Symplectic Embeddings of Balls and the Mapping Problem

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Abstract of Dissertation

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The symplectic mapping problem is studied for open, infinite volume subsets of $(\mathbb{R}^4, \omega_0)$. The spaces considered are symplectically convex and are generalizations of the space underlying the symplectic camel problem. Spaces are symplectically distinguished by using holomorphic techniques to study symplectic embeddings of balls.

To Paul

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List of Mathematical Symbols

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Chapter 1

Overview

A *symplectic manifold* is a smooth, even dimensional manifold M^{2n} together with a closed, non-degenerate differential 2-form ω. This means ω satisfies

$$dw = 0, \qquad \omega^n = \omega \wedge \cdots \wedge \omega \neq 0.$$

$\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$ has a standard symplectic form given by

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Any open subset U of $\mathbb{R}^{2n}$ has a symplectic form induced by ω_0.

A *symplectic diffeomorphism* (or *symplectomorphism*) is a diffeomorphism between two symplectic manifolds

$$\psi\colon (M_1, \omega_1) \to (M_2, \omega_2)$$

such that $\psi^*\omega_2 = \omega_1$. Two symplectic manifolds are *symplectically equivalent,* or *symplectomorphic*, if there exists a symplectic diffeomorphism between them. A basic problem in symplectic geometry is to understand when two symplectic manifolds are symplectically equivalent. The *symplectic mapping problem* is this equivalence question addressed to open subsets of

$(\mathbb{R}^{2n}, \omega_0)$: *When are two open, diffeomorphic subsets of* $\mathbb{R}^{2n}$ *symplectically equivalent?*

Since ω_0^n is a volume form on $\mathbb{R}^{2n}$, two open subsets can be symplectomorphic only if they have the same volume. However, having the same volume does not guarantee symplectic equivalence. For example, if $D(r) \subset \mathbb{R}^2$ is the open 2-dimensional disc of radius r, the polydiscs

$$D(1) \times D(1) \qquad \text{and} \qquad D(1/2) \times D(2)$$

have the same volume. In fact, there exists a *volume-preserving* diffeomorphism, i.e. a diffeomorphism ϕ satisfying $\phi^*(\omega_0 \wedge \omega_0) = \omega_0 \wedge \omega_0$, between them. However in [G], Gromov proved that these polydiscs are not symplectically equivalent. Using the theory of capacities, the mapping problem has been studied for higher dimensional polydiscs and other open subsets of $\mathbb{R}^{2n}$. (See [EkH1], [EkH2], [H1], [H2].) In the following chapters, holomorphic techniques are applied to give new examples of open subsets of $\mathbb{R}^4$ which are equivalent with respect to volume preserving diffeomorphisms but are not symplectically equivalent.

Holomorphic techniques were introduced into symplectic geometry by Gromov in [G]. The theory uses the fact that any symplectic manifold M can be viewed as an almost complex manifold where the almost complex structure, J, satisfies certain "compatibility" conditions with the symplectic structure. A J-holomorphic map of a surface into M is a natural generalization of a holomorphic map and studying the images of such maps can lead to a great deal of information about the underlying symplectic manifold. (See [M4].)

In Chapter 2, J-holomorphic maps of the disc into $\mathbb{R}^4$ are studied. A key result which is used to prove many of the main results in Chapters 2 through 5 is a "sphere filling" theorem. Below is a paraphrase of this result. For a precise statement see (2.3.1)

Sphere Filling Theorem. *Given a 2-dimensional sphere S in $\mathbb{R}^4$, under a "convexity" condition on the almost complex structure, there exists a 1-parameter family of J-holomorphic discs whose boundaries foliate S and whose union, $F(J)$, is diffeomorphic to a 3-dimensional ball.*

The filling technique is applied, in Chapter 3, to study the space of symplectic embeddings of a closed ball into open subsets of $\mathbb{R}^4$ whose boundaries contain 2-dimensional spheres and which are "symplectically convex." In particular, it is shown that for special open subsets, the space of embeddings has more than one path component (Theorem 3.2.1) and a criterion is found for determining when a given embedding is in a standard path component (Theorem 3.3.3).

In the remaining chapters, the mapping problem is considered for open, connected subsets which are the union of half spaces and open 3-dimensional balls. The 3-dimensional balls can be thought of as "holes" and the results of Chapter 3 are applied to study the obstruction formed by these holes.

In Chapter 4, spaces referred to as C spaces are investigated. $C(\lambda) \subset \mathbb{R}^4$ is defined as

$$C(\lambda) = \{y_1 < 0\} \cup \{y_1 > 0\} \cup \{x_1^2 + x_2^2 + y_2^2 < \lambda^2, y_1 = 0\}.$$

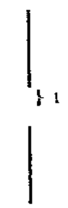

FIGURE 1.1. $C(1)$

The first main result is:

Theorem 4.3.2 (McDuff and Traynor, [MT]**).** *$C(\lambda)$ is symplectomorphic to $C(\mu)$ if and only if $\lambda = \mu$.*

However it is possible to change the shape of $C(\lambda) \cap \{y_1 = 0\}$ without producing a symplectically different space (Theorem 4.3.3).

The C spaces can be thought of as spaces having one wall which has one hole. The C spaces are generalized, in Chapter 5, to W spaces which are spaces with multiple one holed walls.

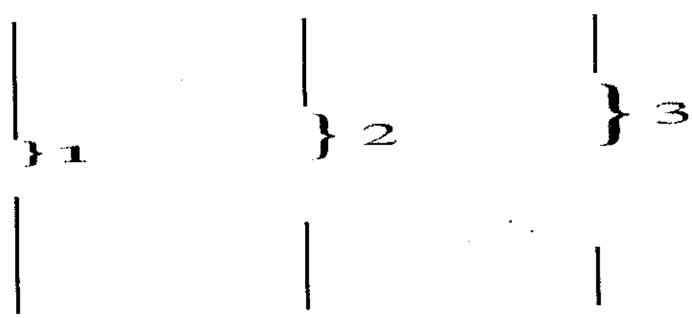

FIGURE 1.2. $W(1, 2, 3)$

By studying symplectic embeddings of balls, spaces with different numbers of holes or different numbers of holes of a given radius can be distinguished.

Theorem 5.2.6. *If $W(\lambda_1, \ldots, \lambda_n)$ is symplectomorphic to $W(\mu_1, \ldots, \mu_m)$ then $m = n$ and, more generally, for all r,*

$$\left|\{j\colon \lambda_j \leq r\}\right| = \left|\{k\colon \mu_k \leq r\}\right|.$$

In addition, "embedding trees" are constructed which distinguish some different orderings of the holes. However these trees do not distinguish other orderings which are believed to correspond to non-symplectically equivalent spaces.

Corollary 5.3.3. *Neither $W(1,2,3)$ nor $W(1,3,2)$ is symplectically equivalent to $W(2,1,3)$. More generally, if $\lambda_1 < \cdots < \lambda_n$ and σ is a permutation of $\{1, \ldots, n\}$ so that $\sigma(1) \notin \{1, n\}$ then $W(\lambda_1, \ldots, \lambda_n)$ and $W\left(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}\right)$ are not symplectically equivalent.*

In Chapter 6, C spaces are generalized to Z spaces which are spaces having one multi-holed wall.

FIGURE 1.3. $Z(1,2)$

Recent results of Eliashberg and Hofer [EH1] suggest that changing the relative positions of the holes is perhaps important (Theorem 6.2.5). Again, by studying symplectic embeddings of balls, it is possible to distinguish spaces with different size holes (Theorem 6.3.6). A slight modification of particular Z spaces produces spaces referred to as P^0 spaces which are symplectically equivalent to the above mentioned W spaces (Theorem 6.4.2).

Chapter 2

Filling Spheres with J-holomorphic Discs

For $\mathbb{R}^4$ with the standard symplectic structure ω_0, there exist almost complex structures J on $\mathbb{R}^4$ which are ω_0*-compatible* in the sense that

$$\omega_0(v, Jv) > 0 \text{ for all } v \neq 0 \quad \text{and} \quad \omega_0(Jv, Jw) = \omega_0(v, w)$$

(see [MS]). This first condition is often referred to as J being ω_0*-tame* and the second as J being ω_0*-calibrated.* $D \subset \mathbb{C}$ will denote the closed unit disc with complex structure i. Then a J*-holomorphic disc* in $(\mathbb{R}^4, \omega_0)$ is the image of a (C^1) map $f\colon D \to \mathbb{R}^4$ with J-linear derivative,

$$df \circ i = J \circ df,$$

where J is ω_0-compatible.

Given a 2-dimensional sphere $S \subset \mathbb{R}^4$, if there exists a 1-parameter family of J-holomorphic discs whose boundaries foliate S and whose union forms a submanifold diffeomorphic to a 3-dimensional ball, there is said to be a *filling*, $F(J)$, of S. Notice that if $S := \{x_1^2 + x_2^2 + y_2^2 = 1, y_1 = 0\}$, $B := \{x_1^2 + x_2^2 + y_2^2 \leq 1, y_1 = 0\}$, each slice $D_c := B \cap \{x_1 = c\}$, $|c| \leq 1$, is a holomorphic (J_0-holomorphic) disc. Thus there exists a filling for the standard almost complex structure J_0. In general, a convexity condition must be imposed on J to guarantee the existence of a filling. To describe this condition, let $\partial\Omega$ be an oriented hypersurface in $(\mathbb{R}^4, J)$, and, for each $x \in \partial\Omega$, let $\xi_{x,J}$ be the maximal J-invariant subspace of the tangent space $T_x\partial\Omega$. $\partial\Omega$ is said to be *J-convex* if for one (and hence any) defining 1-form α of ξ, $d\alpha(v, Jv) > 0$ for all non-zero $v \in \xi_{x,J}$. Since $\xi = \ker\alpha$, it follows that ξ is a contact structure on $\partial\Omega$. Eliashberg has proven, although not published all details of his proof, that an embedded copy S of S^2 in $\mathbb{R}^4$ has a filling corresponding to an integrable, ω_0-tame J if S is contained in a J-convex hypersurface, (see [E]). In other words, there exists a filling of S by J-holomorphic discs if S can be extended to a hypersurface $\partial\Omega$ such that $\partial\Omega$ is J-convex. Moreover, this convexity condition cannot be completely removed since Eliashberg and Harlamov have constructed an example of an embedded S^2 and a complex structure J for which there does not exist a filling [E].

In this chapter, a filling result is proven for the case where S is a "standard" 2-dimensional sphere and J, $\partial\Omega$ are "standard" near S. These conditions are general enough for the applications in the remaining chapters. In addition to showing the existence of a 1-parameter family of discs, it is shown that the discs in the filling are disjoint and, away from the "poles" of S, embedded. This will be important when studying symplectic embeddings of balls in Chapter 3.

A large portion of this chapter appears in [MT].

2.1 Set-Up

$\mathcal{J}$ will be the set of smooth, ω_0-compatible almost complex structures on $\mathbb{R}^4$, $J_0 \in \mathcal{J}$ will denote the standard almost complex structure, and $\Omega \subset \mathbb{R}^4$, $\Omega = \{\varphi \leq 1\}$ will denote a region satisfying the following properties:

(1) $\pi_2(\Omega) = 0$;

(2) there exists a vector field η, transverse to $\partial\Omega$, satisfying

$$\mathcal{L}_\eta \omega_0 = 2\omega_0;$$

(3) there exists $\beta > 0$ and $0 < h \leq 2$ such that **(2.1.1)**

(a) $\varphi|_{\{|y_1|<\beta\}} = h\, x_1^2 + (2-h)\, y_1^2 + x_2^2 + y_2^2$;

(b) on $\{|y_1| < \beta\} \cap \partial\Omega$,

$$\begin{aligned} \eta &= \frac{1}{2}\operatorname{grad}\varphi \\ &= hx_1\frac{\partial}{\partial x_1} + (2-h)y_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}. \end{aligned}$$

For convenience, in this chapter it will be assumed that $h = 1$. However all arguments easily generalize. Notice then that (3)(a) implies $S = \{x_1^2 + x_2^2 + y_2^2 = 1, y_1 = 0\} \subset \partial\Omega$ and moreover that $B = B(\beta) = \{x_1^2 + y_1^2 + x_2^2 + y_2^2 \leq 1, |y_1| < \beta\} \subset \Omega$. Consider $\alpha = i(\eta)\omega_0$. Condition (2) implies $d\alpha = 2\omega_0$. Then since η is transverse, it follows that α is a contact form on $\partial\Omega$. Moreover, $\partial\Omega$ is J-convex for any ω_0-tame J such that $\ker \alpha|_{\partial\Omega}$ is J-invariant. Let $\mathcal{C}_\Omega$ be the (non-empty) open subset of $\mathcal{J}$ consisting of those J which make $\partial\Omega$ J-convex. Fix $0 < \varepsilon < 1$ and let

$$\mathcal{J}_\Omega = \big\{J \in \mathcal{C}_\Omega \colon J = J_0 \text{ on } \{\varphi > 1 - \varepsilon\} \cap \{|y_1| < \beta\}\big\}.$$

By (3)(b), on $\partial\Omega \cap \{|y_1| < \beta\}$, $J_0\eta \in \ker \omega_0|_{\partial\Omega}$ and thus $\mathcal{J}_\Omega$ is non-empty.

Next, given δ such that $1 - \varepsilon < \delta < 1$, put

$$S_\delta = \{(x_1, 0, x_2, y_2) \in S \colon |x_1| < \delta\}.$$

Then S_δ is diffeomorphic to an open annulus and, since the elements of $\mathcal{J}_\Omega$ are standard near S, for each $J \in \mathcal{J}_\Omega$, S_δ is a totally real surface sitting inside a J-convex $\partial\Omega$. (A submanifold M of the almost complex manifold (V, J) is said to be *totally real* if $TM \cap JTM = \{0\}$.) Let D be the unit disc in $\mathbb{C}$ with almost complex structure i and let A be the generator of $\pi_2(\mathbb{R}^4, S_\delta)$ which is represented by

$$(D, \partial) \to (\mathbb{R}^4, S_\delta)$$
$$(x, y) \mapsto (0, 0, x, y).$$

Notice that by the construction of $\mathcal{J}_\Omega$, for each c such that $1 - \varepsilon < |c| < 1$,

$$f_c(x, y) = (c, 0, \sqrt{1 - c^2}x, \sqrt{1 - c^2}y)$$

is a J-holomorphic map representing A for all $J \in \mathcal{J}_\Omega$. Denote the images of these flat maps at height c by D_c. Finally, for $J \in \mathcal{J}_\Omega$, a *J-holomorphic A-disc* is defined to be the image of a J-holomorphic map

$$f \colon (D, \partial) \to (\mathbb{R}^4, S_\delta),$$

which represents $A \in \pi_2(\mathbb{R}^4, S_\delta)$ and is in the connected component of the flat discs. Thus each J-holomorphic A-disc, $\operatorname{Im} f$, may be joined to a flat disc, D_c, by a path $\operatorname{Im} f_t$ of J_t-holomorphic A-discs, where $J_t \in \mathcal{J}_\Omega$.

2.2 Properties of J-Holomorphic A-Discs

Technically, a J-holomorphic map of the disc, f, is only required to have J-linear derivative on the closed disc. However, since the boundary of the disc is sent to S_δ, more can be said because of the following.

Lemma 2.2.1. *For some neighborhood U of S_δ in $(\mathbb{R}^4, J_0)$ there is an antiholomorphic reflection $\sigma\colon U \to U$ in S_δ.*

Proof. Consider the cylinder $C = \{(x_1, 0, x_2, y_2)\colon |x_1| < 1 - \delta, x_2^2 + y_2^2 = 1\}$. There exists a neighborhood U_1 of C in $\mathbb{C}^2$ and $\sigma_1\colon U_1 \to U_1$ an antiholomorphic reflection in C given by $\sigma_1(z_1, z_2) = \left(\overline{z_1}, \frac{1}{\overline{z_2}}\right)$. Choosing U_1 sufficiently small, let U be the image of U_1 under the biholomorphism $\varphi(z_1, z_2) = \left(z_1, \sqrt{1 - z_1^2}\, z_2\right)$ and define σ on U by $\sigma = \varphi \circ \sigma_1 \circ \varphi^{-1}$. $\square$

Thus, by the Schwarz reflection principle, f has a unique J-holomorphic extension to a neighborhood of the closed disc.

The next proposition is well-known. A proof is included for completeness.

Proposition 2.2.2. *All interior points of a J-holomorphic A-disc are contained in $\operatorname{Int}\Omega$.*

Proof. Suppose there exists an f such that $\operatorname{Int} f$ is not contained in $\operatorname{Int}\Omega$. Since f, by definition, is in the connected component of the flat discs, there exists a family f_t of J_t-holomorphic discs such that, for some t_0, $\operatorname{Im} f_t \subset \Omega$, for $t \leq t_0$, and $\operatorname{Im} f_{t_0}$ is tangent to $\partial\Omega$ at $p = f_{t_0}(z_0)$. This can be broken into two cases.

(i) $z_0 \in \operatorname{Int} D$. The following is a reproduction of the proof from [M5, Lemma 2.4].

Recall $\partial\Omega = \{\varphi = 1\}$. Let $\xi_p = \ker \alpha(p)|_{\partial\Omega}$, $\alpha = i(\eta)\omega_0$. If $v \in \xi_p$, for any $J \in \mathcal{J}_\Omega$, $Jv \in \xi_p$ and thus

$$\ker J^* d\varphi|_{\partial\Omega} = \ker \alpha|_{\partial\Omega}.$$

This implies there exist functions $\mu > 0$, λ defined near $\partial\Omega$ such that

$$J^* d\varphi(p) = \mu\alpha(p) + \lambda d\varphi(p)$$

for all $p \in \partial\Omega$. Thus $dJ^*d\varphi = \mu d\alpha$ on ξ_p and so will be positive on all J-complex lines which are sufficiently close to ξ_p in $T_p\partial\Omega$.

Since f_{t_0} is J-holomorphic,

$$i^* d(\varphi \circ f_{t_0}) = (f_{t_0})^*(d\varphi \circ J) = (f_{t_0})^*(J^* d\varphi).$$

By hypothesis, $\operatorname{Im} f_{t_0}$ is tangent to ξ_p and thus

$$d(i^* d(\varphi \circ f_{t_0}))(z_0) = f_{t_0}^*(dJ^* d\varphi)(z_0) = f_{t_0}^*(\mu d\alpha)(z_0)$$

and so $d(i^* d(\varphi \circ f_{t_0}))$ is a positive area form near $z_0 \in D$. This implies that the function $\varphi \circ f_{t_0}$ is subharmonic is some neighborhood of z_0 and has a maximum at z_0. However this contradicts the maximum principle.

(ii) $z_0 \in \partial D$. By construction $J = J_0$ near $p \in S_\delta$, and $\partial\Omega$ coincides with S^3 near S_δ. Let $T_p^{\mathbb{C}} S^3$ be the complex part of the tangent space to $\partial\Omega$ at p and $(T_p^{\mathbb{C}} S^3)^\perp$ its ω_0 perpendicular. Then, if π_p is the orthogonal projection of $\mathbb{R}^4$ onto the complex line $(T_p^{\mathbb{C}} S^3)^\perp$, the composite $\pi_p \circ f_{t_0}$ is holomorphic in a neighborhood of ∂D in $\mathbb{C}$. If it had a zero of infinite order at z_0, it would be constant near ∂D, which would contradict the fact that f_{t_0} represents A. Thus it must have a zero of finite order at z_0 and, by hypothesis, must have a zero of order at least two. Choosing a complex coordinate w near z_o such that neighborhoods of z_o in D are identified with the half-discs $\{w \in \mathbb{C} \colon |w| < \epsilon, \operatorname{Re} w \geq 0\}$, the map $\pi_p \circ f_{t_0}$ has the form $w \mapsto aw^k$ + higher order terms, for some $k \geq 2$. Thus near z_o, $\pi_p \circ f_{t_0}(w) \approx aw^k$ which implies for $k \geq 2$ that sufficiently small neighborhoods V of z_o in D are mapped onto all of $(T_p^{\mathbb{C}} S^3)^{\perp} \equiv \mathbb{C}$. But this is impossible since $\operatorname{Im} f_{t_0} \subset \Omega$ and π_p takes a neighborhood of p in Ω into a half space in $\mathbb{C}$. $\square$

As in the above proof, let $\xi = \ker \alpha$. Because S_δ is totally real, $TS_\delta \cap \xi$ is a real, orientable line field on S_δ. By the standardization of Ω and J near S, it is not hard to check that this line field is vertical, that its integral curves are the restrictions to S_δ of the great arcs on S through the poles $\{(\pm 1, 0, 0, 0)\}$. Thus Proposition 2.2.2 implies

Corollary 2.2.3. *$f|_{\partial D}$ is never vertical, i.e.*

$$T_p(f|_{\partial D}) \neq T_p S_\delta \cap \xi(p).$$

Further important properties of the discs will be derived by a "doubling argument." Let $N = U \cup \operatorname{Int} \Omega$ where U is as in Lemma 2.2.1. Let N_1, N_2 be two copies of N and let U_i be the copy of U in N_i. Form the double

$$V = N_1 \cup_\sigma N_2 = \frac{N_1 \amalg N_2}{p \sim \sigma(p)} \quad \text{for } p \in U_1.$$

One can check V is a manifold and further that

$$T_pV = \begin{cases} T_pN_i, & p \notin U_i \\ \dfrac{T_pN_1 \amalg T_{\sigma(p)}N_2}{(p,v) \sim (\sigma(p), \sigma_* v)}, & p \in U_1. \end{cases}$$

Consider

$$J^V(p) = \begin{cases} J(p), & p \in N_1 \\ -J(p), & p \in N_2. \end{cases}$$

Since σ is anti-holomorphic and J is smooth, it follows that J^V is a well-defined, smooth almost complex structure on V.

Given a J-holomorphic map $f\colon (D, \partial) \to (\Omega, S_\delta)$, a J^V-holomorphic map $f^V\colon (S^2, i) = (\mathbb{C} \cup \{\infty\}, i) \to (V, J^V)$ can be constructed as follows. Let f_1, f_2 be copies of f into N_1, N_2 respectively and let $r\colon \mathbb{C}^2 \cup \{\infty\} \to \mathbb{C}^2 \cup \{\infty\}$ be the antiholomorphic reflection through S^1 given by $r(z) = 1/\overline{z}$. Define $f^V\colon \mathbb{C} \cup \{\infty\} \to V$ by

$$f^V(z) = \begin{cases} f_1(z), & |z| \leq 1 \\ f_2 \circ r(z), & |z| > 1. \end{cases}$$

It is not hard to verify that f^V is a smooth, J^V-holomorphic map of the sphere.

Proposition 2.2.4. *The J-holomorphic A-discs are embedded and disjoint.*

Proof. Let f be a J-holomorphic map representing A, and Z the image of its double. It will be shown that Z is embedded and is disjoint from the image Z' of the double of any other J-holomorphic A-disc. Let D_c be a flat J-holomorphic A-disc and S_c its double. Then Z and S_c are homologous J^V-holomorphic 2-spheres and S_c is embedded. But it is shown in [M2, Theorem 1.3] that there is a homological criterion for a J-holomorphic 2-sphere C to be embedded: namely, C is embedded iff its virtual genus $g(C)$ is 0, where $g(C) = 1 + \frac{1}{2}(C \cdot C - c(C))$. (Here $c \in H^2(V;\mathbb{Z})$ is the first Chern class of the complex vector bundle (TV, J).) Since $g(Z) = g(S_c) = 0$, it follows that Z is embedded. Further

$$Z \cdot Z' = S_c \cdot S_c = 0.$$

But, by [M2, Theorem 1.1], every intersection point of Z with Z' contributes positively to the algebraic intersection number $Z \cdot Z'$. Hence Z and Z' are disjoint. □

Corollary 2.2.5. *For all J-holomorphic A maps f,*

$$\omega_0\text{-area of } f = \int_{f(D)} \omega_0 \in [0, \pi].$$

Proof. Since $\partial f \subset S \subset \{y_1 = 0\}$ and f represents A,

$$0 \leq \int_{f(D)} \omega_0 = \int_{\partial f} x_2\, dy_2 \leq \pi. \quad \square$$

2.3 *Filling the Sphere*

In this section it is shown that the unit 2-sphere S in the hyperplane $\{y_1 = 0\}$ has a J-filling $F(J)$ for all $J \in \mathcal{J}_\Omega$. More precisely:

(2.3.1) Sphere Filling Theorem. *For all $J \in \mathcal{J}_\Omega$, there is a 1-parameter family of disjoint, J-holomorphic discs whose boundaries foliate S and whose union, $F(J)$, is diffeomorphic to the 3-dimensional ball B^3.*

Note that $F(J)$ contains two degenerate discs at the poles $(\pm 1, 0, 0, 0)$ of S.

Only a sketch of the proof will be given since it follows by standard Fredholm theory from the results of Section 2.2. The first step is to define suitable Banach manifolds of almost complex structures and maps.

The results used in Proposition 2.2.4 require that J be smooth. Therefore, the following procedure of Floer [F] will be used to construct a Banach manifold $N(J) \subset \mathcal{J}_\Omega$ containing a given J. The tangent space $T_J\mathcal{J}_\Omega$ to $\mathcal{J}_\Omega$ at J consists of the space $C^\infty(S_J)$ of smooth sections j of the bundle $\mathrm{End}\,(T_x\mathbb{R}^4)$ such that $jJ + Jj = 0$, $\langle j\alpha, \beta\rangle + \langle \alpha, j\beta\rangle = 0$ (where $\langle\ ,\ \rangle$ is the standard inner product), and $j(x) = 0$ on $\{\varphi > 1-\varepsilon\} \cap \{|y_1| < \beta\}$. Let $\bar{\varepsilon} = (\varepsilon_i)_{i\in\mathbb{N}}$ be any sequence of positive real numbers. Then

$$\|j\|_{\bar{\varepsilon}} = \sum_{k\in\mathbb{N}} \varepsilon_k \max_{x\in\mathbb{R}^4} |D^k j(x)|$$

is a norm on the linear space

$$C^{\bar{\varepsilon}}(S_J) = \{j \in C^\infty(S_J) : \|j\|_{\bar{\varepsilon}} < \infty\}.$$

Further, one can check $C^{\bar{\varepsilon}}(S_J)$ is a Banach space. As Floer has shown [F, Lemma 5.1], it is possible to choose $\bar{\varepsilon}$ so that $C^{\bar{\varepsilon}}(S_J)$ is dense in $T_J\mathcal{J}_\Omega$ with respect to the L^2-norm. Choose $r > 0$ small enough so that, for $\|j\|_{\bar{\varepsilon}} < r$, the exponential map is injective. Let $N'(J) = \{j \in C^{\bar{\varepsilon}}(S_J) : \|j\|_{\bar{\varepsilon}} < r\}$. Then $N'(J)$ is an open set of a Banach space and thus a Banach manifold. Define $N(J)$ to be the image of $N'(J)$ under the exponential map diffeomorphism.

Fix $s > 1$ and let $\mathcal{F} = \mathcal{F}^\delta_{A,s}$ be the Sobolev space $H^{s+1}(D, \partial; \mathbb{R}^4, S_\delta)$ of all maps $f\colon (D, \partial) \to (\mathbb{R}^4, S_\delta)$ whose $(s+1)^{th}$ derivative is L^2 and which represent $A \in \pi_2(\mathbb{R}^4, S_\delta)$.

For $J_1 \in \mathcal{J}_\Omega$, define

$$\mathcal{M} = \mathcal{M}(N(J_1)) = \{(f, J) \in \mathcal{F} \times N(J_1) : \bar{\partial}_J f = df + J \circ df \circ i = 0\}.$$

Proposition 2.3.2. *$\mathcal{M}$ is a Banach submanifold of $\mathcal{F} \times N(J_1)$.*

Proof. This can be proved locally using the procedure of McDuff [M1, §4]. There are two points to note. First observe that, because S_δ is totally real, the boundary conditions imposed on the maps are elliptic (see [BB]). Thus, in the notation of [M1, §4], L_J is still Fredholm and $\operatorname{Im} d\Phi$ is closed and of finite dimension. The proof that 0 is a regular value of $d\Phi_{(f,J)}$ goes through as before, provided that $\operatorname{Im} f$ intersects the part of $\mathbb{R}^4$ where the elements of $\mathcal{J}_\Omega$ are allowed to vary. However, one must also consider the possibility that $\operatorname{Im} f$ is contained in the region $\{\varphi \geq 1-\varepsilon\} \cap \{|y_1| \leq \beta\}$ where J is constrained to equal J_0. But then the integrability tensor of J vanishes on $\operatorname{Im} f$. Thus L_J is just the usual $\bar{\partial}$-operator and so is surjective, as required. □

It follows as in McDuff [M1, Proposition 4.2] that the projection map

$$P_A = P_{A,J_1} : \mathcal{M}(N(J_1)) \to N(J_1)$$

is Fredholm. By the Sard-Smale Theorem, there is a subset of second category $N(J_1)_{reg} \subset N(J_1)$ which consists of regular elements. For these J, the inverse image

$$P_A^{-1}(J) =: \mathcal{M}_p(J, A, \delta)$$

is a manifold. Let $(\mathcal{J}_\Omega)_{reg}$ be the union of all the $N(J)_{reg}$, $J \in \mathcal{J}_\Omega$. Since this is an uncountable union, the set $(\mathcal{J}_\Omega)_{reg}$ need not be of second category. However, it is dense.

Since the index of $P_{A,J}$ is determined by the symbol of the elliptic operator L_J, one easily sees that it is independent of $J \in \mathcal{J}_\Omega$, and so may be calculated by considering a special J. Thus, suppose that $J = J_0$ near the unit ball $B^3 \subset \{y_1 = 0\}$. Then the flat maps f_c, for $|c| < 1 - \varepsilon$ are J-holomorphic. Since the boundaries of these flat discs completely fill S_δ and, by Proposition 2.2.4, distinct discs must be disjoint, it follows that these flat discs are the only J-holomorphic A-discs. Thus, there is a 4-parameter family of flat J-holomorphic A-discs. (Note that three of these dimensions correspond to the reparametrization group $PSL(2,\mathbb{R})$.) Further, since J is integrable near the image of each flat disc, L_J is the usual $\bar{\partial}$ operator and it is easy to see that (f_c, J) is a regular point for all c. It follows that J is a regular value for $P_{A,J}$. Hence the index of $P_{A,J}$ is 4 and the following has been shown:

Proposition 2.3.3. *For all J in a dense set $(\mathcal{J}_\Omega)_{reg}$ of $\mathcal{J}_\Omega$, $\mathcal{M}_p(J, A, \delta)$ is a smooth 4-manifold.*

The next step is to compactify the manifolds $\mathcal{M}_p(J_i, A, \delta)$. It is not enough simply to quotient out by the noncompact reparametrization group $G = PSL(2,\mathbb{R})$ of all biholomorphisms of the disc, because S_δ is itself not compact. To fix this, consider the closure $\overline{S}_\delta$ of S_δ. Let $\overline{\mathcal{M}}_p(J, A, \delta)$ consist of all J-holomorphic discs with image in $(\Omega, \overline{S}_\delta)$.

Lemma 2.3.4. $\overline{\mathcal{M}}_p(J, A, \delta) = \mathcal{M}_p(J, A, \delta) \cup (f_\delta)_G \cup (f_{-\delta})_G$, *where $(f_\delta)_G$ is the G-orbit of the flat disc at height δ and similarly for $(f_{-\delta})_G$.*

Proof. The inclusion $\supset$ is clear. Conversely, suppose there exists an f such that $\operatorname{Im} f$ is a non-flat A-disc which intersects one of the circles of $\overline{S}_\delta \setminus S_\delta$.

By construction of $\mathcal{J}_\Omega$ and definition of δ, there are flat J-holomorphic discs whose boundaries lie on these circles. Thus $\operatorname{Im} f$ must intersect one of these flat discs. However this contradicts Proposition 2.2.4. □

Proposition 2.3.5. *The space of unparametrized J-holomorphic A-discs, $\overline{\mathcal{M}}_p(J, A, \delta)/G$ is diffeomorphic to a compact interval, for all $J \in \mathcal{J}_\Omega$.*

Proof. Suppose $J \in (\mathcal{J}_\Omega)_{reg}$. It is easy to see that in this case $\overline{\mathcal{M}}_p(J, A, \delta)/G$ is a 1-dimensional manifold with two boundary points given by $[f_{\pm\delta}]$. Therefore, it suffices to prove that it is compact and connected.

To see that this space is compact, double all the discs so that they become J^V-holomorphic A^V-spheres. Construct a metric g_J^V on V by piecing together $g_1(v, w) = \omega_0(v, J^V w)$ on N_1 and $g_2(v, w) = -\omega_0(v, J^V w)$ on N_2, and observe that, by Corollary 2.2.5, all the J^V-holomorphic A^V-spheres are uniformly bounded in the associated H^1 Sobolev norm. Thus one can apply the standard compactness theorem for closed J-holomorphic spheres. (See, for example, [M4, (2.4)].) Clearly, any limiting cusp-curve is the double of a J-holomorphic "cusp-disc" in Ω. The most general form for such a cusp-disc is a connected union of J-holomorphic components, some of which are discs and some of which are "bubbles" (i.e. J-holomorphic spheres). In the case being considered, Corollary 2.2.3 implies that the boundary of the cusp-disc is transverse to the vertical, and so only one of its components can be a disc. Further, there can be no bubbles since $\pi_2(\Omega) = 0$. Hence, the limit must consist of a single disc. This proves compactness. (An alternative proof, which does not use doubling, may be constructed by the methods of [O] or [Y].)

It follows that one connected component of $\overline{\mathcal{M}}_p(J, A, \delta)$ is diffeomorphic to a compact interval. Since this component is non-empty and contains the flat discs $f_{\pm\delta}$, the boundaries of the discs in this component must fill out the whole of S_δ. But distinct discs are disjoint. Therefore there can be no other discs and $\overline{\mathcal{M}}_p(J, A, \delta)$ is connected, as claimed.

Now consider an arbitrary J. The first claim is that there is a J-holomorphic disc through each point of S_δ. This follows from the compactness theorem, because it is true for regular J and because regular elements are dense in $\mathcal{J}_\Omega$. Next observe that, if α is a vertical arc in S going from one pole to the other, Corollary 2.2.3 implies that each disc intersects α exactly once. The result now follows easily. □

(2.3.6) Proof of Theorem 2.3.1.

Let

$$\mathcal{M}_p(J, A) = \overline{\mathcal{M}}_p(J, A, \delta) \cup \{f_c \circ \gamma \colon \delta < |c| \leq 1, \gamma \in G\},$$

and let $F(J)$ be the image of the evaluation map

$$e_D(J) \colon \mathcal{M}_p(J, A) \times_G (D, \partial) \to (\mathbb{R}^4, S).$$

In the proof of Proposition 2.3.5., it was shown that $\partial F(J) = S$. Since G acts freely away from the degenerate discs at the poles, $\overline{\mathcal{M}}_p(J, A, \delta) \times_G (D, \partial)$ is a fiber bundle over the interval $\overline{\mathcal{M}}_p(J, A, \delta)/G$ with fiber (D, ∂). Thus $\mathcal{M}_p(J, A) \times_G (D, \partial)$, with its obvious smooth structure, is diffeomorphic to a 3-ball. By Proposition 2.2.4, $e_D(J)$ is injective and restricts to an embedding on each disc. The fact that it is a diffeomorphism onto its image $F(J)$ may be proved by the argument in [M3, Lemma 3.5]. □

Chapter 3

Symplectic Embeddings of Balls

Throughout this chapter, $B(r) \subset (\mathbb{R}^4, \omega_0)$ will denote the closed 4-dimensional ball of radius r,

$$B(r) = \{x_1^2 + y_1^2 + x_2^2 + y_2^2 \leq r^2\}.$$

Given a symplectic manifold (U, ω), an embedding $g \colon B(r) \to U$ is *symplectic* if $g^*\omega = \omega_0$ for all $p \in B(r)$. $\mathcal{E}(r, U)$ will denote the space of symplectic embeddings of a ball of radius r into U.

By the non-degeneracy of a symplectic structure, it is easy to check that the images of all elements in $\mathcal{E}(r, U)$ must have the same volume. However there exist regions in $(\mathbb{R}^4, \omega_0)$ which are diffeomorphic to and have the same volume as $B(r)$ but are not the symplectic image of any ball. This is a consequence of the squeezing theorem.

Squeezing Theorem (Gromov). *Let $D(R) \subset \mathbb{R}^2$ be the open disc of radius R. If there exists a symplectic embedding g of $B(r)$ into $D(R) \times \mathbb{R}^2$ then $r < R$.*

Gromov proved this theorem, in [G], using the theory of pseudo-holomorphic curves. In this chapter, the filling results from the Chapter 2 will be used to further study symplectic embeddings of balls into open subsets of $\mathbb{R}^4$. In particular, the filling technique will be used to study, $\pi_0(\mathcal{E}(r,U))$, the path components of $\mathcal{E}(r,U))$. Recall that $[f]=[g]\in\pi_0(\mathcal{E}(r,U))$ if and only if f and g are *symplectically isotopic.* This means there exists a 1-parameter family of symplectic embeddings $g_t\colon B(r)\to U$ such that $g_0=f$, $g_1=g$. An *inclusion*, $i\in\mathcal{E}(r,U)$, is an embedding such that $i_*=id$.

For later comparison, it is useful to point out that all symplectic embeddings of a ball of radius r into $\mathbb{R}^4$ are symplectically isotopic. As proof, given an arbitrary embedding g it suffices to construct a symplectic isotopy between g and an inclusion i. By applying a translation, it is possible to assume $g(0)=0$. Consider

$$g_{1-t}(v)=\lim_{s\to t}\frac{g(sv)}{s}=\lim_{s\to t}\frac{(g\,(sv)-g\,(0))}{s},\qquad t\in[0,1].$$

Then $g_0=g$, $g_1=dg(0)\in Sp(4)$, where $Sp(4)$ is the symplectic linear group of $\mathbb{R}^4$. Since $Sp(4)$ is connected, there exists a path h_t in $Sp(4)$, $t\in[1,2]$ so that $h_1=dg(0)$, $h_2=i$.

3.1 Symplectic Convexity

A vector field η on $(\mathbb{R}^{2n},\omega_0)$ is said to be *expanding* if

$$\mathcal{L}_\eta\omega_0=2\omega_0.$$

(Compare [EG, 1.2].) Recall that given a 2-dimensional sphere $S\subset\mathbb{R}^4$, one hypothesis for the existence of a filling is that there exists a region Ω, $S\subset\partial\Omega$ and an expanding vector field η which is transverse to $\partial\Omega$. Thus the following class of open manifolds will be of interest.

Definition 3.1.1. (Compare [EG, 1.7].) An open set $U \subset (\mathbb{R}^{2n}, \omega_0)$ is *symplectically convex* if there exists a sequence of compact domains, Ω_k, such that

$$U = \cup_k \operatorname{Int} \Omega_k$$

and for all k, there exists an expanding vector field η_k which is transverse to $\partial\Omega_k$.

The radial vector field

$$\eta = x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}$$

on $\mathbb{R}^4$ is expanding and thus any star convex subset of $\mathbb{R}^4$ is symplectically convex. The focus in this chapter will be on "special" symplectically convex subsets; namely, subsets to which the filling technique of Chapter 2 can be applied.

Definition 3.1.2. Let $U \subset \mathbb{R}^4$ be an open subset whose boundary ∂U contains 2-dimensional spheres $S_1, \ldots, S_n$ of the form

$$S_i = \left\{ h_i(x_1 - a_i)^2 + (x_2 - b_i)^2 + (y_2 - c_i)^2 = \lambda_i^2,\, y_1 = d_i \right\},$$

for $a_i, b_i, c_i, d_i \in \mathbb{R}$, $0 < h_i \leq 2$. The spheres S_i are said to be *fillable holes of U* if

$$U = \cup \operatorname{Int} \Omega_k$$

where (Ω_k) is a sequence of regions satisfying conditions analogous to (2.1.1). More specifically, (3) is replaced by

(3′) for all i, there exists β_i and ϵ_i (depending on k) such that on

$$N_i = \left\{ \begin{array}{ll} |x_1 - a_i| < \dfrac{\lambda_i}{\sqrt{h_i}} + \epsilon_i, & |y_1 - d_i| < \beta_i \\ |x_2 - b_i| < \lambda_i + \epsilon_i, & |y_2 - c_i| < \lambda_i + \epsilon_i \end{array} \right\},$$

$$\Omega_k = \left\{ h_i(x_1 - a_i)^2 + (2 - h_i)(y_1 - d_i)^2 + (x_2 - b_i)^2 + (y_2 - c_i)^2 \leq \lambda_i \right\}$$

and

$$\eta_k = h_i(x_1 - a_i)\frac{\partial}{\partial x_1} + (2 - h_i)(y_1 - d_i)\frac{\partial}{\partial y_1} + (x_2 - b_i)\frac{\partial}{\partial x_2} + (y_2 - c_i)\frac{\partial}{\partial y_2}.$$

It is clear that given a manifold U with fillable holes, for any Ω_k and $J \in \mathcal{J}_{\Omega_k}$, there exists a filling $F(J) = F^1 \cup F^2 \cup \cdots \cup F^n$ of the n spheres $S_1, \ldots, S_n$.

Lemma 3.1.3. *$F^i \cap F^j = \emptyset$ for $i \neq j$.*

Proof. If the discs $D^r \in F^r$ and $D^s \in F^s$ intersect, they must do so at a point which is in the interior of each disc. But this is impossible by positivity of intersections: see Proposition 2.2.4. □

The following lemma will be extremely useful. The argument is very similar to those given in [G], [M4, (2.5.2)].

(3.1.4) Extension Lemma. *Let U be an open set with fillable holes. Let X be a subset of $\mathbb{R}^4$ and $E(X,r) = \cup_{z \in X} B(r,z)$ where $B(r,z)$ is the ball of radius r centered at z. Suppose there exists an embedding $g\colon X \to U$ which extends to a symplectic embedding $\tilde{g}\colon E(X,r) \to \operatorname{Int}\Omega_k \subset U$. For all $J \in \mathcal{J}_{\Omega_k}$ which equal $\tilde{g}_* J_0$ on $\operatorname{Im}\tilde{g}$ and all i such that $\lambda_i \leq r$,*

$$F^i \cap g(X) = \emptyset.$$

Proof. Suppose there exists a disc C in $F(J)$ whose boundary lies on S_i and a point $z \in X$ such that $C \cap g(z) \neq \emptyset$. Consider $\tilde{g}_z = \tilde{g}|_{B(z,r)}$ and let C' be the pull-back of the disc C by $\tilde{g}_z$. It follows that C' is a J_0-holomorphic curve through the center of the ball $B(z,r)$ with boundary on $\partial B(z,r)$. Thus C' is a minimal surface and, by the monotonicity theorem, must have area $\geq \pi r^2$ with respect to the standard metric. On the other hand, this area can be calculated by integrating ω_0 over C' and hence is strictly bounded above by the integral of ω_0 over C. However, by Corollary 2.2.5, $\omega_0(C) \leq \pi\lambda_i^2$. □

3.2 Embeddings in Different Path Components

The Extension Lemma (3.1.4) will first be applied to the case where X is a point to show that there exist connected regions U of $\mathbb{R}^4$ such that $\mathcal{E}(r, U)$ has more than one path component.

If S_i is a fillable hole of U then S_i bounds a 3-dimensional ball denoted by $\mathcal{D}_i$. The argument below is given by Eliashberg and Gromov in their proof of the "Camel Lemma" ([EG, 3.4.B]).

Theorem 3.2.1. *Let S_i be a fillable hole in U of radius λ_i and suppose $U \backslash \mathcal{D}_i = \amalg U^{\pm}$. If $r \geq \lambda_i$ and there exist inclusions $i_r^{\pm}$ such that $\mathrm{Im}\, i_r^{\pm} \subset U^{\pm}$, then then i_r^- and i_r^+ are not symplectically isotopic.*

Proof. Any isotopy $g_t, 1 \leq t \leq 2$ from i_r^- to i_r^+ is contained in a compact subset of U and hence in $\mathrm{Int}\, \Omega = \mathrm{Int}\, \Omega_k$ for some k. Let $J_t \in \mathcal{J}_\Omega$ be a smooth family of almost complex structures chosen so that $J_t = (g_t)_* J_0$ on

$\operatorname{Im} g_t$. Because each filling $F(J_t)$ disconnects Ω, the set

$$Y = \{(t,x) : x \in F(J_t), 1 \le t \le 2\}$$

disconnects $[1,2]\times\Omega$. Further, since $(g_t)_* J_0 = J_0$ when $t = 1, 2$, it is possible to assume that J_1 and J_2 equal J_0 near the 3-ball $\mathcal{D}_i$. Then $F(J_t) = \mathcal{D}_i$ when $t = 1, 2$ and so the points $(1, g_1(0))$ and $(2, g_2(0))$ lie in distinct components of $([1,2]\times\Omega)\setminus Y$. Therefore, the path $(t, g_t(0))$, $1 \le t \le 2$, must intersect Y. In other words, for some t, there is a J_t-holomorphic A-disc C through $g_t(0)$. However by the Extension Lemma (3.1.4), this is impossible. □

Example 3.2.2. Let

$$DB(a) := \{(x_1 - a)^2 + y_1^2 + x_2^2 + y_2^2 < 1\} \cup \{(x_1 + a)^2 + y_1^2 + x_2^2 + y_2^2 < 1\}$$

where a is chosen so that $DB(a)$ is connected. See Figure 3.2.3.

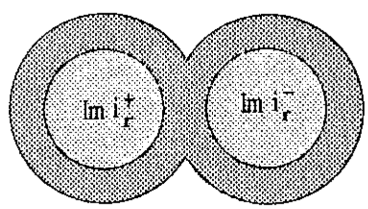

FIGURE 3.2.3.

Using the ideas in the proof of Proposition 4.1.1, it can be shown that $S := \partial DB(a) \cap \{y_1 = 0\}$ is a fillable hole of U of radius $\sqrt{1-a^2}$. Thus Theorem 3.2.1 implies that inclusions $i_r^{\pm}\colon B(r) \to DB(a)$ of the ball of radius $r \ge \sqrt{1-a^2}$ into each half of $DB(a)$ are not isotopic.

Example 3.2.4. Let

$$H(a) := \{y_1 < 0\} \cup \{x_1^2 + (y_1 - a)^2 + x_2^2 + y_2^2 < 1\}$$

where a is chosen so that H is connected. Again using the ideas in the proof of Proposition 4.1.1, $S := \partial(H(a) \cap \{y_1 = 0\})$ is a fillable hole of radius $\sqrt{1-a^2}$. Thus inclusions $i_r^{\pm}\colon B(r) \to H(a)$ of the ball of radius $r = \sqrt{1-a^2}$ into $H(a) \cap \{y_1 < 0\}$ and $H(a) \cap \{y_1 > 0\}$ are not isotopic for $r \geq \sqrt{1-a^2}$. See Figure 3.2.5.

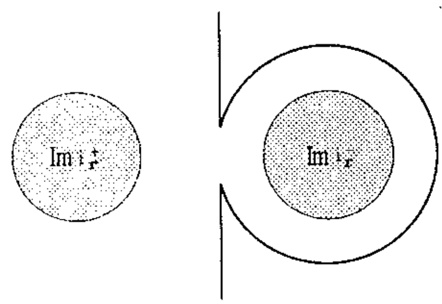

FIGURE 3.2.5.

3.3 Criterion for Standard Isotopy Class

In this section, the Extension Lemma (3.1.4) will be applied to the case where $X = B(r)$ for U which are "special" subsets with fillable holes.

Definition 3.3.1. Let U be open set with n fillable holes. Let W^j, $j = 1, \ldots, m$ be the boundary components of U which contain the fillable spheres. Suppose there exists constants $c_1, \ldots, c_m$ such that,

(1) $W^j \subset \{y_1 = c_j\}$;

(2) $\left(U \setminus \cup_j \{y_1 = c_j\}\right) = \amalg_{k=1}^{m} V_k$ where V_k is symplectomorphic to $\mathbb{R}^4$.

Then U is said to have (m) *flat boundaries.*

Example 3.3.2. Let

$$C = \{y_1 < 0\} \cup \{y_1 > 0\} \cup \{x_1^2 + x_2^2 + y_2^2 < 1, y_1 = 0\}.$$

In the Chapter 4, it will be shown that C has a fillable hole and a flat boundary.

Given an open set U with m flat boundaries, there exist symplectic diffeomorphisms $i^k \colon \mathbb{R}^4 \to V_k$, $k = 0, \ldots, m$. Let $i_r^k := i^k|_{B(r)}$. If j^k is a different symplectic parameterization of V_k, it is easy to check that $[j_r^k] = [i_r^k] \in \pi_0(\mathcal{E}(r, U))$.

The remainder of this chapter is devoted to proving the following theorem.

Theorem 3.3.3. *Let U have fillable holes of radii $\lambda_1, \ldots, \lambda_n$ and flat boundaries. Let $\lambda_{\max} = \max\{\lambda_i\}$. If $g \colon B(r) \to U$ is an embedding which has an extension to a symplectic embedding $g' \colon B(r + \lambda_{\max}) \to U$ then g is symplectically isotopic to i_r^k, for some k.*

This theorem is a a slight generalization of the "Extendable Embeddings Lemma" from [MT]. The following results and proofs are completely analogous to those given in Section 6 of [MT].

Proposition 3.3.4. *(Compare* [MT], *Proposition 6.2.) Let U have n fillable holes and m flat boundaries. Fix Ω, $J \in \mathcal{J}_\Omega$, and let F^{c_i} be the fillings of the fillable spheres contained in $\{y_1 = c_i\}$. Then $F(J) = \cup_{i=1}^m F^{c_i}$ can be ε-perturbed so that each F^{c_i} fits together with W^i to form a 3-manifold Q^{c_i} so that there is a symplectomorphism Ψ of $\mathbb{R}^4$ which takes Q^{c_i} to $\{y_1 = c_i\}$.*

Before proving this proposition, it will be shown how, together with Lemma 3.1.4, it proves Theorem 3.3.3. If an embedding g has an extension to the ball of radius $r + \lambda_{\max}$, Lemma 3.1.4 implies $g(B(r)) \cap F^{c_i} = \emptyset$, for all i. Since ε may be chosen sufficiently small so that $Q^{c_i} \cap g(B(r)) = \emptyset$, this means $g(B(r))$ lies in a subset of U which is symplectomorphic to $\mathbb{R}^4$. However, as mentioned in the beginning of this chapter, the space of symplectic embeddings of a ball into $\mathbb{R}^4$ is connected. Hence g is isotopic to i_r^k as claimed.

In the course of proving Proposition 3.3.4, a "parameterized family of fillings" will be used. Consider the following 2-spheres and 4-balls of radius γ:

$$S(\gamma, s) = \{x_1^2 + x_2^2 + y_2^2 = \gamma^2, y_1 = s\},$$
$$B(\gamma, s) = \{x_1^2 + (y_1 - s)^2 + x_2^2 + y_2^2 \leq \gamma^2\}.$$

If an almost complex structure J is chosen such that $J = J_0$ outside the 4-ball $B(\mu) = B(\mu, 0)$ and choose $\gamma > 2\mu$ so that $B(\mu) \subset B(\gamma, s)$ for $-\mu \leq s \leq \mu$ then, for each s, $B(\gamma, s)$ may play the role of Ω, and so it is possible to fill $S(\gamma, s)$ with respect to J. Denote the filling by $\widetilde{F}^s$. By Proposition 2.2.2, $\widetilde{F}^r \subset B(\gamma, r)$ for all r. Since $B(\gamma, r) \cap S(\gamma, s) = \emptyset$ when $r \neq s$ the proof of Lemma 3.1.3 proves the following.

Lemma 3.3.5. *$\widetilde{F}^r \cap \widetilde{F}^s = \emptyset$ when $r \neq s$.*

Proof of Proposition 3.3.4. First the hypersurfaces Q^{c_i}, $i = 1, \dots, m$ are constructed. Each Q^{c_i} will be defined as the union of a 1-parameter family of symplectic 2-manifolds, each of which consists of a disc in the filling F^{c_i} extended so as to join together with a flat 2-plane $\{x_1 = \text{constant},\ y_1 = c_i\}$ outside some large ball.

Let α^{c_i} be a sequence of vertical arcs on the fillable spheres in $\{y_1 = c_i\}$. By Corollary 2.2.3, the discs in F^{c_i} meet α^{c_i} once transversally. Let $D_\nu^{c_i} = \operatorname{Im} f_\nu^{c_i}$ be the disc in F^{c_i} which intersects this arc α^{c_i} at $x_1 = \nu$. Then

$$
\begin{aligned}
(f_\nu^{c_i})^* \omega_0 &= (f_\nu^{c_i})^* (dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\
&= (\pi_1 \circ f_\nu^{c_i})^* (dx_1 \wedge dy_1) + (\pi_2 \circ f_\nu^{c_i})^* (dx_2 \wedge dy_2),
\end{aligned}
$$

where π_i denotes the projection onto the (x_i, y_i)-plane. Both terms here are ≥ 0 near ∂D since $\pi_i \circ f_\nu^{c_i}$ is J_0-holomorphic near ∂D. Moreover, the second term is strictly positive, because the boundary of $D_\nu^{c_i}$ is transverse to the vertical arcs. Thus, by flattening the y_1 coordinate of $f_\nu^{c_i}$ near $\partial D_\nu^{c_i}$, the disc $D_\nu^{c_i}$ can be perturbed to a disc $\widetilde{D}_\nu^{c_i} = \operatorname{Im} \widetilde{f}_\nu^{c_i}$ which is infinitely tangent to the hyperplane $\{y_1 = c_i\}$ along its boundary and which is still symplectically embedded. Clearly, this perturbation can be done smoothly with respect to ν in such a way that all the discs $\widetilde{D}_\nu^{c_i}$ are disjoint. Note that this perturbation does not alter the flat discs $D_\nu^{c_i}$ near the x_1-poles of the spheres being filled.

Recall that by the definition of fillable holes, each sphere $S_j^{c_i} \subset \{y_1 = c_i\}$ being filled is the boundary of a 3-dimensional ball $\mathcal{D}_j^{c_i}$. Let $P_\nu^{c_i}$ be the portion of the flat plane $\{(x_1, y_1) = (\nu, c_i)\}$ which lies outside $\cup\mathcal{D}_j^{c_i}$, i.e. $P_\nu^{c_i} = \{(\nu, c_i, x_2, y_2) \notin \cup_j \mathcal{D}_j^{c_i}\}$. Perturb each $P_\nu^{c_i}$ inside the hyperplane $\{y_1 = c_i\}$ to a 2-dimensional space $\widetilde{P}_\nu^{c_i}$ which is still a graph over the (x_2, y_2) plane and which joins smoothly with $\partial\widetilde{D}_\nu^{c_i}$. Then $L_\nu^{c_i} = \widetilde{D}_\nu^{c_i} \cup \widetilde{P}_\nu^{c_i}$ is a smoothly embedded 2-manifold which equals $P_\nu^{c_i}$ outside a compact set. It is symplectic since both $\omega_0|_{\widetilde{D}_\nu^{c_i}}$ and $\omega_0|_{\widetilde{P}_\nu^{c_i}}$ are positive. Again, it may be assumed that the $L_\nu^{c_i}$ vary smoothly with respect to ν, and that they coincide with the flat planes $\{(x_1, y_1) = (\nu, c_i)\}$ outside some compact subset X^{c_i} of the strip $|x_1| <$ const $\in \mathbb{R}^4$.

Let $Q^{c_i} = \cup_\nu L_\nu^{c_i}$. By construction, Q^{c_i} is foliated by the symplectic 2-manifolds $L_\nu^{c_i}$. Furthermore, by Lemma 3.1.3, it can be assumed that $Q^{c_i} \cap Q^{c_j} = \emptyset$, $c_i \neq c_j$. A symplectomorphism of $\mathbb{R}^4$ is now constructed which takes Q^{c_i} to $\{y_1 = c_i\}$, $i = 1, \ldots, m$.

Choose a J' which equals J_0 outside some large 4-ball $B(\mu)$ containing the compact subset $X = \cup X^{c_i}$ defined above and such that the leaves of the foliation of Q^{c_i} are J'-holomorphic. With γ as in Lemma 3.3.5, let $S(\gamma, s)$ be the 2-sphere in $\{y_1 = s\}$ with filling $\widetilde{F}^s$. Note that the discs in $\widetilde{F}^{c_i}$ are just the intersections of the leaves $L_\nu^{c_i}$ with the ball $B(\gamma, c_i)$. By Lemma 3.3.5, the fillings $\widetilde{F}^s$ are mutually disjoint. Thus, by repeating the above argument, the $\widetilde{F}^s$ can be perturbed so they fit together with the hyperplanes $\{y_1 = s\}$ to form a foliation of $\mathbb{R}^4$ with leaves Q^s, each of which is foliated by symplectic 2-planes. Further, outside of a compact subset of $\mathbb{R}^4$, Q^s coincides with $\{y_1 = s\}$ and is foliated by the planes $\{x_1 = \text{constant}\}$.

Let ξ^s be a non-vanishing vector field on Q^s satisfying $\xi^s \in \ker \omega_0|_{Q^s}$. The integral curves of any such vector field form the characteristic foliation of Q^s. It is easy to check that the leaves of this foliation are transverse to the symplectic leaves and agree with $\{y_1 = s, (x_2, y_2) = \text{ constant }\}$ outside a large ball. Therefore, the characteristic flow ϕ_t^s on Q^s may be parametrized smoothly in t, s so that it preserves the foliation of each Q^s and has tangent $\frac{\partial}{\partial x_1}$ outside a compact set. Choose $K \gg 0$ so that the plane $\{x_1 = -K, y_1 = s\}$ is a symplectic leaf in Q^s for all s. Then define $\Phi \colon \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Phi(p) = (-K + t, s, x_2, y_2)$$

where $p \in Q^s$ is the image of the point $(-K, s, x_2, y_2)$ under the map ϕ_t^s. This takes Q^s to the hyperplane $\Pi^s = \{y_1 = s\}$, for all s, but is not quite symplectic. Thus to complete the proof of Proposition 3.3.4, it remains to show

Lemma 3.3.6. *There exists a diffeomorphism h of $\mathbb{R}^4$ which is the identity on Q^{c_i}, $i = 1, \ldots, m$, and satisfies $h^*\Phi^*\omega_0 = \omega_0$.*

Proof. Let ξ^s be the vector field on Q^s whose flow is ϕ_t^s. By construction, $\Phi_*(\xi^s) = \frac{\partial}{\partial x_1}$. It follows that if $q = (x_1, s, x_2, y_2)$ where $x_1 < -K$, then for all t, $\Phi \circ \phi_t^s(q) = \tau_t \circ \Phi(q)$ where $\tau_t(x_1, y_1, x_2, y_2) = (x_1 + t, y_1, x_2, y_2)$. Since the restriction of ω_0 to Q^s is invariant under ϕ_t^s, $\Phi^*\omega_0|_{Q^s} = \omega_0|_{Q^s}$ for all s. It follows that the forms $\omega_r = r\Phi^*\omega_0 + (1 - r)\omega_0$ are non-degenerate on $\mathbb{R}^4$ when $0 \leq r \leq 1$. Suppose there exists a 1-form β which vanishes at all points of $\cup_{i=1}^m Q^{c_i}$ and satisfies $d\beta = \Phi^*\omega_0 - \omega_0$. Then if the vector field v_r defined by

$$i(v_r)\omega_r + \beta = 0$$

is integrable, the standard Moser method argument shows that the integral h_r of v_r satisfies $h_r^*\omega_r = \omega_0$ (see [MS]). Since $h_r = id$ on Q^{c_i} the desired diffeomorphism is then given by h_1.

Consider $\beta = \Phi^*(x_1dy_1 + x_2dy_2) - (x_1dy_1 + x_2dy_2)$. Outside a compact set K_1,

$$\Phi(x_1, y_1, x_2, y_2) = (x_1, y_1, \psi_{y_1}(x_2, y_2))$$

where $\psi_{y_1} : \mathbb{R}^2 \to \mathbb{R}^2$ is independent of x_1, and equals the identity outside a compact set K_2 in the (y_1, x_2, y_2)-space. It follows that outside K_1,

$$\omega_r = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + r(h_1\ dy_1 \wedge dx_2 + h_2\ dy_1 \wedge dy_2)$$

where $|h_1|$ and $|h_2|$ are bounded functions of y_1, x_2, y_2. It is then easy to check that given any bounded 1-form β, the associated vector field v_r defined by the equation $i(v_r)\omega_r + \beta = 0$ has bounded growth and is thus integrable. The β constructed above is in fact bounded. However since it does not vanish on Q^{c_i}, $i = 1, \ldots, m$, the following modifications are made.

$\Phi^*\omega_0|_{Q^{c_i}} = \omega_0|_{Q^{c_i}}$ implies that $\beta|_{Q^{c_i}}$ is closed. From the description of Φ outside the compact set K_1, it is easy to check $\beta|_{Q^{c_i}}$ is bounded. Thus $\beta|_{Q^{c_i}} = df^{c_i}$ where f^{c_i} is a function on Q^{c_i}. These functions can simultaneously be extended to a function $\widetilde{f}$ on $\mathbb{R}^4$ so that $d\widetilde{f}$ is bounded. Then $\beta' = \beta - d\widetilde{f}$ is cohomologous to β, is still bounded, and vanishes for all vectors in TQ^{c_i}. Lastly β' is modified to a form β'' which equals zero at all points of Q^{c_i}. To do this, first choose coordinates x_1', y_1', x_2', y_2' on $\mathbb{R}^4$ so that $Q^{c_i} = \{y_1' = c_i\}$ and so that these are the standard coordinates on $\mathbb{R}^4$ outside a compact set. Suppose

$$\beta' = \sum f_i dx_i' + g_i dy_i'.$$

Then f_1, f_2 and g_2 vanish on $\cup_{i=1}^m Q^{c_i}$ by construction. Further, outside a compact set, $g_1 = \psi^1 \frac{\partial \psi^2}{\partial y_1}$ where $\psi_{y_1}(x_2, y_2) = (\psi^1, \psi^2)$. Hence g_1 is independent of x_1 and has compact support with respect to the other variables. Choose a function $h(y_1')$ so that $h(y_1') = y_1' - c_i$ near Q^{c_i} for $i = 1, \ldots, m$. Then $d(g_1 h)$ is bounded and $\beta'' = \beta' - d(g_1 h)$ satisfies all the desired properties. □

This completes the proof of Proposition 3.3.4. □

Chapter 4

C Spaces

Consider $\mathbb{R}^4$ with the standard symplectic structure ω_0. For $\lambda > 0$, let $U(\lambda)$ be the following open subset of $(\mathbb{R}^4, \omega_0)$:

$$U(\lambda) = \{y_1 < 0\} \cup \{y_1 > 0\} \cup \{x_1 \neq 0, y_1 = 0\} \cup \{x_1 = y_1 = 0, x_2^2 + y_2^2 < \lambda^2\}.$$

Alternately,

$$\mathbb{R}^4 \backslash U(\lambda) = \{x_2^2 + y_2^2 \geq \lambda^2, x_1 = y_1 = 0\}.$$

In [M6], McDuff proves that for all λ, $U(\lambda)$ is symplectomorphic to $\mathbb{R}^4$.

In this chapter, an open subset $C(\lambda)$ of $U(\lambda)$ will be examined. $C(\lambda)$ is defined as

$$C(\lambda) = \{y_1 < 0\} \cup \{y_1 > 0\} \cup H(\lambda)$$

where $H(\lambda) := \{(x_1, 0, x_2, y_2) \colon x_1^2 + x_2^2 + y_2^2 < \lambda^2\} \subset \{y_1 = 0\}$. As a comparison to $U(\lambda)$,

$$\mathbb{R}^4 \backslash C(\lambda) = \{x_1^2 + x_2^2 + y_2^2 \geq \lambda^2, y_1 = 0\}.$$

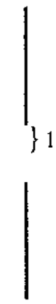

FIGURE 4.0. $C(1)$

$C(\lambda)$ contains two half spaces, each symplectically equivalent to $\mathbb{R}^4$. For a path $\gamma\colon [0,1] \to C(\lambda)$ satisfying $\gamma(0) \in \{y_1 < 0\}$, $\gamma(1) \in \{y_1 > 0\}$, there must exist a t such that $\gamma(t)$ is an element of the "hole" $H(\lambda)$. The space $C(1)$ was first introduced by Gromov. It is the space which underlies the symplectic camel problem. (See Section 4.2, [A], [EG], [MT], [V].)

4.1 Convexity of C Spaces

It is easy to check that $C(\lambda)$ is star convex with respect to the origin and thus $C(\lambda)$ is symplectically convex (see Definition 3.1.1). In order to apply the filling technique from Chapter 2, it is necessary to prove that the exhausting regions can be chosen to be "standard" near $\partial H(\lambda)$ (see (2.1.1)). Recall the notion of fillable holes from Definition 3.1.2.

Proposition 4.1.1. *$\partial H(\lambda)$ is a fillable hole of $C(\lambda)$.*

Proof. The exhausting regions Ω_k will be constructed by centering large 4-balls of radius k at $(0, \pm k, 0, 0)$ and smoothing the union via a "solid cylinder" which agrees with $B^4(\lambda)$ near $\{y_1 = 0\}$. More precisely, let

$$B^{\pm}(k) = \{x_1^2 + (y_1 \mp k)^2 + x_2^2 + y_2^2 \leq k^2\}$$

$$B(\lambda) = \{x_1^2 + y_1^2 + x_2^2 + y_2^2 \leq \lambda^2\}$$

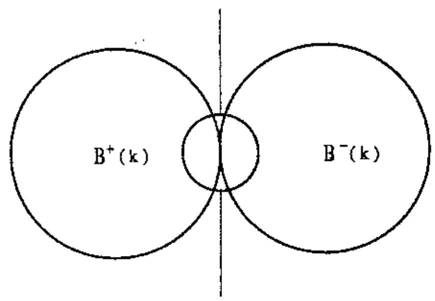

FIGURE 4.1.2.

Notice that the radial vector field

$$\eta = x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}$$

is transverse to $\partial B^{\pm}(k) \backslash \{0\}$ and $\partial B(\lambda)$. Thus it follows that η is transverse to the non-smooth boundary $\partial \widetilde{\Omega}_k$ for

$$\widetilde{\Omega}_k = B^-(k) \cup B(\lambda) \cup B^+(k).$$

It is possible to smooth $\widetilde{\Omega}_k$ on $\{|y_1 \pm \frac{2}{k}| < \frac{1}{k}\}$ to a region Ω_k with smooth boundary so that η is transverse to $\partial\Omega_k$ and $\cup_k \operatorname{Int}\Omega_k = C(\lambda)$. $\square$

4.2 Embedding Balls into the C Spaces

Let $\mathcal{E}(r, C(\lambda))$ be the space of symplectic embedddings of $B(r)$, the closed ball of radius r in $(\mathbb{R}^4, \omega_0)$, into $C(\lambda)$. For any strictly increasing diffeomorphism $f\colon \mathbb{R} \to \mathbb{R}^+$, the diffeomorphism $i^+\colon \mathbb{R}^4 \to \{y_1 > 0\}$ given by

$$i^+(x_1, y_1, x_2, y_2) = \left(\frac{x_1}{f'(y_1)}, f(y_1), x_2, y_2\right)$$

is symplectic. A similar procedure can be done to construct a symplectic diffeomorphism $i^-\colon \mathbb{R}^4 \to \{y_1 < 0\}$. Then $i_r^\pm := i^\pm|_{B(r)} \in \mathcal{E}(r, C(\lambda))$. Notice that given any r, it is possible to construct the symplectic maps $i^\pm$ so that $i_r^\pm$ are inclusions, i.e. $(i_r^\pm)_* = id$, of the ball of radius r into $C(\lambda)$.

When r is small, meaning $r < \lambda$, i_r^+ and i_r^- are in the same path component of $\mathcal{E}(r, C(\lambda))$. In fact, in this case it is possible to construct a 1-parameter family of inclusions $\tau_t\colon B(r) \to C(\lambda)$, $t \in [0,1]$ so that $\tau_0 = i_r^-$, $\tau_1 = i_r^+$. When $r \geq \lambda$, it is easy to explicitly construct a 1-parameter family of volume-preserving embeddings $v_t\colon B(r) \to C(\lambda)$, $t \in [0,1]$, so that $v_0 = i_r^-$, $v_1 = i_r^+$. However, the situation is much different with respect to symplectic diffeomorphisms. Using Proposition 4.1.1 in combination with Theorem 3.2.1, immediately produces

(4.2.1) Symplectic Camel. *For all $r \geq \lambda$, i_r^+ and i_r^- are in different path components of $\mathcal{E}(r, C(\lambda))$.*

Remark 4.2.2. Using the symplectic isotopy extension theorem (see [MS]), the Symplectic Camel can alternately be stated as follows: *When $r \geq \lambda$, there does not exist a symplectic isotopy, g_t of $C(\lambda)$, $t \in [0,1]$, so that $g_1 \circ i_r^- = i_r^+$.*

As mentioned in Chapter 3, a proof of this result is sketched by Eliashberg and Gromov in [EG, 3.4.B]. It is interesting to note that Viterbo in [V] gives an alternate proof of the Symplectic Camel using generating functions.

If $g : B(r) \to C(\lambda)$ is a symplectic embedding so that

$$\mathrm{Vol}(\,\mathrm{Im}\, g \cap \{y_1 > 0\}) = \mathrm{Vol}\,\mathrm{Im}\, g,$$

then it is easy to see that $\mathrm{Im}\, g \subset \{y_1 \geq 0\}$ and thus g is isotopic to i_r^+. However it is interesting to note that

Theorem 4.2.3. *For all r, $\varepsilon > 0$, there exists a symplectic embedding $g : B(r) \to C(\lambda)$ such that*

$$Vol\,(\,Im\, g \cap \{y_1 > 0\}) > Vol(\,Im\, g) - \varepsilon$$

and g is isotopic to i_r^-.

To prove this proposition, the following result, proved by McDuff and Polterovich in [MP], will be applied.

(4.2.4) Sphere Packing Theorem. *Given any compact symplectic manifold M, for all $\epsilon > 0$ there exists r,k and a symplectic embedding g of k disjoint balls of radius ρ into M,*

$$g\colon\ \amalg_{i=1}^{k} B_i(\rho) \to M,$$

so that $Vol(M \setminus Im\, g) < \epsilon$.

In addition, they have proved that for $M = B(r)$, for all ϵ and k, there exists ρ and an embedding

$$g\colon\ \amalg_1^{k^2} B_i(\rho) \to B(r)$$

so that $\mathrm{Vol}\,(B(r)\setminus \mathrm{Im}\, g) < \epsilon$.

Before applying this theorem, it will be useful to have the following lemma.

Lemma 4.2.5. *Suppose* $g, h\colon \amalg_{i=1}^{k} B(r_i) \to \mathbb{R}^{2n}$ *are symplectic embeddings. Then there exists a symplectic isotopy* Ψ_t *of* $\mathbb{R}^{2n}$, $t \in [0,1]$ *so that* $\Psi_1 \circ g = h$.

Proof. Fix j_{r_i}, $i = 1, \ldots, k$ to be disjoint inclusions of the ball of radius r_i into $\mathbb{R}^{2n}$ and let $g_{r_i} := g|_{B(r_i)}$. It suffices to show that there exists a symplectic isotopy, Ψ_t, of $\mathbb{R}^{2n}$ so that $\Psi_1 \circ g_{r_i} = j_{r_i}$.

Suppose $\mathrm{Im}\, g_{r_i}$, $i = 1, \ldots, k$ are sufficiently far apart in the sense that there exist regions V_{r_i} symplectomorphic to $\mathbb{R}^{2n}$, $V_{r_i} \cap V_{r_j} = \emptyset$ for $i \neq j$, where each V_{r_i} contains the image of g_{r_i} and the image of a standard inclusion i_{r_i}. Then there exists an isotopy ϕ_t of $\mathbb{R}^{2n}$, $t \in [0, 2]$ so that $\phi_1 \circ g_{r_i} = i_{r_i}$, $\phi_2 \circ g_{r_i} = j_{r_i}$, $i = 1, \ldots, k$. Thus to construct Ψ_t, first a symplectic isotopy ψ_t is constructed so that for some $t_0 \gg 0$, $\mathrm{Im}\, \psi_{t_0} \circ g_{r_i}$, $i = 1, \ldots, k$ are sufficiently far apart.

η_{r_i} will denote the radial vector field on $B(r_i) \subset \mathbb{R}^{2n}$ and $\rho_{r_i} = (g_{r_i})_* \eta_{r_i}$ will denote the induced vector field on $\mathrm{Im}\, g_{r_i}$. Let $\varphi_t^{r_i}$ be the backward flow of ρ_{r_i}:

$$\frac{d}{dt}\varphi_t^{r_i}(p) = -\rho_{r_i}(p).$$

It follows that for $t \geq 0$, $\varphi_t^{r_i}(\mathrm{Im}\, g_{r_i}) \subset \mathrm{Im}\, g_{r_i}$. Let ∂ be the (integrable) radial vector field on $\mathbb{R}^{2n}$ centered at $p \notin \mathrm{Im}\, g$ and let χ_t be its flow. Thus χ_t is a diffeomorphism of $\mathbb{R}^{2n}$ for all t. Notice that

$$\mathcal{L}_{-\rho_{r_i}}\omega_0 = -2\omega_0, \qquad \mathcal{L}_{\partial}\omega_0 = 2\omega_0.$$

œIt follows that $\varphi_t^*\omega_0 = c(t)\omega_0$, $\chi_t^*\omega_0 = \frac{1}{c(t)}\omega_0$, for a non-vanishing function c and thus $\psi_t = \chi_t \circ \varphi_t$ is symplectic for all t. In addition, it is easy to check that for some t_0 sufficiently large, $\operatorname{Im}\psi_{t_0} \circ g_{r_i}$ must all be sufficiently far apart and thus the argument in the above paragraph can be applied. □

Proof of Theorem 4.2.3. Apply the sphere packing to fill $\operatorname{Im} i_r^-$ with k^2 balls of radius ρ, $\rho < \lambda$. By Lemma 4.2.5, it is possible to construct a symplectic isotopy of $\{y_1 < 0\}$ which takes the i^{th} ball to a standard inclusion centered at $(x_1, y_1, x_2, y_2) = (0, -i2\lambda, 0, 0)$. By the symplectic isotopy extension theorem, [MS], the symplectic isotopy defined by translating all these balls to the space $\{y_1 > 0\}$ has an extension to $C(\lambda)$. □

The Symplectic Camel provides information about standard elements $i_r^\pm \in \mathcal{E}(r, C(\lambda))$ but nothing about an arbitrary element of $\mathcal{E}(r, C(\lambda))$. Since $C(\lambda)$ has a flat boundary (see Definition 3.3.1), Proposition 4.1.1 implies that Theorem 3.3.3 can be applied to give

(4.2.6) Extendable Embeddings ([MT]). *If $g\colon B(r) \to C(\lambda)$ has a symplectic extension to $g'\colon B(r+\lambda) \to C(\lambda)$, then g is symplectically isotopic to* i_r^+ or i_r^-.

The hypothesis that g has an extension g' is closely tied to a notion of completeness for symplectic manifolds. Eliashberg and Gromov have explored this idea of completeness in [EG].

Definition 4.2.7. (Compare [EG, 1.8].) A symplectically convex manifold is *complete* if it admits an integrable expanding vector field.

It is easy to check that the radial vector field η on $\mathbb{R}^{2n}$ has an integrable flow and thus $\mathbb{R}^{2n}$ is complete. The completeness of $\mathbb{R}^{2n}$ guarantees that all symplectic embeddings of a ball of radius r into $\mathbb{R}^{2n}$ have extensions to embeddings of balls of arbitrarily large radii. This is proved in [EG, 2.1.B]. Their proof, included below for completeness, proves the more general statement.

Lemma 4.2.8. *Suppose $D \subset \mathbb{R}^{2n}$ is a star convex region and W is a complete, convex symplectic $2n$-dimensional manifold. Then every symplectic embedding of D into W has an extension to $\mathbb{R}^{2n}$.*

Proof. Assume $g\colon D \to W$ is a symplectic embedding and η is the given integrable, expanding vector field on W. Let ∂_1, defined on $\operatorname{Im} g$, be the image of the radial vector field under g_*. Choose a compact subset K of W so that $\operatorname{Im} g \subset \operatorname{Int} K$ and let ∂_2 be the restriction of η to the complement of K.

Since η is expanding, $\omega = d\lambda$ for $2\lambda = i(\eta)\omega$. Since ∂_1 is also expanding and $H^1 D = 0$, there exists a function f_1 on $\operatorname{Im} g$ so that

$$i(\partial_1)\omega = 2\lambda + df_1.$$

Extend f_1 to a function f on W which is identically 0 on the complement of K. The non-degeneracy of ω guarantees there exists a unique vector field ∂ on W satisfying

$$i(\partial)\omega = 2\lambda + df$$

and it is easy to see that ∂ is an expanding vector field which extends ∂_1, ∂_2. Since, outside the compact set K, ∂ agrees with the integrable vector field ∂_2,

∂ is integrable. Let φ_t be the flow of this vector field and define $\psi\colon \mathbb{R}^{2n} \to W$ by $\psi(tp) = \varphi_t(g(p))$, where $p \in \partial D$. It is easy to check that ψ is symplectic and extends g. $\square$

Proposition 4.2.9. *$C(\lambda)$ is not complete.*

Proof. Choose $\mu > \lambda$ and let

$$B^{\pm}(\mu, a) = \{x_1^2 + (y_1 \mp a)^2 + x_2^2 + y_2^2 \leq \mu^2\}$$

where $0 < a < \mu$ is chosen so that the interior of $DB(\mu, a) := B^+(\mu, a) \cup B^-(\mu, a)$ is connected and contained in $C(\lambda)$. See Figure 4.2.10. Suppose $B^{\pm}(\mu, a) = \operatorname{Im} j_\mu^{\pm}$. By the Symplectic Camel (4.2.1), $j_\mu^{\pm}$ are not symplectically isotopic.

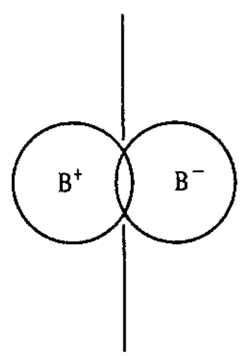

FIGURE 4.2.10. $DB(\mu, a) \subset C(\lambda)$

Suppose $C(\lambda)$ is complete. Then since $DB(\mu, a)$ is star convex, Lemma 4.2.8 implies there exists an embedding g of $\mathbb{R}^4$ into $C(\lambda)$ so that $DB(\mu, a) \subset \operatorname{Im} g$. However since $g^{-1} \circ j_\mu^{\pm}$ are symplectically isotopic this then implies that $j_\mu^{\pm}$ must be in the same path component. # $\square$

There is no known example of a symplectic embedding $g\colon B(r) \to C(\lambda)$ which does not have a symplectic extension to $g'\colon B(r+\lambda) \to C(\lambda)$. However, there do exist embeddings of star convex regions which do not have symplectic "λ" extensions.

Lemma 4.2.11. *There exist symplectic embeddings of convex regions D into $C(\lambda)$ which do not extend to symplectic embeddings of $E(D,\lambda) := \cup_{z\in D} B(z,\lambda)$, where $B(z,\lambda)$ is the closed ball of radius λ centered at z.*

Proof. Choose $\mu > \lambda$ and let $B^{\pm}(\mu, a) = \operatorname{Im} j_{\mu}^{\pm}$ and $DB(\mu, a)$ be as in the proof of Proposition 4.2.9. Suppose there exists a symplectic embedding $g\colon E(D,\lambda) \to C(\lambda)$. By Proposition 4.1.1, it is possible to construct a region Ω so that $g(E) \subset \operatorname{Int}\Omega$, choose $J \in \mathcal{J}_{\Omega}$ so that $J = g_* J_0$ on $\operatorname{Im} E$, and construct the filling $F(J)$. By the Extension Lemma (3.1.4), it follows that $F(J) \cap \operatorname{Im} j_{\mu}^{\pm} = \emptyset$. However by Proposition 3.3.4, this would then imply that $\operatorname{Im} j_{\mu}^{\pm}$ are both contained in a region which is symplectically equivalent to $\mathbb{R}^4$ and thus $j_{\mu}^{\pm}$ are symplectically isotopic. # □

The completeness of $\mathbb{R}^{2n}$ was used in proving that k symplectic embeddings of balls could be isotoped to standard positions. Although $C(\lambda)$ is no longer complete, the Extendable Embeddings can be used to prove a similar result.

Lemma 4.2.12. *Suppose $g, h\colon \amalg_{i=1}^{k} B(r_i) \to C(\lambda)$ are symplectic embeddings which have symplectic extensions $g', h'\colon \amalg_{i=1}^{k} B(r_i+\lambda) \to C(\lambda)$. If $[g|_{B(r_i)}] = [h|_{B(r_i)}] \in \pi_0\left(\mathcal{E}(r_i, C(\lambda))\right)$, $i = 1, \ldots, k$, then there exists a symplectic isotopy ψ_t of $C(\lambda)$ so that $\psi_1 \circ g|_{B(r_i)} = h|_{B(r_i)}$.*

Proof. Suppose $[g|_{B(r_i)}] = [h|_{B(r_i)}] = [i^s_{r_i}]$, $s = s(i) \in \{+,-\}$, then it suffices to construct a symplectic isotopy ψ_t of $C(\lambda)$ so that $\psi_1 \circ g|_{B(r_i)} = i^s_{r_i}$.

By Proposition 4.1.1, it is possible to choose Ω so that $\operatorname{Im} g' \subset \Omega$, choose $J \in \mathcal{J}_\Omega$ so that $J = (g')_* J_0$ on $\operatorname{Im} g'$ and construct a filling $F(J)$. By the Extension Lemma (3.1.4), $F(J)$ does not intersect $\operatorname{Im} g$. Proposition 3.3.4 implies that each g_i lies in a region H^- or H^+ which is symplectically equivalent to $\mathbb{R}^4$. Thus it is possible to apply Lemma 4.2.5 and construct a symplectic isotopy $\varphi_t\colon \amalg_{i=1}^k B(r_i) \to C(\lambda)$ so that $\varphi_0|_{B(r_i)} = g_{r_i}$ $\varphi_1|_{B(r_i)} = j_{r_i}$ where j_{r_i} are disjoint, standard inclusions of a ball of radius r_i into $H^\pm$. By the symplectic isotopy extension theorem, φ_t extends to an isotopy, $\tilde{\varphi}_t$, of $C(\lambda)$. Since j_{r_i} are inclusions, it is easy to construct an isotopy ψ_t of $C(\lambda)$, $t \in [1,2]$ so that $\psi_1 \circ j_{r_i} = i^s_{r_i}$. □

The Symplectic Camel says $\pi_0(\mathcal{E}(r, C(\lambda))$ has more than one element when $r \geq \lambda$. Next an example is given to show how fillings can be used to study, to some extent, higher homotopy groups of $\mathcal{E}(r, C(\lambda))$. Let $\pi_1\big(\mathcal{E}(r, C(\lambda)), +\big)$ be the fundamental group of $\mathcal{E}(r, C(\lambda))$ with basepoint i^+_r. Notice that for all r, elements of $\pi_1\big(\mathcal{E}(r+\lambda, C(\lambda)), +\big)$ induce a subgroup G of $\pi_1\big(\mathcal{E}(r, C(\lambda), +)\big)$ by restriction.

Proposition 4.2.13. $G \simeq \mathbb{Z}$.

Proof. Assume the symplectic diffeomorphism $i^+\colon \mathbb{R}^4 \to \{y_1 > 0\}$ is chosen so that i^+_r is an inclusion. Let $Sp(4)$ be the symplectic linear group of $\mathbb{R}^4$. If $M \in Sp(4)$, $M_r := i^+ \circ M|_{B(r)} \in \mathcal{E}(r, C(\lambda))$ has an extension to $M_{r+\lambda}$. Thus it is easy to see that $\mathbb{Z} = \pi_1(Sp(4), +) \subset G$.

Suppose $\alpha \in \pi_1\big(\mathcal{E}(r,\{y_1 > 0\}),+\big)$. Then α can be homotoped to a loop $\alpha^0 \in \pi_1\big(\mathcal{E}(r,\{y_1 > 0\}),+\big)$ such that $\alpha^0_t(0) = i^+_r(0)$. As described in the introduction, there is an isotopy of $\{y_1 > 0\}$ which takes each embedding α^0_t to the linear map $d\alpha^0_t(0)$. Thus it is easy to see that α^0 is homotopic to a loop of linear embeddings and so $\pi_1\big(\mathcal{E}(r,\{y_1 > 0\}),+\big) \subset \mathbb{Z}$.

Suppose $\gamma \in G$. Choose Ω so that $\operatorname{Im}\gamma_t \subset \operatorname{Int}\Omega$. It follows that there exists $J \in \pi_1(\mathcal{J}_\Omega)$ and a corresponding loop of fillings $F(J_t)$ such that $F(J_t) \cap \operatorname{Im}\gamma_t = \emptyset$, for all t. By Proposition 3.3.4, it follows that there exists a homotopy between γ and $\beta \in \pi_1\big(\mathcal{E}(r,\{y_1 > 0\}),+\big)$ and thus $G \subset \mathbb{Z}$. □

4.3 C Spaces and the Mapping Problem

Recall that the space $U(\lambda)$ introduced at the beginning of the chapter is symplectically equivalent to $\mathbb{R}^4$. Since all symplectic embeddings of a ball of radius r into $\mathbb{R}^4$ are symplectically isotopic, the Symplectic Camel (4.2.1) implies:

Theorem 4.3.1. *For all λ, $C(\lambda)$ is not symplectically equivalent to $\mathbb{R}^4$.*

Now the effect of changing the radius of the hole is investigated. First notice that $C(\lambda)$ and $C(\mu)$ are equivalent with respect to volume preserving diffeomorphisms since

$$\Phi(x_1, y_1, x_2, y_2) = \left(\frac{\mu}{\lambda}x_1, \frac{\lambda^3}{\mu^3}y_1, \frac{\mu}{\lambda}x_2, \frac{\mu}{\lambda}y_2\right)$$

is a volume preserving diffeomorphism of $\mathbb{R}^4$ which sends $C(\lambda)$ to $C(\mu)$. However,

Theorem 4.3.2 ([MT]). *$C(\lambda)$ is symplectomorphic to $C(\mu)$ if and only if $\lambda = \mu$.*

Proof. Suppose $\psi\colon C(\lambda) \to C(\mu)$ is a symplectomorphism with $\lambda < \mu$, and r is chosen so that $\lambda < r < \mu$. By the Symplectic Camel (4.2.1), the embeddings $i_r^{\pm}$ are not symplectically isotopic and thus $\psi \circ i_r^{\pm}$ are not symplectically isotopic. However since $\psi \circ i_r^{\pm}$ have $\psi \circ i_{r+\mu}^{\pm}$ as extensions to the ball of radius $r + \mu$, the Extendable Embeddings (4.2.6) implies that both $\psi \circ i_r^+$ and $\psi \circ i_r^-$ must be symplectically isotopic to either i_r^+ or i_r^- in $\mathcal{E}(r, C(\mu))$. However since $r < \mu$, this implies that $\psi \circ i_r^+$ and $\psi \circ i_r^-$ must be symplectically isotopic. # □

The fact that changing the radius of the hole produces symplectically different spaces makes these C spaces seem very "rigid". However, as the following theorem shows, the rigidity is only in the (x_2, y_2)-directions. More precisely, let $\pi\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the projection to the (x_2, y_2)-plane. For $D \subset \mathbb{R}^2$,

$$\operatorname{area} D := \left| \int_D dx_2 \wedge dy_2 \right|.$$

Theorem 4.3.3. *Let $G(\lambda) \subset \{y_1 = 0\}$ be an open region diffeomorphic to $H(\lambda)$ such that*

(1) *$\operatorname{area} \pi(G(\lambda)) = \operatorname{area} \pi(H(\lambda))$;*

(2) *for all $p \in \pi(G(\lambda))$, $\pi^{-1}(p) \cap G(\lambda)$ is a connected subset of $\mathbb{R}$.*

Then $C'(\lambda) := \{y_1 < 0\} \cup \{y_1 > 0\} \cup G(\lambda)$ is symplectically equivalent to $C(\lambda)$.

Proof. It is possible to assume $G(\lambda) \subset \{|x_1| < 1\}$ by the following argument. Suppose $F(\lambda)$, the hole in $C''(\lambda)$, satisfies (1) and (2) but $F(\lambda) \not\subset \{|x_1| <$

1}. The results of Greene and Shiohama, [GS], imply there exists an area preserving embedding, α, of the (x_1, y_1)-plane into itself which takes $\{y_1 = 0\}$ to $\{|x_1| < 1, y_1 = 0\}$ and which preserves the sets $\{y_1 < 0\}$ and $\{y_1 > 0\}$. Then $\psi = \alpha \times id$ is a symplectomorphism of $C''(\lambda)$ to a space $C'(\lambda)$ whose hole $G(\lambda) \subset \{|x_1| < 1, y_1 = 0\}$ satisfies the hypotheses (1) and (2).

Furthermore, by (1), there exists an area preserving diffeomorphism β of the (x_2, y_2)-plane such that $\beta \circ \pi(G(\lambda)) = \pi(H(\lambda))$. Thus by applying the symplectomorphism $\psi = id \times \beta$, it is possible to assume $\pi(G(\lambda)) = \pi(H(\lambda))$.

To construct a symplectomorphism between $C'(\lambda)$ and $C(\lambda)$, let v be a vector field of the form

$$v := \kappa \frac{\partial}{\partial x_1}$$

where κ is a non-vanishing function which is identically 1 outside a compact set. Let φ_t be its flow. Choose $K \ll 0$ and consider

$$\Phi(x_1, y_1, x_2, y_2) = (K + t, y_1, x_2, y_2)$$

where (x_1, y_1, x_2, y_2) is the image of the point (K, y_1, x_2, y_2) under the map φ_t. By conditions (1), (2), and the facts that $G(\lambda), H(\lambda) \subset \{|x_1| < 1, y_1 = 0\}$ and $\pi(G(\lambda)) = \pi(H(\lambda))$, it is possible to construct the function κ so that Φ takes $G(\lambda)$ to $H(\lambda)$ and so that Φ is the identity outside a compact subset of $\mathbb{R}^4$.

As in the proof of Lemma 3.3.6, a Moser argument is now applied to modify Φ so it is symplectic. It is easy to explicitly check that the forms $\omega_r = r\Phi^*\omega_0 + (1-r)\omega_0$ are non-degenerate on $\mathbb{R}^4$ when $0 \leq r \leq 1$. Suppose there exists a 1-form β which vanishes at all points of $\{y_1 = 0\}$ and satisfies

$d\beta = \Phi^*\omega_0 - \omega_0$. Then if β has compact support, it follows that the vector field v_r defined by

$$i(v_r)\omega_r + \beta = 0$$

is integrable, and the standard Moser method argument then shows that the integral h_r of v_r satisfies $h_r^*\omega_r = \omega_0$. Since $h_r = id$ on $\{y_1 = 0\}$ the desired symplectomorphism between $C'(\lambda)$ and $C(\lambda)$ is then given by $\Psi = \Phi \circ h_1$.

Consider $\beta' = \Phi^*(x_1dy_1 + x_2dy_2) - (x_1dy_1 + x_2dy_2)$. Then, by construction of Φ,

$$\beta' = f dy_1$$

where f has compact support and thus $\beta = \beta' - d(fy_1)$ has all the desired properties. □

In particular, if $G(\lambda)$ is any of the following sets

(1) $\{(\frac{x_1}{a})^2 + (\frac{x_2}{b})^2 + (by_2)^2 < \lambda^2, y_1 = 0\}$ (Ellipsoid);

(2) $\{x_2^2 + y_2^2 < \lambda^2, |x_1| < 1, y_1 = 0\}$ (Bounded Cylinder);

(3) $\{x_2^2 + y_2^2 < \lambda^2, y_1 = 0\}$ (Infinite Cylinder);

then $C'(\lambda) = \{y_1 < 0\} \cup \{y_1 > 0\} \cup G(\lambda)$ is symplectically equivalent to $C(\lambda)$. See Figure 4.3.4.

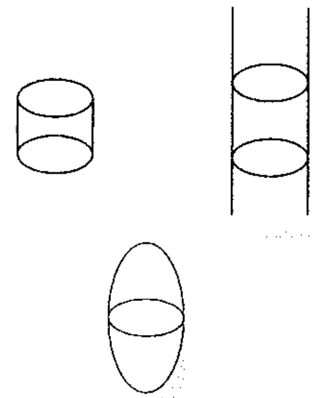

FIGURE 4.3.4. Alternate possible hole shapes.

It is clear that condition (1) in the above theorem is necessary. It is not clear whether or not condition (2) is necessary. For example:

Question 4.3.5. *For $\lambda > 1$, let $G(\lambda)$ be the "dumbbell"*

$$G(\lambda) = \cup\{(x_1 \pm 1)^2 + x_2^2 + y_2^2 < \lambda^2, y_1 = 0\}.$$

See Figure 4.3.6. Then is $D(\lambda) := \{y_1 < 0\} \cup \{y_1 > 0\} \cup G(\lambda)$ symplectically equivalent to $C(\lambda)$?

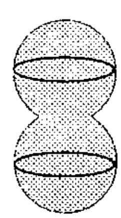

FIGURE 4.3.6.

Another natural question, suggested by Lalonde, is whether or not a "thicker" hole would correspond to a different space. More precisely,

Question 4.3.7. *If*

$$T(\lambda) = \{x_1^2 + x_2^2 + y_2^2 < \lambda^2, |y_1| < 1\},$$

is $U(\lambda) := \{y_1 < -1\} \cup \{y_1 > 1\} \cup T(\lambda)$ *symplectically equivalent to* $C(\lambda)$*?*

FIGURE 4.3.8. $U(1)$

Chapter 5

W Spaces

In Chapter 4, the space $C(\lambda)$ which is the union of the two half-spaces $\{y_1 < 0\}$, $\{y_1 > 0\}$ and the open 3-ball $H(\lambda) = \{x_1^2 + x_2^2 + y_2^2 < \lambda^2, y_1 = 0\}$ was examined. $C(\lambda)$ can be thought of as a subset of $\mathbb{R}^4$ having one wall which has one hole. As natural generalizations, spaces with multiple one-holed walls, W spaces, will now be investigated. Given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$,

$$W(\lambda_1, \lambda_2, \ldots, \lambda_n) := \{y_1 < 1\} \cup H_1 \cup \{1 < y_1 < 2\} \cup \cdots \cup H_n \cup \{n < y_1\}$$

where $H_i = \{2x_1^2 + x_2^2 + y_2^2 < \lambda_i^2, y_1 = i\}$.

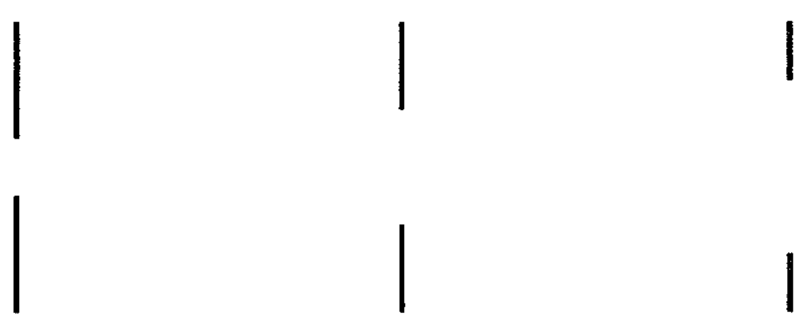

FIGURE 5.0. $W(1,2,3)$

H_i, $i = 1, \ldots, n$, will be referred to as the "holes" of $W(\lambda_1, \ldots, \lambda_n)$ and the hyperplanes $\{y_1 = i\}$ which contain these holes will be referred to as the "walls" of $W(\lambda_1, \ldots, \lambda_n)$.

5.1 Non-important Specifications

The placements of the walls and holes in these W spaces have been described very precisely but in fact do not need to be specified. To investigate different positions of the walls, given $k_1 < k_2 < \cdots < k_n$, consider $W(\lambda_1(k_1), \ldots, \lambda_n(k_n))$ which is analogous to $W(\lambda_1, \ldots, \lambda_n)$ except its walls are centered at $y_1 = k_1, \ldots, k_n$.

Lemma 5.1.1. *$W(\lambda_1, \ldots, \lambda_n)$ and $W(\lambda_1(k_1), \ldots, \lambda_n(k_n))$ are symplectically equivalent.*

Proof. Since $k_1 < \cdots < k_n$, there exists a smooth, strictly increasing function $g(y_1)$ with image $(-\infty, \infty)$ so that $g(y_1)$ agrees with $y_1 - i + k_i$ near $\{y_1 = i\}$, $i = 1, \ldots, n$. Then define $f(x_1, y_1) = x_1 / \frac{dg}{dy_1}$ and consider

$$\Phi(x_1, y_1, x_2, y_2) = \big(f(x_1, y_1),\, g(y_1),\, x_2,\, y_2\big).$$

Φ is a symplectic diffeomorphism of $\mathbb{R}^4$ which descends to a map between $W(\lambda_1, \ldots, \lambda_n)$ and $W(\lambda_1(k_1), \ldots, \lambda_n(k_n))$. $\square$

In the definition of $W(\lambda_1, \ldots, \lambda_n)$, it was specified that the holes are "aligned" in the sense that all the open 3-balls, H_i, have centers at the same (x_1, x_2, y_2)-coordinates. To investigate the different centerings of the holes, given $\{(a_i, b_i, c_i)\}_1^n$, consider $W(\lambda_1(a_1, b_1, c_1), \ldots, \lambda_n(a_n, b_n, c_n))$ which is analogous to $W(\lambda_1, \ldots, \lambda_n)$ except its i^{th} hole of radius λ_i in $\{y_1 = i\}$ is centered at $(x_1, x_2, y_2) = (a_i, b_i, c_i)$.

Lemma 5.1.2. *For any* $\{(a_i, b_i, c_i)\}_1^n$, $W(\lambda_1, \ldots, \lambda_n)$ *is symplectomorphic to* $W(\lambda_1(a_1, b_1, c_1), \ldots, \lambda_n(a_n, b_n, c_n))$.

Proof. Choose $0 < \epsilon < 1/2$ and let $\rho_i = \rho_i(y_1)$, $i = 1, \ldots, n$ be a family of smooth functions so that ρ_i is identically 1 near $y_1 = i$ and identically 0 for $|y_1 - i| > \epsilon$. Then consider

$$\Psi(x_1, y_1, x_2, y_2) =$$
$$\left(x_1 + \sum_1^n \left(a_i\rho_i + b_i \frac{d\rho_i}{dy_1} y_2 - c_i \frac{d\rho_i}{dy_1} x_2\right),\ y_1,\ x_2 + \sum_1^n b_i\rho_i,\ y_2 + \sum_1^n c_i\rho_i\right).$$

Ψ is a symplectic diffeomorphism of $\mathbb{R}^4$ which descends to a map between $W(\lambda_1, \ldots, \lambda_n)$ and $W(\lambda_1(a_1, b_1, c_1), \ldots, \lambda_n(a_n, b_n, c_n))$. □

Thus the notation $W(\lambda_1, \ldots, \lambda_n)$ is well chosen since the relative positions of the holes and of the walls is unimportant.

Lastly, notice that in passing from the C spaces to the W spaces, there has been a slight modification in the shape of the hole. This will prove to be an easier shape to work with when applying the filling technique. In fact, this space is the same as the analogous space with spherical holes. More generally, if $\pi\colon \mathbb{R}^4 \to \mathbb{R}^2$ is the projection to the (x_2, y_2)-plane, the following is true.

Theorem 5.1.3. *Let $G_i \subset \{y_1 = i\}$ be an open region diffeomorphic to H_i so that*

(1) $\operatorname{area} \pi(G_i) = \operatorname{area} \pi(H_i)$;

(2) $\pi^{-1}(p) \cap G_i$ *is a connected subset of* $\mathbb{R}$ *for all* $p \in \pi(G_i)$.

Then

$$W'(\lambda_1, \lambda_2, \ldots, \lambda_n) := \{y_1 < 1\} \cup G_1 \cup \{1 < y_1 < 2\} \cup \cdots \cup G_n \cup \{n < y_1\}$$

is symplectically equivalent to $W(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof. By Lemma 5.1.2, it is possible to assume that $\pi(H_i) \cap \pi(H_j) = \emptyset = \pi(G_i) \cap \pi(G_j)$, $i \neq j$. Then by (1), there must exist an area preserving diffeomorphism θ of the (x_2, y_2)-plane so that $\theta \circ \pi(G_i) = \pi(H_i)$, $i = 1, \ldots, n$. The remainder of the proof easily follows using the method in the proof of Theorem 4.3.3. □

5.2 Varying the Radii of the Holes

To examine the effect of varying the radii of the holes, it will be important to have analogues of the Symplectic Camel (4.2.1) and the Extendable Embeddings (4.2.6) for the W spaces. Thus it is important to know that ∂H_i are fillable holes in terms of Definition 3.1.2. In particular, it must be shown that these W spaces are symplectically convex. Recall that this means there exists a sequence of regions Ω_k and expanding vector fields η_k transverse to $\partial\Omega_k$ so that $W(\lambda_1, \ldots, \lambda_n) = \cup \operatorname{Int} \Omega_k$, $\partial H_i \subset \partial\Omega_k$ for all i, k.

Recall also that to apply the filling result from Chapter 2, $\partial\Omega_k, \eta_k$ should be "standard" near ∂H_i. By the definition of H_i, this means Ω_k should be a "solid cylinder" near ∂H_i (see (2.2.1)). The $C(\lambda)$ from the previous chapter is star convex and since the radial vector field is expanding, it is easy to prove the C spaces are symplectically convex. To show $W(\lambda_1, \dots, \lambda_n)$ has fillable holes, exhausting regions Ω_k will be constructed by piecing together handles and convex regions. The associated expanding vector field η_k will be constructed by piecing together expanding vector fields which are transverse to the handle and convex bodies.

Proposition 5.2.1. *$\partial H_1, \dots, \partial H_n$ are fillable holes of $W(\lambda_1, \dots, \lambda_n)$.*

Proof. Assume the hole H_1 of radius λ_1 is contained in $\{y_1 = 0\}$ and the hole H_2 of radius λ_2 is contained in $\{y_1 = 2\}$. It will be shown how it is possible to construct $\Omega_k \cap \{0 \le y_1 < 2\}$. Analogous procedures can be applied to construct the remaining portions of Ω_k.

Given $k > 0$, let

$$\mathcal{H} = \mathcal{H}_k = \left\{ 2x_1^2 + x_2^2 + y_2^2 \le \lambda_1^2, \quad |y_1| < \frac{1}{k} \right\}$$

and let

$$v_h = 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}.$$

It then follows that, with respect to v_h, $\mathcal{H}$ satisfies condition (3) of (2.1.1).

For functions $k_1(y_1), k_2(y_1), k_3(y_1)$, let

$$\mathcal{C} = \mathcal{C}_k = \left\{ k_1^2 x_1^2 + (y_1 - 1)^2 + k_2^2 x_2^2 + k_3^2 y_2^2 \le 1 \right\},$$
$$v_c = x_1 \frac{\partial}{\partial x_1} + (y_1 - 1) \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}.$$

It is clear that the functions k_i can be chosen so that $\{0 < y_1 < 2\} \subset \cup_k \mathcal{C}_k$ and so that $\mathcal{H} \cup C$ can be smoothed to a region $\Omega = \Omega_k$ on $\mathcal{N} := \{|y_1 - \frac{1}{k}| < \delta\}$, for some δ.

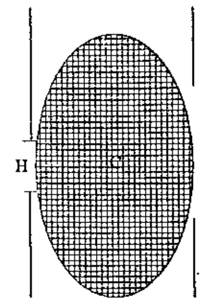

FIGURE 5.2.2.

In the following, an expanding vector field η_k is constructed which is transverse to $\partial\Omega_k$, agrees with v_h on $\mathcal{H}\backslash\mathcal{H} \cap \mathcal{N}$, and agrees with v_c on $\mathcal{C}\backslash\mathcal{C} \cap \mathcal{N}$.

It is easy to check that v_c and v_h are transverse to both $\partial\mathcal{C} \cap \mathcal{N}$ and $\partial\mathcal{H} \cap \mathcal{N}$ and thus can be assumed to be transverse to $\partial\Omega_k \cap \mathcal{N}$. Choose a smooth function $\rho = \rho_k(y_1)$ so that ρ is identically 1 on $\mathcal{H}\backslash\mathcal{H} \cap \mathcal{N}$ and identically 0 on $\mathcal{C}\backslash\mathcal{C} \cap \mathcal{N}$. By definition of $\mathcal{N}$, it is possible to choose ρ so that $\frac{d\rho}{dy_1} \leq 0$.

Let χ_ρ be the symplectic gradient of ρ; $i(\chi_\rho)\omega_0 = d\rho$ and notice that

$$i(v_h - v_c)\omega_0 = df$$

where $f = x_1(y_1 - 1)$. It is easy to check that

$$\eta_k = \rho v_h + (1 - \rho)v_c + f\chi_\rho$$

is expanding. It remains to check that η_k is transverse to $\partial\Omega_k \cap \mathcal{N}$. Since v_c and v_h are both transversal to both $\partial\mathcal{H}$ and $\partial\mathcal{C}$, it suffices to check that $f\chi_\rho$ is transverse (or tangent) to $\partial\mathcal{H}$ and $\partial\mathcal{C}$. Where $d\rho \neq 0$ (and thus where χ_ρ does not vanish), f has the form $f = -\gamma^2 x_1$. Then by the specifications on ρ,

$$f\chi_\rho = -\gamma^2 x_1 \frac{d\rho}{dy_1}\frac{\partial}{\partial x_1} = x_1\tilde{\gamma}^2\frac{\partial}{\partial x_1}.$$

Thus it follows that η_k is transverse to $\partial\Omega_k$. □

Notice that $W(\lambda_1, \ldots, \lambda_n)$ has flat boundaries. (See Definition 3.3.1.) As in Section 3.3, fix $i^k \colon \mathbb{R}^4 \to W \setminus \cup H_i$, for $k = 0, 1, \ldots n$, as symplectic parameterizations of the remaining spaces where the numbering is chosen so that $\operatorname{Im} i^{k-1} \cup H_k \cup \operatorname{Im} i^k$ is connected, $k = 1, \ldots, n$. Let $\mathcal{E}(r, W(\lambda_1, \ldots, \lambda_n))$ denote the space of symplectic embeddings of a closed ball of radius r into $W(\lambda_1, \ldots, \lambda_n)$ and let $i^k_r = i^k|_{B(r)} \in \mathcal{E}(r, W(\lambda_1, \ldots, \lambda_n))$ where $B(r)$ is the closed ball of radius r.

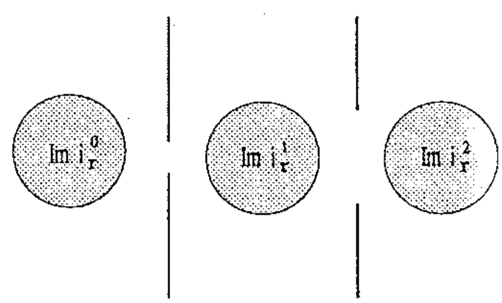

FIGURE 5.2.3.

Theorems 3.2.1 and 3.3.3 imply

(5.2.4) Symplectic Camel for W Spaces. *i_r^k and i_r^ℓ represent different elements of $\pi_0\big(\mathcal{E}(r, W(\lambda_1, \ldots, \lambda_n))\big)$ when $r \geq \min\{\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_\ell\}$.*

(5.2.5) Extendable Embeddings in W Spaces. *Let g be an embedding of the closed ball of radius r into $W(\lambda_1, \ldots, \lambda_n)$ which has an extension to a symplectic embedding g' of the closed ball of radius $r + \max\{\lambda_1, \ldots, \lambda_n\}$ into $W(\lambda_1, \ldots, \lambda_n)$. Then g is symplectically isotopic to i_r^k, for some $k \in \{0, \ldots, n\}$.*

These results can then be used to prove

Theorem 5.2.6. *If $W(\lambda_1, \ldots, \lambda_n)$ is symplectomorphic to $W(\mu_1, \ldots, \mu_m)$ then $m = n$ and, more generally, for all r,*

$$\big|\{j\colon \lambda_j \leq r\}\big| = \big|\{k\colon \mu_k \leq r\}\big|.$$

Proof. Suppose $\psi\colon W(\lambda_1, \ldots, \lambda_n) \to W(\mu_1, \ldots, \mu_m)$ is a symplectomorphism and there exists an r such that

$$\ell := \big|\{j\colon \lambda_j \leq r\}\big| \neq \big|\{k\colon \mu_k \leq r\}\big| =: \nu.$$

Assume, without loss of generality, that $\ell > \nu$. By the choice of r and the above Symplectic Camel, (5.2.4), $\{i_r^j\}_{j=0}^n$ represent $\ell + 1$ distinct path components of $\mathcal{E}(r, W(\lambda_1, \ldots, \lambda_n))$. Thus $\{\psi \circ i_r^j\}_{j=0}^n$ represent $\ell + 1$ distinct path components in $\mathcal{E}(r, W(\mu_1, \ldots, \mu_n))$. However for each j, $\psi \circ i_r^j$ extends to the symplectic embedding $\psi \circ i_{r+\mu_{\max}}^j$ of the ball of radius $r + \mu_{\max}$, where $\mu_{\max} = \max\{\mu_1, \ldots, \mu_m\}$. Thus (5.2.5) and (5.2.4) imply $\{\psi \circ i_r^j\}_{j=0}^n$ represent at most $\nu + 1$ distinct path components. Thus $\ell + 1 \leq \nu + 1$ which is a contradiction. $\square$

Corollary 5.2.7. *If $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_m$, then $W(\lambda_1, \ldots, \lambda_n)$ is symplectomorphic to $W(\mu_1, \ldots, \mu_m)$ if and only if $n = m$ and $\lambda_i = \mu_i$.*

5.3 Symplectic Embedding Trees

In Corollary 5.2.7, it was specified that $\lambda_1 \leq \cdots \leq \lambda_n$. Now the equivalence question is examined without this condition. First notice that via a symplectomorphism of the form $\psi = id \times \theta$, where $\theta(x_2, y_2)$ is a rotation in the (x_2, y_2)-plane, $W(\lambda_1, \ldots, \lambda_n)$ is symplectically equivalent to $W\left(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}\right)$ for $\sigma(1) = n, \sigma(2) = n-1, \ldots, \sigma(n) = 1$. To investigate other permutations of the holes, "embedding trees" will be associated to each W space.

Let $\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_n\}$. By the Symplectic Camel, $\left[i^k_{\lambda_{\max}}\right]$ and $\left[i^j_{\lambda_{\max}}\right]$, $j \neq k$, are distinct elements of $\pi_0\left(\mathcal{E}(\lambda_{\max}, W(\lambda_1, \ldots, \lambda_n))\right)$. More generally, (5.2.4) implies that for any j and k, $\left[i^k_{\lambda_j}\right]$ is contained in an $m+1$ element subset I_{λ_j} of $\pi_0\left(\mathcal{E}(\lambda_j, W(\lambda_1, \ldots, \lambda_n))\right)$ where m is the number of λ_i which are less than or equal to λ_j. Suppose $\{\lambda_1, \ldots, \lambda_n\} = \{\mu_1, \ldots, \mu_d\}$ where $\mu_1 < \mu_2 < \cdots < \mu_d$. Then a d-level tree can be associated to $W(\lambda_1, \ldots, \lambda_n)$ by placing $m+1$ vertices on the ℓ^{th} level where m is the number of λ_i which are less than or equal to μ_ℓ. In other words, each vertex corresponds to an element of I_{μ_ℓ}. A vertex corresponding to $\left[i^k_{\mu_\ell}\right] \in I_{\mu_\ell}$ is then connected to a vertex corresponding to $\left[i^j_{\mu_{\ell+1}}\right] \in I_{\mu_{\ell+1}}$ if

$$\left[i^j_{\mu_{\ell+1}}|_{B(\mu_\ell)}\right] = \left[i^k_{\mu_\ell}\right].$$

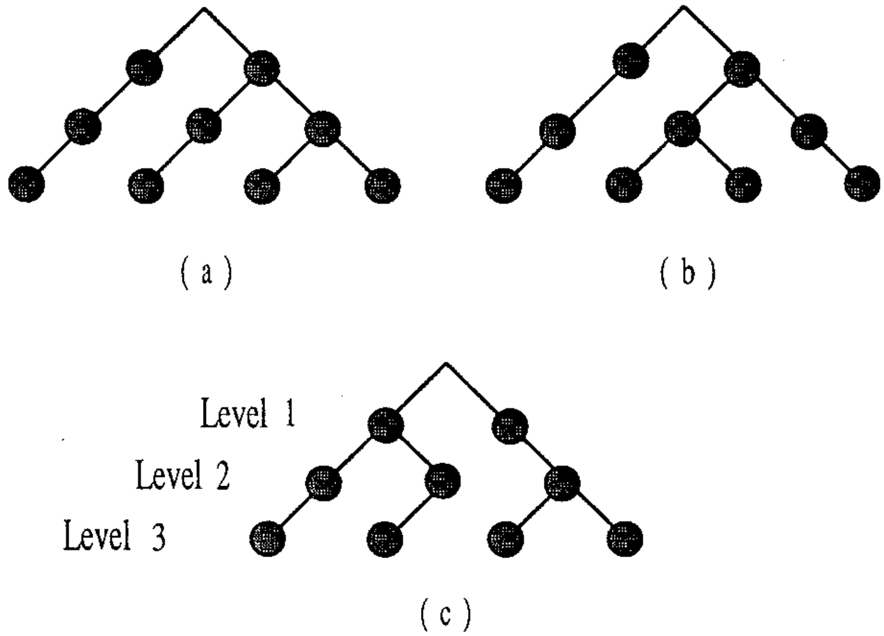

FIGURE 5.3.1. Trees associated to $W(1,2,3)$, $W(1,3,2)$ and $W(2,1,3)$ are given by (a), (b), (c) respectively.

Two trees T_1, T_2 are *isomorphic* if there is a level-preserving bijection between the vertices of T_1 and those of T_2 which preserves the connections between the levels. If there exists a symplectomorphism $\psi : W \to \widetilde{W}$, then by the above Extendable Embeddings (5.2.5), ψ induces a bijection between the vertices of the lowest levels of the trees. This in turn determines a bijection between the other levels of the trees. Thus,

Theorem 5.3.2. *If two W spaces are symplectomorphic then their associated trees must be isomorphic.*

As an easy consequence, it is immediate that

Corollary 5.3.3. *Neither $W(1,2,3)$ nor $W(1,3,2)$ is symplectically equivalent to $W(2,1,3)$. More generally, if $\lambda_1 < \cdots < \lambda_n$ and σ is a permutation of $\{1,\ldots,n\}$ so that $\sigma(1) \notin \{1,n\}$ then $W(\lambda_1,\ldots,\lambda_n)$ and $W\left(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(n)}\right)$ are not symplectically equivalent.*

Notice that the trees associated to $W(1,2,3)$ and $W(1,3,2)$ are isomorphic and thus Theorem 5.3.2 does not imply that these spaces are symplectically different. However, the trees indicate that a symplectomorphism between $W(1,2,3)$ and $W(1,3,2)$ would be "wildly behaved" near the end. To describe this more precisely, $\operatorname{Im} i^k, \operatorname{Im} i^\ell \subset W(\lambda_1,\ldots,\lambda_n)$ will be called ***adjacent*** if there exists a path $\alpha(t)$, $t \in [0,1]$ such that $\alpha(0) \in \operatorname{Im} i^k$, $\alpha(1) \in \operatorname{Im} i^\ell$, and α passes through precisely one hole. $\left[i_r^k\right], \left[i_r^\ell\right] \subset \pi_0\left(\mathcal{E}(r, W(\lambda_1,\ldots,\lambda_n))\right)$ will be called ***adjacent*** if $\operatorname{Im} i^k$ and $\operatorname{Im} i^\ell$ are adjacent. Notice that if there is a symplectomorphism ψ between $W(1,2,3)$ and $W(1,3,2)$ then it must take adjacent isotopy classes of $\pi_0\left(\mathcal{E}(3, W(1,2,3))\right)$ to non-adjacent isotopy classes in $\pi_0\left(\mathcal{E}(3, W(1,3,2))\right)$. This implies, for example,

Lemma 5.3.4. *If there exists a symplectomorphism $\psi\colon W(1,2,3) \to W(1,3,2)$ then $\psi(H_2)$ is not contained in a compact set $K \subset \mathbb{R}^4$. Furthermore,*

$$\psi(H_2) \cap K^c \neq \{y_1 = constant\} \cap K^c,$$

where K^c is the complement of K.

Proof. Suppose there exists a symplectomorphism $\psi\colon W(1,2,3) \to W(1,3,2)$ so that $\psi(H_2)$ is contained in the interior of a compact set K. Assume

$i^k\colon \mathbb{R}^4 \to W(1,2,3)\backslash \cup H_i$ and $j^k\colon \mathbb{R}^4 \to W(1,3,2)\backslash \cup H_i$ where $\operatorname{Im} i^0 = \{y_1 < 1\} = \operatorname{Im} j^0$. By the embedding trees, it is easy to check that

$$\psi_* \left[i_3^0\right] = \left[j_3^0\right], \qquad \psi_* \left[i_3^1\right] = \left[j_3^3\right].$$

Then there must exist a symplectomorphism $\psi'\colon W(1,2,3) \to W(1,3,2)$ so that $\psi'(H_2) = \psi(H_2)$, $\psi' \circ i_3^0 = j_3^0$, $\psi' \circ i_3^1 = j_3^3$ where $\operatorname{Im} j_3^0 \subset W(1,3,2) \cap K^c$, $\operatorname{Im} j_3^3 \subset W(1,3,2) \cap K^c$. It follows that $\operatorname{Im} j_3^0$ and $\operatorname{Im} j_3^3$ must be in separate components of $W(1,3,2)\backslash\psi'(H_2)$. Notice that there exists a path $\gamma\colon [0,1] \to W(1,2,3)\backslash H_2$ such that $\gamma(0) \in \operatorname{Im} i_3^0$, $\gamma(1) \in \operatorname{Im} i_3^3$. It then follows that $\psi' \circ \gamma$ is a path such that $\psi' \circ \gamma(0) \in \operatorname{Im} j_3^0$, $\psi' \circ \gamma(1) \in \operatorname{Im} j_3^3$, and $\psi' \circ \gamma$ does not cross $\psi'(H_2)$. However, this is a contradiction.

Next suppose $\psi(H_2)$ agrees with $\{y_1 = c\}$ on K^c. Then it is possible to construct a symplectomorphism ψ' so that

$$\psi' \circ i_3^0 = j_3^0, \quad \psi' \circ i_3^1 = j_3^1, \quad \psi'(H_2) = \{y_1 = c\}.$$

However, it is easy to check that this is impossible for any c. □

This gives support to the following conjecture.

Conjecture 5.3.5. *If $\left[i_r^k\right]$ and $\left[i_r^\ell\right]$ are adjacent isotopy classes in W and $\psi\colon W \to \widetilde{W}$ is a symplectomorphism, then $\left[\psi \circ i_r^k\right]$ and $\left[\psi \circ i_r^\ell\right]$ are adjacent isotopy classes in $\widetilde{W}$.*

If this conjecture is true, it would imply the following.

Conjecture 5.3.6. *$W(\lambda_1, \dots \lambda_n)$ is symplectomorphic to $W(\mu_1, \dots \mu_n)$ if and only if $\lambda_i = \mu_i$ or $\lambda_i = \mu_{n-i}$ for all i.*

Chapter 6

Z Spaces

In the previous chapter, spaces with multiple 1-holed walls, W spaces, were investigated. In this chapter, spaces with 1 multi-holed wall, Z spaces, are examined. The focus will be on spaces consisting of the two half spaces $\{y_1 < 0\}$, $\{y_1 > 0\}$ and two open 3-dimensional holes H_1, H_2 in $\{y_1 = 0\}$. Notice that these Z spaces have a fundamental group equal to $\mathbb{Z}$.

FIGURE 6.0. $Z(1,2)$

6.1 Notation for the Z Spaces

Although the relative positions of the holes in the W spaces were unimportant, the question is more subtle when both holes are contained in the hyperplane $\{y_1 = 0\}$.

Let $\pi\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the projection of $\mathbb{R}^4$ to the (x_2, y_2)-plane. Let $H_{allow}(\lambda_1, \lambda_2)$ be the set of all pairs (H_1, H_2) such that

$$H_1 = \left\{2x_1^2 + x_2^2 + y_2^2 < \lambda_1^2, \quad y_1 = 0\right\},$$

$$H_2 = \left\{2(x_1 - a)^2 + (x_2 - b)^2 + (y_2 - c)^2 < \lambda_2^2, \quad y_1 = 0\right\},$$

where $\overline{H}_1 \cap \overline{H}_2 = \emptyset$ and $\partial(\pi(H_1)) \cap \partial(\pi(H_2))$ is not a one point set. See Figure 6.1.1.

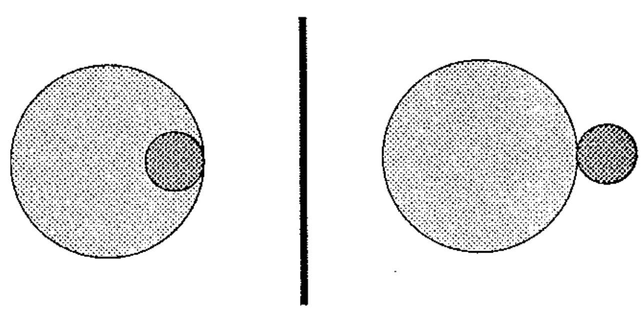

FIGURE 6.1.1. Forbidden projections of H_1 and H_2.

Then given $\lambda_1, \lambda_2 \in (0, \infty)$, fix $\alpha \in [0, \max]$, $\max := \pi \min\{\lambda_1^2, \lambda_2^2\}$, and let

$$Z^\alpha(\lambda_1, \lambda_2) = \left\{\{y_1 < 0\} \cup \{y_1 > 0\} \cup H_1 \cup H_2\right\}$$

where $(H_1, H_2) \in H_{allow}(\lambda_1, \lambda_2)$ and $\operatorname{area}(\pi(H_1) \cap \pi(H_2)) = \alpha$.

Proposition 6.1.2. *Any two elements of $Z^\alpha(\lambda_1, \lambda_2)$ are symplectically equivalent.*

Thus the notation $Z^\alpha(\lambda_1, \lambda_2)$ will be used to denote any element of the corresponding set. $Z^0(\lambda_1, \lambda_2)$ can be thought of as a space with *completely non-aligned* holes. The notation $Z(\lambda_1, \lambda_2)$ without any superscript will denote the space $Z^{\max}(\lambda_1, \lambda_2)$ with *completely aligned* holes.

Proof. Fix $\lambda_1, \lambda_2, \alpha$ and as temporary notation let Z, Z' denote the elements of $Z^\alpha(\lambda_1, \lambda_2)$ with the holes H_1, H_2 of Z centered at $(0,0,0)$ and (a,b,c) and the holes H_1', H_2' of Z' centered at $(0,0,0)$ and (a',b',c'). It is possible to assume $a, a' \geq 0$ by applying the symplectomorphism $\psi(x_1, y_1, x_2, y_2) = (-x_1, -y_1, x_2, y_2)$ if necessary.

To construct a symplectomorphism between Z and Z', first notice that by the definition of (H_1, H_2) and (H_1', H_2'), the results of Greene and Shiohama [GS] can be used to construct an area preserving diffeomorphism θ of the (x_2, y_2)-plane so that $\theta \circ \pi(H_i) = \pi(H_i')$, $i = 1, 2$. Then $\psi_1 = \theta \times id$ is a symplectomorphism of Z into $\mathbb{R}^4$.

If area $(\pi(H_1) \cap \pi(H_2)) = 0$, it is possible to construct θ so that it is the identity on $H_1 = H_1'$ and so that it is a translation between H_2 and H_2'. Thus, in fact, ψ_1 is a symplectomorphism between Z and Z'.

Similarly, if $0 <$ area $(\pi(H_1) \cap \pi(H_2)) <$ max, by the symmetrical shapes of the projections, it is possible to choose θ to be a rotation of the (x_2, y_2)-plane. Thus again, ψ_1 is a symplectomorphism between Z and Z'.

If area $(\pi(H_1) \cap \pi(H_2)) =$ max, then ψ_1 will distort the shapes of the holes. However notice that $\psi_1(H_i)$ are still diffeomorphic to H_i, and for all

$p \in \pi(\psi_1(H_i))$, $\pi^{-1}(p) \cap \psi_1(H_i)$ is connected. Thus since $aa' \geq 0$, using the proof of Theorem 4.3.3, it is possible to construct a symplectomorphism ψ_2 of $\mathbb{R}^4$ which takes $\psi_1(Z)$ to Z'. $\square$

In addition, the following theorem shows that the actual shape of the holes is often not important. The proof is similar to the above proof.

Theorem 6.1.3. *Let $G_i \subset \{y_1 = 0\}$ be an open region diffeomorphic to H_i such that*

(1) $\overline{G}_1 \cap \overline{G}_2 = \emptyset$;

(2) *there exists a diffeomorphism θ of $\mathbb{R}^2$ such that $\theta \circ \pi(G_i) = H_i$, $i = 1, 2$;*

(3) *area $(\pi(G_1) \cap \pi(G_2)) = \alpha$ and area $\pi(G_i) =$ area $\pi(H_i)$, $i = 1, 2$;*

(4) *for each i, $\pi^{-1}(p) \cap G_i$ is a connected subset of $\mathbb{R}$ for all $p \in \pi(G_i)$.*

Then

$$\widetilde{Z}^\alpha(\lambda_1, \lambda_2) := \{y_1 < 0\} \cup \{y_1 > 0\} \cup G_1 \cup G_2$$

is symplectically equivalent to $Z^\alpha(\lambda_1, \lambda_2)$.

6.2 Changing the Overlapping Parameter

For this section, λ_1, λ_2 are fixed and the notation Z^α is used while the dependence on the overlapping parameter α is investigated. This overlapping parameter is closely tied to symplectic boundary behavior in the following way. Any hypersurface Σ has an invariant foliation, the *characteristic foliation,* associated to it. To define this foliation, choose any smooth, non-vanishing vector field ξ, $\xi(p) \in \ker \omega_0|_{\Sigma(p)}$. The integral curves of any such vector field form the characteristic foliation of Σ. $\mathcal{L}_\Sigma(p)$ will denote the leaf of the characteristic foliation through p.

The boundary of Z^α consists of the hyperplane $\{y_1 = 0\} = \mathbb{R}^3$ minus two 3-balls, $H_\pm$, and the lines $\{(t, 0, c, d), t \in \mathbb{R}\}$ make up the characteristic foliation of $\{y_1 = 0\}$. $D_\pm$ will denote the discs on the boundaries of the holes which are connected by the characteristic foliation of $\{y_1 = 0\}$. More precisely, by Proposition 6.1.2, it is possible to assume that the holes H_1, H_2, denoted in the following by $H_\pm$, are centered at $x_1 = \pm a$. Then let

$$D_- = \pi^{-1}\left(\pi\left(\overline{H}_-\right) \cap \pi\left(\overline{H}_+\right)\right) \cap \left(\partial H_- \cap \{x_1 \geq -a\}\right)$$
$$D_+ = \pi^{-1}\left(\pi\left(\overline{H}_-\right) \cap \pi(\overline{H}_+)\right) \cap \left(\partial H_+ \cap \{x_1 \leq a\}\right)$$

which implies

$$\alpha = \operatorname{area} \pi(D_-) = \operatorname{area} \pi(D_+).$$

See Figure 6.2.1. Notice that

$$D_- = \left\{p \in \partial H_- \cap \{x_1 > -a\} \colon \mathcal{L}_{\{y_1=0\}}(p) \cap \partial H_+ \neq \emptyset\right\}.$$

Assume ξ is chosen so that if $p \in D_-$, $h_1(p) \in D_+$, where h_t is the flow of ξ. Then let $T \subset \partial Z^\alpha$ be the portion of the boundary which contains the portion of the characteristic flow which connects $D_- \subset \partial H_-$ and $D_+ \subset \partial H_+$. In other words,

$$T = \{h_t(p) \colon p \in D_-, t \in [0,1]\}.$$

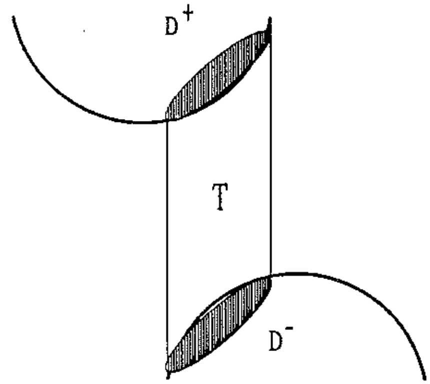

FIGURE 6.2.1.

Lemma 6.2.2. *For $\alpha > \beta$, there does not exist a symplectomorphism of $Z^\alpha \cup T$ into $\mathbb{R}^4$ which sends Z^α to Z^β and $D_\pm \subset T$ into $\partial H'_\pm$ where $H'_\pm$ are the holes of Z^β.*

Proof. Suppose $\alpha > \beta$ and there exists a symplectomorphism $\psi\colon Z^\alpha \cup T \to \mathbb{R}^4$ so that $\psi(Z^\alpha) = Z^\beta$, $\psi(D_\pm) \subset \partial H'_\pm$. If p and q are on the same characteristic line in T, i.e. $q = p + (a, 0, 0, 0)$, it is easy to check that $\psi(p)$ and $\psi(q)$ must be on the same characteristic line in $\psi(T)$. Since $\psi(T) \subset \{y_1 = 0\}$, this implies that $\psi(q) = \psi(p) + (b, 0, 0, 0)$. Thus $\pi(\psi(D_+)) = \pi(\psi(D_-))$ and

$$\int_{\pi(D_\pm)} dx_2 \wedge dy_2 = \int_{D_\pm} \omega_0 = \int_{\psi(D_\pm)} \omega_0 = \int_{\pi(\psi(D_\pm))} dx_2 \wedge dy_2.$$

However it then follows that

$$\alpha = \text{area}\,(\pi(\psi(D_-)) \cap \pi(\psi(D_+))) \leq \text{area}\left(\pi\left(\overline{H}'_-\right) \cap \pi\left(\overline{H}'_+\right)\right) = \beta$$

which is a contradiction. □

Thus the open sets Z^α, Z^β augmented with certain pieces of the boundary become symplectically non-equivalent. It is still unknown whether or not Z^α and Z^β are symplectically equivalent. However Z^α and Z^β augmented with "less" of the boundary are still non-equivalent. In fact, the obstruction formed by the characteristic flow on $T \subset \{y_1 = 0\}$ can be "measured" by including other portions of the boundary. This can be shown by using very recent results of Eliashberg and Hofer, namely their energy-capacity inequality [EH1], [EH2]. To explain their result, it is necessary to first introduce the following notions. For convenience, these notions will be defined in the 4-dimensional case. Analogous definitions and results hold for higher dimensions.

A hypersurface $\Sigma \subset \mathbb{R}^4$ is said to be *asymptotically flat* if it is diffeomorphic to $\mathbb{R}^3$ and there exists a compact subset K of $\mathbb{R}^4$ so that $\Sigma \backslash K = \{y_1 = 0\} \backslash K$. Let $\mathcal{D}$ be the group of compactly supported symplectomorphisms of $\mathbb{R}^4$. Then the *energy* or *symplectic width* of Σ, $e(\Sigma)$, is defined as follows:

$$e(\Sigma) = \inf_{\psi \in \mathcal{D}} \{2a(c^+ - c^-) \colon \psi(\Sigma) \subset \{y_1 = 0\} \cup \{x_1 \in (-a, a),\, y_1 \in (c^-, c^+)\}\}.$$

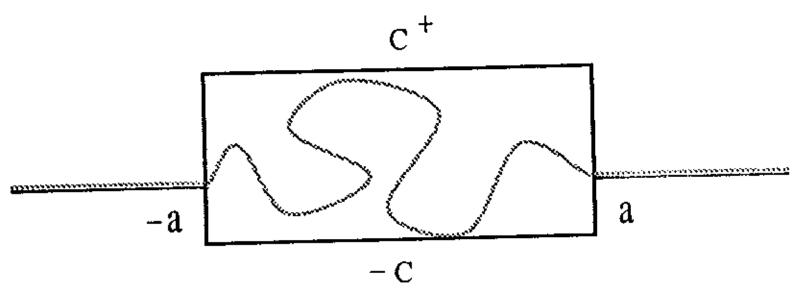

FIGURE 6.2.3.

Also for such asymptotically flat hypersurfaces, it is possible to introduce the following notion of *holonomy*. For $a \gg 0$, let

$$F_a^{\pm} = \{x_1 = \pm a, y_1 = 0\} \subset \Sigma.$$

For $p \in F_a^-$, let $L_\Sigma(p)$ be the leaf through p associated to the characteristic foliation of Σ. Then let $U_\Sigma = \{p : L_\Sigma(p) \cap F_a^+ \neq \emptyset\}$. The *holonomy map* $\varphi_\Sigma \colon U_\Sigma \to \mathbb{R}^2$ is defined by $\varphi_\Sigma(p) = L_\Sigma(p) \cap F_a^+$. One can verify that $U_\Sigma \subset \mathbb{R}^2$ is open and dense and that φ_Σ is area preserving and equal to the identity outside a compact set.

See [EH1], [EH2] for a higher dimensional analogue of the following result.

(6.2.4) Energy-Capacity Inequality (Eliashberg – Hofer). *Let Σ be an asymptotically flat hypersurface in $\mathbb{R}^4$ with holonomy $\varphi_\Sigma \colon U_\Sigma \to \mathbb{R}^2$ and energy $e(\Sigma)$. If $\Omega \subset \mathbb{R}^2$ is an open domain with $\overline{\Omega} \subset U_\Sigma$ and*

$$\operatorname{area} \Omega > e(\Sigma)$$

then

$$\varphi_\Sigma\left(\overline{\Omega}\right) \cap \overline{\Omega} \neq \emptyset.$$

Returning to the Z spaces, applying the above result gives some understanding of the effect of changing the overlapping parameter.

Theorem 6.2.5. *Let $S \subset \partial Z^\alpha$ be diffeomorphic to a 2-sphere and assume $D_\pm$ are contained in the bounded component of $\partial Z^\alpha \backslash S$. Let $N_\pm$ be neighborhoods of $D_\pm$ in ∂Z^α. For $\alpha > \beta$, there does not exist a symplectomorphism of $Z^\alpha \cup N_\pm$ into $\mathbb{R}^4$ which extends continuously to S and sends Z^α to Z^β and $D_\pm$ into $\partial H_\pm$.*

Proof. Suppose $\alpha > \beta$ and there exists a symplectomorphism ψ of $Z^\alpha \cup N_\pm$ into $\mathbb{R}^4$ which is defined and continuous on S and such that $\psi(Z^\alpha) = Z^\beta$ and $\psi(D_\pm) \subset \partial H'_\pm$.

There exists a sequence of hypersurfaces T_i in the domain of ψ so that $S \cup N_\pm \subset T_i$, $\lim_i T_i = \{y_1 = 0\}$, and so that for all $p \in D_-$, $\mathcal{L}_i(p) \cap D_+ \neq \emptyset$ where $\mathcal{L}_i(p)$ is the leaf through p of the characteristic flow associated to T_i. The following procedure can be used to construct such a sequence of T_i. Using the method in the proof of Theorem 4.3.3, it is easy to check that there exists a symplectomorphism g of $\mathbb{R}^4$ such that g preserves $\{y_1 = 0\}$, $g(D_\pm) \subset \{x_1 = \pm a\}$, and $\pi(g(D_\pm)) = \pi(D_\pm)$. Let $D = \pi(D_+) = \pi(D_-) \subset \mathbb{R}^2$ and construct $\widehat{T}_i$ to be of the form $\widehat{T}_i = \{y_1 = h_i(x_1, x_2, y_2)\}$ where $h_i \geq 0$, $h_i|_{(-a,a)\times D} = f_i(x_1)$, $f_i(-a) \equiv 0 \equiv f_i(a)$, and $h_i|_S = 0$. $\widehat{T}_i$ then contains S and agrees with $\{y_1 = 0\}$ near $g(D_\pm)$. In addition, on $\{(-a, a) \times D\}$, the characteristic flow of $\widehat{T}_i$, $\widehat{h}^i_t$, can be parameterized so that

$$\frac{d}{dt}\widehat{h}^i_t = \frac{\partial}{\partial x_1} + \frac{df_i}{dx_1}\frac{\partial}{\partial y_1}.$$

Since $f(a) = f(-a) = 0$, it follows that for all $p \in g(D_-)$, $\mathcal{L}_i(p) \cap g(D_+) \neq \emptyset$. (In fact, the holonomy map restricted to $g(D_-)$ is the identity.) In this way, $\widehat{T}_i$ can be constructed so that $T_i := g^{-1}(\widehat{T}_i)$ will have the desired properties.

Let $T_i' = \psi(T_i)$. The characteristic flow on T_i' can be parameterized so that it is given by $\left(h_t^i\right)' = \psi \circ h_t^i \circ \psi^{-1}$, where h_t^i is the characteristic flow on T_i. Then points originating from $\psi(D_-)$ arrive at $\psi(D_+)$ at $t = 1$. Let

$$(T_i^c)' = \left\{ \left(h_t^i\right)'(p) \colon p \in \psi(D_-),\ t \in [0,1] \right\}.$$

To study the relative positions of $\psi(D_-)$ and $\psi(D_+)$, $(T_i')^c$ will be extended to an asymptotically flat hypersurface, Σ_i, whose holonomy can be studied via the energy-capacity inequality.

Let T_i^b denote the closure of the bounded component of $T_i \backslash S$ and then let $\left(T_i^b\right)' = \psi(T_i^b) \subset T_i'$. Then $\lim_i \left(T_i^b\right)' \subset \partial Z^\beta \cup \psi(H_\pm)$. To see this, it is first easy to check that, since ψ is defined and continuous on the 2-dimensional sphere S and $\psi(S) \subset \left(T_i^b\right)'$, $\left(T_i^b\right)'$ converge. Suppose there exists $p \in \lim_i \left(T_i^b\right)'$, $p \notin \partial Z^\beta \cup \psi(H_\pm)$. Thus $p \in Z^\beta$ and it follows that for $q := \psi^{-1}(p)$, $q \in Z^\alpha$, $q \notin H_\pm$. However it is easy to check that since ψ is a diffeomorphism between Z^α and Z^β, q must be in $\lim_i T_i^b$ and thus there is a contradiction.

Since $\lim_i \left(T_i^b\right)' \subset \partial Z^\beta \cup \psi(H_\pm)$, it follows that $\lim_i (T_i^c)' \subset \{y_1 = 0\}$. Thus $(T_i^c)'$ can be extended to an asymptotically flat hypersurface Σ_i, $\lim_i \Sigma_i \subset \{y_1 = 0\}$. In addition, since ψ must send $N_\pm \cap \partial Z^\alpha$ into $\{y_1 = 0\}$, it is possible to assume that the characteristic flow of Σ_i, $h_t^{\Sigma_i}$, satisfies for $p \in \psi(D_-)$, $h_t^{\Sigma_i}(p) = \left(h_t^i\right)'(p)$ when $t \in [0,1]$ and $\frac{d}{dt}\left(h_t^{\Sigma_i}(p)\right)$ has only a $\frac{\partial}{\partial x_1}$ component when $t \notin [0,1]$. Let $\Omega_\pm' = \pi(\psi(D_\pm))$. As in the proof of Lemma 6.2.2,

$$\alpha = \operatorname{area} \pi(D_\pm) = \operatorname{area} \pi(\psi(D_\pm)) = \operatorname{area} \Omega_\pm'.$$

By construction of Σ_i, $\Omega'_- = \overline{\Omega}'_- \subset U_{\Sigma_i}$ and the holonomy map φ_{Σ_i} satisfies $\varphi_{\Sigma_i}(\Omega'_-) = \Omega'_+$. Since $\lim_i \Sigma_i = \{y_1 = 0\}$, $e(\Sigma_i) \to 0$. It is easy to check that $\varphi_{\Sigma_i}|_{\Omega'_-}$ does not depend on i and thus (6.2.4) implies that $\varphi_{\Sigma_i}|_{\Omega'_-}$ must be the identity. Since

$$\Omega'_- \cap \Omega'_+ = \pi(\psi(D_-)) \cap \pi(\psi(D_+)) \subset \pi\left(\overline{H}'_1\right) \cap \pi\left(\overline{H}'_2\right),$$

it then follows that

$$\alpha = \operatorname{area}\Omega'_- \leq \operatorname{area}\left(\pi\left(\overline{H}'_1\right) \cap \pi\left(\overline{H}'_2\right)\right) = \operatorname{area}\left(\pi\left(H'_1\right) \cap \pi\left(H'_2\right)\right) = \beta.$$

However $\alpha > \beta$. This contradiction completes our proof. □

Remark 6.2.6. This result is still in a preliminary stage. The hypothesis requiring the extension to S seems to be necessary to apply the Energy-Capacity Theorem for it is this condition which guarantees the "thinness" of the image hypersurfaces. However, the hypothesis that neighborhoods of $D_\pm$ must be contained in the domain of the symplectic map can perhaps be weakened or even eliminated.

Although these statements do not imply that Z^α and Z^β are symplectically non-equivalent when $\alpha \neq \beta$, they give support to the following

Conjecture 6.2.7. *$Z^\alpha(\lambda_1, \lambda_2)$ is symplectically equivalent to $Z^\beta(\lambda_1, \lambda_2)$ iff $\alpha = \beta$.*

6.3 Changing the Radii of the Holes

In this section, the effect of changing the radii of the holes in the completely aligned spaces, $Z(\lambda_1, \lambda_2)$, is investigated. As before, analogues of the symplectic camel and extendable embeddings results will be used to investigate this question. To apply the theorems from Chapter 3, it is necessary to construct a sequence of regions Ω_k, and expanding vector fields η_k so that $Z^\alpha(\lambda_1, \lambda_2) = \cup \operatorname{Int} \Omega_k$, η_k is transverse to $\partial\Omega_k$, and $\partial\Omega_k, \eta_k$ are standard near ∂H_i.

Proposition 6.3.1. *$\partial H_1, \partial H_2$ are fillable holes of $Z(\lambda_1, \lambda_2)$.*

Proof. It is possible to assume the holes $H_\pm$ of $Z^\alpha(\lambda_-, \lambda_+)$ are centered at $(x_1, x_2, y_2) = (-a, 0, 0)$ and $(a, 0, 0)$, $a > 2\max\{\lambda_-, \lambda_+\}$. In the following, regions Ω_k are constructed whose interiors exhaust $Z(\lambda_-, \lambda_+) \cap \{y_1 \geq 0\}$. An analogous procedure can be applied to construct regions Ω'_k whose interiors exhaust $Z(\lambda_-, \lambda_+) \cap \{y_1 \leq 0\}$ and which smoothly patch together with Ω_k.

Given $k > 0$, let

$$\mathcal{H}^\pm = \mathcal{H}_k^\pm = \left\{ 2\,(x_1 \mp a)^2 + x_2^2 + y_2^2 \leq \lambda_\pm^2, \quad |y_1| < \frac{1}{k} \right\}$$

and let v_h be the vector field on $\mathcal{H}^+ \cup \mathcal{H}^-$ satisfying

$$v_h|_{\mathcal{H}^\pm} = 2(x_1 \mp a)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}.$$

Let $\mathcal{N}^\pm$ be a convex neighborhood of $\mathcal{H}^\pm \cap \{y_1 = \frac{1}{k}\}$. Let $\mathcal{C} = \mathcal{C}_k \subset \{y_1 > 0\}$ be a region such that $\mathcal{C} \cap \{y_1 = \frac{1}{k}\} \subset \mathcal{N}^\pm$, $(x_1 \mp a)\frac{\partial}{\partial x_1}$ is transverse or tangent to $\partial\mathcal{C}|_{\mathcal{N}^\pm}$, and $(y_1-k)\frac{\partial}{\partial y_1}$ is transverse or tangent to $\partial\mathcal{C}|_{\{|x_1|<\frac{1}{k}\}}$. In addition, if

$$v_c^\pm = (x_1 \mp a)\frac{\partial}{\partial x_1} + (y_1 - k)\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2},$$

it is possible to construct $\mathcal{C}$ so that v_c^+ is transverse to $\partial\mathcal{C} \cap \{x_1 > -\frac{1}{k}\}$, v_c^- is transverse to $\partial\mathcal{C} \cap \{x_1 < \frac{1}{k}\}$, and so that $\cup_k \mathcal{C}_k = \{y_1 > 0\}$.

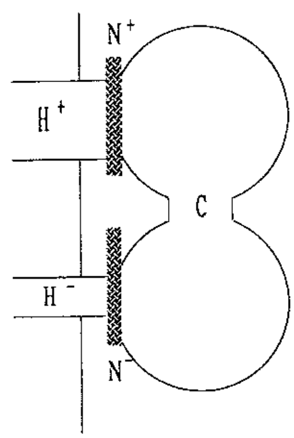

FIGURE 6.3.2.

As a first step, it is shown that there exists an expanding vector field v_c on $\mathcal{C}$ which is transverse to $\partial\mathcal{C}$ and so that on $\mathcal{C} \cap \{x_1 > \frac{1}{k}\}$, $v_c = v_c^+$ and on $\mathcal{C} \cap \{x_1 < -\frac{1}{k}\}$, $v_c = v_c^-$.

Let $\rho = \rho(x_1)$ be a decreasing function which is identically 1 when $x_1 < -\frac{1}{k}$ and identically 0 when $x_1 > \frac{1}{k}$. Let χ_ρ satisfy $i(\chi_\rho)\omega_0 = d\rho$. Notice that

$$i(v^- - v^+)\omega_0 = df$$

where $f = 2a(y_1 - k)$. Then it is easy to check that

$$v_c = \rho v_c^- + (1-\rho)v_c^+ + f\chi_\rho$$

is expanding. It remains to check that v_c is transverse to $\partial\mathcal{C}$. Since v^- and v^+ are both transversal to both $\partial\mathcal{C} \cap \{|x_1| < \frac{1}{k}\}$, it suffices to check that $f\chi_\rho$ is transverse (or tangent) to $\partial\mathcal{C} \cap \{|x_1| < \frac{1}{k}\}$. By the specifications on ρ,

$$f\chi_\rho = 2a\,(y_1 - k)\left(-\frac{d\rho}{dx_1}\right)\frac{\partial}{\partial y_1} = \delta^2(y_1 - k)\frac{\partial}{\partial y_1}$$

and thus it follows that v_c is transverse to $\partial\mathcal{C}$.

$\mathcal{H}^{\pm} \cup \mathcal{C}$ can be smoothed to a region $\Omega = \Omega_k$ on $\mathcal{N}^{\pm}$. Next an expanding vector field η_k is constructed which agrees with v_h on $\mathcal{H}^{\pm} \backslash (\mathcal{H}^{\pm} \cap \mathcal{N}^{\pm})$ and with v_c on $\mathcal{C} \backslash (\mathcal{C} \cap \mathcal{N}^{\pm})$.

Since $\mathcal{C} \cap \{y_1 = \frac{1}{k}\} \subset \mathcal{N}^{\pm}$, by shrinking $\mathcal{N}_{\pm}$ if necessary, there exists $\rho^h = \rho^h(y_1)$, a smooth, decreasing function, such that ρ^h equals 1 on $\mathcal{H} \backslash (\mathcal{H} \cap \mathcal{N}^{\pm})$ and which equals 0 on $\mathcal{C} \backslash (\mathcal{C} \cap \mathcal{N}^{\pm})$. Let χ_{ρ^h} satisfy $i(\chi_{\rho^h})\omega_0 = d\rho^h$ and let f be a function which agrees with $(x_1 \mp a)(y_1 - k)$ on $\mathcal{N}^{\pm}$. Then

$$\eta_k = \rho^h v_h + (1 - \rho^h)v_c + f\chi_{\rho^h}$$

is expanding. Furthermore, since on $N^{\pm}$,

$$f\chi_{\rho^h} = (x \mp a)(y_1 - k)\frac{d\rho^h}{dy_1}\frac{\partial}{\partial x_1} = \delta^2(x \mp a)\frac{\partial}{\partial x_1}$$

it is easy to see that η_k will be transverse to $\partial\Omega_k$. $\square$

Remark 6.3.3. Using the Lemma 6.4.3 from the next section, it is easy to prove that ∂H_i are fillable holes in the completely non-aligned space $Z^0(\lambda_1, \lambda_2)$. It is easy to check that $Z^\alpha(\lambda_1, \lambda_2)$ is symplectically convex for $0 < \alpha < \max$ by using the method in the above proof except choosing the vector fields $v_h, v_c^{\pm}$ to vanish when $x_2 = c$, where $(c, 0) \in \pi(H_1) \cap \pi(H_2)$.

However this procedure does not construct η which is standardized near the holes. (See condition (3′) from Definition 3.1.1.) Thus the following results do apply to Z^0 but not yet to Z^α, $0 < \alpha < \max$.

Let $i^{\pm}\colon \mathbb{R}^4 \to Z(\lambda_1, \lambda_2) \cap \{y_1 \in \mathbb{R}^{\pm}\}$ be symplectic diffeomorphisms and let $i_r^{\pm} = i^{\pm}|_{B(r)}$. Theorems 3.2.1 and 3.3.3 then imply

(6.3.4) Symplectic Camel for Z spaces. *If $r \geq \max\{\lambda_1, \lambda_2\}$, i_r^+ and i_r^- are not symplectically isotopic.*

(6.3.5) Extendable Embeddings into Z spaces. *Let g be an embedding of the closed ball of radius r into $Z(\lambda_1, \lambda_2)$ which has an extension to a symplectic embedding*

$$g'\colon B(r + \max\{\lambda_1, \lambda_2\}) \to Z(\lambda_1, \lambda_2).$$

Then g is symplectically isotopic to i_r^+ or i_r^-.

Theorem 6.3.6. *If $\{\lambda_1, \lambda_2\} \neq \{\mu_1, \mu_2\}$, then $Z(\lambda_1, \lambda_2)$ is not symplectomorphic to $Z(\mu_1, \mu_2)$.*

Proof. Let $\lambda_{\min} = \min\{\lambda_1, \lambda_2\}$, $\mu_{\min} = \min\{\mu_1, \mu_2\}$; $\lambda_{\max} = \max\{\lambda_1, \lambda_2\}$, $\mu_{\max} = \max\{\mu_1, \mu_2\}$.

First consider the case where $\lambda_{\min} \neq \mu_{\min}$. Suppose there exists a symplectomorphism $\psi\colon Z(\lambda_1, \lambda_2) \to Z(\mu_1, \mu_2)$ with $\lambda_{\min} > \mu_{\min}$. There exists a non-trivial loop γ, $0 \neq [\gamma] \in \pi_1(Z(\lambda_1, \lambda_2)) = \mathbb{Z}$, so that there exists an inclusion of a ball of radius r, $\mu_{\min} < r < \lambda_{\min}$ around each point of γ. Let $T(\gamma)$ be the union of all these balls and consider $\psi(T(\gamma)) \subset Z(\mu_1, \mu_2)$. By Proposition 6.3.1, there exists an Ω, $\psi(T(\gamma)) \subset \operatorname{Int}\Omega$ and $J \in \mathcal{J}_\Omega$ satisfying

$J = \psi_*(J_0)$ on $\operatorname{Im}\psi(T(\gamma))$. By Lemma 3.1.4, it follows that $F(J) \cap \psi(\gamma) = \emptyset$ where $F(J)$ is the filling of the sphere of radius $\mu_{\min}$. However since $F(J)$ is diffeomorphic to a closed 3 dimensional ball, this contradicts the fact that $0 \neq [\psi(\gamma)] \in \pi_1(Z(\mu_1, \mu_2))$.

Next consider the case where $\lambda_{\min} = \mu_{\min}$. Suppose $\psi\colon Z(\lambda_1, \lambda_2) \to Z(\mu_1, \mu_2)$ is a symplectomorphism with $\lambda_{\max} < \mu_{\max}$. Choose $\lambda_{\max} < r < \mu_{\max}$. By (6.3.4), $i_r^{\pm}\colon B(r) \to Z^{\alpha}(\lambda_1, \lambda_2)$ are non-isotopic and thus $\psi(i_r^{\pm})$ must also be non-isotopic. However, $\psi(i_r^{\pm})$ have $\psi(i_{r+\mu_{\max}}^{\pm})$ as extensions to ball of radius $r + \mu_{\max}$. (6.3.5) then implies that $\psi(i_r^{\pm})$ must be isotopic to $i_r^{\pm}\colon B(r) \to Z^{\beta}(\mu_1, \mu_2)$ and thus to each other since $r < \mu_{\max}$. # $\square$

6.4 A Modified Completely Non-Aligned Space

Next, the completely non-aligned space $Z^0(\lambda_1, \lambda_2)$ will be slightly modified by introducing a "partition wall" of the form $\{x_1 = a/2\}$ in the half space $\{y_1 > 0\}$. More precisely, given $\lambda_1, \lambda_2 \in (0, \infty)$ and $a > 2\max\{\lambda_1, \lambda_2\}$, let

$$P^0(\lambda_1, \lambda_2) = \{y_1 < 0\} \cup \{y_1 > 0, x_1 > a/2\} \cup \{y_1 > 0, x_1 < a/2\} \cup H_1 \cup H_2$$

where

$$H_1 = \left\{2x_1^2 + x_2^2 + y_2^2 < \lambda_1^2, \quad y_1 = 0\right\},$$
$$H_2 = \left\{2(x_1 - a)^2 + (x_2 - b)^2 + (y_2 - c)^2 < \lambda_2^2, \quad y_1 = 0\right\}.$$

See Figure 6.4.1.

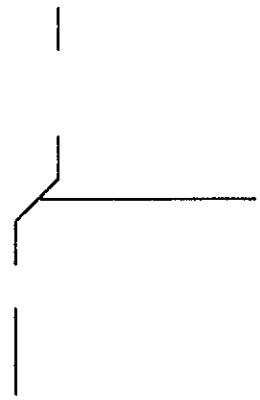

FIGURE 6.4.1. $P^0(1,2)$

By the proof of Proposition 6.1.2, this is a symplectically well-defined space. Notice that, in contrast to $Z^0(\lambda_1, \lambda_2)$, $P^0(\lambda_1, \lambda_2)$ has a trivial fundamental group.

Theorem 6.4.2. *$P^0(\lambda_1, \lambda_2)$ is symplectically equivalent to $W(\lambda_1, \lambda_2)$.*

Recall $W(\lambda_1, \lambda_2)$ was explored in Chapter 5. This theorem can be easily proven using the following lemma.

Lemma 6.4.3. *Assume $b > \lambda_1+\lambda_2$. Choose ϵ such that $b-(\lambda_1+\lambda_2) > 2\epsilon > 0$ and let*

$$A = \left\{ \begin{matrix} |x_1| < \lambda_1 + \epsilon, & -\epsilon < y_1 < 0 \\ |x_2| < \lambda_1 + \epsilon, & |y_2| < \lambda_1 + \epsilon \end{matrix} \right\}$$
$$B = \left\{ \begin{matrix} |x_1 - a| < \lambda_2 + \epsilon, & -\epsilon < y_1 < 0 \\ |x_2 - b| < \lambda_2 + \epsilon, & |y_2| < \lambda_2 + \epsilon \end{matrix} \right\}.$$

Then there exists a symplectic diffeomorphism

$\Psi\colon \{y_1 < 0\} \to \{-1 < y_1 < 0\}$ *so that*

$$\Psi|_A = id \qquad \Psi|_B = (-x_1 + a, -1 - y_1, x_2, y_2).$$

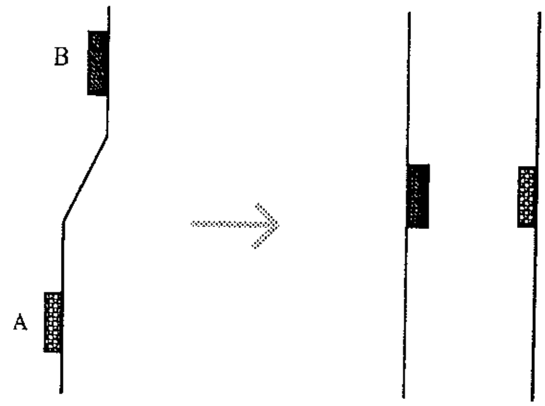

FIGURE 6.4.4.

Remark 6.4.5. Notice that the hypothesis $b > \lambda_1 + \lambda_2$ implies that $\pi\left(\overline{A}\right) \cap \pi\left(\overline{B}\right) = \emptyset$.

Proof. There exists a symplectic diffeomorphism $\psi_0 \colon \{y_1 < 0\} \to \{-1 < y_1 < 0\}$ so that $\psi_0|_{\{y_1 > -\epsilon\}} = id$. Thus $\psi_0(A) = A$, $\psi_0(B) = B$.

Let $pr \colon \mathbb{R}^4 \to \mathbb{R}^2$ be the projection of $\mathbb{R}^4$ to the (x_1, y_1)-plane. $pr(A)$ and $pr(B)$ are joined by the 2-cell $C = \{\lambda_1 + \epsilon \leq x_1 \leq a - \lambda_2 - \epsilon, -\epsilon < y_1 < 0\}$. See Figure 6.4.6. It is impossible to construct an area preserving diffeomorphism g of the 2-dimensional region $\{-1 < y_1 < 0\}$ which fixes $pr(A)$ and takes $pr(B)$ into $\{-1 < y_1 < -1 + \epsilon\}$ for g would necessarily take the cell C which has finite area to a region with infinite area.

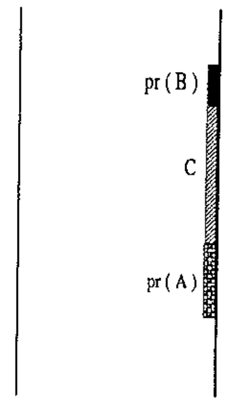

FIGURE 6.4.6.

Thus the strategy will be to first construct a symplectic diffeomorphism ψ_1 of $\{-1 < y_1 < 0\}$ so that the corresponding cell bounded by $pr \circ \psi_1(A)$ and $pr \circ \psi_1(B)$ has infinite area.

To construct ψ_1, first notice that by definition of ϵ, there exists a smooth function $h(y_1, x_2)$ such that when $|x_2 - b| < \lambda_2 + \epsilon$, $h \equiv 0$ and when $|x_2| < \lambda_1 + \epsilon$, $h = h(y_1)$. Then consider

$$f(y_1, x_2) = \int_0^{y_1} \left(s \, \frac{\partial h}{\partial s} \right)_{(s, x_2)} ds$$

and let $\psi_1 \colon \{-1 < y_1 < 0\} \to \{-1 < y_1 < 0\}$ be given by

$$\psi_1(x_1, y_1, x_2, y_2) = \left(x_1 - h(y_1, x_2),\ y_1,\ x_2,\ y_2 + y_1 \frac{\partial h}{\partial x_2} - \frac{\partial f}{\partial x_2} \right).$$

It is easy to check that ψ_1 is symplectic. Moreover, for $|x_2 - b| < \lambda_2 + \epsilon$, ψ_1 is the identity and for $|x_2| < \lambda_1 + \epsilon$, $\psi_1(x_1, y_1, x_2, y_2) = (x_1 - h, y_1, x_2, y_2)$. The above function h can be constructed so that $pr \circ \psi_1(A) \cap pr \circ \psi_1(B) = \emptyset$ and so that the region $C \subset \{-\epsilon < y_1\}$ in the (x_1, y_1)-plane bounded by $pr \circ \psi_1(A)$ and $pr \circ \psi_1(B)$ has infinite area.

Applying the results of Greene and Shiohama [GS], there exists an area preserving diffeomorphism g such that $g|_{pr(\psi_1(B))} = (-x_1 + a, -1 - y_1)$, $g|_{pr(\psi_1(A))} = (x_1 + h(b, y_1), y_1)$, for h as constructed above. $\psi_2 = g \times id$ is a symplectic diffeomorphism.

$\Psi = \psi_2 \circ \psi_1 \circ \psi_0$ is the desired symplectomorphism. □

Remark 6.4.7. Notice that the hypothesis $b > \lambda_1 + \lambda_2$, or more generally that $\pi\left(\overline{A}\right) \cap \pi\left(\overline{B}\right) = \emptyset$, is crucial to the definition of ψ_1 in the above proof. If an attempt is made to apply the above proof to that case $\pi\,(A) \cap \pi\,(B) \neq \emptyset$, there is an "area obstruction" as described prior to the construction of ψ_1. The author believes that this obstacle is extremely difficult to avoid. Yet at this point in time, there appears to be no known invariants which could be used to measure this "obstruction". Notice that if there is no obstruction and an analogue of Lemma 6.4.3 holds when $\pi\,(A) \cap \pi\,(B) \neq \emptyset$, this, together with the proof of Theorem 6.4.2, would imply that $Z^{\alpha}(\lambda_1, \lambda_2)$ is symplectically equivalent to $W(\lambda_1, \lambda_2)$ and thus to $Z^{\beta}(\lambda_1, \lambda_2)$ for all α, β. (Compare to Section 6.2.)

Proof of Theorem 6.4.2. By the proof of Proposition 6.1.2, it is possible to assume the holes H_1, H_2 of $P^0(\lambda_1, \lambda_2)$ have centers at (x_1, x_2, y_2) coordinates $(0, 0, 0)$ and $(a, b, 0)$ where $b > \lambda_1 + \lambda_2$ and a is chosen large enough so that for A and B from the previous lemma, $A \cap \{x_1 = a/2\} = \emptyset = B \cap \{x_1 = a/2\}$.

Then let A^+, B^+ be extension of A, B in $\{y_1 \geq 0\}$:

$$A^+ = \begin{Bmatrix} |x_1| < \lambda_1 + \epsilon, & 0 \leq y_1 < \epsilon \\ |x_2| < \lambda_1 + \epsilon, & |y_2| < \lambda_1 + \epsilon \end{Bmatrix}$$

$$B^+ = \begin{Bmatrix} |x_1 - a| < \lambda_2 + \epsilon, & 0 \leq y_1 < \epsilon \\ |x_2 - b| < \lambda_2 + \epsilon, & |y_2| < \lambda_2 + \epsilon \end{Bmatrix}.$$

Let $\Psi^1\colon \{y_1 < 0\} \to \{-1 < y_1 < 0\}$ be the symplectomorphism given by Lemma 6.4.3. It is easy to check that there exists a symplectic diffeomorphism Ψ^0 of $\{x_1 > a/2, y_1 > 0\} \cup B^+$ into $\{y_1 \leq -1\}$ such that $\Psi^0(\{x_1 > a/2, y_1 > 0\}) = \{y_1 < -1\}$ and $\Psi^0|_{B+} = (-x_1 + a, -1 - y_1, x_2, y_2)$. Similarly, there exists a symplectomorphism Ψ^2 of $\{x_1 < a/2, y_1 > 0\} \cup A^+$ into $\{y_1 \geq 0\}$ so that $\Psi^2(\{x_1 < a/2, y_1 > 0\}) = \{y_1 > 0\}$ and $\Psi|_{A+} = id$. Thus there exists a symplectomorphism Ψ defined by

$$\Psi = \begin{cases} \Psi^0, & \text{on} \quad \{x_1 > a/2, y_1 > 0\} \cup B^+ \\ \Psi^1, & \text{on} \quad \{y_1 < 0\} \cup A \cup B \\ \Psi^2, & \text{on} \quad \{x_1 < a/2, y_1 > 0\} \cup A^+. \end{cases}$$

Ψ descends to define a symplectic map between $P^0(\lambda_1, \lambda_2)$ and $W(\lambda_2, \lambda_1)$ which is symplectomorphic to $W(\lambda_1, \lambda_2)$. □

Corollary 6.4.8. *$P^0(\lambda_1, \lambda_2)$ is symplectomorphic to $P^0(\lambda_2, \lambda_1)$.*

For later comparisons, it will also be interesting to consider

$$R^0(\lambda_1, \lambda_2) = \Upsilon(P^0(\lambda_1, \lambda_2))$$

where Υ is a reflection in $\{y_1 = 0\}$:

$$\Upsilon(x_1, y_1, x_2, y_2) = (x_1, -y_1, x_2, y_2).$$

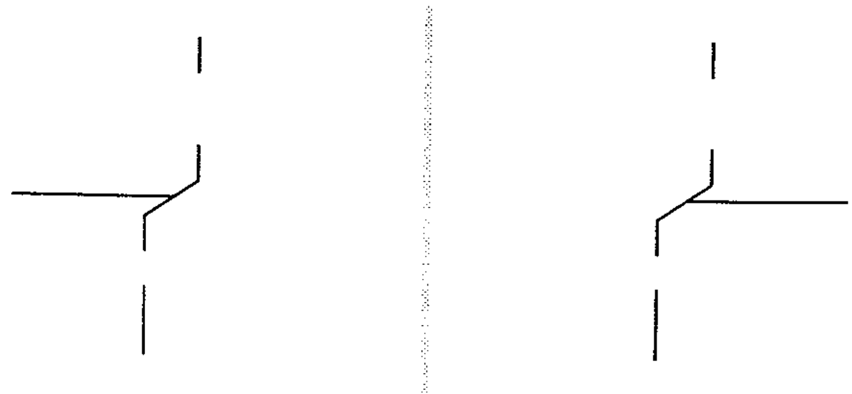

FIGURE 6.4.9. $R^0(1,2)$ vs. $P^0(1,2)$

Corollary 6.4.10. *$R^0(\lambda_1, \lambda_2)$ is symplectomorphic to $P^0(\lambda_1, \lambda_2)$.*

Proof. Notice that via a rotation in the (x_1, y_1)-plane, $R^0(\lambda_1, \lambda_2)$ is symplectically equivalent to $P^0(\lambda_2, \lambda_1)$ which, by Corollary 6.4.8, is symplectomorphic to $P^0(\lambda_1, \lambda_2)$. □

6.5 Multi-holed Z Spaces

Two natural choices for generalizing the two holed space $Z(\lambda_1, \lambda_2)$ are introduced in this section. Neither of these generalizations has been explored to any large degree. However, these generalizations do raise some interesting new questions.

The first generalization of $Z(\lambda_1, \lambda_2)$ is to increase the number of holes in the wall, keeping them all aligned. Given $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$, let

$$F(\lambda_1, \ldots, \lambda_n) = \{y_1 < 0\} \cup \{y_1 > 0\} \cup H_1 \cup \cdots \cup H_n$$

where $H_i = \{2(x_1 - na)^2 + x_2^2 + y_2^2 < \lambda_i^2,\, y_1 = 0\}$, $a \gg \max\{\lambda_i\}$. Note that the fundamental group of $F(\lambda_1, \ldots, \lambda_n)$ is the free product of n copies of $\mathbb{Z}$.

FIGURE 6.5.1. $F(1,2,3)$

It is easy to prove that ∂H_i are fillable holes and thus there are the natural analogues of the symplectic camel and extendable embeddings results for these F spaces. The arguments in the proof of Theorem 6.3.6 can be used to prove

Theorem 6.5.2. *If $F(\lambda_1, \ldots, \lambda_n)$ is symplectomorphic to $F(\mu_1, \ldots, \mu_m)$ then $n = m$ and for all r,*

$$\big|\{j\colon \lambda_j \le r\}\big| = \big|\{k\colon \mu_k \le r\}\big|.$$

By a symplectomorphism of the form $\theta \times id$ where θ is a rotation in the (x_1, y_1)-plane, it is easy to see that $F(1,2,3)$ is symplectically equivalent to $F(3,2,1)$.

Question 6.5.3. *Is $F(1,2,3)$ symplectically equivalent to either $F(2,1,3)$ or $F(1,3,2)$?*

In Chapter 5, the embedding trees were useful for distinguishing different orderings of holes in the W spaces. It is easy to check that these embedding trees give no insight to an answer for Question 6.5.3.

An alternate generalization to the $Z(\lambda_1, \lambda_2)$ space is to consider a multi-holed space with fundamental group $\mathbb{Z}$. More precisely, let (r, θ) be polar coordinates on the (x_1, y_1)-plane and define

$$Z(\lambda_1, \ldots, \lambda_n) = \left\{0 < \theta < \frac{2\pi}{n}\right\} \cup \cdots \cup \left\{(n-1)\frac{2\pi}{n} < \theta < 2\pi\right\} \cup H_1 \cdots \cup H_n$$

where $H_i = \{(r - r_0)^2 + x_2^2 + y_2^2 < \lambda_i, \theta = i\frac{2\pi}{n}\}$, $r_0 > \max\{\lambda_i\}$. See Figure 6.5.4.

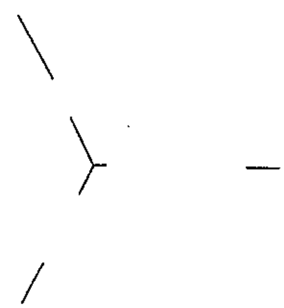

FIGURE 6.5.4. $Z(1,2,3)$

Using the proof of Proposition 6.3.1, it is easy to see that ∂H_i are fillable holes. $Z(\lambda_1, \ldots, \lambda_n)$ does not quite have flat boundaries according to Definition 3.3.1. However the statement and proof of Proposition 3.3.4 can easily be modified for these spaces. In this case, the fillings F^i of H_i can be

ε-perturbed to fit together with $W^i \subset \{\theta = i\frac{2\pi}{n}\}$ to form a 3 dimensional manifold Q^i so that there is a symplectomorphism Ψ of $\mathbb{R}^4$ which takes Q^i to $\{\theta = i\frac{2\pi}{n}\}$. The proof follows much as before. The following few points should be noted. The "parameterized family of fillings" mentioned in the paragraph preceding Lemma 3.3.5 can be replaced in the following manner. Consider the holomorphic map $\phi\colon \mathbb{R}\times(\mathbb{R}/(2\pi\mathbb{Z}))\times\mathbb{R}^2 \to \mathbb{R}^2\backslash\{0\}\times\mathbb{R}^2$ defined by

$$\phi(x_1, y_1, x_2, y_2) = (e^{x_1}\cos y_1,\ e^{x_1}\sin y_1,\ x_2,\ y_2).$$

Define $T(\gamma) = \phi(\mathcal{C}(\gamma))$ where $\mathcal{C}(\gamma) = \{x_1^2 + x_2^2 + y_2^2 < \gamma^2\}$. It is easy to check that $\partial\mathcal{C}(\gamma)$ is J_0-convex. Then since ϕ is holomorphic, $\partial T(\gamma)$ is J_0-convex. Using the terminology of Section 2.1, $T(\lambda) = \{\varphi \le 1\}$ can play the role of Ω and given $\varepsilon > 0$, define

$$\mathcal{J}_\Omega = \{J \in \mathcal{J}\colon J = J_0 \text{ on } \{\varphi > 1-\varepsilon\}\}.$$

Using the method of Chapter 2, it is easy to prove that $S(\gamma, s) := T(\gamma)\cap\{\theta = s\}$ is a fillable for any $J \in \mathcal{J}_\Omega$. Furthermore, as before, the fillings $\widetilde{F}^s$ and $\widetilde{F}^t$ are disjoint when $s \neq t$. The map Φ described preceding Lemma 3.3.5 should be modified as follows. Choose $\delta > 0$ sufficiently small so that the plane $\{r = \delta, \theta = s\}$ is a symplectic leaf in Q^s, for all s. Notice that for all points (r_0, t, x_2, y_2) with $r_0 < \delta$, Φ will be the identity. Thus Φ is a well-defined map on $\mathbb{R}^4$.

Thus analogues of the symplectic camel and extendable embeddings results hold for these multi-holed Z spaces. The arguments in the proof of Theorem 6.3.6 then prove

Theorem 6.5.5. *If $Z(\lambda_1, \ldots, \lambda_n)$ is symplectomorphic to $Z(\mu_1, \ldots, \mu_m)$ then $m = n$ and, more generally, for all r,*

$$\left|\{j : \lambda_j \leq r\}\right| = \left|\{k : \mu_k \leq r\}\right|.$$

Whereas the orderings of the holes in the W spaces were defined up to reflection, the orderings of the holes in these Z spaces are defined up to a cyclic permutation. Thus, for example, $Z(1,2,1,2)$ is symplectomorphic to $Z(2,1,2,1)$. In contrast to the above F spaces, the embedding trees from Section 5.3 can be used to distinguish some different orderings. In particular Figure 6.5.6 gives the embedding tree for $Z(1,1,2,2)$ in (a) and the embedding tree for $Z(1,2,1,2)$ in (b). Thus $Z(1,1,2,2)$ and $Z(1,2,1,2)$ are not symplectically equivalent.

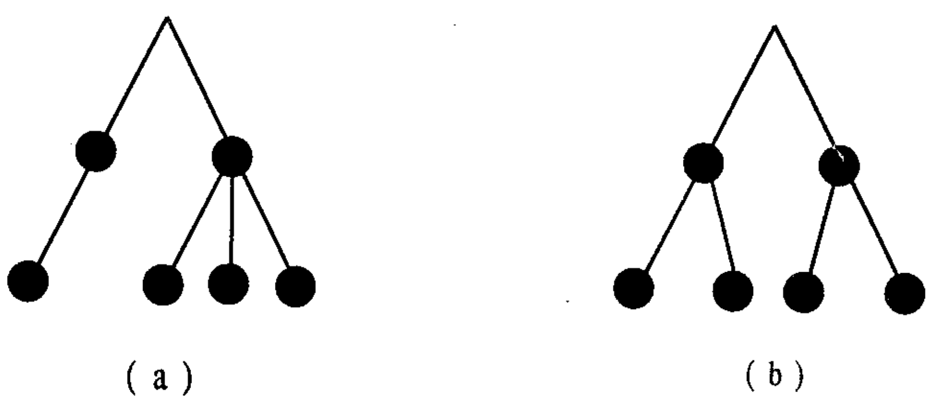

FIGURE 6.5.6. (a) Tree corresponding to $Z(1,1,2,2)$; (b) Tree corresponding to $Z(1,2,1,2)$.

It is easy to check that the embedding trees are isomorphic for $Z(1,2,3)$ and $Z(1,3,2)$. In fact, these spaces are anti-symplectically equivalent since the map

$$\Upsilon\colon Z(1,2,3) \to Z(1,3,2)$$

defined by $\Upsilon(x_1, y_1, x_2, y_2) = (x_1, -y_1, x_2, -y_2)$ satisfies $\Upsilon^*\omega_0 = -\omega_0$. Thus it is interesting to compare the next and final question with Corollary 6.4.6.

Question 6.5.7. *Are* $Z(1,2,3)$ *and* $Z(1,3,2)$ *symplectically equivalent?*

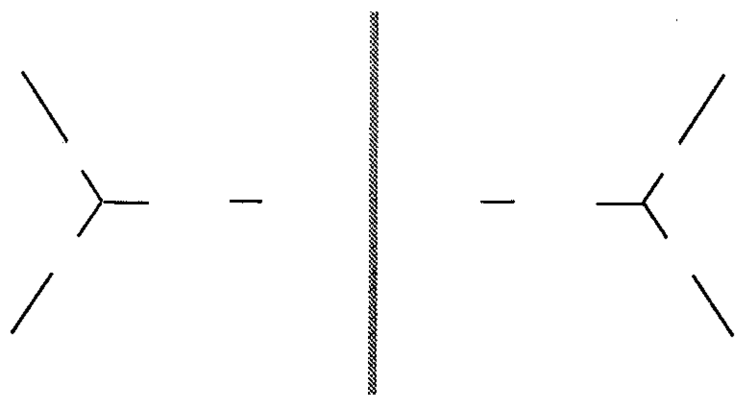

FIGURE 6.5.8. $Z(1,2,3)$ vs. $Z(1,3,2)$

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