^m)) \tag{3.10}$$

and

$$i_* \circ \beta : K_0(C(S^{m-1})) \to K_1(C_0(\mathbf{R}^m)). \tag{3.11}$$

Proposition 3.12 *Let m be a positive odd integer, and identify $K_0(C(S^{m-1}))$ with $\mathbf{Z} \oplus \mathbf{Z}$ as in Section 6.1. Write*

$$i_* \circ \beta : K_0(C(S^{m-1})) \to K_1(C_0(\mathbf{R}^m)),$$

as

$$i_* \circ \beta : \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}.$$

Then

$$0 \to \mathbf{Z} \oplus \{0\} \to \mathbf{Z} \oplus \mathbf{Z} \stackrel{i_* \circ \beta}{\to} \mathbf{Z} \to 0$$

is a short exact sequence. (The map $\mathbf{Z} \oplus \{0\} \to \mathbf{Z} \oplus \mathbf{Z}$ *is inclusion.) That is* $(i_* \circ \beta)(n, 0) = 0$ *for every integer* n*, and the restricted map*

$$i_* \circ \beta : \{0\} \oplus \mathbf{Z} \to \mathbf{Z}$$

is an isomorphism.

Proof: Consider the extension

$$0 \to \Omega(C(S^{m-1})) \xrightarrow{i} C_0(\mathbf{R}^m) \xrightarrow{q} \mathbf{C} \to 0$$

where $q : C_0(\mathbf{R}^m) \to \mathbf{C}$ is evaluation at $0 \in \mathbf{R}^m$. That is, $q(f) = f(0)$. We get the six-term exact sequence

$$\begin{array}{ccccc} K_1(\Omega(C(S^{m-1}))) & \xrightarrow{i_*} & K_1(C_0(\mathbf{R}^m)) & \xrightarrow{q_*} & K_1(\mathbf{C}) \\ {\scriptstyle \partial_0}\uparrow & & & & \downarrow \\ K_0(\mathbf{C}) & \xleftarrow{q_*} & K_0(C_0(\mathbf{R}^m)) & \xleftarrow{i_*} & K_0(\Omega(C(S^{m-1}))). \end{array}$$

Now, $K_1(\mathbf{C}) = 0$ and, since m is odd, $K_0(C_0(\mathbf{R}^m)) = 0$. By exactness of the above sequence, it follows that $K_0(\Omega(C(S^{m-1}))) = 0$. So the above sequence is the same as

$$\begin{array}{ccccc} K_1(\Omega(C(S^{m-1}))) & \xrightarrow{i_*} & K_1(C_0(\mathbf{R}^m)) & \longrightarrow & 0 \\ {\scriptstyle \partial_0}\uparrow & & & & \downarrow \\ K_0(\mathbf{C}) & \xleftarrow{q_*} & K_0(C_0(\mathbf{R}^m)) & \longleftarrow & 0 \end{array} \qquad (3.13)$$

which reduces to the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\partial_0} K_1(\Omega(C(S^{m-1}))) \xrightarrow{i_*} \mathbf{Z} \to 0. \qquad (3.14)$$

Claim 3.15 *We have*

$$\partial_0([1]) = \beta([1]),$$

or

$$\partial_0([1]) = \beta(1,0),$$

where [1] *in the expression* $\partial_0([1])$ *is the element of* $K_0(\mathbf{C})$ *determined by* $1 \in \mathbf{C}$, *and* [1] *in the expression* $\beta([1])$ *is the element* $[1] = (1,0)$ *of* $K_0(C(S^{m-1}))$ *determined by the constant function* 1 *on* S^{m-1}.

Proof of Claim 3.15: Let

$$g : [0,\infty) \to (0,1]$$

be any decreasing continuous function such that $g(0) = 1$, and $g(t) \to 0$ as $t \to \infty$. The function g gives an isomorphism

$$g^* : C_0((0,1), C(S^{m-1})) \to C_0((0,\infty), C(S^{m-1})).$$

given by $g^*(f) = f \circ g$ for every $f \in C_0((0,1), C(S^{m-1}))$. We now identify $\Omega(C(S^{m-1}))$ with $C_0((0,1), C(S^{m-1}))$. With this identification, and using the isomorphism g^*, the definition (3.4) of $i : \Omega(C(S^{m-1})) \to C_0(\mathbf{R}^m)$ translates to the formula

$$(if)(v) = \begin{cases} f(g(\|v\|))(\frac{v}{\|v\|}), & \text{if } v \neq 0 \\ 0, & \text{if } v = 0. \end{cases} \tag{3.16}$$

Now, if 1 is the constant function 1 on S^{m-1} then $\beta(1)(t) = e^{2\pi it}\,1 + 0 = e^{2\pi it} \cdot 1 \in U_1(\Omega(C(S^{m-1})))$ for every $t \in (0,1)$. That is,

$$\beta(1)(t) = e^{2\pi it} \cdot 1$$

for every $t \in (0,1)$.

Now, let $f \in C_0(\mathbf{R}^m)$ be the function

$$f(v) = \begin{cases} g(\|v\|), & \text{if } v \neq 0 \\ 1, & \text{if } v = 0 \end{cases} .$$

Since $f(0) = 1$, then $q(f) = 1$. So, by the definition of the index map ∂_0, $\partial_0([1]) = [e^{2\pi i f}] \in K_1(i(\Omega(C(S^{m-1})))) \cong K_1(\Omega(C(S^{m-1})))$.

Now, for every $v \in \mathbf{R}^m, v \neq 0$, $(e^{2\pi i f})(v) = e^{2\pi i f(v)} = e^{2\pi i g(\|v\|)} = (iu)(v)$ where $u \in U_1(\Omega(C(S^{m-1})))$ is the unitary satisfying

$$u(t) = e^{2\pi i t} \cdot 1, \ \forall t \in [0,1]$$

where 1 is the constant function 1 on S^{m-1}. That is, $e^{2\pi i f} = i(u)$, which means that $\partial_0([1]) = [e^{2\pi i f}]$ in $K_1(i(\Omega(C(S^{m-1})))$ corresponds to $[u]$ in $K_1(\Omega(C(S^{m-1})))$. That is, $\partial_0([1]) = [u]$.

We showed above that $\beta(1) = u$. Hence, $\beta([1]) = [u] = \partial_0([1])$, which proves Claim 3.15. ♠ **(End Proof of Claim 3.15)**

Proof of Proposition 3.12 (cont'd):

By Claim 3.15, $\beta(1,0) = \partial_0([1])$. By exactness of Diagram 3.13, $(i_* \circ \beta)(1,0) = i_*(\partial_0([1])) = 0$. Thus, $(i_* \circ \beta)(n,0) = 0$ for every $n \in \mathbf{Z}$.

Also, by exactness of (3.14), $i_* : K_1(\Omega(C(S^{m-1}))) \to \mathbf{Z}$ is onto. Since the Bott map $\beta : K_0(C(S^{m-1})) \to K_1(\Omega(C(S^{m-1})))$ is an isomorphism, it follows that $i_* \circ \beta : \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}$ is also onto. From this and the fact that $(i_* \circ \beta)(n,0) = 0$ for all $n \in \mathbf{Z}$, it is clear that $i_* \circ \beta$ is an isomorphism from $\{0\} \oplus \mathbf{Z}$ onto $\mathbf{Z}$. The proposition is therefore true. ♠

Corollary 3.17 *Let m be an odd positive number. Let p be a fundamental projection on S^{m-1}. Then $(i_* \circ \beta)([p])$ in $K_1(C_0(\mathbf{R}^m)) \cong \mathbf{Z}$ is a generator of* $\mathbf{Z}$.

Proof: Since p is fundamental, $[p] = (n, 1)$ or $(n, -1)$ for some integer n. So, by Claim 3.12, $(i_* \circ \beta)([p])$ is a generator of $\mathbf{Z}$. ♠

6.4 The Kunneth Map

Let N denote the class of "nice" C*-algebras defined in 22.3.4 of [**Bla**]. If A and B are C*-algebras and $A \in N$, let

$$\kappa : (K_0(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B)) \to K_0(A \otimes B), \qquad (4.1)$$

and

$$\kappa : (K_0(A) \otimes K_1(B)) \oplus (K_1(A) \otimes K_0(B)) \to K_1(A \otimes B), \qquad (4.2)$$

be the injective group homomorphisms given by the Kunneth Theorem for tensor products (Theorem 23.1.3 of [**Bla**]). We also use κ to denote the restricted maps

$$\kappa : K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes B),$$

which are also injective group homomorphisms. All of these maps κ will be referred to as **Kunneth maps**. However, we have the following definition.

Definition 4.3 *The maps κ in (4.1) and (4.2) will be called the* **full**, *or the* **total Kunneth maps for $A \otimes B$**.

Theorem 4.4 *The total Kunneth maps κ in (4.1) and (4.2) are isomorphisms if either $K_*(A)$ or $K_*(B)$ is torsion free.*

Proof: This is part of Theorem 23.1.3 of [**Bla**]. ♠

If A is any C^*-algebra, we know that $K_i(A) \cong K_{i+1}(\Omega(A))$. The Bott map $\beta : K_0(A) \to K_1(\Omega(A))$ gives the isomorphism between $K_0(A)$ and $K_1(\Omega(A))$. The symbol θ is used in [**Bla**] for the isomorphism

$$\theta : K_1(A) \xrightarrow{\cong} K_0(\Omega(A)).$$

Notation 4.5 *In this thesis, the symbol* $\boldsymbol{\beta}$ *and the term* **Bott map** *will also be used for the isomorphism* $K_1(A) \overset{\cong}{\to} K_0(\Omega(A))$ *given above. If we need to distinguish this Bott map*

$$\beta : K_1(A) \to K_0(\Omega(A))$$

from the other Bott map $\beta : K_0(A) \to K_1(\Omega(A))$ *we will use the subscript* i *for the Bott map*

$$\beta_i : K_i(A) \to K_{i+1}(\Omega(A)).$$

Proposition 4.6 *The Kunneth maps (4.1) and (4.2) are natural and respect suspension and index maps (or boundary maps).*

To respect suspension means that if $A, B \in N$ *then the diagrams*

$$\begin{array}{ccc} K_i(A) \otimes K_j(B) & \xrightarrow{\kappa} & K_{i+j}(A \otimes B) \\ \downarrow{\scriptstyle Id_* \otimes \beta} & & \downarrow{\scriptstyle \beta} \\ K_j(A) \otimes K_{j+1}(\Omega(B)) & \xrightarrow{\kappa} & K_{i+j+1}(A \otimes \Omega(B)) \\ & & \cong K_{i+j+1}(\Omega(A \otimes B)) \end{array}$$

and

$$\begin{array}{ccc} K_i(A) \otimes K_j(B) & \xrightarrow{\kappa} & K_{i+j}(A \otimes B) \\ \downarrow{\scriptstyle \beta \otimes Id_*} & & \downarrow{\scriptstyle \beta} \\ K_{i+1}(\Omega(A)) \otimes K_j(B) & \xrightarrow{\kappa} & K_{i+j+1}(\Omega(A) \otimes B) \\ & & \cong K_{i+j+1}(\Omega(A \otimes B)) \end{array}$$

commute.

To respect index maps means that if

$$0 \to J \xrightarrow{i} B \xrightarrow{q} E \to 0 \qquad (4.7)$$

is an extension of C^*-algebras in N, and if A is another C^*-algebra in N such that the sequence

$$0 \to A \otimes J \overset{Id \otimes i}{\to} A \otimes B \overset{Id \otimes q}{\to} A \otimes E \to 0 \tag{4.8}$$

is exact, and if

$$\partial_i : K_i(E) \to K_{i+1}(J)$$

are the index maps of (4.7) and

$$\partial_{A,i} : K_i(A \otimes E) \to K_{i+1}(A \otimes J)$$

are the index maps of (4.8), then the diagram

$$\begin{array}{ccc} K_i(A) \otimes K_j(E) & \overset{\kappa}{\longrightarrow} & K_{i+j}(A \otimes E) \\ \downarrow{\scriptstyle Id_* \otimes \partial_j} & & \downarrow{\scriptstyle \partial_{A,i+j}} \\ K_i(A) \otimes K_{j+1}(J) & \overset{\kappa}{\longrightarrow} & K_{i+j+1}(A \otimes J) \end{array}$$

commutes.

Proof: See page 446 of [**Sch**] or Theorem 5.1 of [**Kar**]. ♠

Remark 4.9 *By definition of κ, if $p \in Proj_k\ (A)$, and $q \in Proj_l\ (B)$, then $[p] \otimes [q]$ in $K_0(A) \otimes K_0(B)$ is sent by κ to*

$$\kappa([p] \otimes [q]) = [p \otimes q] \tag{4.10}$$

in $K_0\ (A \otimes B)$.

Lemma 4.11 *The Kunneth map*

$$\kappa : K_0(\mathbf{C}) \otimes K_0(\mathbf{C}) \to K_0(\mathbf{C} \otimes \mathbf{C}) \tag{4.12}$$

is an isomorphism.

Proof: Since $K_1(\mathbf{C}) = 0$, one of the total Kunneth maps in this case is the map (4.12). But, since $K_0(\mathbf{C}) \cong \mathbf{Z}$, and $K_1(\mathbf{C}) = 0$ are both torsion free, the total Kunneth maps are isomorphisms. Therefore the map (4.12) is an isomorphism. ♠

If we view $\Omega(\mathbf{C})$ as $\{f \in C(S^1) : f(1) = 0\}$, then clearly

$$\Omega(\mathbf{C})^+ = C(S^1).$$

Thus the inclusion $i : \Omega(\mathbf{C}) \to C(S^1)$ induces an isomorphism

$$i_* : K_1(\Omega(\mathbf{C})) \stackrel{\cong}{\to} K_1(C(S^1)).$$

Composing i_* with the Bott map

$$\beta : K_0(\mathbf{C}) \to K_1(\Omega(\mathbf{C}))$$

gives an isomorphism

$$i_* \circ \beta : K_0(\mathbf{C}) \to K_1(C(S^1))$$

determined by the maps

$$i_* \circ \beta : Proj_n(\mathbf{C}) \to U_n(C(S^1))$$

with the property that

$$(i_* \circ \beta)(1) = z \tag{4.13}$$

where $z \in C(S^1)$ is the identity function $z \mapsto z$ on S^1.

Now, we observe that $\Omega(C(X)\otimes C(Y)) \cong \Omega(C(X))\otimes C(Y)$. Also, the Kunneth map respects suspensions. What this means is that the diagram

$$\begin{array}{ccc} K_0(C(X))\otimes K_0(C(Y)) & \xrightarrow{\kappa} & K_0(C(X)\otimes C(Y)) \\ \downarrow{\scriptstyle \beta\otimes Id_*} & & \downarrow{\scriptstyle \beta} \\ K_1(\Omega(C(X)))\otimes K_0(C(Y)) & \xrightarrow{\kappa} & K_1(\Omega(C(X)\otimes C(Y))) \\ & & \cong K_1(\Omega(C(X))\otimes C(Y)) \end{array}$$

commutes.

Hence, the diagram

$$\begin{array}{ccc} K_0(\mathbf{C})\otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_0(\mathbf{C}\otimes \mathbf{C}) \\ \downarrow{\scriptstyle \beta\otimes Id_*} & & \downarrow{\scriptstyle \beta} \\ K_1(\Omega(\mathbf{C}))\otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_1(\Omega(\mathbf{C})\otimes \mathbf{C}) \\ & & \cong K_1(\Omega(\mathbf{C}\otimes \mathbf{C})) \end{array}$$

commutes. By naturality of κ, the diagram

$$\begin{array}{ccc} K_1(\Omega(\mathbf{C}))\otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_1(\Omega(\mathbf{C})\otimes \mathbf{C}) \\ \downarrow{\scriptstyle i_*\otimes Id_*} & & \downarrow{\scriptstyle (i\otimes Id)_*} \\ K_1(C(S^1))\otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_1(C(S^1)\otimes \mathbf{C}) \end{array}$$

also commutes. Putting these diagrams together gives the commutative diagram

$$\begin{array}{ccc} K_0(\mathbf{C})\otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_0(\mathbf{C}\otimes \mathbf{C}) \\ \downarrow{\scriptstyle (i_*\circ\beta)\otimes Id_*} & & \downarrow{\scriptstyle (i\otimes Id)_*\circ\beta} \\ K_1(C(S^1))\otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_1(C(S^1)\otimes \mathbf{C}). \end{array} \tag{4.14}$$

Lemma 4.15 *The Kunneth map*

$$\kappa : K_1(C(S^1))\otimes K_0(\mathbf{C}) \to K_1(C(S^1)\otimes \mathbf{C})$$

is an isomorphism which satisfies

$$\kappa([z]\otimes[1]) = [z\otimes 1].$$

Proof: By (4.13), $(i_* \circ \beta)(1) = z$. Thus,

$$((i_* \circ \beta) \otimes Id_*)([1] \otimes [1]) = [z] \otimes [1]$$

in $K_1((C(S^1)) \otimes K_0(\mathbf{C})$. By commutativity of Diagram 4.14, it follows that

$$\kappa([z] \otimes [1]) = ((i \otimes Id_*)_* \circ \beta)(\kappa([1] \otimes [1]))$$

$$\begin{aligned} &= ((i \otimes Id)_* \circ \beta)([1 \otimes 1]) \\ &= (i \otimes Id)_*([\beta(1 \otimes 1)]) \end{aligned}$$

where $[\beta(1 \otimes 1)] \in K_1(\Omega(\mathbf{C} \otimes \mathbf{C})) \cong K_1(\Omega(\mathbf{C}) \otimes \mathbf{C})$.

Now, $\beta(1 \otimes 1)$ is the map $z \mapsto z(1 \otimes 1) = (z \otimes 1)$ from S^1 to $\mathbf{C} \otimes \mathbf{C}$. We may therefore write

$$\beta(1 \otimes 1) = z \otimes 1$$

in $U_1(\Omega(\mathbf{C}) \otimes \mathbf{C})$. Thus

$$(i \otimes Id)_*([\beta(1 \otimes 1)])$$

$$\begin{aligned} &= (i \otimes Id)_*([z \otimes 1]) \\ &= [i(z) \otimes 1] \\ &= [z \otimes 1] \end{aligned}$$

in $K_1(C(S^1) \otimes \mathbf{C})$.

Therefore $\kappa([z] \otimes [1]) = [z \otimes 1]$.

Since $[z \otimes 1]$ is a generator of $K_1(C(S^1) \otimes \mathbf{C}) \cong \mathbf{Z}$, and $[z] \otimes [1]$ is a generator of $K_1(C(S^1)) \otimes K_0(\mathbf{C}) \cong \mathbf{Z}$, it follows that the map κ is an isomorphism in this case. ♠

If a is an element of a C^*-algebra, we will use $\boldsymbol{C^*(a)}$ to denote the C^*-algebra generated by a.

Now take $p \in Proj_l\ (B)$ and $v \in U_k\ (A)$. Assume $p \neq 0$.

The spectrum of v is a subset of S^1. So, for every $f \in C(S^1)$, we have by the functional calculus an element $f(v)$ in $C^*(v) \subseteq M_k(A)$. Let

$$s : C(S^1) \to M_k(A)$$

be the $*$-homomorphism given by the formula

$$s(f) = f(v).$$

Next, we note that $C^*(p) \cong \mathbf{C}$, the isomorphism being the map

$$j : \mathbf{C} \to C^*(p)$$

with the formula

$$j(\lambda) = \lambda p.$$

Now, the diagram

$$\begin{array}{ccc} K_1(C(S^1)) \otimes K_0(\mathbf{C}) & \xrightarrow{\kappa} & K_1(C(S^1) \otimes \mathbf{C}) \\ \Big\downarrow{\scriptstyle s_* \otimes j_*} & & \Big\downarrow{\scriptstyle (s \otimes j)_*} \\ K_1(A) \otimes K_0(B) & \xrightarrow{\kappa} & K_1(A \otimes B) \end{array} \tag{4.16}$$

commutes (by naturality of κ).

Proposition 4.17 *The Kunneth map*

$$\kappa : K_1(A) \otimes K_0(B) \to K_1(A \otimes B)$$

satisfies

$$\kappa([v] \otimes [p]) = [v \otimes p + 1_k \otimes 1_l - 1_k \otimes p]$$

in $K_1(A \otimes B)$.

Proof: Note that $s(z) = z(v) = v$, so that $s_*([z]) = [v]$ in $K_1(A)$. Also, $s(1) = 1(v) = 1_k$ (since $v \in U_k A$). Furthermore, $j(1) = p$, which implies that $j([1]) = [p]$ in $K_0(B)$. So $[v] \otimes [p] = (s_* \otimes j_*)([z] \otimes [1])$. By commutativity of Diagram 4.16, it follows that

$$\begin{aligned}
\kappa([v] \otimes [p]) &= \kappa((s_* \otimes j_*)([z] \otimes [1])) = (s \otimes j)_*(\kappa([z] \otimes [1])) \\
&= (s \otimes j)_*([z \otimes 1]) \\
&\quad \text{(by Lemma 4.15)} \\
&= [(s \otimes j)(z \otimes 1) + 1_k \otimes 1_l - (s \otimes j)(1 \otimes 1)] \\
&= [s(z) \otimes j(1) + 1_k \otimes 1_l - s(1) \otimes j(1)] \\
&= [v \otimes p + 1_k \otimes 1_l - 1_k \otimes p]. \spadesuit
\end{aligned}$$

6.5 The Diagonal Map and Cup Product

Let X be a compact Hausdorff space. Let $\pi : \mathcal{B} \to X$ be a continuous C*-algebra bundle over X. Assume that $\mathcal{B}$ is a B-bundle where B is a fixed C*-algebra. Let $\mathcal{A}$ be a C*-subalgebra of the C*-algebra $C(\mathcal{B})$ of continuous sections of $\mathcal{B}$. Define the $*$-homomorphism

$$\rho : C(X, \mathcal{A}) \to C(\mathcal{B})$$

by the formula

$$\rho(f)(x) = f(x)(x) \in \mathcal{B}_x, \ \forall x \in X,$$

for every $f \in C(X, \mathcal{A})$. This map ρ will be called **the diagonal map on $C(X, \mathcal{A})$**.

Definition 5.1 *$\mathcal{A}$ is said to be* **preserved on the diagonal of X** *if $\rho(f)$ belongs to $\mathcal{A}$ for every $f \in C(X, \mathcal{A})$.*

Remark 5.2 *If $\mathcal{A}$ is preserved on the diagonal of X, then the diagonal map ρ on $C(X, \mathcal{A})$ is actually a $*$-homomorphism*

$$\rho : C(X, \mathcal{A}) \to \mathcal{A}$$

from the C-algebra $C(X, \mathcal{A})$ to the C*-algebra $\mathcal{A}$.*

Identify $C(X, \mathcal{A})$ with $C(X) \otimes \mathcal{A}$. If $\varphi \in C(X)$ and $a \in \mathcal{A}$, the object $\varphi \otimes a$, as an element of $C(X, \mathcal{A})$, has the formula

$$(\varphi \otimes a)(x) = \varphi(x) \cdot a \ \text{ in } \mathcal{A}$$

for every $x \in X$.

Note that if $\varphi \in C(X)$ and $a \in \mathcal{A}$, then we can multiply a by φ on the left to get an element $\varphi \cdot a$ of $C(\mathcal{B})$. For $x \in X$, we have

$$(\varphi \cdot a)(x) = \varphi(x) \cdot a(x).$$

Lemma 5.3 *If $\varphi \in C(X)$ and $a \in \mathcal{A}$, then*

$$\rho(\varphi \otimes a) = \varphi \cdot a$$

Proof: By definition of ρ, we have, for every $x \in X$, $\rho(\varphi \otimes a)(x) = (\varphi \otimes a)(x)(x) = (\varphi(x) \cdot a)(x) = \varphi(x) \cdot a(x) = (\varphi \cdot a)(x)$. That is, $\rho(\varphi \otimes a) = \varphi \cdot a$. ♠

Proposition 5.4 *$\mathcal{A}$ is preserved on the diagonal of X if and only if $\varphi \cdot a \in \mathcal{A}$ for every $\varphi \in C(X)$ and every $a \in \mathcal{A}$, that is, if and only if $\mathcal{A}$ is* **$C(X)$-invariant**.

Proof: Suppose $\mathcal{A}$ is preserved on the diagonal of X. Let $\varphi \in C(X)$, $a \in \mathcal{A}$. We want to show that $\varphi \cdot a \in \mathcal{A}$. But $\varphi \cdot a$ is equal to $\rho(\varphi \otimes a)$ (by Lemma 5.3) which belongs to $\mathcal{A}$ by our assumption that $\mathcal{A}$ is preserved on the diagonal of X. So, the proposition is true in one direction.

Now, suppose that $\varphi \cdot a \in \mathcal{A}$ for every $\varphi \in C(X)$ and $a \in \mathcal{A}$. We want to show that $\rho(f) \in \mathcal{A}$ for every $f \in C(X, \mathcal{A})$. Identify $C(X, \mathcal{A})$ with $C(X) \otimes \mathcal{A}$. Since $C(X) \otimes \mathcal{A}$ is generated by the set of all $\varphi \otimes a$ such that $\varphi \in C(X)$ and $a \in \mathcal{A}$, it suffices to show that $\rho(\varphi \otimes a) \in \mathcal{A}$ for every $\varphi \in C(X)$ and $a \in \mathcal{A}$. But for every $\varphi \in C(X)$ and $a \in \mathcal{A}$, $\rho(\varphi \otimes a) = \varphi \cdot a$ (by Lemma 5.3) which belongs to $\mathcal{A}$ by assumption. The proposition is therefore true. ♠

Definition 5.5 *If $\mathcal{A}$ is $C(X)$ invariant, then, for every $a \in K_i(C(X))$ and $b \in K_j(\mathcal{A})$, we can define the* **cup product $a \cup b$** *in $K_{i+j}(\mathcal{A})$ by the formula*

$$a \cup b \stackrel{\text{def}}{=} \rho_*(\kappa(a \otimes b))$$

where

$$\kappa : K_i(C(X)) \otimes K_j(\mathcal{A}) \to K_{i+j}(C(X, \mathcal{A})) \cong K_{i+j}(C(X) \otimes \mathcal{A})$$

is the Kunneth map, and $\rho : C(X, \mathcal{A}) \to \mathcal{A}$ is the diagonal map (which maps into $\mathcal{A}$ by Remark 5.2 and Proposition 5.4).

Remark 5.6 *If $\mathcal{A}$ is $C(X)$-invariant, we will also regard the cup product as the map*

$$\cup = \rho_* \circ \kappa : K_i(C(X)) \otimes K_j(\mathcal{A}) \to K_{i+j}(\mathcal{A}).$$

Remark 5.7 *Let D be a C^*-algebra. If $A \in M_k(\mathbf{C}), B \in M_l(D)$, $A = (a_{ij}), B = (b_{ij})$, we identify $A \otimes B$ with the element*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1k}B \\ \vdots & & & \vdots \\ a_{k1}B & a_{k2}B & \cdots & a_{kk}B \end{pmatrix}$$

of $M_{kl}(D)$.

Each element A of $M_k(C(X))$ may be regarded as a continuous function from X to $M_k(\mathbf{C})$. So $A(x) \in M_k(\mathbf{C})$ for every $x \in X$, if $A \in M_k(C(X))$.

Since $\mathcal{A}$ is a C*-subalgebra of $C(\mathcal{B})$, then $M_l(\mathcal{A})$ can be considered a C*-subalgebra of $C(M_l(\mathcal{B}))$. So for every $B \in M_l(\mathcal{A})$, $x \in X$, $B(x)$ is an element of $M_l(\mathcal{B}_x)$.

We have

$$M_k(C(X)) \otimes M_l(\mathcal{A}) \cong C(X, M_{kl}(\mathcal{A})). \tag{5.8}$$

So $M_k(C(X)) \otimes M_l(\mathcal{A})$ may be viewed as a C*-subalgebra of $C(X, C(M_{kl}(\mathcal{B})))$.

Remark 5.9 *If $A \in M_k(C(X)), B \in M_l(\mathcal{A})$ and if A is thought of as a function from X to $M_k(\mathbf{C})$ and B is thought of as an element of $C(M_l(\mathcal{B}))$, then we identify $A \otimes B$ with the element of $C(X, M_{kl}(\mathcal{A}))$ given by the formula*

$$(A \otimes B)(x)(y) = A(x) \otimes B(y) \in M_{kl}(\mathcal{B}_y) \tag{5.10}$$

(for every $x, y \in X$), where $A(x) \otimes B(y)$ can be regarded as in Remark 5.7.

Definition 5.11 *Suppose $\mathcal{A}$ is $C(X)$-invariant. If $A \in M_k(C(X)), B \in M_l(\mathcal{A})$, we define **$A \odot B$** in $M_{kl}(\mathcal{A})$ by the formula*

$$(A \odot B)(x) = A(x) \otimes B(x) \in M_{kl}(\mathcal{B}_x)$$

for every $x \in X$, where $A(x) \otimes B(x)$ is given by Remark 5.7.

Remark 5.12 *In Definition 5.11, $A \odot B$ belongs to $M_{kl}(\mathcal{A})$ because $\mathcal{A}$ is $C(X)$-invariant.*

Lemma 5.13 *Suppose $\mathcal{A}$ is $C(X)$-invariant. If $A \in M_k(C(X)), B \in M_l(\mathcal{A})$, then*

$$(A \odot B)(x) = (A \otimes B)(x)(x)$$

for every $x \in X$*. Therefore*

$$\rho_*(A \otimes B) = A \odot B$$

in $M_{kl}(\mathcal{A})$.

Proof: Take $A \in M_k(C(X))$ and $B \in M_l(\mathcal{A})$.

By (5.10),

$$(A \otimes B)(x)(x) = A(x) \otimes B(x).$$

So

$$\begin{aligned} (A \odot B)(x) &\stackrel{\text{def}}{=} A(x) \otimes B(x) \\ &= (A \otimes B)(x)(x) \\ &= \rho_*(A \otimes B)(x). \spadesuit \end{aligned}$$

Lemma 5.14 *Assume* $\mathcal{A}$ *is* $C(X)$*-invariant and has a unit. Suppose* $q \in Proj_l(C(X)), v \in U_l(C(X))$*, and* $p \in Proj_k(\mathcal{A})$*. These represent elements* $[q] \in K_0(C(X)), [v] \in K_1(C(X))$*, and* $[p] \in K_0(\mathcal{A})$*. Then*

$$[q] \cup [p] = [q \odot p]$$

in $K_0(\mathcal{A})$*, and*

$$\begin{aligned} [v] \cup [p] &= [v \odot p + 1_l \odot 1_k - 1_l \odot p] \\ &= [v \odot p + 1_{lk} - 1_l \odot p] \end{aligned}$$

in $K_1(\mathcal{A})$*, where* 1_{lk} *is the* $lk \times lk$ *identity matrix, regarded as a constant section of* $\mathcal{A}$*.*

Proof: By Remark 4.9,

$$\kappa([q] \otimes [p]) = [q \otimes p]$$

in $K_0(C(X, \mathcal{A}))$. Hence

$$\begin{aligned} [q] \cup [p] &= (\rho_* \circ \kappa)([q] \otimes [p]) \\ &= \rho_*([q \otimes p]) \\ &= [\rho_*(q \otimes p)] \\ &= [q \odot p] \end{aligned}$$

in $K_0(\mathcal{A})$, by Lemma 5.13.

Also, by Proposition 4.17,

$$\kappa([v] \otimes [p]) = [v \otimes p + 1_l \otimes 1_k - 1_l \otimes p]$$

in $K_1(C(X, \mathcal{A}))$. Hence

$$\begin{aligned} [v] \cup [p] &= \rho_*(\kappa([v] \otimes [p])) \\ &= \rho_*([v \otimes p + 1_l \otimes 1_k - 1_l \otimes p]) \\ &= [\rho_*(v \otimes p + 1_l \otimes 1_k - 1_l \otimes p)] \\ &= [v \odot p + 1_l \odot 1_k - 1_l \odot p] \end{aligned}$$

by Lemma 5.13.♠

Remark 5.15 *Note that the algebra $C(X)$ may be considered the algebra of continuous sections of the trivial* **C** *C^*-algebra bundle over X. Thus for every $A \in M_k(C(X))$, $B \in M_k(C(X))$, $A \odot B$ in $M_{kl}(C(X))$ is well-defined and we have*

$$(A \odot B)(x) = A(x) \otimes B(x) \tag{5.16}$$

in $M_{kl}(\mathbf{C})$ *where* $A(x) \otimes B(x)$ *is regarded as in Remark 5.7. Also,*

$$\rho_*(A \otimes B) = A \odot B$$

where $\rho : C(X, C(X)) \to C(X)$ *is the diagonal map defined by the formula*

$$(\rho f)(x) = f(x, x),$$

and $A \otimes B$ *is considered an element of* $C(X, M_{kl}(C(X)))$ *with formula*

$$(A \otimes B)(x)(y) = A(x) \otimes B(y)$$

as in Remark 5.9

The following definition stems from (5.16).

Definition 5.17 *If* Y *is a set,* A *is a* $k \times k$ *matrix-valued function on* Y*, and* B *is an* $l \times l$ *matrix-valued function on* Y*, we define the* $kl \times kl$ *matrix valued function* $\boldsymbol{A \odot B}$ *on* Y *by the formula*

$$(A \odot B)(y) = A(y) \otimes B(y)$$

for every $y \in Y$*, where* $A(y) \otimes B(y)$ *is defined in Remark 5.7.*

Lemma 5.18 *If* $\mathcal{B}' \to X$ *is another* C^**-algebra bundle over* X *and* $\mathcal{A}'$ *is a* $C(X)$ *invariant* C^**-subalgebra of* $C(\mathcal{B}')$*, then* $\mathcal{A} \oplus \mathcal{A}'$ *is a* $C(X)$*-invariant* C^**-subalgebra of* $C(\mathcal{B} \oplus \mathcal{B}') \cong C(\mathcal{B}) \oplus C(\mathcal{B}')$*, and if we make the identification*

$$C(X, \mathcal{A} \oplus \mathcal{A}') \cong C(X, \mathcal{A}) \oplus C(X, \mathcal{A}')$$

$$K_i(\mathcal{A} \oplus \mathcal{A}') \cong K_i(\mathcal{A}) \oplus K_i(\mathcal{A}'),$$

then, for every $c \in K_i(C(X))$, $a \in K_j(\mathcal{A})$, *and* $b \in K_j(\mathcal{A}')$, *we have*

$$c \cup (a, 0) = (c \cup a, 0)$$

and

$$c \cup (0, b) = (0, c \cup b)$$

in $K_{i+j}(\mathcal{A}) \oplus K_{i+j}(\mathcal{A}')$.

Proof: Follows easily from the definitions and the corresponding facts about the Kunneth and diagonal maps. ♠

Naturality.

Definition 5.19 *If* $\mathcal{B}' \to X$ *is another* C^**-algebra bundle over* X *and* $\mu : \mathcal{B} \to \mathcal{B}'$ *is a homomorphism of* C^**-algebra bundles over* X, *we let*

$$\mu_* : C(\mathcal{B}) \to C(\mathcal{B}'),$$

and

$$\mu_{**} : C(X, C(\mathcal{B})) \to C(X, C(\mathcal{B}')),$$

be the $*$*-homomorphism induced by* μ. *That is, we let*

$$\mu_*(b) = \mu \circ b$$

for every $b \in C(\mathcal{B})$, *and*

$$\mu_{**}(f) = \mu_* \circ f$$

for every $f \in C(X, C(\mathcal{B}))$.

Proposition 5.20 *If $\mu : \mathcal{B} \to \mathcal{B}'$ is a homomorphism of C^*-algebra bundles over X (as in Definition 5.19), then, for every $f \in C(\mathcal{B})$ and $\varphi \in C(X)$, we have*

$$\mu_*(\varphi \cdot f) = \varphi \cdot \mu_*(f)$$

in $C(\mathcal{B}')$.

Proof:

$$\begin{aligned} \mu_*(\varphi \cdot f)(x) &= (\mu \circ (\varphi \cdot f))(x) \\ &= \mu((\varphi \cdot f)(x)) \\ &= \mu(\varphi(x) \cdot f(x)) \\ &= \varphi(x) \cdot \mu(f(x)) \end{aligned}$$

by linearity of μ on the fibers. Hence

$$\begin{aligned} \mu_*(\varphi \cdot f)(x) &= \varphi(x) \cdot (\mu \circ f)(x) \\ &= \varphi(x)(\mu_* f)(x) \\ &= (\varphi \cdot (\mu_* f))(x) \end{aligned}$$

which implies $\mu_*(\varphi \cdot f) = \varphi \cdot \mu_*(f)$. ♠

Lemma 5.21 *If $\mu : \mathcal{B} \to \mathcal{B}'$ is a homomorphism of C^*-algebra bundles (as in Definition 5.19), and if $\mathcal{A}$ is $C(X)$- invariant, then $\mu_*(\mathcal{A})$ is also $C(X)$-invariant.*

Proof: Take $b \in \mu_*(\mathcal{A})$ and $\varphi \in C(X)$. We want to show that $\varphi \cdot b$ belongs to $\mu_*(\mathcal{A})$. Now, $b = \mu_*(a)$ for some $a \in \mathcal{A}$. So $\varphi \cdot b = \varphi \cdot \mu_*(a)$

$\mu_*(\varphi \cdot a)$ by Proposition 5.20. But $\mathcal{A}$ is $C(X)$ invariant. So $\varphi \cdot a \in \mathcal{A}$. Hence $\varphi \cdot b \in \mu_*(\mathcal{A})$. ♠

Proposition 5.22 *If $\mathcal{B}' \to X$ is another C^*-algebra bundle over X and $\mu : \mathcal{B} \to \mathcal{B}'$ is a homomoprhism of C^*-algebra bundles over X, then the diagram*

$$\begin{array}{ccc} C(X, C(\mathcal{B})) & \xrightarrow{\rho} & C(\mathcal{B}) \\ \big\downarrow{\scriptstyle \mu_{**}} & & \big\downarrow{\scriptstyle \mu_*} \\ C(X, C(\mathcal{B}')) & \xrightarrow{\rho} & C(\mathcal{B}') \end{array} \tag{5.23}$$

commutes.

If, in addition, $\mathcal{A}$ is $C(X)$ invariant, $\mathcal{A}'$ is a $C(X)$ invariant C^*-subalgebra of $C(\mathcal{B}')$, and $\mu_*(A) \subseteq \mathcal{A}'$, then the diagrams

$$\begin{array}{ccc} C(X, \mathcal{A}) & \xrightarrow{\rho} & \mathcal{A} \\ \big\downarrow{\scriptstyle \mu_{**}} & & \big\downarrow{\scriptstyle \mu_*} \\ C(X, \mathcal{A}') & \xrightarrow{\rho} & \mathcal{A}' \end{array}, \tag{5.24}$$

$$\begin{array}{ccc} K_i(C(X, \mathcal{A})) & \xrightarrow{\rho_*} & K_i(\mathcal{A}) \\ \big\downarrow{\scriptstyle \mu_*} & & \big\downarrow{\scriptstyle \mu_*} \\ K_i(C(X, \mathcal{A}')) & \xrightarrow{\rho_*} & K_i(\mathcal{A}') \end{array}, \tag{5.25}$$

and

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(\mathcal{A}) & \xrightarrow{\cup} & K_{i+j}(\mathcal{A}) \\ \big\downarrow{\scriptstyle Id_* \otimes \mu_*} & & \big\downarrow{\scriptstyle \mu_*} \\ K_i(C(X)) \otimes K_j(\mathcal{A}') & \xrightarrow{\cup} & K_{i+j}(\mathcal{A}') \end{array} \tag{5.26}$$

commute, and thus, if $a \in K_i(C(X))$ and $b \in K_j(\mathcal{A})$, then

$$\mu_*(a \cup b) = a \cup \mu_*(b)$$

in $K_{i+j}(\mathcal{A}')$.

Proof: Suppose $f \in C(X, C(\mathcal{B}))$ and $x \in X$. Then

$$\begin{aligned}
\rho(\mu_{**}(f))(x) &= (\mu_{**}f)(x)(x) \\
&= (\mu_* \circ f)(x)(x) \\
&= \mu_*(f(x))(x) \\
&= (\mu \circ f(x))(x) \\
&= \mu(f(x)(x)) \\
&= \mu(\rho(f)(x)) \\
&= (\mu \circ (\rho f))(x) \\
&= \mu_*(\rho(f))(x).
\end{aligned}$$

So $\rho(\mu_{**}(f)) = \mu_*(\rho(f))$, which implies that $\rho \circ \mu_{**} = \mu_* \circ \rho$. That is, Diagram 5.23 commutes.

Commutativity of Diagram 5.24 follows from that of Diagram 5.23. Commutativity of Diagram 5.25 then follows from that of Diagram 5.24.

Naturality of the Kunneth map κ gives commutativity of

$$\begin{array}{ccc}
K_i(C(X)) \otimes K_j(\mathcal{A}) & \xrightarrow{\kappa} & K_{i+j}(C(X, \mathcal{A})) \\
\big\downarrow{\scriptstyle Id_* \otimes \mu_*} & & \big\downarrow{\scriptstyle \mu_*} \\
K_i(C(X) \otimes K_j(\mathcal{A}') & \xrightarrow{\kappa} & K_{i+j}(C(X, \mathcal{A}')).
\end{array}$$

Putting this together with Diagram 5.25 (after first replacing i with $i + j$ in (5.25)), and using the fact that $\rho_* \circ \kappa$ is the cup product, gives commutativity of Diagram 5.26. ♠

Respect for Suspensions.

Recall that $\mathcal{B} \to X$ is assumed to be a B-bundle where B is a fixed C*-algebra.

Definition 5.27 *The* **suspension of $\mathcal{B}$** *denoted by*

$$\pi : \Omega(\mathcal{B}) \to X$$

is the $\Omega(B)$ C^*-algebra bundle derived from the bundle $\mathcal{B} \to X$ with fiber

$$\Omega(\mathcal{B})_x = \Omega(\mathcal{B}_x) \cong \Omega(B)$$

for each $x \in X$.

Definition 5.28 *Define*

$$i : \Omega(C(\mathcal{B})) \to C(\Omega(\mathcal{B}))$$

as follows. If

$$f \in C_0\ ((0,1), C(\mathcal{B}))$$

is an element of $\Omega(C(\mathcal{B}))$ and $x \in X$, define $i(f)(x) \in \Omega(\mathcal{B}_x) = \Omega(\mathcal{B})_x$ by the formula

$$i(f)(x)(t) = f(t)(x) \in \mathcal{B}_x\ , \tag{5.29}$$

for every $t \in (0,1)$.

Claim 5.30 *The map $i : \Omega(C(\mathcal{B})) \to C(\Omega(\mathcal{B}))$ defined above is an embedding, that is, an injective $*$-homomoprhism, of $\Omega(C(\mathcal{B}))$ into $C(\Omega(\mathcal{B}))$.*

Proof: Easy. ♠

Since $\Omega(\mathcal{A})$ is a C*-subalgebra of $\Omega(C(\mathcal{B}))$ this embedding $i : \Omega(C(\mathcal{B})) \to C(\Omega(\mathcal{B}))$ also gives an embedding

$$i : \Omega(\mathcal{A}) \to C(\Omega(\mathcal{B}))$$

of $\Omega(\mathcal{A})$ into $C(\Omega(\mathcal{B}))$. So $\Omega(\mathcal{A})$ may be regarded as a C*-subalgebra of $C(\Omega(\mathcal{B}))$.

Definition 5.31 *If* $f \in \Omega(C(\mathcal{B})) = C_0\ ((0,1), C(\mathcal{B}))$ *and* $\varphi \in C(X)$ *define* $\varphi \cdot f \in \Omega(C(\mathcal{B}))$ *by letting*

$$(\varphi \cdot f)(t) = \varphi \cdot (f(t)),\ \forall t \in C(0,1).$$

Lemma 5.32 *If* $\mathcal{A}$ *is* $C(X)$ *invariant and* $f \in \Omega(\mathcal{A}) = C_0((0,1), \mathcal{A})$*, then* $\varphi \cdot f$ *belongs to* $\Omega(\mathcal{A})$*.*

Proof: By definition,

$$(\varphi \cdot f)(t) = \varphi \cdot (f(t))$$

for every $t \in (0,1)$. Since $\mathcal{A}$ is $C(X)$ invariant and since $f(t) \in \mathcal{A}$ for every $t \in (0,1)$, then $\varphi \cdot f(t) \in \mathcal{A}$ for $t \in (0,1)$. Hence $(\varphi \cdot f)(t) = \varphi \cdot (f(t))$ belongs to $\mathcal{A}$ for every $t \in (0,1)$. Therefore $\varphi \cdot f \in \Omega(\mathcal{A})$. ♠

Lemma 5.33

$$i(\varphi \cdot f) = \varphi \cdot i(f) \quad in \quad C(\Omega(\mathcal{B}))$$

for every $f \in \Omega(C(\mathcal{B}))$*.*

Proof:

$$\begin{aligned} i(\varphi \cdot f)(x)(t) &= (\varphi \cdot f)(t)(x) \\ &= (\varphi \cdot (f(t)))(x) \\ &= \varphi(x) \cdot (f(t)(x)) \\ &= \varphi(x) \cdot (i(f)(x)(t)) \\ &= (\varphi \cdot i(f))(x)(t). \end{aligned}$$

Hence $i(\varphi \cdot f) = \varphi \cdot i(f)$. ♠

Proposition 5.34 *If $\mathcal{A}$ is $C(X)$ invariant then so is the C^*-subalgebra $i(\Omega(\mathcal{A}))$ of $C(\Omega(\mathcal{B}))$. That is, $\Omega(\mathcal{A})$, regarded as a C^*-subalgebra of $C(\Omega(\mathcal{B}))$, is $C(X)$ invariant if $\mathcal{A}$ itself is $C(X)$ invariant.*

Proof: Take $g \in i(\Omega(\mathcal{A}))$ and $\varphi \in C(X)$. Then $g = i(f)$ for some $f \in \Omega(\mathcal{A})$. So, by Lemma 5.33, $\varphi \cdot g = \varphi \cdot i(f) = i(\varphi \cdot f)$ in $C(\Omega(\mathcal{B}))$. Since $\mathcal{A}$ is $C(X)$-invariant and $f \in \Omega(\mathcal{A})$, then, by Lemma 5.32, $\varphi \cdot f$ belongs to $\Omega(\mathcal{A})$. Therefore $\varphi \cdot g = i(\varphi \cdot f)$ belongs to $i(\Omega(\mathcal{A}))$. This shows that $\varphi \cdot g \in i(\Omega(\mathcal{A}))$ for every $g \in i(\Omega(\mathcal{A}))$. Hence, $i(\Omega(\mathcal{A}))$ is $C(X)$ invariant. ♠

If $\mathcal{A}$ is $C(X)$ invariant, then, by Proposition 5.34, $i(\Omega(\mathcal{A}))$ is also $C(X)$-invariant, and therefore the diagonal map on $C(X, i(\Omega(\mathcal{A})))$ is a map

$$\rho : C(X, i(\Omega(\mathcal{A}))) \to i(\Omega(\mathcal{A}))$$

which we also consider as a map

$$\rho : C(X, \Omega(\mathcal{A})) \to \Omega(\mathcal{A}) \tag{5.35}$$

after identifying $\Omega(\mathcal{A})$ with $i(\Omega(\mathcal{A}))$. If we now make the identification

$$C(X, \Omega(\mathcal{A})) \cong \Omega(C(X, \mathcal{A}))$$

then this map can be considered as a map

$$\rho : \Omega(C(X, \mathcal{A})) \to \Omega(\mathcal{A}).$$

Proposition 5.36 *Suppose $\mathcal{A}$ is $C(X)$ invariant. Then the diagonal map*

$$\rho_\Omega : C(X, i(\Omega(\mathcal{A}))) \to i(\Omega(\mathcal{A})),$$

considered as a map

$$\rho_\Omega : \Omega(C(X,\mathcal{A})) \to \Omega(\mathcal{A}),$$

is the suspension of the diagonal map

$$\rho : C(X,\mathcal{A}) \to \mathcal{A}.$$

That is,

$$\rho_\Omega = \Omega(\rho).$$

Another way to express this is to say that the diagonal map ρ **respects suspensions.**

Proof: If

$$\mu : \Omega(C(X,\mathcal{A})) \overset{\cong}{\to} C(X,\ i(\Omega(\mathcal{A})))$$

is the identification map, we want to show that the diagram

$$\begin{array}{ccc} C(X,i(\Omega(\mathcal{A}))) & \xrightarrow{\rho_\Omega} & i(\Omega(\mathcal{A})) \\ \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle i} \\ \Omega(C(X,\mathcal{A})) & \xrightarrow{\Omega(\rho)} & \Omega(\mathcal{A}) \end{array} \tag{5.37}$$

commutes.

Note first that $f \in \Omega(C(X,\mathcal{A}))$ is mapped by μ to $\mu(f)$ where

$$\mu(f)(x)(y)(t) = f(t)(x)(y) \tag{5.38}$$

for every $x, y \in X$ and $t \in (0,1)$. So

$$\begin{aligned} \rho_\Omega(\mu(f))(x)(t) &\overset{\text{def}}{=} \mu(f)(x)(x)(t) \\ &= f(t)(x)(x) \ \ \text{(by (5.38))} \end{aligned}$$

$$
\begin{aligned}
&= \rho(f(t))(x) \\
&= (\Omega\rho)(f)(t)(x) \\
&= i(\Omega\rho)(x)(t) \quad \text{(by (5.29))},
\end{aligned}
$$

which implies that $(\rho_\Omega \circ \mu)(f) = i \circ (\Omega\rho)$. Diagram 5.37 therefore commutes. ♠

Corollary 5.39 *Identify $\Omega(\mathcal{A})$ as a C^*-subalgebra of $C(\Omega(\mathcal{B}))$, and assume $\mathcal{A}$ is $C(X)$ invariant. Then the diagrams*

$$
\begin{array}{ccc}
K_{i+1}(C(X,\Omega(\mathcal{A}))) & \xrightarrow{\rho_*} & K_{i+1}(\Omega(\mathcal{A})) \\
\cong K_{i+1}(\Omega(C(X,\mathcal{A}))) & & \\
\uparrow \beta & & \uparrow \beta \\
K_i(C(X,\mathcal{A})) & \xrightarrow{\rho_*} & K_i(\mathcal{A})
\end{array}
\tag{5.40}
$$

and

$$
\begin{array}{ccc}
K_i(C(X)) \otimes K_j(\mathcal{A}) & \xrightarrow{\cup} & K_{i+j}(\mathcal{A}) \\
\downarrow Id_* \otimes \beta & & \downarrow \beta \\
K_i(C(X)) \otimes K_{j+1}(\Omega(\mathcal{A})) & \xrightarrow{\cup} & K_{i+j+1}(\Omega(\mathcal{A}))
\end{array}
\tag{5.41}
$$

commute.

Hence, if $a \in K_i(C(X))$ and $b \in K_j(\mathcal{A})$, then

$$\beta(a \cup b) = a \cup \beta(b)$$

$$\beta(a \cup b) = a \cup \beta(b)$$

in $K_{i+j+1}(\Omega(\mathcal{A}))$.

Proof: Commutativity of (5.40) is a consequence of (Proposition 5.36) and the fact that the Bott map $\beta : K_i(A) \to K_{i+1}(A)$ respects suspensions.

Since the Kunneth map κ respects suspension, we also have commutativity of

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(\mathcal{A}) & \xrightarrow{\kappa} & K_{i+j}(C(X,\mathcal{A})) \\ \big\downarrow{\scriptstyle Id_* \otimes \beta} & & \big\downarrow{\scriptstyle \beta} \\ K_i(C(X)) \otimes K_{j+1}(\Omega(\mathcal{A})) & \xrightarrow{\kappa} & K_{i+j+1}(C(X,\Omega(\mathcal{A}))). \end{array}$$

Putting this together with Diagram 5.40 (after first replacing i with $i+j$ in (5.40)) and using $\cup = \rho^* \circ \kappa$, gives commutativity of Diagram 5.41. ♠

Respect for Index Maps.

Let $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{A}_1 = \mathcal{A}$ and suppose that $\mathcal{B}_0 \to X$ and $\mathcal{B}_2 \to X$ are two other C*-algebra bundles over X. Let

$$\mathcal{B}_0 \xrightarrow{\eta} \mathcal{B}_1 \xrightarrow{\mu} \mathcal{B}_2$$

be a sequence of C*-algebra bundle homomorphisms and let $\mathcal{A}_0 \subseteq C(\mathcal{B}_0)$ and $\mathcal{A}_2 \subseteq C(\mathcal{B}_2)$ be C*-subalgebras such that

$$\mu_*(\mathcal{A}_1) = \mathcal{A}_2,$$

$$\eta_*(\mathcal{A}_0) \subseteq \mathcal{A}_1,$$

and such that the sequence

$$0 \to \mathcal{A}_0 \xrightarrow{\eta_*} \mathcal{A}_1 \xrightarrow{\mu_*} \mathcal{A}_2 \to 0 \tag{5.42}$$

is exact.

Exactness of (5.42) implies exactness of

$$0 \to C(X,\mathcal{A}_0) \xrightarrow{\eta_{**}} C(X,\mathcal{A}_1) \xrightarrow{\mu_{**}} C(X,\mathcal{A}_2) \to 0.$$

Proposition 5.43 *Assume that $\mathcal{A}_0$ and $\mathcal{A}_1$ are $C(X)$ invariant. (By Lemma 5.21, it follows that $\mathcal{A}_2 = \mu_*(\mathcal{A}_1)$ is also $C(X)$ invariant.) Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(X,\mathcal{A}_0) & \xrightarrow{\eta_{**}} & C(X,\mathcal{A}_1) & \xrightarrow{\mu_{**}} & C(X,\mathcal{A}_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \\ 0 & \longrightarrow & \mathcal{A}_0 & \xrightarrow{\eta_*} & \mathcal{A}_1 & \xrightarrow{\mu_*} & \mathcal{A}_2 & \longrightarrow & 0 \end{array}$$

commutes.

Proof: Follows from Proposition 5.22. ♠

Corollary 5.44 *Assume $\mathcal{A}_0$ and $\mathcal{A}_1$ are $C(X)$ invariant. (By Lemma 5.21, $\mathcal{A}_2$ is also $C(X)$ invariant.) Then the diagrams*

$$\begin{array}{ccc} K_i(C(X,\mathcal{A}_2)) & \xrightarrow{\rho_*} & K_i(\mathcal{A}_2) \\ \downarrow{\scriptstyle\partial_i} & & \downarrow{\scriptstyle\partial_i} \\ K_{i+1}(C(X,\mathcal{A}_0)) & \xrightarrow{\rho_*} & K_{i+1}(\mathcal{A}_0) \end{array} \qquad (5.45)$$

and

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(\mathcal{A}_2) & \xrightarrow{\cup} & K_{i+j}(\mathcal{A}_2) \\ \downarrow{\scriptstyle Id_*\otimes\partial_j} & & \downarrow{\scriptstyle\partial_{i+j}} \\ K_i(C(X)) \otimes K_{j+1}(\mathcal{A}_0) & \xrightarrow{\cup} & K_{i+j+1}(\mathcal{A}_0) \end{array} \qquad (5.46)$$

commute.

Hence, if $a \in K_i(C(X))$ and $b \in K_j(\mathcal{A}_2)$, then

$$\partial_{i+j}(a \cup b) = a \cup \partial_j(b).$$

Proof: Commutativity of Diagram 5.45 follows from Proposition 5.43 and naturality of the index maps.

Since the Kunneth map respects index maps, the diagram

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(\mathcal{A}_2) & \xrightarrow{\kappa} & K_{i+j}(C(X,\mathcal{A}_2)) \\ \big\downarrow {\scriptstyle Id_* \otimes \partial_j} & & \big\downarrow {\scriptstyle \partial_{i+j}} \\ K_i(C(X)) \otimes K_{j+1}(\mathcal{A}_0) & \xrightarrow{\kappa} & K_{i+j+1}(C(X,\mathcal{A}_0)) \end{array}$$

also commutes. Putting this together with Diagram 5.45 (after first replacing i with $i+j$ in (5.45)) and using $\cup = \rho_* \circ \kappa$, we get commutativity of Diagram 5.46. ♠

6.6 The Thom Isomorphism

Let X be a compact Hausdorff space and $E \xrightarrow{\pi} X$ a real continuous vector bundle of dimension m over X.

We let $\boldsymbol{\mathcal{E}} \xrightarrow{\pi} \boldsymbol{X}$ be the $C_0(\mathbf{R}^m)$ C*-algebra bundle over X whose fiber at each $x \in X$ is the commutative C*-algebra $C_0(E_x)$. The bundle $\mathcal{E} \xrightarrow{\pi} X$ will be called **the commutative C*-algebra bundle over $\boldsymbol{X}$ associated to the vector bundle $\boldsymbol{E} \xrightarrow{\pi} \boldsymbol{X}$.**

The C*-algebra $C(\mathcal{E})$ of continuous sections of $\mathcal{E}$ is of course isomorphic to the C*-algebra $C_0(E)$ of continuous functions on E vanishing at ∞. Under this isomorphism $\varphi \in C_0(E)$ corresponds to the continuous section $x \mapsto \varphi|_{E_x} \in C_0(E_x)$ for $x \in X$.

We will often view $C_0(E)$ as the algebra $C(\mathcal{E})$.

With this identification, we have the diagonal map

$$\rho : C(X, C_0(E)) \to C_0(E)$$

and the cup product

$$\cup = \rho_* \circ \kappa,$$

$$\cup : K_i(C(X)) \otimes K_j(C_0(E)) \to K_{i+j}(C_0(E)).$$

That is,

$$a \cup b = \rho_*(\kappa(a \otimes b))$$

for $a \in K_i(C(X)), b \in K_j(C_0(E))$.

If $x \in X$, let $i_x : E_x \to E$ denote the inclusion of E_x into E.

Definition 6.1 *A* **Thom class for E,** *or a* **K-theory orientation for E,** *is a class $u \in K^m(E)$ with the property that $(i_x)^*(u)$ in $K^m(E_x)$ is a generator for $K^m(E_x) \cong K^m(\mathbf{R}^m) \cong \mathbf{Z}$ for every x in X.*

If E has a K-theory orientation, we also say that E is **K-theory oriented.** *If $[u] \in K^m(E)$ is a Thom class represented by some unitary $u \in U_k(C_0(E))$, then we call u a* **Thom element** *for E.*

Definition 6.2 *If M is a compact C^∞-manifold, a Thom class for TM will also be called a* **Thom class for M** *or a* **K-theory orientation for M.** *If M has a K-theory orientation we will say that* **M is K-theory oriented.** *A Thom element for TM will also be called a* **Thom element for M.**

Theorem 6.3 (Thom Isomorphism Theorem) *If $u \in K^m(E)$ is a Thom class for E, then the map*

$$\Phi_u : K_i(C(X)) \to K_{m+i}(C_0(E)),$$

which sends a in $K_i(C(X))$ to

$$\Phi_u(a) = a \cup u$$

in $K_{m+i}(C_0(E))$, is an isomorphism, which will be called a **Thom isomorphism for E.**

Proof: This is Theorem 3.2 of [**Kam**]. See also [**Swi**].♠

6.7 The Thom Isomorphism and the Bott Map

Now, let $\pi : E \to X$ be a riemannian, real, vector bundle of dimension m over a compact Hausdorff space X. That is, there is a euclidean metric on each fiber E_x, which varies continuously with x. Let

$$S = \{v \in E : \|v\| = 1\}.$$

Then $\pi : S - X$ will be called **the (unit) sphere bundle of E** over X.

Definition 7.1 *The suspension $\Omega(C(S))$ of $C(S)$ can be imbedded into $C_0(E)$ by first identifying $\Omega(C(S))$ with $C_0((0,\infty), C(S))$ and then letting*

$$i : \Omega(C(S)) \to C_0(E)$$

be the map

$$(if)(v) = \begin{cases} f(\|v\|)(\frac{v}{\|v\|}), & \text{if } \|v\| \neq 0 \\ \\ 0, & \text{if } \|v\| = 0 \end{cases} \tag{7.2}$$

for every $f \in \Omega(C(S)), v \in E$.

This embedding induces another embedding

$$i : U_k(\Omega(C(S))) \to U_k(C_0(E)) \tag{7.3}$$

and a map

$$i_* : K_1(\Omega(C(S))) \to K_1(C_0(E)). \tag{7.4}$$

For each x in X, we have embeddings

$$E_x \to E, \quad S_x \to S,$$

corresponding restriction maps

$$r_x : C_0(E) \to C_0(E_x)$$

and

$$r_x : C(S) \to C(S_x),$$

and the induced maps

$$r_x : Proj_k(C(S)) \to Proj_k(C(S_x)),$$

$$r_x : U_k(C_0(E)) \to U_k(C_0(E_x)),$$

$$(r_x)_* : \Omega(C(S)) \to \Omega(C(S_x)),$$

and

$$(r_x)_* : U_k(\Omega(C(S))) \to U_k(\Omega(C(S_x))).$$

The last two maps will also be denoted by

$$r_x : \Omega(C(S)) \to \Omega(C(S_x))$$

and

$$r_x : U_k(\Omega(C(S))) \to U_k(\Omega(C(S_x))).$$

Now imbed $\Omega(C(S_x))$ into $C_0(E_x)$ in the same way that $\Omega(C(S^{m-1}))$ was embedded into $C_0(\mathbf{R}^m)$. This gives us, for every $x \in E$, an embedding

$$i_x : \Omega(C(S_x)) \to C_0(E_x) \tag{7.5}$$

satisfying the formula

$$(i_x f)(v) = \begin{cases} f(\|v\|)(\frac{v}{\|v\|}), & \text{if } \|v\| \neq 0 \\ 0, & \text{if } \|v\| = 0 \end{cases}$$

for f in $\Omega(C(S_x)), v \in E_x$, an embedding

$$i_x : U_k(\Omega(C(S_x))) \to U_k(C_0(E_x)) \tag{7.6}$$

and a map

$$(i_x)_* : K_1(\Omega(C(S_x))) \to K_1(C_0(E_x)). \tag{7.7}$$

The maps (7.5), (7.6) and (7.7) are essentially the same as the maps (3.3), (3.5), and (3.6), once E_x is identified with $\mathbf{R}^m$.

From the definitions, it is not hard to see that, for every $x \in X$, the diagram

$$\begin{array}{ccc} \Omega(C(S)) & \xrightarrow{i} & C_0(E) \\ \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} \\ \Omega(C(S_x)) & \xrightarrow{i_x} & C_0(E_x) \end{array} \tag{7.8}$$

commutes. It follows that the diagram

$$\begin{array}{ccc} U_k(\Omega(C(S))) & \xrightarrow{i} & U_k(C_0(E)) \\ \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} \\ U_k(\Omega(C(S_x))) & \xrightarrow{i_x} & U_k(C_0(E_x)) \end{array} \tag{7.9}$$

commutes, and from this we get commutativity of

$$\begin{array}{ccc} K_1(\Omega(C(S))) & \xrightarrow{i_*} & K_1(C_0(E)) \\ \downarrow{\scriptstyle (r_x)_*} & & \downarrow{\scriptstyle (r_x)_*} \\ K_1(\Omega(C(S_x))) & \xrightarrow{(i_x)_*} & K_1(C_0(E_x)). \end{array} \tag{7.10}$$

We also have the Bott map

$$\beta : K_0(C(S)) \to K_1(\Omega(C(S)))$$

determined by the Bott maps

$$\beta : Proj_k(C(S)) \to U_k(\Omega(C(S))),$$

and, for every x in X, we have the Bott map

$$\beta : K_0(C(S_x)) \to K_1(\Omega(C(S_x))) \tag{7.11}$$

determined by the Bott maps

$$\beta : Proj_k(C(S_x)) \to U_k(\Omega(C(S_x))). \tag{7.12}$$

The maps (7.11) and (7.12) are essentially the same as the maps (3.8) and (3.9) once E_x is identified with $\mathbf{R}^m$.

From the definitions, or from naturality of the Bott map, we see that the diagram

$$\begin{array}{ccc} Proj_k(C(S)) & \xrightarrow{\beta} & U_k(\Omega(C(S)) \\ \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} \\ Proj_k(C(S_x)) & \xrightarrow{\beta} & U_k(\Omega(C(S_x))) \end{array}$$

commutes for every $x \in X$, which gives commutativity of

$$\begin{array}{ccc} K_0(C(S)) & \xrightarrow{\beta} & K_1(\Omega(C(S))) \\ \downarrow{\scriptstyle (r_x)_*} & & \downarrow{\scriptstyle (r_x)_*} \\ K_0(C(S_x)) & \xrightarrow{\beta} & K_1(\Omega(C(S_x))) \end{array}$$

for every x in X. Putting these last two diagrams together with Diagrams 7.9 and 7.10 gives us commutativity of

$$\begin{array}{ccccc} Proj_k(C(S)) & \xrightarrow{\beta} & U_k(\Omega(C(S))) & \xrightarrow{i} & U_k(C_0(E)) \\ \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} \\ Proj_k(C(S_x)) & \xrightarrow{\beta} & U_k(\Omega(C(S_x))) & \xrightarrow{i_x} & U_k(C_0(E_x)) \end{array}$$

and

$$\begin{array}{ccccc} K_0(C(S)) & \xrightarrow{\beta} & K_1(\Omega(C(S))) & \xrightarrow{i_*} & K_1(C_0(E)) \\ \downarrow{\scriptstyle (r_x)_*} & & \downarrow{\scriptstyle (r_x)_*} & & \downarrow{\scriptstyle (r_x)_*} \\ K_0(C(S_x)) & \xrightarrow{\beta} & K_1(\Omega(C(S_x))) & \xrightarrow{(i_x)_*} & K_1(C_0(E_x)) \end{array}$$

which gives commutativity of

$$\begin{array}{ccc} Proj_k(C(S)) & \xrightarrow{i\circ\beta} & U_k(C_0(E)) \\ \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} \\ Proj_k(C(S_x)) & \xrightarrow{i_x\circ\beta} & U_k(C_0(E_x)) \end{array} \tag{7.13}$$

and

$$\begin{array}{ccc} K_0(C(S)) & \xrightarrow{i_*\circ\beta} & K_1(C_0(E)) \\ \downarrow{\scriptstyle (r_x)_*} & & \downarrow{\scriptstyle (r_x)_*} \\ K_0(C(S_x)) & \xrightarrow{(i_x)_*\circ\beta} & K_1(C_0(E_x)). \end{array} \tag{7.14}$$

By comments made earlier, we remark that the maps

$$i_x \circ \beta : Proj_k(C(S_x)) \to U_k(C_0(E_x))$$

and

$$(i_x)_* \circ \beta : K_0(C(S_x)) \to K_1(C_0(E_x))$$

are essentially the same as the maps

$$i \circ \beta : Proj_k(C(S^{m-1})) \to U_k(C_0(\mathbf{R}^m))$$

and

$$i_* \circ \beta : K_0(C(S^{m-1})) \to K_1(C_0(\mathbf{R}^m))$$

of (3.10) and (3.11), once E_x is identified with $\mathbf{R}^m$.

Theorem 7.15 *Suppose E is odd-dimensional and that p is a projection orientation for S. Say $p \in Proj_k(C(S))$. Then $(i \circ \beta)(p) \in U_k(C_0(E))$ is a Thom element for E.*

Proof: If $x \in X$, let S_x be the fiber of S at x, let $p_x = p|_{s_x} = r_x(p)$, let

$$u = i(\beta(p))$$

in $U_k(C_0(E))$, and let $u_x = u|_{C_0(E_x)} = r_x(u)$ in $U_k(C_0(E_x))$.

By commutativity of Diagram 7.13, $u_x = r_x(u) = r_x(i(\beta(p))) = (i_x \circ \beta)(r_x p) = (i_x \circ \beta)(p_x)$ for every $x \in X$. Since p is a projection orientation for S, p_x is a fundamental projection on the even sphere S_x for every $x \in X$. By Corollary 3.17 and by the fact that the map

$$i_x \circ \beta : Proj_k(C(S_x)) \to U_k(C_0(E_x))$$

is the same as the map

$$i \circ \beta : Proj_k(C(S^{m-1})) \to U_k(C_0(\mathbf{R}^m)),$$

once E_x is identified with $\mathbf{R}^m$, it follows that $[u_x] = [(i_x \circ \beta)(p_x)] \in K_1(C_0(E_x)) \cong \mathbf{Z}$ is a generator of $K_1(C_0(E_x))$ for every $x \in X$. Therefore, u is, by definition, a Thom element for E. ♠

Now, let

$$\pi : \mathcal{S} \to X$$

denote the $C(S^{m-1})$ C*-algebra bundle over X whose fiber at each x in X is the C*-algebra $C(S_x)$. The bundle $\mathcal{S} \xrightarrow{\pi} X$ is **the commutative C*-algebra bundle associated to the sphere bundle $S \xrightarrow{\pi} X$.**

Of course we have

$$C(S) \cong C(\mathcal{S}),$$

the isomorphism being the map which sends each $\varphi \in C(S)$ to the continuous section $x \mapsto \varphi|_{S_x}$ of $C(\mathcal{S})$.

Identifying $C(S)$ with $C(\mathcal{S})$, we obtain the diagonal map

$$\rho : C(X, C(S)) \to C(S)$$

and the cup product

$$\cup : K_i(C(X)) \otimes K_j(C(S)) \to K_{i+j}(C(S))$$

where $\cup = \rho_* \circ \kappa$.

Definition 7.16 *Define the C^*-algebra bundle homomorphism*

$$\hat{i} : \Omega(\mathcal{S}) \to \mathcal{E}$$

as the map which, on each fiber $\Omega(\mathcal{S})_x = \Omega(C(S_x))$, is the embedding

$$i_x : \Omega(C(S_x)) \to C_0(E_x) = \mathcal{E}_x$$

defined in (7.5).

That is,

$$\hat{i}|_{\Omega(\mathcal{S})_x} : \Omega(\mathcal{S})_x \to \mathcal{E}_x$$

is the same as the map

$$i_x : \Omega(C(S_x)) \to C_0(E_x).$$

Now since $C(S)$ can be considered the same as $C(\mathcal{S})$, then $\Omega(C(S))$ can be identified with $C(\Omega(\mathcal{S}))$.

The identification is the map which sends each f in $\Omega(C(S))$ to the section

$$x \mapsto r_x(f) \in \Omega(C(S_x))$$

where

$$r_x : \Omega(C(S)) \to \Omega(C(S_x))$$

is the map induced by the restriction map

$$r_x : C(S) \to C(S_x)$$

which sends φ to $\varphi|_{S_x}$.

Now, by commutativity of Diagram 7.8, we have that

$$r_x(i(f)) = i_x(r_x(f))$$

for every $f \in \Omega(C(S))$. Thus, considered as a map

$$i : C(\Omega(\mathcal{S})) \to C(\mathcal{E})$$

on sections of C*-algebra bundles, we see that i is the C*-algebra bundle homomorphism $\hat{i}_*$. So the following is true.

Lemma 7.17 *The embedding*

$$i : \Omega(C(S)) \to C_0(E)$$

is the same as

$$\hat{i}_* : C(\Omega(\mathcal{S})) \to C(\mathcal{E})$$

after making the appropriate identifications.

We can therefore apply Proposition 5.22 (naturality of the cup product) to get commutativity of the diagram

$$\begin{array}{ccc} K_i(C(X)) \otimes K_{j+1}(\Omega(C(S))) & \xrightarrow{\cup} & K_{i+j+1}(\Omega(C(S))) \\ \Big\downarrow{\scriptstyle Id_* \otimes i_*} & & \Big\downarrow{\scriptstyle i_*} \\ K_i(C(X)) \otimes K_{j+1}(C_0(E)) & \xrightarrow{\cup} & K_{i+j+1}(C_0(E)) \end{array} \qquad (7.18)$$

By Corollary 5.39 (respect for suspensions of the cup product), we also have commutativity of the diagram

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(C(S)) & \xrightarrow{\cup} & K_{i+j}(C(S)) \\ \big\downarrow{\scriptstyle Id_* \otimes \beta} & & \big\downarrow{\scriptstyle \beta} \\ K_i(C(X)) \otimes K_{j+1}(\Omega(C(S))) & \xrightarrow{\cup} & K_{i+j+1}(\Omega(C(S))) \end{array}$$

Putting this diagram above Diagram 7.18 gives us the following proposition.

Proposition 7.19 *The diagram*

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(C(S)) & \xrightarrow{\cup} & K_{i+j}(C(S)) \\ \big\downarrow{\scriptstyle Id_* \otimes (i_* \circ \beta)} & & \big\downarrow{\scriptstyle i_* \circ \beta} \\ K_i(C(X)) \otimes K_{j+1}(C_0(E)) & \xrightarrow{\cup} & K_{i+j+1}(C_0(E)) \end{array}$$

commutes. That is,

$$(i_* \circ \beta)(a \cup b) = a \cup (i_* \circ \beta)(b)$$

for every $a \in K_i(C(X))$ and $b \in K_j(C(S))$.

Now, suppose $A \in M_l(C(X))$ and $B \in M_k(C(S))$. Then we have $\pi^*(A) \in M_l(C(S))$ where $\pi : S \to X$ is the bundle projection map. We have the product $A \odot B$ in $M_{kl}(C(S))$ (Definition 5.11), and the product $\pi^*(A) \odot B$ in the same space $M_{kl}(C(S))$ (see Remark 5.15). These two elements are the same as asserted in the next lemma.

Lemma 7.20 *If $A \in M_l(C(X))$ and $B \in M_k(C(S))$, then*

$$A \odot B = \pi^*(A) \odot B$$

in $M_{kl}(C(S))$.

Proof: $A \odot B$ is defined with B considered as the section $x \mapsto B|_{S_x} \in C(S_x)$. By definition

$$(A \odot B)(x) = A(x) \otimes (B|_{S_x}) \in M_{lk}(C(S_x)).$$

Thus, for $v \in S_x$,

$$(A \odot B)(x)(v) = A(x) \otimes B(v) \in M_{lk}(\mathbf{C}).$$

That is, considered as an element of $M_{lk}(C(S))$, for every $v \in S$,

$$\begin{aligned} (A \odot B)(v) &= A(\pi v) \otimes B(v) \\ &= (\pi^* A)(v) \otimes B(v). \end{aligned}$$

But

$$(\pi^*(A) \odot B)(v) = (\pi^* A)(v) \otimes B(v)$$

by Remark 5.15, or by Definition 5.17. Thus $A \odot B = (\pi^* A) \odot B$. ♠

Lemma 7.21 *Suppose $q \in Proj_l(C(X))$, $v \in U_l(C(X))$, and $p \in Proj_k(C(S))$. Then*

$$[q] \cup [p] = [(\pi^* q) \odot p]$$

in $K_0(C(S))$, and

$$\begin{aligned} [v] \cup [p] &= [(\pi^* v) \odot p + 1_l \odot 1_k - 1_l \odot p] \\ &= [(\pi^* v) \odot p + 1_{lk} - 1_l \odot p] \end{aligned}$$

in $K_1(C(S))$, where 1_{lk} is the $lk \times lk$ identity matrix, regarded as a constant function.

Proof: By Lemma 7.20,

$$(\pi^* q) \odot p = q \odot p$$

and

$$(\pi^* v) \odot p = v \odot p.$$

Thus Lemma 7.21 follows from Lemma 5.14. ♠

Corollary 7.22 *Suppose E is odd-dimensional (dimension m) and that p is a projection orientation for S. Let $u = (i \circ \beta)(p)$ be the Thom element for E corresponding to p. (By Theorem 7.15 , u is a Thom element.) Then the Thom isomorphism*

$$\Phi_u : K_i(C(X)) \to K_{i+1}(C_0(E))$$

sends each $a \in K_i(C(X))$ to

$$\Phi_u(a) = (i_* \circ \beta)(a \cup [p]) \tag{7.23}$$

in $K_{i+1}(C_0(E))$.

Moreover, if $q \in Proj_l(C(X))$ and $v \in U_l(C(X))$, then

$$\Phi_u([q]) = (i_* \circ \beta)([(\pi^* q) \odot p]) \tag{7.24}$$

in $K_1(C_0(E))$, and

$$\begin{aligned} \Phi_u([v]) &= (i_* \circ \beta)([(\pi^* v) \odot p + 1_l \odot 1_k - 1_l \odot p]) \\ &= (i_* \circ \beta)([(\pi^* v) \odot p + 1_{lk} - 1_l \odot p]). \end{aligned} \tag{7.25}$$

Proof: By Theorem 6.3, $\Phi_u(a) = a \cup [u]$ for every a in $K_i(C(X))$. Since $u = (i \circ \beta)(p)$, then $[u] = (i_* \circ \beta)([p])$. Hence,

$$\begin{aligned}\Phi_u(a) &= a \cup [u] \\ &= a \cup (i_* \circ \beta)([p]) \\ &= (i_* \circ \beta)(a \cup [p])\end{aligned}$$

by Proposition 7.19 (respect for suspensions, of the cup product). This proves (7.23).

Now take $q \in Proj_l(C(X))$ and $v \in U_l(C(X))$. Then

$$\begin{aligned}\Phi_u([q]) &= (i_* \circ \beta)([q] \cup [p]) \\ &\quad \text{(by (7.23))} \\ &= (i_* \circ \beta)([(\pi^* q) \odot p])\end{aligned}$$

by Lemma 7.21, which proves (7.24).

Similarly,

$$\begin{aligned}\Phi_u([u]) &= (i_* \circ \beta)([v] \cup [p]) \\ &= (i_* \circ \beta)([(\pi^* v) \odot p + 1_{lk} - 1_l \odot p])\end{aligned}$$

which proves (7.25). ♠

6.8 Thom Extensions and the Thom Isomorphism

In this section, M is a compact riemannian manifold and $S \to M$ is the unit sphere bundle over M.

Recall (from Proposition 3.1.9) that if p is a nonzero continuous $k \times k$ projection on the sphere bundle over M, there are two Thom extensions, the $C(p)$ Thom extension

$$0 \to M_k(C_0(TM)) \xrightarrow{i} C(p) \xrightarrow{l} C(M) \to 0$$

and the $SC(p)$ Thom extension

$$0 \to M_k(C_0(TM)) \xrightarrow{i} SC(p) \xrightarrow{l} C(M) \to 0.$$

Theorem 8.1 *Suppose M is odd-dimensional and that p is a fundamental $k \times k$ projection on the sphere bundle over M. Let $u = (i \circ \beta)(p)$ be the Thom element for M determined by p. Then the index maps*

$$\partial_i : K_i(C(M)) \to K_{i+1}(C_0(TM))$$

determined by the $C(p)$ Thom extension are the same as the Thom isomorphism maps

$$\Phi_u : K_i(C(M)) \to K_{i+1}(C_0(TM)).$$

That is,

$$\partial_i = \Phi_u$$

on $K_i(C(M))$.

Proof of Case 1: First look at

$$\partial_1 : K_1(C(M)) \to K_0(C_0(TM)).$$

Take $v \in U_l(C(M))$.

So far we have used 1_n to denote the $n \times n$ identity matrix. Let us now extend its use to denote any matrix of the form

$$1_n = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots, 0)$$

Now, take any w in $C([0, \infty), U_{2kl}(C(S)))$ such that

$$w(0) = 1_{2kl} \quad ,$$

$w(t)$ approaches a limit $w(\infty) \in U_{2kl}(C(S))$ as $t \to \infty$, and

$$w(\infty) = \begin{pmatrix} (\pi^* v) \odot p + 1_{lk} - 1_l \odot p & 0 \\ 0 & ((\pi^* v) \odot p + 1_{lk} - 1_l \odot p)^* \end{pmatrix}$$

$$= \begin{pmatrix} (\pi^* v) \odot p + 1_{lk} - 1_l \odot p & 0 \\ 0 & (\pi^* v)^* \odot p + 1_{lk} - 1_l \odot p \end{pmatrix}.$$

Let

$$q(t) = w(t) \ 1_{kl} \ w(t)^*$$

for every $t \in (0, \infty)$. That is, let

$$q = w \ 1_{kl} \ w^*.$$

Then q is an element of $Proj_{2kl}(\Omega(C(S))^+)$ and by the definition of the Bott map

$$\beta : K_1(C(S)) \to K_0(\Omega(C(S))),$$

we have

$$\beta([(\pi^* v) \odot p + 1_{lk} - 1_l \odot p]) = [q] - [1_{kl}] \tag{8.2}$$

in $K_0(\Omega(C(S)))$.

Now, we find an expression for $\partial_1([v])$ in $K_0(C_0(TM))$. Define

$$iw \in C(TM, U_{2kl}(\mathbf{C}))$$

by letting

$$(iw)(a) = \begin{cases} w(\|a\|)(\frac{a}{\|a\|}), & \text{if } a \neq 0 \\ 1_{2kl}, & \text{if } a = 0 \end{cases}$$

For $a \in TM, a \neq 0$, we have

$$\begin{aligned} (iw)(a) &= w(\|a\|)(r(a)) \\ &= r^*(w(\|a\|))(a) \end{aligned}$$

where $r : TM\backslash M \to S$ is the retraction map. (Originally, i was defined on matrices over $\Omega(C(S))$. This definition extends the domain of this original i to include w.) As $\|a\| \to \infty, (iw)(a) = r^*(w(\|a\|))(a) \to r^*\ (w(\infty))(a)$. That is, $i(w)$ is equal to $r^*(w(\infty))$

$$= \begin{pmatrix} r^*((\pi^* v) \odot p + \pi^*(1_l) \odot (1_l - p)) & 0 \\ 0 & r^*((\pi^* v)^* \odot p + \pi^*(1_l) \odot (1_k - p)) \end{pmatrix}$$

$$= \begin{pmatrix} (\pi^*v) \odot (r^*p) + \pi^*(1_l) \odot (1_k - r^*p) & 0 \\ 0 & \pi^*(v^*) \odot r^*p + \pi^*(1_l) \odot (1_k - r^*p) \end{pmatrix}$$

at infinity.

It follows that

$$i(w) \in U_{2l}(C(p))$$

(i.e. $i(w)$ is a unitary in $M_{2l}(C(p)^+)$ equivalent to 1_{2l} modulo $M_{2l}(C(p))$) and that

$$l(i(w)) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$$

in $U_{2l}(C(M))$ (where

$$l : C(p)^+ \to C(M)^+$$

is the natural extension of the limiting map

$$l : C(p) \to C(M)$$

in the $C(p)$ Thom extension, and $l(iw)$ in $U_{2l}(C(M)^+)$ is identified in a standard way with an element of $U_{2l}(C(M))$).

So, by definition,

$$\begin{aligned} \partial_1([v]) &= [(iw)1_{kl}(iw)^*] - [1_{kl}] \\ &= [i(w \cdot 1_{kl}\ w^*)] - [1_{kl}] \\ &= [i(q)] - [1_{kl}] \\ &= i_*([q] - [1_{kl}]) \end{aligned}$$

in $K_0(C_0(TM))$. (Note: $[q]$ is in $K_0(\Omega(C(S))^+)$. So $[i(q)]$ is in $K_0(C_0(TM)^+)$.)

From (8.2), it follows that

$$\partial_1([v]) = (i_* \circ \beta)([(\pi^* v) \odot p + 1_{lk} - 1_l \odot p]).$$

By Corollary 7.22, this is equal to $\Phi_u([v])$. So $\partial_1([v]) = \Phi_u([v])$. ♠ **(End Proof of Case 1)**

Proof of Case 2: Now look at the second case.

$$\partial_0 : K_0(C(M)) \to K_1(C_0(TM)).$$

Take $q \in Proj_l(C(M))$.

Let $g(t) = e^{-\frac{1}{t}}$ for $t \in (0, \infty)$. Then $g(t)$ increases from 0 to 1. By definition, the Bott map

$$\beta : Proj_{kl}(C(S)) \to U_{kl}(\Omega(C(S)))$$

sends each $a \in Proj_{kl}(C(S))$ to $\beta(a)$ in $U_{kl}(\Omega(C(S))) \cong \{v \in C([0,\infty), U_{kl}(C(S)))$ $: v(0) = 1_{kl}$ and $v(t) \to 1_{kl}$ as $t \to \infty\}$, where

$$\beta(a)(t) = e^{2\pi i g(t)} a + (1_{kl} - a)$$

for $t \in (0, \infty)$. Therefore

$$\beta((\pi^* q) \odot p)(t) = e^{2\pi i g(t)}(\pi^* q) \odot p + 1_{kl} - (\pi^* q) \odot p \qquad (8.3)$$

for $t \in (0, \infty)$.

Now we get an expression for $\partial_0([q])$ in $K_1(C_0(TM))$. Define

$$f(t) = g(t)((\pi^* q) \odot p) \in C(S),$$

for $t \in (0,\infty)$. Note that $f \in C([0,\infty), C(S))$. Since $g(0)=0$, $g(\infty)=1$, we have that

$$f(0)=0, \text{ and } f(\infty) = (\pi^* q) \odot p.$$

Look at $i(f) \in C(TM)$ where $i(f)$ is by definition the function

$$i(f)(v) = \begin{cases} f(\|v\|)(\frac{v}{\|v\|}), & \text{if } v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

As before, we have that $i(f) \in C(TM)$ is equal to $r^*(f(\infty)) = r^*((\pi^* q) \odot p) = (\pi^* q) \odot (r^* p)$ at infinity.

Since $q \in Proj_l(C(M))$, it follows that $i(f)$ belongs to $M_l(C(p))$ and that

$$l(i(f)) = q \in Proj_l(C(M)).$$

By definition of the index map ∂_0, we therefore have that

$$\partial_0([q]) = [e^{2\pi i \cdot i(f)}] \tag{8.4}$$

in $K_1(C_0(TM))$. (Note to the reader: In the expression $e^{2\pi i \cdot i(f)}$ the left i is the complex number $i = \sqrt{-1}$, the right i is an inclusion. They were not supposed to meet like this. For the time being, we will use $\sqrt{-1}$ for the complex number i.)

Observe that

$$e^{2\pi\sqrt{-1}i(f)} = i(e^{2\pi\sqrt{-1}f}) \tag{8.5}$$

and that

$$(e^{2\pi\sqrt{-1}f})(t) \quad = \quad e^{2\pi\sqrt{-1}f(t)}$$

$$
\begin{aligned}
&= e^{2\pi\sqrt{-1}g(t)((\pi^* q)\odot p)} \\
&= e^{2\pi\sqrt{-1}g(t)}((\pi^* q)\odot p) + 1_{kl} - (\pi^* q)\odot p \\
&\quad \text{(since } (\pi^* q)\odot p \text{ is a projection)} \\
&= \beta((\pi^* q)\odot p)(t) \\
&\quad \text{(by (8.3).}
\end{aligned}
$$

(For any projection P in a C^*-algebra with identity 1, and for any $t \in \mathbf{R}$, $e^{2\pi itP} = e^{2\pi it}\, P + (1 - P)$.) That is,

$$e^{2\pi\sqrt{-1}f} = \beta((\pi^* q)\odot p).$$

Together with (8.5) and (8.4) this gives us that

$$
\begin{aligned}
\partial_0([q]) &= [(i\circ\beta)((\pi^* q)\odot p)] \\
&= (i_*\circ\beta)([(\pi^* q)\odot p]) \\
&= \Phi_u([q])
\end{aligned}
$$

by Corollary 7.22. Therefore $\partial_0 = \Phi_u$ on $K_0(C(M))$. ♠ **(End Proof of Case 2 and of Theorem 8.1).**

Lemma 8.6 *Let A be a C^*-algebra with unit. Suppose $p, q \in Proj_k(A)$, and $\|p - q\| < 1$. Then*

$$[p] = [q] \textit{ in } K_0(A).$$

Proof: This follows from Proposition 4.3.2 of [**Bla**]. ♠

The following lemma is the corresponding fact about unitaries.

Lemma 8.7 *Let A be a C^*-algebra. Suppose $u, v \in U_k(A)$ and that $\|u - v\| < 1$. Then $[u] = [v]$ in $K_1(A)$.*

Proof: Since $u \in U_k(A)$, then u is an element of $M_k(A^+)$ equivalent to 1_k modulo $M_k(A)$. The same is true for v. Hence $u - v$ is equivalent to 0 modulo $M_k(A)$. Let $u_t = tv + (1-t)u$, for $0 \leq t \leq 1$. Then $u_0 = u$, $u_1 = v$, and

$$u - u_t = t(u - v)$$

for $t \in [0,1]$. Since $u - v$ is equivalent to 0 modulo $M_k(A)$, the same is true for $t(u-v)$. Hence $u - u_t$ is equivalent to 0 mod $M_k(A)$. Since $u \equiv 1_k \bmod M_k(A)$, it follows that

$$u_t \equiv 1_k \bmod M_k(A)$$

for all $t \in [0,1]$.

Moreover $\|u - u_t\| = |t| \, \|u - v\| < |t| \leq 1$ for every $t \in [0,1]$. That is, $\|u - u_t\| < 1$, $\forall\, t \in [0,1]$. Since u is invertible, it follows that u_t is invertible for every $t \in [0,1]$.

So u_t is an invertible element of $M_k(A^+)$ equivalent to 1_k modulo $M_k(A)$ for every $t \in [0,1]$. In other words, $u_t \in GL_k(A)$ for every $t \in [0,1]$. Since $u_0 = u$, $u_1 = v$, it follows that $[u] = [v]$ in $K_1(A)$. ♠

We now give the $SC(p)$ analogue of Theorem 8.1.

Theorem 8.8 *Suppose M is odd-dimensional and that p is a fundamental projection on the sphere bundle over M. Let $u = (i \circ \beta)(p)$ be the Thom element for M determined by p. Then the index maps*

$$\partial_i : K_i(C(M)) \to K_{i+1}(C_0(TM))$$

determined by the $SC(p)$ Thom extension are the same as the Thom isomorphism maps

$$\Phi_u : K_i(C(M)) \to K_{i+1}(C_0(TM)).$$

That is,

$$\partial_i = \Phi_u$$

on $K_i(C(M))$.

Proof: We will use in this proof the subscript A on the limiting map

$$l_A : C(p) \to C(M)$$

of the $C(p)$ Thom extension to distinguish it from the limiting map

$$l : SC(p) \to C(M)$$

of the $SC(p)$ Thom extension.

Also, in this proof, let $\partial_{A,i} : K_i(C(M)) \to K_{i+1}(C_0(TM))$ denote the index map determined by the $C(p)$ Thom extension. By Theorem 8.1, $\partial_{A,i} = \Phi_u$ on $K_i(C(M))$. So to prove this theorem, it suffices to show that $\partial_i = \partial_{A,i}$.

Proof of Case 1: First, we show that $\partial_1 = \partial_{A,1}$. Take $v \in U_l(C(M))$. Let

$$V = (\exp^* v) \odot (r^*p) + 1_l \odot (1_k - r^*p)$$

which is an $lk \times lk$ unitary on $TM \backslash M$. Then

$$V^* = \exp^*(v^*) \odot (r^*p) + 1_l \odot (1_k - r^*p).$$

Similarly, we let

$$W = (\pi^* v) \odot (r^*p) + 1_l \odot (1_k - r^*p)$$

which is an $lk \times lk$ unitary on $TM \backslash M$. Then

$$W^* = \pi^*(v^*) \odot (r^*p) + 1_l \odot (1_k - r^*p).$$

For $t \in [0,1]$, let

$$\alpha(t) = \begin{pmatrix} \cos(t\frac{\pi}{2}) & -\sin(t\frac{\pi}{2}) \\ \sin(t\frac{\pi}{2}) & \cos(t\frac{\pi}{2}) \end{pmatrix} \odot 1_{lk}$$

$$= \begin{pmatrix} \cos(t\frac{\pi}{2}) \cdot 1_{lk} & -\sin(t\frac{\pi}{2}) \cdot 1_{lk} \\ \sin(t\frac{\pi}{2}) \cdot 1_{lk} & \cos(t\frac{\pi}{2}) \cdot 1_{lk} \end{pmatrix}.$$

and, for $t \geq 1$, let

$$\alpha(t) = \alpha(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \odot 1_{lk}.$$

If $s > 0$ we let

$$\alpha_s(t) = \alpha(\frac{t}{s}), \text{ for } t \geq 0.$$

We also define $\gamma_s \in U_{2lk}(SC(p))$ and $\gamma_{A,s} \in U_{2lk}(C(p)$ by letting

$$\gamma_s(w) = \begin{cases} \operatorname{diag}(V(w), 1_{lk}) \cdot \alpha_s(\|w\|)^* \cdot \operatorname{diag}(V(w)^*, 1_{lk}) \cdot \alpha_s(\|w\|), & \text{if } w \neq 0 \\ 1_{2lk}, & \text{if } w = 0 \end{cases} \tag{8.9}$$

$$\gamma_{A,s}(w) = \begin{cases} \operatorname{diag}(W(w), 1_{lk}) \cdot \alpha_s(\|w\|)^* \cdot \operatorname{diag}(W(w)^*, 1_{lk}) \cdot \alpha_s(\|w\|), & \text{if } w \neq 0 \\ 1_{2lk}, & \text{if } w = 0 \end{cases} \tag{8.10}$$

for every $w \in TM$.

Since V, W, and $\alpha_s(\|w\|)$ are all unitaries, then γ_s and $\gamma_{A,s}$ are continuous unitaries on TM for every $s > 0$.

Suppose $s > 0$. If $w \in TM, \|w\| \geq s$ then $\frac{\|w\|}{s} \geq 1$ which implies that $\alpha(\frac{\|w\|}{s}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \odot 1_{lk}$. So

$$\alpha_s(\|w\|) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \odot 1_{lk} \text{ whenever } \|w\| \geq s.$$

It follows that,

$$\gamma_s(w) = \begin{pmatrix} V(w) & 0 \\ 0 & V(w)^* \end{pmatrix} \text{ if } \|w\| \geq s \tag{8.11}$$

and

$$\gamma_{A,s}(w) = \begin{pmatrix} W(w) & 0 \\ 0 & W(w)^* \end{pmatrix} \text{ if } \|w\| \geq s. \tag{8.12}$$

Hence γ_s is equal to $\begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$ at infinity and $\gamma_{A,s}$ is equal to $\begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix}$

at infinity. Since

$$V = (\exp^* v) \odot (r^* p) + 1_l \odot (1_k - r^* p) \tag{8.13}$$

and

$$W = (\pi^* v) \odot (r^* p) + 1_l \odot (1_k - r^* p) \tag{8.14}$$

it follows that

$$\gamma_s \in M_{2l}(SC(p)^+),$$

$$\gamma_{A,s} \in M_{2l}(C(p)^+),$$

and that γ_s is equivalent to 1_{2l} modulo $M_{2l}(SC(p))$, and $\gamma_{A,s}$ is equivalent to 1_{2l} modulo $M_{2l}(C(p))$. Since both γ_s and $\gamma_{A,s}$ are continuous unitaries on TM, we have

$$\gamma_s \in U_{2l}(SC(p)^+),$$

and

$$\gamma_{A,s} \in U_{2l}(C(p)^+).$$

Since γ_s is also equivalent to 1_{2l} mod $M_{2l}(SC(p))$ and $\gamma_{A,s}$ is equivalent to 1_{2l} mod $M_{2l}(C(p))$, then we actually have $\gamma_s \in U_{2l}(SC(p))$ and $\gamma_{A,s} \in U_{2l}(C(p))$.

Moreover, from the expression (8.13) and (8.14) for V and W, and from the fact that γ_s equals $\begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$ and $\gamma_{A,s}$ equals $\begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix}$ at infinity, it follows that

$$l(\gamma_s) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix} \text{ and } l_A(\gamma_{A,s}) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}.$$

So, from the definitions of ∂_1 and $\partial_{A,1}$, we have

$$\partial_1([v]) = [\gamma_s \cdot 1_{kl} \cdot \gamma_s^*] - [1_{kl}]$$

and

$$\partial_{A,1}([v]) = [\gamma_{A,s} \cdot 1_{kl} \cdot \gamma_{A,s}{}^*] - [1_{kl}]$$

in $K_0(C_0(TM))$, for all $s > 0$.

We want to show that $\partial_1([v]) = \partial_{A,1}([v])$. From above, it suffices to show that, for some $s > 0$,

$$[\gamma_s \cdot 1_{kl} \cdot \gamma_s^*] = [\gamma_{A,s} \cdot 1_{kl} \cdot \gamma_{A,s}{}^*]$$

in $K_0(C_0(TM)^+)$. By Lemma 8.6, it suffices to show that for some $s > 0$, we have

$$\|\gamma_s \; 1_{kl} \; \gamma_s^* - \gamma_{A,s} \; 1_{kl} \; \gamma_{A,s}{}^*\|_\infty < 1.$$

By (8.11) and (8.12),

$$\gamma_s(w) = \begin{pmatrix} V(w) & 0 \\ 0 & V(w)^* \end{pmatrix}$$

and

$$\gamma_{A,s}(w) = \begin{pmatrix} W(w) & 0 \\ \\ 0 & W(w)^* \end{pmatrix}$$

when $\|w\| \geq s$. It follows that

$$(\gamma_s \cdot 1_{kl} \cdot \gamma_s^*)(w) = 1_{kl} = (\gamma_{A,s}\ 1_{kl}\ \gamma_{A,s}{}^*)(w)$$

whenever $\|w\| \geq s$. That is

$$\|(\gamma_s \cdot 1_{kl} \cdot \gamma_s^*)(w) - (\gamma_{A,s} 1_{kl} \gamma_{A,s})(w)\| = 0 \text{ whenever } \|w\| \geq s \qquad (8.15)$$

For any $w \in TM$, we have

$$\begin{aligned} & \|(\gamma_s \cdot 1_{kl} \cdot \gamma_s^*)(w) - (\gamma_{A,s} \cdot 1_{kl} \cdot \gamma_{A,s}{}^*)(w)\| \\ \\ \leq\ & \|(\gamma_s(w) - \gamma_{A,s}(w)) \cdot 1_{kl} \cdot \gamma_s(w)^*\| \\ & + \|\gamma_{A,s}(w) \cdot 1_{kl}(\gamma_s(w)^* - \gamma_{A,s}(w)^*)\| \\ & \text{(since } \gamma_s(w) \text{ and } \gamma_{A,s}(w) \text{ are unitaries)} \\ \leq\ & \|\gamma_s(w) - \gamma_{A,s}(w)\| + \|\gamma_s(w)^* - \gamma_{A,s}(w)^*\| \\ =\ & 2\|\gamma_s(w) - \gamma_{A,s}(w)\|. \end{aligned}$$

That is

$$\|(\gamma_s \cdot 1_{kl} \cdot \gamma_s^*)(w) - (\gamma_{A,s} \cdot 1_{kl} \cdot \gamma_{A,s}{}^*)(w)\| \leq 2\|\gamma_s(w) - \gamma_{A,s}(w)\| \qquad (8.16)$$

for all $w \in TM$.

From the definition (8.9) and (8.10) of γ_s and $\gamma_{A,s}$, we get that if $w \neq 0$, then $\|\gamma_s(w) - \gamma_{A,s}(w)\|$

$$\begin{aligned}
&\leq \|\mathrm{diag}(V(w) - W(w), 0_{lk})\| \cdot \|\alpha_s(\|w\|)^*\mathrm{diag}(V(w)^*, 1_{lk}) \cdot \alpha_s(\|w\|)\| \\
&\quad + \|\mathrm{diag}(W(w), 1_{lk})\| \cdot \|\alpha_s(\|w\|)^*\mathrm{diag}(V(w)^* - W(w)^*, 0_{lk})\alpha_s(\|w\|)\| \\
&= \|V(w) - W(w)\| + \|V(w)^* - W(w)^*\| \\
&\quad \text{(since } V(w), W(w), \text{ and } \alpha_s(\|w\|) \text{ are all unitaries)} \\
&= 2\|V(w) - W(w)\|.
\end{aligned}$$

That is,

$$\|\gamma_s(w) - \gamma_{A,s}(w)\| \leq 2\|V(w) - W(w)\| \tag{8.17}$$

for all $w \neq 0$. But, from the definitions of V and W we have that

$$V - W = [\exp^*(v) - \pi^*(v)] \odot (r^*p)$$

from which it follows that

$$\begin{aligned}
\|V(w) - W(w)\| &\leq \|(\exp^* v)(w) - (\pi^* v)(w)\| \\
&= \|v(\exp(w)) - v(\pi(w))\|
\end{aligned} \tag{8.18}$$

Since v continuous on the compact set M, v is uniformly continuous. So there is a $\delta > 0$ such that $\|v(x) - v(y)\| < \frac{1}{4}$ whenever $d(x, y) < \delta$. So, if $w \in TM$ and $\|w\| < \delta$, then $d(\exp(w), \pi(w)) = \|w\| < \delta$, which implies that $\|v(\exp(w)) - v(\pi(w))\| < \frac{1}{4}$. By (8.18), it follows that

$$\|V(w) - W(w)\| < \frac{1}{4} \text{ if } \|w\| < \delta.$$

From (8.17) it then follows that

$$\|\gamma_s(w) - \gamma_{A,s}(w)\| < \frac{1}{2} \text{ if } \|w\| < \delta.$$

This implies, by (8.16), that

$$\|(\gamma_s \cdot 1_{kl} \cdot \gamma_s^*)(w) - (\gamma_{A,s} \cdot 1_{kl} \cdot \gamma_{A,s}{}^*)(w)\| < 1 \ \text{ if } \ \|w\| < \delta. \tag{8.19}$$

From (8.15) and (8.19), it follows that

$$\|\gamma_s \cdot 1_{kl} \cdot \gamma_s^* - \gamma_{A,s} \cdot 1_{kl} \cdot \gamma_{A,s}{}^*\|_\infty < 1 \ \text{ if } \ 0 < s < \delta.$$

This completes the proof that $\partial_1 = \partial_{A,1}$. ♠ **(End Proof of Case 1)**

Proof of Case 2: Now, we show that $\partial_0 = \partial_{A,0}$. Take $q \in Proj_l(C(M))$. We want to show that $\partial_0([q]) = \partial_{A,0}([q])$ in $K_1(C_0(TM))$.

Let g be any function in $C([0,\infty))$ such that $0 \le g \le 1$, $g(0) = 0$, and $g(t) = 1$ for all $t \ge 1$. If $s > 0$, let

$$g_s(t) = g(\,\frac{1}{s}\,t)$$

for all $t \in [0,\infty)$. Then

$$g_s(t) = 1 \text{ if } t \ge s. \tag{8.20}$$

Now define continuous $lk \times lk$ matrices X_s and Y_s on TM by letting

$$X_s(v) = \begin{cases} g_s(\|v\|) \cdot ((\exp^* q) \odot (r^* p))(v), & \text{if } v \ne 0 \\ \\ 0, & \text{if } v = 0 \end{cases}$$

and

$$Y_s(v) = \begin{cases} g_s(\|v\|)((\pi^* q) \odot (r^* p))(v), & \text{if } v \ne 0 \\ \\ 0, & \text{if } v = 0 \end{cases}$$

for v in TM.

Since $g_s(t) = 1$ for all $t \geq s$, it follows that

$$X_s(v) = (\exp^* q)(v) \odot (r^*p)(v), \text{ if } \|v\| \geq s \tag{8.21}$$

and

$$Y_s(v) = (\pi^* q)(v) \odot (r^*p)(v), \text{ if } \|v\| \geq s. \tag{8.22}$$

As a consequence, we have that X_s is equal to $(\exp^* q) \odot (r^*p)$ at infinity, and Y_s is equal to $(\pi^* q) \odot (r^*p)$ at ∞. It follows that $X_s \in M_l(SC(p))$, and that $l(X_s) = q \in Proj_l(C(M))$, and similarly, $Y_s \in M_l(C(p))$ and $l_A(Y_s) = q$. From the definitions of ∂_0 and $\partial_{A,0}$, it follows that $\partial_0([q]) = [e^{2\pi i X_s}]$ and that $\partial_{A,0}([q]) = [e^{2\pi i Y_s}]$ in $K_1(C_0(TM))$.

To show $\partial_0([q]) = \partial_{A,0}([q])$ it therefore suffices to show that $[e^{2\pi i X_s}] = [e^{2\pi i Y_s}]$ in $K_1(C_0(TM))$ for some $s > 0$.

By Lemma 8.7, we only have to show that

$$\|e^{2\pi i X_s} - e^{2\pi i Y_s}\|_\infty < 1 \tag{8.23}$$

for some $s > 0$. But

$$\begin{aligned} e^{2\pi i X_s}(v) &= e^{2\pi i X_s(v)} \\ &= e^{2\pi i g_s(\|v\|)}(\exp^* q)(v) \odot (r^*p)(v) \\ &\quad +1_{lk} - (\exp^* q)(v) \odot (r^*p)(v) \end{aligned}$$

since $(\exp^* q)(v) \odot (r^*p)(v)$ is a projection and $X_s(v) = g_s(\|v\|)\ ((\exp^* q)(v) \odot (r^*p)(v))$. Similarly,

$$\begin{aligned} (e^{2\pi i Y_s})(v) &= e^{2\pi i g_s(\|v\|)}(\pi^* q)(v) \odot (r^*p)(v) \\ &\quad +1_{lk} - (\pi^* q)(v) \odot (r^*p)(v). \end{aligned}$$

Therefore

$$\|(e^{2\pi i X_s})(v) - (e^{2\pi i Y_s})(v)\|$$

$$\begin{aligned} &\leq |e^{2\pi i g_s(\|v\|)}| \ \|((\pi^* q)(v) - (\exp^* q)(v)) \odot (r^* p)(v)\| \\ &\quad + \|((\exp^* q)(v) - (\pi^* q)(v)) \odot (r^* p)(v)\| \\ &\leq 2\|(\exp^* q)(v) - (\pi^* q)(v)\|. \end{aligned}$$

That is

$$\|(e^{2\pi i X_s})(v) - (e^{2\pi i Y_s})(v)\| \leq 2\|(\exp^* q)(v) - (\pi^* q)(v)\| \tag{8.24}$$

for all $v \in TM$.

Now, by uniform continuity of q on M, there exists $\delta > 0$ such that

$$\|q(x) - q(y)\| < \frac{1}{2} \text{ when } d(x, y) < \delta.$$

If $v \in TM$ and $\|v\| < \delta$, then, of course, $d(\exp(v), \pi v) = \|v\| < \delta$. Therefore $\|(\exp^* q)(v) - (\pi^* q)(v)\| = \|q(\exp(v)) - q(\pi v)\| < \frac{1}{2}$ if $\|v\| < \delta$. By (8.24), it follows that

$$\|(e^{2\pi i X_s})(v) - (e^{2\pi i Y_s})(v)\| < 1 \text{ whenever } \|v\| < \delta. \tag{8.25}$$

Now, from (8.21) and (8.22), $X_s(v)$ and $Y_s(v)$ are projections when $\|v\| \geq s$. Therefore $e^{2\pi i X_s(v)} = e^{2\pi i Y_s(v)} = 1_l$ when $\|v\| \geq s$. This implies that

$$\|(e^{2\pi i X_s})(v) - (e^{2\pi i Y_s})(v)\| = 0 \text{ when } \|v\| \geq s. \tag{8.26}$$

From (8.26) and (8.25), we get that, if $0 < s < \delta$, then $\|e^{2\pi i X_s} - e^{2\pi i Y_s}\|_\infty < 1$. This proves (8.23) and the proof of Case 2 is complete. ♠ **(End Proof of Case 2)** ♠ **(End Proof of Theorem 8.8)**

Chapter 7

Index Maps of the Dirac Extension of $C_0(TM)$

7.1 Triviality of Hilbert Bundles

In this thesis, **every Hilbert space is assumed to be separable.**

If V is an infinite dimensional separable Hilbert space, and X is a compact Hausdorff space, we let

$$\pi_v : X \times V \to X \tag{1.1}$$

denote the trivial hilbert bundle given by the formula

$$\pi_V(x, v) = x$$

for every (x, v) in $X \times V$. This bundle will be called **the trivial hilbert bundle over X associated to V.**

If A is a C*-algebra, we have similarly defined the trivial bundle

$$\pi_A : X \times A \to X$$

called **the trivial C*-algebra bundle over X associated to A.** Note that the C*-algebra bundle

$$B(X \times V) \to X$$

associated to the trivial hilbert bundle

$$\pi_V : X \times V \to X$$

is isomorphic to the trivial C*-algebra bundle

$$\pi_{B(V)} : X \times B(V) \to X$$

in an obvious way, and, similarly, the $\mathcal{K}$ C*-algebra bundle

$$K(X \times V) \to X$$

over X associated to $\pi_V : X \times V \to X$ is isomorphic to the trivial C*-algebra bundle

$$\pi_{K(V)} : X \times K(V) \to X.$$

Theorem 1.2 *Every continuous infinite-dimensional Hilbert bundle $h \to X$ over a compact metric space X is trivial. That is, for any infinite-dimensional Hilbert space V, there is a hilbert bundle isomorphism U from the bundle $h \to X$ to the trivial hilbert bundle $X \times V \to X$ associated to V.*

Proof: Let $h \to X$ be a hilbert bundle over a compact metric space X. Let V be a Hilbert space. The pair $\mathcal{E} = ((h_x)_{x \in X},\ C(h))$ is a continuous field of Hilbert spaces over X as defined in 10.1.2 of [**Dix**].

Now, since $h \to X$ is a hilbert bundle, the field $\mathcal{E}$ is locally trivial. Also, since X is a compact metric space, it is a separable metric space. Moreover,

h_x is a separable Hilbert space by assumption for each $x \in X$. By Proposition 10.2.7 of [**Dix**], it follows that the field $\mathcal{E}$ is a separable continuous field of Hilbert space over X (as defined in 10.2.1 of [**Dix**]). This, together with compactness of X, implies (by Lemma 10.8.7 of [**Dix**]) that $\mathcal{E}$ is trivial, which is the same as saying that the hilbert bundle $h \to X$ is trivial. ♠

Let $h \to X$ be a continuous infinite dimensional hilbert bundle over a compact Hausdorff space X, let V be an infinite dimensional Hilbert space, and let $U : h \to X \times V$ be the hilbert bundle isomorphism (given by Theorem 1.2) between the hilbert bundles $h \to X$ and $\pi_V : X \times V \to X$. For each $x \in X$, we let

$$U_x : h_x \to V$$

be the restriction

$$U|_{h_x} : H_x \to \{x\} \times V \cong V.$$

Each U_x is a unitary.

U induces an isomorphism

$$U_* : B(h) \to X \times B(V)$$

between the C*-algebra bundles, $B(h) \to X$ and

$$\pi_{B(V)} : X \times B(V) \to X$$

and is given by the formula

$$U_*(a) = (x, U_x \cdot a \cdot U_x{}^*)$$

for every $a \in B(h)_x = B(h_x)$.

Because conjugation of a compact operator by a unitary gives another compact operator, we have that U_* restircts to an isomorphism

$$U_* : K(h) \to X \times K(V)$$

between the C*-algebra bundles $K(h) \to X$ and the trivial bundle

$$\pi_{K(V)} : X \times K(V) \to X.$$

This isomorphism in turn induces a C*-algebra isomorphism

$$U_* : C(K(h)) \to C(X, K(V))$$

between the algebra $C(K(h))$ of continuous sections of $K(h)$ and the algebra $C(X, K(V))$ of continuous functions from X to $K(V)$. The formula for this map is

$$(U_* f)(x) = U_*(f(x)).$$

We state this formally.

Theorem 1.3 *Let $h \to X, V$, and U be as in Theorem 1.2. Then the induced C*-algebra bundle isomorphism*

$$U_* : B(h) \to X \times B(V)$$

between $B(h) \to X$ and $\pi_{B(V)} : X \times B(V) \to X$, restricts to a C-algebra bundle isomorphism*

$$U_* : K(h) \to X \times K(V)$$

between $K(h) \to X$ and $\pi_{K(V)} : X \times K(V) \to X$.

This, in turn, induces a C-algebra isomorphism*

$$\begin{aligned} U_* : C(K(h)) \longrightarrow \quad & C(X, K(V)) \\ & \cong C(X) \otimes K(V). \end{aligned}$$

Let X be a compact Hausdorff space, and let $\mathcal{K}$ denote the algebra of compact operators. Let

$$o \to \mathcal{K} \xrightarrow{i} \mathcal{A} \xrightarrow{q} C_0(\mathbf{R}) \to 0 \tag{1.4}$$

be any C*-algebra extension of $C_0(\mathbf{R})$ by $\mathcal{K}$ whose index map

$$\partial : K_1(C_0(\mathbf{R})) \to K_0(\mathcal{K})$$

is an isomorphism. The extension (1.4) induces the extension

$$0 \to C(X,\mathcal{K}) \xrightarrow{i_*} C(X,\mathcal{A}) \xrightarrow{q_*} C(X,C_0(\mathbf{R})) \to 0. \tag{1.5}$$

Let

$$\partial_{X,i} : K_i(C(X,C_0(\mathbf{R}))) \to K_{i+1}(C(X,\mathcal{K}))$$

denote the index maps determined by the extension (1.5).

Since the Kunneth map respects index maps, the diagram

$$\begin{array}{ccc} K_i(C(X)) \otimes K_1(C_0(\mathbf{R})) & \xrightarrow{\kappa} & K_{i+1}(C(X,C_0(\mathbf{R})) \\ \big\downarrow{\scriptstyle Id_*\otimes\partial} & & \big\downarrow{\scriptstyle \partial_{X,i+1}} \\ K_i(C(X)) \otimes K_0(\mathcal{K}) & \xrightarrow{\kappa} & K_i(C(X,\mathcal{K})) \end{array} \tag{1.6}$$

commutes.

Now, both Kunneth maps in this diagram are isomorphisms. For $K_0(C_0(\mathbf{R})) = 0$ and $K_1(\mathcal{K}) = 0$ imply that the two Kunneth maps are both total. (See Definition 6.4.3). Since $K_*((C_0(\mathbf{R}))$ and $K_*(\mathcal{K})$ are both torsion free, it then follows that the two Kunneth maps in the above diagram are isomorphisms.

Since $\partial : K_1(C_0(\mathbf{R})) \to K_0(\mathcal{K})$ is assumed to be an isomorphism, then $Id_* \otimes \partial$ in the diagram is also an isomorphism.

So, all the maps in Diagram 1.6 are isomorphisms except possibly the map $\partial_{X,i+1}$. Since the diagram commutes, this map must also be an isomorphism. That is, the following is true.

Lemma 1.7 *The index map*

$$\partial_{X,i} : K_i(C(X, C_0(\mathbf{R}))) \to K_{i+1}(C(X, \mathcal{K}))$$

is an isomorphism.

Definition 1.8 *If $\mathcal{B} \to X$ is a C^*-algebra bundle over X, and $x \in X$, we let*

$$r_x : C(\mathcal{B}) \to \mathcal{B}_x$$

denote the $*$-homomorphism which sends each $f \in C(\mathcal{B})$ to $f(x) \in \mathcal{B}_x$.

Definition 1.9 *If $k \to X$ is a $\mathcal{K}$ C^*-algebra bundle over X, then $b \in K_0(C(k))$ is called a* **Thom class for $C(k)$** *if, for every $x \in X$,*

$$(r_x)_*(b) \in K_0(k_x) \cong K_0(K(H)) \cong \mathbf{Z}$$

is a generator of $\mathbf{Z}$.

Now regard

$$X \times \mathcal{K} \stackrel{Id \times i}{\to} X \times \mathcal{A} \stackrel{Id \times q}{\to} X \times C_0(\mathbf{R}) \qquad (1.10)$$

as a sequence of bundle homorphisms of trivial C*-algebra bundles over X. The C*-algebras $C(X, \mathcal{K}), C(X, \mathcal{A})$, and $C(X, C_0(\mathbf{R}))$ may be regarded as algebras of continuous sections of the bundles $X \times \mathcal{K}$, $X \times \mathcal{A}$ and $X \times C_0(\mathbf{R})$ respectively. Viewed in this way, we see that the extension (1.5) is induced by the sequence (1.10).

We can therefore apply Corollary 5.44 to the extension (1.5) to obtain the following.

Lemma 1.11 *The diagram*

$$\begin{array}{ccc} K_i(C(X)) \otimes K_j(C(X, C_0(\mathbf{R}))) & \xrightarrow{\cup} & K_{i+j}(C(X, C_0(\mathbf{R}))) \\ \downarrow{\scriptstyle Id_* \otimes \partial_{X,j}} & & \downarrow{\scriptstyle \partial_{X,i+j}} \\ K_i(C(X)) \otimes K_{j+1}(C(X, \mathcal{K})) & \xrightarrow{\cup} & K_{i+j+1}(C(X, \mathcal{K})) \end{array} \tag{1.12}$$

commutes.

Now it is clear that, for every $x \in X$, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(X, \mathcal{K}) & \longrightarrow & C(X, \mathcal{A}) & \longrightarrow & C(X, C_0(\mathbf{R})) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} & & \downarrow{\scriptstyle r_x} & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{A} & \longrightarrow & C_0(\mathbf{R}) & \longrightarrow & 0 \end{array}$$

commutes. It follows from naturality of index maps that

$$\begin{array}{ccc} K_1(C(X, C_0(\mathbf{R}))) & \underset{\cong}{\xrightarrow{\partial_{X,1}}} & K_0(C(X, \mathcal{K})) \\ \downarrow{\scriptstyle (r_x)_*} & & \downarrow{\scriptstyle (r_x)_*} \\ K_1(C_0(\mathbf{R})) & \underset{\cong}{\xrightarrow{\partial}} & K_0(\mathcal{K}) \end{array} \tag{1.13}$$

commutes.

Proposition 1.14 *If $a \in K_1(C(X, C_0(\mathbf{R})))$ is a Thom class, then $b = \partial_{X,1}(a)$ is a Thom class in $K_0(C(X, \mathcal{K}))$. Conversely if $b \in K_0(C(X, \mathcal{K}))$ is a Thom class, then $(\partial_{X,1})^{-1}(b) \in K_1(C(X, C_0(\mathbf{R})))$ is a Thom class.*

Proof: Suppose $a \in K_1(C(X, C_0(\mathbf{R})))$ and $b = \partial_{X,1}(a)$. By commutativity of Diagram 1.13, we have that

$$(r_x)_*(b) = \partial((r_x)_*(a))$$

for every $x \in X$. Hence b is a Thom class if and only if $\partial((r_x)_*(a))$ is a generator of $K_0(\mathcal{K})$ for every $x \in X$. But ∂ is an isomorphism. Hence $\partial((r_x)_*(a))$ is a generator of $K_0(\mathcal{K})$ for every x if and only if $(r_x)_*(a)$ is a generator of $K_1(C_0(M))$ for every x. Thus, b is a Thom class if and only if a is a Thom class.

This completes the proof since the map $\partial_{X,1}$ is an isomorphism.(Lemma 1.7). ♠

Definition 1.15 *If $\mathcal{B}' \to X$ is a C^*-algebra bundle, $\mathcal{A}'$ is a $C(X)$-invariant C^*-subalgebra of $C(\mathcal{B}')$, and $b \in K_j(\mathcal{A}')$, we define*

$$\Phi_b : K_i(C(X)) \to K_{i+j}(\mathcal{A}')$$

by the formula

$$\Phi_b(c) = c \cup b.$$

Lemma 1.16 *If $a \in K_1(C(X, C_0(\mathbf{R})))$ and $b = \partial_{X,1}(a)$, then*

$$\Phi_b = \partial_{X,i+1} \circ \Phi_a.$$

Proof: If $c \in K_i(C(X))$ then, by Lemma 1.11, $\Phi_b(c) = c \cup b = c \cup (\partial_{X,1}(a)) = \partial_{X,i+1}(c \cup a) = \partial_{X,i+1}\ (\Phi_a(c))$. Therefore $\Phi_b = \partial_{X,i+1} \circ \Phi_a$. ♠

Lemma 1.17 *Let $b \in K_0(C(X, \mathcal{K}))$ be a Thom class. Then*

$$\Phi_b : K_i(C(X)) \to K_i(C(X, \mathcal{K}))$$

is an isomorphism.

Proof: Let $a = (\partial_{X,1})^{-1}(b)$. Then, by Proposition 1.14, a is a Thom class. By the Thom Isomorphism Theorem (Theorem 6.6.3), it follows that

$$\Phi_a : K_i(C(X)) \to K_{i+1}(C(X, C_0(\mathbf{R})))$$

is an isomorphism. Since

$$\Phi_b = \partial_{X,i+1} \circ \Phi_a$$

(Lemma 1.16) and since $\partial_{X,i+1}$ is an isomorphism, it follows that Φ_b is an isomorphism. Lemma 1.17 is therefore true. ♠

Corollary 1.18 *Let $k \to X$ be a $\mathcal{K}$ C^*-algebra bundle over X, and let $b \in K_0(C(k))$ be a Thom class. Then*

$$\Phi_b : K_i(C(X)) \to K_i(C(k))$$

is an isomorphism.

Proof: By Theorem 1.3, $k \to X$ is isomorphic to $X \times \mathcal{K} \to X$ as C*-algebra bundles. Corollary 1.18 therefore follows from Lemma 1.17. ♠

7.2 Index Maps of the Dirac Extension with Periodic Multipliers

Let M be a **compact riemannian spin manifold of nonpositive curvature.** We assume in this section all the notation of Sections 5.1.

We consider now the adjoined Dirac extension

$$0 \to C(\mathcal{K}) \xrightarrow{i} \mathcal{A}_{01}(M) \xrightarrow{q} C_0(TM) \oplus C_0(TM) \to 0 \tag{2.1}$$

of $C_0(TM)$ (Definition 3.41). Note that $\mathcal{A}_{01}(M)$ is considered here a C*-subalgebra of $C(\mathcal{B})$.

Let

$$\partial_i : K_i(C_0(TM)) \oplus K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

denote the index maps obtained from this extension.

Define

$$\partial_{i,l} : K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

and

$$\partial_{i,r} : K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

by the formulas

$$\partial_{i,l}(a) = \partial_i(a, 0)$$

and

$$\partial_{i,r}(a) = \partial_i(0, a).$$

In this section we will show that the maps

$$\partial_{i,l} : K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

and

$$\partial_{i,r} : K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

are isomorphisms, and that

$$\partial_{i,l} = -\partial_{i,r}.$$

Let

$$0 \to \mathcal{K} \xrightarrow{i} \mathcal{T} \xrightarrow{q} C_0(\tilde{M}) \oplus C_0(\tilde{M}) \to 0$$

be the Toeplitz extension of $C_0(\tilde{M})$ given in Definition 2.7.13. Recall from Remark 2.11.16 that this is the same as the Dirac extension of $C_0(\tilde{M})$

$$0 \to \mathcal{K} \to \mathcal{D}_{C_0(\tilde{M})} + \mathcal{K} \to C_0(\tilde{M}) \oplus C_0(\tilde{M}) \to 0$$

with compacts adjoined. Recall that the bundle $\mathcal{K} \to M$ is the $\mathcal{K}$ C*-algebra bundle

$$K(H) \to M.$$

We will show that

$$\mathcal{A}_{01}(M) \cong C(t)$$

where $t \to M$ is a $\mathcal{T}$ C*-algebra bundle over M.

Recall that $B(\tilde{H}) \to \tilde{M}$ is isomorphic to the trivial C*-algebra bundle

$$\tilde{M} \times B(L^2(\tilde{\Delta})) \to \tilde{M},$$

the isomorphism being the map

$$\pi \times exp : B(\tilde{H}) \xrightarrow{\cong} \tilde{M} \times B(L^2(\tilde{\Delta})).$$

(See Section 4.1.7.)

Definition 2.2 *We define* **the bundle $\tilde{\mathcal{t}} \to \tilde{M}$ of Toeplitz operators** *as the C^*-algebra subbundle of $B(\tilde{H}) \to \tilde{M}$ corresponding to the sub-bundle*

$$\tilde{M} \times \mathcal{T} \to \tilde{M}$$

of the bundle $\tilde{M} \times B(\tilde{H}) \to \tilde{M}$. That is, we let

$$\tilde{\mathcal{t}} = (\pi \times exp)^{-1}(\tilde{M} \times \mathcal{T}).$$

Note that $\tilde{\mathcal{t}}$ is a $\mathcal{T}$ C^*-algebra bundle over $\tilde{M}$.

Lemma 2.3 *If $T \in \mathcal{T}$ and $h \in \Gamma = \pi_1(M)$, then $h \cdot T$ also belongs to $\mathcal{T}$. So Γ acts on $\mathcal{T}$.*

Moreover, if $q : \mathcal{T} \to C_0(\tilde{M}) \oplus C_0(\tilde{M})$ is the map in the Toeplitz extension, then for every $h \in \Gamma$, $T \in \mathcal{T}$, we have

$$q(h \cdot T) = h \cdot q(T).$$

That is q is invariant under the action by Γ.

Proof: Γ of course leaves $K(L^2(\tilde{\Delta}))$ invariant (since Γ acts by inner automorphisms). It also clearly leaves $\{M_\varphi : \varphi \in C_0(\tilde{M})\}$ invariant, and $h \cdot M_\varphi = M_{h \cdot \varphi}$ for every $\varphi \in C_0(\tilde{M})$ and $h \in \Gamma$. Also, by Proposition 5.2.3 $g \cdot f(\tilde{D}) = f(\tilde{D})$ for all $g \in \Gamma$ and $f \in$ Flip. It follows that for every $f \in$Flip$_l$, $g \in$Flip$_r$, $\varphi \in C_0(\tilde{M}), \eta \in C_0(\tilde{M})$, $K \in \mathcal{K}$, and $h \in \Gamma$, we have

$$h \cdot (f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta + K) = f(\tilde{D})M_{h \cdot \varphi} + g(\tilde{D})M_{h \cdot \eta} + h \cdot K$$

which remains in $\mathcal{T}$. Since every element of $\mathcal{T}$ is of the form $f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta + K$ as above, it follows that $\mathcal{T}$ is invariant under the action of Γ.

Moreover, if $T = f(\tilde{D})M_\varphi + g(\tilde{D}) \cdot M_\eta + K$ as above, then

$$\begin{aligned} q(h \cdot T) &= q(f(\tilde{D} \cdot M_{h\cdot\varphi} + g(\tilde{D}) \cdot M_{h\cdot\eta} + h \cdot K) \\ &= (h \cdot \varphi,\ h \cdot \eta) \\ &= h \cdot (\varphi,\ \eta) \\ &= h \cdot q(f(\tilde{D}) \cdot M_\varphi + g(\tilde{D})M_\eta + K) \\ &= h \cdot q(T). \ \spadesuit \end{aligned}$$

The action of Γ on $\mathcal{T}$ induces an action α of Γ on $\tilde{M} \times \mathcal{T}$, given by,

$$g \cdot (x,\ T) = (g \cdot x,\ g \cdot T)$$

which gives an action α of Γ on $\tilde{t}$ making the diagram

$$\begin{array}{ccc} \tilde{t} & \xrightarrow{\alpha(g)} & \tilde{t} \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{\alpha(g)} & \tilde{M} \end{array}$$

commute. We can therefore take a quotient $\tilde{t}/\Gamma$ to get a $\mathcal{T}$ C*-algebra bundle over M.

Definition 2.4 *Define* **the bundle $t \to M$ of Toeplitz operators** *as the C*-algebra bundle*

$$t = \tilde{t}/\Gamma \to M.$$

Now, we have a sequence

$$\tilde{M} \times \mathcal{K} \overset{Id\times i}{\to} \tilde{M} \times \mathcal{T} \overset{Id\times q}{\to} \tilde{M} \times (C_0(\tilde{M}) \oplus C_0(\tilde{M})) \tag{2.5}$$

of C*-algebra bundle homomorphisms derived from the Toeplitz extension

$$0 \to \mathcal{K} \xrightarrow{i} \mathcal{T} \xrightarrow{q} C_0(\tilde{M}) \oplus C_0(\tilde{M}) \to 0.$$

This gives a sequence (using $\pi \times exp$)

$$\tilde{\mathcal{K}} \xrightarrow{\hat{i}} \tilde{t} \xrightarrow{\hat{q}} \bigcup_{x\in\tilde{M}} (C_0(T_x\tilde{M}) \oplus C_0(T_x\tilde{M})) \tag{2.6}$$

of bundle homomorphisms on bundles over $\tilde{M}$.

Since i is the inclusion it, like q, is also invariant under the action of Γ. The sequence (2.6) therefore induces a sequence

$$\mathcal{K} \xrightarrow{\hat{i}} t \xrightarrow{\hat{q}} \bigcup_{x\in M} (C_0(T_xM) \oplus C_0(T_xM))$$

of bundle homomorphisms on the quotient bundles over M.

This in turn induces a sequence

$$C(\mathcal{K}) \xrightarrow{\hat{i}_*} C(t) \xrightarrow{\hat{q}_*} C_0(TM) \oplus C_0(TM) \tag{2.7}$$

of $*$-homomorphisms on algebras of sections.

Remark 2.8 *So $C_0(TM) \oplus C_0(TM)$ should be viewed as the algebra of continuous sections of the bundle $\bigcup_{x\in M}(C_0(T_xM)\oplus C_0(T_xM))$.*

Alternatively, let $\boldsymbol{Per}(\tilde{t})$ denote the C*-algebra of periodic sections of $\tilde{t}$, and view $\tilde{C}_0(TM)$ as the algebra of periodic sections of the bundle

$$\bigcup_{x\in\tilde{M}} (C_0(T_x\tilde{M}) \oplus C_0(T_x\tilde{M})).$$

Then (2.6) induces a sequence

$$Per(\tilde{\mathcal{K}}) \xrightarrow{\hat{i}_*} Per(\tilde{t}) \xrightarrow{\hat{q}_*} \tilde{C}_0(TM) \oplus \tilde{C}_0(TM)$$

of $*$ homomorphisms. This sequence is the same as (2.7) up to isomorphism.

Proposition 2.9 *$\mathcal{A}_{01}(M) = C(\tilde{t})$ or $\mathcal{A}_{01}(M) = Per(\tilde{t})$, depending on how one views $\mathcal{A}_{01}(M)$, and the map*

$$q : \mathcal{A}_{01} \to \tilde{C}_0(TM) \oplus \tilde{C}_0(TM)$$

is the same as the map

$$\hat{q}_* : Per(\tilde{t}) \to \tilde{C}_0(TM) \oplus \tilde{C}_0(TM).$$

Proof: For the proof, we will view $\mathcal{A}_{01}(M)$ as a C*-subalgebra of $Per(\tilde{\mathcal{B}})$. Since the sequence

$$0 \to Per(\tilde{\mathcal{K}}) \xrightarrow{i} \mathcal{A}_{01}(M) \xrightarrow{q} \tilde{C}_0(TM) \oplus \tilde{C}_0(TM) \to 0$$

is exact, then each T in $\mathcal{A}_{01}(M)$ can be written as

$$T = f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta + L$$

for some $f \in$Flip$_l$, $g \in$Flip$_r$, $\varphi, \eta \in \tilde{C}_0(TM)$ and $L \in Per(\tilde{\mathcal{K}})$. If $x \in \tilde{M}$, the x-component T_x of such a T is the operator

$$T_x = f(\tilde{D})M_{\varphi_x} + g(\tilde{D})M_{\eta_x} + L_x$$

in $B(L^2(\tilde{\Delta}))$. Since φ_x, $\eta_x \in C_0(\tilde{M})$, and $L_x \in K(L^2(\tilde{\Delta}))$, then $T_x \in \mathcal{T}$ for all $x \in \tilde{M}$ and for all $T \in \mathcal{A}_{01}(M)$. Thus T is a continuous section of $\tilde{t}$. We already know that elements of $\mathcal{A}_{01}(M)$ are periodic. Hence $\mathcal{A}_{01}(M) \subseteq Per(\tilde{t})$.

Now, take $T \in Per(\tilde{t})$. Let $\hat{q}_*(T) = (\varphi, \eta)$ where $\varphi, \eta \in \tilde{C}_0(TM)$. Take any $f \in$Flip$_l$ and $g \in$Flip$_r$. Let

$$A = f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta.$$

Then A is an element of $\mathcal{A}_{01}(M)$. Now, since $\hat{q}_*(T) = (\varphi,\ \eta)$, we have that

$$q(T_x) = (\varphi_x, \eta_x)$$

in $C_0(\tilde{M}) \oplus C_0(\tilde{M})$, where q is the map

$$q : \mathcal{T} \to C_0(\tilde{M}) \oplus C_0(\tilde{M})$$

in the Toeplitz extension. It follows that for every $x \in \tilde{M}$,

$$\begin{aligned} T_x &= f(\tilde{D})M_{\varphi_x} + g(\tilde{D})M_{\eta_x} + K_x \\ &= A_x + K_x \end{aligned}$$

for some $K_x \in K(L^2(\tilde{\Delta}))$. Since T_x and A_x are continuous in x, so is K_x.

Let K be the continuous section of $\tilde{\mathcal{K}}$ where the x-component is K_x. Then

$$T = A + K.$$

So $K \in Per(\tilde{\mathcal{K}}) \subseteq \mathcal{A}_{01}(M)$. Since $T = A + K$ and $A, K \in \mathcal{A}_{01}(M)$ it follows that $T \in \mathcal{A}_{01}(M)$. Thus $\mathcal{A}_{01}(M) = Per\ \tilde{t}$.

Now to show $\hat{q}_* = q$, take a typical element

$$T = f(\tilde{D})M_\varphi + g(\tilde{D})M_\eta + K$$

in $\mathcal{A}_{01}(M)$, where $f \in \mathrm{Flip}_l$, $g \in \mathrm{Flip}_r$, $\varphi,\ \eta \in \tilde{C}_0\ (TM)$, and $K \in Per(\tilde{\mathcal{K}})$. Of course, $q(T) = (\varphi, \eta)$.

On the other hand $\hat{q}_*(T)$ is such that

$$\begin{aligned} (\hat{q}_*T)_x &= q(T_x) \\ &= q(f(\tilde{D}) \cdot M_{\varphi_x} + g(\tilde{D}) \cdot M_{\eta_x} + K_x) \\ &= (\varphi_x, \eta_x) \end{aligned}$$

for all $x \in \tilde{M}$. Thus $\hat{q}_*(T) = (\varphi, \eta) = q(T)$. ♠

Of course the map

$$i : Per(\tilde{\mathcal{K}}) \to \mathcal{A}_{01}(M)$$

is the same as

$$\hat{i}_* : Per(\tilde{\mathcal{K}}) \to Per(\tilde{t}),$$

since they are both inclusion maps. Thus, we have proved the following.

Proposition 2.10 *The adjoined Dirac extension*

$$0 \to Per(\tilde{\mathcal{K}}) \xrightarrow{i} \mathcal{A}_{01}(M) \xrightarrow{q} \tilde{C}_0(TM) \oplus \tilde{C}_0(TM) \to 0$$

which is the same as

$$0 \to C(\mathcal{K}) \xrightarrow{i} \mathcal{A}_{01}(M) \xrightarrow{q} C_0(TM) \oplus C_0(TM) \to 0$$

is induced by the sequence

$$\mathcal{K} \xrightarrow{\hat{i}} t \xrightarrow{\hat{q}} \bigcup_{x \in M} (C_0(T_xM) \oplus C_0(T_xM))$$

of C^*-algebra bundle homomorphisms.

Proposition 2.11 *Assume M is odd dimensional. If $a \in K_1(C_0(TM))$ is a Thom class, then $b = \partial_{1,r}(a) \in K_0(C(\mathcal{K}))$ is also a Thom class.*

Proof: By Proposition 2.10, we have, for every $x \in M$, commutativity of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\mathcal{K}) & \xrightarrow{\hat{i}_*} & C(t) & \xrightarrow{\hat{q}_*} & C_0(TM) \oplus C_0(TM) & \to & 0 \\ & & \downarrow{r_x} & & \downarrow{r_x} & & \downarrow{r_x} & & \\ 0 & \longrightarrow & \mathcal{K}_x & \xrightarrow{\hat{i}} & t_x & \xrightarrow{\hat{q}} & C_0(T_xM) \oplus C_0(T_xM) & \to & 0. \end{array}$$

This implies, by naturality of the index maps, commutativity of

$$\begin{array}{ccc} K_1(C_0(TM)) & \xrightarrow{\partial_{1,r}} & K_0(C(\mathcal{K})) \\ \big\downarrow{\scriptstyle (r_x)_*} & & \big\downarrow{\scriptstyle (r_x)_*} \\ K_1(C_0(T_xM)) & \xrightarrow{\partial_{x,r}} & K_0(\mathcal{K}_x) \end{array} \tag{2.12}$$

for every $x \in M$.

Now, from the definitions, the extension

$$0 \to \mathcal{K}_x \xrightarrow{\hat{i}} t_x \xrightarrow{\hat{q}} C_0(T_xM) \oplus C_0(T_xM) \to 0$$

is isomorphic to the Toeplitz extension

$$0 \to \mathcal{K} \xrightarrow{i} \mathcal{T} \xrightarrow{q} C_0(\tilde{M}) \oplus C_0(\tilde{M}) \to 0.$$

Since M is odd-dimensional by assumption, then, by Theorem 2.9.2, the index map

$$\partial_{\mathcal{T},r} : K_1(C_0(\tilde{M})) \to K_0(\mathcal{K})$$

of the Toeplitz extension is an isomorphism. Therefore the index map

$$\partial_{x,r} : K_1(C_0(T_xM)) \to K_0(\mathcal{K}_x)$$

is also an isomorphism.

Suppose now that $a \in K_1(C_0(TM))$ is a Thom class. Then, for every $x \in M$, $(r_x)_*(a) \in K_1(C_0(T_xM))$ is a generator of $\mathbf{Z}$. Since $\partial_{x,r}$ is an isomorphism, it follows that $\partial_{x,r}((r_x)_*(a)) \in K_0(\mathcal{K}_x)$ is a generator of $\mathbf{Z}$ for all $x \in M$. By commutativity of 2.12, it follows that

$$(r_x)_*(\partial_{1,r}(a)) = \partial_{x,r}((r_x)_*(a))$$

is a generator of $K_0(\mathcal{K}_x)$ for every $x \in M$. Therefore $\partial_{1,r}(a) \in K_0(C(\mathcal{K}))$ is a Thom class. ♠

Proposition 2.13 *If* $a \in K_1(C_0(TM))$ *and* $b = \partial_{1,r}(a)$ *in* $K_0(C(\mathcal{K}))$ *then*

$$\Phi_b = \partial_{i+1,r} \circ \Phi_a.$$

where

$$\Phi_b : K_i(C(M)) \to K_i(C(\mathcal{K}))$$

and

$$\Phi_a : K_i(C(M)) \to K_{i+1}(C_0(TM)).$$

Proof: It follows from Proposition 2.10, Corollary 6.5.44, and Lemma 5.18 that the diagram

$$\begin{array}{ccc} K_i(C(M)) \otimes K_1(C_0(TM)) & \xrightarrow{\cup} & K_{i+1}(C_0(TM)) \\ \Big\downarrow{\scriptstyle Id_* \otimes \partial_{1,r}} & & \Big\downarrow{\scriptstyle \partial_{i+1,r}} \\ K_i(C(M)) \otimes K_0(C(\mathcal{K})) & \xrightarrow{\cup} & K_i(C(\mathcal{K})) \end{array}$$

commutes. If $a \in K_1(C_0(TM))$, $b = \partial_{1,r}(a)$, it follows that, for every $c \in K_i(CM))$, $\partial_{i+1,r}(c \cup a) = c \cup b$. That is, $\partial_{i+1,r}(\Phi_a(c)) = \Phi_b(c)$, which gives us that $\Phi_b = \partial_{i+1,r} \circ \Phi_a$. ♠

Theorem 2.14 *If* M *is odd dimensional, then both*

$$\partial_{i,r} : K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

and

$$\partial_{i,l} : K_i(C_0(TM)) \to K_{i+1}(C(\mathcal{K}))$$

are isomorphisms. In fact, if $a \in K_1(C_0(TM))$ *is a Thom class and* $b = \partial_{1,r}(a) \in K_0(C(\mathcal{K}))$, *then both*

$$\Phi_a : K_i(C(M)) \to K_{i+1}(C_0(TM))$$

and

$$\Phi_b : K_i(C(M)) \to K_i(C(\mathcal{K}))$$

are isomorphisms,

$$\partial_{i+1,r} = \Phi_b \circ {\Phi_a}^{-1}.$$

and

$$\partial_{i+1,l} = -\partial_{i+1,r}$$

Proof: Since M is spin, there exists a Thom class $a \in K_i(C_0(TM))$. By Proposition 2.11, $b \stackrel{\text{def}}{=} \partial_{1,r}(a) \in K_0(C(\mathcal{K}))$ is also a Thom class. This implies, by Corollary 1.18, that the map

$$\Phi_b : K_i(C(M)) \to K_i(C(\mathcal{K}))$$

is an isomorphism. Now, $\Phi_b = \partial_{i+1,r} \circ \Phi_a$, by Proposition 2.13, where $\Phi_a : K_i(C(M)) \to K_{i+1}(C_0(TM))$. Since $a \in K_1(C_0(TM))$ is a Thom class then Φ_a is a Thom isomorphism. Since both Φ_b and Φ_a are isomorphisms, it follows that $\partial_{i+1,r} = \Phi_b \circ {\Phi_a}^{-1}$ is an isomorphism.

The rest follows from

$$\partial_{i,l} = -\partial_{i,r}$$

which is given below in Proposition 2.16. ♠

Note that if $\varphi \in C_0(TM)$, then M_φ is an element of $\mathcal{A}_{01}(M)$ and

$$q(M_\varphi) = (\varphi,\ \varphi).$$

Let

$$\mu : C_0(TM) \to \mathcal{A}_{01}(M)$$

be the map

$$\mu(\varphi) = M_\varphi.$$

Then $q \circ \mu = (Id,\ Id)$. That is, $q(\mu(\varphi)) = (\varphi,\ \varphi)$ for every $\varphi \in C_0(TM)$. It follows that

$$q_*(\mu_*(a)) = (a, a)$$

in $K_i(C_0(TM)) \oplus K_i(C_0(TM))$ for every $a \in K_i(C_0(TM))$. That is, $(a,\ a) \in Im(q_*)$ for every $a \in K_i(C_0(TM))$. By exactness of

$$K_i(\mathcal{A}_{01}(M)) \xrightarrow{q_*} K_i(C_0(TM)) \oplus K_i(C_0(TM)) \xrightarrow{\partial_i} K_{i+1}(C(\mathcal{K}))$$

it follows that

$$\partial_i(a,\ a) = 0$$

for every $a \in K_i(C_0(TM))$. This implies that

$$\begin{aligned} \partial_{i,l}(a) + \partial_{i,r}(a) &= \partial_i(0, a) + \partial_i(a, 0) \\ &= \partial_i(a, a) \\ &= 0 \end{aligned}$$

or that

$$\partial_{i,l} = -\partial_{i,r}.$$

If ∂_i instead denoted the index map

$$\partial_i : K_i(C(M)) \oplus K_i(C(M)) \to K_{i+1}(\mathcal{L}_{Per(M)})$$

of the Dirac extension

$$0 \to \mathcal{L}_{Per(M)} \to \mathcal{D}_{Per(M)} \to Per(M) \oplus Per(M) \to 0 \qquad (2.15)$$

with periodic multipliers, and if again

$$\partial_{i,r}(a) = \partial_i(0, a)$$

$$\partial_{i,l}(a) = \partial_i(a, 0),$$

then

$$\partial_{i,l} = -\partial_{i,r}$$

is again true and can be proved by the same argument given above.

The following proposition is therefore true.

Proposition 2.16 *Let ∂_i, $\partial_{i,r}$, and $\partial_{i,l}$ (defined above) be the index maps resulting from either the extension*

$$0 \to C(\mathcal{K}) \to \mathcal{A}_{01}(M) \to C_0(TM) \oplus C_0(TM) \to 0$$

or the extension (2.15). Then $\partial(a, a) = 0$, and $\partial_l = -\partial_r$.

Now, let us return to Diagram 5.4.63,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{Per(M)} & \xrightarrow{i} & \mathcal{D}_{Per(M)} & \xrightarrow{q} & C(M) \oplus C(M) & \longrightarrow & 0 \\
 & & \uparrow l & & \uparrow l & & \uparrow l\oplus l & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \xrightarrow{i} & \mathcal{A}_{11}(p) & \xrightarrow{q} & SC(p) \oplus SC(p) & \longrightarrow & 0 \\
 & & \uparrow i & & \uparrow i & & \uparrow i\oplus i & & \\
0 & \longrightarrow & M_k(C(\mathcal{K})) & \xrightarrow{i} & M_k(\mathcal{A}_{01}(M)) & \xrightarrow{q} & M_k(C_0(TM)) \oplus M_k(C_0(TM)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (2.17)$$

of Chapter 5.

This diagram is commutative and exact at every point. Let us now use ∂_N for the index maps resulting from the top extension

$$0 \to \mathcal{L}_{Per(M)} \to \mathcal{D}_{Per(M)} \to C(M) \oplus C(M) \to 0,$$

∂_W for the index maps of the left extension

$$0 \to \mathcal{A}_{00}(M)_k \to \mathcal{A}_{10}(p) \to \mathcal{A}_{20}(M) \to 0,$$

∂_S for the index maps of the bottom extension

$$0 \to M_k(C(\mathcal{K})) \to M_k(\mathcal{A}_{01}(M)) \to M_k(C_0(TM)) \oplus M_k(C_0(TM)) \to 0$$

and ∂_E for the index maps resulting from the right extension

$$0 \to \mathcal{A}_{02}(M)_k \to \mathcal{A}_{12}(p) \to \mathcal{A}_{22}(M) \to 0,$$

(which is the double of the $SC(p)$ Thom extension). Let

$$\begin{aligned}
\partial_{N,l}(a) &= \partial_N(a, o) \\
\partial_{N,r}(a) &= \partial_N(0, a) \\
\partial_{E,l}(a) &= \partial_E(a, 0) \\
\partial_{E,r}(a) &= \partial_E(0, a) \\
\partial_{S,l}(a) &= \partial_S(a, 0) \\
\partial_{S,r}(a) &= \partial_S(0, a).
\end{aligned}$$

By naturality of the index maps, we have

$$\partial_W \circ \partial_N = \partial_S \circ \partial_E.$$

This splits into

$$\partial_W \circ \partial_{N,l} = \partial_{S,l} \circ \partial_{E,l}$$

and

$$\partial_W \circ \partial_{W,r} = \partial_{S,r} \circ \partial_{E,r}. \tag{2.18}$$

Theorem 2.19 *Suppose M is odd dimensional. Then both*

$$\partial_{N,r} : K_i(C(M)) \to K_{i+1}(\mathcal{L}_{Per(M)})$$

and

$$\partial_{N,l} : K_i(C(M)) \to K_{i+1}(\mathcal{L}_{Per(M)})$$

are 1 - 1 *(and $\partial_{N,r} = -\partial_{N,l}$).*

Moreover, if p is a fundamental projection on the sphere bundle S over M, if $a \in K_1(C_0(TM))$ is the Thom class ($a = (i_* \circ \beta)[p]$) determined by the fundamental projection p, if $b \in K_0(C(\mathcal{K}))$ is the Thom class $b = \partial_{S,r}(a)$, if

$$\Phi_a : K_i(C(M)) \to K_{i+1}(C_0(TM))$$

$$\Phi_b : K_i(C(M)) \to K_i(C(\mathcal{K}))$$

are the corresponding isomorphisms given by cup product on the right by a and b respectively, then

$$\begin{aligned}\partial_W \circ \partial_{N,r} &= \Phi_b \\ &= \partial_{S,r} \circ \Phi_a \\ &= \partial_{S,r} \circ \partial_{E,r}.\end{aligned}$$

Thus, $\partial_W \circ \partial_{N,r}$ is an isomorphism equal to Φ_b.

Proof: Let p be a fundamental projection on the sphere bundle S over M. By (2.18)

$$\partial_W \circ \partial_{N,r} = \partial_{S,r} \circ \partial_{E,r}. \tag{2.20}$$

By Theorem 6.8.8,

$$\partial_{E,r} = \Phi_a$$

is the Thom isomorphism. By Theorem 2.14, $\partial_{S,r}$, is an isomorphism. Thus, from (2.20) $\partial_{N,r}$ is $1-1$. For if $\partial_{N,r}(c) = 0$, then $(\partial_{S,r} \circ \partial_{E,r})(s) = 0$ by (2.20). Since $\partial_{S,r}$ and $\partial_{E,r}$ are both isomorphisms, we would then have $c = 0$. Hence, $\partial_{N,r}$ is $1-1$.

Moreover, since $\partial_{E,r} = \Phi_a$, (2.20) implies that

$$\partial_W \circ \partial_{N,r} = \partial_{S,r} \circ \Phi_a.$$

By Theorem 2.14, this is equal to Φ_b. ♠

Remark 2.21 *It seems reasonable to interpret*

$$\partial_W \circ \partial_{N,r} : K_i(C(M)) \to K_i(C(\mathcal{K}))$$

as some sort of analytic index map, and to interpret

$$\Phi_b : K_i(C(M)) \to K_i(C(\mathcal{K}))$$

as a topological index map. The statement

$$\partial_W \circ \partial_{N,r} = \Phi_b$$

may then be interpreted as a rough index theorem, a statement that analytic index is equal to topological index.

In the next section, we look at the special case where M is equal to an odd-dimensional flat torus. In this case, it is shown that the index map $\partial_{N,r}$ is actually an isomorphism.

7.3 The Case $M = \mathbf{T}^m$

We now look at the Dirac extension

$$0 \to \mathcal{L}_{Per(M)} \to \mathcal{D}_{Per(M)} \to Per(M) \oplus Per(M) \to 0 \tag{3.1}$$

with periodic multipliers (see (2.11.17)) in the special case where $M = \mathbf{T}^m$ and m is odd. The metric on $\mathbf{T}^m$ is assumed to be the flat metric induced by the euclidean metric on $\mathbf{R}^m$. More precisely, if we let $\mathbf{Z}^m$ act on $\mathbf{R}^m$ by translations and view $\mathbf{T}^m$ as $\mathbf{R}^m/\mathbf{Z}^m$, then the riemannian metric on $\mathbf{T}^m$ is the flat metric on $\mathbf{R}^m/\mathbf{Z}^m$ induced by the euclidean metric on $\mathbf{R}^m$. We will refer to $\mathbf{T}^m$ with this metric as **the flat m-torus.**

For convenience, the Dirac extension

$$0 \to \mathcal{L}_{Per(\mathbf{T}^m)} \to \mathcal{D}_{Per(\mathbf{T}^m)} \to Per(\mathbf{T}^m) \oplus Per(\mathbf{T}^m) \to 0$$

will be denoted simply by

$$0 \to \mathcal{L} \to \mathcal{D} \to Per \oplus Per \to 0.$$

In addition $\mathcal{L}_{Per(\mathbf{T}^m)}{}'$ will be denoted by $\mathcal{L}'$.

$\mathbf{R}^m$ is of course the universal cover of $\mathbf{T}^m$. We will use $\Delta \to \mathbf{R}^m$ (not $\tilde{\Delta}$) for the bundle of spinors over $\mathbf{R}^m$. The algebras $\mathcal{L}$ and $\mathcal{D}$ are then C*-subalgebras of $B(L^2(\Delta))$.

Let CLIF_m denote the complex Clifford algebra of $\mathbf{R}^m$ and let Δ_m denote the complex spinor module over CLIF_m. The bundle $\Delta \to \mathbf{R}^m$ is isomorphic to the trivial bundle $\mathbf{R}^m \times \Delta_m \to \mathbf{R}^m$. So $L^2(\Delta)$ may be regarded as the space of L^2-functions from $\mathbf{R}^m$ to Δ_m. After choosing an orthonormal basis for Δ_m,

$L^2(\Delta)$ may also be identified with the Hilbert space $L^2(\mathbf{R}^m)^k$ where k is the dimension of Δ_m. Each $f \in L^2(\Delta)$ can be written

$$f = (f_1,\ f_2,\ \dots, f_k)$$

for some $f_i \in L^2(\mathbf{R}^m)$, $1 \le i \le k$. We can therefore define a Fourier transform

$$U : L^2(\Delta) \to L^2(\Delta)$$

by the formula

$$U(f) = (\hat{f}_i,\ \hat{f}_2,\ \dots, \hat{f}_k),$$

where $\hat{f}_i$ is the Fourier transform of f_i. The operator U is unitary.

One can show that

$$UDU^* = M_\sigma \tag{3.2}$$

where

$$\sigma : \mathbf{R}^m \to B(\Delta_m)$$

satisfies

$$\sigma(v)(s) = \sigma_v(s) \stackrel{\text{def}}{=} iv \cdot s \tag{3.3}$$

for $v \in \mathbf{R}^m$ and $s \in \Delta_m$ (where $iv \cdot s$ is Clifford multiplication of s by iv), and M_σ is the multiplication operator on $L^2(\Delta)$ with symbol σ. That is, if $w \in L^2(\Delta)^k$, then

$$(UDU^*)(w)(v) = \sigma_v(w(v)) = iv \cdot w(v)$$

for every $v \in \mathbf{R}^m$. (σ_v was introduced in 2.3.6.)

By (3.2), we also have

$$Uf(D)U^* = M_{f(\sigma)}. \tag{3.4}$$

for every $f \in C_0(\mathbf{R})$.

For each $x \in \mathbf{R}^m$ let T_x denote the translation operator on $L^2(\Delta) \cong L^2(\mathbf{R}^m)^k$ given by

$$T_x(f)(y) = f(y - x)$$

for every $f \in L^2(\mathbf{R}^m)^k$.

Definition 3.5 *Define $\mathcal{N}'$ to be the C^*-subalgebra of $B(L^2(\Delta)) \cong B(L^2(\mathbf{R}^m)^k)$ generated by the set of all M_φ such that $\varphi \in Per$ and the set of all $U^* M_\eta U \in B(L^2(\mathbf{R}^m)^k)$ such that $\eta \in M_k(C_0(\mathbf{R}^m))$.*

Then define $\mathcal{N} \subseteq \mathcal{N}'$ to be the ideal of $\mathcal{N}'$ generated by the set of all $U^* M_\eta U$ such that $\eta \in M_k(C_0(\mathbf{R}^m))$.

Lemma 3.6 *If $f \in C_0(\mathbf{R})$, then*

$$f(\sigma) \in M_k(C_0(\mathbf{R}^m)).$$

Proof: Suppose $f \in C_0(\mathbf{R})$. From Remarks 2.3.6, for every $v \in \mathbf{R}^m$, $\sigma(v)$ is self-adjoint with spectrum contained in $\{-\|v\|, \|v\|\}$. Hence, for every $v \in \mathbf{R}^m$, $f(\sigma)(v) = f(\sigma(v))$ has spectrum contained in $\{f(-\|v\|), f(\|v\|)\}$. Since $f \in C_0(\mathbf{R})$, it follows that, for every $\epsilon > 0$, there exists $R > 0$ such that $\|f(\sigma)(v)\| < \epsilon$ when $\|v\| > R$. Therefore $f(\sigma) \in M_k(C_0(\mathbf{R}^m))$. ♠

Lemma 3.7

$$\mathcal{L}' \subseteq \mathcal{N}' \text{ and } \mathcal{L} \subseteq \mathcal{N}.$$

Proof: Suppose $f \in C_0(\mathbf{R})$. From 3.2, we have

$$f(D) = U^* M_{f(\sigma)} U, \tag{3.8}$$

and, by Lemma 3.6, $f(\sigma)$ belongs to $M_k(C_0(\mathbf{R}^m))$. So, for every $f \in C_0(\mathbf{R}), f(D) = U^* M_\eta U$ for some $\eta \in M_k(C_0(\mathbf{R}^m))$. Since $\mathcal{L}'$ is generated by M_φ and $f(D)$ such that $\varphi \in Per$ and $f \in C_0(\mathbf{R})$, and since $\mathcal{N}'$ is generated by M_φ and $U^* M_\eta U$ such that $\varphi \in Per$ and $\eta \in M_k(C_0(\mathbf{R}^m))$, it is then clear that $\mathcal{L}' \subseteq \mathcal{N}'$.

Since $\mathcal{L}$ is the ideal of $\mathcal{L}'$ generated by $f(D)$ such that $f \in C_0(\mathbf{R})$, and since $\mathcal{N}$ is the ideal of $\mathcal{N}'$ generated by $U^* M_\eta U$ such that $\eta \in M_k(C_0(\mathbf{R}^m))$, then it is also clear that $\mathcal{L} \subseteq \mathcal{N}$. ♠

Lemma 3.9 *Let $x \in \mathbf{R}^m$ and let*

$$\varphi(y) = e^{ix \cdot y}, \ \forall\, y \in \mathbf{R}^m.$$

Then

$$U \cdot M_\varphi U^* = T_x.$$

Proof: This is well-known. ♠

Lemma 3.10 *If $n \in \mathbf{Z}^m$, then the translation operator T_n belongs to $U\mathcal{L}'U^*$.*

Proof: Suppose $n \in \mathbf{Z}^m$ and let

$$\varphi(x) = e^{in \cdot x}, \ \forall\, x \in \mathbf{R}^m.$$

Then $\varphi \in Per$ which implies that $M_\varphi \in \mathcal{L}'$. Also, by Lemma 3.9, $UM_\varphi U^* = T_n$. Therefore $T_n \in U\mathcal{L}'U^*$. ♠

Definition 3.11 *If η is a function from $\mathbf{R}^m$ to $M_l(\mathbf{C})$, and $n \in \mathbf{Z}^m$, define the translation $T_n(\eta)$ of η by letting*

$$T_n(\eta)(x) = \eta(x - n)$$

for every $x \in \mathbf{R}^m$.

Lemma 3.12 *If η is a function from $\mathbf{R}^m$ to $M_l(\mathbf{C})$, and $n \in \mathbf{Z}^m$, then*

$$M_{T_n(\eta)} = T_n \; M_\eta \; T_{-n}.$$

Proof: Easy. ♠

Lemma 3.13 *If $\eta \in M_k(C_0(\mathbf{R}^m))$ and $M_\eta \in U\mathcal{L}'U^*$, then, for every $n \in \mathbf{Z}^m$, $T_n(\eta) \in M_k(C_0(\mathbf{R}^m))$ and $M_{T_n(\eta)} \in U\mathcal{L}'U^*$.*

Proof: It is obvious that $T_n(\eta)$ belongs to $M_k(C_0(\mathbf{R}^m))$.

By Lemma 3.12, $M_{T_n(\eta)} = T_n M_\eta T_{-n}$, and, by Lemma 3.10, T_n and T_{-n} both belong to $U\mathcal{L}'U^*$. Since M_η also belongs to $U\mathcal{L}' \; U^*$ by assumption, it follows that $M_{T_n(\eta)} = T_n M_\eta T_{-n}$ belongs to $U\mathcal{L}'U^*$. ♠

Now, let

$$\pi : \; \mathrm{CLIF}_m \to B(\Delta_m)$$

be the representation of CLIF_m on Δ_m given by Clifford multiplication on the left. That is, if $w \in \mathrm{CLIF}_m$ and $s \in \Delta_m$,

$$\pi(w)(s) = w \cdot s.$$

Proposition 3.14 *The representation $\pi : CLIF_m \to B(\Delta_m)$ is irreducible. That is,*

$$\pi(CLIF_m) = B(\Delta_m).$$

Proof: This is Proposition I.5.15 of [L&M]. ♠

Definition 3.15 *Let*

$$r : \mathbf{R}^m \backslash \{0\} \to S^{m-1}$$

be the retraction map

$$r(v) = \frac{v}{\|v\|}.$$

Lemma 3.16 *If $v \in \mathbf{R}^m$, there exists $\eta \in M_k(C_0(\mathbf{R}^m))$ such that $M_\eta \in U\mathcal{L}'U^*$ and $\eta(v) = \sigma_v$.*

Proof: Let v be a vector in $\mathbf{R}^m$. Let f be any function in $C_0(\mathbf{R})$ such that $f(\|v\|) = \|v\|$ and $f(-\|v\|) = -\|v\|$. As in the proof of Lemma 3.6, $\sigma_v \in B(\Delta_m)$ is self-adjoint with spectrum contained in $\{-\|v\|, \|v\|\}$. From the assumptions on f, it follows that $f(\sigma_v) = \sigma_v$. Let $\eta = f(\sigma)$ which belongs to $M_k(C_0(\mathbf{R}^m))$ (by Lemma 3.6). Since $Uf(D)U^* = M_{f(\sigma)} = M_\eta$ (by 3.2), it follows that $M_\eta \in U\mathcal{L}'U^*$. Moreover, $\eta(v) = f(\sigma)(v) = f(\sigma_v) = \sigma_v$. The lemma is therefore true. ♠

Lemma 3.17 *If $x \in \mathbf{R}^m$, $v \in \mathbf{R}^m$ and there exists $\eta \in M_k(C_0(\mathbf{R}^m))$ such that $M_\eta \in U\mathcal{L}'\ U^*$ and $\eta(x) = \sigma_v$.*

Proof: Take $x \in \mathbf{R}^m$ and let $\mathcal{A}$ be the set of all $\eta(x) \in B(\Delta_m) \cong M_k(\mathbf{C})$ such that $\eta \in M_k(C_0(\mathbf{R}^m))$ and $M_\eta \in U\mathcal{L}'\ U^*$. Then $\mathcal{A}$ is a C*-subalgebra of $B(\Delta_m)$. We want to show that $\sigma_v \in \mathcal{A}$ for all $v \in \mathbf{R}^m$.

We may assume without loss of generality that $\|v\| = 1$. So, suppose $\|v\| = 1$.

Take a sequence $t_n \in \mathbf{R}$ such that $t_n > 0$ and $t_n \to \infty$ as $n \to \infty$. Then choose $w_n \in x + \mathbf{Z}^m$ such that $\|w_n - t_n \cdot v\| \le \sqrt{2}$. It is then clear that

$$r(w_n) \to v \text{ as } n \to \infty. \tag{3.18}$$

Also, it follows from Lemma 3.16 that, for each n, there is an $\eta_n \in M_k(C_0(\mathbf{R}^m))$ such that $\eta_n(w_n) = \sigma_{r(w_n)}$ and $M_{\eta_n} \in U\mathcal{L}'\,U^*$. By assumption,

$$w_n = x + l_n$$

for some $l_n \in \mathbf{Z}^m$. By Lemma 3.13, $T_{-l_n}(\eta_n) \in M_k(C_0(\mathbf{R}^m))$ and $M_{T_{-l_n}(\eta_n)} \in U\mathcal{L}'\,U^*$. Furthermore, $T_{-l_n}(\eta_n)(x) = \eta_n(x + l_n) = \eta_n(w_n) = \sigma_{r(w_n)}$. Therefore, $\sigma_{r(w_n)} \in \mathcal{A}$ for all n.

Since $r(w_n) \to v$ as $n \to \infty$ (by (3.18)), we have $\sigma_{r(w_n)} \to \sigma_v$ as $n \to \infty$. Therefore σ_v also belongs to $\mathcal{A}$. ♠

Lemma 3.19 *If $x \in \mathbf{R}^m$ and $T \in B(\Delta_m) \cong M_k(\mathbf{C})$, then there exists $\eta \in M_k(C_0(\mathbf{R}^m))$ such that $M_\eta \in U\mathcal{L}'\,U^*$ and $\eta(x) = T$.*

Proof: As in the proof of Lemma 3.17, let $\mathcal{A}$ be the C*-algebra of all $\eta(x) \in B(\Delta_m)$ such that $\eta \in M_k(C_0(\mathbf{R}^m))$ and $M_\eta \in U\mathcal{L}'\,U^*$. We want to show that $\mathcal{A} = B(\Delta_m)$.

By Lemma 3.17, $\sigma_v \in \mathcal{A}$ for all $v \in \mathbf{R}^m$. Let $\pi : \text{CLIF}_m \to B(\Delta_m)$ be the representation given by Clifford multiplication on the left. Note that $\sigma_v = \pi(iv)$. Of course, CLIF_m is generated by the set of all the iv such that $v \in \mathbf{R}^m$. Thus, the $\sigma_v = \pi(iv)$, such that $v \in \mathbf{R}^m$ generate $\pi(\text{CLIF}_m)$. Since $\sigma_v \in \mathcal{A}$ for all $v \in \mathbf{R}^m$, it follows that $\pi(\text{CLIF}_m) \subseteq \mathcal{A} \subseteq B(\Delta_m)$.

But $\pi(\text{CLIF}_m) = B(\Delta_m)$ (by Proposition 3.14). Therefore $\mathcal{A} = B(\Delta_m)$. ♠

Lemma 3.20 *Let $\tilde{\rho}$ be an element of $C_0([0,\infty))$. Let $\rho \in C_0(\mathbf{R}^m)$ be such that $\rho(v) = \tilde{\rho}(\|v\|)$ for every $v \in \mathbf{R}^m$. Let $f \in C_0(\mathbf{R})$ be the function satisfying $f(t) = \tilde{\rho}(|t|)$ for all $t \in \mathbf{R}$. Then $M_\rho = M_{f(\sigma)} = Uf(D)U^*$. So $M_\rho \in U\mathcal{L}U^*$.*

Proof: From Remarks 2.3.6, for every $v \in \mathbf{R}^m$, $\sigma(v)$ is self adjoint with spectrum contained in $\{-\|v\|, \|v\|\}$. It follows that $f(\sigma(v)) = \tilde{\rho}(\|v\|) \cdot Id_{\Delta_m}$ $= \rho(v) \cdot Id_{\Delta_m}$. That is $f(\sigma(v)) = \rho(v) \cdot Id_{\Delta_m}$ for every $v \in \mathbf{R}^m$ which implies that

$$f(\sigma) = \rho \cdot Id_{\Delta_m}.$$

It follows that $M_{f(\sigma)} = M_\rho$. From (3.4), we get that $Uf(D)U^* = M_\rho$. ♠

Lemma 3.21 *Let $x,\ y \in \mathbf{R}^m,\ \ x \neq y$, and let $T \in B(\Delta_m) \cong M_k(\mathbf{C})$. Then there exists $\eta \in M_k(C_0(\mathbf{R}^m))$ such that $M_\eta \in U\mathcal{L}'\ U^*$, $\eta(x) = T$ and $\eta(y) = 0$.*

Proof: We may assume that $\|x\| \neq \|y\|$. Otherwise, if $\|x\| = \|y\|$, we find $n \in \mathbf{Z}^m$ such that $\|x - n\| \neq \|y - n\|$. If we can find $\mu \in M_k(C_0(\mathbf{R}^m))$ such that $M_\mu \in U\mathcal{L}'\ U^*$, $\mu(x-n) = T$ and $\mu(y-n) = 0$, and if we let $\eta = T_n(\mu)$, then, by Lemma 3.13, we have $\eta \in M_k(C_0(\mathbf{R}^m))$ and $M_\eta \in U\mathcal{L}'\ U^*$. Moreover, we have that $\eta(x) = T$ and $\eta(y) = 0$.

Thus, the proof is reduced to the case where $\|x\| \neq \|y\|$.

Assume $\|x\| \neq \|y\|$. Choose any $\tilde{\rho} \in C_0([0,\infty)$ such that $\tilde{\rho}(\|x\|) = 1$ and $\tilde{\rho}(\|y\|) = 0$. Let $\rho(v) = \tilde{\rho}(\|v\|)$ for every $v \in \mathbf{R}^m$. By Lemma 3.20, $\rho \in C_0(\mathbf{R}^m)$ and $M_\rho \in U\mathcal{L}'\ U^*$. Moreover, we have $\rho(x) = 1$ and $\rho(y) = 0$.

Now, by Lemma 3.19, there exists $\mu \in M_k(C_0(\mathbf{R}^m))$ such that $M_\mu \in U\mathcal{L}'\ U^*$ and $\mu(x) = T$. Let $\eta = \rho \cdot \mu$. Clearly $\eta \in M_k(C_0(\mathbf{R}^m))$. Since ρ and μ both belong to $U\mathcal{L}'\ U^*$, then so does $\eta = \rho\mu$. Furthermore, we have $\eta(x) = T$ and $\eta(y) = 0$. Lemma 3.21 is therefore true. ♠

Lemma 3.22 *Let $x,\ y \in \mathbf{R}^m, x \neq y$, and let $S, T \in B(\Delta_m) \cong M_k(\mathbf{C})$. Then there exists $\eta \in M_k(C_0(\mathbf{R}^m))$ such that $M_\eta \in U\mathcal{L}'\ U^*$, $\eta(x) = S$ and $\eta(y) = T$.*

Proof: Find η_1, η_2, as in Lemma 3.21, such that $\eta_1(x) = S$, $\eta_1(y) = 0$, $\eta_2(x) = 0$, and $\eta_2(y) = T$. Then let $\eta = \eta_1 + \eta_2$. ♠

Lemma 3.23 *$M_\eta \in U\mathcal{L}'\, U^*$ for every $\eta \in M_k(C_0(\mathbf{R}^m))$.*

Proof: Let $\mathcal{A}$ equal the set of all $\eta \in M_k(C_0(\mathbf{R}^m))$ such that $M_\eta \in U\mathcal{L}'U^*$. Then $\mathcal{A}$ is a C*-subalgebra of $M_k(C_0(\mathbf{R}^m))$ with the property that, for every x, $y \in \mathbf{R}^m$ such that $x \neq y$, and for every $S, T \in M_k(\mathbf{C})$, there exists $\eta \in \mathcal{A}$ such that $\eta(x) = S$ and $\eta(y) = T$ (by Lemma 3.22). By a Stone-Weirstrass Theorem (Corollary 11.5.3 of [**Dix**]), it follows that $\mathcal{A} = M_k(C_0(\mathbf{R}^m))$. ♠

Proposition 3.24

$$\mathcal{L}' = \mathcal{N}'.$$

Proof: We already know that $\mathcal{L}' \subseteq \mathcal{N}'$ (Lemma 3.7). So it suffices to show that $\mathcal{N}' \subseteq \mathcal{L}'$.

Now, $\mathcal{N}'$ is generated by M_φ and $U^*M_\eta U$ such that $\varphi \in Per$ and $\eta \in M_k(C_0(\mathbf{R}^m))$. From the definition of $\mathcal{L}'$, we know that $M_\varphi \in \mathcal{L}'$ for all $\varphi \in Per$, and by Lemma 3.23, we have that $U^*M_\eta U \in \mathcal{L}'$ for all $\eta \in M_k(C_0(\mathbf{R}^m))$. Hence, $\mathcal{N}'$ is generated by elements in $\mathcal{L}'$. Therefore $\mathcal{N}' \subseteq \mathcal{L}'$. ♠

Lemma 3.25 *$U^*M_\eta U \in \mathcal{L}$ for all $\eta \in M_k(C_0(\mathbf{R}^m))$.*

Proof: Let $\eta \in M_k(C_0(\mathbf{R}^m))$. Suppose η has compact support.

By Lemma 3.23, we have

$$U^*\, M_\eta\, U \in \mathcal{L}'.$$

Since η has compact support, there exists $N > 0$ such that η has support inside $B(0, N)$. Let $\tilde{\rho} \in C_0([0, \infty))$ be such that

$$\tilde{\rho} \equiv 1 \text{ on } [0, N].$$

Define $\rho \in C_0(\mathbf{R}^m)$ by letting

$$\rho(x) = \tilde{\rho}(\|x\|).$$

Then $\rho \equiv 1$ on $B(0, N)$, which implies that

$$\rho \cdot \eta = \eta,$$

since η has support in $B(0, N)$. By Lemma 3.20, $M_\rho \in U\mathcal{L}U^*$ which implies that

$$U^* M_\rho U \in \mathcal{L}.$$

Since $\mathcal{L}$ is an ideal of $\mathcal{L}'$ and since $U^* M_\eta U \in \mathcal{L}'$ (from above), it follows that

$$\begin{aligned} U^* \, M_\eta \, U &= U^* M_{\rho\eta} \, U \\ &= (U^* M_\rho U)(U^* M_\eta U) \end{aligned}$$

belongs to $\mathcal{L}$. ♠

Proposition 3.26 $\mathcal{L} = \mathcal{N}$.

Proof: We have already seen that $\mathcal{L} \subseteq \mathcal{N}$ (Lemma 3.7) and that $\mathcal{L}' = \mathcal{N}'$ (Proposition 3.24). We need to show therefore that $\mathcal{N} \subseteq \mathcal{L}$.

Now $\mathcal{N}$ is, by definition, the ideal of $\mathcal{N}' = \mathcal{L}'$ generated by the set of all $U^* \, M_\eta \, U$ such that $\eta \in M_k(C_0(\mathbf{R}^m))$. But such operators are in $\mathcal{L}$ by Lemma 3.25. So $\mathcal{N}$ is an ideal of $\mathcal{L}'$ generated by objects in $\mathcal{L}$. Since $\mathcal{L}$ is an ideal of $\mathcal{L}'$, it follows thta $\mathcal{N} \subseteq \mathcal{L}$. Hence $\mathcal{L} = \mathcal{N}$. ♠

Definition 3.27 *Define $\mathcal{N}_1'$ to be the C^*-subalgebra of $B(L^2(\mathbf{R}^m))$ generated by the set of all M_φ such that $\varphi \in Per$ and the set of all $U^* M_\eta U \in B(L^2(\mathbf{R}^m))$ such that $\eta \in C_0(\mathbf{R}^m)$.*

Then define $\mathcal{N}_1 \subseteq \mathcal{N}_1'$ to be the ideal of $\mathcal{N}_1'$ generated by the set of all $U^* M_\eta U$ such that $\eta \in C_0(\mathbf{R}^m)$.

It is a simple excercise to show that

$$\mathcal{N}' = M_k(\mathcal{N}_1') \tag{3.28}$$

and

$$\mathcal{N} = M_k(\mathcal{N}_1). \tag{3.29}$$

As a consequence, we have from Proposition 3.25, the following.

Proposition 3.30 *$\mathcal{L}' = M_k(\mathcal{N}_1')$, and $\mathcal{L} = M_k(\mathcal{N}_1)$.*

Now, consider the covariant representation $(\pi,\ \beta,\ L^2(\Delta))$ where

$$\pi : Per \to B(L^2(\mathbf{R}^m))$$

is the map

$$\pi(\varphi) = M_\varphi,\ \forall\, \varphi \in Per,$$

and

$$\beta : \mathbf{R}^m \to U(L^2(\mathbf{R}^m))$$

is the map defined by

$$\beta(x) = T_x$$

where, as before, T_x is the translation operator $(T_x f)(y) = f(y-x)$, for $y \in \mathbf{R}^m$ and $f \in L^2(\Delta)$.

This induces a representation

$$\tilde{\pi} : Per \times_{\alpha} \mathbf{R}^m \to B(L^2(\mathbf{R}^m))$$

where α is the action of $\mathbf{R}^m$ on Per by translations.

Definition 3.31 *Let $\mathcal{P}$ equal $\tilde{\pi}(Per \times_{\alpha} \mathbf{R}^m)$ in $B(L^2(\mathbf{R}^m))$.*

Remark 3.32 *Note that if $A \in \mathcal{P}$ and $\varphi \in Per$, then $M_{\varphi}A$ and $A \cdot M_{\varphi}$ belong to $\mathcal{P}$. We express this by saying that $\mathcal{P}$ is a module over Per.*

Proposition 3.33 *If $\eta \in \mathcal{S}(\mathbf{R}^m)$ and $\hat{\eta}$ has compact support, then*

$$U^* M_{\eta}\, U = \int_{\mathbf{R}^m} \hat{\eta}(x)\, T_x\, dx.$$

Proof: This follows from a simple computation using the formulas for the Fourier transform U and its inverse U^*. ♠

Lemma 3.34 *If $\eta \in C_0(\mathbf{R}^m)$, then $U^* M_{\eta} U$ belongs to $\mathcal{P}$.*

Proof: If $\eta \in \mathcal{S}(\mathbf{R}^m)$ and $\hat{\eta}$ has compact support, then, by Proposition 3.33,

$$U^* M_{\eta} U = \int_{\mathbf{R}^m} \hat{\eta}(x) T_x.$$

So $U^* M_{\eta} U = \tilde{\pi}(g)$ where g is the element in the convolution algebra $C_c^{\infty}(\mathbf{R}^m, Per)$ defined by $g(x) = \hat{\eta}(x) \cdot 1$. Thus, $U^* M_{\eta} U \in \mathcal{P}$ in this case. Since the set of all $\eta \in \mathcal{S}(\mathbf{R}^m)$ such that $\hat{\eta}$ has compact support, is dense in $C_0(\mathbf{R}^m)$, it follows that $U^* M_{\eta} U \in \mathcal{P}$ for all $\eta \in C_0(\mathbf{R}^m)$. ♠

Lemma 3.35 $\mathcal{N}_1 = \mathcal{P}$.

Proof: By Proposition 3.34, $U^*M_\eta U \in \mathcal{P}$ for all $\eta \in C_0(\mathbf{R}^m)$. Since $\mathcal{P}$ is a module over Per (Remark 3.32), since $\mathcal{N}_1{}'$ is the algebra generated by M_φ and $U^*M_\eta U$ such that $\varphi \in Per$ and $\eta \in C_0(\mathbf{R}^m)$, and since $\mathcal{N}_1$ is the ideal of $\mathcal{N}_1{}'$ generated by the $U^*M_\eta U$ such that $\eta \in C_0(\mathbf{R}^m)$, it folows that $\mathcal{N}_1 \subseteq \mathcal{P}$.

Now, to show $\mathcal{P} \subseteq \mathcal{N}_1$, it suffices to show that, if $g \in C_c^\infty(\mathbf{R}^m, Per)$ and if we let $g(x) = \varphi_x \in Per$, then

$$\int_{\mathbf{R}^m} M_{\varphi_x} \cdot T_x \, dx \in \mathcal{N}_1.$$

To do this, we note that if $\epsilon > 0$, we can choose a C^∞ partition of unity ρ_i, $1 \leq i \leq n$, of the compact support of g, and x_i in the support of ρ_i such that

$$\| \sum_{i=1}^{n} \int_{\mathbf{R}^m} M_{\varphi_{x_i}} \cdot \rho_i(x) \; T_x \, dx - \int_{\mathbf{R}^m} M_{\varphi_x} \; T_x \, dx \| < \epsilon.$$

Then we note that each

$$A_i \stackrel{\text{def}}{=} \int_{\mathbf{R}^m} M_{\varphi_{x_i}} \; \rho_i(x) \; T_x \, dx = M_{\varphi_{x_i}} \int_{\mathbf{R}^m} \rho_i(x) T_x \, dx.$$

Now let $\eta_i \in \mathcal{S}(\mathbf{R}^m)$ be such that $\hat{\eta}_i = \rho_i$. Then

$$\begin{aligned} A_i &= M_{\varphi_{x_i}} \int_{\mathbf{R}^m} \hat{\eta}_i \; (x) \; T_x \, dx \\ &= M_{\varphi_{x_i}} (U^* M_{\eta_i} \; U) \end{aligned}$$

A_i is therefore an element of $\mathcal{N}_1$. It follows that $\int_{\mathbf{R}^m} M_{\varphi_x} \; T_x \, dx$ belongs to $\mathcal{N}_1$. Hence $\mathcal{P} \subseteq \mathcal{N}_1$, and the proof is complete. ♠

As a corollary, we have, by Proposition 3.30, the following.

Proposition 3.36 $\mathcal{L} = M_k(\mathcal{P})$.

Proposition 3.37 *The representation*

$$\tilde{\pi} : Per \times_\alpha \mathbf{R}^m \to B(L^2(\mathbf{R}^m))$$

gives an isomorphism from $Per \times_\alpha \mathbf{R}^m$ *to its image* $\mathcal{P}$. *So*

$$\mathcal{P} \cong Per \times_\alpha \mathbf{R}^m.$$

Proof: Let

$$\begin{aligned} \tilde{\pi}_2 : Per \times_\alpha \mathbf{R}^m \longrightarrow \ & B(L^2(\mathbf{R}^m),\ L^2(\mathbf{R}^m)) \\ & \cong B(L^2(\mathbf{R}^m \times \mathbf{R}^m)) \end{aligned}$$

be the regular representation of $Per \times_\alpha \mathbf{R}^m$ as given in 7.7.1 in **[Ped]**. By definition, this is the representation induced by the covariant representation $(\pi_2,\ \lambda,\ L^2(\mathbf{R}^m \times \mathbf{R}^m))$ which is defined as follows.

If $f \in L^2(\mathbf{R}^m \times \mathbf{R}^m)$ is considered a function $f(x,y)$ of $(x,y) \in \mathbf{R}^m \times \mathbf{R}^m$, if we let

$$f_x(y) = f(x,\ y),$$

and if $\varphi \in Per$, then

$$\begin{aligned} \pi_2(\varphi)(f)(x,\ y) &\stackrel{\text{def}}{=} (M_{T_{-x}(\varphi)}(f_x))(y) \\ &= (T_{-x}(\varphi) \cdot f_x)(y) \\ &= \varphi(x+y) \cdot f(x,y), \end{aligned}$$

and

$$\lambda(z)(f)(x,\ y) = f(x-z,\ y)$$

for $z \in \mathbf{R}^m$.

Note, we may view the representation π_2 of $Per(M)$ in a different way. We may write

$$\begin{aligned} \pi_2(\varphi)(f)(x,\ y) &= (M_{T_{-y}(\varphi)})(f_{-,y})(x) \\ (\text{where } (f_{-,y})(x) &= f(x,\ y)). \end{aligned}$$

This is true because

$$\pi_2(\varphi)(f)(x,\ y) = \varphi(x+y) \cdot f(x,y).$$

Now, consider the unitary map

$$V : L^2(\mathbf{R}^m \times \mathbf{R}^m) \to L^2(\mathbf{R}^m \times \mathbf{R}^m)$$

given by

$$(Vf)(x,\ y) = f(x+y,\ y).$$

Let

$$\pi_3(\varphi) = V^* \pi_2(\varphi) V$$

for every $\varphi \in Per$. Let

$$\lambda_3(x) = V^* \lambda(x) V.$$

Since V is unitary, the covariant representation $(\pi_3,\ \lambda_3,\ L^2(\mathbf{R}^m \times \mathbf{R}^m))$ is isomorphic to $(\pi_2,\ \lambda,\ L^2(\mathbf{R}^m \times \mathbf{R}^m))$.

Now, one checks easily that

$$\pi_3(\varphi)(f)(x,\ y) = \varphi(x) f(x,y)$$

and

$$\begin{aligned} \lambda_3(z)(f)(x,y) &= f(x-z,\ y) \\ &= \lambda(z)(f)(x,\ y). \end{aligned}$$

(That is $\lambda_3 = \lambda$.)

From this, it is clear that the representation of $Per \times_\alpha \mathbf{R}^m$ induced by $(\pi_3,\ \lambda_3,\ L^2(\mathbf{R}^m \times \mathbf{R}^m))$ is isomorphic to the representation

$$\tilde{\pi} : Per \times_\alpha \mathbf{R}^m \to B(L^2(\Delta))$$

of $Per \times_\alpha \mathbf{R}^m$ induced by $(\pi, \beta,\ L^2(\mathbf{R}^m))$.

Since $(\pi_3,\ \lambda_3,\ L^2(\mathbf{R}^m \times \mathbf{R}^m))$ is isomorphic to $(\pi_2,\ \lambda,\ L^2(\mathbf{R}^m \times \mathbf{R}^m))$, then the corresponding representations of $Per \times_\alpha \mathbf{R}^m$ are also isomorphic.

Hence the representation

$$\tilde{\pi} : Per \times_\alpha \mathbf{R}^m \to B(L^2(\Delta))$$

is isomorphic to the regular representation of $Per \times_\alpha \mathbf{R}^m$. Since $\mathbf{R}^m$ is amenable, it follows from Section 7.7 of [**Ped**] that the regular representation of $Per \times_\alpha \mathbf{R}^m$ is an isomorphism onto its image. Therefore

$$\tilde{\pi} : Per \times_\alpha \mathbf{R}^m \to \mathcal{P}$$

is a $*$-isomorphism. Thus $\mathcal{P} \cong Per \times_\alpha \mathbf{R}^m$. ♠

Corollary 3.38

$$\begin{aligned} \mathcal{L} &\cong M_k(Per \times_\alpha \mathbf{R}^m) \\ &\cong M_k(C(\mathbf{T}^m) \times_\alpha \mathbf{R}^m). \end{aligned}$$

Proof: Follows from Propositions 3.36 and 3.37. ♠

Corollary 3.39 $K_i(\mathcal{L}) \cong \mathbf{Z}^{2^{m-1}}$ *for both* $i = 0$ *and* $i = 1$.

Proof: $\mathbf{R}^m$ is a simply connected, solvable Lie group, of odd dimension, and α is a continuous action of $\mathbf{R}^m$ on Per. Hence, by Corollary 19.3.9 of [**Bla**], it follows that $Per \times_\alpha \mathbf{R}^m$ is KK-equivalent to the suspension $\Omega(Per)$ of Per which is isomorphic to $\Omega(C(\mathbf{T}^m))$. Therefore

$$K_i(Per \times_\alpha \mathbf{R}^m) \cong K_{i+1}(C(\mathbf{T}^m)).$$

One can use the Kunneth formula, for example, to show that

$$K_{i+1}(C(\mathbf{T}^m)) \cong \mathbf{Z}^{2^{m-1}}.$$

Therefore

$$K_i(Per \times_\alpha \mathbf{R}^m) \cong \mathbf{Z}^{2^{m-1}}.$$

Since $\mathcal{L} \cong M_k(Per \times_\alpha \mathbf{R}^m)$ (Corollary 3.38), it follows that $K_i(\mathcal{L}) \cong \mathbf{Z}^{2^{m-1}}$ for both $i = 0$ and $i = 1$. ♠

We return now to the Dirac extension

$$0 \to \mathcal{L} \xrightarrow{i} \mathcal{D} \xrightarrow{q} Per \oplus Per \to 0$$

with periodic multipliers on $\mathbf{R}^m$. Again, we are interested in the K-theory index maps of this extension. We use the notation $\partial_{N,r}$ and $\partial_{N,l}$, employed in Theorem 2.19 for the index maps of this extension.

Theorem 3.40 *The index maps*

$$\partial_{N,r} : K_i(C(\mathbf{T}^m)) \to K_{i+1}(\mathcal{L})$$

and

$$\partial_{N,l} : K_i(C(\mathbf{T}^m)) \to K_{i+1}(\mathcal{L})$$

are isomorphisms, and $\partial_{N,r} = -\partial_{N,l}$.

Proof: By Theorem 2.19, we know that $\partial_{N,r}$ and $\partial_{N,l}$ are 1 - 1, and that $\partial_{N,r} = -\partial_{N,l}$. Since $\partial_{N,l} = -\partial_{N,r}$, then, to complete the proof, we only need to show that $\partial_{N,r}$ is onto.

Now $K_{i+1}(\mathcal{L}) \cong \mathbf{Z}^{2^{m-1}}$ by Corollary 3.39. Let us pick 2^{m-1} generators $a_1, a_2, \ldots, a_{2^{m-1}}$ of $K_{i+1}(\mathcal{L})$ such that

$$K_{i+1}(\mathcal{L}) = \mathbf{Z} \cdot a_1 \oplus \mathbf{Z} \cdot a_2 \oplus \ldots \oplus \mathbf{Z} \cdot a_{2^{m-1}}.$$

To complete the proof, it suffices to show that all of the a_i lie in the range of $\partial_{N,r}$.

Suppose this were not true. Say, for example, that

$$a_1 \notin Im(\partial_{N,r}).$$

Then by exactness of

$$K_i(C(\mathbf{T}^m)) \overset{\partial_{N,r}}{\to} K_{i+1}(\mathcal{L}) \overset{i_*}{\to} K_{i+1}(\mathcal{D}), \tag{3.41}$$

we have that $i_*(a_1) \in K_{i+1}(\mathcal{D})$ is not zero.

Note that some nonzero element of $\mathbf{Z}a_1$ must lie in the image of $\partial_{N,r}$. Otherwise, since $\partial_{N,r}$ is 1 - 1, we would have $\mathbf{Z}^{2^{m-1}}$ isomorphic to a subgroup of $\mathbf{Z}^{2^{m-2}}$ contradicting the fundamental theorem for finitely generated abelian groups.

So,

$$l \cdot a_1 \in Im(\partial_{N,r})$$

for some $l \in \mathbf{Z},\ l \neq 0$. By exactness of (3.41), it follows that

$$l \cdot i_*(a_1) = 0 \in K_{i+1}(\mathcal{D}).$$

Hence $i_*(a_1)$ is a torsion element of $K_{i+1}(\mathcal{D})$.

Now choose a fundamental projection on the sphere bundle over $\mathbf{T}^m$ and let us reproduce Diagram (2.17), which, in this case, looks like

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L} & \xrightarrow{i} & \mathcal{D} & \xrightarrow{q} & C(\mathbf{T}^m)\oplus C(\mathbf{T}^m) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l\oplus l} & & \\
0 & \longrightarrow & \mathcal{A}_{10}(p) & \xrightarrow{i} & \mathcal{A}_{11}(p) & \xrightarrow{q} & SC(p)\oplus SC(p) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i\oplus i} & & \\
0 & \longrightarrow & M_k(C(\mathcal{K})) & \xrightarrow{i} & M_k(\mathcal{A}_{01}(\mathbf{T}^m)) & \xrightarrow{q} & M_k(C_0(T\mathbf{T}^m))\oplus M_k(C_0(T\mathbf{T}^m)) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that since the index maps $\partial_{E,r}$, $\partial_{E,l}$, $\partial_{S,r}$, $\partial_{S,l}$ are isomorphisms (from previous theorems), then

$$K_j(SC(p)) = 0$$

$$K_j(M_k(\mathcal{A}_{01}(\mathbf{T}^m)) = 0$$

for both $j = 0$ and $j = 1$. Therefore the map

$$l_* : K_{i+1}(\mathcal{A}_{11}(p)) \stackrel{\cong}{\to} K_{i+1}(\mathcal{D})$$

is an isomorphism. $K_{i+1}(\mathcal{A}_{11}(p))$ therefore must also contain torsion elements (for $K_{i+1}(\mathcal{D})$ has torsion $\neq 0$).

Similarly the map

$$i_* : K_{i+1}(\mathcal{A}_{10}(p)) \stackrel{\cong}{\to} K_{i+1}(\mathcal{A}_{11}(p))$$

is an isomorphism, and so $K_{i+1}(\mathcal{A}_{10}(p))$ also has nonzero torsion.

By commutativity of the diagram

$$\begin{array}{ccc} K_{i+1}(\mathcal{L}) & \xrightarrow{i_*} & K_{i+1}(\mathcal{D}) \\ \uparrow l_* & & \cong \uparrow l_* \\ K_{i+1}(\mathcal{A}_{10}(p) & \xrightarrow[\cong]{i_*} & K_{i+1}(\mathcal{A}_{11}(p)) \end{array}$$

it follows that l_* is 1-1. Therefore $K_{i+1}(\mathcal{L})$ also has nonzero torsion! This is a contradiction, for $K_{i+1}(\mathcal{L}) \cong \mathbf{Z}^{2^{m-1}}$ is free.

Therefore all of tha a_i lie in the image of $\partial_{N,r}$. It follows that $\partial_{N,r}$ is onto. Hence, $\partial_{N,r}$ is an isomorphism. ♠

7.4 Conclusion

It is of interest to know for which M the index map $\partial_{N,r}$ of Theorem 2.19 is an isomorphism. A better understanding of the index map ∂_W would certainly help. However, as is clear from the example of the flat odd dimensional torus, if one knows enough about the K-groups of $C(M)$ and $\mathcal{L}_{Per(M)}$, one may have enough to conclude that $\partial_{N,r}$ is an isomorphism.

A special case of a manifold of nonpositive curvature is the manifold G/K where G is semisimple Lie group and K is a maximal compact subgroup. Hyperbolic spaces are examples of such manifolds. By taking quotients of G/K by certain discrete actions, one can generated compact manifolds M of nonpositive curvature. Since we have more structure in this case, we may be able to describe $\mathcal{L}_{Per(M)}$ more concretely as we did in the case of the m-torus.

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