The Intersection Dold-Thom Theorem

A Dissertation Presented

by

Paweł Gajer

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August 1993

State University of New York
at Stony Brook

The Graduate School

Paweł Gajer

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Blaine Lawson
Professor of Mathematics
Dissertation Advisor

Marie-Louise Michelsohn
Professor of Mathematics
Chair of Defense

Lowell Jones
Professor of Mathematics

Anita Wasilewska
Associate Professor of Computer Science
SUNY at Stony Brook
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation

The Intersection Dold-Thom Theorem

by

Paweł Gajer

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1993

The classical Dold-Thom theorem asserts that for every connected polyhedron X there is an isomorphism between the integral singular homology of X and the homotopy groups of the free abelian topological group $AG(X)$ generated by the points of X. We show that an analogous result holds for intersection homology. We introduce intersection homotopy groups of filtered spaces and prove that for every startified polyhedron X the intersection homology groups of X are isomorphic to the intersection homotopy groups of $AG(X)$ equipped with some finite filtration induced by a stratification of X. We also show that there exist long exact sequences of intersection homotopy groups for filtered pairs and filtered fibrations, and that the intersection homotopy groups of a pl stratification of a polyhedron X depend on the pl structure of X and in certain cases only on the homeomorphism type of X. It is proved that intersection homotopy groups can be thought of as a functor on the category of polyhedra with morphisms - either pl allowable maps, or placid homotopy classes of continuous placid maps.

Dla Basieńki

Contents

Acknowledgements

I want to express my deepest gratitude to Blaine Lawson for many extremely stimulating conversations, helpful advise, and constant encouragement. I feel very fortunate to have as an advisor not only one of the greatest mathematicians but also a wonderful man.

I would like to thank Emili Bifet for many excellent conversations. His joy of life and enthusiasm towards mathematics shaped my optimism and helped me to survive many difficult moments.

I thank all participants of the 1991/92 seminar on intersection homology: Blaine, Lam, Carlos, Javier, and especially Alastair for helping me to learn this beautiful subject.

I am grateful to Pablo Ares for his personal help.

I would like to thank Bogusław Hajduk and Marek Lewkowicz, from Wrocław University, for their help and guidance in the first years of my mathematical career.

Chciałbym również podziękować moim rodzicom za pomoc i wsparcie jakiego mi zawsze udzielali. Ich miłość i pogoda ducha miała dla mnie szczególną wartość tu na obczyźnie.

I also thank my daughter Ola for being a joy of my life. Sorry that I could not spend with you as much time as I would like to.

The very special thanks are for my beloved wife for her love and support. This is to her that this work is dedicated.

CHAPTER I

Introduction

The classical Dold-Thom Theorem asserts that for every connected polyhedron X there is an isomorphism

$$H_*(X, \mathbb{Z}) \cong \pi_*(AG(X)) \tag{1}$$

between the integral singular homology of X and the homotopy groups of the free abelian topological group $AG(X)$ generated by the points of X [DT56]. The importance of the Dold-Thom Theorem comes not only from the fact that it describes a relationship between homology and homotopy groups, but also (and maybe foremost) because it served as an inspiration for such fundamental results as the Spanier-Whitehead duality, the Almgren isomorphism (see the next paragraph), and the Lawson suspension theorem for L-homology [Spa59, FJA62, Law89, Fri91].

The Almgren isomorphism is a natural generalization of the Dold-Thom Theorem that stemmed from the following observation. For every polyhedron

X the topological group $AG(X)$ can be identify with the space of 0-cycles of X. Thus, the Dold-Thom Theorem gives an explicit description of the homotopy groups of the space of 0-cycles of a polyhedron X. It is natural to ask if a similar result holds for higher dimensional cycles. An affirmative answer to the above question, posed by H. Federer, was given by F. Almgren in 1962 [FJA62]. He proved that for every euclidean polyhedron X, and $k \geq r \geq 0$, there is an isomorphism

$$H_k(X, \mathbb{Z}) \cong \pi_{k-r}(Z_r(X))$$

where $Z_r(X)$ is the space of integral Federer r-cycles with the weak topology.

The isomorphisms of Dold-Thom and Almgren served as motivation for Blaine Lawson in his work on homotopy groups of spaces of algebraic cycles [Law89]. Lawson's results were generalized and extended by Eric Friedlander, who introduced the notion of L-homology [Fri91]. L-homology assigns to an arbitrary complex projective variety X the groups

$$L_r H_k(X) \stackrel{def}{=} \pi_{k-2r}(\mathcal{Z}_r(X))$$

where $\mathcal{Z}_r(X)$ is the naïve group completion of the union of Chow varieties of the effective algebraic r-cycles of the variety X [Law89, Fri91]. From the Dold-Thom Theorem and the fact that $\mathcal{Z}_0(X) = AG(X)$ it follows that the L-homology $L_r H_k(X)$ of an algebraic variety X specializes to the ordinary homology $H_k(X,\mathbb{Z})$ of X for $r=0$. On the other hand the groups $L_*H_*(X)$ are sensitive to the algebraic structure of X. For example, for each $r \geq 0$, the group $L_rH_{2r}(X)$ is isomorphic to the group $\mathcal{A}_r(X)$ of algebraic r-cycles on X, modulo algebraic equivalence.

One of the most fundamental properties of ordinary and L-homology is the existence of intersection pairings

$$H_k(M) \times H_l(M) \longrightarrow H_{k+l-n}(M) \tag{2}$$

$$L_rH_k(M) \times L_sH_l(M) \longrightarrow L_{r+s-n}H_{k+l-2n}(M) \tag{3}$$

where M in (2) is a pl manifold of dimension n and M in (3) is a complex quasi-projective manifold of dimension n. The classical Poincare duality asserts that the pairing (2) is non-degenerate over $\mathbb{Q}$[1]. If M is a singular polyhedron of dimension n, then usually the pairing (2) cannot be defined, and the groups $H_k(M)\otimes\mathbb{Q}$ and $\mathrm{Hom}(H_{n-k}(M)\otimes\mathbb{Q},\mathbb{Q})$ are not isomorphic. This is one of the reasons for which ordinary homology is not the most suitable tool for studying singular spaces (an another reason is that ordinary homology is a homotopy invariant, and a homotopy equivalence may essentially change the singular structure of a space).

In 1980 R. MacPherson and M. Goresky defined a natural generalization of

[1]It induces an isomorphism $H_k(M)\otimes\mathbb{Q} \cong \mathrm{Hom}(H_{n-k}(M)\otimes\mathbb{Q},\mathbb{Q})$.

ordinary homology called *intersection homology* [GM80]. Intersection homology assigns to every stratified polyhedron[2] X a family $\{I_{\bar{p}}H_*(X)\}$ of graded groups indexed by perversities[3]. It is equipped with a pairing of intersection type and is not a homotopy type invariant. The groups $I_{\bar{p}}H_*(X)$ are the homology groups of a complex of pl chains of X, whose intersections with strata of X are controlled by the perversity $\bar{p}$. Intersection homology specializes to ordinary homology and cohomology. Actually, if X is a normal pseudomanifold of dimension n, then

$$I_{\bar{p}}H_*(X) = \begin{cases} H_*(X) & \text{for } \bar{p}(k) = k - 2 \\ H^{n-*}(X) & \text{for } \bar{p}(k) = 0. \end{cases}$$

The remarkable feature of intersection homology is the existence of Poincare duality for singular spaces. There is also a Künneth Theorem, a long exact sequence of a pair, a Mayer-Vietoris Theorem, and many other properties characteristic for homology theory.

---

[2]A *stratified polyhedron* is a pair $(X, \mathfrak{X})$ consisting of a polyhedron X and a *filtration*

$$\mathfrak{X}: \quad X = X^0 \supset X^1 \supset X^2 \supset \cdots \supset X^n \supset \emptyset$$

of X by its subpolyhedra (called *skeleta* of $\mathfrak{X}$) satisfying certain extra conditions (for details see Chapter 2). In the sequel, by abuse of notation we will often write X when referring to a filtered (pl stratified) space $(X, \mathfrak{X})$.

[3]A perversity is a function $\bar{p} : \mathbb{Z}_+ \to \mathbb{Z}_+$ with the property that for every $k \geq 0$ we have $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.

If we think of L-homology as an extension of ordinary homology on the category of projective algebraic varieties, then it is natural to ask if it is possible to find an analogous extension for intersection homology.

The problem of finding an intersection version of L-homology is a generalization of the old problem (as old as intersection homology) of finding an intersection version of the groups $\mathcal{A}_r(X)$ of algebraic r-cycles on X, modulo algebraic equivalence. The existence of such a theory would give us an intersection theory for singular varieties, whose existence is still problematic.

By analogy with L-homology, the intersection L-homology $I_{\bar{p}}L_rH_k(X)$ of a variety X should specialize, for $r = 0$, to the intersection homology $I_{\bar{p}}H_k(X)$ of X, and the isomorphism

$$I_{\bar{p}}H_k(X, \mathbb{Z}) \cong I_{\bar{p}}L_0H_k(X)$$

should be an analogue of the Dold-Thom Theorem for intersection homology. One of the main difficulties in establishing such an isomorphism was lack of a good candidate for an intersection analogue of the right hand side of the Dold-Thom isomorphism (1). In this thesis we introduce the notion of intersection homotopy groups and prove the following result.

Theorem I.1. *For every perversity $\bar{p}$ and every stratified connected polyhedron X there is an isomorphism*

$$I_{\bar{p}}H_*(X,\mathbb{Z}) \cong I_{\bar{p}}\pi_*(AG(X)).$$

The definition of the right hand side of the above isomorphism is given below.

Recall, that the homotopy groups of a topological space X are canonically isomorphic with the simplicial homotopy groups of the simplicial set $S(X)$ of singular simplices of X. The following definition of intersection homotopy groups was motivated by the above isomorphism.

We define the *intersection homotopy groups $I_{\bar{p}}\pi_*(X)$ of a filtered space X* as the homotopy groups of a simplicial set $I_{\bar{p}}S(X)$ of $\bar{p}$-singular simplices of X that can be described as follows. A continuous map σ from the standard n-simplex Δ_n into X is a *$\bar{p}$-singular simplex of X* if for every skeleton X^k of X the inverse image $\sigma^{-1}(X^k)$ is contained in a subpolyhedron of Δ_n of dimension $\leq n-k+\bar{p}(k)$, and a similar condition holds for every face of σ (for more details see Chapter 2). The archetypes of our $\bar{p}$-singular simplices are H. King's intersection singular simplices [Kin85].

Immediately from the definition of intersection homotopy groups it follows that for every filtered space X there is an isomorphism

$$I_{\bar{p}}\pi_*(X) \cong \pi_*(X)$$

where $\bar{p}(k) = k$. We will show that many fundamental properties of ordinary homotopy groups have "intersection" analogues. For example, there is an intersection Hurewicz Theorem and there exist long exact sequences of intersection homotopy groups for filtered pairs and filtered fibrations. On the other hand, contrary to homotopy invariance of ordinary homotopy groups, the intersection homotopy groups of a stratified polyhedron X depend only on the pl structure of X and in certain cases on the homeomorphism type of X. We also prove that intersection homotopy groups can be thought of as a functor on the category of polyhedra with morphisms - either pl allowable maps, or placid homotopy classes of continuous placid maps.

The reason we define intersection homotopy groups in the context of finitely filtered spaces (not just pl stratified ones) is that we want to define intersection homotopy groups of $AG(X)$ which is in general infinite dimensional and hence does not carry any stratification structure.

If we identify the elements of $AG(X)$ with functions of finite support from X to $\mathbb{Z}$, then the finite filtration $AG(\mathfrak{X}) = \{AG(\mathfrak{X})^k\}$, with respect to which the intersection homotopy groups of $AG(X)$ are computed, is defined so that $n : X \to \mathbb{Z}$ belongs to the skeleton $AG(\mathfrak{X})^k$ of $AG(\mathfrak{X})$ whenever $n(x) \neq 0$ for some element x of the skeleton X^k of $\mathfrak{X}$.

Theorem I.1 is proved by showing that the natural equivalence of homology theories

$$\varphi_* : H_* \longrightarrow \pi_* \circ AG$$

extends to a natural equivalence of intersection homology theories

$$I_{\bar{p}}\varphi_* : I_{\bar{p}}H_* \longrightarrow I_{\bar{p}}\pi_* \circ AG.$$

In order to make the last statement more precise we will describe in more detail the transformations φ_* and $I_{\bar{p}}\varphi_*$.

The assignment $X \mapsto AG(X)$ induces a functor AG from the category of polyhedra and continuous maps to the category of free abelian topological groups and continuous homomorphisms. There exists a simplicial counterpart of this functor (that we will also denote by AG) that assigns to every simplicial set S a free abelian simplicial group $AG(S)$ [Spa59]. Let $S(X)$ be the simplicial set of singular simplices of X. For every topological space X the simplicial group $AG(S(X))$ can be identified with the complex of singular chains of X and, under this identification, the singular homology groups $H_*(X, \mathbb{Z})$ correspond to the simplicial homotopy groups $\pi_*(AG(S(X)))$ [May82]. On the other hand, for every space X the homotopy groups of X are naturally isomorphic to the simplicial homotopy groups of the complex $S(X)$. In particular,

$$\pi_*(AG(X)) \cong \pi_*(S(AG(X))).$$

Using the above identifications φ_* can be rewritten in the form

$$\varphi_* = \pi_*(\varphi) : \pi_* \circ AG \circ S \longrightarrow \pi_* \circ S \circ AG$$

where φ is the natural transformation

$$\varphi : AG \circ S \longrightarrow S \circ AG \tag{4}$$

$$\varphi(\sum n_i \cdot \sigma_i) = (s \mapsto \sum n_i \cdot \sigma_i(s)).$$

An intersection analogue

$$I_{\bar{p}}\varphi : AG \circ I_{\bar{p}}S \longrightarrow I_{\bar{p}}S \circ AG$$

of the transformation (4) is obtained by replacing S by a functor $I_{\bar{p}}S$ with values in $\bar{p}$-singular simplicial complexes, and by extending AG to a *finitely filtered* free abelian group functor, that assigns to every finitely filtered space $(X, \mathfrak{X})$ the group $AG(X)$ with the finite filtration $AG(\mathfrak{X})$.

From the definition of intersection homotopy groups and the fact that for every stratified polyhedron X the homology of $I_{\bar{p}}S(X)$ is isomorphic to the intersection homology $I_{\bar{p}}H_*(X, \mathbb{Z})$ of X, the transformation $I_{\bar{p}}\varphi_*$ induced by

$I_{\bar{p}}\varphi$

$$(5) \qquad I_{\bar{p}}\varphi_* : \pi_* \circ AG \circ I_{\bar{p}}S \longrightarrow \pi_* \circ I_{\bar{p}}S \circ AG$$

can be rewritten in the form

$$I_{\bar{p}}\varphi_* : I_{\bar{p}}H_* \longrightarrow I_{\bar{p}}\pi_* \circ AG.$$

$I_{\bar{p}}\varphi_*$ is an isomorphism on the class of stratified polyhedra, because both $I_{\bar{p}}H_*$ and $I_{\bar{p}}\pi_* \circ AG$ are intersection pl homology functors (see Chapter 3 for the definition), and the transformation $I_{\bar{p}}\varphi_*$ satisfies conditions of the uniqueness theorem (Theorem IV.1) for intersection pl homology.

CHAPTER II

Stratified polyhedra

This chapter contains a brief discussion of basic properties of stratified spaces.

A *filtered space* $(X, \mathfrak{X})$ is a space X with a *filtration*

$$\mathfrak{X} : X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

of X by closed (at least when X is of finite dimension) subspaces of X. All filtrations considered in this paper are finite, that is $n < \infty$. The elements X_k of $\mathfrak{X}$ will be called *skeleta* and the differences $S_k = X_k \setminus X_{k-1}$ *strata* of the filtration $\mathfrak{X}$. We will sometimes refer to the top dimensional stratum S_n as the *regular stratum* and denote it by $\mathrm{reg}(X)$. In the sequel we will assume that all filtrations have a non-empty regular stratum. The subscript index k of X_k will be called the *formal dimension* of X_k. Very often it is more convenient to index filtrations by *formal codimension* of skeleta. We follow a standard convention where the skeleton X_{n-k} of $\mathfrak{X}$ of formal codimension k is denoted by X^k.

If a space X is a polyhedron and a filtration $\mathfrak{X}$ of X is given by subpolyhedra, then the pair $(X, \mathfrak{X})$ is called a *filtered polyhedron.*

Let X be a polyhedron with a filtration $\mathfrak{X}$ of length n. The *open cone*

$$c(X) = \begin{cases} X \times [0,1)/X \times 0 & \text{if } X \neq \emptyset \\ \{pt\} & \text{if } X = \emptyset \end{cases}$$

on X has a natural filtration $c(\mathfrak{X})$ with the skeleta

$$(c(X))^k = \begin{cases} c(X^k) & \text{for } 0 \leq k \leq n \\ \text{the vertex of the cone} & \text{for } k = n+1 \\ \emptyset & \text{for } k > n+1 \end{cases}$$

Let X and Y be polyhedra with filtrations $\mathfrak{X} = \{X^k\}$ and $\mathfrak{Y} = \{Y^k\}$ respectively. A pl map $f : X \to Y$ is *cofiltered with respect to* $\mathfrak{X}$ *and* $\mathfrak{Y}$ if for every $k \geq 0$ one has $X^k \supset f^{-1}(Y^k)$. A cofiltered pl map $f : (X, \mathfrak{X}) \to (Y, \mathfrak{Y})$ is a *filtered isomorphism* of filtered spaces if $f : X \to Y$ is a pl homeomorphism and for every $k \geq 0$ one has $f(X^k) = Y^k$.

A *locally cone-like filtration* of a polyhedron X is a filtration $\mathfrak{X}$ of X so that for each point $x \in S_k = X_k \setminus X_{k-1}$ there is a neighborhood U of x in S_k, a compact filtered polyhedron $(L, \mathfrak{L})$, called a *link* of x in X, and a filtered isomorphism of $U \times c(L)$ onto an open neighborhood of x in X, where $U \times c(L)$ carries the product filtration with skeleta $(U \times c(L))^k = U \times (c(L))^k$.

A *pl stratification* of a polyhedron X is a locally cone-like filtration of X whose strata are pl manifolds (without boundary) [Sie72]. A *stratified polyhedron* is a pair $(X, \mathfrak{X})$ consisting of a polyhedron X and a pl stratification $\mathfrak{X}$ of X. In the sequel, by abuse of notation we will often write X when referring to a filtered (or pl stratified) space $(X, \mathfrak{X})$.

For every pl stratification $\mathfrak{X}$ of a polyhedron X the natural filtration $c(\mathfrak{X})$ of the cone $c(X)$ is a pl stratification of $c(X)$. Every triangulation of a polyhedron X induces a pl stratification of X by its skeleta.

We say that a stratification $\mathfrak{X}$ of X *coarsens* another stratification $\mathfrak{X}'$ of X if every stratum of $\mathfrak{X}$ is a union of connected components of strata of $\mathfrak{X}'$. For example, if K' is a subcomplex of a simplicial complex K, then the stratification induced by K coarsens the stratification induced by K'. Thus, in a sense, the operation of coarsening of stratification is reverse to the subdivision of triangulations.

The subdivision of triangulations is an ordering relation and divides the class of all triangulations of a polyhedron into partially ordered subclasses called pl structures of the polyhedron. Also, the coarsening of stratifications is an ordering relation and it divides all stratifications of a stratified space into partially ordered subclasses. There is an essential difference between these two ordering relations. Contrary to the subdivision of simplicial complexes, the coarsening of pl stratifications of a polyhedron X (with a fixed pl structure) has a terminal object. It is called the *intrinsic pl stratification of* X. Let $I_k(X)$ denote the k-dimensional skeleton of the intrinsic pl stratification $\mathfrak{I}(X)$ of X. The intrinsic pl stratification of a polyhedron X is constructed as follows.

With every point x of a polyhedron X we associate an integer $d(x, X)$ called the *intrinsic dimension of* X *at* x that is defined as the greatest integer t satisfying one of the following equivalent conditions:

(1) The link of x in X is a t-fold suspension. That is there exists a polyhedron Y so that the link of x in X is the join of the $(t-1)$-dimensional sphere and Y.

(2) There exists an embedding $f : \mathbb{R}_t \times c(W) \to X$ so that $f(\mathbb{R}_t \times c(W))$ is a neighborhood of x in X.

(3) There exists a triangulation of X so that x lies in the interior of a simplex of dimension t.

For the proof of equivalence of the above three conditions see [Aki69]. The *intrinsic pl i-skeleton of* X is

$$I_i(X) = \{x \in X \mid d(x, X) \leq i\}.$$

Part 3 of the definition of the intrinsic dimension implies that

$$I_i(X) = \bigcap\{t(|K_i|) \mid (K,t) \text{ is a triangulation of } X\}$$

where K_i is the i-dimensional skeleton of K. Thus $I_i(X)$ is a closed subpolyhedron of X. Moreover, for every i the difference $I_i(X) \setminus I_{i-1}(X)$ is a pl manifold of dimension i [Aki69]. Hence, by the part 2 of the definition of the intrinsic dimension, the filtration $\mathfrak{I}(X) = \{I_i(X)\}$ is a pl stratification of X.

If $c(N)$ is an open cone over a pl manifold N and X is a locally trivial fibration over a pl-manifold M with fiber $c(N)$, then

$$I_k(X) = \begin{cases} \emptyset & \text{for } k < \dim(M) \\ M & \text{for } \dim(M) \leq k < \dim(X) \\ X & \text{for } k = \dim(X) \end{cases}$$

In particular, if M is a pl manifold of dimension n, then $I_n(M) = M$ and $I_k(M) = \emptyset$ for $k < \dim(M)$.

CHAPTER III

Intersection Homotopy Groups

In this chapter it will be shown that to every filtered space X one can assign a Kan complex $I_{\bar{p}}S(X)$ whose homology coincide with Goresky-MacPherson intersection pl homology of X, when X is a stratified polyhedron. We define the intersection homotopy groups $I_{\bar{p}}\pi_*(X)$ of X as the simplicial homotopy groups of $I_{\bar{p}}S(X)$. We introduce notions of filtered pairs and filtered fibrations and show that for every filtered pair or fibration there exist long exact sequences of intersection homotopy groups. It will be shown that the intersection homotopy groups of a stratified polyhedron X depend on the pl structure of X and in some cases only on the homeomorphism type of X. Finally, we will see that intersection homotopy groups can be thought of as a functor on the category of polyhedra with morphisms either pl allowable maps, or placid homotopy classes of continuous placid maps (see Section 2.2).

Our basic reference for homotopy theory of simplicial sets is [May82].

1. Intersection Homotopy Groups of Filtered Spaces

Let

$$\Delta_k = \{(t_0, t_1, \cdots, t_k) \in \mathbb{R}_{k+1} \,|\, 0 \le t_i \le 1, \sum t_i = 1\}$$

be the standard k-simplex in $\mathbb{R}_{k+1}$. A singular k-simplex of a topological space X is a continuous map $\sigma : \Delta_k \longrightarrow X$. Let $S_k(X)$ be the set of all singular k-simplices of X. The graded set $S(X) = \coprod S_k(X)$ becomes a simplicial set if we define face operators

$$\partial_i : S_k(X) \longrightarrow S_{k-1}(X), \qquad 0 \le i \le k$$

by

$$\partial_i \sigma(t_0, \ldots, t_{k-1}) = \sigma(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1})$$

and degeneracy operators

$$s_i : S_k(X) \longrightarrow S_{k+1}(X), \qquad 0 \le i \le k+1$$

by

$$s_i \sigma(t_0, \ldots, t_{k+1}) = \sigma(t_0, \ldots, t_i + t_{i+1}, \ldots, t_{k+1}).$$

The simplicial set $S(X)$ is called the *singular complex of* X.

A *perversity* is a function $\bar{p} : \mathbb{Z}_+ \to \mathbb{Z}_+$ so that for every $k \ge 0$

$$\bar{p}(k) \le \bar{p}(k+1) \le \bar{p}(k) + 1.$$

Let X be a filtered space. A singular simplex $\sigma : \Delta_k \to X$ is $\bar{p}$ *allowable with respect to* $\mathfrak{X}$ if for every skeleton X^s of X the subset $\sigma^{-1}(X^s)$ is contained in a subpolyhedron of Δ_k of dimension less than or equal to $k - s + \bar{p}(s)$.

A singular simplex $\sigma : \Delta_k \to X$ is of *perversity* $\bar{p}$ *with respect to a filtration* $\mathfrak{X}$ of X, or *perversity* $\bar{p}$ (rel $\mathfrak{X}$), if σ and all its faces $\partial_{i_1} \circ \partial_{i_2} \circ \cdots \circ \partial_{i_l}(\sigma)$ are $\bar{p}$ allowable with respect to $\mathfrak{X}$. It is easy to see that if σ is of perversity $\bar{p}$ (rel $\mathfrak{X}$), then every degenerated simplex $s_{j_1} \circ s_{j_2} \circ \cdots \circ s_{j_l}(\sigma)$ is of perversity $\bar{p}$ (rel $\mathfrak{X}$). Hence, all perversity $\bar{p}$ (rel $\mathfrak{X}$) singular simplices of X constitute a simplicial set $I_{\bar{p}}S(X)$ that will be called *perversity* $\bar{p}$ *singular complex of the filtered space* X. Sometimes, in order to specify a filtration $\mathfrak{X}$ of X, with respect to which the complex $I_{\bar{p}}S(X)$ is computed, we will write $I_{\bar{p}}S(X, \mathfrak{X})$ for $I_{\bar{p}}S(X)$.

If X is a stratified polyhedron, then the complex $I_{\bar{p}}S(X)$ is an intersection analogue of $S(X)$. Actually, we have the following result.

THEOREM III.1. *For every stratified polyhedron X the homology groups of the complex $I_{\bar{p}}S(X)$ are isomorphic to Goresky-MacPherson intersection pl homology of X.*

Theorem III.1 can be proved in exactly the same way as an analogous statement for King's singular intersection homology [Kin85].

If $f : X \to Y$ is a cofiltered map, then f induces a morphism

$$f_\sharp : I_{\bar p}S(X) \to I_{\bar p}S(Y)$$

of simplicial sets. In particular, if an embedding $i : A \hookrightarrow X$ is a cofiltered map, then $I_{\bar p}S(A)$ is a simplicial subset of $I_{\bar p}S(X)$. A pair (X, A) is called a *filtered pair* if the embedding map $i : A \hookrightarrow X$ is cofiltered. For example, if $x \in X$, then (X, x) is a filtered pair if and only if $x \in \text{reg}(X)$. A triple (X, A, B) is called a *filtered triple* if both pairs (X, A), (A, B) are filtered pairs.

The isomorphisms

$$\begin{aligned} \pi_k(X, x) &\cong \pi_k(S(X), S(x)) \\ \pi_k(X, A, x) &\cong \pi_k(S(X), S(A), S(x)) \end{aligned}$$

motivate the following definition. The *k-th perversity $\bar p$ intersection homotopy group* $I_{\bar p}\pi_k(X, A, x)$ of a filtered triple (X, A, x) is the simplicial homotopy group of the simplicial triple $(I_{\bar p}S(X), I_{\bar p}S(A), I_{\bar p}S(x))$. In particular, if $X = A$, then

$$I_{\bar p}\pi_k(X, x) \stackrel{def}{=} \pi_k(I_{\bar p}S(X), I_{\bar p}S(x)).$$

In terms of perversity $\bar p$ singular simplices of X the group $I_{\bar p}\pi_k(X, x)$ can be described as follows. We say that two perversity $\bar p$ singular k−simplices σ_0, σ_1 of X are *$\bar p$-homotopic* if there is a perversity $\bar p$ singular $(k+1)$−simplex σ so that $\partial_k \sigma = \sigma_0, \partial_{k+1}\sigma = \sigma_1$, and for all $0 < i < k$ the simplex $\partial_i \sigma$ is the constant map into x. The group $I_{\bar{p}}\pi_k(X,x)$ consists of $\bar{p}$-homotopy classes of perversity $\bar{p}$ singular simplices $\sigma : \Delta_k \to X$ so that for every $0 \le i \le k$ the singular simplex $\partial_i \sigma$ maps Δ_{k-1} into the point x.

When working with different filtrations of a space it is convenient to specify with respect to which filtration the intersection homotopy groups are computed. In this situations the perversity $\bar{p}$ intersection homotopy groups of a filtered space $(X, \mathfrak{X})$ will be denoted by $I_{\bar{p}}\pi_*(X, \mathfrak{X})$.

Recall, that the simplicial homotopy groups of a simplicial set S are well defined only if S satisfies the following Kan extension condition. For every $0 \le i \le k$ and a simplicial map $s : \Delta[k,i] \to S$ there is an extension of s to a simplicial map $\Delta[k] \to S$, where $\Delta[k]$ is the standard simplicial k-simplex and $\Delta[k,i]$ is the simplicial subset of $\Delta[k]$ generated by the simplices

$$\partial_0 \sigma_k, \ldots, \partial_{i-1}\sigma_k, \partial_{i+1}\sigma_k, \ldots, \partial_k \sigma_k$$

with σ_k being the non-degenerated k-simplex of $\Delta[k]$.

Thus, the definition of intersection homotopy groups makes sense if for every filtered space X the complex $I_{\bar{p}}S(X)$ satisfies the Kan extension condition. In other words, one has to prove that if $V(k,i)$ is the union of all but i^{th} $(k-1)$-dimensional faces of Δ_k and $\sigma : V(k,i) \to X$ is a continuous map so that its restriction to every simplex is a perversity $\bar{p}$ map, then σ extends to a perversity

$\bar{p}$ singular k-simplex. The required extension is obtained by the composition of σ with the the radial projection $r_i : \Delta_k \rightarrow V(k,i)$ from the baricenter of the i^{th} face of Δ_k. The map $\sigma \circ r_i$ is a perversity $\bar{p}$ singular simplex because for every subpolyhedron P of $V(k,i)$ the inverse image $r_i^{-1}(P)$ is of dimension $\dim P + 1$.

Examples.

1. Let M be a pl manifold with the trivial filtration $\mathfrak{M} = \{\emptyset \subset M\}$ and let $(X, \mathfrak{X})$ be a filtered space. For every perversity $\bar{p}$ and a non-negative integer k

$$I_{\bar{p}}\pi_k(X \times M) \cong I_{\bar{p}}\pi_k(X) \times \pi_k(M)$$

where $X \times M$ is equipped with the product filtration

$$\mathfrak{X} \times \mathfrak{M} : \emptyset \subset X^n \times M \subset X^{n-1} \times M \subset \cdots \subset X^1 \times M \subset X \times M.$$

2. Let $\mathfrak{X}$ be a filtration of X of length n, then for every perversity $\bar{p}$ and a non-negative integer k

$$I_{\bar{p}}\pi_k(c(X)) \cong \begin{cases} I_{\bar{p}}\pi_k(X) & \text{for } k < n - \bar{p}(n+1) \\ 0 & \text{for } k \geq n - \bar{p}(n+1). \end{cases}$$

3. Let X be a polyhedron of dimension n, with a finite set Σ of isolated singularities. For a stratification

$$\mathfrak{X} : X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

of X so that $X_i = \Sigma$ for $0 \leq i < n$

$$I_{\bar{p}}\pi_k(X) \cong \begin{cases} \pi_k(X \setminus \Sigma) & \text{for } k+1-n+\bar{p}(n) < 0 \\ \operatorname{im}(\pi_k(X \setminus \Sigma) \to \pi_k(X)) & \text{for } k+1-n+\bar{p}(n) = 0 \\ ? & \text{for } k+1-n+\bar{p}(n) > 0. \end{cases}$$

Since every cofiltered map of filtered pairs $f : (X, x) \to (Y, y)$ induces a simplicial map $f_\sharp : I_{\bar{p}}S(X) \to I_{\bar{p}}S(Y)$, it induces also a homomorphism of corresponding homotopy groups $f_* : I_{\bar{p}}\pi_k(X, x) \to I_{\bar{p}}\pi_k(Y, y)$.

For every Kan triple (T, S, v) there is a long exact sequence of homotopy groups

$$\cdots \to \pi_k(S, v) \to \pi_k(T, v) \to \pi_k(T, S, v) \to \pi_{k-1}(S, v) \to \cdots$$

Hence, for every filtered triple (X, A, x) there is a long exact sequence of intersection homotopy groups

$$\cdots \to I_{\bar{p}}\pi_k(A, x) \to I_{\bar{p}}\pi_k(X, x) \to I_{\bar{p}}\pi_k(X, A, x) \to I_{\bar{p}}\pi_{k-1}(A, x) \to \cdots$$

Now we are going to define an intersection counterpart of the notion of Serre fibration. Our definition is motivated by the fact that a continuous map $\pi : E \to B$ is a Serre fibration if and only if the induced simplicial morphism $S(\pi) : S(E) \to S(B)$ of singular complexes is a simplicial fibration. The above equivalence follows directly from the definitions of Serre and simplicial fibrations. Recall that a simplicial map $\pi : T \to S$ is a *simplicial fibration* if π is a surjection and for every $k \geq 0$ and $0 \leq i \leq k$ the extension problem

$$\begin{array}{ccc} \Delta[k,i] & \longrightarrow & T \\ \big\downarrow & \overset{\bar{G}}{\nearrow} & \big\downarrow \pi \\ \Delta[k] & \longrightarrow & S \end{array}$$

has a solution $\bar{G}$.

A map $\pi : E \to B$ of filtered spaces will be called a *filtered fibration* if for every perversity $\bar{p}$ it induces a simplicial fibration $\pi_{\bar{p}} : I_{\bar{p}}S(E) \to I_{\bar{p}}S(B)$. Equivalently, $\pi : E \to B$ is a filtered fibration if for every $\bar{p}$ it induces a surjective map $\pi_{\bar{p}} : I_{\bar{p}}S(E) \to I_{\bar{p}}S(B)$ and the following extension problem

$$\begin{array}{ccc} V(k,i) & \xrightarrow{g} & E \\ \big\downarrow & \overset{\bar{G}}{\nearrow} & \big\downarrow \pi \\ \Delta^k & \xrightarrow[G]{} & B \end{array}$$

where $V(k,i)$ is the geometric realization of $\Delta[k,i]$, and the maps G and g are of perversity $\bar{p}$ with respect to the filtrations of B and E respectively, has a solution $\bar{G}$ that is of perversity $\bar{p}$ with respect to the filtration of E.

Since a simplicial fibration of Kan complexes induces a long exact sequence of homotopy groups, for every filtered fibration $\pi : E \to B$ there is a long exact sequence of intersection homotopy groups

$$\cdots \to \pi_k(\mathrm{fib}(\pi_{\bar{p}})) \to I_{\bar{p}}\pi_k(E,e) \to I_{\bar{p}}\pi_k(B,b) \to \pi_{k-1}(\mathrm{fib}(\pi_{\bar{p}})) \to \cdots$$

where $\mathrm{fib}(\pi_{\bar{p}})$ represents the fiber of $\pi_{\bar{p}} : I_{\bar{p}}S(E) \to I_{\bar{p}}S(B)$.

THEOREM III.2. *Let $\pi : E \to B$ be a filtered fibration so that for some $b \in$ reg(B) the fiber $F = \pi^{-1}(b)$ is a filtered subspace of E. Then there exists a long exact sequence of intersection homotopy groups*

$$\cdots \to I_{\bar{p}}\pi_k(F,e) \to I_{\bar{p}}\pi_k(E,e) \to I_{\bar{p}}\pi_k(B,b) \to I_{\bar{p}}\pi_{k-1}(F,e) \to \cdots \tag{6}$$

The proof of Theorem III.2 is exactly the same as the proof of the existence of long exact sequences for Serre fibrations.

THEOREM III.3. *Let $\pi : E \to B$ be a locally trivial fibration with a fiber F and let E, B and F be filtered by $\mathfrak{E} = \{E^s\}, \mathfrak{B} = \{B^s\}$ and $\mathfrak{F} = \{F^s\}$ respectively, so that for every $b \in B$ there is a neighborhood U of b in B and a trivialization $\varphi : \pi^{-1}(U) \to U \times F$ so that for every $s \geq 0$ and $U^s = U \cap B^s$*

$$\varphi(\pi^{-1}(U) \cap E^s) = U^s \times F \ \cup \ U \times F^s. \tag{7}$$

Then π is a filtered fibration. Moreover, if for some $b \in$ reg(B) the fiber $F = \pi^{-1}(b)$ is a filtered subspace of E, then the sequence (6) is exact.

A straightforward consequence of Theorem III.3 is the following corollary.

Corollary III.4. *If M is a pl manifold and $c(L)$ is a cone over a filtered space L, then every locally trivial fibration $\pi : E \to M$ with fiber $c(L)$ is a filtered fibration with respect to the trivial filtration of M and a filtration of $\mathfrak{E} = \{E^s\}$ of E so that for every open subset U of M for which $\pi^{-1}(U)$ is isomorphic to $U \times c(L)$ the intersection $E^s \cap \pi^{-1}(U)$ is isomorphic to $U \times c(L)^s$.*

Proof of Theorem III.3. First note that for every perversity $\bar{p}$ (rel $\mathfrak{E}$) simplex $\sigma : \Delta_k \to E$ there is a triangulation T of Δ_k so that for every simplex Δ of T there exists an open subset U of B so that $\sigma(\Delta) \subset \pi^{-1}(U)$ and $\pi^{-1}(U)$ has a trivialization satisfying the condition (7) . Moreover, the triangulation T can be chosen so that the restriction $\sigma|_\Delta$ is of perversity $\bar{p}$ (rel $\mathfrak{E}$). Every such triangulation will be called *triangulation of σ transversal to $\mathfrak{E}$*. The triangulation T can be obtained from an arbitrary triangulation S of Δ_k by moving (using induction on the dimension of skeleta of S) the skeleta of S into a general position with a pl filtration $\{A^s\}$ of Δ_k so that $\sigma^{-1}(E^s) \subset A^s$.

Lemma III.5. *Let $\pi : E \to B$ and $\varphi : \pi^{-1}(U) \to U \times F$ be as in Theorem* III.3. *Then a singular simplex $\sigma : \Delta_k \to \pi^{-1}(U)$ is of perversity $\bar{p}$ (rel $\mathfrak{E}$) if and only if the simplices $\pi_1 \circ \varphi \circ \sigma$ and $\pi_2 \circ \varphi \circ \sigma$ are of perversity $\bar{p}$ with respect to $\mathfrak{B}$ and $\mathfrak{F}$ respectively, where $\pi_1 : U \times F \to U, \pi_2 : U \times F \to F$ are the standard projections.*

Proof. The proof follows from the sequence of equalities

$$\sigma^{-1}(\pi^{-1}(U) \cap E^s) = (\varphi \circ \sigma)^{-1}(\varphi(\pi^{-1}(U) \cap E^s)) =$$

$$(\varphi \circ \sigma)^{-1}(U^s \times F \ \cup \ U \times F^s) =$$

$$(\sigma_1, \sigma_2)^{-1}(U^s \times F \ \cup \ U \times F^s) =$$

$$(\sigma_1)^{-1}(U^s) \cup (\sigma_2)^{-1}(F^s)$$

where $\sigma_i = \pi_i \circ \varphi \circ \sigma$ for $i = 1.2$. $\square$

The projection π will induce a map $\pi_{\bar{p}} : I_{\bar{p}}S(E) \to I_{\bar{p}}S(B)$ if for an arbitrary perversity $\bar{p}$ (rel $\mathfrak{E}$) simplex $\sigma : \Delta_k \to E$ the composition $\pi \circ \sigma$ is of perversity $\bar{p}$ (rel $\mathfrak{B}$). Let T be a triangulation of σ transversal to $\mathfrak{E}$. For every $\Delta \in T$ the restriction $\sigma|_\Delta$ is a perversity $\bar{p}$ (rel $\mathfrak{E}$) simplex that sends Δ into some $\pi^{-1}(U)$ that admits a trivialization satisfying the condition (7) of Theorem III.3. Hence, by Lemma III.5 the map $\pi_1 \circ \varphi \circ \sigma|_\Delta$ is of perversity $\bar{p}$ (rel $\mathfrak{B}$). Since this is true for every $\Delta \in T$, the map $\pi_1 \circ \varphi \circ \sigma = \pi \circ \sigma$ is of perversity $\bar{p}$ (rel $\mathfrak{B}$).

Now we will see that $\pi_{\bar{p}} : I_{\bar{p}}S(E) \to I_{\bar{p}}S(B)$ is surjection. Let $\sigma \in I_{\bar{p}}S(E)$. Again, by referring to a triangulation of σ transversal to $\mathfrak{B}$, we can reduce the construction to the case when σ is a perversity $\bar{p}$ (rel $\mathfrak{B}$) simplex that sends Δ_k into some U so that the inverse image $\pi^{-1}(U)$ admits a trivialization φ satisfying the condition (7) of Theorem III.3. Let τ be the constant k-simplex mapping Δ_k into a non-singular point of F. Then, by Lemma III.5, $\varphi^{-1}\circ(\sigma,\tau)$ is the required lifting of σ to a perversity $\bar{p}$ (rel $\mathfrak{C}$) simplex of E.

Now we will show that every extension problem

$$\begin{array}{ccc} V(k,i) & \xrightarrow{g} & E \\ \big\downarrow & \overset{\bar{G}}{\nearrow} & \big\downarrow \pi \\ \Delta^k & \xrightarrow[G]{} & B \end{array}$$

with G and g being of perversity $\bar{p}$ with respect to $\mathfrak{B}$ and $\mathfrak{C}$ respectively has a solution $\bar{G}$ that is of perversity $\bar{p}$ with respect to $\mathfrak{C}$.

Using again an appropriate triangulation T of Δ_k one can reduce this problem to one in which the fibration $\pi : E \to B$ is trivial. Therefore we have to consider the following extension problem

$$(8) \qquad \begin{array}{ccc} V(k,i) & \xrightarrow{g} & U \times F \\ \big\downarrow & \overset{\bar{G}}{\nearrow} & \big\downarrow \pi \\ \Delta^k & \xrightarrow[G]{} & U \end{array}$$

Let r be a linear projection from the i^{th} face of Δ_k onto $V(k,i)$. Define $\bar{G} = (G, \pi_2 \circ g \circ r)$, where $\pi_2 : U \times F \to F$ is the projection on the second factor. The map $\bar{G}$ makes the diagram (8) commute, so one has to only check that it is of perversity $\bar{p}$ (rel $\mathfrak{C}$) i.e. that the inverse image $\bar{G}^{-1}((U \times F)^s)$ is contained in a subpolyhedron of Δ_k of dimension less of equal to $k - s + \bar{p}(s)$. We have the following sequence of equalities

$$\bar{G}^{-1}((U \times F)^s) = \bar{G}^{-1}(U^s \times F \cup U \times F^s) = G^{-1}(U^s) \cup (\pi_2 \circ g \circ r)^{-1}(F^s) =$$

$$G^{-1}(U^s) \cup r^{-1} \circ g^{-1} \circ \pi_2^{-1}(F^s) = G^{-1}(U^s) \cup r^{-1} \circ g^{-1}(U \times F^s)$$

By the assumption, g is of perversity $\bar{p}$. Hence, $g^{-1}(U \times F^s)$ is contained in a subpolyhedron of $V(k,i)$ of dimension $\leq k-1-s+\bar{p}(s)$. The retraction r has the property that for an arbitrary subpolyhedron P of $V(k,i)$ of dimension l the inverse image $r^{-1}(P)$ is of dimension $l+1$. Hence, $r^{-1} \circ g^{-1}(U \times F^s)$ is contained in a subpolyhedron of Δ_k of dimension $\leq k-s+\bar{p}(s)$. Finally, G is of perversity $\bar{p}$ and therefore $G^{-1}(U^s)$ is contained in a subpolyhedron of Δ_k of dimension $\leq k-s+\bar{p}(s)$. $\square$

2. Intersection Homotopy Groups of Stratified Spaces

2.1. Invariance of Intersection Homotopy Groups. Goresky and MacPherson proved [GM83] (see also [Kin85]) that their intersection pl homology of a stratified polyhedron X depends only on the homeomorphism type of X. It is natural to conjecture that the same is true for the intersection homotopy groups. We will show that the intersection homotopy groups of a stratified polyhedron X depend only on the pl structure of X and under certain extra conditions they are homeomorphism invariants of X.

The pl invariance of the intersection homotopy groups of a pl stratification is a consequence of the following theorem.

THEOREM III.6. *Let $(X, \mathfrak{X})$ be a stratified polyhedron and let $\mathfrak{I}(X)$ be the intrinsic pl stratification of X. Then the natural homomorphism*

$$I_{\bar{p}}\pi_*(X, \mathfrak{X}) \to I_{\bar{p}}\pi_*(X, \mathfrak{I}(X))$$

is an isomorphism.

The proof of Theorem III.6 is based on the following stratified general position theorem for pl maps and the filtered simplicial approximation theorem.

THEOREM III.7 (STRATIFIED GENERAL POSITION FOR PL MAPS). *Let $f : X \to Y$ be a pl map, $B \subset A$ a pair of subpolyhedra of Y, and $\mathfrak{Y}$ a pl stratification of Y. Then there exists a pl isotopy h of Y so that for every stratum S^l of $\mathfrak{Y}$ we have*

$$\operatorname{codim}_{\tilde{f}^{-1}(S^l)} \tilde{f}^{-1}(A \cap S^l) \geq \operatorname{codim}_{S^l} A \cap S^l$$

where $\tilde{f} = h_1 \circ f$. Moreover, if

$$\operatorname{codim}_{f^{-1}(S^l)} f^{-1}(B \cap S^l) \geq \operatorname{codim}_{S^l} B \cap S^l,$$

then h can be chosen to fix B.

In order to prove Theorem III.7 move the image of f in every stratum S^l of $\mathfrak{Y}$ to a general position with B (using McCrory's stratified general position theorem for subpolyhedra). Then triangulate f and use the fact that every linear epimorphism preserves the codimension of linear subspaces.

THEOREM III.8 (FILTERED SIMPLICIAL APPROXIMATION THEOREM). *Let $\mathfrak{X} = \{X^s\}$ and $\mathfrak{Y} = \{Y^s\}$ be pl filtrations of polyhedra X and Y respectively and let $f : (X, \mathfrak{X}) \to (Y, \mathfrak{Y})$ be a continuous cofiltered map . Then there is a simplicial approximation $g : K \to L$ of f so that $g^{-1}(L^s) \subset K^s$ where K^s and L^s are subcomplexes of K and L respectively so that $|K^s| = X^s$ and $|L^s| = Y^s$.*

Proof. Let K and L be triangulations of X and Y respectively, compatible with the filtrations $\mathfrak{X}$ and $\mathfrak{Y}$. That is K and L have filtrations by subcomplexes

$$\emptyset \subset K^n \subset K^{n-1} \subset \cdots \subset K^1 \subset K^0 = K$$

$$\emptyset \subset L^m \subset L^{m-1} \subset \cdots \subset L^1 \subset L^0 = L$$

so that $|K^s| = X^s$ and $|L^s| = Y^s$ for every $s \geq 0$. Note that for every $s \geq 0$ we have the following equalities

$$X \setminus X^s = \bigcup_{v \in C(K^s, K)} st(v), \qquad Y \setminus Y^s = \bigcup_{v \in C(L^s, L)} st(v)$$

where $st(v)$ denotes the open star of the vertex v and

$$C(K', K) = \{\Delta \in K : |\Delta| \cap |K'| = \emptyset\}$$

for any subcomplex K' of K.

Since f is a cofiltered map, for every $s \geq 0$ we have an inclusion $f(X \setminus X^s) \subset$

$Y \setminus Y^s$. The standard baricentric subdivision argument assures the existence of a subdivision $(K_1, K_1^1, K_1^2, \dots, K_1^n)$ of $(K, K^1, K^2, \dots, K^n)$ so that for every $s \geq 0$ and $v \in C(K_1^s, K_1)$ there exists $g(v) \in C(L^s, L)$ so that $f(st(v)) \subset st(g(v))$.

Note that $\Delta \in g^{-1}(L^s)$ is and only if for every vertex v of Δ the image $g(v)$ is in L^s. From the construction of g it follows that every $\Delta \in g^{-1}(L^s)$ has no vertex in $C(K_1^s, K_1)$. Therefore, $g^{-1}(L^s) \subset K_1^s$. for every $s \geq 0$. □

Proof of Theorem III.6. By the filtered simplicial approximation theorem, we can assume that all perversity $\bar{p}$ singular simplices are pl maps.

Let $\mathfrak{X}$ be a pl stratification of X. The proof is by an induction on the dimension of skeleta S_X^i of $\mathfrak{X}$.

Let $\sigma : \Delta_n \rightarrow X$ be a singular simplex (that is a pl map) of perversity $\bar{p}$ with respect to the intrinsic pl stratification $\mathfrak{I}(X)$ of X. Suppose, that σ has been already modified so that for every $k > i$ we have

$$\operatorname{codim}_{\Delta_n}(\sigma)^{-1}(X^k) \leq k - \bar{p}(k).$$

By the stratified general position theorem for pl maps, there exists a pl isotopy h of X so that h fixes X^{i+1} and

$$\begin{aligned}\dim \tilde{\sigma}^{-1}(X^i \cap S^j) &\leq \dim \tilde{\sigma}^{-1}(S^i) + \dim X^i \cap S^j - \dim S^i \\ &\leq n - j + \bar{p}(j) + (\dim X - i) - (\dim X - j) \\ &\leq n - i + \bar{p}(j)\end{aligned}$$

where $\tilde{\sigma} = h_1 \circ \sigma$ and S^j denotes the codimension j intrinsic stratum of $\mathfrak{I}(X)$. From the definition of the intrinsic stratification it follows that $S_X^i \subset S^i \cup S^{i-1} \cup \cdots \cup S^0$. Therefore,

$$\begin{aligned}\dim \tilde{\sigma}^{-1}(S_X^i) &= \tilde{\sigma}^{-1}(S_X^i \cap (S^i \cup S^{i-1} \cup \cdots \cup S^0)) \\ &\leq \max_{0 \leq j \leq i} \dim \tilde{\sigma}^{-1}(X^i \cap S^j) \\ &\leq \max_{0 \leq j \leq i} (n - i + \bar{p}(j)) \\ &\leq n - i + \bar{p}(i).\end{aligned}$$

From the definition of perversity it follows that

$$\dim \tilde{\sigma}^{-1}(X^i) \leq n - i + \bar{p}(i).$$

Hence, every pl representative $\sigma : (\Delta_n, \partial\Delta_n) \rightarrow (X, x_0)$ of $I_{\bar{p}}\pi_*(X, \mathfrak{I}(X))$ is homotopic to a pl map $\tilde{\sigma}$ that is of perversity $\bar{p}$ with respect to $\mathfrak{X}$. Since h fixes the strata of $\mathfrak{I}(X)$, the homotopy $\sigma \circ h_t$ between $\sigma$and $\tilde{\sigma}$ is of perversity $\bar{p}$ with respect to $\mathfrak{I}(X)$.

Thus we have proved that the map

$$I_{\bar{p}}\pi_*(X, \mathfrak{X}) \to I_{\bar{p}}\pi_*(X, \mathfrak{I}(X))$$

is an epimorphism. The monomorphism is proved in a similar way. □

The conjecture on homeomorphism invariance of the intersection homotopy groups of a pl stratification is supported by the following result.

THEOREM III.9. *If X is a stratified polyhedron X so that $I_{\bar{p}}\pi_1(X) = 0$, then the intersection homotopy groups $I_{\bar{p}}\pi_*(X)$ depend only on the homeomorphism type of X.*

Proof. Let $\mathfrak{I}^{top}(X)$ denotes the coarsest cone-like stratification of X [Kin85]. Since $\mathfrak{I}^{top}(X)$ coarsens every cone-like stratification $\mathfrak{X}$ of X, the identity map of X is a cofiltered map with respect to $\mathfrak{X}$ and $\mathfrak{I}^{top}(X)$. Hence, it induces a simplicial map

$$i : I_{\bar{p}}S(X, \mathfrak{X}) \to I_{\bar{p}}S(X, \mathfrak{I}^{top}(X))$$

and a homomorphism

$$i_* : I_{\bar{p}}\pi_*(X, \mathfrak{X}) \to I_{\bar{p}}\pi_*(X, \mathfrak{I}^{top}(X)).$$

It is easy to see that if $I_{\bar{p}}\pi_1(X, \mathfrak{X}) = 0$, then for every coarsening $\mathfrak{X}'$ of $\mathfrak{X}$ we have $I_{\bar{p}}\pi_1(X, \mathfrak{X}') = 0$. In particular, $I_{\bar{p}}\pi_1(X, \mathfrak{I}^{top}(X)) = 0$. By the topological invariance of the singular intersection homology and the Hurewicz theorem for simplicial sets [May82], the homomorphism i_* is an isomorphism. □

An easy consequence of Theorem III.9 is the following result.

COROLLARY III.10. *If a polyhedron X admits a pl stratification $\mathfrak{X}$ so that the fundamental group of the regular stratum of $\mathfrak{X}$ is trivial, then for every cone-like stratification $\mathfrak{X}'$ that coarsens $\mathfrak{X}$ there is an isomorphism*

$$I_{\bar{p}}\pi_*(X,\mathfrak{X}) \rightarrow I_{\bar{p}}\pi_*(X,\mathfrak{X}').$$

In particular, the intersection homotopy groups $I_{\bar{p}}\pi_*(X,\mathfrak{X}')$ depend only on the homeomorphism type of X.

2.2. Functoriality of Intersection Homotopy Groups. A continuous map $f : X \rightarrow Y$ is called *pl placid* if there are pl stratifications $\mathfrak{X}$, $\mathfrak{Y}$ of X and Y respectively so that $f^{-1}(Y^k) \subset X^k$ for every $k \geq 0$.

LEMMA III.11. *The following conditions are equivalent*

(1) *$f : X \rightarrow Y$ is a pl placid map.*

(2) *There exists a pl stratification $\mathfrak{X} = \{X^k\}$ of X so that for every $k \geq 0$ one has $f^{-1}(I^k(Y)) \subset X^k$.*

(3) *For every $k \geq 0$ the inverse image $f^{-1}(I^k(Y))$ is contained in a subpolyhedron of X of codimension greater or equal to k.*

(4) *There exists a pl stratification $\mathfrak{Y} = \{Y^k\}$ of Y so that for every $k \geq 0$ the inverse image $f^{-1}(Y^k)$ is contained in a subpolyhedron of X of codimension greater or equal to k.*

The proof of Lemma III.11 is an easy exercise.

From the definition of intersection homotopy groups it follows that every pl placid map $f : X \to Y$ induces a homomorphism $f_{X,Y} : I_{\bar{p}}\pi_*(X) \to I_{\bar{p}}\pi_*(Y)$. Since for every pl stratification $\mathfrak{X}$ of X the simplicial map

$$i_X : I_{\bar{p}}S(X, \mathfrak{X}) \to I_{\bar{p}}S(X, \mathfrak{I}(X))$$

induces an isomorphism of homotopy groups, we define the *pl intersection homotopy groups* $I_{\bar{p}}\pi_*^{pl}(X)$ *of a polyhedron* X as the intersection homotopy groups $I_{\bar{p}}\pi_*(X, \mathfrak{I}(X))$. For every pl placid map $f : X \to Y$ a homomorphism

$$f_* : I_{\bar{p}}\pi_*^{pl}(X) \to I_{\bar{p}}\pi_*^{pl}(Y)$$

is given by the composition

$$I_{\bar{p}}\pi_*(X, \mathfrak{I}(X)) \xrightarrow[(i_X)_*^{-1}]{} I_{\bar{p}}\pi_*(X, \mathfrak{X}) \xrightarrow[f_{X,Y}]{} I_{\bar{p}}\pi_*(Y, \mathfrak{Y}) \xrightarrow[(i_Y)_*]{} I_{\bar{p}}\pi_*(Y, \mathfrak{I}(Y)).$$

It is easy to see that the homomorphism f_* does not depend on the choice of the pl stratifications of X and Y.

The composition of two pl placid maps is not necessary pl placid. Thus polyhedra and pl placid maps do not form a category. There are basically two approaches to cure this problem. We can either look for a subclass of pl placid maps that is closed under the composition, or impose certain equivalence relation on pl placid maps that will make their equivalence classes closed under the composition.

We say that a map $f : X \to Y$ is *pl allowable* if for every $k \geq 0$ we have $f^{-1}(I^k(Y)) \subset I^k(X)$. It is obvious from the definition of pl allowability that every identity map is pl allowable and the class of pl allowable maps is closed under the composition. It is also easy to see that if $f : X \to Y$ and $g : Y \to Z$ are pl allowable, then $(g \circ f)_* = g_* \circ f_*$. Thus, pl intersection homotopy groups constitute a functor on the class of polyhedra and pl allowable maps.

Directly from the definition of intrinsic pl skeleta it follows that a map $f : X \to Y$ is pl allowable if and only if for every $y \in Y$ and $x \in f^{-1}(y)$ there is an inequality $cd(x, X) \geq cd(y, Y)$, where $cd(x, X) = \dim(X) - d(x, X)$. Thus pl allowability is a local property.

Examples. 1. Let M be a pl manifold and X an arbitrary polyhedron. Every map $X \to M$ is pl allowable, but $f : M \to X$ is pl allowable if and only if the image $f(M)$ is contained in the nonsingular locus of Y.

In the next example we will see that every normally nonsingular pl map [FM81, Gor81] is pl allowable.

2. A pl embedding $i : Y \to X$ is pl allowable if for every $k \geq 0$ we have $I^k(Y) \supset I^k(X) \cap Y$.

A pl embedding $i : Y \to X$ is called *normally nonsingular inclusion* if Y has an open neighborhood N in X and a retraction $r : N \to Y$ which is a pl vector bundle over Y. Directly from the definition of the normally nonsingular pl embedding it follows that $I^k(Y) = I^k(X) \cap Y$. Hence every normally nonsingular pl embedding is pl allowable.

A pl fiber bundle $p : E \to B$ is *normally nonsingular fiber bundle* if its fiber is a pl manifold. Since the fiber of a normally nonsingular fiber bundle $p : E \to B$ is nonsingular we have $p^{-1}(I^k(B)) = I^k(E)$. Hence, every normally nonsingular pl fiber bundle is pl allowable.

A *normally nonsingular pl map* is one that can be factored as a composition of a normally nonsingular pl embedding followed by a normally nonsingular pl fiber bundle. Since every normally nonsingular pl embedding and fiber bundle is pl allowable the same is true for every normally nonsingular pl map.

Remark. PL allowable maps have its counterparts in other classes of stratifications. For example, we say that a map $f : X \to Y$ of cone-like stratified spaces is *allowable* if for every $k \geq 0$ we have $f^{-1}(I^k_{top}(Y)) \subset I^k_{top}(X)$ where $I^k_{top}(X)$ is the codimension k skeleton of the coarsest cone-like stratification of X. If intersection homotopy groups (on the class of cone-like stratifications) are homeomorphism invariants, then they constitute a functor on the class of cone-like stratified spaces and allowable maps.

The class of pl allowable maps forms a narrow subclass of pl placid maps. Therefore, it is better to view pl intersection homotopy groups as a functor on the class of pl placid maps, modulo certain equivalence relation.

We say that maps $f_0, f_1 : X \to Y$ are *pl placid homotopic* if f_0 and f_1 are homotopic by a pl placid map. Let $[f]_p$ denotes the pl placid homotopy class of a pl placid map f and let $[X, Y]_p$ be the set of pl placid homotopy classes of pl placid maps from X to Y.

THEOREM III.12. *PL intersection homotopy groups constitute a functor on the class $\mathcal{P}$ of polyhedra and pl placid homotopy classes of pl placid continuous maps.*

Proof. First, we have to prove that $\mathcal{P}$ is a category. It will be shown the class $\mathcal{PP}$ of polyhedra and pl placid homotopy classes of pl placid *pl maps* constitutes a category, and there is a bijective correspondence between pl placid homotopy classes of *pl placid pl maps* and pl placid homotopy of all (continuous) pl placid maps. The last statement is a consequence of the following lemma.

LEMMA III.13. *Let $\mathfrak{X} = \{X^s\}$ and $\mathfrak{Y} = \{Y^s\}$ be pl filtrations of polyhedra X and Y respectively and let $f : (X, \mathfrak{X}) \to (Y, \mathfrak{Y})$ be a continuous cofiltered map.*

Then there exists a homotopy F of f to a cofiltered (with respect to $\mathfrak{X}$ and $\mathfrak{Y}$) pl map so that for every $s \geq 0$ there is an inclusion $F^{-1}(Y^s) \subset X^s \times I$.

Proof. Let f and g be as in Theorem III.8. We know that for every vertex v of K_1 there is an inclusion $f(st(v)) \subset st(g(v))$. Hence, for every point $x \in X$ there is a simplex of L that contains both $f(x)$ and $g(x)$. Therefore, for every $x \in X$ the formula

$$F(x,t) = t \cdot g(x) + (1-t) \cdot f(x)$$

determines a homotopy between f and g. Note, that $(x,t) \in F^{-1}(Y^s)$ if and only if $t \cdot g(x) + (1-t) \cdot f(x) \in L^s$ and this implies that both $g(x)$ and $f(x)$ belong to a simplex of L^s. Since f and g are cofiltered with respect to $\mathfrak{X}$ and $\mathfrak{Y}$ the point (x,t) has to belong to $X^s \times I$. □

The bijection between $\mathcal{P}$ and $\mathcal{PP}$ will be used to define a composition pairing on $\mathcal{P}$ with respect to which $\mathcal{P}$ and $\mathcal{PP}$ become an equivalent categories. Let $[X,Y]_{pp}$ be the set of pl placid homotopy classes of pl placid pl maps between X and Y, and let $[f]_{pp}$ denotes the class of a pl map $f : X \to Y$ in $[X,Y]_{pp}$. The following lemma shows that $\mathcal{PP}$ is a category.

LEMMA III.14. *There exists a pairing*

$$\circ : [X,Y]_{pp} \times [Y,Z]_{pp} \longrightarrow [X,Z]_{pp}$$

so that:

(1) $\circ$ *is associative.*

(2) *For every* $\alpha \in [X,Y]_{pp}$ *we have* $\alpha \circ [id_X]_{pp} = \alpha = [id_Y]_{pp} \circ \alpha$.

(3) *If* $f \in \alpha \in [X,Y]_{pp}$ *and* $g \in \beta \in [Y,Z]_{pp}$ *are so that the composition* $g \circ f$ *is a pl placid pl map, then* $\beta \circ \alpha = [g \circ f]_{pp}$.

Proof. For any pair of placid pl maps $f : X \to Y$ and $g : Y \to Z$ we will construct a placid pl map $f' : X \to Y$ so that the composition $g \circ f'$ is placid and f' is pl placid homotopic to f. Then, it will be shown that the operation $(f,g) \mapsto g \circ f'$ induces the required composition on pl placid homotopy classes of placid pl maps.

The proof is based on the following consequence of the stratified general position theorem for maps.

LEMMA III.15. *If* $f : X \to Y$ *is a pl placid map, then for every stratification* $\mathfrak{Y}$ *of* Y *there is a pl isotopy* h *of* Y *so that for every skeleton* Y^k *of* $\mathfrak{Y}$

$$\operatorname{codim}_X (h_1 \circ f)^{-1}(Y^k) \geq k.$$

Proof. Let $\mathfrak{Y}$ be an arbitrary pl stratification of Y. We will construct the required isotopy be an induction on the dimension of skeleta of $\mathfrak{Y}$.

Let B and A, from Theorem III.7, be equal to Y^{k+1} and Y^k respectively, and suppose that we have already constructed $\tilde{h}$ so that

$$\operatorname{codim}_{(\tilde{h}_1 \circ f)^{-1}(S^l)}(\tilde{h}_1 \circ f)^{-1}(A \cap S^l) \geq \operatorname{codim}_{S^l}(A \cap S^l)$$

for all $s > k$. There exists a pl isotopy h of Y so that for every $l \geq 0$

$$\operatorname{codim}_{(h_1 \circ \tilde{f})^{-1}(S^l)}(h_1 \circ \tilde{f})^{-1}(Y^k \cap S^l) \geq \operatorname{codim}_{S^l}(Y^k \cap S^l)$$

where $\tilde{f} = \tilde{h}_1 \circ f$ and S^l is a stratum of the intrinsic pl stratification of Y.

Since $\operatorname{codim}_{S^l}(Y^k \cap S^l) \geq k - l$ we have

$$\operatorname{codim}_{(h_1 \circ \tilde{f})^{-1}(S^l)}(h_1 \circ \tilde{f})^{-1}(Y^k \cap S^l) \geq k - l$$

Since h fixes stratifications $\dim \tilde{f}^{-1}(S^l) = \dim(h_1 \circ \tilde{f})^{-1}(S^l)$ and

$$\operatorname{codim}_X(h_1 \circ \tilde{f})^{-1}(Y^k \cap S^l) =$$

$$\operatorname{codim}_X \tilde{f}^{-1}(S^l) + \operatorname{codim}_{(h_1 \circ \tilde{f})^{-1}(S^l)}(h_1 \circ \tilde{f})^{-1}(Y^k \cap S^l) \geq l + (k - l) = k$$

Thus $\operatorname{codim}_X(h_1 \circ \tilde{f})^{-1}(Y^k) \geq k$.

Repeating this process finite number of times we get a pl isotopy h of Y so that $f' = h_1 \circ f$ is the required modification of f. Since every pl isotopy preserves the intrinsic stratification of Y, the map $h \circ (f \times id)$ is pl placid. Thus f' is pl placid homotopic to f. □

Let us define the composition of the pl placid homotopy classes of pl placid pl maps $\alpha \in [X, Y]_{pp}$, $\beta \in [Y, Z]_{pp}$ by the formula $\beta \circ \alpha = [g \circ f']_{pp}$ where $g \in \beta$

and f' is a pl map pl placid homotopic to $f \in \alpha$ so that $g \circ f'$ is pl placid. The existence of such f' follows from Lemma III.15.

If $g_0, g_1 \in \alpha \in [X, Y]_{pp}$, then we write $g_0 \sim_{pp} g_1$. In order to prove that the composition is well defined we have to show that:

(1) If $g_0 \sim_{pp} g_1$ and f_0, f_1 are modifications of f so that $g_0 \circ f_0$ and $g_1 \circ f_1$ are pl placid, then $g_0 \circ f_0 \sim_{pp} g_1 \circ f_1$.

(2) If $f_0 \sim_{pp} f_1$ and f_0', f_1' are deformations of f_0 and f_1 respectively so that $g \circ f_0'$ and $g \circ f_1'$ are pl placid, then $g \circ f_0' \sim_{pp} g \circ f_1'$.

Proof of (1). Let G be a pl placid homotopy between g_0 and g_1. Applying our modification process to $f \times id : X \times I \to Y \times I$ and $G : Y \times I \to Z$ we get a pl placid map $F : X \times I \to Y \to Y \times I$ so that $F_{X \times i} = f_i$ for $i = 0, 1$. The composition $G \circ F$ is the required pl placid homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$. □

Proof of (2). Let F, F_0, F_1 be pl placid homotopies between f_0 and f_1, f_0' and f_0, and f_1 and f_1' respectively. Applying our modification process to $F_0 \cup F \cup F_1$ and g we get the required pl placid homotopy. □

The associativity and the property $\alpha \circ [id_X]_p = \alpha = [id_Y]_p \circ \alpha$ follow directly from the definition of the composition operation. The last property of

Lemma III.14 is a consequence of the property (2). □

This ends the proof of the fact that $\mathcal{PP}$, and hence $\mathcal{P}$, is a category. It is easy to see that every two pl placid homotopic pl placid maps induce the same homomorphism on intersection homotopy groups, and $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ for $\alpha \in [X, Y]_p$ and $\beta \in [Y, Z]_p$. Hence, pl intersection homotopy groups constitute a functor on $\mathcal{P}$. □

CHAPTER IV

Intersection PL Homology Theories

In this chapter we present a set of axioms of intersection pl homology theory. They are intersection analogues of the Eilenberg-MacLane axioms modified so that the functor $I_{\bar{p}}\pi_* \circ AG$ (defined in the next chapter) is an example of intersection pl homology. A modification of King's intersection homology characterization theorem (see Theorem 10 in [Kin85]), gives a uniqueness theorem for theories satisfying the above axioms.

A pair (X, A) of filtered polyhedra is called a *closed pl filtered pair* if A is a closed subpolyhedron of X so that for every skeleton X^k of X the intersection $A \cap X^s$ is the skeleton of A of formal codimension k. A closed pl filtered pair (X, A) is called a *closed pl filtered NDR pair* if there exist: an open neighborhood U of A in X and a retraction $r : U \rightarrow A$ so that for every $i \geq 0$

$$r(U \cap S^i_X) \subset S^i_A$$

where S^i_A and S^i_X are formal codimension i strata of A and X respectively.

If (X, A) is a closed pl filtered NDR pair, then A is called a *closed pl filtered NDR of* X.

Let $\mathcal{PLF}$ be the category of closed pl filtered NDR pairs with morphisms strata preserving pl homeomorphisms and compositions of embeddings $A \hookrightarrow X$ of closed pl filtered neighborhood deformational retracts. Let $\mathcal{G}_*$ be the category of graded abelian groups. An *intersection pl homology theory* is a family (indexed by perversities $\bar{p}$) of covariant functors $I_{\bar{p}}H_* : \mathcal{PLF} \to \mathcal{G}_*$ and natural transformations $I_{\bar{p}}\partial_k : I_{\bar{p}}H_k \to I_{\bar{p}}H_{k-1} \circ R$, where $R(X, A) = (A, \emptyset)$, satisfying the following properties:

A1: For every filtered polyhedron X and an interior point b of Δ_n the inclusion map $X \times b \hookrightarrow X \times \Delta_n$ induces an isomorphism

$$I_{\bar{p}}H_*(X \times b) \to I_{\bar{p}}H_*(X \times \Delta_n)$$

where $X \times \Delta_n$ is equipped with the product filtration.

A2: (*Exactness axiom*) For every closed pl filtered NDR pair (X, A) there is a long exact sequence

$$\cdots \to I_{\bar{p}}H_k(A) \to I_{\bar{p}}H_k(X) \to I_{\bar{p}}H_k(X, A) \overset{I_{\bar{p}}\partial_k}{\to} I_{\bar{p}}H_{k-1}(A) \to \cdots$$

An inclusion $j : (X, A) \hookrightarrow (Y, B)$ of closed pl filtered NDR pairs is a *filtered excision map* if $j|_{X \setminus A}$ is a filtration preserving pl homeomorphism.

A3: (*Excision axiom*) Every filtered excision map $j:(X,A)\hookrightarrow(Y,B)$ of closed pl filtered NDR pairs induces and isomorphism

$$I_{\bar{p}}H_*(X,A)\to I_{\bar{p}}H_*(X,B).$$

A4: If X is a filtered polyhedron and $D_0\subset D_1\subset\cdots$ is a sequence of closed pl filtered NDR subpolyhedra of X so that $X=\bigcup D_i$, then the natural map

$$\varinjlim I_{\bar{p}}H_*(D_i)\to I_{\bar{p}}H_*(X)$$

is an isomorphism.

Examples.

1. The Goresky-MacPherson oriented pl intersection homology [GM80] is an intersection pl homology theory. One can define an ordered pl intersection homology for pairs of polyhedra and prove, using Theorem IV.1, its isomorphism with the oriented pl intersection theory.

2. The Goresky-MacPherson sheaf intersection cohomology with compact support [GM83] is an intersection pl homology theory.

3. The singular intersection homology, due to C.H. King (see [Kin85]) is also an example of intersection pl homology theory. King defines a perversity $\bar{p}$ singular k-simplex of a filtered space X as a continuous map $\sigma:\Delta_k\to X$ so

that $\sigma^{-1}(X^s)$ is contained in a $(k-s+\bar{p}(s))$-dimensional skeleton of Δ_k .

The following theorem is a modification of the characterization theorem for intersection homology theories on the class of topologically cone-like stratified spaces due to King (see Theorem 10 in [Kin85]).

THEOREM IV.1. *Let* $\varphi : I_{\bar{p}}H_* \to I_{\bar{p}}H'_*$ *be a natural transformation of intersection pl homology theories so that*

(1) $\varphi(pt) : I_{\bar{p}}H_*(pt) \to I_{\bar{p}}H'_*(pt)$ *is an isomorphism.*

(2) *If* X *is a filtered compact polyhedron and* $\varphi(X) : I_{\bar{p}}H_*(X) \to I_{\bar{p}}H'_*(X)$ *is an isomorphism, then* $\varphi(c(X)) : I_{\bar{p}}H_*(c(X)) \to I_{\bar{p}}H'_*(c(X))$ *is an isomorphism as well.*

Then $\varphi(X) : I_{\bar{p}}H_*(X) \to I_{\bar{p}}H'_*(X)$ *is an isomorphism for any pl stratified euclidean polyhedron* X.

In the next chapter we will show that every filtration $\mathfrak{X}$ of a space X induces a filtration $AG(\mathfrak{X})$ of $AG(X)$. Thus to every filtered polyhedron X we can assign a family of the intersection homotopy groups $I_{\bar{p}}\pi_*(AG(X), AG(\mathfrak{X}))$. Theorem I.1 is a corollary of the following result.

THEOREM IV.2. *The functor $I_{\bar{p}}\pi_* \circ AG$ defines an intersection pl homology theory and there is a natural transformation*

$$\varphi_* : I_{\bar{p}}H_* \rightarrow I_{\bar{p}}\pi_* \circ AG,$$

satisfying the conditions of Theorem IV.1.

Proof of Theorem IV.1. is a modification of the proof of Theorem 10 from [Kin85].

Let $I_{\bar{p}}H_*$ be an intersection homology theory. The exactness and excision axioms imply the following Mayer-Vietoris principle.

PROPOSITION IV.3. *Let X be a filtered polyhedron and let $A, B, A \cap B$ be subpolyhedra of X with the filtrations induced from X. If $X = A \cup B$ and the pairs (X, B) and $(A, A \cap B)$ are closed pl filtered NDR pairs, then there is an exact Mayer-Vietoris sequence*

$$\cdots \rightarrow I_{\bar{p}}H_k(A \cap B) \rightarrow I_{\bar{p}}H_k(A) \oplus I_{\bar{p}}H_k(B) \rightarrow I_{\bar{p}}H_k(X) \rightarrow I_{\bar{p}}H_{k-1}(A \cap B) \rightarrow \cdots$$

Let $\varphi : I_{\bar{p}}H_* \rightarrow I_{\bar{p}}H'_*$ be a natural transformation from Theorem IV.1. First, we will show that from any compact pl manifold M with the trivial stratification the homomorphism $\varphi(M)$ is an isomorphism.

By the axiom A1 and the fact that $\varphi(pt)$ is an isomorphism, for every $n \geq 0$

the homomorphism $\varphi(\Delta_n)$ is an isomorphism. Using inductively the Mayer-Vietoris principle we can show that for every $n \geq 0$ and n-dimensional sphere S_n the homomorphisms $\varphi(S_n)$ are isomorphisms. By the axiom A1, for every $n, m \geq 0$ the homomorphism $\varphi(S_n \times \Delta_m)$ is an isomorphism. If M is a compact pl manifold with a handle decomposition

$$H_0 \cup H_2 \cup \cdots \cup H_k$$

then for every $i \in \{1, 2, \ldots, k\}$ the intersection

$$(H_0 \cup H_1 \cup \cdots \cup H_{i-1}) \cap H_i$$

is pl homeomorphic to $S_n \times \Delta_m$. Hence, by the Mayer-Vietoris principle the homomorphism $\varphi(M)$ is an isomorphism.

In a similar way we can prove that if $\varphi(X)$ is an isomorphism, then for any compact pl manifold M the homomorphism $\varphi(M \times X)$ is an isomorphism.

Let X be a stratified polyhedron. The following part of the proof is by induction on the number of non-empty strata of X. A stratified polyhedron with only one non-empty stratum is a pl manifold, and we have proved that in this case Theorem IV.1 is true.

Let X be a stratified polyhedron with $i+1$ non-empty strata and let Y be the lowest dimensional non-empty stratum of X. Then we may write

$$X = (X \setminus Y) \cup \bigcup_{j=0}^{\infty} A_j$$

where each A_j is a filtered subpolyhedron of X filtered isomorphic to $M_j \times \bar{c}(L_j)$ where M_j is a compact pl manifold and L_j is a stratified polyhedron with j or less non-empty strata. Let

$$D_j = (X \setminus Y) \cup \bigcup_{j=0}^{j-1} A_j$$

and let $B_j = A_j \cap D_j$. It is easy to see that A_j is a closed pl filtered NDR of D_j and B_j (equipped with the filtrations induced from X). The homomorphism $\varphi(B_j)$ is an isomorphism, because B_j is filtered isomorphic to $N \times \bar{c}(L_j)$ where N is a compact pl manifold and by inductive assumption $\varphi(\bar{c}(L_j))$ is an isomorphism. Applying the Mayer-Vietoris principle to the pair D_j, A_j we see that for every $j \geq 0$ the homomorphism $\varphi(D_{j+1})$ is an isomorphism. Hence by the axiom A4 the homomorphism $\varphi(X)$ is an isomorphism. □

CHAPTER V

Proof of the Intersection Dold-Thom Theorem

Let $\mathcal{PL}$ be the category of polyhedra with morphisms - continuous maps. The naïve group completion AG of the infinite symmetric product functor SP constitutes a functor from $\mathcal{PL}$ to the category free abelian topological groups and continuous homomorphisms [DT56]. In Section 1 it will be shown that AG extends to a functor (that we will denote by the same symbol AG) from the category $\mathcal{PLF}$ of closed pl filtered NDR pairs and strata preserving pl homeomorphisms and compositions of embeddings of closed pl filtered neighborhood deformational retracts to the category of finitely filtered free abelian topological groups and continuous cofiltered homomorphisms. Thus the composition $I_{\bar{p}}\pi_* \circ AG$ is a functor from $\mathcal{PLF}$ to $\mathcal{G}_*$.

In Section 2 we will show that $I_{\bar{p}}\pi_* \circ AG$ is an intersection pl homology theory and in Section 3 we will construct a natural transformation

$$\varphi_* : I_{\bar{p}}H_* \rightarrow I_{\bar{p}}\pi_* \circ AG$$

that satisfies the conditions of Theorem IV.1.

1. L-filtrations

The *q-th symmetric product* $SP^q(X)$ *of a space* X is created from the q-product $X^q = X \times \cdots \times X$ (q factors) by identifying points which differ only by the order of components. We denote by $< x_1, \ldots, x_q >$ the point of $SP^q(X)$ determined by $(x_1, \ldots, x_q) \in X^q$.

For example, the q-th symmetric product $SP^q(\mathbb{CP}_1)$ of the complex projective line $\mathbb{CP}_1$ is homeomorphic to $\mathbb{CP}_q$. The homeomorphism is given by assigning to an unordered q-tuple $z_1, z_2, \ldots, z_q$ of points of $\mathbb{CP}_1$ the unique homotety class of a homogeneous polynomial of degree q that vanishes at $z_1, z_2, \ldots, z_q$.

If X is a polyhedron, then the disjoint union $SP(X) = \coprod_{q>0} SP^q(X)$ is a topological abelian monoid with respect to the addition

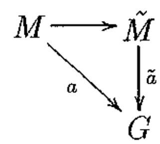
$$< x_1, x_2, \ldots, x_q > + < y_1, y_2, \ldots, y_r > \stackrel{def}{=} < x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_r > .$$

In particular

$$< x_1, x_2, \ldots, x_q > = x_1 + x_2 + \cdots + x_q = \sum_{i=1}^{q} x_i$$

For example, $SP(\mathbb{CP}_1)$ can be identified with the disjoint union of homotety classes of homogeneous polynomials of different degrees. The addition in $SP(\mathbb{CP}_1)$ corresponds, under the above identification, to the multiplication of polynomials.

A *group completion* of a monoid M is a group $\tilde{M}$ together with a monomorphism of monoids $M \to \tilde{M}$ satisfying the following universal property. For every group G and a monoid homomorphism $a : M \to G$ there exists a unique group homomorphism $\tilde{a} : \tilde{M} \to G$ so that the diagram



commutes. If M is an abelian monoid with the cancellation law, then $\tilde{M}$ can be defined as the quotient of the product $M \times M$ with respect to the equivalence relation that identifies (m, n) with (m', n') if and only if $m + n' = m' + n$.

For example, if M is the set of natural numbers, then its group completion is the group of integers. If M is the monoid of vector bundles over X, then $\tilde{M}$ is the Grothendieck K-group of X.

In general, from the fact that M is a topological monoid does not follow that $\tilde{M}$ is a topological group, but for $M = SP(X)$ with X being a polyhedron, the group completion $AG(X)$ is a topological group. For a generalization of this statement see [LF].

The elements of $AG(X)$ can be represented in the form $\sum n_i \cdot x_i$ where n_i are integers, only finite number of which is different from zero. It is convenient to think of the elements of $AG(X)$ as functions from X to $\mathbb{Z}$. Actually, to every formal sum $s = \sum n_i \cdot x_i$ one can assign a function $n_s : X \to \mathbb{Z}$ so that

$$n_s(x) = \sum_{x_i = x} n_i.$$

McCord showed in [McC69] that if X is a polyhedron, then the group $B(\mathbb{Z}, X)$ of functions of finite support from X to $\mathbb{Z}$ can be equipped with a topology so that the natural map

$$n : AG(X) \to B(\mathbb{Z}, X)$$

becomes an isomorphism of topological groups.

In the case of the complex projective line the group $AG(\mathbb{CP}_1)$ can be identified with the space of homogeneous rational functions (of two variables over $\mathbb{C}$) modulo the equivalence relation that identifies two functions that differ by a nonzero constant factor. The addition in $AG(\mathbb{CP}_1)$ is induced by the multiplication of rational functions.

If X is a polyhedron, then $AG(X)$ is a CW-complex (see Theorem 4.6 in [DT56]). Actually, if X is a polyhedron, then for every $q > 0$ the the q-symmetric product $SP^q(X)$ is a polyhedron. The cells of $AG(X)$ are images under the quotient map

$$\eta : SP(X) \times SP(X) \rightarrow AG(X)$$

of the interiors of simplexes of

$$SP(X) \times SP(X) = \coprod_{q,r \geq 0} SP^q(X) \times SP^r(X)$$

whose all point are non-degenerated, where a point

$$(< x'_1, x'_2, \ldots, x'_q >, < x''_1, x''_2, \ldots, x''_q >) \in SP^q(X) \times SP^r(X)$$

is *degenerated* if the sets $\{x'_1, x'_2, \ldots, x'_q\}$ and $\{x''_1, x''_2, \ldots, x''_q\}$ have a common element.

A continuous map $f : X \rightarrow X'$ induces a continuous homomorphism

$$\tilde{f} : AG(X) \rightarrow AG(X')$$

defined by the formula

$$\tilde{f}(\sum n_i \cdot x_i) = \sum n_i \cdot f(x_i).$$

Moreover, the homotopy type of $AG(X)$ depends only on the homotopy type of X.

Let X a polyhedron. If A is a closed subpolyhedron of X, then $AG(A)$ is a closed (normal) subgroup of $AG(X)$ and we define $AG(X, A)$ to be the quotient group $AG(X)/AG(A)$ with the quotient topology. If $A = \emptyset$, then

$AG(A)$ can be identified with the neutral element 0 of $AG(X)$ and we put $AG(X, \emptyset) = AG(X)$. Let

$$\pi : AG(X) \rightarrow AG(X, A)$$

be the quotient map. The image $\pi(x)$ of an element x of $AG(X)$ in $AG(X, A)$ will be denoted in the sequel by $[x]$.

Let (X, A) be a closed pl filtered pair. A *L-filtration*

$$AG(\mathfrak{X}, \mathfrak{A}) = \{AG(\mathfrak{X}, \mathfrak{A})^k\}$$

of $AG(X, A)$ is defined in the following way. $[\sum n_i \cdot x_i] \in AG(\mathfrak{X}, \mathfrak{A})^k$ if there exists a nondegenerate representative $\sum m_j \cdot x_j$ of $[\sum n_x \cdot x]$ so that $x_j \in X^k$ for some $m_j \neq 0$. In other words, an element s of $AG(X, A)$ belongs to $AG(\mathfrak{X}, \mathfrak{A})^k$ if the induced by s function $n_s : X \rightarrow \mathbb{Z}$ is non-zero on some element of X^k. By definition, the zero element 0 of $AG(X, A)$ is in the regular stratum of $AG(X, A)$. Thus for every $k > 0$ if $AG(\mathfrak{X}, \mathfrak{A})^k \neq \emptyset$, then the skeleton $AG(\mathfrak{X}, \mathfrak{A})^k$ is not closed in $AG(X, A)$.

Let $f : (X, A) \rightarrow (Y, B)$ be a continuous map of pairs of polyhedra. The formula

$$\tilde{f}([\sum n_i \cdot x_i]) = [\sum n_i \cdot f(x_i)]$$

defines a continuous map

$$\tilde{f} : AG(X, A) \rightarrow AG(Y, B).$$

Note that if f is a cofiltered map, then $\tilde{f}$ is cofiltered. Moreover, if f and g are cofiltered maps of pairs of filtered polyhedra, then $\tilde{g} \circ \tilde{f} = \widetilde{g \circ f}$. Thus AG is a functor from $\mathcal{PLF}$ to the category of finitely filtered free abelian topological groups and continuous cofiltered homomorphisms.

Let $I_{\bar{p}}\pi_*(AG(X,A))$ denotes perversity $\bar{p}$ intersection homotopy groups of the filtered pair $((AG(X,A),0),(AG(\mathfrak{X},\mathfrak{A}),0))$ where 0 denotes the neutral element of $AG(X,A)$ or the trivial filtration on 0. By definition, $I_{\bar{p}}\pi_*(AG(X,A))$ is the group of $\bar{p}$ -homotopy classes of maps $\sigma : \Delta_k \rightarrow AG_0(X,A)$ so that for every $0 \leq i \leq k$ the simplex $\partial_i\sigma$ maps Δ_{k-1} to 0, where $AG_0(X,A)$ is the connected component of 0 in $AG(X,A)$ that is the set of elements of $AG(X,A)$ of degree 0.

If $f : (X,A) \rightarrow (Y,B)$ is a cofiltered map, then $\tilde{f}$ is cofiltered and hence it induces a homomorphism

$$\tilde{f}_* : I_{\bar{p}}\pi_*(AG(X,A)) \rightarrow I_{\bar{p}}\pi_*(AG(Y,B)).$$

Moreover, if f and g are cofiltered maps of pairs of filtered polyhedra, then

$$\tilde{g}_* \circ \tilde{f}_* = (\tilde{g} \circ \tilde{f})_* = (\widetilde{g \circ f})_*.$$

Thus the assignment

$$\begin{aligned}\mathrm{Ob}(\mathcal{PLF}) \ni (X, A) &\mapsto I_{\bar{p}}\pi_*(AG(X, A)) \in \mathrm{Ob}(\mathcal{G}_*)\\ \mathrm{Mor}(\mathcal{PLF}) \ni f &\mapsto \tilde{f}_* \in \mathrm{Mor}(\mathcal{G}_*)\end{aligned}$$

is a covariant functor form $\mathcal{PLF}$ into $\mathcal{G}_*$.

2. $I_{\bar{p}}\pi_* \circ AG$: An Intersection PL Homology Theory

In this section we will show that $I_{\bar{p}}\pi_* \circ AG$ satisfies the axioms of intersection pl homology theory.

The axiom A1 is satisfied by the following lemma.

LEMMA V.1. *Let X be a space with a filtration $\mathfrak{X}$ and let $X \times \Delta_n$ be equipped with the product filtration $\mathfrak{X} \times \Delta_n$. Then for any interior point b of Δ_n and every perversity $\bar{p}$ the inclusion map $X \times b \hookrightarrow X \times \Delta_n$ induces an isomorphism*

$$I_{\bar{p}}\pi_*(AG(X \times b)) \rightarrow I_{\bar{p}}\pi_*(AG(X \times \Delta_n))$$

Proof. It is enough to show that every perversity $\bar{p}$ k-simplex

$$\sigma : \Delta_k \rightarrow AG(X \times \Delta_n)$$

is $\bar{p}$ -homotopic to a simplex $\sigma' : \Delta_k \rightarrow AG(X \times b)$.

Let us embed Δ_n into $\mathbb{R}_n$ so that b is send into the origin of $\mathbb{R}_n$ and let $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n$ be induced on Δ_n coordinate system.

The map

$$H : AG(X \times \Delta_n) \times I \longrightarrow AG(X \times \Delta_n)$$

$$H(\sum n_i \cdot (x_i, \bar{t}_i)\,,\, t) = \sum n_i \cdot (x_i, \bar{t}_i \cdot t)$$

is a homotopy between the identity map on $AG(X \times \Delta_n)$ and the retraction

$$AG(X \times \Delta_n) \rightarrow AG(X \times 0)$$

$$\sum n_i \cdot (x_i, \bar{t}_i) \mapsto \sum n_i \cdot (x_i, 0)$$

Immediately from the definition of H it follows that

$$H^{-1}(AG(\mathfrak{X} \times \Delta_n)^s) \subset AG(\mathfrak{X} \times \Delta_n)^s \times I.$$

Hence, H is a perversity $\bar{p}$ map and the composition $(\sigma \times \mathrm{id}) \circ \hat{H}$ is the required $\bar{p}$ -homotopy between σ and σ'. $\square$

Essentially the same proof gives the following result.

LEMMA V.2. *Let X be a space with a filtration $\mathfrak{X}$ and let $X \times \mathbb{R}_n$ be equipped with the product filtration $\mathfrak{X} \times \mathbb{R}_n$. Then for every perversity $\bar{p}$ the inclusion map $X \times 0 \hookrightarrow X \times \mathbb{R}_n$ induces an isomorphism*

$$I_{\bar{p}}\pi_*(AG(X \times 0)) \rightarrow I_{\bar{p}}\pi_*(AG(X \times \mathbb{R}_n))$$

The proof of the exactness axiom for $\pi_* \circ AG$ (see [DT56, LF]) is based on the observation that for an arbitrary CW-complex X and its subcomplex A the projection $\pi : AG(X) \to AG(X)/AG(A)$ is a fibration (it is a locally trivial fibration with fiber AG(A)). Actually, if this is the case, then the long exact sequence of homotopy groups of this fibration is the long exact sequence of the pair (X, A) for the functor $\pi_* \circ AG$. The following result is the intersection counterpart of the above observation.

THEOREM V.3. *Let (X, A) be a closed pl filtered NDR pair. Then the projection $\pi : AG(X) \to AG(X, A)$ is a filtered fibration with respect to the filtrations $AG(\mathfrak{X})$ and $AG(\mathfrak{X}, \mathfrak{A})$ of $AG(X)$ and $AG(X, A)$ respectively. Moreover, the fiber $AG(A)$ of π over 0 is a filtered subspace of $AG(X)$ and hence π induces a long exact sequence of intersection homotopy groups.*

The last statement of Theorem V.3 follows directly form the definition of AG filtrations. The first part of Theorem V.3 is a consequence of Theorem III.3 and the following lemma.

LEMMA V.4. *Let (X, A) be a closed pl filtered NDR pair. Then there exists an open covering $\mathcal{U}$ of $AG(X, A)$ so that for every element U of $\mathcal{U}$ there exists* *a trivialization homeomorphism* $\varphi : \pi^{-1}(U) \to U \times AG(A)$ *so that*

$$\varphi(\pi^{-1}(U) \cap AG(\mathfrak{X})^s) = U^s \times AG(A) \cup U \times AG(\mathfrak{A})^s \tag{9}$$

where $\mathfrak{X}$ *is a filtration of* X*,* $\mathfrak{A}$ *is the induced from* $\mathfrak{X}$ *filtration of* A*, and* $U^s = U \cap AG(\mathfrak{X}, \mathfrak{A})^s$.

Proof. The following proof is a modification of the proof of Satz 5.4 from [DT56].

Let G be a group and let H be its closed subgroup. Then the quotient map $\pi : G \to G/H$ is a principal H-bundle if there exists an open neighborhood U of $[H]$ in G/H so that one of the following equivalent conditions holds.

(1) There exists a section $s : U \to G$ of π.

(2) There exists an H-equivariant map $\sigma : \pi^{-1}(U) \to H$.

If (X, A) is a closed pl pair, then $AG(X)$ is a topological group, $AG(A)$ is its closed subgroup, and the projection

$$\pi : AG(X) \to AG(X, A)$$

is a quotient map. Dold and Thom proved that there exists a local section of π and hence π is a principal $AG(A)$-fibration.

We will show that there exist: an open neighborhood U of 0 in $AG(X, A)$

and a $AG(A)$-equivariant map

$$\sigma : \pi^{-1}(U) \rightarrow AG(A)$$

so that the the trivialization

$$\varphi = (\pi, \sigma) : \pi^{-1}(U) \rightarrow U \times AG(A)$$

satisfies the condition (9). Then we will see that U and σ induce: an open covering $\{U_\alpha\}$ and a family of $AG(A)$-equivariant maps

$$\sigma_\alpha : \pi^{-1}(U_\alpha) \rightarrow AG(A)$$

for which the trivialization $\varphi_\alpha = (\pi, \sigma_\alpha)$ satisfy the condition (9).

Let

$$AG^k(X) = \rho(\coprod_{q+r \leq k} SP^q(X) \times SP^r(X))$$

be the subset of $AG(X)$ whose elements have at most k different from 0 components. We set $AG^k(X, A) = \pi(AG^k(X))$. It is easy to see that a subset V of $AG(X)$ (or $AG(X, A)$) is open if and only if for every $k \geq 0$ the intersection $V \cap AG^k(X)$ (or $V \cap AG^k(X, A)$) is open in $AG^k(X)$ (or $AG^k(X, A)$). We will construct: a family $\{U_k\}$ of open neighborhoods of 0 in $AG^k(X, A)$ and a family $\{\sigma_k\}$ of $AG(A)$-equivariant maps

$$\sigma_k : \pi^{-1}(U_k) \rightarrow AG(A)$$

so that $U_{k+1}|_{AG^k(X,A)} = U_k$ and $\sigma_{k+1}|_{U_k} = \sigma_k$. Then we set $U = \bigcup_{k \geq 0} U_k$ and $\sigma|_{U_k} = \sigma_k$.

Let W be an open neighborhood of A in X for which there is a strata preserving retraction $r : W \to A$. Then $AG(W)$ is an open neighborhood of $AG(A)$ in $AG(X)$ and hence for every $k \geq 0$ the intersection $W_k = AG(W) \cap AG^k(X)$ is open in $AG^k(X)$. The set W_k consists of elements of $AG(W)$ with at most k different from 0 components. We will write them as follows

$$\sum_{\sum |n_i| \leq k} n_i \cdot w_i$$

For every $k \geq 0$ the set $U_k = \pi(W_k)$ is open in $AG^k(X, A)$ and $\pi^{-1}(U_k)$ consists of elements of $AG(X)$ of the form

$$\sum_{\substack{\sum |n_i| \leq k \\ w_i \in W \setminus A}} n_i \cdot w_i + \sum_{a_j \in A} n_j \cdot a_j$$

Let us define σ_k by the formula

$$\sigma_k\Big(\sum_{\substack{\sum |n_i| \leq k \\ w_i \in W \setminus A}} n_i \cdot w_i + \sum_{a_j \in A} n_j \cdot a_j \Big) = \sum_{\substack{\sum |n_i| \leq k \\ w_i \in W \setminus A}} n_i \cdot r(w_i) + \sum_{a_j \in A} n_j \cdot a_j$$

It is easy to see that it is a continuous $AG(A)$-equivariant map and moreover for every $k \geq 0$ we have $U_{k+1}|_{\widetilde{SP}_k(X,A)} = U_k$ and $\sigma_{k+1}|_{U_k} = \sigma_k$.

We have to show that

$$(10) \qquad (\pi, \sigma_k)(\pi^{-1}(U_k) \cap AG(\mathfrak{X})^s) = (U_k)^s \times AG(A) \cup U \times AG(\mathfrak{A})^s$$

where $(U_k)^s = U_k \cap AG(\mathfrak{X},\mathfrak{A})^s$.

The intersection $\pi^{-1}(U_k) \cap AG(\mathfrak{X})^s$ consists of the points

$$\sum_{\substack{\sum |n_i| \le k \\ w_i \in W\setminus A}} n_i \cdot w_i \; + \; \sum_{a_j \in A} n_j \cdot a_j$$

so that:

1. $w_i \in (W \setminus A) \cap X^s$ for some $n_i \neq 0$.

or

2. $a_j \in A \cap X^s$ for some $n_j \neq 0$.

In the first case $\pi(z) \in (U_k)^s$. In the second case $\sigma_k(x) \in AG(\mathfrak{A})^s$, because r satisfies the condition

$$r(W \cap S_X^k) \subset S_A^k.$$

Since for every pl stratification $\mathfrak{X}$ of X the regular set of $\mathfrak{X}$ is dense in X, the regular set of $AG(\mathfrak{X},\mathfrak{A})$ is also dense in $AG(X,A)$ and hence every point of $AG(X,A)$ has a neighborhood of the form $y + U$ where $y \in$reg$(AG(\mathfrak{X},\mathfrak{A}))$. Let $\bar{y} \in \sigma(X)$ be so that $\pi(\bar{y}) = y$ and $\bar{y}$ does not have any components in A. Then it is easy to see that the trivialization

$$\varphi_y = (\pi, \sigma_y) : \pi^{-1}(y+U) \rightarrow (y+U) \times AG(A)$$

where $\sigma_y(x) = \sigma(x - \bar{y})$ satisfies the condition (9). $\square$

The following lemma takes care for the axiom A3.

LEMMA V.5. *A filtered excision map $j : (X, A) \hookrightarrow (Y, B)$ of filtered pairs with filtrations $(\mathfrak{X}, \mathfrak{A})$ and $(\mathfrak{X}, \mathfrak{B})$ induces a filtered isomorphism of topological groups*

$$\tilde{j} : AG(X, A) \rightarrow AG(Y, B)$$

with respect to the filtrations $AG(\mathfrak{X},\mathfrak{A})$ and $AG(\mathfrak{Y},\mathfrak{B})$ of $AG(X, A)$ and $AG(Y, B)$ respectively.

Proof. Lima-Filho ([LF]) proved that $\tilde{j}$ is an isomorphism of the topological groups. Note, that the filtration $AG(\mathfrak{X},\mathfrak{A})$ induced by $(\mathfrak{X}, \mathfrak{A})$ is determined by the filtration induced by $\mathfrak{X}$ on $X \setminus A$. In particular, if $j : (X, A) \hookrightarrow (Y, B)$ is a filtered excision map, then $\tilde{j}(AG(\mathfrak{X},\mathfrak{A})^k) = AG(\mathfrak{Y},\mathfrak{B})^k$ for every k. That is $\tilde{j}$ is a filtered isomorphism of topological groups. $\square$

$I_{\bar{p}}\pi_* \circ AG$ satisfies the last A4 axiom, because if $S_0 \subset S_1 \subset \cdots$ is a sequence of Kan subcomplexes of a Kan complex S so that $S = \bigcup S_i$, then for every non-negative integer the natural map

$$\varinjlim \pi_k(S_i) \longrightarrow \pi_k(S)$$

is an isomorphism.

3. Natural Transformation $\varphi_* : I_{\bar{p}}H_* \to I_{\bar{p}}\pi_* \circ AG$.

We will show that there is a natural transformation

$$\varphi_* : I_{\bar{p}}H_* \to I_{\bar{p}}\pi_* \circ AG$$

of intersection pl homology theories satisfying the conditions of Theorem IV.1.

Let X be a polyhedron with a filtration $\mathfrak{X}$ and let $\sum n_i\sigma_i$ be a singular k-chain of X. The assignment $s \mapsto \sum n_i\sigma_i(s)$ defines a map $\varphi(\sum n_i\sigma_i) : \Delta_k \to AG(X)$. This map is continuous because $AG(X)$ is a topological group. Hence we have a map $\varphi : AG(S(X)) \to S(AG(X))$ where $AG(S(X))$ stands for the free abelian simplicial set generated by $S(X)$. φ is a simplicial map because

$$\partial_i(\varphi(\sum n_i\sigma_i))(s) = \varphi(\sum n_i\sigma_i)(\delta_i(s)) = \sum n_i\sigma_i(\delta_i(s)) =$$

$$\sum n_i(\partial_i\sigma_i)(s) = \varphi(\sum n_i(\partial_i\sigma_i))(s)$$

In the same way one can prove that φ commutes with degeneracy operators.

It is easy to see that φ is a natural transformation between $AG \circ S$ and $S \circ AG$ (the action of φ on morphisms is irrelevant for validity of Theorem IV.1). We will see that φ restricts to a natural transformation from $AG \circ I_{\bar{p}}S$ into $I_{\bar{p}}S \circ AG$.

Consider a singular k-chain $\sum n_i\sigma_i$ of perversity $\bar{p}$ in $(X, \mathfrak{X})$. The singular simplex $\varphi(\sum n_i\sigma_i)$ is of perversity $\bar{p}$ with respect to $AG(\mathfrak{X})$ because

$$\varphi(\sum n_i\sigma_i)^{-1}(AG(\mathfrak{X})^l) = \{s \in \Delta_k \mid \exists\, i \text{ so that } \sigma_i(s) \in X^l\} = \bigcup(\sigma_i)^{-1}(X^l)$$

and every simplex σ_i is of perversity $\bar{p}$.

Since φ is a simplicial map and for every j the simplex $\partial_j\sigma_i$ is of perversity $\bar{p}$, the map $\partial_j(\varphi(\sum n_i\sigma_i))$ is of perversity $\bar{p}$ as well. We have proved that $\varphi : AG(I_{\bar{p}}S(X)) \rightarrow I_{\bar{p}}S(AG(X))$ is a natural simplicial map. Hence, applying simplicial homotopy groups functor, we get the required natural transformation $\varphi_* : I_{\bar{p}}H_* \rightarrow I_{\bar{p}}\pi_* \circ AG$.

It is easy to see that $\varphi(pt)$ is an isomorphism. The second condition of Theorem IV.1 is satisfied by the following lemma.

LEMMA V.6. *If X is a filtered space with a filtration of length n, then for every perversity $\bar{p}$ and positive integer k there is an isomorphism*

$$I_{\bar{p}}\pi_k(AG(c(X))) \cong \begin{cases} I_{\bar{p}}\pi_k(AG(X)) & \text{for } k < n - \bar{p}(n+1) \\ 0 & \text{for } k \geq n - \bar{p}(n+1) \end{cases}$$

Recall that if X is as in Lemma V.6, then there is an isomorphism

$$I_{\bar{p}}H_k(c(X)) \cong \begin{cases} I_{\bar{p}}H_k(X) & \text{for } k < n - \bar{p}(n+1) \\ 0 & \text{for } k \geq n - \bar{p}(n+1) \end{cases}$$

Hence, if $\varphi_*(X) : I_{\bar{p}}H_*(X) \to I_{\bar{p}}\pi_*(AG(X))$ is an isomorphism, then $\varphi_*(c(X))$ is an isomorphism.

Proof of Lemma V.6. Let $\mathfrak{X}$ be a filtration of length n on X and let $c(\mathfrak{X})$ be the induced filtration on the cone $c(X)$. A simplex $\sigma : \Delta_l \to AG(c(X))$ is of perversity $\bar{p}$ if $\sigma^{-1}(AG(c(\mathfrak{X}))^s)$ is of dimension $\leq l - s + \bar{p}(s)$. Recall that $AG(c(\mathfrak{X}))^{n+1}$ consists of those elements of $AG(c(X))$ whose nondegenerated representatives contain the vertex of $c(X)$. Hence, if $(k+1)-(n+1)+\bar{p}(n+1) < 0$, then for every: $l \leq k+1$, a simplex $\sigma : \Delta_l \to AG(c(X))$, and a point $s \in \Delta_l$ the nondegenerated representative of the element $\sigma(s)$ does not contain the vertex of $c(X)$. That is $\sigma : \Delta_l \to AG(c(X) \setminus (*)) = AG(X \times \mathbb{R})$. Therefore, for $(k+1) - (n+1) + \bar{p}(n+1) < 0$ one has an isomorphism

$$I_{\bar{p}}\pi_*(AG(c(X))) \cong I_{\bar{p}}\pi_*(AG(X \times \mathbb{R}))$$

and by Lemma V.2 the last group is isomorphic to $I_{\bar{p}}\pi_*(AG(X))$.

Let now $(k+1) - (n+1) + \bar{p}(n+1) \geq 0$. We will see that every k-simplex of perversity $\bar{p}$ is $\bar{p}$-homotopic to the constant map map to 0. Let H be a deformation of the identity map on $c(X)$ to the retraction of $c(X)$ onto $(*)$.

H induces a map $\hat{H} : AG(c(X)) \times I \rightarrow AG(c(X))$ between the identity map on $AG(c(X))$ and the projection of $AG(c(X))$ onto 0. It is easy to see that $\hat{H}$ is of perversity $\bar{p}$ and the composition $(\sigma \times id) \circ \hat{H}$ gives the required $\bar{p}$-homotopy between σ and the constant map to 0. $\square$

Bibliography

[Aki69] E. Akin, *Manifold phenomena in the theory of polyhedra*, Trans. Amer. Math. Soc. **143** (1969), 413–473.

[DT56] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische produkte*, Ann. of Math. (2) **67** (1956), 230–281.

[FJA62] Jr. F. J. Almgren, *Homotopy groups of the integral cycle groups*, Topology **1** (1962), 257–299.

[FM81] W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc., vol. 243, AMS, 1981.

[Fri91] E. Friedlander, *Algebraic cycles, Chow varieties and Lawson homology*, Compositio Math. **77** (1991), 55–93.

[GM80] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[GM83] M. Goresky and R. MacPherson, *Intersection homology II*, Invent. Math. **72** (1983), 77–130.

[Gor81] R. M. Goresky, *Whitney stratified chains and cochains*, Trans. Amer. Math. Soc. **267** (1981), 175–196.

[Kin85] J. King, *Topological invariance of intersection homology without sheaves*, Topology and its Applications **20** (1985), 146–160.

[Law89] H. B. Lawson, Jr., *Algebraic cycles and homotopy theory*, Ann. of Math. (2) **129** (1989), 253–291.

[LF] P. C. Lima-Filho, *Completions and fibrations for topological monoids*, To appear in Trans. Amer. Math. Soc.

[May82] J. P. May, *Simplicial objects in algebraic topology*, Univ. of Chicago Press, Chicago, IL, 1982.

[McC69] M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. **146** (1969), 273–298.

[Sie72] L. C. Siebenmann, *Deformation of homeomorphisms on stratified sets*, Comment. Math. Helv. **47** (1972), 123–163.

[Spa59] E. H. Spanier, *Infinite symmetric products, function spaces, and duality*, Ann. of Math. (2) **69** (1959), 143–198.