On Teichmüller spaces of b-groups with torsion

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Abstract of the Dissertation

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We compute coordinates (called horocyclic coordinates) for the Teichmüller spaces of terminal regular b-groups uniformizing orbifolds of finite hyperbolic type. We extend the results of the torsion free case obtained by Irwin Kra to the general case; the main tool is the existence of an equivalence between pairs of (orbifolds,maximal partitions) and weighted graphs. As an application of our results, we give explicit formulae for the Patterson Isomorphisms in horocyclic coordinates. In the last part we prove that the Bers-Greenberg Isomorphism between Teichmüller spaces of orbifolds of

the same type splits into a product mapping in the Maskit embedding.

To all my friends, but specially to Manuel Tejera and Miguel Alvarez.

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Chapter 1

Background and Statements of Main Results

In this chapter we introduce the background needed in this dissertation and state the main results of it. Section 1 contains the terminology and basic facts about orbifolds and Kleinian groups. In section 2 deals with the deformation spaces. Section 3 contains the main results of our work.

1.1 Orbifolds and Kleinian Groups

1.1.1. It is a well known fact that the full set of complex automorphisms of the Riemann sphere $\hat{\mathbb{C}}$ is the Möbius group, $Mob(\hat{\mathbb{C}})$, which consists of the transformations of the form $z \to \frac{az+b}{cz+d}$, with $ad - bc \neq 0$. Multiplying all the coefficients by a complex number we can assume that $ad - bc = 1$, and then we can identify $Mob(\hat{\mathbb{C}})$ with $PSL(2, \mathbb{C})$, whose elements will be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Any Möbius transformation not equal to the identity can be conjugated in $PSL(2,\mathbb{C})$ to one of the following types:

i) **parabolic:** $z \to z+1$, which has only one fixed point in $\hat{\mathbb{C}}$;

ii) **elliptic:** $z \to \lambda z$ with $|\lambda| = 1$;

iii) **loxodromic:** $z \to \lambda z$ with $|\lambda| \neq 1$; those transformations with λ real and positive are also known as **hyperbolic**. Elliptic and loxodromic (hyperbolic included) have two fixed points in $\hat{\mathbb{C}}$.

Suppose T is an elliptic element of finite order q. Let H be the subgroup of $PSL(2,\mathbb{C})$ generated by T, $H = \{T^n / n \in \mathbb{Z}\}$. We conjugate by a Möbius transformation so that the fixed points of the transformation T are 0 and ∞. Then the elements of H are of the form $z \mapsto e^{2\pi i p/q} z$. The mappings $z \mapsto e^{\pm 2\pi i/q} z$ are called **geometric** generators of H (or simply geometric transformations).

We can consider the group $SL(2,\mathbb{C})$ as a subset of $\mathbb{C}^4$ and give to it the subset topology. Taking the quotient by its center, $\pm I$, we have a topology on $PSL(2,\mathbb{C})$, and therefore a concept of discrete subgroups.

1.1.2. Given a subgroup G of $PSL(2,\mathbb{C})$ and a point z of $\hat{\mathbb{C}}$, we will say that G **acts discontinuously** at z if the following conditions are satisfied:

i) the stabilizer of z in G, $Stab(z,G) = \{g \in G / g(z) = z\}$ is finite; and

ii) there exists a neighborhood U of z such that $g(U) \cap U = \emptyset$ for all $g \in G - Stab(z,G)$.

The set of points where G acts discontinuously is called the **regular set**, or **region of discontinuity**, and it will be denoted by $\Omega = \Omega(G)$. Its complement on the Riemann sphere is known as the **limit set**, $\Lambda = \Lambda(G)$. A group of Möbius transformations is called **Kleinian** if its regular set is not empty. A Kleinian group is always discrete, but the converse statement is not true.

A Kleinian group that leaves invariant some disc is called a **Fuchsian** group. The limit set of such a group is contained in the circle that bounds the disc.

1.1.3. The limit set of a Kleinian group has a peculiar behaviour: either it has less than three points or it is uncountable. In the former case we will say that the group is **elementary**. These groups are very well known, and the reader can find more information about them in [Bea83] or [Mas88].

1.1.4. Given an element g of G, we will say that it is **primitive** if there are no roots of it in the group, or more precisely if $h \in G$ is such that $h^n = g$ then $n = \pm 1$.

1.1.5. Given a subset of the sphere, say X, and a subgroup $H \leq G$, we will say that X is **precisely invariant** under H in G if $H = Stab(X, G)$ and $g(X) \cap X = \emptyset$ for all $g \in G - H$. This simply means that the action of the whole group on the set X is reduced to the action of the subgroup H on X

1.1.6. A **2-orbifold** is a generalization of a manifold: it is a Hausdorff topological space where every point has a neighborhood homeomorphic to either the unit disc or the unit disc quotiented by a finite subgroup of rotations. The covering from the disc to the orbifold is in this case locally n-to-1, assuming that the group of rotations has order n. If x is the center of the above disc, we will say that x is a **ramification** or **branch point** with **ramification value** n (and branch value n-1). An orbifold where every point has ramification value one is just a Riemann surface.

Remark: topologically, any 2-orbifolds is a manifold; what changes is the complex structure (see below).

A **puncture** in an orbifold $\mathcal{S}$ is a open subset conformally equivalent to the punctured disc $\{z/\,|z| < 1\}$ and such for any sequence $z_n \to 0$, the corresponding sequence in $\mathcal{S}$ has no limit. We can identify the puncture with the point $z = 0$ (see [FK92]). In this setting, a puncture is a point removed from the orbifold; by abuse of notation, we will say that a puncture is a point with ramification value ∞, even though the point is not in the orbifold.

1.1.7. The orbifolds we will work with are of **finite conformal type**, which means that the underlying topological space is a closed manifold of genus p, with the possible exception of some punctures, say $x_1, \ldots, x_m$ and there are some points, say $x_{m+1}, \ldots, x_n$, with ramification values $\nu_{m+1}, \ldots, \nu_n$. We will write all this information as $(p, n; \infty, \ldots, \infty, \nu_{m+1}, \ldots, \nu_n)$, and we call it the **signature** of the orbifolds. It should be remarked here that the ramification values in the signature can be in any order (we do not require that the punctures appear first). The pair (p, n) is called the **type** of the orbifold.

1.1.8. The signature of an orbifold $\mathcal{S}$ determines its universal branched covering space as follows. If the signature is $(p, n; \nu_1, \ldots, \nu_n)$, then

$$2p - 2 + n - \sum_{j=1}^{n} \frac{1}{\nu_j}$$

is negative, zero or positive if and only if the universal branched covering space of $\mathcal{S}$ is (conformally equivalent to) the Riemann sphere, the complex plane or the upper half plane $\mathsf{H} = \{z/Im(z) > 0\}$. Other authors called this covering the **universal covering orbifold** ([Thu79]). The conformal structure on $\mathcal{S}$

comes from its universal covering by natural projection.

1.1.9. It is an easy exercise to prove that for a Kleinian group G, the quotient space Ω/G is a union of (possibly infinitely many) orbifolds. An important result in the theory of Kleinian groups, Alhfors' finiteness theorem ([Ahl64], [Kra72a]), states that if G is finitely generated, then Ω/G is a finite union of orbifolds of finite conformal type. We will call the space Ω/G the **quotient orbifold(s)** (associated to G).

1.1.10. A **triangle group** is an elementary group with quotient orbifold of type (0,3) or a Fuchsian group with two quotient orbifolds of type (0,3). We will divide the triangle groups into **elliptic**, **parabolic** or **hyperbolic** depending whether the region of discontinuity is (conformally equivalent to) the Riemann sphere, the complex plane or two discs, respectively. Similarly we will call the orbifolds elliptic, parabolic or hyperbolic depending on their universal branched covering space.

1.1.11. Another important result of Ahlfors is about the punctures: suppose G is Kleinian and Ω/G has (at least) one puncture. Then we consider a simple small loop around the puncture, bounding a punctured disc; this loop will be represented by some element of G. Ahlfors' Lemma tells us that the representative must be a parabolic element.

Similarly, if we consider a simple small loop bounding a disc whose center is a point with finite ramification value n, then the representative must be an elliptic element of order n.

1.1.12. Suppose G is a Kleinian group and let Δ be a simply connected invariant component of $\Omega(G)$. Being invariant by the group G means that

$g(\Delta) = \Delta$, $\forall g \in G$. Let $\mathbb{H}$ be the upper half plane, and let $\omega : \mathbb{H} \to \Delta$ be a Riemann mapping. Let us assume that T is a parabolic element of G. We will say that T is **accidental parabolic** if the Möbius transformation $\omega^{-1} \circ T \circ \omega$ is hyperbolic. This definition is independent of the choice of the Riemann map ω, since two such maps differ by composition with a Möbius transformation. More information about accidental parabolics can be found in [Mas70] or [Mas88].

1.1.13. One of the most important tools to construct Kleinian groups from smaller ones are the Klein-Maskit Combination Theorems. The first combination theorem says roughly speaking the following ([Mas92]). Suppose G_1 and G_2 are two Keinian groups, and suppose that they share a common proper subgroup J. Let W be a curve dividing the Riemann sphere into two topological disc, B_1 and B_2, such that B_i is precisely invariant under J in G_1. Then, if certains conditions are satisfied, the First Combination Theorem tell us that the group generated by G_1 and G_2, say G, is also a Kleinian group. G is known as the **Amalgamated Free Product**, or the AFP, of G_1 and G_2 across J. We will write $G = G_1 *_{<A>} G_2$. A smart choice of fundamental domains for G_1 and G_2 will give that their intersection is a fundamental domain for G.

The situation for the other theorem is slightly different, in the sense that we start only with a Kleinian group G. Suppose that H_1 and H_2 are subgroups of G, and that A is a Möbius transformation, $A \notin G$, that conjugates H_1 into H_2, that is, the mapping $B \mapsto ABA^{-1}$ is an isomorphism between H_1 and H_2. Then the group generated by G and A, say K, is called an **HNN Extension** of G, and it is denoted by $K = G*_{<A>}$. In the Second Combination Theorem we find a set of conditions that, if they are met by the group G, then we can say that the HNN extension of that group, K, is also Kleinian. A way of producing a fundamental domain for K from a fundamental domain for G is also given in that theorem.

The technical statements of these theorems can be found in Maskit's book, [Mas88], or in the coming paper [Mas92].

1.1.14. The following lemma gives conditions on the elements of a discrete group that contains a parabolic transformation.

Lemma 1 (Shimizu-Leutbecher) *Let Γ be a discrete subgroup of $Mob(\hat{\mathbb{C}})$, where Γ contains the transformation $f(z) = z + t, t \in \hat{\mathbb{C}}$. Then for every*

$$g \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \textit{ either } c = 0 \textit{ or } |c| \geq \frac{1}{|t|}.$$

For a proof see [Mas88].

1.1.15. One last definition we need concerns the cross ratio on the Riemann sphere. This is a measure of the relative position in $\hat{\mathbb{C}}$ of four distinct points, but since there are six different possible definitions, we state here the one we will use thorough this dissertation:

Definition 1 *Given four distinct points on $\hat{\mathbb{C}}$, say a, b, c, d, the* **cross ratio** *of them is defined by*

$$cr(a, b, c, d) = \frac{d-b}{d-a}\,\frac{c-a}{c-b},$$

and by continuity if one of the points is ∞

Observe that $cr(\infty, 0, 1, z) = z$.

1.2 Teichmüller and Riemann spaces. The Maskit embedding

1.2.1. Let G be a non-elementary Kleinian group. An isomorphism $\theta : G \to PSL(2, \mathbb{C})$ is called **geometric** if there exists a quasiconformal mapping w of the complex sphere such that $\theta(g) = w \circ g \circ w^{-1}$, $\forall g \in G$. We will say that two isomorphisms θ_1 and θ_2 are **equivalent** if there exists a Möbius transformation A such that $\theta_1(g) = A \circ \theta_2(g) \circ A^{-1}$. The set of equivalence classes of geometric isomorphisms is called the **Teichmüller** or **deformation space** of G, $\mathrm{T}(G)$. In order to put a topology on this space, we can consider a different approach: a quasiconformal mapping ω of the Riemann sphere is **compatible** with G if $\omega G \omega^{-1}$ is a subgroup of $PSL(2, \mathbb{C})$. Then we can normalize a compatible mapping by choosing three points in the limit set of G, say x_1, x_2, x_3 and ask for the mapping to fix them pointwise. Two normalized mappings are equivalent if and only if they have the same values on the limit set $\Lambda(G)$. In this setting, the Teichmüller space of a group G can be described as the restrictions to the limit set $\Lambda(G)$ of normalized compatible mappings; then the complex structure of $\mathrm{T}(G)$ is given by requiring that, for each limit point $x \in \Lambda$, the map $T(G) \ni \omega \to \omega(x) \in \mathbb{C}$ be holomorphic (see for example [Kra72b] or [Kra88]).

1.2.2. Let QC denote the space of quasiconformal homeomorphisms of the Riemann sphere. We define the normalizer of G in QC as the set:

$$N_{qc}(G) = \{\omega \in QC;\ \omega G \omega^{-1} = G\}.$$

The modular group of G, Mod(G) is the group of geometric automorphisms of G quotiented by the group of inner automorphisms of G. Since G is not elementary, it does not have center, and therefore the group of inner automorphisms of G is isomorphic to G. So the modular group is isomorphic to $Mod(G) \cong N_{qc}(G)/G$, where $h \in G$ acts on G by $G \ni g \mapsto h \circ g \circ h^{-1} \in G$.

If ω is an element of $N_{qc}(G)$, then it induces a group automorphism of G by the rule $G \ni g \mapsto \omega \circ g \circ \omega^{-1} \in G$. We denote this mapping by θ_ω. We want next to define an action of the modular group on Teichmüller space. To do it, we consider an element $\omega \in N_{qc}(G)$ and the induced mapping θ_ω. This mapping gives rise to a mapping on $T(G)$, that we will denote by θ^*_ω, defined as follows: suppose that u is a representative of a point of $T(G)$, then $\theta^*_\omega(u)$ is the mapping on G induced by $u \circ \omega^{-1}$; that is,

$$(\theta^*_\omega(u))(g) = (u \circ \omega^{-1}) \circ g \circ (u \circ \omega^{-1})^{-1},$$

for all elements $g \in G$. The group of inner automorphisms of G acts trivially, since we conjugate by Möbius transformations, which does not change the Teichmüller class. This means that the modular group acts on Teichmüller space. The quotient space $T(G)/Mod(G)$ is known as the **Riemann** or **moduli space** of G, $R(G)$, and it represents the $PSL(2,\mathbb{C})$-conjugacy classes of Kleinian groups quasiconformally equivalent to G.

1.2.3. Let $\mathcal{S}$ be an orbifold with signature $(p,n;\nu_1,\ldots,\nu_n)$ and let $\mathcal{S}'$ be the surface with punctures constructed by removing from $\mathcal{S}$ all the ramification points, i.e.

$$\mathcal{S}' = \mathcal{S} - \{x_j;\ 1 \le j \le n, \nu_j < \infty\}.$$

A **MAXIMAL PARTITION**, $\mathcal{C}$, on $\mathcal{S}$ is a set of $3p-3+n$ simple unoriented closed curves in $\mathcal{S}'$ such that:

1.- no two curves on the partition are freely homotopically equivalent on $\mathcal{S}'$;

2.- no curve on the partition is homotopically trivial on $\mathcal{S}'$;

3.- no curve on the partition is contractible to one of the punctures of $\mathcal{S}'$,

The maximal part comes from the fact that any set of curves satisfying the above three conditions has 3p-3+n or *less* elements. Existence of maximal partitions is a well known topological fact that can be found in [Str82].

1.2.4. A **terminal regular b-group** G is a geometrically finite (it has a finited sided fundamental polygon for its action on hyperbolic 3-space, [Mas88]) Kleinian group with a simply connected invariant component $\Delta \subset \Omega$, and so that $(\Omega - \Delta)/G$ is a union of orbifolds of type (0,3). Given an orbifold $\mathcal{S}$ with hyperbolic signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$, and maximal partition $\mathcal{C}$, we will say that the terminal regular b-group G **uniformizes** the triple $(\mathcal{S}, \sigma, \mathcal{C})$, if the following conditions are satisfied:

1.- the quotient of the invariant component Δ by the group G is conformally equivalent to the orbifold $\mathcal{S}$;

2.- for each element a_j of the partition $\mathcal{C}$, there is a curve $\tilde{a_j}$ in Δ, precisely invariant under an accidental parabolic subgroup $< A_j >$ of G, and such that $\Delta \supseteq \tilde{a_j} \xrightarrow{\pi} \pi(\tilde{a_j}) = a_j \subseteq \mathcal{S}$, where $\pi : \Delta \to \mathcal{S}$ is the natural projection.

The following theorem of Maskit tells us that such an uniformization is possible:

Theorem 1 (Maskit [Mas70], [Mas75]) *Given a triple as above, there exists a terminal regular b-group G such that:*

1.- G uniformizes the triple in the above sense (in the invariant component Δ);

2.- G is unique up to conjugation in $PSL(2,\mathbb{C})$;

3.- $(\Omega(G)-\Delta)/G$ is the union of the orbifolds of type (0,3) obtained by squeezing each curve of $\mathcal{C}$ to a puncture and discarding all orbifolds of signature $(0,3;2,2,\infty)$ that appear.

1.2.5. This theorem gives us a way of decomposing the orbifold S into 'smaller' surfaces (whose deformation spaces are simpler) as follows: let T_k ($1 \leq k \leq 3p-3+n$) be the connected component containing a_k obtained by cutting S along $a_j, j \neq k$, that is T_k is the connected subset of $S - \{a_j/a_j \in \mathcal{C}, j \neq k\}$ containing a_k. The orbifolds $T_1, \ldots, T_{3p-3+n}$ are called the **modular parts** of S. Let D_k be a component of $\pi^{-1}(T_k)$ and let G_k be the stabilizer of D_k in G. These subgroups are known as the **modular subgroups** of G. They are terminal regular b-groups of type (1,1) or (0,4). Geometrically they represent the surfaces T_k where we have attached to each boundary curve a punctured disc. Note that the modular parts are equipped with a maximal partition consisting of a_k. The group G is generated by the union $G_1 \cup \ldots \cup G_{3p-3+n}$ in $PSL(2,\mathbb{C})$. There is an algorithm to construct the modular groups, and therefore the group G, for which we refer the reader to the paper by I. Kra [Kra88].

1.2.6. We will work with a set of coordinates for Teichmüller space given by B. Maskit in [Mas74] and I. Kra in [Kra88]. Consider a triple $(S, \sigma, \mathcal{C})$ uniformized by the terminal regular b-group G in the simply connected invariant component Δ. The partition $\mathcal{C}$ gives a decomposition of Γ into modular subgroups $G_1, \ldots, G_{3p-3+n}$ (as explained above), whose deformation spaces $T(G_j)$ are one-dimensional. It is clear that any deformation of G supported on Δ induces a deformation of G_j, and this gives a mapping from $T(G)$ to $\prod_{j=1}^{3p-3+n} T(G_j)$. The following theorem states that this mapping is actually injective, providing a set of coordinates on the deformation space $T(\Gamma)$.

Theorem 2 *1.-([Mas74], [Kra88]) The mapping defined above*

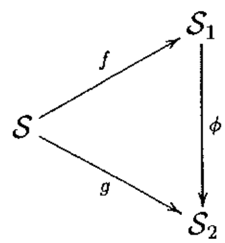
$$T(G) \hookrightarrow \prod_{j=1}^{3p-3+n} T(G_j)$$

is holomorphic and injective with open image.

2.- (Gentilesco [Gen79]) The image of this mapping is a proper subset of $\prod_{j=1}^{3p-3+n} T(G_j)$*, except in the case of 3p-3+n=1.*

We will call the mapping of this theorem the **Maskit embedding** of $T(G)$.

1.2.7. We also need the definition of deformation spaces of orbifolds, rather than deformation spaces of groups. Let $\mathcal{S}$ be an orbifold of finite conformal type with hyperbolic signature $\sigma = (p, n; \nu_1 \ldots, \nu_n)$. Let $f : \mathcal{S} \to \mathcal{S}_1$ and $g : \mathcal{S} \to \mathcal{S}_2$ be quasiconformal homeomorphisms that 'preserves' the complex structure of $\mathcal{S}$, in the sense that they map ramification points to ramification points with the same ramification value. Then we will say that f and g are **Teichmüller equivalent** if and only if there is a biholomorphism $\phi : \mathcal{S}_1 \to \mathcal{S}_2$, that 'preserves' the complex structures and such that the following diagram commutes up to homotopy:



By commuting up to homotopy we mean that $g^{-1} \circ \phi \circ f$ is homotopic to the identity map of S.

The **Teichmüller space** of S is the set of equivalence classes of quasiconformal deformations. The intuitive idea behind this definition of deformation spaces is that two quasiconformal deformations of the orbifold S are in the same Teichmüller class if we can pass from one to the other by composing with a mapping homotopic to the identity and with a conformal homeomorphism.

1.2.8. Now suppose $\mathcal{S}$ is given with a maximal partition, $\mathcal{C}$. By theorem 1 in §1.2.4, there exists a terminal regular b-group G uniformizing $(\mathcal{S}, \sigma, \mathcal{C})$ on its invariant component Δ. We have that the spaces $T(G)$ and $T(\mathcal{S})$, as defined in §1.2.1 and above respectively, are equivalent. This last fact can be found in [Nag88] and [Kra72b]; in the first of these two references it is proven that the deformation space of an orbifold, $T(\mathcal{S})$, is equivalent to the deformation space of a Fuchsian group uniformizing it, while in the second paper Kra proves that the deformation space of a (terminal regular) b-group is equivalent to the deformation space of the Fuchsian group uniformizing the orbifold.

The deformation space of the orbifold $\mathcal{S}$ will be denoted by $T(p, n; \nu_1 \ldots, \nu_n)$. By the above paragraph, if G is a terminal regular b-group uniformizing $(\mathcal{S}, \sigma, \mathcal{C})$, then we have $T(\mathcal{S}) = T(G) = T(p, n; \nu_1 \ldots, \nu_n)$.

Remark: in Kra's work, [Kra72b], a more general result about equivalences among deformation spaces is proven, but for our purposes what we have told suffices.

1.3 Statements of Main Results

1.3.1. Our work deals with the problem of finding good coordinates for the deformation spaces of (terminal regular) b-groups. The uniformization theorem of Maskit (§1.2.4), tells us that the study of the deformation spaces of terminal regular b-groups and the study of the deformation spaces of orbifolds is equivalent. Ideally, we would like, given a complex structure on an orbifold S, to be able to find its position in the corresponding deformation space $T(S)$, or equivalently in the deformation space of a terminal regular b-group, Γ, uniformizing S. A possible solution to this problem is the one given by Irwin Kra in [Kra90]: to find coordinates on $T(\Gamma)$ from which is possible to construct explicitly Kleinian groups. The case of surfaces (orbifolds **without** ramification points) of hyperbolic finite type was completely solved in the quoted paper. Our work is a generalization of his in the sense that we compute similar coordinates for the deformation spaces of terminal regular b-groups *with torsion.*

1.3.2. The b-groups we work with are constructed from triangle groups by use of Maskit Combination Theorems; that is, by AFP's and HNN extensions. Therefore, first of all we need to study these basic triangle groups. Given two such groups with the same signature, we know from the classical theory of Kleinian groups that they are conjugate in $PSL(2,\mathbb{C})$. Since a Möbius transformation is determined by the images of three points, to know a triangle group all we need is three complex parameters, besides its signature. This is the content of our first result:

Proposition 1(§2.3.7): Given three distinct points (a,b,c) in the Riemann sphere, and a hyperbolic signature $(0,3;\nu_1,\nu_2,\nu_3)$, there exists a triangle group $\Gamma(\nu_1,\nu_2,\nu_3;a,b,c)$ of signature $(0,3;\nu_1,\nu_2,\nu_3)$, generated by a unique pair of Möbius transformations (A,B) determined by the parameters (a,b,c). The transformations A and B are called canonical generators of the group for the given parameters (a,b,c).

The definition of canonical generators is given in §2.3.6. For this section it suffices to say that what that definition does is to relate the parameters (a,b,c) to the Möbius transformations (A,B) in a geometric way.

A similar result for the parabolic groups with signature $(0,3;\infty,2,2)$ is in the proposition 5 of paragraph §2.4.3.

1.3.3. One can consider two triangle groups and apply Maskit First Combination Theorem to obtain a terminal regular b-group G uniformizing a surface of type (0,4). To do it, we start with $\Gamma(\infty,\nu_1,\nu_2;a,b,c)$ and $\Gamma(\infty,\nu_3,\nu_4;a,b',c')$, with a choice of parameters so that they share a parabolic element, say A, and we then construct the group G by an AFP:

$G=\Gamma(\infty,\nu_1,\nu_2;a,b,c)*_{<A>}\Gamma(\infty,\nu_3,\nu_4;a,b',c')$. The coordinate for the deformation space of G is given as a cross ratio:

Theorem 3(§3.2.5): $\alpha = cr(a,b,c,b')$ is a global coordinate for $T(G)$, and we have the following inclusions:

$$\{\alpha / Im(\alpha) > y_1\} \subset T(0,4;\nu_1,\nu_2,\nu_3,\nu_4) \subset \{\alpha / Im(\alpha) > y_2\},$$

where

$$y_1 = \frac{1}{q_1+q_2} + \frac{1}{q_3+q_4},\ y_2 = max(\frac{1}{q_1+q_2}, \frac{1}{q_3+q_4}),$$

and $q_i = cos(\pi/\nu_i)$.

1.3.4. If we start with only one triangle group, $\Gamma(\infty,\infty,\nu;a,b,c)$ and we do a HNN extension, then we will obtain a group uniformizing an orbifold of signature $(1,1;\nu)$. The result about the coordinate for the corresponding deformation space is in §3.2.9 (theorem 5).

1.3.5. By use of Maskit embedding we obtain coordinates for the deformation spaces of orbifolds of finite hyperbolic type:

Theorem 7(§3.2.12): Let $\mathcal{S}$ be an orbifold of finite hyperbolic type with signature $\sigma = (p,n;\nu_1,\ldots,\nu_n)$ and let $\mathcal{C}$ be a maximal partition on $\mathcal{S}$. Then there exists a set of (global) coordinates for the deformation space $T(p,n;\nu_1,\ldots,\nu_n)$, say $(\alpha_1,\ldots,\alpha_d)$, where $d = 3p-3+n$, and a set of complex numbers, $(y_1^1,\ldots,y_1^d,y_2^1,\ldots,y_2^d)$, that depend on the signature σ and the partition $\mathcal{C}$, such that

$$\{(\alpha_1,\ldots,\alpha_d) \in \mathbb{C}^d / Im(\alpha_i) > y_1^i\ , \forall\ 1 \le i \le d\} \subset T(p,n;\nu_1,\ldots,\nu_n)$$

and

$$T(p,n;\nu_1,\ldots,\nu_n) \subset \{(\alpha_1,\ldots,\alpha_d) \in \mathbb{C}^d / Im(\alpha_i) > y_2^i\ , \forall\ 1 \le i \le d\}.$$

1.3.6. A result of Patterson states that the only isomorphisms between Teichmüller spaces of orbifolds of different type (with $2p-2+n>0$) are $T(2,0)\cong T(0,6)$, $T(1,2)\cong T(0,5)$, $T(1,1)\cong T(0,4)$. In chapter 4 we compute this isomorphisms explicitly in the coordinates given by the above theorem:

Theorem 12(§4.2.2); The mapping

$$(\tau_1,\tau_2,\tau_3)\mapsto(\frac{\tau_1}{2},1+\tau_2,1+\frac{\tau_3}{2})$$

gives an isomorphism between $T(2,0)$ and $T(0,6;2,2,2,2,2,2)$. If we consider the cases of $\tau_3=0$ and $\tau_3=\tau_2=0$, then we obtain the isomorphisms $T(1,2;\infty)\cong T(0,5;\infty,2,2,2,2)$ and $T(1,1;\infty)\cong T(0,4;\infty,2,2,2)$, respectively.

1.3.7. Another type of isomorphism between deformation spaces are those addressed in the Bers-Greenberg theorem: two Teichmüller spaces of orbifolds of the same type, but different signature, are equivalent. One may ask what happens when we look at this theorem in the Maskit embedding. That is the content of the main theorem of chapter 5:

Theorem 13(§5.1.2): Let Γ_0 and Γ be terminal regular b-groups uniformizing orbifolds of signatures $(p,n;\infty,\dots,\infty)$ and $(p,n;\nu_1,\dots,\nu_n)$ respectively. Let

$$T(\Gamma_0)\hookrightarrow\prod_{j=1}^{3p-3+n}T(\Gamma_j)\qquad \textit{and} \qquad T(\Gamma)\hookrightarrow\prod_{j=1}^{3p-3+n}T(\Gamma_j')$$

be the Maskit embeddings. Then there exist isomorphisms

$$h_j^*:T(\Gamma_j)\to T(Gamma_j'),$$

for $1 \leq j \leq 3p-3+n$, such that the restriction

$$(h_1^*, \ldots, h_{3p-3+n}^*)|_{T(\Gamma_0)} : T(\Gamma_0) \to T(\Gamma)$$

is an isomorphism.

This theorem shows that the Bers-Greenberg Isomorphism is a consequence of the fundamental structure of the groups. The proof uses the fact that deformations on an orbifold can be made conformal on neighborhoods of the punctures, staying in the same Teichmüller class:

Lemma 5(Deformation Lemma) (§5.4.12): Suppose S is an orbifold with (at least) n punctures. Let $U_1, \ldots, U_n$ be punctured discs around the punctures of S, with disjoint closures. Then any quasiconformal deformation of S is equivalent to a quasiconformal deformation that is conformal on the sets $U_1, \ldots, U_n$.

Chapter 2

Triangle Groups

2.1 Introduction

2.1.1. The basic building blocks for hyperbolic compact surfaces are pairs of pants (spheres with three discs removed); a surface of genus g(≥ 2) can be obtained by gluing 2g−2 spheres with three holes along their boundaries. The pairs of pants can be constructed from spheres with three punctures by removing 'neighborhoods' of the punctures, and then gluing along the boundaries. The size of these neighborhoods ('radius' of the punctured discs, in a good coordinate) and how twisted the boundaries identifications are (twisting angle) produce a set of complex coordinates in the deformation (Teichmüller) space of the surface (remark: this is not quite correct, as the construction is independent of the 'radius', but it is the right philosophy. For a better explanation of this technique, and how it can be understood, the reader is referred to Irwin Kra's paper [Kra90]). With this method we can construct surfaces with punctures as well.

One can generalize the above technique to construct orbifolds (see chapter 1 for the definition). All we need to do is to start with orbifolds with signature $(0,3;\nu_1,\nu_2,\nu_3)$, where the numbers ν_i could be ∞ (punctures), or integers, with $\nu_j \geq 2$ (ramification points).

2.1.2. The only Kleinian groups uniformizing orbifolds of type (0,3) are the triangle groups. The classical theory tells us that two such groups with the same signature are conjugate in $PSL(2,\mathbb{C})$. Since a Möbius transformation is determined by the images of three points, all we need to know a triangle group is three different points in $\hat{\mathbb{C}}$ (besides its signature, of course). This is the main objective of this chapter: to give presentations for triangle groups based on three points of $\hat{\mathbb{C}}$ that we call **parameters**, and to study the dependence of the generators on the paramenters. We will study hyperbolic and parabolic groups; the results about the elliptic groups will appear somewhere else in the future.

The chapter is divided as follows: we motivate our work in the second section by studying the torsion free case solved by I. Kra in [Kra90]; the third section deals with hyperbolic triangle groups; section 4 is dedicated to the parabolic groups; section 5 studies the geometry of the corresponding quotient orbifolds, and we finish with a section about the automorphism groups of some of these orbifolds.

2.2 The torsion free case

2.2.1. In this section we will study the case of torsion-free triangle groups (these are Fuchsian groups with signature $(0,3;\infty,\infty,\infty)$), based on the work of Irwin Kra, [Kra90]. The reason for including these results here is two fold: first as a motivation for all our later work, and second for the sake of completeness.

2.2.2. Let (a,b,c) three distinct ordered points in $\hat{\mathbb{C}}$. We want to find Möbius transformations A and B such that they generate a group with signature $(0,3;\infty,\infty,\infty)$ and they are somehow related to (a,b,c).

The three given points determine a unique oriented circle on the Riemann sphere $\hat{\mathbb{C}}$. Here we take circles in the general sense, where a line is simply a circle passing through ∞ (think of $\hat{\mathbb{C}}$ as S^2). That circle bounds two discs, one of them lying to the left. This is a well defined concept:

Definition 2 *If z does not belong to the oriented circle determined by a, b, c, then z is* **to the left** *of that circle if and only if $cr(a,b,c,z)$ has positive imaginary part.*

Since Möbius transformations preserve cross ratios and the family of circles on the Riemann sphere, the concept of being to the left of an oriented circle is $PSL(2,\mathbb{C})$-invariant, therefore it is a good definition for our purposes.

Going back to our problem, on the disc to the left of (a,b,c) we consider the metric of constant curvature -1. What we are going to do is to find a fundamental domain the group, that will consist on two triangles whose sides are relates to (a,b,c), and the transformations we are looking for, A and B, will be the side identifications (in the spirit of Poincaré's theorem). But in order to make things clear, we will first consider the case $a=\infty$, $b=0$ and

$c = 1$.

The circle determined by these points is the extended real line $\hat{\mathbb{R}}$, with its usual orientation. $\mathbb{H} = \{z \in \mathbb{C} / \ Im(z) > 0\}$ is the disc to the left of the real line. The metric is the known Poincaré's metric, $\frac{|dz|}{y}$ $(z = x + i\,y)$.

2.2.3. Before computing the matrix expressions of the elements A and B, let us stop for a moment to see what we want. A parabolic transformation (given by the ∞ in the signature) can be understood as the limiting case of an elliptic element when the order goes to infinity. Elliptic elements of $PSL(2, \mathbb{C})$ of finite order are rotations with two fixed points in $\hat{\mathbb{C}}$; if the transformation fixes our circle, then it must have one fixed point inside each disc. A geodesic ending at the fixed point will be rotated by certain angle (inversely proportional to the order of the element). As a parabolic element is the limiting case when the order goes to infinity, the 'rotation angle' will be zero. If the element had a fixed point in the interior of one of the discs, a geodesic abutting at that point should be rotated by a zero degrees angle, and therefore this transformation would be the identity. So the fixed point should be in the circle at infinity, where it is possible for two <u>different</u> geodesics to meet with zero angle.

A natural requirement to ask is that A fix ∞ and B fix 0. We will also ask for their product AB to fix the third point 1; and we know that these three transformations, A, B and AB, have to be parabolic. To compute their formulae, let us consider a triangle with its vertices at $\infty, 0$ and 1, and the reflections on its sides as shown in the figure 2.1.

The formulae of these reflections are:

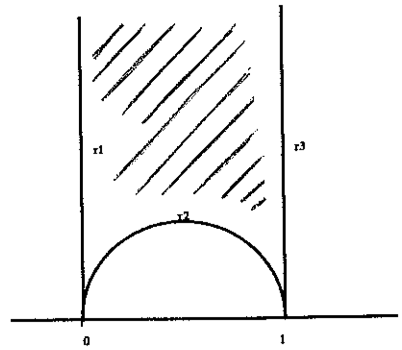


Figure 2.1: A triangle for the torsion free case

$$r_1(z) = -\overline{z},\ r_2(z) = \frac{\overline{z}}{2\overline{z}-1},\ r_3(z) = 2 - \overline{z}.$$

2.2.4. These reflections generate a discrete group of isometries of $(\mathbb{H}, \frac{|dz|}{y})$ containing orientation reversing maps. There is an index two subgroup consisting of the orientation preserving mappings, which are those mapping obtained by composing an even number of these reflections. Two possible generators are:

$$r_3 r_1(z) = A(z) = z + 2,$$

and

$$r_1 r_2(z) = B(z) = \frac{-z}{2z-1}.$$

The group $\Gamma =< A, B >$ is a triangle group with signature $(0, 3; \infty, \infty, \infty)$. It is easy to compute the inverse of the product of these generators, namely

$$C(z) = (AB)^{-1}(z) = r_2 r_3(z) = \frac{z-2}{2z-3},$$

which fixes 1 as we wanted. A fundamental domain for the action of Γ on $\Omega(\Gamma) = \mathbb{C} - \mathbb{R}$ can be obtained by considering the above triangle and applying first the reflection r_3, and then a reflection on the real axis; such fundamental domain is shown in figure 2.2.

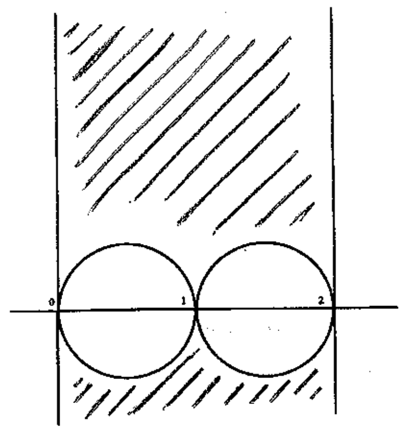


Figure 2.2: A fundamental domain for Γ.

In [Kra90], the group Γ generated by A and B is denoted by $F(\infty, 0, 1)$. In order to include the signature, we will adopt the notation of $\Gamma(\infty, \infty, \infty; \infty, 0, 1)$ for the above triangle group.

2.2.5. It should be remarked that the order of the composition of the r_i is extremely important, because if we take $r_1 r_3 = A^{-1}$ as the first generator, and we keep B as the second, then the inverse of the product $A^{-1}B$ is NOT parabolic (and therefore it will not represent the third puncture).

2.2.6. The general case of parameters (a, b, c) is obtained by finding a Möbius transformation that takes these points to $\infty, 0$ and 1, respectively, and then conjugating by this transformation the group $\Gamma(\infty, \infty\infty; \infty, 0, 1)$ previously obtained. This new group will be denoted by $\Gamma(\infty, \infty, \infty; a, b, c)$.

2.3 Hyperbolic groups

2.3.1. In order to generalize the result of the previous section, we will start by considering the case of a group Γ_ν with signature $(0,3;\infty,\infty,\nu)$, where $\nu \in \mathbf{Z}, \nu \geq 2$. A good generalization should be such that as ν goes to infinity, the generators of Γ_ν approach those of the group Γ in the previous section (assuming we keep the same set of parameters). This makes sense, since we 'understand' parabolic elements as the limiting case of elliptic transformations. A way of getting the group Γ_ν is by constructing a triangle that generates it (in the sense of the previous section) and whose sides approach to those of the triangle of the previous section, as ν goes to ∞. To fix ideas what we will do is to consider the case with parameters $(\infty, 0, 1)$ as before. Our triangle must have two zero angles with vertices on the (extended) real axis, and one interior vertex, where two geodesics should meet making an angle of π/ν. We will ask for this vertex to be on the line $\{z/Re(z) = 1\}$. It is easy to see that this is possible, and only in one way (cfr. [Bea83]). See figure 2.3.

Computing as before we get the following reflections:

$$r_1(z) = -\overline{z}, \quad r_2(z) = \frac{1}{(1+q)^2\overline{z} - (1+q)} + \frac{1}{1+q},$$
$$r_3(z) = 2 - \overline{z}.$$

The orientations preserving subgroup of index two is then generated by:

$$r_3r_1(z) = A(z) = z + 2, \quad r_1r_2(z) = B(z) = \frac{-z}{(1+q)z - 1},$$

where $q = cos(\frac{\pi}{\nu})$. The inverse of their product is given by

$$(AB)^{-1} = C(z) = \frac{z-2}{(1+q)z - 1 - 2q}.$$

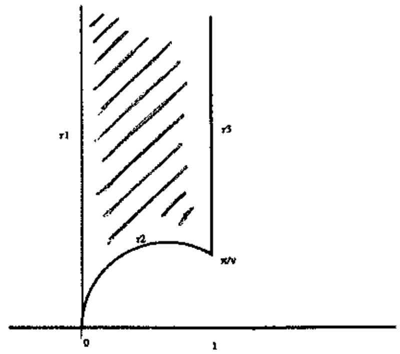


Figure 2.3: A triangle for the group of section 2.3.1

We can see that the transformations B and C converge, as ν goes to infinity, to the corresponding transformations studied in section 1. In our notation, this group will be described as $\Gamma(\infty, \infty, \nu; \infty, 0, 1)$.

2.3.2. If the group has signature $(0,3;\infty,\nu_1,\nu_2)$ with ν_1 and ν_2 finite integers, then we can construct a unique triangle with a vertex at infinity, and two vertices inside $\mathbf{H}$ on the lines $\{z/Re(z) = 0\}$ and $\{z/Re(z) = 1\}$ (see [Bea83]). This will give us a group generated by

$$A(z) = z + 2, B(z) = \frac{-q_1 z + b}{(q_1 - q_2)z - q_1},$$

where $q_i = cos(\frac{\pi}{\nu_i})$ and $b = \frac{q_1^2 - 1}{q_1 - q_2}$ (in this way the matrix corresponding to B has determinant equal to 1).

2.3.3. Suppose now that the group has no parabolic transformations; (that is, the signature is $(0,3;\nu_1,\nu_2,\nu_3)$ with $2 \leq \nu_i < \infty$). We then construct a triangle on the upper half plane, with angles $\frac{\pi}{\nu_i}$, $i = 1,2,3$, and such that two of its vertices lie on the line $\{z \,/\, Re(z) = 0\}$ and the other on $\{z \,/\, Re(z) = 1\}$.

To generalize the previous results we need to take the first two vertices (angles $\frac{\pi}{\nu_1}$ and $\frac{\pi}{\nu_2}$) on the imaginary axis, and the third vertex on the other vertical line. But under those conditions it is possible to find TWO different triangles (see 2.3.4). The goal is to get the parabolic case as the limiting case of elliptic, so we make a choice: the vertex corresponding to A (angle $\frac{\pi}{\nu_1}$) has bigger imaginary part than the one corresponding to B (angle $\frac{\pi}{\nu_2}$); or loosely speaking, the vertex corresponding to A is 'closer to $i\infty$' than the vertex corresponding to B. Later we will give a conjugacy invariant definition of this ordering. Let us remark now that these conditions give us a triangle like the one in figure 2.4, and that the group we are looking for is generated by

$$A(z) = \frac{-q_1 z - kp_1}{k^{-1}p_1 z - q_1}, \;\; B(z) = \frac{-q_2 z + b}{cz - q_2},$$

where $q_i = cos(\frac{\pi}{\nu_i})$,$k = \frac{q_2 - q_1 q_3 + q_1 l}{p_1 l}$, $c = \frac{q_1 q_2 - q_3 + l}{p_1 k}$, $b = \frac{q_2^2 - 1}{c}$, and $l = \sqrt{q_1^2 + q_2^2 + q_3^2 - 2q_1 q_2 q_3 - 1}$. The inverse of the product of the two generators is given by

$$C(z) = (AB)^{-1} = \frac{(k^{-1}bp_1 + q_1 q_2)z + bq_1 - kp_1 q_2}{(k^{-1}p_1 q_2 + cq_1)z + q_1 q_2 - kp_1 c}.$$

2.3.4. Let us stop for a moment in order to explain how we obtained the triangle of the figure 2.4. First, we construct any triangle with angles π/ν_1, π/ν_2 and π/ν_3 and vertices v_1, v_2 and v_3 respectively. A basic fact from hyperbolic geometry is that two triangles with the same angles are congruents (one can be mapped by an isometry into the other); therefore there is, essentially, only one triangle with the given angles. Then we map v_1 to a point in the imaginary axis, say i. By use of a rotation, we can map the side v_1, v_2 into

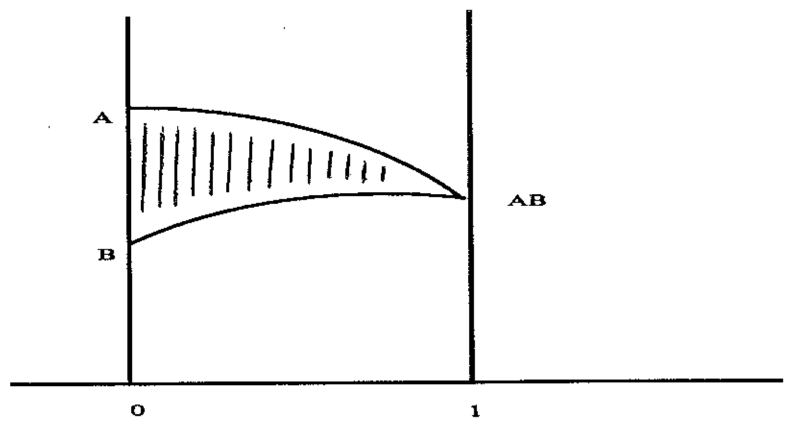


Figure 2.4: A triangle for the group of section 2.3.3

the imaginary axis. If necessary, we apply the reflection $z \mapsto -\bar{z}$ so that the third vertex has positive real part. If this vertex has real part equal to 1, we are done; if not, we can apply a dilatation $z \mapsto \lambda z$, with $\lambda > 0$, to get its real part exactly equal to one. This will give the triangle we want.

But we can see that there are two possible triangles, depending whether the imaginary part of the second vertex is bigger than the imaginary part of the first or not. We will deal with this problem later (see definition 3 in the next subsection).

2.3.5. We will now give invariant definitions for the particular cases studied above. We start with three distinct points in $\hat{\mathbb{C}}$, say a, b, c. Let Λ be the circle determined by them, and orient it so that a, b, c follow each other in the positive orientation. Let L be the circle orthogonal to Λ and passing through a and b. Similarly, let L' be the circle orthogonal to Λ and passing through a and c. For the case of $a = \infty$, $b = 0$ and $c = 1$ we obtain the real axis, the imaginary axis and the line $\{z/ \, Re(z) = 1\}$, respectively.

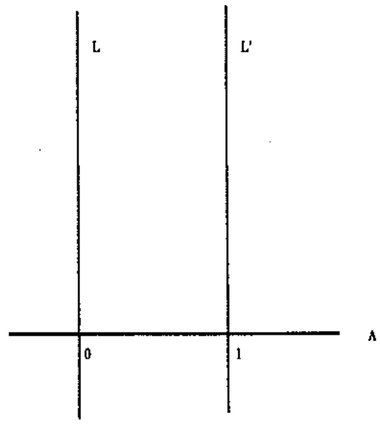


Figure 2.5: The circles for parameters $(\infty, 0, 1)$.

Let Δ denote the disc to the left of Λ; that is,

$$\Delta = \{z \in \hat{\mathbb{C}};\ Im(cr(a,b,c,z)) > 0\};$$

and let $\overline{\Delta}$ denote its closure in the Riemann sphere. Since Möbius transformations preserve angles and the family of circles on the Riemann sphere, we have that the following definition is a $PSL(2,\mathbb{C})$-conjugacy invariant choice of triangles, similar to that of 2.3.3. We need that choice in order to obtain the uniqueness statements of the proposition of §2.3.7.

Definition 3 *Let z_1 and z_2 be two distinct points in $L \cap \overline{\Delta}$. We will say that they are WELL ORDERED, with respect to a and b, if one of the following set of conditions is satisfied (they are not mutually exclusive):*

1) $z_1 = a$,

2) $z_2 = b$,

3) $z_1 \neq a$, $z_2 \neq b$ and $cr(a, z_1, z_2, b)$ is real and strictly bigger than 1.

If $a = \infty, b = 0$ and $c = 1$, the third case happens when $z_1 = \lambda i$ and $z_2 = \mu i$, with $\lambda > \mu$, which agrees with our previous work.

2.3.6. For an elliptic or parabolic Möbius transformation, A, let $|A|$ denote its order, where order equal to infinity means that the transformation is parabolic.

Definition 4 *Let (a, b, c) be three distinct points of $\hat{\mathbb{C}}$, and let Λ, Δ, L and L' be as in §2.3.5. Suppose that Γ is a triangle group with hyperbolic signature $(0, 3; \nu_1, \nu_2, \nu_3)$ and whose limit set is Λ. Let A and B be elements of Γ. We will say that (A, B) are CANONICAL GENERATORS of Γ for the parameters (a, b, c) if they generate the group Γ and the following conditions are satisfied:*

1) $|A| = \nu_1$, $|B| = \nu_2$, $|AB| = \nu_3$,

2)A and B have their fixed points on L, and AB on L',

3) if z_1 and z_2 are the fixed points of A and B on $L \cap \overline{\Delta}$, then they are well ordered with respect to a and b,

4) A and B are primitive elements, and geometric whenever elliptic.

It is clear that this definition is conjugacy invariant in the following sense: if Γ is a group with canonical generators (A, B) and T is a Möbius transformation, then (TAT^{-1}, TBT^{-1}) are canonical generators for $T\Gamma T^{-1}$. This is so because conjugation preserves the order of transformations, and the elements of $PSL(2, \mathbb{C})$ preserve cross ratios.

We already have computed the canonical generators for parameters $\infty, 0$ and 1. To find canonical generators for a group Γ with parameters (a, b, c), all we need to do is to find a transformation that takes these three points to $\infty, 0, 1$, respectively, and conjugate by that transformation. The only IMPORTANT detail left is to prove the uniqueness of the generators for the case of parameters $(\infty, 0, 1)$. That is what we will do next.

2.3.7. We now state our previous results in the form of a proposition. The Möbius transformations will be given as (representative of classes of) matrices of $PSL(2, \mathbb{C})$.

Proposition 1 *Given three different points (a, b, c) in the Riemann sphere, and a hyperbolic signature $(0, 3; \nu_1, \nu_2, \nu_3)$, there exists a triangle group $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$ with a unique pair generators (A, B) canonical for the given parameters.*

In the case $(a = \infty, b = 0, c = 1)$, these generators are given by:

1)Signature $(0, 3; \infty, \infty, \nu)$

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -1 & 0 \\ 1+q & -1 \end{bmatrix}.$$

Here $q = cos(\pi/\nu)$ with $q = 0$ if $\nu = \infty$.

2)Signature $(0, 3; \infty, \nu_1, \nu_2)$, $\nu_i < \infty$

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -q_1 & b \\ q_1 + q_2 & -q_1 \end{bmatrix},$$

$q_i = cos(\pi/\nu_i)$ and $b = \frac{q_1^2 - 1}{q_1 + q_2}$.

3)Signature $(0,3;\nu_1,\nu_2,\nu_3)$*, where all the* ν_i *are finite.*

$$A=\begin{bmatrix} -q_1 & -kp_1 \\ k^{-1}p_1 & -q_1. \end{bmatrix},\ B=\begin{bmatrix} -q_2 & b \\ c & -q_2 \end{bmatrix}.$$

We have

$$q_i=cos(\pi/\nu_i), p_i=sin(\pi/\nu_i),$$

$$k=\frac{q_2+q_1q_3+q_1l}{p_1l},\ c=\frac{q_1q_2+q_3+l}{kp_1},\ b=\frac{q_2^2-1}{c},$$

$$\text{where } l=\sqrt{q_1^2+q_2^2+q_3^2+2q_1q_2q_3-1}$$

and the square root is chosen to be positive.

For any other set of parameters, (a,b,c)*,the generators are given by conjugating the above generators by the unique Möbius transformation* T *that maps* a,b,c *to* $\infty,0,1$ *respectively.*

A few words about this result are needed. First of all, lemma 2 guarantees that the expression under the square root is positive. The existence of the generators was established in the previous sections. The uniqueness will be proven in proposition 3 (§2.3.10).

2.3.8. Next we prove a lemma showing that the term under the square root in the above proposition is positive.

Lemma 2 *Let* Γ *be a triangle group with signature* $(0,3;\mu_1,\mu_2,\mu_3)$*, where* $\mu_i\in\mathbb{Z}\cup\{\infty\},\mu_i\geq 2$*, and let* $q_i=cos(\pi/\mu_i)$*. Let*

$$l^2=q_1^2+q_2^2+q_3^2+2q_1q_2q_3-1.$$

Then l^2 is positive, zero or negative if and only if Γ is hyperbolic, parabolic or elliptic, respectively.

Before proving this lemma we have to say that we think this is a well known fact, but we could not find it anywhere in the literature. As usual, along the proof, $q_i = \cos(\pi/\nu_i)$.

Proof. The elliptic triangle groups have signatures $(0,3;2,2,\nu)$, finite ν, or $(0,3;2,3,\mu)$ with $\mu = 3,4,5$. The values of l^2 is for each of these signatures are

$$(cos(\pi/\nu))^2 - 1, \text{ and } \frac{1}{4} + cos^2(\pi/\mu) - 1,$$

respectively which are all negative numbers.

The parabolic signatures are $(0,3;2,3,6)$, $(0,3;2,4,4)$ $(0,3;3,3,3)$, and $(0,3;\infty,2,2)$. The value of l^2 is zero for these cases, as it easily checked.

In the hyperbolic case we have that the partial derivative of l^2 with respect to q_1 is equal to

$$\frac{\partial l^2}{\partial q_1} = 2q_1 + 2q_2q_3.$$

This expression is zero if and only if two of the cosines are zero; but this would imply that the signature has two ramification values equal to 2, and it would not be hyperbolic. So we can conclude that for hyperbolic signatures, the value of l^2 is increasing with q_1, or equivalently with ν_1. Since the expression is symmetric with respect to the q_i, we have that the same increasing behaviuor of l^2 with respect to the other cosines. Therefore it will be enough to compute the values of l^2 for 'small' signatures. Because of the mentioned symmetry, we can also assume, without lost of generality, that $\nu_1 \leq \nu_2 \leq \nu_3$. There are several cases to be studied:

Case 1: $\nu_1 = 2$. The smallest values of the ramification numbers occur in the case of signature $(0,3;2,3,7)$. For this case we have that l^2 is positive ($l^2 = \cos^2(\pi/7) - 3/4 \simeq 0.0617449$); and this completes the argument in this case.

Case 2: $\nu_1 = 3$. We have that the smallest signature is $(0,3;3,3,4)$, for which the value of l^2 is the positive number $\sqrt{2}/4$. This proves this case.

Case 3: $\nu_1 \geq 4$. For the case of $(0,3;4,4,4)$ we obtain $l^2 = \frac{1\pm\sqrt{2}}{2}$, which is positive. This solves this case and completes the proof of the lemma. □

2.3.9. The following proposition is a technical result that we need in order to prove the uniqueness of the generators of the group $\Gamma(\nu_1,\nu_2,\nu_3;a,b;c)$ in §2.3.7.

Proposition 2 *Let A and B be canonical generators for the hyperbolic group $\Gamma(\nu_1,\nu_2,\nu_3;\infty,0,1)$. Assume that A and B have liftings to $SL(2,\mathbb{C})$ with negative traces. Then their product AB has negative trace.*

Proof. First of all, a matter of natation: along the proof we will use q_i and p_i to denote the $\sin(\pi/\nu_i)$ and $\cos(\pi/\nu_i)$,respectively.

We start with the observation that the ramification values should all be bigger than 2, since if not the proposition does not make sense as the trace of an involution is zero.

Suppose first that our group has signature $(0,3;\infty,\nu_1,\nu_2)$. Let A and B be canonical generators for the parameters $(\infty,0,1)$, with matrices representatives with negative trace, and so that their product has a matrix representative with positive trace. The discreteness of the group forces the signature to be $(0,3;\infty,\infty,3)$, but we will check that there is an element in the group with order two, reaching a contradiction.

So let us assume that the canonical generators are A and B. Let $q_i = \cos(\pi/\nu_i)$, $i = 1,2$.. The element A is parabolic and fixes ∞. We also have that the matrix representative for B has negative trace equal to $-2q_1$. This gives the following matrices:

$$A = \begin{bmatrix} -1 & -\alpha \\ 0 & -1. \end{bmatrix}, \ B = \begin{bmatrix} a & b \\ c & -a-2q_1 \end{bmatrix}.$$

There are three more conditions that have to be satisfied, namely AB has positive trace equal to $2q_2$, B has its fixed points on the line $\{z/\ Re(z) = 0\}$ and AB has the fixed points on $\{z/\ Re(z) = 1\}$. This translates to:

$$\begin{cases} -2q_1 - \alpha c = 2q_2 \\ \dfrac{2a - 2q_1}{c} = 0 \\ \dfrac{-\alpha c - 2q_1 - 2a}{-2c} = 1 \end{cases}$$

Solving these equations we get the following matrices:

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -q_1 & b \\ q_1 - q_2 & -q_1 \end{bmatrix},$$

with $b = \frac{q_1^2-1}{q_1-q_2}$. The Schimizu-Leutbecher lemma (see chapter 1) implies that $|q_1 - q_2| \geq 1/2$ or $q_1 - q_2 = 0$. This last case is not possible since the group would be elementary. In the other case, the only option we have is $q_1 = 1$ and $q_2 = 1/2$ (or viceversa), since $\cos(\frac{\pi}{n}) \geq \frac{1}{2}$ for $n \geq 3$. This implies that the group would have signature $(0, 3; \infty, \infty, 3)$. But considering the element ABA we see this is NOT true, because $(ABA)^2 = I$. Therefore A and B are not the desired generators.

Suppose now that the group has parabolic elements, but the signature is $(0, 3; \nu_1, \infty, \infty)$, with ν_1 finite. The group $\Gamma(\nu_1, \infty, \infty; \infty, 0, 1)$ is generated by the elliptic element A and the parabolic B, whose product is parabolic. The fixed point of B is 0; the fixed points of A have real part equal to 0, and the fixed point of AB is 1. Take $\alpha = 0$, $\beta = \infty$ and $\gamma = 1$. Then the fixed point of B is α, the fixed points of A are in the line that joins α to γ, and the fixed point of AB is β. This means that B and $(AB)^{-1}$ are canonical generators for $\Gamma(\infty, \infty, \nu_1; \alpha, \beta, \gamma) = \Gamma(\nu_1, \infty, \infty; \infty, 0, 1)$, and we are in the case already covered.

We can think similarly in the other cases with parabolic elements. For example, if our group is $\Gamma(\nu_1, \infty, \nu_3; \infty, 0, 1)$, with ν_1 and ν_3 finite, take $\alpha = 0$, β the end point of the geodesic that joins 0 to the fixed point of AB in the upper half plane, and $\gamma = \infty$. Then $\Gamma(\nu_1, \infty, \nu_3; \infty, 0, 1) = \Gamma(\infty, \nu_3, \nu_1; \alpha, \beta, \gamma)$. For the group $\Gamma(\nu_1, \nu_2, \infty; \infty, 0, 1)$, with ν_i finite, we take $\alpha = 1$, β the end point of the geodesic joining 1 to the fixed point of A (in the upper half plane), and γ will be the end point of the geodesic that goes from 1 to the fixed point of AB in the upper half plane. Then we get $\Gamma(\nu_1, \nu_2, \infty; \infty, 0, 1) = \Gamma(\infty, \nu_1, \nu_2; \alpha, \beta, \gamma)$. We see then that the all the cases of signatures with parabolic elements are covered by the one already studied.

Consider now the case of signature $(0, 3; \nu_1, \nu_2, \nu_3)$, where all the ramification values are finite, and assume that A and B are generators with negative traces but that the product has positive trace. By a geometrical argument we will see that ABA and B share the fixed points, and this will give that $\nu_3 = 0$, a contradiction with our first hypothesis.

Computations similar to the previous case give liftings for A and B as follows:

$$A = \begin{bmatrix} -q_1 & -mp_1 \\ m^{-1}p_1 & -q_1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_2 & b \\ c & -q_2 \end{bmatrix},$$

where

$$m = \frac{q_2 - q_1q_3 + q_1r}{p_1r}, c = \frac{q_1q_2 - q_3 + r}{p_1m}, b = \frac{q_2^2 - 1}{c},$$

$$r = \sqrt{q_1^2 + q_2^2 + q_3^2 - 2q_1q_2q_3 - 1}.$$

Observe the change of signs with respect to the formulae obtained before, since we are requiring that AB have positive trace.

We have two cases to study:

<u>CASE 1</u>: $m > 0$, which is equivalent to say that $q_2 - q_1q_3 + q_1r > 0$.

In this situation we have that the point $A(\infty) = \frac{-q_1}{p_1m}$ is negative.

We also get that $q_2m^{-1}p_1 + q_1c > 0$. To see this, multiply the left hand term by the positive number mp_1 to get

$$q_2m^{-1}p_1mp_1 + q_1cmp_1 = p_1^2q_2 + q_1(q_1q_2 - q_3 + r) = p_1^2q_2 + q_1^2q_2 - q_1q_3 + q_1r =$$

$$= q_2 - q_1q_3 + q_1r,$$

and this last number is positive by hypothesis.

This will give that $AB(\infty) < 1$. In fact, an easy computation shows that

$$AB(\infty) = \frac{q_1q_2 - mp_1c}{-q_2m^{-1}p_1 - q_1c} = \frac{r - q_3}{q_2m^{-1}p_1 + q_1c},$$

and then this number will be less that 1 if and only if

$$r - q_3 < q_2m^{-1}p_1 + q_1c.$$

Multiply both term by the positive quantity p_1mr to get

$$r < q_3 + q_2m^{-1}p_1 + q_1c \Leftrightarrow$$

$$rp_1mr < q_3p_1mr + q_2m^{-1}p_1^2mr + q_1cp_1mr \Leftrightarrow$$

$$r(q_2 - q_1q_3 + q_1r) < q_3(q_2 - q_1q_3 + q_1r) + q_2p_1^2r + q_1r(q_1q_2 - q_3 + r) \Leftrightarrow$$

$$q_2r - q_1q_3r + q_1r^2 < q_2q_3 - q_1q_3^2 + q_1q_3r + q_2r(p_1^2 + q_1^2) - q_1q_3r + q_1r^2 \Leftrightarrow$$

$$0 < q_2q_3 - q_1q_3^2 + q_1q_3r = q_3(q_2 - q_1q_3 + r),$$

which is the product of two positive numbers. Observe that we have used the fact that $1 = \sin^2(\pi/\nu_1) + \cos^2(\pi/\nu_1) = p_1^2 + q_1^2$.

Consider the triangle of the figure 2.6 with vertices v_1, v_2 and v_3, which are fixed by the transformations A, B and AB respectively. Reflect that triangle on the geodesic that joins v_1 to v_3 to get a fundamental domain for the group

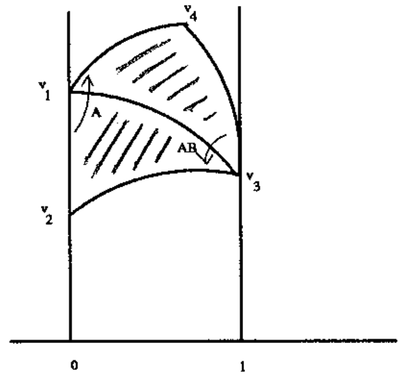


Figure 2.6: A possible fundamental domain for a group with no parabolics.

generated by A and B. The actions of A and AB are also marked in that figure.

The angle at the vertex v_1 is $2\pi/\nu_1$, and the distance from v_1 to v_2 is equal to the distance from v_1 to v_4. Since Möbius transformations are isometries, we have that A maps v_2 to v_4. Similarly, the angle at v_3 is $2 = pi/\nu_3$, and the distance from v_4 to v_3 is the same that the distance from v_3 to v_2. Therefore, AB maps v_4 back to v_2, and we have that ABA fixes v_2, which is also fixed by B. This implies that ABA and B have the same fixed points (both of them) and therefore $ABA = B^n$, for some integer n. If $n = 0$ then $B = A^{-2}$ which is not possible as A and B generate a triangle group, not a cyclic one. Since $\nu_3 \geq 3$, if $n \neq 0$ we have

$$I = (AB)^{\nu_3} \Rightarrow AB = (B^{-1}A^{-1})^{\nu_3 - 1} \Rightarrow$$

$$\Rightarrow B^n = ABA = (B^{-1}A^{-1})^{\nu_3-1}A = B^{-1}(A^{-1}B^{-1})^{\nu_3-2}A^{-1}A =$$

$$= (B^{-1}A^{-1})^{\nu_3-2}B^{-1} \Rightarrow$$

$$\Rightarrow (AB)^{2-\nu_3} = B^{n+1}.$$

But this would imply that $\nu_3 = 2$ contrary to the first assumption.

CASE 2: $m < 0$. In this case, with a computation like the one in the first case , we will get that $A(\infty) > 0$ and $AB(\infty) > 1$. This means that $A^{-1}B^{-1}A^{-1}$ and B share a fixed point. But taking inverses we will have that ABA and B share a fixed point, which is the situation proven in the first case.

This completes the proof of proposition 2.3.9. □

2.3.10. In this paragraph we will provide a proof of the uniqueness of generators for given parameters.

Proposition 3 *The canonical generators A and B for the triangle groups* $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$ *are uniquely determined by the parameters* (a, b, c).

Proof. Our definition of canonical generators is invariant under conjugation by elements of $PSL(2, \mathbb{C})$. So if we can prove the uniqueness of generators for a particular triple of parameters, we have then proven uniqueness for any set of parameters. Therefore we will conjugate by a Möbius transformation to get as parameters the points 0, 1 and ∞. Without lost of generality we can assume as well that $\nu_1 \geq \nu_2, \nu_3$, since this can be achieved by another conjugation by an element of $PSL(2, \mathbf{C})$. The proof is divided into two different cases, depending on whether or not the group contains parabolic transformations. Our line of thought will be as follows: we choose matrices representing the elements A and B with negative (or zero) trace (or zero, if they are involutions), and then we use the definition of canonical generators (§2.3.6) to compute those matrices explicitly. One result that we will use in the proof is proposition 2, in §2.3.9, that tells us that the matrix representing AB has negative (or zero) trace.

<u>CASE 1</u>: the signature is $(0,3;\infty,\nu_1,\nu_2)$. As in the proof of proposition 2, we see that this case covers all the signatures with parabolic elements. Let us look first at the element A; this is a parabolic transformation fixing ∞. Choose a matrix representing it,

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

A fixes ∞ if and only if $\gamma = 0$. The negative trace condition is expressed by $\alpha + \delta = -2$. The determinant of the matrix is 1, which gives the equation $\alpha\delta = 1$. These two equations together give $\alpha = \delta = -1$.

Consider now the element B, with matrix representative

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

Then we have the following

$$\begin{cases} \text{trace}(B) = -2q_1 \leq 0 & \Leftrightarrow \quad e + h = -2q_1 \\ \text{the fixed points of B have real part } 0 & \Leftrightarrow \quad e - h = 0 \end{cases}$$

These conditions give $e = h = -q_1$. Thus

$$A = \begin{bmatrix} -1 & \beta \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -q_1 & f \\ g & -q_1 \end{bmatrix}.$$

We also have three more conditions; namely, (i) AB has non-positive trace equal to $-2q_3$, (ii) the fixed points of this transformation have real part equal to 1 and (iii) the determinant of B is also equal to 1. Computations give that these conditions are equivalent to:

$$\begin{cases} q_1 - g\beta + q_1 = -2q_2 \\ \dfrac{q_1 + g\beta - q_1}{-2g} = 1 \\ q_1^2 - fg = 1 \end{cases}$$

By solving these equations we get the matrices of proposition §2.3.7.

<u>CASE 2</u>: the signature is $(0, 3; \nu_1, \nu_2, \nu_3)$, where all the ramification values are finite. We have to compute as in the first case 1, but this time A has two fixed points with real part equal to 0. So let us start again with a matrix representing A, say

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

The trace has to be non-positive, which implies $\alpha + \delta = -2q_1$; the fixed points have real part equal to 0, which means $\alpha - \delta = 0$. These two equations give $\alpha = \delta = -q_1$.

Now, for the transformation B we have a similar situation, that is, its fixed points have real part equal to 0 and the trace of the matrix is negative or zero. So if we choose a matrix representing B, say

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

then we have $e = h = -q_2$.

Other necessary conditions that have to be satisfied are that AB has non-positive trace equal to $-2q_3$, the fixed points of AB have real part equal to 1, and the matrices of A and B have determinant equal to 1. These four statements are given by the following four equations:

$$\begin{cases} q_1q_2 + g\beta + f\gamma + q_1q_2 = -2q_3 \\ \dfrac{q_1 q_2 + g\beta - f\gamma - q_1 q_2}{2(-q_2\gamma - q_1 g)} = 1 \\ q_1^2 - \beta\gamma = 1 \\ q_2^2 - fg = 1 \end{cases}$$

To solve these equations, we also use that the absolute value of the imaginary part of the fixed points of A is greater than that of the fixed points of B. (This last fact is part of the definition of canonical generators in the case of parameters $(\infty, 0, 1)$, see definition 4 in §2.3.6). Then the solutions are the elements of the proposition in §2.3.7. □

2.3.11. Our next result is about the possible pairs of canonical generators a group can have. Assuming we fix the signature and one generator, say A; what is the form of any other element of the group, B, so that (A, B) is a canonical pair of generators? We will need this information later, when we will study the uniqueness of certain geodesics on orbifolds (see subsection 2.5.3).

Proposition 4 *If (A, B) and (A, D) are two pairs of canonical generators for a hyperbolic triangle group with signature $(0, 3; \infty, \nu_1, \nu_2)$ then there exists an integer number, n, such that $D = A^{n/2}BA^{-n/2}$.*

Proof. By conjugation we may assume that

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1. \end{bmatrix}, B = \begin{bmatrix} -q_1 & b \\ q_1 + q_2 & -q_1 \end{bmatrix}.$$

By proposition 2 in 2.3.9 we have that the transformation D has a matrix representative with trace $-2q_1$; we also have that their product AD has negative trace. Computations shown then that under these conditions D is given by

$$\begin{bmatrix} \alpha & \beta \\ q_1 + q_2 & -2q_1 - \alpha \end{bmatrix}.$$

The fixed point of AD are

$$\frac{2\alpha + 2q_1 \pm 2ip_1}{2q_1 + 2q_2}.$$

The real part equals

$$\frac{2\alpha + 2q_1}{2q_1 + 2q_2} = \frac{\alpha + q_1}{q_1 + q_2}.$$

Let h be equal to $\frac{\alpha+q_1}{q_1+q_2}$. The transformation

$$T = \begin{bmatrix} -1 & -h \\ 0 & -1 \end{bmatrix},$$

conjugates B into D; that is, $TBT^{-1} = D$. So T belongs to the normalizer of Γ in $PSL(2, \mathbb{C})$, and induces an automorphism of the quotient surface that fixes one puncture (since $TAT^{-1} = A$). This means that either $T \in \Gamma$ or $T^2 \in \Gamma$, giving us $T = A^n$ or $T = A^{n/2}$ as desired. □

2.4 Parabolic groups

2.4.1. In this section we will give definitions similar to those in the previous one, but for triangle groups with one limit point (also called Euclidean groups, since their region of discontinuity is conformally equivalent to the plane). We will have to deal with two different cases, depending on whether or not the quotient surface has punctures. The same phenomenon already occurred in the case of hyperbolic groups, where we have to put an 'extra' condition (choice of triangles) in the case the quotient surface was compact.

As we did in the previous section, we will conjugate by a Möbius transformations in order to reduce all the computations to the case of parameters $(\infty, 0, 1)$.

2.4.2. The only parabolic signature of type (0,3) whose quotient orbifold has some punctures is $(0, 3; \infty, 2, 2)$, up to order of the ramification values. It should be remarked that the fixed point of the parabolic element corresponding to the puncture is also fixed by any other element of the group (see [Mas88] for a proof of this simple fact about parabolic groups). The definition of canonical generators for these cases is as follows:

Definition 5 *Let Γ be a triangle group with signature $(0, 3; \nu_1, \nu_2, \nu_3)$, where one of the ramification values is equal to ∞ and the other two are equal to 2. Let A and B be two generators of the group. We will say that they are CANONICAL for the parameters (a, b, c) if the following conditions are satisfied:*

1) $|A| = \nu_1$, $|B| = \nu_2$ and $|AB| = \nu_3$,

2) $A(a) = a$, $B(b) = b$ and $AB(c) = c$,

3) A and B are primitive.

2.4.3. With this definition we can prove the existence of canonical generators:

Proposition 5 *Given three different points on $\hat{\mathbb{C}}$, (a, b, c), there exist a unique pair of canonical generators for a group with signature $(0, 3; \nu_1, \nu_2, \nu_3)$, as described in the previous definition, and for the parameters (a, b, c). This group will be denoted by $\Gamma(0, 3; \nu_1, \nu_2, \nu_3; a, b, c)$.*

Proof. As we have already said, we can conjugate by a Möbius transformation so that we have to prove the uniqueness of canonical generators of the group $\Gamma(\infty, 2, 2; \infty, 0, 1)$. In this case, the transformation A is parabolic and fixes ∞, so $A(z) = z + \alpha$, where α is a complex number to be determined. The mapping B is an involution that fixes ∞ and 0. This implies $B(z) = -z$. Now, the product of these two mappings is $AB(z) = -z + \alpha$. We have that the fixed points of AB are ∞ and 1. Therefore $\alpha = 2$, and this completes the case of signature $(0, 3; \infty, 2, 2)$.

We have to more cases to deal with, namely, $\Gamma(2, \infty, 2; \infty, 0, 1)$ and $\Gamma(2, 2, \infty; \infty, 0, 1)$. But these cases are obtained from the previous one by conjugating by a Möbius transformation. For example, in the case $(0, 3; 2, \infty, 2)$, the parabolic element B fixes 0, and therefore so does any other element in the group. The involution A corresponding to the first 2 of the signature will fix ∞, and the product $AB = (AB)^{-1}$ fixes 1. We consider the Möbius transformation

$$T(z) = \frac{-1}{z-1},$$

which maps $\{\infty, 0, 1\}$ to $\{0, \infty, 1\}$ pointwise. Conjugating by it we will obtain the generators of this new group, which are

$$A(z) = -z, \quad B(z) = \frac{z}{-2z+1}, \quad AB(z) = \frac{-z}{-2z+1}.$$

In a similar way, if we want to obtain the parabolic group with signature $(0, 3; 2, 2, \infty)$, what we need to do is to consider the transformation

$$S(z) = \frac{z-1}{z},$$

and conjugate by it to obtain the generators, which are

$$A(z) = \frac{z-2}{2z-3}, \quad B(z) = -z+2, \quad (AB)^{-1}(z) = \frac{-z}{-2z+1}.$$

□

In this particular case we have an explicit formula for the covering map from the complex plane to the quotient orbifold $\mathcal{S}$. To compute that covering map. let us consider the group $\Gamma(\infty, 2, 2; \infty, 0, 1)$, generated by the Möbius transformations $A(z) = z + 2$ and $B(z) = -z$. If $\phi : \mathbb{C} \to \mathcal{S}$ is the natural projection mapping, than we must have that $\phi(z+2) = \phi(z)$. This means that the function ϕ has a Fourier expansion of the form $\phi(z) = \sum_{-\infty}^{+\infty} a_n e^{\pi i z}$. We also have that $\phi(-z) = \phi(z)$, which is translated to $a_n = -a_n$. Choosing the simplest case; that is, $a_0 = 0 = a_n$, $n > 1$ and $a_1 = a_-1 = 1/2$ we obtain the function $\phi(z0 = \cos(\pi i z)$. It is a classical result that any other covering map $\mathbb{C} \to \mathcal{S}$ is a rational function of our ϕ (see [For51], p. 144).

2.4.4. The parabolic groups with compact quotient orbifolds have signatures $(0,3;2,3,6)$, $(0,3;3,3,3)$ or $(0,3;2,4,4)$, up to order of the ramification values. We have a slightly different definition of canonical generators for these cases:

Definition 6 *Let Γ be parabolic triangle group with signature $(0,3;\nu_1,\nu_2,\nu_3)$, all ν_i's finite, and let A and B two elements generating this group. Let (a,b,c), be three different points of $\hat{\mathbb{C}}$. We will say that A and B are CANONICAL GENERATORS of Γ for the given parameters if the following conditions are satisfied:*

1) $|A| = \nu_1$, $|B| = \nu_2$, $|AB| = \nu_3$, *2) $A(a) = a$, $A(b) = b$,*

3) $B(a) = a$, $B(c) = c$,

4) A and B are geometric primitive.

2.4.5. The result about existence and uniqueness is given in the following proposition:

Proposition 6 *Let $(0,3;\nu_1,\nu_2,\nu_3)$ be a parabolic signature where all the ν_i are finite. Let (a,b,c) be three distinct points of the Riemann sphere. Then there exists two pairs of canonical generators for the group $\Gamma(\nu_1,\nu_2,\nu_3;a,b,c)$. More precisely, if (A,B) is one such pair, then the other is given by (A^{-1},B^{-1}).*

Proof. Conjugating by a Möbius transformation we reduce the proof to the case of the groups $\Gamma(\nu_1,\nu_2,\nu_3;\infty,0,1)$. In this case, the transformation A fixes ∞ and 0, so it is of the form

$$A(z) = \lambda z,$$

where $\lambda = exp(\pm\frac{2\pi i}{\nu_1})$, since A has to be primitive. For the transformation B we have the formula

$$B(z) = \mu(z-1)+1,$$

with $\mu = exp(\pm\frac{2\pi i}{\nu_2})$. Then their product is

$$AB(z) = \lambda\mu(z-1)+\lambda.$$

We have to find λ and μ so that AB has order ν_3. To do it, first we see that the fixed points of AB are ∞ and $z_0 = \frac{\lambda\mu-\lambda}{\lambda\mu-1}$; so we take the translation $T(z) = z - z_0$ and conjugate by it to obtain $T(AB)T^{-1}(z) = (\lambda\mu)z$. The order of this transformation has to be ν_3, since conjugation preserves the order. We have several cases, depending on our choices for λ and μ.

CASE 1: $\lambda = exp(+\frac{2\pi i}{\nu_1})$ and $\mu = exp(+\frac{2\pi i}{\nu_2})$. Then

$$\lambda\mu = exp(\frac{2\pi i}{\nu_1} + \frac{2\pi i}{\nu_2}) = exp(2\pi i(1 - \frac{1}{\nu_3})) = exp(\frac{-2\pi i}{\nu_3}),$$

which implies that AB has order ν_3 as we wanted. Here we have used the fact that for parabolic signatures $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} = 1$.

CASE 2: $\lambda = exp(-\frac{2\pi i}{\nu_1})$ and $\mu = exp(-\frac{2\pi i}{\nu_2})$. It is solved as the previous case, but this time we get that the generators are the inverses of those of case 1, proving the non-uniqueness stated in the proposition.

CASE 3: $\lambda = exp(+\frac{2\pi i}{\nu_1})$ and $\mu = exp(-\frac{2\pi i}{\nu_2})$. Multiply these two numbers to get

$$\lambda\mu = exp(2\pi i(\frac{1}{\nu_1} - \frac{1}{\nu_2})).$$

We have to remark at this moment that if $\nu_1 = 2$ or ν_2=2 there is nothing to prove, as $exp(\frac{2\pi i}{2}) = exp(-\frac{2\pi i}{2})$. So we are left with the following signatures $(0, 3; 3, 6, 2)$, $(0, 3; 6, 3, 2)$, $(0, 3; 3, 3, 3)$ and $(0, 3; 4, 4, 2)$. Computing $\frac{1}{\nu_1} - \frac{1}{\nu_3}$ in all four signatures we get $\frac{1}{6}$, $-\frac{1}{6}$, 0 and 0, respectively, but none of these gives the correct order of AB.

CASE 4: $\lambda = exp(-\frac{2\pi i}{\nu_1})$ and $\mu = exp(+\frac{2\pi i}{\nu_2})$. This case is like the previous one.

This completes the proof of the proposition. □

2.5 The geometry of the quotient orbifolds

2.5.1. This section is dedicated to study the geometry of (some of) the quotient orbifolds corresponding to the groups of the previous sections. Before getting involved in the particular cases, some general ideas about the subject are needed. We will work with orbifolds of type (0,3), where the universal branched covering space is (conformally equivalent to) either the upper half plane or the complex plane. In these two cases, the spaces carry metrics that are preserved by the corresponding groups (they become groups of isometries), and therefore we can push down the metrics to the quotient spaces. In the case of the upper half plane we are talking about Poincaré's metric $\frac{|dz|}{y}$, while the case of the complex plane is the usual Euclidean metric, $|dz|$.

With this in mind, we project the metric of the covering space to the orbifold, obtaining a Riemannian structure, and therefore a metric structure, with a well defined concept of distance.

2.5.2. Let Γ be a triangle group acting on the upper half plane (or some bounded disc which is conformally equivalent to the upper half plane) or on the complex plane. Let denote either one of these sets by $\mathcal{U}$. Put on each the corresponding metric of constant curvature -1 or 0 that makes Γ a group of isometries. Let the quotient space $\mathcal{U}/\Gamma$ be denoted by $\mathcal{S}$. A geodesic on the quotient orbifold is just a curve that lifts to a geodesic on $\mathcal{U}$. Formally:

Definition 7 *A geodesic on $\mathcal{S}$ is the projection by $\pi : \mathcal{U} \to \mathcal{S}$ of a geodesic on $\mathcal{U}$.*

2.5.3. The goal of this section is to find geodesics on S and use them to compute coordinates. We will do it by mapping those geodesics to certain subintervals of the real line, with a metric so that the mapping is an isometry.

The geodesics we are looking for are especial, in the sense that they 'start' at the punctures, or at some ramification value, and therefore they can be used to produce coordinates on the orbifold related to these special points. We do this because, later on this thesis, we will need to cut and paste neighborhoods of those points, and a good set of local coordinates will make our computations much easier.

Our first result is for hyperbolic groups whose quotient orbifold has punctures.

Proposition 7 *Let $\mathcal{S}$ be a hyperbolic orbifold with signature $(0,3;\infty,\nu_1,\nu_2)$ and with covering $\mathcal{U}$ conformally equivalent to the upper half plane (thus $\frac{1}{\nu_1}+\frac{1}{\nu_2}<1$). Let $P\notin\mathcal{S}$ be the puncture corresponding to the first ∞ in the signature, and let P^1 the puncture or branched point corresponding to ν_1. Then there exists a unique simple geodesic $c: I\longrightarrow\mathcal{S}$ joining P and P^1 such that if c is parametrized by arc length s, we have the following:*

$$\text{1) if } \nu_1=\infty, \text{ then } I=\mathbf{R} \text{ and } \begin{cases} lim_{s\to+\infty}c(s)=P \\ lim_{s\to-\infty}c(s)=P^1 \end{cases}$$

$$\text{2) if } \nu_1<\infty, \text{ then } I=[0,+\infty) \text{ and } \begin{cases} lim_{s\to+\infty}c(s)=P \\ c(0)=P^1 \end{cases}$$

and the image of c does not contain the point corresponding to ν_2.

Proof. The existence part is easy. These orbifolds have no moduli, so we can assume that the covering space is $\mathcal{U}=\mathsf{H}$, and the covering group has parameters $(\infty, 0, 1)$. As a fundamental domaion for our group we can choose (depending on the signature of the group) one of those in figure 2.2 below.

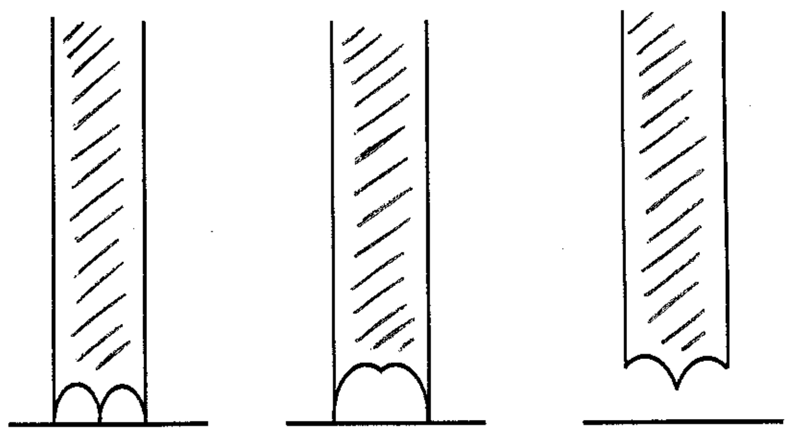


Figure 2.7: Fundamental domains for hyperbolic triangle groups

The projection of the part of the imaginary axis that lies in the boundary of that fundamental domain gives a geodesic on $\mathcal{S}$ that satisfies the conditions of the statement of the proposition.

For the uniqueness part, let us assume that there is another geodesic, satisfying the properties stated in the proposition. We lift it to $\mathbf{H}$ and we can assume that the lifting is a vertical line (or half-line, depending upon the signature). We want to prove that this second vertical line is simply a translate of the imaginary axis under a power of A, and therefore the projection of the two lines will be the same geodesic in the orbifold.

The vertical line that ends at x_0 must have a point corresponding to P^1 (x_0 itself if the point is a puncture, a point inside the upper half plane otherwise), which will be the fixed point of some element B_1. Now, if we remove on the orbifold the point corresponding to ν_2 (if $\nu_2 = \infty$, then we do not have to remove anything, since punctures are not in the orbifold), we are in a situation like the torsion free case, and we get that A and B_1 generate the group Γ (see [Kra90]). Therefore A and B will be canonical generators for some parameters. By the last proposition of the subsection 2.3.11, we have that there is an integer $n \in \mathbb{Z}$ such that $A^{n/2}BA^{-n/2} = B_1$. Our proof will be complete is we show that n is integer.

Suppose $A^{1/2}BA^{-1/2}$ is conjugate to B in the group Γ. Then the element $A^{1/2}$ belongs to the normalizer of Γ in $PSL(2, \mathbb{R})$ and induces an automorphism of the quotient orbifold that fixes at least one puncture (the one represented by A). Since $A^{1/2}$ does not belong to Γ, the induced automorphism is not the identity, and therefore that automorphism has to interchange the other two ramification points. This implies that B and AB are elliptic transformations of the same order. Then it is easy to compute that $A^{1/2}BA^{-1/2} = (AB)^{-1}$. This would imply that B and $(AB)^{-1}$ are conjugate in the group Γ, which is not true since they correspond to different branch points. Therefore x_0 is an even integer and the geodesics are the same. □

2.5.4. Let us consider the case of a parabolic group whose quotient orbifold, $\mathcal{S}$, has a puncture. If $P \notin \mathcal{S}$ is the puncture and P^1 is one of the branch points, what we want is to find a geodesic c joining P^1 to P. Conjugating by a Möbius transformation we can assume that the group is $\Gamma(\infty, 2, 2; \infty, 0, 1)$, whose region of discontinuity is $\mathbb{C}$. Then a geodesic on S will lift to a straight line from 0 to ∞. The problem is that there are infinitely many such lines. To understand what extra conditions we have to impose on c to guarantee uniqueness, let us look at two such lines. Take $c_1(s) = is$ and

$c_2(s) = \frac{1}{\sqrt{2}}(1+i)s$, with $s \in [0, +\infty)$. Both lines have been parametrized by the arc length. Now, the distance from 0 to $2i$ along c_1 is 2; the distance from 0 to $2+2i$ along c_2 is $2\sqrt{2}$. But since the mapping $A(z) = z + 2$ belongs to the group $\Gamma(\infty, 2, 2; \infty, 0, 1)$, the points 2 and $2+2i$ are equivalent. This means that on the quotient orbifold $\mathcal{S}$, the projection of c_1 is 'shorter' than the projection of c_2. This example suggest that the condition to guarantee uniqueness is that the geodesic has to minimize distances between any two points on it.

Proposition 8 *Let $\mathcal{S}$ be an orbifold of signature $(0, 3; \infty, 2, 2)$. Let $P \notin \mathcal{S}$ be the puncture, and let P^1 be one of the branched points. Then there exists a unique geodesic $c : [0, \infty) \longrightarrow \mathcal{S}$, such that $c(0) = P^1$, $lim_{s\to\infty} c(s) = P$. For s the arc length parametrization, c realizes the distance between any two points on it.*

Proof. We should first note that we can take the group that has parameters $(\infty, 0, 1)$, and assume that P^1 lifts to 0. By our definition of geodesics, any straight line joining zero and infinity will project onto a geodesic of the orbifold. Suppose then that c lifts to the line given by the equation $y = mx$, with m real. The slope cannot be zero, because the real axis projects onto a closed curve that connects the two ramification points, but it does not go through the puncture. The point $2 + 2mi$ belongs to the line. Along it, the distance from the origin to that point is $2\sqrt{1+m^2}$. But this point is equivalent to $2mi$, which is at distance $2|m|$ from the zero. Since $2|m| < 2\sqrt{1+m^2}$, this geodesic is not distance minimizing. The only case left is the geodesic whose lift is the imaginary axis. It is trivial now to see that the imaginary axis projects to the orbifold onto the geodesic that satisfies the required properties.

Suppose that we have another geodesic satisfying the required properties, and consider its lift to $\mathbb{C}$ as in the proof of the previous proposition. We are in same situation of 2.5.3, and the proofs applies verbatim to this case. $\square$

2.5.5. We want to use the special coordinate around the puncture to produce a coordinate patch on the orbifold. To fix ideas, let us consider the case of a hyperbolic orbifold $\mathcal{S}$. Assume that $\mathcal{S}$ is uniformized by the group $\Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1)$. The choice of parameters is not an issue, since the orbifolds of type $(0,3)$ have no moduli, and our statement is related to the geometry of $\mathcal{S}$, not to the particular group. As before, let $P \notin \mathcal{S}$ be the puncture corresponding to ∞ in the signature, and let P^1 be the ramification point or puncture corresponding to ν_1. We have a special geodesic c joining P to P^1. What we will do is to find a coordinate around P such that the geodesic c is mapped into the positive real axis, and a neighborhood of P (punctured disc) is mapped into a neighborhood of the origin in the punctured unit disc. Let

$$\rho : \mathbf{H} \longrightarrow \mathcal{S}$$

be the universal covering orbifold. Then the function defined by

$$f(z) = e^{\pi i \rho^{-1}(z)}, z \in \mathcal{S},$$

maps the geodesic onto a segment of the real axis. More precisely, the image of the geodesic is contained on the unit interval. This function can be extended to a punctured disc around P, and it will map that punctured disc into the unit disc. If we consider in this last set the Poincaré metric of curvature -1, then f is an isometry. The germ of the holomorphic function defined by f is uniquely determined by the fact that a portion of the geodesic c is mapped isometrically into the real axis in the punctured disc. We will call this germ of functions a **horocyclic coordinate**, centered at P and relative to P^1. The proof of this uniqueness statement for the torsion free case can be found in [Kra90], and it works in our case as well.

2.5.6. The case of ramification points is similar, and the formula is easy to get. To understand it, all we have to do is understand what happens with the punctures: they are realized as the fixed points of parabolic elements, which are conjugate to translations; therefore to get something invariant under a parabolic element, the most natural function is the exponential, since it is invariant under translations. Similarly, the elliptic elements are conjugate to euclidean rotations; the most natural functions invariant under rotations are powers. So to get coordinates around the elliptic fixed points we should first conjugate the element to a rotation that fixes the origin and ∞, and then raise to the correct power. To fix ideas, let us consider the case of a hyperbolic group with signature $(0,3;\infty,\nu_1,\nu_2)$, where ν_1 is finite. Consider the geodesic c that joins the puncture P (corresponding to the ∞ in the signature) with the point P^1, of ramification value ν_1. As we did in the previous paragraph, we can assume that the uniformizing group of $\mathcal{S}$ is $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$. The element of order ν_1 is

$$B = \begin{bmatrix} -q_1 & b \\ q_1 + q_2 & -q_1 \end{bmatrix}, \; q_i = cos(\pi/\nu_i), \; b = \frac{q_1^2 - 1}{q_1 + q_2}.$$

Its fixed point inside the upper half plane is $z_0 = \frac{ip_1}{q_1+q_2}$, where $p_1 = sin(\pi/\nu_1)$. Consider the transformation

$$M = \begin{bmatrix} 1 & -z_0 \\ 1 & z_0 \end{bmatrix} = \begin{bmatrix} 1 & -z_0 \\ 1 & -\overline{z}_0 \end{bmatrix}.$$

This function takes the upper half plane onto the unit disc, mapping the points $\infty, 0, z_0$ to $1, -1, 0$ respectively. We also have that $MBM^{-1}(z) = e^{2\pi i/\nu_1}z$. The coordinate is then given by

$$f(z) = (\frac{\rho^{-1}(z) - z_0}{\rho^{-1}(z) + z_0})^{\nu_1}.$$

As before, ρ represents the covering from the upper half plane onto the orbifold. Let $\mathbb{D}$ denote the unit disc with the standard Poincaré metric $\frac{2|dz}{1-|z|^2}$. If we take the quotient of $\mathbb{D}$ by the group of transformations $< z \mapsto e^{2\pi i/\nu_1}z >$, we get an orbifold, say $\mathbb{D}_{\nu_1}$, with covering mapping

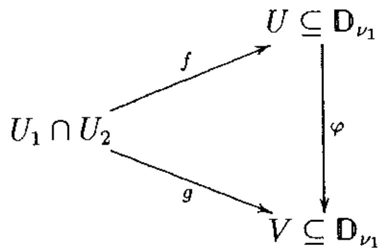
$$\begin{array}{ccc} \mathbb{D} & \rightarrow & \mathbb{D}_{\nu_1} \\ z & \mapsto & \zeta = z^n \end{array}$$

and in $\mathbb{D}_{\nu_1}$ we have the quotient metric

$$\frac{2}{n|\zeta|^{(n-1)/n}(1 - |\zeta|^{2/n})}.$$

Then the mapping f is an isometry from a neighborhood of P^1 into a neighborhood of 0 in $\mathbb{D}_{\nu_1}$. f maps a piece of the geodesic c into the real axis (contained in $\mathbb{D}_{\nu_1}$).

To prove uniqueness, we suppose there are two coordinate mappings, say $f : U_1 \to \mathbb{D}_{\nu_1}$ and $g : U_2 \to \mathbb{D}_{\nu_1}$, defined in neighborhoods of the branch point P^1, and so that they map a piece of the geodesic c into the positive real axis. Form the function $\varphi = g \circ f^{-1}$:



Then $\varphi : U \to V$, where $U = f(U_1 \cap U_2)$ and $V = g(U_1 \cap U_2)$ are neighborhoods of 0 in $\mathbb{D}_{\nu_1}$, and φ maps the positive real axis into itself. We also know that φ has a simple zero at that origin. Due to the fact that f and g are isometries in the corresponding metrics, we get that φ is an isometry on the metric of $\mathbb{D}_{\nu_1}$. This means, in the distance on $\mathbb{D}_{\nu_1}$ induced by the above metric, we have $dist(\varphi(x), \varphi(y)) = dist(x, y)$ for $0 < y < x < \epsilon$, for some small ϵ. We have the explicit formula of the metric, so we compute and obtain

$$log(\frac{1+\varphi(x)^{1/n}}{1-\varphi(x)^{1/n}}\frac{1-\varphi(y)^{1/n}}{1+\varphi(y)^{1/n}}) = log(\frac{1+x^{1/n}}{1-x^{1/n}}\frac{1-y^{1/n}}{1+y^{1/n}}).$$

This implies

$$\frac{1+\varphi(x)^{1/\nu_1}}{1-\varphi(x)^{1/\nu_1}} = (constant)\frac{1+x^{1/\nu_1}}{1-x^{1/\nu_1}},$$

for x real and small enough. Substituting x for 0 we get that the constant should be 1 and therefore $\varphi(x)^{1/n} = x^{1/n}$, which gives $\varphi(x) = x$. This is equivalent to say that $f = g$ in the overlap $U_1 \cap U_2$.

The germ of holomorphic functions defined by f is called a **horocyclic coordinate** centered at P^1 and relative to P.

We now state the results of §2.5.5 and §2.5.6 in a proposition as follows:

Proposition 9 *Let $\Gamma(\nu_1, \nu_2, \nu_3; a, b, c)$ be a hyperbolic triangle group such that either $\nu_1 = \infty$ or $\nu_2 = \infty$. Let $\mathcal{S}$ be an orbifold uniformized by this group, and suppose that $\mathcal{S}$ has the metric of constant curvature -1 that comes from its universal (branched) covering space. Let P^1 and P^2 be the points corresponding to ν_1 and ν_2 respectively. Let c be the geodesic on $\mathcal{S}$ joining these two points. Then there exists a local biholomorphism z, defined in a neighborhood N of P^1, such that $z(P^1) = 0$ and z maps isometrically the portion of c inside N into the positive real axis, with the metric of the punctured disc if $\nu_1 = \infty$, or the metric of $\mathbf{D}_{\nu_1}$ if $\nu_1 < \infty$. The germ of the biholomorphic function defined by z is unique.*

2.5.7. All these results can be extended to the case of compact orbifolds, but we will not include them here, since they do not have new ideas but the computations become quite messy.

2.5.8. The parabolic case is handled in the same way. For the puncture we have that one possible coordinate is given by the function

$$f(z) = e^{\pi i \rho^{-1}(z)}.$$

For the elliptic fixed point that lifts to 0 we have that $f(z) = (\rho^{-1}(z))^2$ is a good coordinate.

2.6 Normalizers

In this section we will give formulae for the normalizers of some of the above groups in $PSL(2,\mathbb{R})$. Γ will denote a hyperbolic triangle group, with parameters $(\infty, 0, 1)$. Since the group is non-elementary and Fuchsian, so is $N(\Gamma)$, its normalizer in $PSL(2,\mathbb{R})$. Moreover, $N(\Gamma)/\Gamma$ is isomorphic to the conformal automorphisms group of the quotient orbifold. Any such automorphism should preserve the ramification values of the branch points, and it extends to the punctures (removable singularities theorem for holomorphic functions). This means that the above quotient of groups is isomorphic to :

(1) S_3, the permutation group of three letters, if the group is torsion free, or it has signature $(0,3;\nu,\nu,\nu)$, $\nu < \infty$;

(2) $\mathbb{Z}_2$ if the signature has exactly two ramification values equal.

(3) the trivial group if the three ramification values are different.

The extra generators (that is, those transformation that together with Γ generate the group $N(\Gamma)$) are given by the following elements, for the case of groups with parameters $(\infty, 0, 1)$:

(1) see [Kra90] for the torsion-free case; otherwise we get

$$A^{1/2} = \begin{bmatrix} -\cos(2\pi/\nu) & -k\sin(2\pi/\nu) \\ k^{-1}\sin(2\pi/\nu) & -\cos(2\pi/\nu) \end{bmatrix},$$

$$B^{1/2} = \begin{bmatrix} \cos(2\pi/\nu) & -\frac{b}{p}\sin(2\pi/\nu) \\ \frac{p}{b}\sin(2\pi/\nu) & -\cos(2\pi/\nu) \end{bmatrix},$$

where $q = \cos(\pi/\nu), p = \sin(\pi/\nu)$ and k and b are given in Proposition 1 (subsection 2.3.7). The quotient group $< A^{1/2}, B^{1/2} > /\Gamma$ is isomorphic to S_3.

(2)

$$(AB)^{1/2} = \begin{bmatrix} \sqrt{2+2q} & -\frac{\sqrt{2+2q}}{1+q} \\ \frac{\sqrt{2+2q}}{2} & 0 \end{bmatrix},$$

in the case of signature $(0, 3; \infty, \infty, \nu)$;

$$(A)^{1/2} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix},$$

for the case of signature $(0, 3; \infty, \nu, \nu)$; and

$$A^{1/2} = \begin{bmatrix} -\cos(\frac{\pi}{2\mu}) & -k\sin(\frac{\pi}{2\mu}) \\ k^{-1}\sin(\frac{\pi}{2\mu}) & -\cos(\frac{\pi}{2\mu}) \end{bmatrix},$$

for the case of signature $(0, 3; \mu, \nu, \nu)$, with $\mu \neq \nu$, μ and ν finite, and k is given in proposition 1 of subparagraph 2.3.7.

(3) there is nothing to be shown.

Conjugating by a Möbius transformation one can get the expressions of the normalizers of these groups for the case of any other parameters.

Chapter 3

Coordinates for the Teichmüller spaces of b-groups with torsion

3.1 Weighted graphs and deformation spaces

3.1.1. The main goal of this chapter is to produce coordinates for the Teichmüller spaces of orbifolds of finite hyperbolic type. To do it, we will use the main philosophy of Maskit embedding: a surface of high genus can be constructed from several 'smaller' (topologically simpler) surfaces whose deformation spaces are one dimensional. A set of complex numbers will tell us how to combine the smaller surfaces to obtain the original 'bigger' one. These complex numbers will serve as parameters (coordinates) for our Teichmüller spaces. Therefore, we first of all need to study the orbifolds of type (0,4) and (1,1), which are the only ones with deformation spaces of dimension one. But before that we will explain how to cut a 'complicated' orbifold into simpler parts.

3.1.2. We start by recalling the definition of maximal partitions. Let S

be an orbifold with signature $(p, n; \nu_1, \ldots, \nu_n)$. Let $\mathcal{S}'$ be the surface (possibly with punctures) that results from removing from $\mathcal{S}$ all the ramification points.

Definition 8 *A* **MAXIMAL PARTITION**, $\mathcal{C}$, *on* $\mathcal{S}$ *is a set of 3p-3+n simple unoriented closed curves in* $\mathcal{S}'$ *such that:*

1.- no two curves in the partition are freely homotopically equivalent on $\mathcal{S}'$*;*

2.- no curve in the partition is homotopically trivial on $\mathcal{S}'$*;*

3.- no curve in the partition is contractible to one of the punctures of $\mathcal{S}'$*,*

The name of maximal comes from the fact that any set of curves satisfying the above three conditions has at most 3p-3+n elements.

If we cut $\mathcal{S}$ along the curves of a maximal partition $\mathcal{C}$ we will get a set of 2p-2+n spheres with ramification points and/or punctures and some discs removed. An example of a maximal partition on an orbifold with signature $(2, 2; 7, 4)$ is given in figure 3.1; the parts that result from cutting that orbifold along the curves of the partition are shown in figure 3.2.

We can attach to these spheres punctured discs to get orbifolds of type (0,3). To recover the original orbifold what one needs to do is the 'inverse' operation: start with a set of orbifolds of type (0,3), remove punctured discs and glue along the boundary curves. A first idea that comes to one's mind is that gluing the curves involves two real parameters, a radius r (in a good coordinate) of the removed punctured disc and a twisting angle θ (the measure of the turning of one curve with respect to the other), and $re^{i\theta}$ is a complex number.

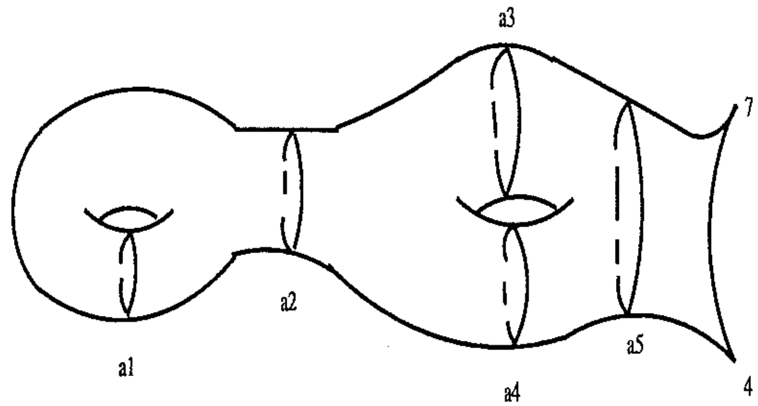


Figure 3.1: A maximal partition of an orbifold

This idea gives a set of coordinates for deformation spaces that could be called the 'complex Fenchel-Nielsen coordinates'.

3.1.3. We see that the above procedure involves a series of distinct mathematical ingredients that we need to control:

analytic aspects: the gluing is done in local coordinates by an identification of the type $zw = t$ (see below for further details);

topological/combinatorial aspects: how many parts and curves, and how they are related. This is due to the fact that there are many homotopically distinct partitions on one orbifold.

The analytic aspect will be controlled by a set of complex numbers that we will called plumbing parameters; see below for details. The topological information will be recorded in a weighted graph, that we define next. Let p and n be two non-negative integers satisfying $2p-2+n > 0$. Let $v = 2p-2+n$ and $d = 3p-3+n$. It is easy to see that then $d \geq 0$.

Definition 9 *A (admissible)* **WEIGHTED GRAPH** *(or graph with weights)*

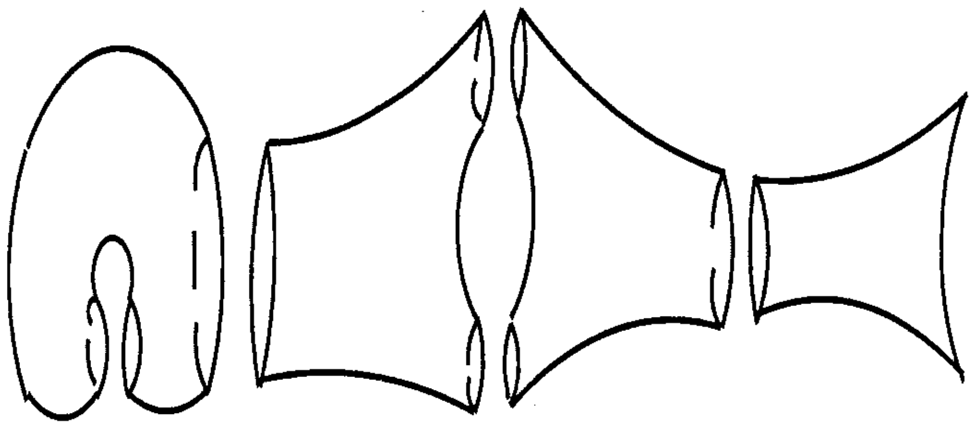


Figure 3.2: Parts of the previous orbifold

is a connected graph with v vertices and $d+n$ edges, satisfying the following conditions:

1.- edges join two different vertices, or a vertex to itself, or they just leave a vertex and do not end at another vertex; edges of this last type are called **phantom** *edges;*

2.- three edges leave every vertex (an edge from a vertex to itself counts as two edges);

3.- phantom edges have assigned numbers belonging to the set $\mathcal{W} = \{q \in \mathbb{Z}/q \geq 2\} \cup \{\infty\}$; these numbers are called **weights***;*

4.- if $\mu_1, \ldots, \mu_n$ is the collection of weights corresponding to phantom edges, then $2p - 2 + n - \sum_{j=1}^{n} \frac{1}{\mu_j} > 0$.

5.- each half of a non-phantom edge has assigned a number in the set $\mathcal{W}$; the two numbers corresponding to the two halves of a non-phantom edge are equal.

An example of an admissible graph is given in the figure 3.3.

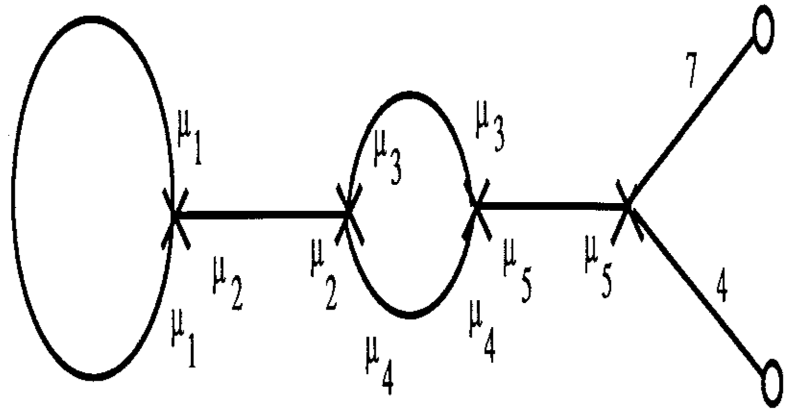


Figure 3.3: Admissible graph

3.1.4. We label the vertices on the graph with v_j, and the edges with e_j. Without loss of generality we assume that the numbering is done in a way that the graph consisting on the first k edges, $\{e_1, \ldots, e_k\}$, and the vertices they join is connected. We can then orient the edges as follows: if e_j joins v_k to v_l, with $k < l$, then e_j is positively oriented from v_k to v_l. If $k = l$, then we give a e_j any orientation of the possible two.

In order to differentiate phanton edges from the non-phantom ones, we will draw a small empty circle at the end of the phantom edges, as it is done in figure 3.3.

3.1.5. Now we will show that there is a correspondence between admissible graphs and pairs (orbifold, maximal partition), if we consider this last set only from a topological point of view. From a graph we will construct an orbifold with a maximal partition; but from an orbifold (with a maximal partition) we can construct a set of different graphs, that will produce different Teichmüller spaces related to the orbifold.

Given a pair $(\mathcal{S},\mathcal{C})$, where $\mathcal{S}$ is an orbifold of finite conformal type with hyperbolic signature σ, and $\mathcal{C}$ is a maximal partition on $\mathcal{S}$, we are going to construct a series of admissible weighted graphs associated to that pair. We start by cutting $\mathcal{S}$ along the curves of $\mathcal{C}$; that is, we consider the disjoint sets $\mathcal{S}_1,\ldots,\mathcal{S}_{2p-2+n}$, such that $\cup_{j=1}^{2p-2+n}\mathcal{S}_j = \mathcal{S} - \{a_k/a_k \in \mathcal{C}\}$. Each $\mathcal{S}_j$ is, topologically, an orbifold of type (0,3). Consider now a curve of the partition, say a_l, which is the common boundary of two holes, say h_l^1 and h_l^2, in the spheres $\mathcal{S}_{j_1}$ and $\mathcal{S}_{j_2}$ respectively. Attached to each of these spheres a disc whose center has ramification value $\mu_l \in \mathcal{W}$ (as usual, if the center has ramification value equal to ∞, the disc is a punctured disc and the 'missing' point is NOT in the orbifold). After this process, we get that each $\mathcal{S}_j$ has been 'completed' to became an orbifold with signature $(0,3;\nu_j^1,\nu_j^2,\nu_j^3)$. Label the special points of $\mathcal{S}_j$ by P_j^1, P_j^2, P_j^3. To each of these spheres, $\mathcal{S}^j$, we assign a vertex, v_j, with three edges, e_j^1, e_j^2, e_j^3, that have weights ν_j^1,ν_j^2,ν_j^3 respectively. The edges are in one-to-one correspondence with the ramification points/punctures of the sphere. See figure 3.4. The edges that correspond to ramification points of the original orbifold are phantom edges, and they end in a empty circle. The edges that correspond to the 'extra' ramification points (added after cutting the original orbifold into parts) are non-phantom edges; or loosely speaking, they are 'half edges'.

Take a non-phantom edge, say e_j^p that starts at the vertex v_j. This edge comes from attaching a (possibly punctured) disc to the sphere $\mathcal{S}_j$. Such disc has been added to the boundary of $\mathcal{S}_j$ determined by certain curve of the partition, say a_l. Therefore there will be another non-phantom edge, e_k^q,

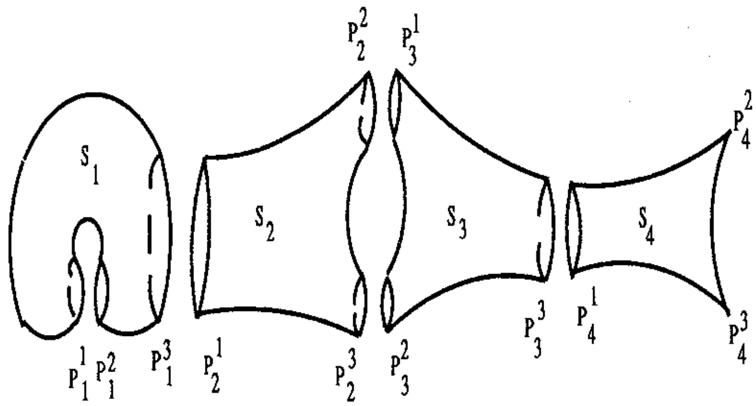


Figure 3.4: Parts of an orbifold with the special points labeled, before adding ramification points

emanating from the vertex v_k, that corresponds to the same partition curve a_l. If ν_j^p and ν_k^q are the weights of e_j^p and e_k^q, we have that $\nu_j^p = \nu_k^q$ by the above construction. j can be equal to k, but in that case we will have $p \neq q$. We then join the edges e_j^p and e_k^q to form one single edge, with weights equal to ν_j^p at each half. Proceeding in the same fashion with all the non-phantom edges we get an admissible graph $\mathcal{G}$, associated to the pair $(\mathcal{S}, \mathcal{C})$. This graph is NOT unique, since we have added arbitrarily the weights of the non-phantom edges. For example, the graph of figure 3.3 is a graph corresponding to the orbifold of figure 3.1.

3.1.6. We now proceed in the inverse order, to get an orbifold from a graph, as follows: consider an admissible graph, $\mathcal{G}$, with vertices $v_1, \ldots, v_{2p-2+n}$ and edges $e_1, \ldots, e_{3p-3+2n}$. Take one vertex, say v_j. From it three edges emanate, e_{j_1}, e_{j_2}, e_{j_3}, $j_1 \leq j_2 \leq j_3$, with weights μ_{j_1}, μ_{j_2}, μ_{j_3}, respectively. Assign to the vertex v_j a sphere $\mathcal{S}_j$ with signature $(0, 3; \mu_{j_1}, \mu_{j_2}, \mu_{j_3})$. We label the special points of $\mathcal{S}_j$ by P_j^1, P_j^2 and P_j^3, so they are in a one-to-one correspondence with the edges e_{j_i}.

Take a non-phantom edge of $\mathcal{G}$, say e_m, and suppose that it connects the vertices v_j and v_k. On the sphere $\mathcal{S}_j$ there is a ramification point P_j^r with ramification value equal to the weight of the half of e_m that starts at v_j. Similarly, on $\mathcal{S}_k$ we have a ramification point P_k^s, corresponding to e_m. The ramification value of P_k^s is the same that the ramification value of P_j^r, since both points correspond to the same non-phantom edge. j can be equal to k, but in that case $r \neq s$. We then cut neighborhoods of P_j^r and P_k^s on $\mathcal{S}_j$ and $\mathcal{S}_k$ respectively, and identify the boundaries of these neighborhoods, to obtain in this way a new 'bigger' surface. After doing the same operation with all non-phantom edges we get a surface of genus p with n ramification points or punctures. The ramification values are given by the weights of the phantom edges.

Observe that two graphs with the same phantom weights induce the same (topological) orbifold.

The analytic process of cutting and pasting around the ramification points or punctures is controlled by a process called **plumbing constructions** that we explain in the next subsection.

3.1.7. The cutting and paste done in subsection 3.1.6 involves only topological aspects; the plumbing constructions are the operations we need in order to work at the level of complex analysis. Here we will follow the work of Irwin Kra in [Kra90].

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two orbifolds with signatures $(0, 3; \mu, \nu_1, \nu_2)$ and

$(0, 3; \mu, \nu_3, \nu_4)$ respectively. Consider on $\mathcal{S}_1$ the horocyclic coordinate z centered at the point with ramification value μ and relative to the point with ramification value ν_1. Similarly, we take the on $\mathcal{S}_2$ horocyclic coordinate w centered at the point with ramification value μ and relative to the point with ramification value ν_3. Suppose t is a complex number small enough so that the closure of the sets $\{z \in \mathcal{S}_1;\ |z| < \sqrt{|t|}\}$ and $\{z \in \mathcal{S}_2;\ |w| < \sqrt{|t|}\}$ are contained in $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Let $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ be the orbifolds with the above (open) sets removed. Consider the quotient surface obtained by identifying the boundaries:

$$\mathcal{S} = (\mathcal{S}_1 \sqcup \mathcal{S}_2)/\sim,$$

where $z \in \mathcal{S}_1^*$ and $w \in \mathcal{S}_2^*$ are equivalent, $z \sim w$, if and only if $|z| = \sqrt{|t|} = |w|$ and $zw = t$ (here $\sqcup$ means the disjoint union of sets). This process is called the **tame plumbing construction**. The 'tame' part comes from the fact that the boundary curves that we identify are given by horocircles (level curves of horocyclic coordinates). We can do this construction along other curves, not horocircles, to obtain the **general plumbing construction**. To see that the resulting surface is a Riemann surface with signature (orbifold), we can do the plumbing construction by identifying annuli (instead of curves), and then prove that the final orbifold is independent of the choice of annuli. This is done in [Kra90]. Let us remark that the proof there is for the case of surfaces with no branch points, but it works as well for orbifolds because the process of gluing takes place in open sets that do not contain ramification points.

If we have only one orbifold $\mathcal{S}$ with signature $(0, 3; \mu, \mu, \nu)$, we can consider horocyclic coordinates around the two points with ramification value μ. Suppose that the coordinates are called z and w. In order to make a plumbing construction we consider the sets $\{z \in \mathcal{S};\ |z| < \sqrt{|t|}\}$ and $\{z \in \mathcal{S};\ |w| < \sqrt{|t|}\}$, but this time we have to choose t so that these two sets are disjoint. Under these conditions we can make the plumbing identification as before, to obtain an orbifold of signature $(1,1;\nu)$. This is again a tame construction; we also have the general plumbing construction, as in the previous case.

These constructions can be extended with no problem to glue any two orbifolds, not necessarily those of type (0,3).

3.1.8. Here we prove a lemma that tells us that the tame plumbing constructions are possible as long as $|t|$ is small and the points we glue have ramification value $\mu = \infty$; that is, we do plumbing constructin removing punctured disc. A similar lemma for points of finite ramification value will appear elsewhere in the near future.

Lemma 3 *Let $\mathcal{S}$ be an orbifold of signature $(0,3;\infty,\nu_1,\nu_2)$ and let z be a horocyclic coordinate around the point corresponding to ∞. Then the image of z contains an open disc of radius*

$$r = r(\infty,\nu_1\nu_2) = e^{\frac{-\pi}{\cos(\pi/\nu_1)+\cos(\pi/\nu_2)}},$$

if the signature is hyperbolic, and of radius 1 *if the signature is parabolic.*

Proof. We can take a triangle group with the above signature and parameters $(\infty,0,1)$. Let us first work out the hyperbolic case. We can take as fundamental domain of this group one of the three drawn in the figure 3.5.

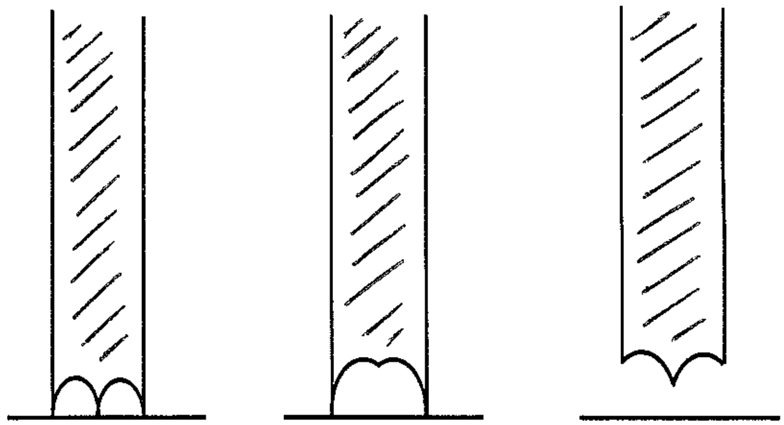


Figure 3.5: Fundamental domains for hyperbolic triangle groups

The left circle of the fundamental domain, whose center is in the real line, is the isometric circle of the transformation

$$B = \begin{bmatrix} -cos(\pi/\nu_1) & b \\ cos(\pi/\nu_1) + cos(\pi/\nu_2) & -cos(\pi/\nu_1) \end{bmatrix}.$$

Its radius is equal to $\frac{1}{cos(\pi/\nu_1)+cos(\pi/\nu_2)}$. Then the horizontal line $Im(z) = \frac{1}{cos(\pi/\nu_1)+cos(\pi/\nu_2)}$ projects on the orbifold onto the circle of the desired radius, and the part of the fundamental domain above that line is our disc. To complete the proof for this case we should remark that the right circle of the above fundamental domains is the isometric circle of the transformation AB, and it has the same radius that the left circle, so we obtain the same bound.

The parabolic case is proven in the same way. □

From this lemma we see that we always have room to perform a tame plumbing construction between two orbifolds of type (0,3). For example, if we have an orbifold with signature $(0, 3; \infty, \nu_1, \nu_2)$ and another with signature

$(0, 3; \infty, \nu_1, \nu_2)$, then we can do a plumbing construction if we take t of §3.1.7 to satisfy $|t| < min(r(\infty, \nu_1, \nu_2),\ r(\infty, \nu_3\nu_4))$. We can also do plumbing construction between two orbifolds of more complicated topology, but then we have to be more careful about the bounds for $|t|$.

3.1.9. Given a weighted graph, $\mathcal{G}$, we are going to construct a deformation space associated to it in a way we now explain. Suppose that $\mathcal{G}$ has d non-phantom edges. We know that to build the orbifold (with maximal partition) associated to the graph $\mathcal{G}$ we have to perform d plumbing constructions, in a way somehow controlled by the non-phantom edges. The set of all n-tuples of complex numbers $(t_1, \ldots, t_d)$ so that it is possible to build an orbifold associated to $\mathcal{G}$ using them as plumbing parameters is called the **DEFORMATION space of the graph** $\mathcal{G}$, and it will be denoted by $\mathbf{D}(\mathcal{G})$.

There is a point that needs to be made clear. If $(t_1, \ldots, t_d)$ is a point of $D(\mathcal{G})$, then we can construct an orbifold with this n-tuple as plumbing parameters. But the plumbing operations require a choice of horocyclic coordinates on the surfaces we are gluing. Our choice is as follows: suppose we want to glue the surfaces $\mathcal{S}_j$ and $\mathcal{S}_k$ by cutting neighborhoods of the punctures P_j^r and P_k^s. Then we take the horocyclic coordinate z in $\mathcal{S}_j$ centered at P_j^r and relative to P_j^{r+1} (where $r+1$ is taken relative mod 3); and on the surface $\mathcal{S}_k$ we consider the horocyclic coordinate w, centered at P_k^s and relative to P_k^{s+1}. Using z and w we make the corresponding plumbing construction (see [Kra90]).

3.1.10. In subsection §3.1.5 we have seen how to build a graph from an orbifold with a maximal partition. One can see that there are two possible operations that occur in that process, namely the union of two disjoint graphs by merging one edge of each graph, or the union of two edges in a single graph. We call these operations the AFP of two graphs or the HNN extension of a graph, for the reason that we explain now. More details can be found in [Kra90].

Suppose we are in the following situation: we have built a graph $\mathcal{G}$ from an orbifold $\mathcal{S}$, and we are now to join that graph with another one, say $\mathcal{G}'$, that correspond to adding a new sphere, $\mathcal{S}'$ to $\mathcal{S}$. The addition of these new graph $\mathcal{G}'$, that consists on only one vertex and three edges, is called the AFP of the graphs $\mathcal{G}$ and $\mathcal{G}'$. At the level of b-groups we have that $\mathcal{S}$ (with its maximal partition) is uniformized by a terminal regular b-group, say Γ. The sphere $\mathcal{S}'$ is uniformized by a triangle group Γ'. Then, the union of the orbifolds $\mathcal{S}$ and $\mathcal{S}'$ is uniformized by the AFP of Γ and Γ'. This is because joining the orbifolds $\mathcal{S}$ and $\mathcal{S}'$ across two ramification points/puncutures means that these two points are given, in Γ and Γ', by the same parabolic/elliptic element. The AFP construction can be generalized to any two disjoint graphs.

The same situation happens with the HNN extension of a graph: suppose $\mathcal{S}$ is an orbifold that gives the graph $\mathcal{G}$, and $\mathcal{S}$ is uniformized by the b-group Γ. If we join two edges of $\mathcal{G}$, at the level of groups we are making an HNN extension of Γ by some element, C. This is the intuitive idea, since joining two edges corresponds to identifying two ramification points/punctures; these special points are represented by two parabolic or elliptic elements, the identification will be conjugating one parabolic/elliptic into the other by a transformation C.

3.1.11. In the next section we will construct the deformation spaces of terminal regular b-groups associated to graphs where the non-phantom edges have weights equal to ∞. The section 3 will deal with the construction of groups associated to graphs of orbifolds with signature $(0, 4, \infty, \infty, \infty, \infty)$, but where the (unique) non-phantom edge has finite weight. More general results will appear elsewhere in the near future.

3.2 Coordinates for deformation spaces

3.2.1. In this section we will study the one-dimensional Teichmüller spaces (the deformation spaces of orbifolds of type (0,4) and (1,1)), and then we will use Maskit embedding theorem to compute coordinates for the deformation spaces of orbifolds of arbitrary finite hyperbolic type.

3.2.2. Let $\mathcal{S}$ be an orbifold with signature $(0, 4; \nu_1, \nu_2, \nu_3, \nu_4)$. A maximal partition on $\mathcal{S}$ consists on one curve, a, that divides $\mathcal{S}$ into two parts, $\mathcal{S}_1$ and $\mathcal{S}_2$. Without loss of generality we can assume that the points on $\mathcal{S}_1$ have ramification values ν_1 and ν_2. Let us orient a so that $\mathcal{S}_1$ lies to its right. Let us assume (for now) that neither of the signatures of $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal to $(0, 3; \infty, 2, 2)$. The situation is pictured in the figure 3.6. We want to reconstruct the orbifold $\mathcal{S}$ from the parts $\mathcal{S}_1$ and $\mathcal{S}_2$ by a plumbing operation; the way this plumbing has to be done will be controlled by a complex number whose logarithm is a coordinate for the deformation space of $\mathcal{S}$.

At the level of graphs, what we do is to take two graphs, each of them with one vertex and three edges, and then we make the AFP of the two graphs (§3.1.10) across one edge to obtain a bigger graph, as is shown in the figure

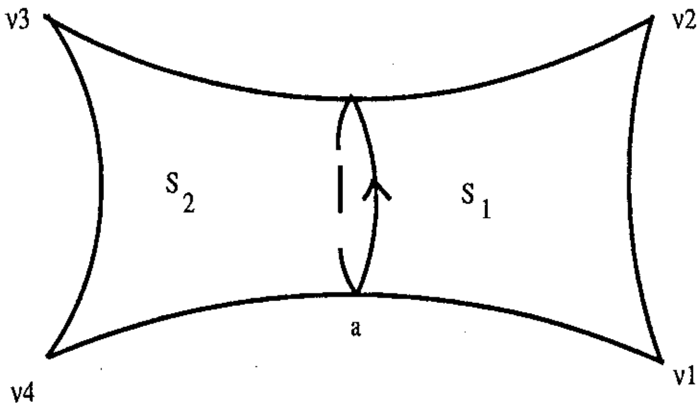


Figure 3.6: An orbifold of type (0,4) with a maximal partition.

3.7.

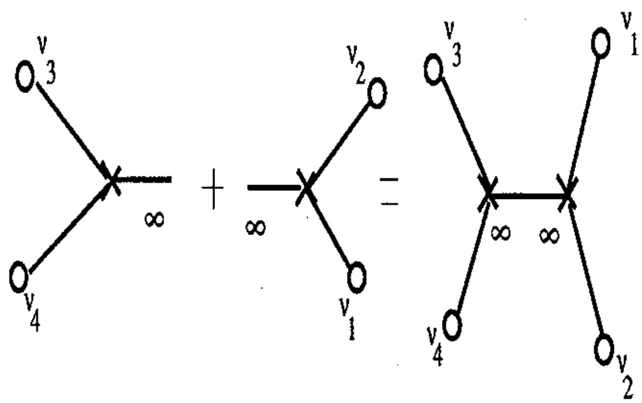


Figure 3.7: The AFP operation at the graph level. $\mu = \infty$

Before starting the technical arguments, we need to say one more thing about choosing generators for triangle groups. Suppose that $\mathcal{S}$ is a hyperbolic orbifold with signature σ, and with at least one puncture. Let a be a small closed loop contractible to a puncture on $\mathcal{S}$. Orient a such that the removed point (puncture) lies to the left of a. Let $\mathcal{F}$ be a triangle group with signature

σ. The region of discontinuity of this group, $\Omega(\mathcal{F})$, consists of two discs, D_1 and D_2. We need some way of choosing one of these two discs as the universal covering space of $\mathcal{S}$. To do so, lift a to each disc: $\tilde{a}_1 \subset D_1$ and $\tilde{a}_2 \subset D_2$, both invariant under the same parabolic element A. Orient $\tilde{a}_i$ so that for any point z in $\tilde{a}_i \subset D_i$ the triple $(z, A(z), A^2(z))$ is positively oriented. Then we will have that in one of the disc, the puncture we are considering lies to the right, while in the other disc, it lies to the left; it is this latter disc the one we choose to uniformize $\mathcal{S}$.

3.2.3. Attaching a punctured disc to the boundary of $\mathcal{S}_1$ we obtain an orbifold of finite hyperbolic type with signature $(0,3;\infty,\nu_1,\nu_2)$. Since we are interested in coordinates on the Teichmüller space, we are free to conjugate by Möbius transformations. This gives us the chance to choose the triangle group $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$ as the uniformizing group of $\mathcal{S}_1$. Its generators, (A,B) are given in section 2.3, but we include them here for the sake of clarity:

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_1 & b \\ q_1+q_2 & -q_1 \end{bmatrix},$$

$q_i = \cos(\pi/\nu_i)$, $b = \frac{1-q_1^2}{q_1+q_2}$.

The element A corresponds to a small loop around a puncture. We will cut a neighborhood of that puncture to make the plumbing construction. This means that A is going to become accidental parabolic after applying Maskit First Combination Theorem ([Mas88]), and it will correspond to the curve a of the maximal partition of the figure 3.6.

3.2.4. Now consider the part $\mathcal{S}_2$ that lies to the left of the curve a. Again add one punctured disc to obtain a complete orbifold, uniformized by a triangle group $\Gamma(\infty, \nu_3, \nu_4; d, e, f)$ where the parameters have to be determined. Due to the orientation of the partition curve, one of the canonical generators of this group, the one corresponding to the puncture used in the plumbing construction, has to be A^{-1}; see §3.2.2. This group can be obtained as follows: the general triangle group with that signature has canonical generators A and $\tilde{B}$, and parameters $(\infty, 0, 1)$. Since we want A^{-1} to be one of the generators, we should change as well to $\tilde{B}^{-1}$, because $A^{-1}\tilde{B}^{-1}$ is parabolic while $A^{-1}\tilde{B}$ is not. That will give parameters $(\infty, 0, -1)$. We want to conjugate (this is the ONLY way to get another triangle group with the desired signature) the group preserving the element A^{-1}. Therefore, the conjugation should be made by an element of the form

$$T_\alpha = \begin{bmatrix} -1 & -\alpha \\ 0 & -1 \end{bmatrix},$$

where α is a complex number. We get that the generators of the group $\Gamma(\infty, \nu_3, \nu_4; d, e, f)$ are

$$A^{-1},\ B_\alpha^{-1} = \begin{bmatrix} -q_3 - \alpha(q_3 + q_4) & -b^* - \alpha^2(q_3 + q_4) \\ -(q_3 + q_4) & -q_3 + \alpha(q_3 + q_4) \end{bmatrix},$$

$q_i = \cos(\pi/\nu_i)$, $b^* = \frac{1-q_3^2}{q_3+q_4}$. This gives that the group uniformizing $\mathcal{S}_2$ (with a punctured disc attached) is $\Gamma(\infty, \nu_3, \nu_4; \infty, \alpha, \alpha - 1)$.

3.2.5. The group generated by the union of $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$ and $\Gamma(\infty,\nu_3,\nu_4;\infty,\alpha,\alpha-1)$, $\Gamma_\alpha =< A, B, B_\alpha >$, is a terminal regular b-group with signature $(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ that uniformizes the orbifold $\mathcal{S}$ on its invariant component, and two orbifold of signature $(0,3;\infty,\nu_1,\nu_2)$ and $(0,3;\infty,\nu_3,\nu_4)$ in the non-invariant components. This can be seen by applying Maskit First Combination Theorem. For example, if α has big imaginary part (see theorem below for a bound) one can use a horizontal line to apply the First Combination Theorem. The same Combination Theorem tells us that Γ_α is the AFP of the two triangle groups across the common subgroup generated by A: $\Gamma_\alpha = \Gamma(\infty,\nu_1,\nu_2;\infty,0,1) *_{<A>} \Gamma(\infty,\nu_3,\nu_4;\infty,\alpha,\alpha-1)$. We also see that the invariant component of Γ_α is contained on the strip $\{z;\ 0 < Im(z) < Im(\alpha)\}$. α is a coordinate for the deformation space of the group (since if we know α we know the group and its marking, and different markings determine different α's). We can express α as something more related to this new group: the parameters for the triangle groups uniformizing $\mathcal{S}_1$ and $\mathcal{S}_2$ are, respectively, $(\infty,0,1)$ and $(\infty,\alpha,\alpha-1)$. Then $\alpha = cr(\infty,0,1,\alpha)$, and this expression is $PSL(2,\mathbb{C})$-conjugacy invariant.

In general, the orbifold $\mathcal{S}$ is uniformized by a group Γ constructed as the AFP of two triangle groups, say $\Gamma = \Gamma(\infty,\nu_1,\nu_2;d,e,f) *_J \Gamma(\infty,\nu_3,\nu_4;d,e',f')$, where J is a common subgroup generated by a parabolic element corresponding to the curve of the partition. Then the point in the Teichmüller space $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ that corresponds to the group Γ is given by the cross ratio $cr(d,e,f,d')$. We would like to remark here that once the first triangle group $\Gamma(\infty,\nu_1,\nu_2;d,e,f)$ is given, we have only one degree of freedom when choosing the parameters of the group $\Gamma(\infty, \nu_3, \nu_4; d, e', f')$; this is so because the two triangle groups must share one parabolic element, so the parameter d is common to both groups, and once e' is chosen, there is only one possible choice of f' that makes that the groups share a parabolic element. This is quite natural, since the Teichmüller spaces of orbifolds of type (0,4) have complex dimension 1.

Choose $\alpha = 2i$; then the group Γ_{2i} is a terminal regular b-group unformizing an orbifold of signature $(0, 4; \nu_1, \nu_2, \nu_3, \nu_4)$ in its invariant component (and the correct number of orbifolds of type (0,3) in the non-invariant sets). The space $T(\Gamma_{2i})$ is then isomorphic to the space $T(0, 4; \nu_1, \nu_2, \nu_3, \nu_4)$. Lemma 3 of the previous section gives us some estimate for the size of the space $T(\Gamma_{2i})$, which are stated in the following result:

Theorem 3 *α is a global coordinate, called* **horociclic** *coordinate, for $T(\Gamma_{2i})$ $\cong T(0, 4; \nu_1, \nu_2, \nu_3, \nu_4)$, and we have the following:*

$$\{\alpha / Im(\alpha) > y_1\} \subset T(\Gamma_{2i}) \subset \{\alpha / Im(\alpha) > y_2\},$$

where

$$y_1 = \frac{1}{q_1 + q_2} + \frac{1}{q_3 + q_4}, \; y_2 = max(\frac{1}{q_1 + q_2}, \frac{1}{q_3 + q_4}),$$

and $q_i = cos(\pi / \nu_i)$.

Proof. If $Im(\alpha)$ is bigger than y_1, we can choose fundamental domains for $\Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1)$ and $\Gamma(\infty, \nu_3, \nu_4; \infty, \alpha, \alpha - 1)$ like those in figure 3.5 (or translated by $T\alpha$). Then we have that there is a horizontal segment in the intersection of the fundamental domains that, under iterations by A, extends to a horizontal line that can be used to apply Maskit First Combination Theorem, [Mas88].

For the second statment we need a general fact about precisely invariant sets. Let us assume that the transformation $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has real coefficients, and $c > 0$. Suppose $k > 0$ and let X be the set

$$X = \{z / \ Im(z) > \ k\}.$$

If $k < \frac{1}{|c|} = \frac{1}{c}$, then $\gamma(X) \cap X \neq \emptyset$. The idea is that the isometric circle of γ intersect X, and therefore its image $\gamma(X)$. To prove it, consider the circle of center $\frac{-d}{c}$ and radius $\frac{1}{c}$. The point with the biggest imaginary part in this circle is $-\frac{d}{c} + \frac{i}{c}$. Then this point is in X and for its image we get

$$\gamma(-\frac{d}{c} + \frac{i}{c}) = \frac{-ad + ia + bc}{ic} = \frac{a}{c} + \frac{i}{c}$$

which is in X, as we claimed.

Returning to the proof of the theorem, we have that the lower half plane has to be precisely invariant under $\Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1)$, and therefore we get that $Im(\alpha) > \frac{1}{q_3+q_4}$. And because the half-plane $\{z / \ Im(z) > \ Im(\alpha)\}$ is precisely invariant under $\Gamma(\infty, \nu_3, \nu_4; \infty, \alpha, \alpha - 1)$, we get $Im(\alpha) > \frac{1}{q_1+q_2}$, completing the proof. □

3.2.6. In [Kra88] it is shown that the trace of $B_\alpha B$ is locally injective on the deformation space we are studying. We see that this trace is equal to

$$2q_1q_3 + b^*(q_1 + q_2) + b(q_3 + q_4) - \alpha^2(q_1 + q_2)(q_3 + q_4),$$

and since α has positive imaginary part, we get that the function $tr(B_\alpha B)$ is GLOBALLY injective on $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$. To see it, suppose that α and β satisfy

$$2q_1q_3 + b^*(q_1+q_2) + b(q_3+q_4) - \alpha^2(q_1+q_2)(q_3+q_4) =$$

$$= 2q_1q_3 + b^*(q_1+q_2) + b(q_3+q_4) - \beta^2(q_1+q_2)(q_3+q_4).$$

This implies

$$\alpha^2(q_1+q_2)(q_3+q_4) = \beta^2(q_1+q_2)(q_3+q_4),$$

but since we are assuming that $\mathcal{S}_1$ and $\mathcal{S}_2$ do NOT have signature equal to $(0,3;\infty,2,2)$, we have that $q_1+q_2 \neq 0$ and $q_3+q_4 \neq 0$. Therefore $\alpha^2 = \beta^2$, and the fact that both numbers have strictly positive imaginary part implies that $\alpha = \beta$, as claimed.

3.2.7. The algebraic AFP construction of the group Γ_α can be translated to a geometric process at the level of orbifolds: amalgamation of the groups $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$ and $\Gamma(\infty,\nu_3,\nu_4;\infty,\alpha,\alpha-1)$ is 'equivalent' to a plumbing construction of the corresponding orbifolds $\mathcal{S}_1$ and $\mathcal{S}_2$. $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$ is generated by the elements A and B. Let z be a horocyclic coordinate on the orbifold determined by $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$, with z centered at the puncture given by A and relative to the special point (either puncture or ramification point) determined by B; then $z = e^{\pi i\zeta}$ for ζ with positive imaginary part, since we are considering the action of $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$ on the upper half plane, due to the orientation of our partition curve. Similarly, consider $\Gamma(\infty,\nu_3,\nu_4;\infty,\alpha,\alpha-1)$ and a horocyclic coordinate w centered to the puncture (given by A) and relative to the special point with ramification value ν_3. Then we have $w = e^{\pi i(\alpha-\zeta)}$ for ζ with imaginary part satisfying $Im(\zeta) < Im(\alpha)$. The identification zw gives a plumbing parameter equal to $zw = e^{\pi i\alpha} = t$. Observe that t is a coordinate for the deformation space of the graph associated to the orbifold of the figure 3.6. That graph can be found in figure 3.7. Lemma 3 in §3.1.8 and theorem 3 in subsection 3.2.5 provide the following result:

Theorem 4 *The orbifold corresponding to the point α in $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ is conformally equivalent to the orbifold constructed by plumbing with parameter t as above. Moreover, we have that*

$$0 < |t| < e^{-\pi y_1},$$

with $y_1 = \frac{1}{q_1+q_2} + \frac{1}{q_3+q_4}$.

3.2.8. Consider now the case that, after cutting the orbifold $\mathcal{S}$ by the curve a, one of the parts has parabolic signature. Without loss of generality, we are in the following situation: $\mathcal{S}_1$ has signature $(0,3;\infty,\nu_1,\nu_2)$, with $\frac{1}{\nu_1}+\frac{1}{\nu_2} < 1$, and $\mathcal{S}_2$ has signature $(0,3;\infty,2,2)$. We can proceed as in the previous case: $\mathcal{S}_1$ is uniformized by the group $\Gamma(\infty,\nu_1,\nu_2;\infty,0,1)$, whose generators are

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -q_1 & b \\ q_1+q_2 & -q_1 \end{bmatrix},$$

with $q_i = \cos(\pi/\nu_i)$, $b = \frac{1-q_1^2}{q_1+q_2}$.

$\mathcal{S}_2$ is uniformized by the group $\Gamma(\infty, 2, 2; \infty, \alpha, \alpha - 1)$, with generators

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 0, -1 & \end{bmatrix}, \; B_\alpha = B_\alpha^{-1} = i \begin{bmatrix} -1 & 2\alpha \\ 0 & 1 \end{bmatrix},$$

and α is a complex number.

The group generated by the AFP of these two triangle groups, $\Gamma_\alpha = \Gamma(\infty, \nu_1, \nu_2; \infty, 0, 1) *_{<A>} \Gamma(\infty, 2, 2; \infty, \alpha, \alpha - 1)$ uniformizes an orbifold of signature $(0, 4; \nu_1, \nu_2, 2, 2)$ in its invariant component and a sphere with signature $(0, 3; \infty, \nu_1, \nu_2)$ in the non-invariant components. The part corresponding to the sphere with signature $(0, 3; \infty, 2, 2)$ is 'missed'. In general, a terminal regular b-group that uniformizes an orbifold of signature $(0, 4; \nu_1, \nu_2, 2, 2)$ is constructed as the AFP of $\Gamma(\infty, \nu_1, \nu_2; c, d, e)$ and $\Gamma(\infty, 2, 2; c, d', e')$ across a common subgroup generated by a parabolic element that fixes the point c. Then the coordinate for such group is given as $\alpha = cr(c, d, e, d')$. As in §3.2.5, we have bounds for the size of the elements of the deformation space of Γ_{2i}:

Theorem 5 *α is a global coordinate for $T(\Gamma_{2i}) \cong T(0, 4; \nu_1, \nu_2, 2, 2)$, and we have the following:*

$$\{\alpha;\ Im(\alpha) > 0\} \subset T(\Gamma_{2i}) \subset \{\alpha;\ Im(\alpha) > y_1\},$$

where

$$y_1 = \frac{1}{q_1 + q_2},$$

and $q_i = cos(\pi/\nu_i)$.

3.2.9. The other orbifolds with one dimensional deformation spaces are those of type (1,1), for which we have similar constructions: we start with an orbifold $\mathcal{S}$ with (necessarily hyperbolic) signature $(0,3;\infty,\infty,\nu)$, we removed punctured discs and make a boundary identification to obtain an orbifold $\mathcal{X}$ with signature $(1,1;\nu)$. To compute the coordinates for the deformation space, let us consider a triangle group with signature $(0,3;\infty,\infty,\nu)$ and parameters $(\infty,0,1)$, which we know is generated by the transformations:

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 1+q & -1 \end{bmatrix},$$

$q = cos(\pi/\nu)$. We want to 'glue' the punctures represented by A and B. This is accomplished by finding a Möbius transformation C such that

$$C < B > C^{-1} =< A > .$$

But we have to be a little careful about this identification: A represents a curve around one of the punctures of the orbifold uniformized by the group $\Gamma(\infty,\infty,\nu;\infty,0,1)$; similarly B corresponds to a curve around another puncture of the same orbifold. Identification of the punctures has to be done preserving the orientation of these two curves, which means that C must satisfy

$CB^{-1}C^{-1} = A$. Then, if $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we must have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1+q & -1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}$$

This gives $b^2 = -\frac{2}{1+q}$, and we choose the positive square root, to get

$$C = i \begin{bmatrix} \tau & \sqrt{\frac{2}{1+q}} \\ \sqrt{\frac{1+q}{2}} & 0 \end{bmatrix}.$$

The group generated by A, B and C is a terminal regular b-group that uniformizes an orbifold with signature $(1,1;\nu)$ in the invariant component, and an orbifold of signature $(0,3;\infty,\infty,\nu)$ in the non-invariant components. This group is constructed as the HNN extension of $\Gamma(\infty,\infty,\nu;\infty,0,1)$ by the element C. This can be seen by applying Maskit Second Combination Theorem. The parameter τ is a coordinate for the deformation space of the group. This parameter can be obtained, in a general setting, as follows: if we want to construct a terminal regular b-group uniformizing an orbifold with signature $(1,1;\nu)$, we have to start with a triangle group, say $\Gamma(\infty,\infty,\nu;d,e,f)$ and add an element C. Then we have that the coordinate of this group in the Teichmüller space $T(1,1;\nu)$ is given as the cross ratio $\sqrt{\frac{2}{1+q}}\, cr(d,e,f,C(d))$, where $q = \cos(\pi/\nu)$.

What the mapping C does geometrically is to send horocircles at 0 to horocircles at ∞ (so we are cutting punctured discs around each puncture and gluing the boundaries in a process similar to the case of orbifolds of type (0,4)). More precisely, we have that

$$C(\{z/|z-ri|=r\}) = \{z/Im(z) = \sqrt{\frac{2}{1+q}}Im(\tau) - \frac{1}{r\sqrt{1+q}}\}.$$

If these two circles are disjoint, Maskit theorem can be applied. Now, the point with the biggest imaginary part in the first circle is $2ri$. Therefore we want (in order to have disjoint circles)

$$\sqrt{\frac{2}{1+q}}Im(\tau) - \frac{1}{r\sqrt{1+q}} > 2r, \text{ or } Im(\tau) > \sqrt{\frac{1+q}{2}}(\frac{1}{r\sqrt{1+q}} + 2r).$$

The minimum value of the last expression is 2. This gives the following result:

Theorem 6 *τ is a global coordinate for $T(1,1;\nu)$, and we have the following:*

$$\{\tau/Im(\tau) > 2\} \subset T(1,1;\nu) \subset \{\tau/Im(\tau) > 0\}.$$

Proof. The only point is the first inclusion. This is due to the fact that the lower half plane is precisely invariant under F_1, so for any point z with negative imaginary part we should have

$$Im(C(z)) = \sqrt{\frac{2}{1+q}}Im(\tau) - \frac{2}{1+q}\frac{Im(z)}{|z|} > 0,$$

giving the desired bound for τ. □

3.2.10. We have checked that C is completely determined by its trace, as it is proven in [Kra88].

3.2.11. As in the case of orbifolds of type (0,4), our next goal is to relate this algebraic HNN extension to a plumbing construction. To do it, let us star by considering the coordinate centered at the puncture corresponding to A and relative to the puncture corresponding to B, where the action of the triangle group is on the upper half plane. That coordinate is $z = e^{\pi i \zeta}$. For the B-puncture we have

$$w = e^{\frac{2\pi i}{1+q}} e^{\frac{-2\pi i}{(1+q)\zeta}}.$$

The identification we make is given by

$$z(C(\zeta))w(\zeta) = z(\sqrt{\frac{2}{1+q}} + \frac{2}{1+q}\frac{1}{\zeta})w(\zeta) = e^{\frac{2\pi i}{1+q}} e^{\sqrt{\frac{2}{1+q}}\pi i \tau} = t.$$

This computation gives the following estimate for t:

Theorem 7 *The orbifold corresponding to the point τ in $T(1,1;\nu)$ is conformally equivalent to the orbifold constructed by plumbing with parameter t as above. Moreover, we have that*

$$0 < |t| < 1.$$

As in the case of orbifolds of type (0,4), we have that the complex number t of the above theorem is a coordinate for the deformation space of the graph associated to the orbifold of signature $(1, 1; \nu)$ constructed in the previous subsections.

3.2.12. Putting together the above results and the Maskit embedding theorem (see chapter 1) we get coordinates for the general Teichmüller spaces of orbifolds of finite conformal type. The previous results will also give us coordinates (plumbing parameters) for the deformation spaces of weighted graphs (see §3.1.9). We have already seen the way to proceed: given an orbifold of finite hyperbolic type and a maximal partition on it, we use the curves of the partition to obtain the modular parts of the orbifold; such modular parts have one dimensional deformation spaces; Maskit embedding gives the desired coordinates.

Theorem 8 *Let $\mathcal{S}$ be an orbifold of finite hyperbolic type with signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$ and let $\mathcal{C}$ be a maximal partition on $\mathcal{S}$, uniformized by the terminal regular b-group Γ. Then there exists a set of (global) coordinates, called* **horocyclic coordinates**, *for the Teichmüller space $T(\Gamma) \cong T(p, n; \nu_1, \ldots, \nu_n)$, say $(\alpha_1, \ldots, \alpha_d)$, where $d = 3p - 3 + n$, and a set of complex numbers, $(y_1^1, \ldots, y_1^d, y_2^1, \ldots, y_2^d)$, that depends on the signature σ and the partition $\mathcal{C}$, such that*

$$\{(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d / Im(\alpha_i) > y_1^i \ , \forall\, 1 \leq i \leq d\} \subset T(\Gamma)$$

and

$$T(\Gamma) \subset \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d / Im(\alpha_i) > y_2^i \ , \forall\, 1 \leq i \leq d\}.$$

Moreover, the surface corresponding to the point $(\alpha_1, \ldots, \alpha_d)$ is conformally equivalent to a surface constructed by plumbing techniques with parameters $(t_1, \ldots, t_d)$ obtained by use of theorems in §3.2.6 and §3.2.9, depending whether the corresponding operation at the graph level is an AFP or an HNN extension.

Proof. The triple $(\mathcal{S}, \sigma, \mathcal{C})$ is uniformized by a terminal regular b-group Γ (§1.2.5). In the first chapter we have explained how to obtained from $\mathcal{S}$ and $\mathcal{C}$

the modular parts $T_1, \ldots, T_{3p-3+n}$ and the modular subgroups $G_1, \ldots, G_{3p-3+n}$. We know that these groups are terminal regular b-groups with one dimensional Teichmüller spaces. Therefore we can apply the theorems 3 (§3.2.5) and 5 (§3.2.9) to obtain coordinates $(\alpha_1, \ldots, \alpha_{3p-3+n})$. Maskit embedding theorem

$$T(\Gamma) \hookrightarrow \prod_{j=1}^{3p-3+n} T(G_j)$$

tells us that we can use this set of complex numbers as coordinates for $T(\Gamma)$

The plumbing parameters t_j are related to the coordinates $(\alpha_1, \ldots, \alpha_d)$ by the formula $t_j = e^{k_j i \alpha_j}$, where k_j is a number that depends on the signature of the modular part corresponding to α_j.

The numbers y_j^1 and y_j^2 are part of the results of the theorems 3 and 5. The plumbing parameters are obtained by application of theorems 4 (§3.2.7) and 6 (§3.2.11). □

3.2.13 In this last part of this section we will explain in a more detailed manner how to pass from a given group to coordinates of its deformation (Teichmüller) space, and viceversa, how to construct a terminal regular b-group from a point in a Teichmüller space.

Let us start by considering a terminal regular b-group, Γ, of hyperbolic signature $(p, n; \nu_1, \ldots, \nu_n)$. We want to compute the coordinates of this group in $T(\Gamma)$. We should remark here that the horocyclic coordinates are not variational. In other sets of coordinates, one starts with a given group, say F_0, and for any other group, F, quasiconformally equivalent to F_0, one computes the coordinates in $T(F_0)$ by seeing how far the quasiconformal mapping is from being conformal. This means that, as the identity is the mapping that takes F_0

to itself, the starting group will be the always the origin of coordinates. One needs then two groups to compute coordinates. In our case the coordinates of a group depend on the modular subgroups, and how they are assembled together to build the bigger group. It makes sense to compute the coordinates of Γ in $T(\Gamma)$ without another group for 'reference'.

After this introduction, let us consider the simply connected invariant component Δ of Γ. This group will uniformize an orbifold, $\mathcal{S} \cong \Delta/\Gamma$ with a maximal partition. Let T_j be one modular, and let D_j be a component of the set $\pi^{-1}(T_j)$, where $\pi : \Delta \to \mathcal{S}$ is the natural projection. Let us look at the stabilizer of D_j in Γ, say $\Gamma_j = Stab(D_j, \Gamma)$. Then Γ_j is a terminal regular b-group uniformizing an orbifold of type (0,4) or (1,1). In subsection §§3.2.5, 3.2.8 and 3.2.9 we have explained how to get the coordinate of groups of type (0,4) or (1,1). Applying the techniques explained there we get the coordinate α_j corresponding to Γ_j. Putting together all modular subgroups we get the coordinates $(\alpha_1, \ldots, \alpha_{3p-3+n})$ of Γ. The reader may ask what happens if we take a different component of $\pi^{-1}(T_j)$, say D'_j. In that case, since $\pi(D_j) = T_j = \pi(D'_j)$, we have that there is a deck transformation $\gamma \in \Gamma$ such that $\gamma(D_j) = D'_j$. If Γ'_j is the stabilizer of D'_j in Γ, we get $\gamma\Gamma_j\gamma^{-1} = \Gamma'_j$. This equality means that Γ_j and Γ'_j are conjugate in $PSL(2, \mathbb{C})$, so they are Teichmüller equivalent. This proves that the coordinate α_j is independent of the component D_j, and our technique is well defined.

3.2.14. We now tackle the inverse problem, given a point $(\alpha_1, \ldots, \alpha_d)$ in the deformation space of the terminal regular b-group Γ, we want to find a group representing that point. We first consider the pair (orbifold, maximal partition) uniformized by Γ and we construct a graph, $\mathcal{G}$, from such pair. In §3.1.5 we have explained how to make that construction, but there was a degree of freedom in the choice of non-phantom weights. Here we put the restriction that any non-phantom edge has weight equal to ∞, since those are the Teichmüller space we are considering in this work. The coordinates α_j are constructed from the graph, and they are in one-to-one correspondence with the non-phantom edges. There are three types of non-phantom edges: (1) those that disconnect the graph, (2) those that do not disconnect the graph and join two distinct vertices and (3) those that join a vertex to itself. The way we will proceed is by induction: suppose we have constructed a group Γ_j corresponding to the point $(\alpha_1, \ldots, \alpha_j)$ and we want to construct from Γ_j a group corresponding to the point $(\alpha_1, \ldots, \alpha_j, \alpha_{j+1})$. The coordinate α_{j+1} corresponds to the edge e_{j+1}. We have three cases, depending on what type of edge e_{j+1} is.

CASE 1: e_{j+1} **disconnects the graph** $\mathcal{G}$.

Suppose that e_{j+1} joins the vertex v_k to v_l, where $k < l$. This means that the vertex v_k corresponds to some sphere with signature $(0, 3; \infty, \nu_1, \nu_2)$, where the ∞ comes from the weight of e_{j+1}. This sphere is uniformized by a triangle group $\Gamma(\infty, \nu_1, \nu_2; c, d, e)$ which is a subgroup of Γ_j. The ∞ in the signature corresponds to the vertex e_{j+1} and it is given by certain parabolic element, say A_{j+1}. The vertex v_l is uniformized by a triangle group $\Gamma(\infty, \nu_3, \nu_4; c', d', e')$, where the ∞ in its signature is given by the same parabolic element A_{j+1}. Then the group Γ_{j+1} is constructed as the AFP of the two smaller groups;

that is,

$$\Gamma_{j+1} = \Gamma_j *_{<A_{j+1}>} \Gamma(\infty, \nu_3, \nu_4; c', d', e').$$

Since A_{j+1} belongs to $\Gamma(\infty, \nu_3, \nu_4; c', d', e')$, we must have $c' = c$. Now, we want that this new group Γ_{j+1} represent certain given point in a defromation space. This means that the choice of parameters for our last triangle group must satisfy $cr(c, d, e, d') = \alpha_{j+1}$. This condition, together with the fact that A_{j+1} belongs to the triangle group, determines $\Gamma(\infty, \nu_3, \nu_4; c, d', e')$ completely.

CASE 2: e_{j+1} **does not disconnect the graph $\mathcal{G}$ but it joins two distinct vertices**

Suppose that v_k and v_l are the two vertices connected by e_{j+1}. These vertices correspond to spheres uniformized by the triangle groups $F_k =$ $= \Gamma(\infty, \nu_1, \nu_2; c, d, e)$ and $F_l = \Gamma(\infty, \nu_3, \nu_4; c', d', e')$ respectively. These two groups are subgroups of Γ_j. Let A_k and A_l be the parabolic elements corresponding to the ∞ in the signature of F_k and F_l respectively (and with the correct orientation requirements given in §3.2.2). Choose a Möbius transformation C such that $CA_kC^{-1} = A_l$ and $\alpha_{j+1} = cr(c, d, e, C(d'))$. Then the group Γ_{j+1} will be the following HNN extension

$$\Gamma_{j+1} = \Gamma_j *_{<C>} .$$

CASE 3: e_{j+1} **joins a vertex of $\mathcal{G}$ to itself**

Suppose that the vertex is v_k. This vertex will correspond to a sphere with signature $(0, 3; \infty, \infty, \nu)$, uniformized by the triangle group $\Gamma(\infty, \infty, \nu; c, d, e)$, which is a subgroup of Γ_j. The group Γ_{j+1} will be constructed by identifying the parabolic elements of this triangle group. More precisely, let A_k and B_k be the parabolic generators of $\Gamma(\infty, \infty, \nu; c, d, e)$. Find a Möbius transformation C such that $CB_k^{-1}C^{-1} = A_k$, and $\sqrt{\frac{2}{1+\cos(\pi/\nu)}}cr(c, d, e, C(d)) = \alpha_{j+1}$. Then

$$\Gamma_{j+1} = \Gamma_j *_{<C>} .$$

3.3 Elliptic Glueing

3.3.1. In this section we will study the deformation spaces of certain Kleinian groups obtained by AFP's across **finite** subgroups. These spaces fit in the general theory as follows: if we consider variations of the complex structure and the fundamental group of a Riemann surface we obtain the Teichmüller space of the surface. Forgetting the homotopy part produces Riemann spaces; that is, classes of biholomorphically equivalent conformal structures. But we may also consider deformations of the complex structure and *some subgroups of the fundamental group.* This process will give the type of deformation spaces we will study here.

These spaces also appear as the Teichmüller spaces of a Kleinian group with a *non-simply connected* invariant component of its region of discontinuity (these groups are known by the name of **function groups**; see [Mas88]). Deforming the group in that region is equivalent to deforming the complex structure of the quotient orbifold, and only part of the fundamental group, as the non-simply connectivity of the covering implies a lost of some topological information.

These deformation spaces are not simply connected, but by results of Lipman Bers, Bernard Maskit and Irwin Kra, we can identify their universal covering spaces with the Teichmüller spaces of the quotient surface. In this part of this chapter we will also consider the problem of identifying the covering group as certain subgroup of the mapping class group of the quotient surface.

3.3.2. After this general setting, let us explain in more detail what is the particular problem we want to look at. Suppose we have two (hyperbolic) triangle groups, F_1 and F_2, with signature $(0,3;\infty,\infty,n)$, where $2 < n < \infty$. We will assume that the groups share an elliptic element, say C, corresponding to the n of the signature. Then (under some additional hypothesis) we can form the AFP of F_1 and F_2 to obtain a Kleinian group Γ with an invariant component Δ. This open set Δ is NOT simply connected. The quotient Δ/Γ is a four times punctured sphere with a central distinguished curve, a, given by the projection of some curve invariant under C. In the non-invariant components, the group Γ represents two surfaces with signature $(0,3;\infty,\infty,n)$ (similar to the case of terminal regular b-groups of the previous section). We will find coordinates for the deformation space of the group Γ, as well as the relation with the deformation space of surfaces with signature $(0,4;\infty,\infty,\infty,\infty)$.

Figure 3.8 shows our AFP at the level of graphs.

Computations for more general AFP operations as well as the study of HNN extensions will appear somewhere else in the future.

3.3.3. This algebraic operation between the groups $\mathcal{F}_1$ and $\mathcal{F}_2$ is translated to the language of orbifolds as follows: each of these two groups uniformizes an orbifold with signature $(0,3;\infty,\infty,n)$, say $\mathcal{S}_1$ and $\mathcal{S}_2$. Consider

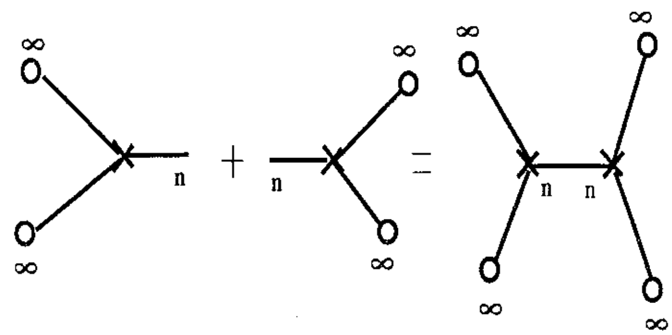


Figure 3.8: Elliptic gluing at the graph level

on each $\mathcal{S}_i$ the ramification point given by the n in the signature. Remove on each orbifold a disc around that point, and make a boundary identification to obtain a surface $\mathcal{S}$ with signature $(0, 4; \infty, \infty, \infty, \infty)$, and a partition given by the boundary identification curve, a. This surface is uniformized by the group $\Gamma = \mathcal{F}_1 *_{<C>} \mathcal{F}_2$ (C is the common elliptic element) on its invariant component Δ. But since this last set is not simply connected, we have that the covering $\Delta \to \Delta/\Gamma$ is not the universal covering. The defining group of this covering is the normal subgroup of the fundamental group of $\mathcal{S}$ generated by the n-th power of a (that is, a does not lift to a loop but a^n does). The deformation space of the group Γ is not the Teichmüller space of $\mathcal{S}$, because Γ is not the universal covering group of $\mathcal{S}$.

3.3.4. So to start to study the problem as stated in §3.3.2, let us consider a triangle group with signature $(0, 3; \infty, \infty, n)$. Our goal is to find coordinates for deformation spaces; this allows us to conjugate by a Möbius transformation (since we will remain in the same Teichmüller class) and assume that our starting group is $\Gamma(\infty,\infty,n;\infty,0,1)$. This group has a presentation given by three elements A, B, and C satisfying $ABC = I$, A and B are parabolic and $C^n = I$ (I is the identity). The matrices corresponding to those elements are

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ 1+q & -1 \end{bmatrix},\ C = \begin{bmatrix} 1 & -2 \\ 1+q & -1-2q \end{bmatrix},$$

where $q = \cos(\pi/n)$. We want to work with the elliptic element C; but computations are easier if we have that C is of the form $z \mapsto e^{2\pi i/n}z$. To get that formula for C all we need to do is conjugate by a Möbius transformation. Consider the fixed points of C, which are $z_0 = 1 + \frac{ip}{1+q}$, and $\overline{z}_0$, with $p = sin(\pi/n)$. Let M be a Möbius transformation taking z_0 to 0 and $\overline{z}_0$ to ∞. We still have one more degree of freedom to determine M, and we use it to require that M maps ∞ to 1 (we have observed that the computations are simpler with this choice). Then M is given by

$$M = \begin{bmatrix} w_0 & -w_0 z_0 \\ w_0 & -z_0 \end{bmatrix},\ w_0 = e^{\pi i/n} = q + ip.$$

Observe that M is given as an element of $PGL(2,\mathbb{C})$, not of $PSL(2,\mathbb{C})$. Conjugate $\Gamma(\infty,\infty,n;\infty,0,1)$ by M to obtain

$$M\Gamma(\infty,\infty,n;\infty,0,1)M^{-1} = \Gamma(\infty,\infty,n;1,w_0,\frac{w_0 - w_0 z_0}{w_0 - z_0}).$$

Denote this last group by $\mathcal{F}_1$, to simplify notation and to agree with the notation of §3.3.2. Then $\mathcal{F}_1$ is generated by

$$A_1 = MAM^{-1} = \begin{bmatrix} -1 + \frac{2w_0}{z_0(w_0-1)} & \frac{-2w_0}{z_0(w_0-1)} \\ \frac{2w_0}{z_0(w_0-1)} & -1 - \frac{2w_0}{z_0(w_0-1)} \end{bmatrix},$$

$$B_1 = MBM^{-1} = \begin{bmatrix} -1 + \frac{z_0(1+q)}{w_0-1} & \frac{-z_0w_0(1+q)}{w_0-1} \\ \frac{z_0(1+q)}{w_0(w_0-1)} & -1 - \frac{z_0(1+q)}{w_0-1} \end{bmatrix},$$

$$C_1 = MCM^{-1} = \begin{bmatrix} -w_0 & 0 \\ 0 & -w_0^{-1} \end{bmatrix}.$$

3.3.5. We want to consider another group $\mathcal{F}_2$, with the same signature, and to do an AFP of $\mathcal{F}_1$ and $\mathcal{F}_2$ across $< C_1 >$ to obtain a group Γ_η, (η is a parameter to be determined) whose deformation space we want to study. To find $\mathcal{F}_2$ we have to conjugate $\mathcal{F}_1$ by a transformation that preserves (commutes with) C_1. Since $n > 2$, the only elements of $PSL(2, \mathbb{C})$ that commute with C_1 are those of the form $z \mapsto \lambda^2 z$. Let $D_\eta = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ be one of those elements, with $\eta = \lambda^2$. Then

$$\mathcal{F}_2 = D_\eta \mathcal{F}_1 D^{-1} = \Gamma(\infty, \infty, n; \eta, \eta w_0, \eta \frac{w_0 - w_0 z_0}{w_0 - z_0})$$

is generated by

$$A_\eta = \begin{bmatrix} -1 - \frac{2w_0}{z_0(w_0-1)} & \frac{2w_0\eta}{z_0(w_0-1)} \\ \frac{-2w_0}{z_0(w_0-1)\eta} & -1 + \frac{2w_0}{z_0(w_0-1)} \end{bmatrix}, \; B_\eta = \begin{bmatrix} 1 - \frac{z_0(1+q)}{w_0-1} & \frac{z_0 w_0(1+q)\eta}{w_0-1} \\ \frac{-z_0(1+q)}{w_0(w_0-1)\eta} & -1 + \frac{z_0(1+q)}{w_0-1} \end{bmatrix},$$

$$C_\eta = C_1.$$

3.3.6. We now want to explain what condition η should satisfy so we can apply Maskit First Combination Theorem to obtain $\Gamma_\eta = \mathcal{F}_1 *_{<C_1>} \mathcal{F}_2$. The limit set of $\mathcal{F}_1$ is S^1, and a fundamental domain is given in the figure 3.8. $\mathcal{F}_2$ behaves like $\mathcal{F}_1$, except that its fundamental domain is modified by η (multiplied by its modulus and twisted by its argument).

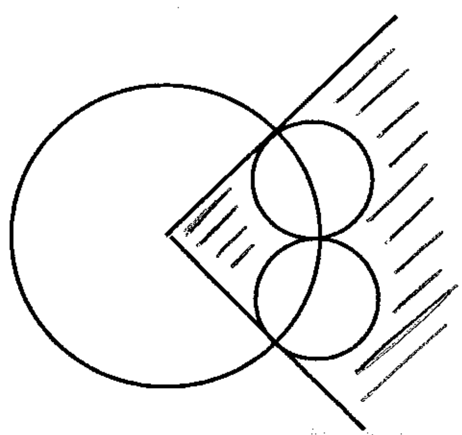


Figure 3.9: A fundamental domain for the group $\mathcal{F}_1$

The groups $\mathcal{F}_i$ uniformize orbifolds $\mathcal{S}_i$ with signature $(0,3;\infty,\infty,n)$ that are going to be glued after removing neighborhoods of the ramification points. The process of gluing is going to produce a new surface, with signature

$(0,4;\infty,\infty,\infty,\infty)$ and a central curve a. Let us assume that after the plumbing of the two orbifolds $\mathcal{S}_1$ and $\mathcal{S}_2$ we have that $\mathcal{S}_1$ lies to the right of a. Now, a lifts to a circle centered at the origin (since the element corresponding to it is the elliptic transformation C_1). We orient that circle by requiring that for any point on it, z, the triple $(z, C_1(z), C_1^2(z))$ is positively oriented; observe that this is possible since the order of C_1 is not 2. Then the surface corresponding to $\mathcal{F}_1$ is to the right of the circle that is the lift of a, while the one corresponding to $\mathcal{F}_2$ is to the left; this forces $|\eta| < 1$ (see subsection 3.2.2).

If the absolute value of η is small enough, we can apply Maskit First Combination Theorem. Among the hypothesis of that theorem, we have the existence of a simple closed curve W that divides the Riemann sphere into two topological discs (§1.1.13). In our case we can take a circle centered at the origin as W. Each of the groups $\mathcal{F}_i$ acts on two circles; in the case of $\mathcal{F}_1$, the action we are interested on takes place in the circle $\{z;\ |z| < 1\}$, while for $\mathcal{F}_2$ we will consider the action on the disc $\{z;\ |z| > \eta\}$. After the application of the Combination Theorem we get $\Gamma_\eta = \Gamma(\infty,\infty,n;1,w_0,\frac{w_0-w_0z_0}{w_0-z_0}) *_{<C_1>} \Gamma(\infty,\infty,n;\eta,\eta w_0,\eta\frac{w_0-w_0z_0}{w_0-z_0})$. This group Γ_η satisfies:

a) Γ_η has an invariant component contained on the annulus $\{z/|\eta| < |z| < 1\}$. The quotient of that component by the group is a four times punctured sphere;

b) the group Γ_η represents two surfaces of signature $(0,3;\infty,\infty,n)$ on its non-invariant components.

In order to estimate the size of those η for which we can apply Maskit theorem, we consider the circle that joins the points 1 and w_0 on the figure

3.8. Its center is at the point

$$c_0 = \frac{e^{\pi i/2n}}{cos(\pi/2n)}$$

and its radius is $r = |1 - c_0| = tan(\pi/2n)$. The closest point to the origin in this circle has modulus given by

$$|e^{\pi i/2n}(|c_0| - r)| = \frac{1 - sin(\pi/2n)}{cos(\pi/2n)}.$$

The furthest point to the origin is at distance

$$|e^{\pi i/2n}(|c_0| + r)| = \frac{1 + sin(\pi/2n)}{cos(\pi/2n)}.$$

For the group $\mathcal{F}_2$ we have the same type of expressions, but multiplied by η. This means that the point in the boundary of a fundamental domain for $\mathcal{F}_2$ with maximum absolute value is

$$\eta\frac{1 + sin(\pi/2n)}{cos(\pi/2n)}.$$

The Second Combination Theorem can be applied if the following inequality is satisfied

$$|\eta|\frac{1 + sin(\pi/2n)}{cos(\pi/2n)} < \frac{1 - sin(\pi/2n)}{cos(\pi/2n)}$$

which is equivalent to

$$|\eta| < \frac{1 - sin(\pi/2n)}{1 + sin(\pi/2n)}.$$

We also have is that the exterior of the unit disc is precisely invariant under F_1 in Γ_η. This gives

$$|\eta|\frac{1 + sin(\pi/2n)}{cos(\pi/2n)} < 1,$$

or equivalently

$$|\eta| < \frac{cos(\pi/2n)}{1 + sin(\pi/2n)}.$$

Also the disc

$$\{z/|z| < |\eta|\}$$

is precisely invariant under F_2 in the group, so $|\eta| < \frac{1-sin(\pi/2n)}{cos(\pi/2n)}$. But we have

$$\frac{1 - sin(\pi/2n)}{cos(\pi/2n)} = \frac{cos(\pi/2n)}{1 + sin(\pi/2n)}$$

.

Let $\eta_0 = \frac{1}{2}\frac{1-sin(\pi/2n)}{1+sin(\pi/2n)}$. Then the group determined by this value Γ_{η_0} satisfies all above conditions. For its deformation space we have the following result:

Theorem 9 *η is a coordinate for the deformation space $T(\Gamma_{\eta_0})$. The following inclusions are satisfied:*

$$\{\eta;\ 0 < |\eta| < \frac{1 - sin(\pi/2n)}{1 + sin(\pi/2n)}\} \subset \mathbf{T}(\mathbf{\Gamma}_{\eta_0}) \subset \{\eta;\ 0 < |\eta| < \frac{cos(\pi/2n)}{1 + sin(\pi/2n)}\}.$$

3.3.7. Our next goal is to understand the AFP as a plumbing construction. For that purpose, we need special coordinates centered at the branch points. For the group $\mathcal{F}_1$, the function $z = \zeta^n$ serves as such coordinate: it is invariant under $\zeta \mapsto C_1(\zeta) = w_0^2\zeta$, and maps the points $1, w_0$ to $1, -1$ respectively. This expression is different from the one obtained in section 2.5 because there the universal branched covering space was the upper half plane, while here the covering space is the unit disc. A simple geodesic, on the orbifold, from the branched point to the puncture represented by A, lifts to the unit interval, which is isometrically mapped, under the z coordinate, into the real axis with the metric

$$\frac{2|dz|}{n|z|^{n-1}(1-|z|^{n/2})},$$

the metric of the unit disc quotiented by a rotation of order n around the origin, as one can naturally expect.

For the group $\mathcal{F}_2$ we need a similar coordinate, but in this case we are interested on the action of the group on the set $\{z/|z|>|\eta|\}$, and we look at the geodesics that joins the branch point given by the ∞ to the fixed point of A^{-1}. That coordinate will be $w=(\eta/\zeta)^n$. We can construct the final orbifold by identification of the coordinates z and w as in the plumbing construction of the previous section . In this case we obtain $zw=t=\eta^n$. Estimates for this plumbing parameter can be found by raising to the n-th power the results of theorem 7 in subsection 3.3.6.

3.3.8. The above construction has given us a four times punctured sphere $\mathcal{S}$ uniformized the group Γ on the non-simply connected open set Δ. In that surface we have a distinguished curve, a. Let ϕ denote the Dehn twist around the curve a, as well as the corresponding element of the mapping class group of $\mathcal{S}$. We first state a result of Bers, Kra and Maskit that tells us what is the universal covering space of $T(\Gamma_{\eta_0})$:

Theorem 10 ([Ber70], [Mas71], [Kra72b]) *The holomorphic universal covering space of $T(\Gamma_{\eta_0})$ is the Teichmüller space $T(0,4)$.*

Given that result, we can identify the covering group.

Theorem 11 *The subgroup of the mapping class group of $\mathcal{S}$ generated by ϕ^n is the covering group of the mapping $T(0,4) \to T(\Gamma_{\eta_0})$*

Proof. We have the covering $\Delta_0 \to \mathcal{S} \cong \Delta_0/\Gamma_{\eta_0}$, where Δ_0 is the invariant component of the group Γ_{η_0}. Let H be the defining subgroup of this covering. Then H is the normal subgroup $\pi_1(\mathcal{S})$ generated by the curve a^n.

The covering group we are looking for, K, is a subgroup of the mapping class group of $\mathcal{S}$. This last group is the set of homeomorphisms of $\mathcal{S}$ modulo homotopy. And we also have that the mapping class group of $\mathcal{S}$ is equivalent to the group of automorphisms of $\pi_1(\mathcal{S})$ modulo inner conjugation. All this is classical, see for example [Nag88]. In [Mas71] (see also [Ear91]) we have a description of K; φ is in K if and only if it has a lifting $\tilde{\varphi}$ to Δ_0,

$$\begin{array}{ccc} \Delta_0 & \xrightarrow{\tilde{\varphi}} & \Delta_0 \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{S} & \xrightarrow{\varphi} & \mathcal{S} \end{array}$$

satisfying $\tilde{\varphi}\gamma(\tilde{\varphi})^{-1} = \gamma$ for all elements γ of Γ_{η_0}.

The mapping φ will have a lifting as above if the induced mapping on the fundamental group of the surface $\mathcal{S}$, φ_*, preserves the group H, that is $\varphi_*(H) = H$. Since this last group is generated by the curve a^n, we will have that $\varphi_*(a^n)$ is conjugated to a^n in $\pi_1(\mathcal{S})$. Inner automorphisms of the fundamental group are identified with trivial mappings in homotopy, so we can assume that $\varphi_*(a^n) = a^n$ (without changing φ as element of the mapping class group), which is equivalent to say that φ preserves the curve a. Since φ_* has to commute with the parabolic elements of Γ_{η_0} representing the punctures, we must have that φ has to preserve the punctures. The curve a divides $\mathcal{S}$ into two parts as the one shown in the figure 3.9 below. Therefore we get that φ has to preserve each part.

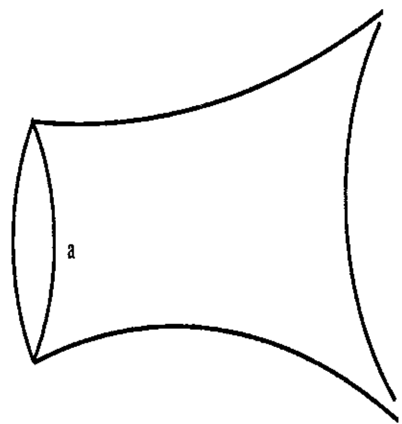


Figure 3.10: A part of an orbifold

Thus we have reduced the problem to characterize the homeomorphisms of $\mathcal{S}$ that preserve a and each of the parts in which a divides the surface, and such that a lifting to Δ_0, $\tilde{\varphi}$ commutes with all the elements of the group Γ_{η_0}. From surface topology (see [Deh87]) we know that the self-mappings of the manifold of the figure 3.9 that preserve the curve a are powers of the half Dehn twist around a. Since we want also to preserve the punctures, we will have that φ is a power of the full Dehn twist, which we have denoted by ϕ. Thus, $\varphi = \phi^k$. Now, if we lift the mapping ϕ to Δ_0, an easy computation shows that the action on Γ_{η_0} is conjugation by C_1. Therefore to preserve the punctures we need to have ϕ^n (since $C_1^n = I$) as claimed. □

3.3.9. We finish this section with a remark about horociclic coordinates and plumgin parameters. Let Γ be a terminal regular b-group of signature $(0, 4; \infty, \infty, \infty, \infty)$. The Teichmüller space $T(\Gamma)$ can be used as a model for

$T(0,4;\infty,\infty,\infty,\infty)$. Let α be a horociclic coordinate in this deformation space. In [Kra90], Irwin Kra showed that the Dehn twist that we have denoted by ϕ is given by $\alpha \mapsto \alpha + 2$. The plumgin parameter corresponding to this surfaces is given by $t = e^{\pi i \alpha}$. Now consider the group Γ_{η_0} of the above theorem. Let η be coordinate for $T(\Gamma_{\eta_0})$. The corresponding plumbing parameter is given by the expression η^n. One would like to have a relation between the two horociclic coordinates. A possible formula is $\eta = e^{\pi i \alpha / n}$. This expression would be good in the sense that behaves well with respect to the plumbing parameters. But this is only a conjecture that we have not been able to prove.

Chapter 4

Patterson Isomorphisms

4.1 Patterson Isomorphisms Theorem

4.1.1 One of the most natural questions one may ask about Teichmüller spaces is under what circumstances are two of these spaces biholomorphically equivalent. One answer was given by Bers and Greenberg, and it says that two Fuchsian groups of the same type (p,n) have isomorphic Teichmüller spaces, independently of their signatures. We will say more about this result in the next chapter.

4.1.2. Another well known case is the equivalence between T(1,0) and T(1,1), the Teichmüller spaces of tori and tori with one puncture. This is due to the fact that the group of automorphisms of the torus acts transitively. More precisely, we have that the Teichmüller space of the torus is the upper half plane, where a point τ in $\mathbf{H}$ represents the torus

$$T_\tau = \mathbb{C}/\{z \mapsto z + n + m\tau; n, m \in \mathbb{Z}\}.$$

Puncture the torus at the point $[z_0]$ (where the brackets mean the equivalence class by the action of the group of translations). Consider the mapping defined on $\mathbb{C}$ by $z \overset{f}{\mapsto} z - z_0$. This mapping induces an automorphisms on the torus T_τ. We also have that f is homotopic to the identity and $f(z_0) = 0$. This means that the torus punctured at $[z_0]$ and the torus punctured at $[0]$ are in the same Teichmüller class; see subsection 1.2.7. Therefore $T(1,0) \cong T(1,1)$. This is a very special case that happens only on the torus, since for surfaces of higher genus Hurtwiz theorem says that the group of conformal automorphisms is finite.

4.1.3. If we exclude this case and the trivial ones, namely when the deformation space is just one point (sphere with zero, one, two or three punctures), then we have a result of Patterson that tells us that there are only three other cases of isometric Teichmüller spaces.

Theorem 12 ([Pat72], [EK74]) *The only biholomorphisms between two distinct Teichmüller spaces T(p,n) (with $2p - 2 + n > 0$) occur precisely for the cases:*

$$T(2,0) \cong T(0,6), \;\; T(1,2) \cong T(0,5), \;\; T(1,1) \cong T(0,4).$$

4.1.4. The above isomorphisms use the fact that all surfaces of genus 2 or 1 (with one or two punctures) are hyperelliptic. This means that they admit a conformal mapping of order 2. In the case of surfaces of genus 2 we have that the hyperelliptic involution has 6 fixed points, the Weierstrass points of the surface. If the surface has signature $(1,2;\infty,\infty)$, then the involution interchanges the two punctures and fixes four other points. For surfaces with signature $(1,1;\infty)$ we have that the hyperelliptic involution fixes the puncture and three more points. Taking the quotient of the surfaces by the corresponding involution we obtain orbifolds with signature (0,6;2,2,2,2,2,2), (0,5;∞,2,2,2,2) and (0,4;∞,2,2,2) respectively; the correspondence between a surface and the quotient orbifold gives the desired isomorphism at the level of Teichmüller spaces.

4.2 The Patterson Isomorphisms in the horocyclic coordinates

4.2.1 As stated in the previous section, the surfaces with signatures $(2,0)$, $(1,2;\infty,\infty)$ and $(1,1;\infty)$ admit an involution that gives rise to an isomorphism at the level of Teichmüller spaces. In this section we will construct the isomorphism explicitly in the coordinates of chapter 3. The idea is to find a Möbius transformation that represents the hyperelliptic involution and compute explicitly in some representation of a group uniformizing the surface. We will work first the case (2,0), and from it we will obtain the the formulae for the other isomorphisms.

4.2.2. In order to write down the main result of this chapter, we need to consider some orbifolds and graphs. In the figure 4.1 we have three graphs corresponding to three surfaces with signatures (2,0), (1,2;∞) and (1,1;∞). Observe that the hyperelliptic involution is given by a rotation of 180 degrees around a horizontal line that goes through the 'middle' of the surfaces. The corresponding graphs associated to the quotient orbifolds are in figure 4.2.

Theorem 7 in section 3.2. gives a set of coordinates for the Teichmüller

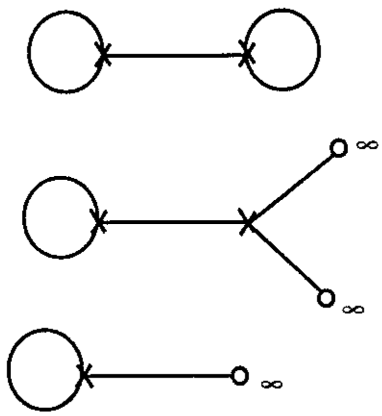


Figure 4.1: Graphs associated to hyperelliptic surfaces

spaces of the surfaces constructed from the graphs of figures 4.1 and 4.2. It it in these particular sets of coordinates in which we will compute the Patterson isomorphisms. The main result of this chapter is then as follows:

Theorem 13 *The mapping*

$$(\tau_1, \tau_2, \tau_3) \mapsto (\frac{\tau_1}{2}, 1 + \tau_2, 1 + \frac{\tau_3}{2})$$

gives an isomorphism between $T(2,0)$ *and* $T(0,6;2,2,2,2,2,2)$. *If we make* $\tau_3 = 0$ *or* $\tau_3 = \tau_2 = 0$, *then we obtain the isomorphisms* $T(1,2;\infty) \cong T(0,5;\infty,2,2,2,2)$ *and* $T(1,1;\infty) \cong T(0,4;\infty,2,2,2)$, *respectively.*

The proof will be as follows: first we will give a presentation for a terminal regular b-group Γ uniformizing a surface of genus 2; we then compute the Möbius transformation $A_2^{1/2}$ that represents the hyperelliptic involution in the quotient surface; we prove that the group generated by Γ and $A_2^{1/2}$, say $\mathcal{G}$, is a terminal regular b-group uniformizing a surface with signature $(0,6;2,2,2,2,2,2)$; the next step is to give a presentation of a terminal regular b-group $\mathcal{F}$ following the

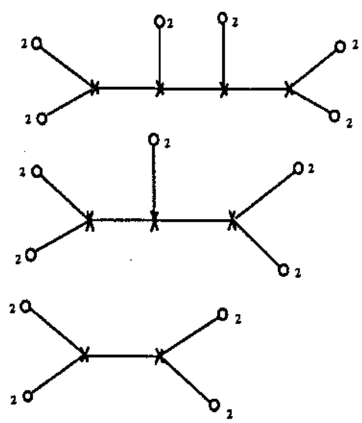


Figure 4.2: Graphs associated to orbifolds

techniques of chapter 3; the last step will be to find a Möbius transformation E such that $E\mathcal{F}E^{-1} = \mathcal{G}$, and this gives the desired isomorphisms.

4.2.3. We start with a surface of genus 2, with no punctures, and a maximal partition given by the curves a_1, a_2 and a_3 in the figure 4.3.

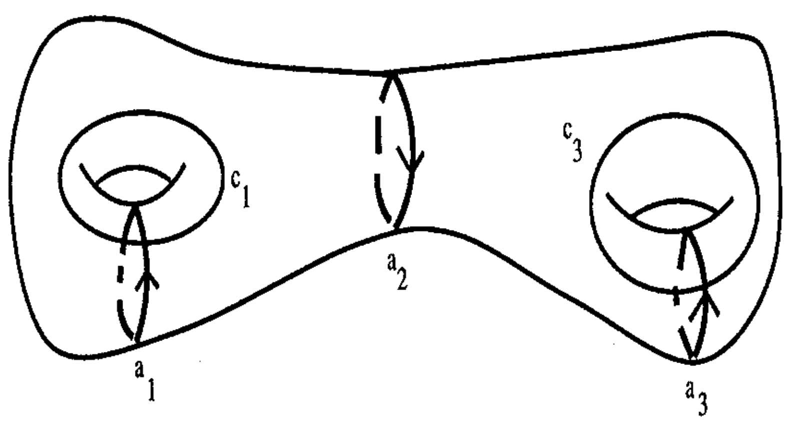


Figure 4.3: A surface of genus 2.

This surface is uniformized by a terminal regular b-group, Γ, in its simply connected invariant component Δ. Γ has the following presentation:

$\Gamma =< A_1,\ C_1,\ A_3,\ C_3;\ A_1,\ A_2 = [C_1^{-1}, A_1],\ A_3$ are accidental parabolic, $[A_1, C_1^{-1}] \circ [A_3^{-1}, C_3^{-1}] = I >$, where $[A,B] = ABA^{-1}B^{-1}$. The elements A_i correspond to the curves a_i, while C_i correspond to c_i. The presentation for Γ and the formulae for the above transformations have been computed by I. Kra in [Kra90]. We will copy them here for the sake of completeness:

$$A_1 = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1-2\tau_2(1-\tau_2) & -2(1-\tau_2)^2 \\ 2\tau_2^2 & -1+2\tau_2(1-\tau_2) \end{bmatrix},\ C_1 = i\begin{bmatrix} \tau_1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C_3 = i\begin{bmatrix} \tau_3\tau_2^2+2(1-\tau_3)\tau_2+\tau_3-2 & -\tau_3\tau_2+(3\tau_3-2)\tau_2-2\tau_3+3 \\ \tau_3\tau_2+(2-\tau_3)\tau_2-1 & -\tau_3\tau_2^{-}2(1-\tau_3)\tau_2+2 \end{bmatrix}.$$

τ_1, τ_2 and τ_3 are complex numbers.

4.2.4. The hyperelliptic involution is given by the transformation

$$A_2^{1/2} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

It is not hard to see that this element conjugates the group Γ into itself. But we have to prove that the group generated by Γ and $A_2^{1/2}$ is terminal regular b-group of signature (0,6;2,2,2,2,2,2):

Proposition 10 *The group $\mathcal{G}$ of Möbius transformations generated by Γ and $A_2^{1/2}$ is a terminal regular b-group of signature* $(0,6;2,2,2,2,2,2)$.

Proof. The proof is in several steps: first we need to show that $\mathcal{G}$ is discrete and geometrically finite; then we will prove that it has a simply connected invariant component; the last part is to prove that outside the invariant component, $\mathcal{G}$ represents orbifolds of type (0,3).

Step 1: $\mathcal{G}$ is Kleinian. This follows from the proposition V.E.10 in page 98 of [Mas88]. We also obtain that $\Omega(\mathcal{G}) = \Omega(\Gamma)$.

Step 2: $\mathcal{G}$ is geometrically finite. Again we have this result in Maskit's book, [Mas88], proposition VI.E.6 in page 132.

Step 3: $\mathcal{G}$ has a simply connected invariant component. Let Δ be the simply connected invariant component of the group Γ. If $A_2^{1/2}(\Delta) = U$, where U is a component of $\Omega(\Gamma) = \Omega(\mathcal{G})$, then for all elements $\gamma \in \Gamma$ we have that $A_2^{1/2}\gamma A_2^{-1/2}(U) = U$. But $A_2^{1/2}\gamma A_2^{-1/2} \in \Gamma$, so we have that U is invariant under Γ, and therefore $U = \Delta$. So both groups, $\mathcal{G}$ and Γ have the same simply connected invariant component.

Since the element $A_2^{1/2}$ is a lifting of the hyperelliptic involution, it is clear that the surface $\Delta/\mathcal{G}$ has signature (0,6;2,2,2,2,2,2).

Step 4: $(\Omega(\mathcal{G}) - \Delta)/\mathcal{G}$ is a union of orbifolds of type (0,3). Let Ω_0 be a component of the set $\Omega(\mathcal{G})-\Delta$. Consider its stabilizer in Γ, $\Gamma_0 = stab(\Omega_0, \Gamma) = \{\gamma \in \Gamma; \gamma(\Omega_0) = \Omega_0\}$. Then, since Γ is a terminal regular torsion free b-group, the orbifold Ω_0/Γ_0 is a surface with signature $(0,3;\infty,\infty,\infty)$. If $A_2^{1/2}(\Omega_0) \neq \Omega_0$, then the groups $\mathcal{G}$ and Γ give the same quotient orbifold, Ω_0/Γ_0.

If to the contrary, $A_2^{1/2}(\Omega_0) = \Omega_0$, we then have $A_2^{1/2}\Gamma_0 A_2^{-1/2} = \Gamma_0$, which means that $A_2^{1/2}$ induces an automorphism on Ω_0/Γ_0. Since $A_2^{1/2}$ belongs to Γ_0, so does $(A_2^{1/2})^2 = A_2$. And because $A_2^{1/2}$ and A_2 commute, we have that the automorphism induced by $A_2^{1/2}$ fixes one of the punctures of Ω_0/Γ_0. Obviously that automorphism is not the identity (because $A_2^{1/2} \notin \Gamma_0$), so it has to interchange the other two punctures of Ω_0/Γ_0; and it has order 2. Therefore

$$\Omega_0/ < \Gamma_0,\ A_2^{1/2} >= (\Omega_0/\Gamma_0)/ < A_2^{1/2} >$$

is a surface with signature $(0,3;\infty,\infty,2)$ as we wanted to prove. □

4.2.5. We have that $\mathcal{G}$ is a terminal regular b-group uniformizing a surface of signature $(0,6;2,2,2,2,2,2)$ in the invariant component Δ. A presentation for this group is given by $\mathcal{G} =< A_1,\ A_3,\ C_1,\ C_3,\ A_2^{1/2};\ A_1,\ A_2^{1/2},\ A_3$ are accidental parabolic, $A_2^{-1/2}C_1^{-1}, C_1A_2^{1/2}A_1,\ A_1^{-1}A_2^{-1/2},\ A_2^{1/2}A_3,\ C_3A_2^{-1/2},\ A_2^{-1/2}C_3^{-1}A_3^{-1}$ are elliptic elements of order 2 and the product of them is the identity $>$.

We will not need the matrices of all these transformation for our computation, so we write only those needed. They correspond to accidental parabolic or branch points as shown in figure 4.4.

The formulae of these elements are the following:

$$(C_1A_2^{1/2}) = i\begin{bmatrix} -1 & 2+\tau_1 \\ 0 & 1 \end{bmatrix},\ A_1^{-1} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix},\ A_2^{1/2} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},$$

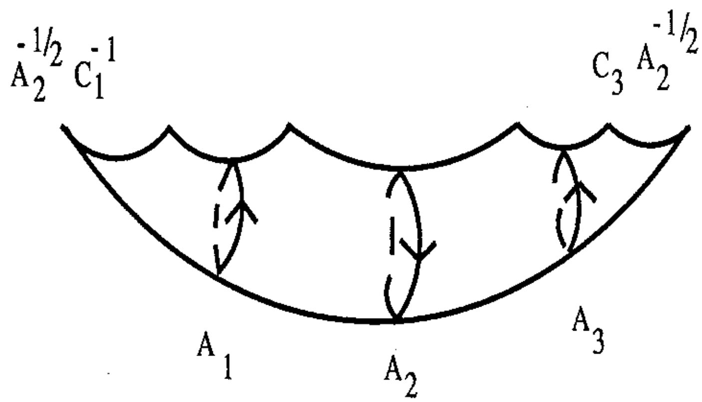


Figure 4.4: Surface of genus 2 quotiented by the hyperelliptic involution

$$A_3^{-1} = \begin{bmatrix} -1+2\tau_2(1-\tau_2) & 2(1-\tau_2)^2 \\ -2\tau_2^2 & -1-2\tau_2(1-\tau_2) \end{bmatrix},$$

$$C_3A_2^{-1/2} = i\begin{bmatrix} -1+2\tau_2-\tau_2\tau_3+\tau_2^2\tau_3 & 2-\tau_3-2\tau_2-\tau_2^2\tau_3+2\tau_2\tau_3 \\ 2\tau_2+\tau_2^2\tau_3 & 1-2\tau_2+\tau_2\tau_3-\tau_2^2\tau_3 \end{bmatrix}.$$

4.2.6. So far we have a group $\mathcal{G}$ uniformizing a surface of signature $(0,6;2,2,2,2,2,2)$. One possible way of finishing the proof of the theorem (for the case of surfaces of genus 2) is by reading from the presentation of the group $\mathcal{G}$ its coordinates in the deformation space $T(0,6;2,2,2,2,2,2)$. Another possible approach is by finding a another terminal regular b-group $\mathcal{F}$, with the signature (0,6;2,2,2,2,2,2), but constructed following the techniques of chapter 3, and a Möbius transformation E such that $E\mathcal{F}E^{-1} = \mathcal{G}$. It is this last point of view the one we will take.

4.2.7. The group $\mathcal{F}$ uniformizes an orbifold as the one in figure 4.5.

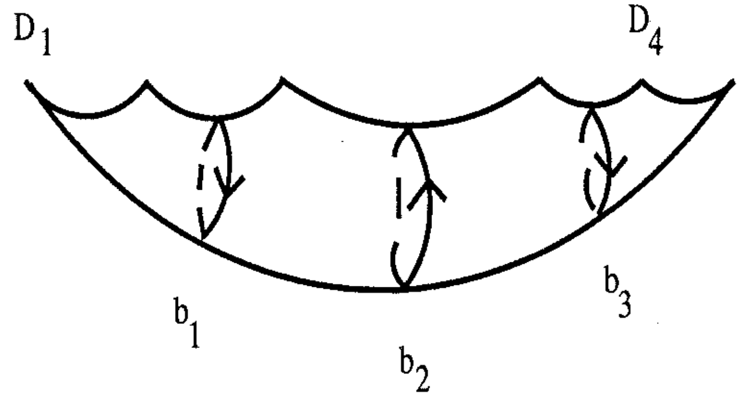


Figure 4.5: A surface with signature (0,6;2,2,2,2,2,2)

The curves b_i are represented by the accidental parabolics B_i, whose formulae are

$$B_1 = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} -1-\alpha & \alpha^2 \\ -1 & -1+\alpha \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -1+2\beta+2\alpha\beta^2 & -2(1+\alpha\beta)^2 \\ 2\beta^2 & -1-2\beta-2\alpha\beta^2 \end{bmatrix}.$$

As it happened with the group $\mathcal{G}$, we do not need to know the matrices corresponding to all the elliptic elements, it suffices with the following two:

$$D_1 = i\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_4 = i \begin{bmatrix} -1 - 2\alpha\beta + 2\alpha\gamma + 2\gamma\beta^{-1} & -2(1+\alpha\beta)(-\alpha\beta^2 + \gamma + \alpha\beta\gamma)\beta^{-2} \\ 2\gamma - 2\beta & 1 + 2\alpha\beta - 2\alpha\gamma - 2\gamma\beta^{-1} \end{bmatrix},$$

these two last elements are elliptic of order 2, and represent small curves around the branch points as marked in the figure. α, β and γ are three complex numbers.

4.2.8 To find our transformation E we look at the figures 4.4 and 4.5. We realize then that E should conjugate $B_1 \in \mathcal{F}$ to $A_1^{-1} \in \mathcal{G}$. This is achieved if E is of the form

$$i \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}.$$

Now, we must have $EB_2E^{-1} = A_2^{1/2}$, which gives $b = -1 - \alpha$.

The accidental parabolics corresponding to the third curve are $B_3 \in \mathcal{F}$ and $A_3 \in \mathcal{G}$. The transformation E has to conjugate the first of these two elements into the inverse of the second element; that is, $EB_3E^{-1} = A_3^{-1}$, or equivalently $EB_3 = A_3^{-1}E$. For the elliptic transformations we have the following identities: $ED_1E^{-1} = A_2^{-1/2}C_1^{-1}$ and $ED_4E^{-1} = C_3A_2^{-1/2}$. We should remark here that all these equalities are identities between Möbius transformations; as equalities in $PSL(2,\mathbb{C})$ we are free to multiply all entries of the matrices by a non-zero constant.

Let us start by considering the identity $ED_1 = A_2^{-1/2}C_1^{-1}E$, which is

$$\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i & -i(2+\tau_1) \\ 0 & -i \end{bmatrix}\begin{bmatrix} -1 & -b \\ 0 & 1 \end{bmatrix}.$$

It is easy to see that if the above identity is satisfied, then

$$\alpha = \frac{\tau_1}{2}.$$

Let us look now at $EB_3 = A_3^{-1}E$, or

$$\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1+2\beta+2\alpha\beta^2 1 & -2(1+\alpha\beta)^2 \\ 2\beta^2 & -1-2\beta-2\alpha\beta^2 \end{bmatrix} =$$

$$\begin{bmatrix} -1+2\tau_2(1-\tau_2) & 2(1-\tau_2)^2 \\ -2\tau_2^2 & -1-2\tau_2(1-\tau_2) \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}.$$

This gives

$$\begin{cases} -1+2\beta+2\alpha\beta^2+2\beta^2 b & = & 1+2\tau_2(1-\tau_2) \\ -2(1+\alpha\beta)^2+b(-1-2\beta-2\alpha\beta^2) & = & -b(-1-2\tau_2+2\tau_2^2)-2(1-\tau_2)^2 \\ -2\beta^2 & = & -2\tau_2^2 \\ 1+2\beta+2\alpha\beta^2 & = & -2\tau_2^2 b-1+2\tau_2(1-\tau_2) \end{cases}$$

The third equation gives $\beta = \pm\tau_2$. It is an easy computation to see that the correct answer is

$$\beta = \tau_2.$$

Finally we have $ED_4 = C_3 A_2^{-1/2} E$, which is

$$\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 - 2\alpha\beta + 2\alpha\gamma + 2\gamma\beta^{-1} & -2(1+\alpha\beta)(-\alpha\beta^2 + \gamma + \alpha\beta\gamma)\beta^{-2} \\ 2\gamma - 2\beta & 1 + 2\alpha\beta - 2\alpha\gamma - 2\gamma\beta^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 2\tau_2 - \tau_2\tau_3 + \tau_2^2\tau_3 & 2 - \tau_3 - 2\tau_2 - \tau_2^2\tau_3 + 2\tau_2\tau_3 \\ 2\tau_2 + \tau_2^2\tau_3 & 1 - 2\tau_2 + \tau_2\tau_3 - \tau_2^2\tau_3 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}.$$

To solve this equality we need to consider only the elements in the first column of both matrices. This is due to the facts the transformations D_4 and $C_3 A_2^{-1/2}$ have zero trace and determinant equal to 1. The equations we need are

$$\begin{cases} -1 - 2\alpha\beta + 2\alpha\gamma + 2\gamma\beta^{-1} + b(2\gamma - 2\beta) & = & -1 + 2\tau_2 - \tau_2\tau_3 + \tau_2^2\tau_3 \\ 2\beta - 2\gamma & = & 2\tau_2 + \tau_2^2\tau_3 \end{cases}$$

The solution to these equations is

$$\gamma = -\frac{\tau_2^2\tau_3}{2}.$$

4.2.9. In [Kra90] it is proven that (τ_1, τ_2, τ_3) are complex coordinates for the space $T(2,0)$. The coordinates for $\mathrm{T}(0,6;2,2,2,2,2,2)$ in terms of the numbers α, β, γ are given by

$$z_1 = \alpha,\ z_2 = 1 + \beta,\ z_3 = 1 - \frac{\gamma}{\beta^2}.$$

Then we have that the isomorphism between the above two deformation spaces is given by

$$(\tau_1, \tau_2, \tau_3) \mapsto (\frac{\tau_1}{2}, 1 + \tau_2, 1 + \frac{\tau_3}{2}).$$

4.2.10 Now we are in conditions to finish the proof of theorem 12. The argument is as follows: to construct the surface of genus 2 with the partition given in the figure 4.3. we have to follow these steps:

1.- start with a three times puncture sphere, S_1;

2.- glue two of the punctures, obtaining a surface S_2 with signature $(1, 1; \infty)$; this involves the coordinate τ_1;

3.- glue to the puncture of S_2 a three times punctured sphere to get a surface S_3 with signature $(1, 2; \infty, \infty)$; we need here the coordinate τ_2;

4.- glue the two punctures of S_3 to obtain the surface of genus 2.

Then we see that after step 2 we have only one coordinate and a torus with a puncture; and after step 3 we have two coordinates and a torus with two punctures. Therefore, if we consider only the mapping involving the first coordinate, that is

$$\tau_1 \mapsto \frac{\tau_1}{2},$$

we have an isomorphism between $T(1, 1; \infty)$ and $T(0, 4; \infty, 2, 2, 2)$. Similarly, the mapping involving two coordinates,

$$(\tau_1, \tau_2) \mapsto (\frac{\tau_1}{2}, 1 + \tau_2)$$

provides the isomorphism $T(1,2;\infty,\infty) \cong T(0,5;\infty,2,2,2,2)$ completing the proof of theorem 12.

Chapter 5

The Bers-Greenberg Isomorphism Theorem

5.1 Statement of the Main Result

5.1.1. In this chapter we will study the Bers-Greenberg Isomorphism Theorem (see §4.1.1), which states that two deformation spaces of Fuchsian groups of the same **type** are isomorphic, independently of the signatures. A purely topological proof of this fact appeared in [Mar69]. Bers and Greenberg provided a version (for Fuchsian groups) in [BG71]; later, Irwin Kra gave a much shorter demonstration using Teichmüller's theorem (see [EK74]). What we want to do here is to relate the Bers-Greenberg theorem and the Maskit coordinates for deformation spaces. We will show that, in the Maskit embedding, the Bers-Greenberg isomorphism is a product map of isomorphisms between one-dimensional Teichmüller spaces. The proof will combine the basic properties of the Maskit embedding and the ideas of [EK74].

5.1.2. We know state the main result of this chapter. For the background and the technical points, see sections §5.2, §5.3 and the first chapter.

Let $\mathcal{S}$ be an orbifold with hyperbolic signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$, and assume that at least one of the ν_j is finite. Let $\mathcal{C}$ be a maximal partition on $\mathcal{S}$. Assume that the triple $(\mathcal{S}, \sigma, \mathcal{C})$ is uniformized by the regular terminal b-group Γ. Remove from $\mathcal{S}$ the ramification points; that is, consider the surface $\mathcal{S}_0 = \mathcal{S} - \{x_j / \ x_j$ is a ramification point with ramification value $\nu_j < \infty, \ 1 \leq j \leq n\}$. Assume that the regular terminal b-group Γ_0 uniformizes the triple $(\mathcal{S}_0, (p, n; \infty, \ldots, \infty), \mathcal{C})$. The statement of our result is the following:

Theorem 14 *Let*

$$T(\Gamma_0) \hookrightarrow \prod_{j=1}^{3p-3+n} T(\Gamma_0^j) \qquad and \qquad T(\Gamma) \hookrightarrow \prod_{j=1}^{3p-3+n} T(\Gamma^j)$$

be the Maskit embeddings of $T(\Gamma_0)$ *and* $T(\Gamma)$ *determined by the above orbifolds and partitions. Then there exist isomorphisms*

$$h_j^* : T(\Gamma_0^j) \to T(\Gamma^j),$$

for $1 \leq j \leq 3p - 3 + n$, *such that the restriction*

$$(h_1^*, \ldots, h_{3p-3+n}^*)|_{T(\Gamma_0)} : T(\Gamma_0) \to T(\Gamma)$$

is an isomorphism.

5.1.3. The rest of this chapter is organized as follows: in the next section we introduce a more analytic approach to deformation theory. In §5.3 we provide the proof of Bers-Greenberg Theorem that appears in [EK74], adapted to the case of b-groups. Finally, in the last section we give the proof of the main result.

5.2 Another way of looking at Teichmüller spaces

5.2.1. Let Γ be a Kleinian group with an simply connected invariant component Δ, and suppose that Δ/Γ is an orbifold of type (p,n). The space of (classes of) essentially bounded measurable functions μ, supported on Δ, and satisfying $(\mu \circ \gamma)\frac{\overline{\gamma'}}{\gamma'} = \mu$, for all elements $\gamma \in \Gamma$, is denoted by $L^{\infty}(\Gamma, \Delta)$. The unit ball in this space is denoted by $M(\Gamma, \Delta)$, and its elements are known as **Beltrami coefficients** for the group Γ. Without loss of generality we can assume that the triple $(0,1,\infty)$ is in the limit set of Γ. Given a Beltrami coefficient μ, we solve the equation $w_{\bar{z}} = \mu w_z$ under the conditions that w fixes $\{0,1,\infty\}$ pointwise. This gives a unique solution, w^{μ}, which is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ onto itself with dilatation μ. It is not difficult to check μ is in $M(\Gamma, \Delta)$ if and only if $w^{\mu} \circ \gamma \circ (w^{\mu})^{-1} \in PSL(2, \mathbb{C})$, for all elements γ of Γ. Two Beltrami coefficients μ_1 and μ_2 will be considered **equivalent** if $w^{\mu_1} \circ \gamma \circ (w^{\mu_1})^{-1} = w^{\mu_2} \circ \gamma \circ (w^{\mu_2})^{-1}$ for all $\gamma \in \Gamma$. The set of equivalence classes is the **deformation** or **Teichmüller** space of the group Γ supported on Δ, and it will be denoted by $T(\Gamma, \Delta)$. If Γ is a terminal regular b-group, then this definition of Teichmüller space is equivalent to the ones of chapter 1 (see [Gar87]).

The space $L^1(\Gamma, \Delta)$ consists of (classes of) measurable functions f supported on Δ such that $(f \circ \gamma)(\gamma')^2 = f$, for all γ in Γ, and with finite norm

$$||f|| = \frac{1}{2} \int \int_{\Delta/\Gamma} |f(z)\, dz \wedge d\bar{z}| < +\infty.$$

There exist a natural pairing between $L^1(\Gamma,\Delta)$ and $L^\infty(\Gamma,\Delta)$ given by

$$(f,\mu) = \frac{1}{2}\int\int_{\Delta/\Gamma} f(z)\mu(z)|dz\wedge d\bar{z}|,$$

which identifies the dual space of $L^1(\Gamma,\Delta)$ with $L^\infty(\Gamma,\Delta)$. The holomorphic functions in $L^1(\Gamma,\Delta)$ form the space of **quadratic differentials**, which is denoted by $Q(\Gamma,\Delta)$. The above pairing identifies $Q(\Gamma,\Delta)^*$ (where the $*$ means the dual space) with $L^\infty(\Gamma,\Delta)/Q(\Gamma,\Delta)^\perp$, where

$$Q(\Gamma,\Delta)^\perp = \{\mu\in L^\infty(\Gamma,\Delta);\ (f,\mu)=0\,,\forall f\in Q(\Gamma,\Delta)\}.$$

5.2.2. Let us assume that F is a Fuchsian group acting on the upper half plane H, and H/F is an orbifold of type (p,n). Then we have two results that give the complex structure of the deformation spaces.

Theorem 15 (Bers [Ber66]) *$T(F,\mathsf{H})$ has a unique complex structure so that the canonical projection $\pi : M(F,\mathsf{H}) \longrightarrow T(F,\mathsf{H})$ is holomorphic with local holomorphic sections. The dimension of $T(F,\mathsf{H})$ is 3p-3+n. The cotangent space to $T(F,\mathsf{H})$ at $\pi(0)$ can be identified with the space of quadratic differentials $Q(F,\mathsf{H})$.*

Theorem 16 (Bers [Ber70], Kra [Kra72b], Maskit [Mas71]) *The space $T(\Gamma,\Delta)$ is naturally isomorphic to $T(F,\mathsf{H})$ for a Fuchsian F group such that F/H is conformally equivalent to Δ/Γ, as orbifolds.*

5.2.3. A **Teichmüller differential** is a Beltrami coefficient of the form $k\frac{\overline{\varphi}}{|\varphi|}$, where $\varphi\in Q(\Gamma,\Delta)$ and $0\le k<1$.

Theorem 17 (Teichmüller, [Ahl54], [Ber60]) *In each class of Beltrami coefficients there is a unique Teichmüller differential.* $T(\Gamma, \Delta)$ *is homeomorphic to the open unit ball of* $\mathbb{R}^{6p-6+2n}$.

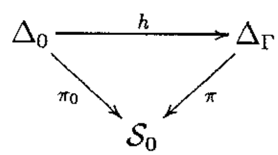
5.3 The proof of the Bers-Greenberg Theorem using terminal b-groups

5.3.1. Let $\mathcal{S}$ be an orbifold with hyperbolic signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$, and maximal partition $\mathcal{C}$ uniformized by the terminal regular b-group Γ. The deformation space of this group on its invariant component Δ will be denoted by $T(\Gamma)$. If we now assume that Γ' is another group with invariant component Δ' and that Δ'/Γ' has hyperbolic signature $\sigma' = (p, n; \infty, \ldots, \infty)$, then we may consider as well the deformation space $T(\Gamma')$. In this context we have the Bers-Greenberg theorem,that says that the important fact is not the particular ramification numbers but the genus and the number of special points. We remark that this theorem was stated for Fuchsian groups, but the results of the previous section allow us to rewrite it in terms of b-groups.

Theorem 18 (Bers-Greenberg [BG71], [EK74]) *The spaces* $T(\Gamma)$ *and* $T(\Gamma')$ *are biholomorphically equivalent.*

5.3.2. Proof. The following proof appears in [EK74] in the Fuchsian group setting, but we will reproduce it for terminal regular b-groups, since it also works in that case. Let us start by considering the orbifold $\mathcal{S}$ with the hyperbolic signature $\sigma = (p, n; \nu_1, \ldots, \nu_n)$ and the maximal partition $\mathcal{C}$

uniformized by the regular terminal b-group Γ on its simply connected invariant set Δ. We will assume that at least one of the $\nu_j' s$ is finite. Let x_j, $j \in \{1, \ldots, n\}$, denote the ramification points of the orbifold $\mathcal{S}$. Let $\mathcal{S}_0 = S - \{x_j;\ 1 \leq j \leq n,\ \nu_j < \infty\}$. Since $\mathcal{C}$ is also a maximal partition for $\mathcal{S}_0$, we have another terminal regular b-group Γ_0 uniformizing the triple $(\mathcal{S}_0, (p, n; \infty, \ldots, \infty), \mathcal{C})$ on its invariant component Δ_0. Consider the set $\Delta_\Gamma = \Delta - \{\text{fixed points of elliptic elements of } \Gamma\}$. Then $\Delta_\Gamma / \Gamma \cong \mathcal{S}_0$. Since Δ_Γ is not simply connected, we have a holomorphic covering map, h, from Δ_0 onto Δ_Γ that makes the following diagram commutative (the mappings π and π_0 are the natural projections):



The mapping h is defined, locally, as $h = \pi^{-1} \circ \pi_0$.

Claim: h induces a group homomorphism $\chi : \Gamma_0 \to \Gamma$ defined by the relation $h \circ \gamma = \chi(\gamma) \circ h, \forall \gamma \in \Gamma_0$.

To see this, suppose that γ belongs to Γ_0; then $\pi_0 \circ \gamma = \pi_0$, which implies that $\pi \circ h \circ \gamma = \pi_0 \circ \gamma = \pi_0 = \pi \circ h$. This means that the functions h and $h \circ \gamma$ are both liftings of the function π_0, in the above diagram. Therefore there exists a deck transformation $\chi(\gamma) \in \Gamma$, which satisfies the relation $h \circ \gamma = \chi(\gamma) \circ h$ as we claimed. ##

It is clear from the topology involved in the above diagram that the group homomorphism χ takes the accidental parabolics of Γ_0 onto those of Γ. With parabolics, which are not accidental, the situation is different: if a parabolic corresponds to a loop around a puncture on S_0 **which was also a puncture of** $\mathcal{S}$, then it will be mapped to a parabolic transformation of Γ; but if the parabolic corresponds to a loop around puncture on $\mathcal{S}_0$ which is given by a **branch point** in $\mathcal{S}$, then the element will be mapped into an elliptic transformation or the identity. From this one can obtain that the kernel of χ is the minimal normal subgroup of Γ_0 containing the set $\{\gamma^n / \gamma$ corresponds to a puncture on $\mathcal{S}_0$ that comes from a ramification point with ramification value n$\}$

Using this mapping we can define the following norm-preserving isomorphisms:

$$h_* : L^1(\Gamma) \to L^1(\Gamma_0), \qquad h^* : L^\infty(\Gamma_0) \to L^\infty(\Gamma),$$

given by

$$h_*(f) = (f \circ h)(h')^2, \qquad (h^*\mu) \circ h = \mu \frac{h'}{\overline{h'}}.$$

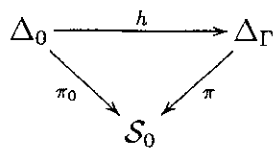
One can check that h^* induces a mapping between $T(\Gamma_0)$ and $T(\Gamma)$ which is holomorphic with respect to the natural complex structures of these two spaces. By an abuse of notation, we will denote by h^* the mapping between the Teichmüller spaces; we hope that it is clear from the context which mapping we are using. The mapping h_* takes $Q(\Gamma)$ onto $Q(\Gamma_0)$, and in this sense is the co-derivative of h^*.

The proof of the Bers-Greenberg isomorphism theorem is completed by observing that the mapping h^* takes Teichmüller differentials to Teichmüller differentials, so h^* is bijective. □

5.4 Proof of the main result

5.4.1. In this section we will prove the main result of this chapter, theorem 14. But first of all we will try to explain the main idea behind the proof. We have a terminal regular b-group, Γ, of type (p,n) with some points of finite ramification number, and we remove all of them to obtain a torsion free group, Γ_0, of the same type. We have the commutative diagram with the universal covering space of subsection 5.3.2. The first step is to decompose both groups into modular subgroups, and to prove that there exist a correspondence between the modular subgroups of Γ_0 and those of Γ: we will prove that each modular subgroup of the torsion free group Γ_0 is mapped onto a modular subgroup of the group Γ. Any deformation of the group Γ_0 induces a deformation of its modular subgroups, which can be pushed forward to a deformation of the modular subgroups of Γ by means of the above described correspondence. The deformations of the modular subgroups induced by deformations of Γ_0 are conformal on fixed neighborhoods of certain punctures. We will see that any deformation of a modular subgroup is equivalent to another of that special kind. This will give a mapping between the Teichmüller spaces of the modular subgroups. A basic proposition about the Maskit embbeding (proposition 11) guarantees that, when we put all these isomorphisms together, they induce an isomorphism from $T(\Gamma_0)$ onto $T(\Gamma)$, completing the proof of our result.

5.4.2. Before starting the technical arguments, let's recall our commutative diagram:



Let T_0 be one of the modular parts of $\mathcal{S}_0$, and let $\pi^{-1}(T_0) = \cup_{j\in J} D_j$ be a decomposition of the pre-image of T_0 into connected components. Apply h^{-1} and use the commutativity of the diagram to get

$$\pi_0^{-1}(T_0) = h^{-1} \circ \pi^{-1}(T_0) = h^{-1}(\cup_{j\in J} D_j) = \cup_{j\in J} h^{-1}(D_j).$$

On the other hand, we also have the decomposition into connected components as follows:

$$\pi_0^{-1}(T_0) = \cup_{k\in K} A_k.$$

Therefore $\forall k \in K, \exists j \in J$ such that $h(A_k) \subseteq D_j$.

5.4.3. <u>Claim</u>: $h(A_k) = D_j$.

Suppose we have A_{k_1} and A_{k_2} such that $h(A_{k_1}) \subseteq D_j$, $h(A_{k_1}) \subseteq D_j$. Since h is an open map and D_j is an open connected set, our claim will be proven if we can establish that the sets $h(A_{k_1})$ and $h(A_{k_2})$ are either disjoint or equal. So let us assume that there is a point y in the intersection of these two sets, $y \in h(A_{k_1}) \cap h(A_{k_2})$. This means that there are points, $x_1 \in A_{k_1}$ and $x_2 \in A_{k_2}$, such that $h(x_1) = h(x_2) = y$. We then have $\pi_0(x_1) = \pi h(x_1) = \pi(y)$ and $\pi_0(x_2) = \pi h(x_2) = \pi(y)$. This implies that there is a deck transformation $\gamma \in \Gamma_0$ such that $\gamma(x_1) = x_2$. Therefore $\gamma(A_1) \cap A_2 \neq \emptyset$. But since γ is a homeomorphism, and $\pi_0\gamma = \pi_0$, we get

$$\cup_{k\in K} A_k = \pi_0^{-1}(T_0) = (\pi_0\gamma^{-1})^{-1}(T_0) = \gamma(\pi_0^{-1}(T_0)) = \gamma(\cup_{k\in K} A_k) = \cup_{k\in K} \gamma(A_k).$$

The sets are disjoint, so we must have $\gamma(A_{k_1}) = A_{k_2}$. We know that $h\gamma(x_1) = h(x_1) = h(x_2) = y$, and $h\gamma(x_1) = \chi(\gamma)(h(x_1))$. This implies $\chi(\gamma)(y) = y$. But y is in D_Γ, where the transformation $\chi(\gamma) \in \Gamma$ has no fixed points, unless it is the identity. So we conclude that $\chi(\gamma) = id$, giving $h(A_{k_2}) = h\gamma(A_{k_1}) = \chi(\gamma)h(A_{k_1}) = h(A_{k_1})$. And this proves the claim. ##

Choose one such pair of sets, D_j and A_k, and rename $A_k = A_j$ for simplicity. We have the following commutative diagram:

$$\begin{array}{ccc} A_j & \xrightarrow{\ h\ } & D_j \\ & {\scriptstyle \pi_0}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & T_0 & \end{array}$$

Let $\Gamma_0^j = Stab(A_j, \Gamma_0)$ and $\Gamma^j = Stab(D_j, \Gamma)$. These groups are modular subgroups of Γ_0 and Γ respectively. We want to prove that the homomorphism χ takes Γ_0^j onto Γ^j. We will do it in several steps, involving algebraic and topological arguments.

5.4.4. <u>Step 1</u>: $\chi(\Gamma_0^j) \subset \Gamma^j$.

This step is obvious, since if $\gamma \in \Gamma_0^j$ then $\gamma(A_j) = A_j$, and therefore $\chi(\gamma)(D_j) = \chi(\gamma) \circ h(A_j) = h \circ \gamma(A_j) = h(A_j) = D_j$. ##

Consider the following diagram:

$$\begin{array}{ccc} A_j & \xrightarrow{\ h\ } & D_j = h(A_j) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ A_j/\Gamma_0^j & \xrightarrow{\ \rho\ } & D_j/\chi(\Gamma_0^j), \end{array}$$

where π_1 and π_2 are the natural projections, and ρ is defined by $\rho(\pi_1(x)) = \pi_2(h(x))$.

5.4.5. Step 2: ρ is well defined.

If $\pi_1(x) = \pi_1(y)$, then $\exists\gamma \in \Gamma_0^j$ such that $\gamma(x) = y$. Therefore $h(y) = h(\gamma(x)) = \chi(\gamma)(h(x))$ which implies $\pi_2(h(y)) = \pi_2(h(x))$, or $\rho(\pi_1(x)) = \rho(\pi_1(y))$. ##

5.4.6. Step 3: ρ is a surjective mapping.

Let $z \in D_j$. Then $\exists w \in A_j$ such that $h(w) = z$. This implies $\rho(\pi_1(w)) = \pi_2(h(w)) = \pi_2(z)$. ##

5.4.7. Step 4: ρ is one-to-one.

$\rho(\pi_1(x)) = \rho(\pi_1(y))$ means $\pi_2(h(x)) = \pi_2(h(y))$. Then $\exists\gamma \in \Gamma_0^j$ such that $h(y) = \chi(\gamma)(h(x)) \Rightarrow h(\gamma(x)) = \gamma(y) \Rightarrow \exists\delta \in H$=covering group of h=$\{\gamma \in \Gamma_0 / h \circ \gamma = h\}$ with $\delta(\gamma(x)) = y$.

$h \circ \delta = h \Rightarrow h \circ \delta(A_j) = h(A_j) = D_j \Rightarrow \delta(A_j)$ is another A_l. But the sets $\{A_l\}$ are disjoint, and we have that $x, \gamma(x)$ and $\delta(\gamma(x))$ belong to A_j, which implies $\delta(A_j) \cap A_j \neq \emptyset$ and therefore $\delta(A_j) = A_j$. So we have that δ is an element of $\Gamma_0^j \cap H$.

From the equation $(\delta \circ \gamma)(x) = y$ we obtain that $\pi_1(x) = \pi_1(y)$, which completes the argument of this step. ##

5.4.8. Step 5: ρ is continuous with respect to the quotient topologies.

ρ is continuous if and only if $\rho \circ \pi_1$ is continuous. But this is obvious since $\rho \circ \pi_1 = \pi_2 \circ h$, which is the composition of two continuous functions. ##

5.4.9. Step 6: ρ is open with respect to the quotient topologies.

Let $U \subset A_j/\Gamma_0^j$ be an open set on the quotient topology. We know that $\pi_1^{-1}(U) = V$ is open in A_j. And this last set is open in Δ_0 (which is open in the Riemann sphere). Since h is a covering map we get that $h(V)$ is an open set in Δ_Γ. Now we have that $\rho(U) = \pi_2(h(V))$ is open $\Leftrightarrow$ $\pi_2^{-1}(\pi_2(h(V)))$ is open. But this set is

$$\pi_2^{-1}(\pi_2(h(V))) = \cup_{\gamma \in \Gamma_0^j} \chi(\gamma)(h(V)).$$

Since $h(V)$ is an open set in the Riemann sphere, and $\chi(\gamma)$ is a homeomorphism of $\hat{\mathbb{C}}$, the set $\chi(\gamma)(h(V))$ is open (for all $\gamma \in \Gamma_0^j$), so $\pi_2^{-1}(\pi_2(h(V)))$ is open, which proves that ρ is a open mapping. ##

5.4.10. So we conclude that $A_j/\Gamma_j^0 \cong h(A_j)/\chi(\Gamma_j^0)$, and remember the fact that $h(A_j) = D_j$.

Step 7: $\chi(\Gamma_0^j) = \Gamma^j$.

We have the following equalities: $A_j/\Gamma_0^j \cong D_j/\Gamma^j$, $A_j/\Gamma_0^j \cong D_j/\chi(\Gamma_0^j)$; and we also know $\chi(\Gamma_0^j) \subseteq \Gamma^j$. Basic set theory gives us the desired conclusion. ##

5.4.11. So we have obtained that the map χ takes modular subgroups of Γ_0 into modular subgroups of Γ. This allows us to define a series of functions h_j^*, $1 \leq j \leq d$, by the rule $(h_j^*\mu) \circ h = \mu \frac{\bar{h}'}{h'}$. The domain of this function will be the set:

$$T(\Gamma_0^j, \Delta_0) = \{[\mu] \in T(\Gamma_0^j);\ supp(\mu) \subset \Delta_0\},$$

and the target will be

$$T(\Gamma^j, \Delta) = \{[\mu] \in T(\Gamma^j);\ supp(\mu) \subset \Delta\}.$$

Here the brackets denote the Teichmüller class of the coefficient μ. What we would like to have is a mapping between the deformation spaces, and not these (apparently proper) subsets. The following lemma states that these subsets are really the whole deformation spaces.

Lemma 4 (Deformation Lemma) *The space* $T(\Gamma^j)$ *is equal to* $T(\Gamma^j, \Delta_0) = \{\mu \in T(\Gamma^j);\ supp(\mu) \subset \Delta_0\}$.

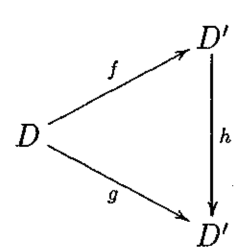
The idea behind the proof of this lemma is the following: the deformations supported on Δ_0 are those of the punctured torus or the four times punctured sphere that are conformal on some fixed neighborhoods of the punctures. Teichmüller space is the set of (quasiconformal) deformations of the complex structure and the fundamental group of the surface. What we are considering is deformations which are conformal on a (set of) puncture(s); but a punctured disc carries only one possible complex structure and the fundamental groups of surfaces with holes coincide with the fundamental groups of surfaces with punctures.

Proof (of the lemma) [Mikhail Lyubich]. First of all, let us look at the meaning of the space $T(\Gamma^j, \Delta_0)$. The group Γ^j is a terminal regular b-group, with invariant component $\Delta(\Gamma^j)$, such that the surface $T = \Delta(\Gamma^j)/\Gamma^j$ has signature $(1, 1; \infty)$ or $(0, 4; \infty, \ldots, \infty)$. Since our proof is essentially the same in both cases, we will assume that the surface is of type (1,1). The set Δ_0 is also invariant under the group Γ^j. We have $\Delta_0 \subset \Delta(\Gamma^j)$ and the quotient Δ_0/Γ^j is a torus with one hole. The difference $\Delta(\Gamma^j)/\Gamma^j - \Delta_0/\Gamma^j$ is a punctured disc that we will denote by D. Therefore, if a Beltrami differential is supported on Δ_0, the corresponding deformation of T is conformal on D. Therefore the lemma is equivalent to prove the following statement:

Let T be a torus with one puncture, and let D be a punctured disc around the missing point. Let $f : T \to T'$ be a quasiconformal deformation of T. Then f is Teichmüller equivalent to a quasiconformal deformation that is conformal on D.

The inclusion $T(\Gamma^j, \Delta_0) \subseteq T(\Gamma^j)$ is trivial. To see the other inclusion, consider a point in $T(\Gamma^j)$, say $[\mu]$. Choose a quasiconformal mapping representing that point, $f : T \to T'$, and consider the punctured disc D. Let D' denote $f(D)$.

The punctured discs D and D' are conformally equivalent; let $g : D \to D'$ be a biholomorphism between the two punctured discs. Since the Teichmüller space of the punctured disc is trivial, we have a conformal mapping $h : D' \to D'$ such that the following diagram is commutative up to homotopy:



Now let us look at the function h. We can extend it to the whole T' so that it is the identity outside a little bit bigger punctured disc $V \supset D'$, an smooth in the annulus $V - D'$. Then it is clear that f and $h \circ f$ are homotopic.

This is not hard to see as follows: on $T' - V$ the function h is the identity, therefore $h \circ f = f$. On the punctured disc V we have that the function h is the identity on the boundary, and basic topology says that then h is homotopic to the identity on the whole V. To see it, since this is a topological statement, we can assume that V is the unit disc $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$, and h is the identity in the boundary of the unit this, S^1. The homotopy between h and the identity function is given by

$$\begin{aligned} H:\ [0,1] \times (\mathbb{D} \cup S^1) &\to \mathbb{D} \cup S^1 \\ (t,z) &\mapsto (1 - t + t|z|)\, h((1-t)z + t\tfrac{z}{|z|}) \end{aligned}$$

So $h \circ f$ is homotopic to f.

Thus we have deformed the function f to another function, $h \circ f$, in the same Teichmüller class, but such that this last function is homotopic to a conformal mapping on D. But this last statement means that the Teichmüller class of $h \circ f$ is trivial on D, that is, it is represented by the a Beltrami differential that is zero outside Δ_0, as we wanted to prove. □

5.4.12. The above proof shows that the result can be generalized as follows:

Lemma 5 (Deformation Lemma) *Suppose S is an orbifold with at least n punctures. Let $U_1, \ldots, U_n$ be punctured discs around the punctures of S, with disjoint closures. Then any quasiconformal deformation of S is equivalent to a quasiconformal deformation that is conformal on the sets $U_1, \ldots, U_n$.*

5.4.13. To recapitulate, we have the covering h that induces a homomorphism χ between the modular subgroups, and a series of mappings between $T(\Gamma_0^j, \Delta_0)$ and $T(\Gamma^j, \Delta)$, called h_j^*. By the previous deformation lemma, we have that h_j^* is really defined between $T(\Gamma_0^j)$ and $T(\Gamma^j)$. To complete the proof of our main result we need one property of the Maskit embedding. But before stating it, let us recall that in the coordinates of chapter 3, the one dimensional deformation spaces are embedded in the upper half plane (§§3.2.5, 3.3.8 and 3.2.9). The statement of the proposition is the following:

Proposition 11 *Let $\mathcal{G}$ be a terminal regular b-group uniformizing an orbifold $\mathcal{S}$ of finite conformal type with a maximal partition $\mathcal{C}$. Let $\mathcal{G}_1, \dots, \mathcal{G}_d$ be (a choice of) the modular subgroups of $\mathcal{G}$, and let $T(\mathcal{G}) \hookrightarrow \prod_{k=1}^{d} T(\mathcal{G}_k)$ be the Maskit embedding of $T(\mathcal{G})$. Let α be in $T(\mathcal{G}_1)$. Then there exist non-negative numbers $(s_\alpha^2, \dots, s_\alpha^d)$ such that the set $\{(\alpha, z_2, \dots, z_d) \in \mathbb{H}^d;\ Im(z_j) > s_\alpha^j\}$ is contained in (the image of) $T(\mathcal{G})$.*

Proof. It is easier to see the proof at the level of orbifolds. Consider the orbifold of type (0,4) or (1,1) given by α. The pair $(\mathcal{S}, \mathcal{C})$ gives an algorithm to construct orbifolds from points in $T(\mathcal{G})$, by a series of plumbing processes. Suppose we have built an orbifold corresponding to the point $(\alpha, z_2, \dots, z_j)$. The next plumbing construction will use either one or two punctures of this last orbifold; it is clear that we can do such construction if we use horocircles of very small radius. But these types of horocircles correspond to coordinates with big imaginary part. □

Observe that a trivial consequence of this proposition (that we will use) is that given α and β in $T(\mathcal{G}_1)$, the corresponding sets obtained in the proposition, $\{(z_2, \dots, z_d) \in \mathsf{H}^{d-1};\ Im(z_j) > s_\alpha^j\}$ and $\{(z_2, \dots, z_d) \in \mathsf{H}^{d-1};\ Im(z_j) > s_\beta^j\}$ have non-empty intersection.

5.4.14. By the above work, we have that a set of 3p-3+n mappings between one-dimensional deformation spaces, and we want to prove that the tuple $(h_1^*, \dots, h_{3p-3+n}^*)$, gives an isomorphism between $T(\Gamma_0)$ and $T(\Gamma)$. Consider now the following diagram:

$$\begin{array}{ccc} T(\Gamma_0) & \xrightarrow{\quad h^* \quad} & T(\Gamma) \\ {\scriptstyle i_0}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \prod_{j=1}^{3p-3+n} T(\Gamma_0^j) & \longrightarrow & \prod_{j=1}^{3p-3+n} T(\Gamma^j), \end{array}$$

where i_0 and i are the Maskit embeddings, and in the lower horizontal arrow we have the function $(h_1^*, \dots, h_{3p-p+n}^*)$. h^* and the h_j^* are defined by the same equation, namely $(h^*\mu) \circ h = \mu \frac{\overline{h'}}{h'}$, where h is the covering map of §5.3.2. This implies that the above diagram is commutative. All we need to complete the proof of the main theorem is to see that each h_j^* is injective and onto.

Injectivity: let us consider h_1^*, and suppose α and β are two points in $T(\Gamma_0^1)$ with $h_1^*(\alpha) = h_1^*(\beta)$. Then there exist open sets V_α and V_β, with non-empty intersection, given by the above proposition. Choose γ in that intersection. We would get that $(h_1^*, \dots, h_{3p-3+n}^*)(\alpha, \gamma) = (h_1^*, \dots, h_{3p-3+n}^*)(\beta, \gamma)$, which contradicts the injectivity of $h^* = (h_1^*, \dots, h_{3p-3+n}^*)$.

Each h_j^* is onto: take α' in $T(\Gamma^1)$, and find β' such that (α', β') belongs to $T(\Gamma)$. Then there exists a point $x = (\alpha, \beta) \in T(\Gamma_0^1) \times \prod_{j=2}^{3p-3+n} T(\Gamma_0^j)$ such that $h^*(\alpha, \beta) = (\alpha', \beta')$, giving $h_1^*(\alpha) = \alpha'$. This completes the proof of theorem 13. □

We would like to finish this chapter with a question posed by Bernard Maskit. We know that two Teichmüller spaces of dimension one are simply connected proper subsets of the complex plane, and therefore conformally equivalent by the Riemann Mapping Theorem. Can we use this Riemann mappings at the one dimensional level to get a proof of the Bers-Geenberg Theorem in higher dimensions?

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