SUMMABILITY OF SUBSEQUENCES

A dissertation presented

by

Francisco J. Lüttecke

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

May 1992

State University of New York
at Stony Brook

The Graduate School

Francisco J. Lüttecke

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Peter Szüsz
Professor of Mathematics
Dissertation Director

Eugene Zaustinsky
Professor of Mathematics
Chairman of Defense

Nicolae Teleman
Professor of Mathematics

Adrián Montoro
Associate Professor of Literature
Department of Hispanic Languages and Literature
State University of New York at Stony Brook
Outside Member

This dissertation is accepted by the Graduate School.

Abstract of the Dissertation

Summability of Subsequences

by

Francisco J. Lüttecke

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1992

Let (a_{nk}) be a summability matrix. Let $\{s_n\}$ be a sequence which is summed by (a_{nk}). That is, if

$$t_n = \sum_{k=1}^{\infty} a_{nk} s_k\,, \qquad n = 1, 2, \ldots$$

then $\lim_{n\to\infty} t_n$ exists. If $t = \sum_{k=1}^{\infty} \varepsilon_k(t) 2^{-k}$ is the non-terminating dyadic expansion of $t \in (0,1)$, we let $\lambda_k(t)$ denote the index of the k^{th} 1 in the sequence $\varepsilon_1(t), \varepsilon_2(t) \ldots$. then the map $t \mapsto \{s_{\lambda_k(t)}\}$ establishes a bijection between $(0,1)$ and the set of proper infinite subsequences of $\{s_n\}$.

We investigate the question of the summability for almost all $t \in (0,1)$ of the sequences $\{s_{\lambda_k(t)}\}$, $k = 1, 2, \ldots$ by the matrix (a_{nk}), that is, the problem of the summability of subsequences of a summable sequence.

A Francisco Jr.

y Janet

a la vuelta

de estos largos años invernales.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I want to thank the Department of Mathematics of the Facultad de Ciencias at the Universidad de Chile for initiating me into Mathematics and the Department of Mathematics of the State University of New York at Stony Brook for allowing me to advance my knowledge in this discipline.

I thank my dissertation director Professor Peter Saïsz for his infinite patience and for sharing his knowledge with me. I also thank my wife and son for enduring this long ordeal. Last, but in no way least, I want to thank Joann Debis, Amy DelloRusso, Pat Gandorf, Lucille Meci, Stella Shivers, and Barbara Wichard for all their help, support and encouragement in these difficult times.

INTRODUCTION

A matrix $A = (a_{nk})$ $(n, k = 1, 2, \dots)$ of real numbers determines a transformation

$$t_n = \sum_{k=1}^{\infty} a_{nk} s_k \,, \qquad n = 1, 2, \dots \tag{1}$$

The sequence $\{s_n\}$ $(n = 1, 2, \dots)$ is called A-summable to s if $t_n \to s$ as $n \to \infty$. The transformation (1) is said to be regular if $\lim_{n\to\infty} s_n = s$ implies that $\lim_{n\to\infty} t_n = s$, that is, $\lim_{n\to\infty} t_n$ exists and equal s. The Silverman-Toeplitz theorem ([5]) characterizes the regular matrices completely: the matrix (a_{nk}) is regular if and only if it satisfies the following three conditions.

$$\lim_{n\to\infty} a_{nk} = 0 \,, \qquad k = 1, 2, \dots$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$$

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| < +\infty \,.$$

If $\{s_n\}$, $(n = 1, 2, \dots)$ is a given sequence, then an infinite proper subsequence $n_1 < n_2 < \cdots$ determines a unique number $t \in (0, 1)$, namely $t = \sum_{k=1}^{\infty} 2^{-n_k}$. The inverse correspondence is clear if we use non-terminating dyadic expansions.

This problem originates in an effort to extend to regular matrices the obvious fact that all the infinite subsequences of a convergent sequence are also convergent to the same limit. However, a theorem of Buck [1] shows that we can not expect the same for regular matrices. He proved that a sequence summable by a matrix A such that all of its infinite subsequences are also summable by A must be convergent. With this limitation, any result of this type is bound to be an "almost everywhere" result and will require some conditions on the matrix and on the sequence. Under these conditions, one can attack the problem using probability theory.

We have used a consequence to the law of the iterated logarithm as applied to the sequence of random variables $\{\lambda_k\}$, ($k = 1, 2, \dots$) plus other conditions investigated by Klee and Szüsz in a joint paper ([7]) to arrive at a set of sufficient conditions that will ensure results which are, in turn, applied to a class of Riesz means and to the Cesàro means of integral order to obtain the summability of $\{s_{\lambda_k(t)}\}$, ($k = 1, 2, \dots$) for almost all $t \in (0, 1)$. In the case of the Riesz means we have a condition on the mean (Theorem 2.1) which doesn't require the sequence $\{p_n\}$ to be monotone as it is assumed in [6]. Also our conditions (1.7), (1.8), (1.7') and (1.8') conform more to the interplay between summability and probability theory than those in [6].

CHAPTER I. GENERALITIES

Definition 1. Let

$$t = 0 \cdot \varepsilon_1(t)\varepsilon_2(t)\ldots \tag{1.1}$$

be the non-terminating dyadic expansion of the number $t \in (0,1)$. For each $k = 1, 2, \ldots$ and each $t \in (0,1)$, let $\lambda_k(t)$ denote the index of the k^{th} 1 in the sequence (1.1).

Lemma 1. (Law of the Iterated Logarithm or LIL) Let P be Lebesgue measure on $(0,1)$. Then

$$P\left(\limsup_{k\to\infty} \frac{\lambda_k - 2k}{2\sqrt{k\log\log k}} = 1\right) = 1 \tag{1.2}$$

$$P\left(\liminf_{k\to\infty} \frac{\lambda_k - 2k}{2\sqrt{k\log\log k}} = -1\right) = 1 \tag{1.3}$$

Proof: See [3], p. 364 with

$$Y_1 = \frac{1}{\sqrt{2}}(\lambda_1 - 2)$$

$$Y_k = \frac{1}{\sqrt{2}}(\lambda_k - \lambda_{k-1} - 2), \qquad k \geq 2$$

which are independent, identically distributed random variables ([9], p. 64).

As a consequence to LIL we obtain that if we write

$$C(t) = \sup_{k\geq 3} \frac{|\lambda_k(t) - 2k|}{\sqrt{k\log\log k}},$$

then $C(t) < +\infty$ almost surely. In other words, for almost all $t \in (0,1)$ we have

$$|\lambda_k(t) - 2k| \leq C\sqrt{k\log\log k}, \qquad k \geq 3 \tag{1.4}$$

where C is a constant that may depend on t, but it is otherwise independent of k.

In what follows, we will frequently drop the t in the notations but it is to be understood that a fixed but arbitrary t satisfying (1.4) has been selected. The letters $C, C_0, C_1, \ldots$ will represent constants that may depend on the fixed t, but that are independent of k. Sometimes we will use the same letter to denote constants that are not necessarily equal in different instances. For example (1.4) implies $\left|\frac{\lambda_k}{2} - k\right| \le C\sqrt{k \log\log k}$ and $\left|\left[\frac{\lambda_k}{2}\right] - k\right| \le C\sqrt{k \log\log k}$, $k \ge 3$ almost surely.

As further implications of the LIL we have

Lemma 2. 1) If $3 \le k \le n$ and $\lambda_k > 2n$, then

$$n - [C\sqrt{n \log\log n}] \le k \le n \tag{1.5}$$

where C is independent of k, n. This certainly holds if $\left[\frac{\lambda_k}{2}\right] > n$.

2) If $n \ge 3$, $k > n$ and $\left[\frac{\lambda_k}{2}\right] \le n$, then

$$n + 1 \le k \le n + [C\sqrt{n \log\log n}]. \tag{1.6}$$

This obviously holds if $\lambda_k \le 2n$ (C is again independent of k, n).

Proof: 1) First

$$\frac{\lambda_k}{2} - k \le \left|\frac{\lambda_k}{2} - k\right| \le C\sqrt{k \log\log k}$$
$$\le C\sqrt{n \log\log n}$$

so $\frac{\lambda_k}{2} - C\sqrt{n \log\log n} \le k$, and then

$$n - C\sqrt{n \log\log n} < \frac{\lambda_k}{2} - C\sqrt{n \log\log n} \le k.$$

Thus

$$n - C\sqrt{n \log\log n} < k$$

and, a fortiori,

$$n - [C\sqrt{n \log\log n}] \le k\,.$$

2) Now $\left[\frac{\lambda_k}{2}\right] \le n$ implies $\lambda_k \le 2n+1$. And since $k \le \lambda_k$ (always) we get $k \le 2n+1$. On the other hand

$$\frac{\lambda_k}{2} < \left[\frac{\lambda_k}{2}\right] + 1 \le n+1 \le k\,.$$

That is $\lambda_k < 2k$, and we can write $\lambda_k = 2k - \mu_k$ where

$$0 < \mu_k \le C\sqrt{k \log\log k}\,.$$

Then

$$\begin{aligned} n &\ge \left[\frac{\lambda_k}{2}\right] = \left[k - \frac{\mu_k}{2}\right] = k + \left[-\frac{\mu_k}{2}\right] \\ &> k - \frac{\mu_k}{2} - 1 \ge k - \frac{C}{2}\sqrt{k \log\log k} - 1 \\ &\ge k - \frac{C}{2}\sqrt{(2n+1)\log\log(2n+1)} - 1 \qquad (k \le 2n+1) \\ &\ge k - C_1\sqrt{n \log\log n}\,. \end{aligned}$$

Therefore,

$$k \le n + C_1\sqrt{n \log\log n}\,.$$

We are interested in the question of when $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a_{nk} s_k = s$ implies

$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} s_{\lambda_k(t)} = s$ almost surely. Now

$$\sum_{k=1}^{\infty} a_{nk} s_{\lambda_k(t)} = a_{n1} s_{\lambda_1(t)} + \sum_{k=2}^{\infty} a_{n,\left[\frac{\lambda_k(t)}{2}\right]} s_{\lambda_k(t)} + \sum_{k=2}^{\infty} (a_{nk} - a_{n,\left[\frac{\lambda_2(t)}{2}\right]}) s_{\lambda_k(t)} .$$

And since $a_{n1} \to 0$ by regularity, in order to have $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} s_{\lambda_k(t)} = s$ almost surely whenever $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} s_k = s$, it suffices to have both

$$\lim_{n\to\infty} \sum_{k=2}^{\infty} a_{n,\left[\frac{\lambda_k(t)}{2}\right]} s_{\lambda_k(t)} = s \quad \text{almost surely} \tag{1.7}$$

and

$$\lim_{n\to\infty} \sum_{k=2}^{\infty} \left| a_{nk} - a_{n,\left[\frac{\lambda_k(t)}{2}\right]} \right| |s_{\lambda_k(t)} = 0 \quad \text{almost surely} \tag{1.8}$$

We will also consider the "continuous" matrix case, that is, the case of a map

$$a \colon \{1, 2, 3, \ldots\} \times (0, +\infty) \to \mathrm{R}$$

such that the matrix (a_{nk}) is regular (here a_{nk} stands for $a(n,k)$. Similarly for $a_{n,\frac{k}{2}}$). In this case we obtain

$$\sum_{k=1}^{\infty} a_{nk} s_{\lambda_k(t)} = \sum_{k=1}^{\infty} a_{n,\frac{\lambda_k(t)}{2}} s_{\lambda_k(t)} + \sum_{k=1}^{\infty} \left(a_{nk} - a_{n,\frac{\lambda_k(t)}{2}} \right) s_{\lambda_k(t)}$$

and $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}s_{\lambda_k(t)} = s$ almost surely is implied by both

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,\frac{\lambda_k(t)}{2}} s_{\lambda_k(t)} = s \qquad \text{almost surely} \tag{1.7'}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} \left|a_{nk} - a_{n,\frac{\lambda_k(t)}{2}}\right| |s_{\lambda_k(t)}| = 0 \qquad \text{almost surely} \tag{1.8'}$$

Theorem 1.1 Let $s_1, s_2, \ldots$ be a sequence of real numbers.

1) If $A = (a_{nk})$ is a regular matrix, $\lim_{n\to\infty}\sum_{k=2}^{\infty} a_{n,[\frac{k}{2}]}s_k = 2s$, and for every $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \exp\left\{-\varepsilon\left[\sum_{k=2}^{\infty} a^2_{n,[\frac{k}{2}]}s_k^2\right]^{-1}\right\} < +\infty \tag{1.9}$$

(Klee – Szüsz condition).

Then, for almost all $t \in (0,1)$,

$$\lim_{n\to\infty}\sum_{k=2}^{\infty} a_{n,\left[\frac{\lambda_k(t)}{2}\right]} s_{\lambda_k(t)} = s\,. \tag{1.10}$$

2) Now suppose $A = (a_{nk})$ is regular and given by a map as explained above. If $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is itself regular, $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,\frac{k}{2}}s_k = 2s$ and for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\varepsilon\left[\sum_{k=1}^{\infty} a^2_{n,\frac{k}{2}}s_k^2\right]^{-1}\right\} < +\infty\,. \tag{1.9'}$$

Then, for almost all $t \in (0,1)$,

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,\frac{\lambda_k(t)}{2}} s_{\lambda_k(t)} = s\,. \tag{1.10'}$$

Proof: 1) First we note that for each $N = 2, 3, \ldots$ the matrix $\left(\frac{1}{N} a_{n,[\frac{k}{N}]}\right)$, $n = 1, 2, \ldots$, $k = N, N+1, \ldots$ is regular if (a_{nk}) is. Now the rest is a direct consequence of [7] because we obtain $\lim_{n\to\infty} \sum_{j=2}^{\infty} a_{n,[\frac{j}{2}]} s_j \varepsilon_j(t) = s$ almost surely. And this in turn implies (1.10) since

$$\{j \mid \varepsilon_j(t) = 1\} = \{\lambda_k(t) \mid k = 1, 2, \ldots\}.$$

2) In view of 1), this is now evident.

As we can deduct from this theorem, 1) yields condition (1.7) and 2) condition (1.7').

Now in order to have $\lim_{n\to\infty} \sum_{k=2}^{\infty} \frac{1}{2} a_{n,[\frac{k}{2}]} s_k = s$ (Thm. 1.1, 1)) or $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2} a_{n,\frac{k}{2}} s_k = s$ (Thm. 1.1, 2)) whenever $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} s_k = s$, we need the concept of strength for summability methods.

Definition 2. 1) Let $A = (a_{nk})$, $B = (b_{nk})$ be regular methods. A is stronger than B if $\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} s_k = s$ implies $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} s_k = s$.

2) A is equivalent to B if A is stronger than B and B is stronger than A (in other words, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} s_k = s$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} s_k = s$).

A most useful criterion is

Lemma 3. 1) If B^{-1} exists, then A is stronger than B if and only if AB^{-1} is regular.

2) If A^{-1} exists, then A is equivalent to convergence if and only if A^{-1} is regular.

Proof: See [11], p. 12.

In view of Definition 2, we need $\left(\frac{1}{2}a_{n,\left[\frac{k}{2}\right]}\right)$ to be stronger than (a_{nk}). We then have the following result.

Theorem 1.2 1) Suppose (a_{nk}) is a regular method and that $\{s_n\}$ is a sequence with $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}s_k = s$. If $\left(\frac{1}{2}a_{n,\left[\frac{k}{2}\right]}\right)$ is stronger than (a_{nk}) and the Klee-Szüsz condition (1.9) is satisfied, then

$$\lim_{n\to\infty}\sum_{k=2}^{\infty} a_{n,\left[\frac{\lambda_k(t)}{2}\right]} s_{\lambda_k(t)} = s$$

almost surely.

2) Now suppose that (a_{nk}) is regular and given by a map. If $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular and stronger than (a_{nk}) and condition (1.9') obtains, then

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,\frac{\lambda_k(t)}{2}} s_{\lambda_k(t)} = s$$

almost surely.

To deal with conditions (1.8) and (1.8') we will make use of condition (1.4) which stems directly from Lemma 1 (LIL). Due to the $\sqrt{\log\log k}$ expression we only consider $k \geq 3$. This is not a loss of generality due to regularity. (1.8) and (1.8') become, respectively,

$$\lim_{n\to\infty}\sum_{k=3}^{\infty}\left|a_{nk} - a_{n,\left[\frac{\lambda_k(t)}{2}\right]}\right| \left|s_{\lambda_k(t)}\right| = 0 \tag{1.11}$$

and

$$\lim_{n\to\infty}\sum_{k=3}^{\infty}\left|a_{nk} - a_{n,\frac{\lambda_k(t)}{2}}\right| \left|s_{\lambda_k(t)}\right| = 0 \tag{1.11'}$$

both almost surely.

Assuming a growth condition on the sequence $\{s_n\}$ we will prove that the "smoothness" condition (1.11) is true for a class of Riesz means while the "smoothness" condition (1.11') holds true for the logarithmic means and the Cesàro methods of integral order. We will also give necessary and sufficient conditions for $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ and $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ to be stronger than (a_{nk}) in the case of Riesz and Nørlund means.

We conclude this chapter with one more definition and some remarks.

Definition 3. $A = (a_{nk})$ is called triangular if $a_{nk} = 0$ when $k > n$, $n = 1, 2, \ldots$.

Lemma 4. If A is triangular, A^{-1} exists if and only if $a_{nn} \neq 0$, $n = 1, 2, \ldots$.

Proof: Using that $a_{nn} \neq 0$, the entries of the inverse may be calculated inductively. Conversely if $A^{-1} = (b_{nk})$, then A^{-1} is also triangular and $a_{nn}b_{nn} = 1$, $n = 1, 2, \ldots$ so $a_{nn} \neq 0$.

Remarks: We will use $\frac{\lambda_k}{2}$. This choice is motivated from the fact that $\lim_{k\to\infty} \frac{\lambda_k(t)}{2k} = 1$ almost surely. This is a consequence of LIL (Lemma 1). It can also be proved by using the Law of Large Numbers. But LIL is more precise, giving also the estimate (1.4) which in turn yields Lemma 2 which is crucial in what follows.

CHAPTER II. THE SIMPLE RIESZ MEANS

Given a sequence $\{p_n\}$, $n = 1,2,3,\ldots$ of positive real numbers, the simple Riesz mean associated with this sequence, in symbols (R,p_n), is the matrix defined by

$$a_{nk} = \begin{cases} \frac{p_k}{P_n}, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{2.1}$$

where $P_n = \sum\limits_{k=1}^{n} p_k$. It is known that (R,p_n) is regular iff $\lim\limits_{n\to\infty} P_n = +\infty$ ([5]).

We also consider the continuous case, that is,

$$a(n,x) = \begin{cases} \frac{f(x)}{F(n)}, & 0 < x \le n \\ 0, & x > n \end{cases} \tag{2.2}$$

where $f: (0,+\infty) \to \mathbb{R}$ has $f(k) > 0$, $k = 1,2,\ldots$ and $F(n) = \sum\limits_{k=1}^{n} f(k)$. Again (a_{nk}) is regular iff $\lim\limits_{n\to\infty} F(n) = +\infty$.

To find conditions for $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ or $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ to be stronger than (a_{nk}), we first need to find the inverse of (a_{nk}) (which exists by Lemma 4, Ch. I). This inverse is

$$a^{\#}_{nk} = \begin{cases} \frac{P_n}{p_n}, & k = n \\ -\frac{P_{n-1}}{p_n}, & k = n-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

for the discrete case (2.1) and

$$a^{\#}_{nk} = \begin{cases} \frac{F(n)}{f(n)}, & k = n \\ -\frac{F(n-1)}{f(n)}, & k = n-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.3'}$$

for the continuous case (2.2).

Next we demand that AB^{-1} be regular where A is either $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ or $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ while $B = (a_{nk})$ (by Lemma 3 of Ch. I). Before continuing we observe that $\sum_{k=1}^{n} a_{nk}^{\#} = 1$, $n = 1, 2, \ldots$ in both (2.3) and (2.3').

The (n, k) entry of AB^{-1} is, by triangularity

$$\frac{1}{2}\sum_{j=2}^{\infty} a_{n,[\frac{j}{2}]}a_{jk}^{\#} = \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}a_{jk}^{\#}$$

or

$$\frac{1}{2}\sum_{j=1}^{2n} a_{n,\frac{j}{2}}a_{jk}^{\#}.$$

Since $a_{jk}^{\#} \neq 0$ only if $j = k, k+1$, the sum above is 0 if $k \geq 2n+2$ and $k \geq 2n+1$, respectively. Therefore, $AB^{-1} = (c_{nk})$ is

$$c_{nk} = \begin{cases} -\frac{p_1 P_1}{2p_2 P_n}, & k = 1 \\ \frac{P_k}{2P_n}\left(\frac{p_{[\frac{k}{2}]}}{p_k} - \frac{p_{[\frac{k+1}{2}]}}{p_{k+1}}\right), & 2 \leq k \leq 2n \\ \frac{p_n P_{2n+1}}{2p_{2n+1}P_n}, & k = 2n+1 \\ 0, & k \geq 2n+2 \end{cases} \tag{2.4}$$

and $AB^{-1} = (d_{nk})$, (continuous case),

$$d_{nk} = \begin{cases} \frac{F(k)}{2F(n)}\left(\frac{f(\frac{k}{2})}{f(k)} - \frac{f(\frac{k+1}{2})}{f(k+1)}\right), & 1 \leq k \leq 2n-1 \\ \frac{f(n)F(2n)}{2f(2n)F(n)}, & k = 2n \\ 0, & k \geq 2n+1 \end{cases} \tag{2.4'}$$

As we said in Theorem 1.1, $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ is regular if (a_{nk}) is. This must be assumed in the continuous case, but before going into that we have the following, useful, Lemma

Lemma 1. Suppose $\{a_n\}$, $\{b_n\}$ are sequences with $b_n > 0$, $n = 1, 2, \ldots$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$ and $\sum_{k=1}^{\infty} b_k = +\infty$. Then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} = \ell .$$

Proof: Consider the Riesz mean (R, b_n). It is regular because $\sum_{k=1}^{\infty} b_k = +\infty$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$, then

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} b_k} \cdot \sum_{k=1}^{n} b_k \cdot \frac{a_k}{b_k} = \ell .$$

That is,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} = \ell .$$

Now we have

Proposition 1. If (a_{nk}) is a regular Riesz mean given by a map as in (2.2), then $\left(\frac{1}{2} a_{n,\frac{k}{2}}\right)$ is regular iff the following two conditions hold:

a) $$\sup_{n\geq 1} \frac{\sum_{k=1}^{n} \left| f\left(k - \frac{1}{2}\right) \right|}{\sum_{k=1}^{n} f(k)} < +\infty$$

b) $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f\left(k - \frac{1}{2}\right)}{\sum_{k=1}^{n} f(k)} = 1$$

If $f \geq 0$ then b) implies a) so, in that case, only b) needs proof.

Proof: We have that

$$\frac{1}{2} a_{n,\frac{k}{2}} = \begin{cases} \frac{f(\frac{k}{2})}{2F(n)}, & 1 \leq k \leq 2n \\ 0, & k \geq 2n+1 \end{cases}$$

where $F(n) = \sum_{k=1}^{n} f(k)$. Since (a_{nk}) is a regular Riesz mean, $\lim_{n\to+\infty} F(n) = +\infty$ so $\lim_{n\to\infty} \frac{1}{2} a_{n,\frac{k}{2}} = 0$, $k = 1, 2, \ldots$. Next,

$$\begin{aligned}\sum_{k=1}^{2n} \frac{1}{2}\left|a_{n,\frac{k}{2}}\right| &= \frac{1}{2F(n)} \sum_{k=1}^{2n} \left|f\left(\frac{k}{2}\right)\right| \\ &= \frac{1}{2F(n)}\left(\sum_{k=1}^{n} |f(k)| + \sum_{k=1}^{n} \left|f\left(k-\frac{1}{2}\right)\right|\right) \\ &= \frac{1}{2} + \frac{\sum_{k=1}^{n} \left|f\left(k-\frac{1}{2}\right)\right|}{2\sum_{k=1}^{n} f(k)}\end{aligned}$$

from which a) follows. Again

$$\begin{aligned}\sum_{k=1}^{2n} \frac{1}{2} a_{n,\frac{k}{2}} &= \frac{1}{2F(n)}\left(\sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} f\left(k-\frac{1}{2}\right)\right) \\ &= \frac{1}{2} + \frac{\sum_{k=1}^{n} f\left(k-\frac{1}{2}\right)}{2\sum_{k=1}^{n} f(k)}\end{aligned}$$

and $\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{1}{2} a_{n,\frac{k}{2}} = 1$ iff b) holds.

Thus the Silverman-Toeplitz conditions are satisfied iff a) and b) hold.

Corollary: If $f \geq 0$ and $\lim_{n\to\infty} \frac{f(n-\frac{1}{2})}{f(n)} = 1$, then $\left(\frac{1}{2} a_{n,\frac{k}{2}}\right)$ is regular.

Proof: Only b) needs verification. Since $\sum_{k=1}^{\infty} f(k) = +\infty$, $f(k) > 0$ and $\lim_{n\to\infty} \frac{f(n-\frac{1}{2})}{f(n)} = 1$, Lemma 1 implies $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(k-\frac{1}{2})}{\sum_{k=1}^{n} f(k)} = 1$, which is b).

Now we turn to the regularity conditions for AB^{-1}.

Theorem 2.1. 1) (c_{nk}) is regular iff the following three conditions hold:

a) $\sup\limits_{n\geq 1} \frac{p_n P_{2n+1}}{p_{2n+1} P_n} < +\infty$

b) $\sup\limits_{n\geq 1} \frac{1}{P_n} \sum\limits_{k=1}^{n} P_{2k} \left| \frac{p_k}{p_{2k}} - \frac{p_k}{p_{2k+1}} \right| < +\infty$

c) $\sup\limits_{n\geq 2} \frac{1}{P_n} \sum\limits_{k=2}^{n} P_{2k-1} \left| \frac{p_{k-1}}{p_{2k-1}} - \frac{p_k}{p_{2k}} \right| < +\infty$

2) (d_{nk}) is regular iff the following three conditions hold (we implicitly assume that $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular):

a) $\sup\limits_{n\geq 1} \frac{f(n)F(2n)}{f(2n)F(n)} < +\infty$

b) $\sup\limits_{n\geq 1} \frac{1}{F(n)} \sum\limits_{k=1}^{n} F(2k-1) \left| \frac{f(k-\frac{1}{2})}{f(2k-1)} - \frac{f(k)}{f(2k)} \right| < +\infty$

c) $\sup\limits_{n\geq 2} \frac{1}{F(n)} \sum\limits_{k=1}^{n-1} F(2k) \left| \frac{f(k)}{f(2k)} - \frac{f(k+\frac{1}{2})}{f(2k+1)} \right| < +\infty$

Proof: We only need to verify the Silverman-Toeplitz conditions for regularity. The first two always hold.

i) $\lim\limits_{n\to\infty} c_{nk} = 0$: For fixed k and $n > \frac{k}{2}$ we obtain

$$c_{nk} = \frac{P_k}{2P_n} \left(\frac{p_{[\frac{k}{2}]}}{p_k} - \frac{p_{[\frac{k+1}{2}]}}{p_{k+1}} \right) \to 0$$

as $n \to \infty$ because $P_n \to +\infty$ by the regularity of (R, p_n).

ii) First $\sum\limits_{k=1}^{2n+1} a_{jk}^{\#} = 1$ if $1 \leq j \leq 2n+1$ and $n = 1, 2, \ldots$ because $a_{jk}^{\#} \neq 0$ only if $k = j-1,\ j$ and $1 \leq j \leq 2n+1$, so $\sum\limits_{k=1}^{2n+1} a_{jk}^{\#} = \sum\limits_{k=1}^{j} a_{jk}^{\#} = 1$.

Therefore,

$$\sum_{k=1}^{2n+1} c_{nk} = \frac{1}{2}\sum_{k=1}^{2n+1}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}a^{\#}_{jk}$$
$$= \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}\sum_{k=1}^{2n+1} a^{\#}_{jk} = \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}$$
$$= \sum_{j=1}^{n} a_{nj} = \frac{1}{P_n}\sum_{j=1}^{n} p_j = 1$$

because $P_n = \sum_{j=1}^{n} p_j$.

iii) $\sup_{n\geq 1}\sum_{k=1}^{2n+1} |c_{nk}| < +\infty$: Since $|c_{n,1}| \to 0$ we require

$$\sum_{k=2}^{2n+1} |c_{nk}| = \frac{1}{2P_n}\sum_{k=2}^{2n} P_k\left|\frac{p_{[\frac{k}{2}]}}{p_k} - \frac{p_{[\frac{k+1}{2}]}}{p_{k+1}}\right| + \frac{p_nP_{2n+1}}{2p_{2n+1}P_n}$$
$$= \frac{1}{2P_n}\sum_{k=1}^{n} P_{2k}\left|\frac{p_k}{p_{2k}} - \frac{p_k}{p_{2k+1}}\right|$$
$$+ \frac{1}{2P_n}\sum_{k=2}^{n} P_{2k-1}\left|\frac{p_{k-1}}{p_{2k-1}} - \frac{p_k}{p_{2k}}\right| + \frac{p_nP_{2n+1}}{2p_{2n+1}P_n}$$

to be bounded. This is obviously equivalent to the conclusion of the theorem.

2) i) $\lim_{n\to\infty} d_{nk} = 0$, $k = 1, 2, \ldots$ because $\lim_{n\to\infty} F(n) = +\infty$.

ii) As in 1) ii), $\sum_{k=1}^{2n} a^{\#}_{jk} = 1$ if $1 \leq j \leq 2n$ so

$$\sum_{k=1}^{2n} d_{nk} = \frac{1}{2}\sum_{j=1}^{2n} a_{n,\frac{j}{2}}\sum_{k=1}^{2n} a^{\#}_{jk} = \frac{1}{2}\sum_{j=1}^{2n} a_{n,\frac{j}{2}} \to 1$$

as $n \to \infty$ due to the assumed regularity of $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$.

iii) This one is, verbatim, 1) iii).

Before continuing, we need the following well-known estimates:

1) If $0 \le \alpha < 1$, then

$$n^{1-\alpha} \le \sum_{k=1}^{n} \frac{1}{k^\alpha} \le \frac{n^{1-\alpha}}{1-\alpha}$$

2) $\log(n+1) \le \sum_{k=1}^{n} \frac{1}{k} \le 1 + \log n$

3) If $\alpha \ge 0$, then

$$\frac{n^{\alpha+1}}{\alpha+1} \le \sum_{k=1}^{n} k^\alpha \le n^{\alpha+1}$$

The Riesz means associated with $\left\{\frac{1}{n^\alpha}\right\}$, $0 \le \alpha \le 1$, and $\{n^\alpha\}$, $\alpha \ge 0$ are all regular.

Theorem 2.2.

1) Suppose $0 \le \alpha < 1$. If $\{s_n\}$ is $\left(R, \frac{1}{n^\alpha}\right)$-summable to s and for constants β, γ with $0 \le \beta < \frac{1}{2}$ and $\gamma \ge 0$ we have

$$\sup_{n \ge 2} \frac{|s_n|}{n^\beta (\log n)^\gamma} < +\infty \tag{2.5}$$

Then almost all subsequences of $\{s_n\}$ are $\left(R, \frac{1}{n^\alpha}\right)$-summable to s.

2) Suppose $\alpha \ge 0$. If $\{s_n\}$ is (R, n^α)-summable to s and (2.4) obtains for $\{s_n\}$, then we get the same conclusion as in 1).

Proof:

1) First, we establish that

$$\sup_{k \ge 2} \frac{|s_{\lambda_k(t)}|}{k^\beta (\log k)^\gamma} < +\infty \qquad \text{almost surely.}$$

Fix $t \in (0,1)$ satisfying (1.4) of Chapter I. Then, for $k \ge 3$

$$|\lambda_k(t)| \le 2k + C\sqrt{k \log\log k}\,.$$

Let $C_0 = \sup_{n\geq 2} \frac{|s_n|}{n^\beta (\log n)^\gamma}$. Then

$$\begin{aligned}
|s_{\lambda_k(t)}| &\leq C_0(\lambda_k(t))^\beta (\log \lambda_k(t))^\gamma \\
&\leq C_0(2k + C\sqrt{k\log\log k})^\beta (\log(2k + C\sqrt{k\log\log k}))^\gamma \\
&= C_0 k^\beta \left(2 + C\sqrt{\frac{\log\log k}{k}}\right)^\beta \left(\log\left(k\left(2 + C\sqrt{\frac{\log\log k}{k}}\right)\right)\right)^\gamma \\
&\leq C_1 k^\beta (\log C_2 k)^\gamma \leq C_3 k^\beta (\log k)^\gamma .
\end{aligned}$$

Next, $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ is stronger than (a_{nk}): we only need to verify the requirements of Theorem 2.1

a)
$$\begin{aligned}
\frac{p_n P_{2n+1}}{p_{2n+1} P_n} &\leq \frac{n^{-\alpha}(2n+1)^{1-\alpha}}{(2n+1)^{-\alpha} n^{1-\alpha}(1-\alpha)} \\
&= \left(2 + \frac{1}{n}\right)\frac{1}{1-\alpha} \leq \frac{3}{1-\alpha}
\end{aligned}$$

b)
$$\begin{aligned}
\frac{1}{P_n}\sum_{k=1}^{n} P_{2k}\left|\frac{p_k}{p_{2k}} - \frac{p_k}{p_{2k+1}}\right| &\leq \frac{1}{n^{1-\alpha}}\sum_{k=1}^{n}\frac{(2k)^{1-\alpha}}{1-\alpha}\left|\frac{k^{-\alpha}}{(2k)^{-\alpha}} - \frac{k^{-\alpha}}{(2k+1)^{-\alpha}}\right| \\
&= \frac{2^{1-\alpha}}{(1-\alpha)n^{1-\alpha}}\sum_{k=1}^{n} k^{1-\alpha}\left|2^\alpha - \left(2 + \frac{1}{k}\right)^\alpha\right| \\
&\leq \frac{2^{1-\alpha}}{(1-\alpha)n^{1-\alpha}}\sum_{k=1}^{n} k^{1-\alpha}\cdot\frac{\alpha}{2^{1-\alpha}k}
\end{aligned}$$

(Mean Value Theorem applied to $t \mapsto t^\alpha$)

$$= \frac{\alpha}{(1-\alpha)n^{1-\alpha}}\sum_{k=1}^{n}\frac{1}{k^\alpha} \leq \frac{\alpha}{(1-\alpha)^2}$$

c)
$$\begin{aligned}
&\frac{1}{P_n}\sum_{k=2}^{n} P_{2k-1}\left|\frac{p_{k-1}}{p_{2k-1}} - \frac{p_k}{p_{2k}}\right| \\
&= \frac{1}{P_n}\sum_{k=1}^{n-1} P_{2k+1}\left|\frac{p_k}{p_{2k+1}} - \frac{p_{k+1}}{p_{2k+2}}\right| \\
&\leq \frac{1}{n^{1-\alpha}}\sum_{k=1}^{n-1}\frac{(2k+1)^{1-\alpha}}{1-\alpha}\left|\frac{k^{-\alpha}}{(2k+1)^{-\alpha}} - \frac{(k+1)^{-\alpha}}{(2k+2)^{-\alpha}}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-\alpha)n^{1-\alpha}} \sum_{k=1}^{n-1} (2k+1)^{1-\alpha} \left| \left(2+\frac{1}{k}\right)^{\alpha} - 2^{\alpha} \right| \\
&\le \frac{1}{(1-\alpha)n^{1-\alpha}} \sum_{k=1}^{n-1} (2k+1)^{1-\alpha} \cdot \frac{\alpha}{2^{1-\alpha}k} \\
&\le \frac{\alpha 3^{1-\alpha}}{(1-\alpha)2^{1-\alpha}} \cdot \frac{1}{n^{1-\alpha}} \sum_{k=1}^{n-1} \frac{1}{k^{\alpha}} \\
&\le \frac{\alpha 3^{1-\alpha}}{(1-\alpha)^2 2^{1-\alpha}}
\end{aligned}
$$

and $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ is stronger than (a_{nk}).

Next, we verify the Klee-Szüsz condition for $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ (see Thm. 1.1)

$$
\begin{aligned}
\sum_{k=2}^{\infty} a^2_{n,[\frac{k}{2}]} s_k^2 = \sum_{k=2}^{2n+1} a^2_{n,[\frac{k}{2}]} s_k^2
&= \frac{1}{P_n^2} \sum_{k=1}^{n} p_k^2 (s_{2k}^2 + s_{2k+1}^2) \\
&\le \frac{C_0^2}{n^{2-2\alpha}} \sum_{k=1}^{n} k^{-2\alpha}((2k)^{2\beta}(\log 2k)^{2\gamma} \\
&\qquad + (2k+1)^{2\beta}(\log(2k+1))^{2\gamma}) \\
&\qquad \left(C_0 = \sup_{n \ge 2} \frac{|s_n|}{n^{\beta}(\log n)^{\gamma}} \right) \\
&\le \frac{C_1}{n^{2-2\alpha}} \sum_{k=1}^{n} k^{2\beta-2\alpha}(\log 2k)^{2\gamma} \\
&\le \frac{C_2(\log n)^{2\gamma}}{n^{2-2\alpha}} \sum_{k=1}^{n} k^{2\beta-2\alpha}
\end{aligned}
$$

Now $-2 < 2\beta - 2\alpha < 1$, and

$$
\sum_{k=1}^{n} k^{2\beta-2\alpha} \le
\begin{cases}
\frac{n^{1-2\alpha+2\beta}}{1-2\alpha+2\beta}, & \beta \le \alpha < \beta + \frac{1}{2} \\
1 + \log n, & \alpha = \beta + \frac{1}{2} \\
\frac{2\alpha-2\beta}{2\alpha-2\beta+1}, & \alpha > \beta + \frac{1}{2} \\
n^{1-2\alpha+2\beta}, & \beta > \alpha
\end{cases}
$$

so

$$\frac{1}{n^{2-2\alpha}}\sum_{k=1}^{n} k^{2\beta-2\alpha} \le \begin{cases} \frac{1}{(1-2\alpha+2\beta)n^{1-2\beta}}, & \beta \le \alpha < \beta+\frac{1}{2} \\ \frac{1+\log n}{n^{2-2\alpha}}, & \alpha = \beta+\frac{1}{2} \\ \frac{2\alpha-2\beta}{2\alpha-2\beta+1}\cdot\frac{1}{n^{2-2\alpha}}, & \alpha > \beta+\frac{1}{2} \\ \frac{1}{n^{1-2\beta}}, & \beta > \alpha \end{cases}$$

In any event, since $0 \le \beta < \frac{1}{2}$, $\gamma \ge 0$, and $0 \le \alpha < 1$, we obtain, for suitable $p \ge 0$, $q > 0$, and C_3, that

$$\sum_{k=2}^{\infty} a^2_{n,[\frac{k}{2}]} s_k^2 \le \frac{C_3(\log n)^p}{n^q}$$
$$\le \frac{C_4}{n^r}$$

for some $r > 0$. Then, if $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \exp\left\{-\varepsilon\left(\sum_{k=2}^{\infty} a^2_{n,[\frac{k}{2}]} s_k^2\right)^{-1}\right\} \le \sum_{n=1}^{\infty}\exp\left\{-\frac{\varepsilon n^r}{C_4}\right\} < +\infty$$

(Choose $\ell > \frac{1}{r}$, ℓ integer. Then $\exp\left\{-\frac{\varepsilon n^r}{C_4}\right\} < \frac{\ell! C_4^\ell}{\varepsilon^\ell}\cdot\frac{1}{n^{r\ell}}$.) Therefore, by Theorem 1.2, we have that

$$\lim_{n\to\infty}\sum_{k=2}^{\infty} a_{n,\left[\frac{\lambda_k(t)}{2}\right]} s_{\lambda_k(t)} = s$$

almost surely where (a_{nk}) is the Riesz mean $\left(R, \frac{1}{n^\alpha}\right)$, $0 \le \alpha < 1$.

To finish the proof we must establish that

$$\lim_{n\to\infty}\sum_{k=3}^{\infty}(a_{nk} - a_{n,\left[\frac{\lambda_k(t)}{2}\right]}) s_{\lambda_k(t)} = 0$$

almost surely (cf. (1.8)). Fix t satisfying (1.4) and for $n \ge 3$, define

$$F_n = \left\{k \mid 3 \le k \le n,\ \left[\frac{\lambda_k(t)}{2}\right] \le n\right\}$$
$$G_n = \left\{k \mid 3 \le k \le n,\ \left[\frac{\lambda_k(t)}{2}\right] > n\right\}$$
$$H_n = \left\{k \mid k > n,\ \left[\frac{\lambda_k(t)}{2}\right] \le n\right\}$$

Since (a_{nk}) is triangular

$$\sum_{k=3}^{\infty}(a_{nk}-a_{n,\left[\frac{\lambda_k(t)}{2}\right]})s_{\lambda_k(t)} = \sum_{k\in F_n}(a_{nk}-a_{n,\left[\frac{\lambda_k(t)}{2}\right]})S_{\lambda_k(t)}$$
$$+\sum_{k\in G_n} a_{nk}s_{\lambda_k(t)} - \sum_{k\in H_n} a_{n,\left[\frac{\lambda_k(t)}{2}\right]}s_{\lambda_k(t)}$$
$$= \Sigma_1+\Sigma_2-\Sigma_3$$

say. We prove $\Sigma_i \to 0$, $i=1,2,3$. For that we need

$$\left|\frac{1}{k^\alpha}-\frac{1}{\left[\frac{\lambda_k(t)}{2}\right]^\alpha}\right| \le \frac{C\sqrt{\log\log k}}{k^{\alpha+\frac{1}{2}}}, \qquad k\ge 3 \tag{2.6}$$

C constant (with respect to k). Assume (2.6) for the moment, and let us complete the proof

i)
$$|\Sigma_1| \le \sum_{k\in F_n}|a_{nk}-a_{n,\left[\frac{\lambda_k(t)}{2}\right]}|\,|s_{\lambda_k(t)}|$$
$$\le \frac{C_0}{n^{1-\alpha}}\sum_{k\in F_n}\left|\frac{1}{k^\alpha}-\frac{1}{\left[\frac{\lambda_k(t)}{2}\right]^\alpha}\right| k^\beta(\log k)^\gamma$$
$$\le \frac{C_1}{n^{1-\alpha}}\sum_{k=3}^{n}\frac{\sqrt{\log\log k}}{k^{\alpha+\frac{1}{2}}}\cdot k^\beta(\log k)^\gamma$$
$$\le \frac{C_1(\log n)^{\gamma+\frac{1}{2}}}{n^{1-\alpha}}\cdot\sum_{k=1}^{n}k^{\beta-\alpha-\frac{1}{2}}$$

and $-\frac{3}{2}<\beta-\alpha-\frac{1}{2}<0$ so

$$\sum_{k=1}^{n}k^{\beta-\alpha-\frac{1}{2}} \le \begin{cases}\frac{2\alpha+1-2\beta}{2\alpha-2\beta-1}, & \alpha>\beta+\frac{1}{2}\\ 1+\log n, & \alpha=\beta+\frac{1}{2}\\ \frac{2n^{\frac{1}{2}-\alpha+\beta}}{1-2\alpha+2\beta}, & \alpha<\beta+\frac{1}{2}\end{cases}$$

Since $0\le\beta<\frac{1}{2}$, we again obtain

$$|\Sigma_1| \le \frac{C_2(\log n)^p}{n^q}$$

for suitable positive constants C_2, p, q and so $\Sigma_1 \to 0$ as $n \to \infty$.

ii) By Lemma 2, Ch. I, we have

$$n - [C\sqrt{n \log\log n}] \le k \le n \quad \text{if} \quad k \in G_n .$$

Now

$$\begin{aligned}
|\Sigma_2| &\le \sum_{k\in G_n} |a_{nk}|\,|s_{\lambda_k(t)}| \\
&\le \frac{C_0}{n^{1-\alpha}} \sum_{k\in G_n} k^{-\alpha} k^{\beta} (\log k)^{\gamma} \\
&\le \frac{C_0}{n}\left(\frac{n}{n - C\sqrt{n\log\log n}}\right)^{\alpha} \cdot n^{\beta} (\log n)^{\gamma} \cdot \sum_{k\in G_n} 1 \\
&\le \frac{C_1(\log n)^{\gamma}}{n^{1-\beta}} \cdot (1 + [C\sqrt{n\log\log n}]) \\
&\le \frac{C_2(\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0 \qquad \left(0 \le \beta < \frac{1}{2}\right)
\end{aligned}$$

iii) Again, by Lemma 2, Ch. I

$$n + 1 \le k \le n + [C\sqrt{n\log\log n}] .$$

If $k \in H_n$

$$\begin{aligned}
|\Sigma_3| &\le \sum_{k\in H_n} |a_{n,\left[\frac{\lambda_k(t)}{2}\right]}|\,|s_{\lambda_k(t)}| \\
&\le \frac{C_0}{n^{1-\alpha}} \sum_{k\in H_n} \left[\frac{\lambda_k(t)}{2}\right]^{-\alpha} k^{\beta} (\log k)^{\gamma} \\
&\le \frac{C_0}{n^{1-\alpha}} \sum_{k\in H_n} \frac{2^{\alpha}}{(k-2)^{\alpha}} \cdot k^{\beta} (\log k)^{\gamma} \\
&\le \frac{C_1}{n^{1-\alpha}} \sum_{k\in H_n} k^{-\alpha} k^{\beta} (\log k)^{\gamma} \\
&\le \frac{C_1}{n^{1-\alpha}} \cdot n^{-\alpha} (n + [C\sqrt{n\log\log n}])^{\beta} (\log(n + [C\sqrt{n\log\log n}]))^{\gamma}
\end{aligned}$$

$$\cdot \sum_{k\in H_n} 1$$

$$\leq \frac{C_2 n^{\beta}(\log n)^{\gamma}[C\sqrt{n}\log\log n]}{n}$$

$$\leq \frac{C_3(\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0\,.$$

This will complete the proof of this part once we establish (2.6): apply the Mean Value Theorem to $t \mapsto t^{-\alpha}$, getting r between k and $\left[\frac{\lambda_k(t)}{2}\right]$ such that

$$\left|\frac{1}{k^{\alpha}} - \frac{1}{\left[\frac{\lambda_k(t)}{2}\right]^{\alpha}}\right| = \frac{\alpha}{r^{\alpha+1}}\left|k - \left[\frac{\lambda_k(t)}{2}\right]\right|.$$

If $r \geq k$, then $\frac{1}{r^{\alpha+1}} \leq \frac{1}{k^{\alpha+1}}$; if $r \geq \left[\frac{\lambda_k(t)}{2}\right]$, then $r \geq \left[\frac{k}{2}\right] > \frac{k}{2} - 1 \geq \frac{k}{6}$, and $\frac{1}{r^{\alpha+1}} \leq \frac{6^{\alpha+1}}{k^{\alpha+1}}$.

In any case

$$\left|\frac{1}{k^{\alpha}} - \frac{1}{\left[\frac{\lambda_k(t)}{2}\right]^{\alpha}}\right| \leq \frac{C_1}{k^{\alpha+1}}\left|k - \left[\frac{\lambda_k(t)}{2}\right]\right|$$

$$\leq \frac{C_2}{k^{\alpha+1}}\sqrt{k\log\log k} = \frac{C_2\sqrt{\log\log k}}{k^{\alpha+\frac{1}{2}}}$$

2) I) Theorem 2.1 is satisfied:

a)

$$\frac{p_n P_{2n+1}}{p_{2n+1}P_n} \leq \frac{n^{\alpha}(2n+1)^{\alpha+1}(\alpha+1)}{(2n+1)^{\alpha}\cdot n^{\alpha+1}}$$

$$= \frac{(\alpha+1)(2n+1)}{n} \leq 3(\alpha+1)$$

b)

$$\frac{1}{P_n}\sum_{k=1}^{n} P_{2k}\left|\frac{p_k}{p_{2k}} - \frac{p_k}{p_{2k+1}}\right|$$

$$\leq \frac{\alpha+1}{n^{\alpha+1}}\sum_{k=1}^{n}(2k)^{\alpha+1}\left|\frac{k^{\alpha}}{(2k)^{\alpha}} - \frac{k^{\alpha}}{(2k+1)^{\alpha}}\right|$$

$$= \frac{(\alpha+1)2^{\alpha+1}}{n^{\alpha+1}} \sum_{k=1}^{n} k^{\alpha+1} \left| \left(\frac{1}{2}\right)^{\alpha} - \left(\frac{k}{2k+1}\right)^{\alpha} \right|$$
$$= \frac{(\alpha+1)2^{\alpha+1}}{n^{\alpha+1}} \sum_{k=1}^{n} k^{\alpha+1} \left| \frac{1}{2^{\alpha}} - \left(\frac{k}{2k+1}\right)^{\alpha} \right|$$
$$\leq \frac{(\alpha+1)2^{\alpha+1}}{n^{\alpha+1}} \sum_{k=1}^{n} k^{\alpha+1} \cdot \frac{3\alpha}{2^{\alpha+1}} \cdot \frac{1}{2k+1}$$
$$\leq \frac{3\alpha(\alpha+1)}{n^{\alpha+1}} \sum_{k=1}^{n} k^{\alpha} \leq 3\alpha(\alpha+1)$$

c)

$$\frac{1}{P_n} \sum_{k=1}^{n-1} P_{2k+1} \left| \frac{p_k}{p_{2k+1}} - \frac{p_{k+1}}{p_{2k+2}} \right|$$
$$\leq \frac{\alpha+1}{n^{\alpha+1}} \sum_{k=1}^{n-1} (2k+1)^{\alpha+1} \left| \frac{k^{\alpha}}{(2k+1)^{\alpha}} - \frac{(k+1)^{\alpha}}{(2k+2)^{\alpha}} \right|$$
$$= \frac{\alpha+1}{n^{\alpha+1}} \sum_{k=1}^{n-1} (2k+1)^{\alpha+1} \left| \left(\frac{k}{2k+1}\right)^{\alpha} - \frac{1}{2^{\alpha}} \right|$$
$$\leq \frac{\alpha+1}{n^{\alpha+1}} \cdot \frac{3\alpha}{2^{\alpha+1}} \sum_{k=1}^{n-1} (2k+1)^{\alpha+1} \cdot \frac{1}{2k+1}$$
$$= \frac{3\alpha(\alpha+1)}{2^{\alpha+1} n^{\alpha+1}} \sum_{k=1}^{n-1} (2k+1)^{\alpha}$$
$$\leq \frac{3\alpha(\alpha+1)}{2^{\alpha+1} n^{\alpha+1}} \cdot (2n-1)^{\alpha}(n-1)$$
$$= \frac{3\alpha(\alpha+1)}{2^{\alpha+1}} \left(2 - \frac{1}{n}\right)^{\alpha} \left(1 - \frac{1}{n}\right)$$
$$\leq \frac{3\alpha(\alpha+1)}{2}.$$

II) The Klee-Szüsz condition is satisfied

$$\sum_{k=2}^{2n+1} a_{n,\left[\frac{k}{2}\right]}^{2} s_k^2 \leq \frac{C_1}{n^{2\alpha+2}} \sum_{k=2}^{2n+1} \left[\frac{k}{2}\right]^{2\alpha} k^{2\beta} (\log k)^{2\gamma}$$
$$\leq \frac{C_2}{n^{2\alpha+2}} \sum_{k=2}^{2n+1} k^{2\alpha+2\beta} (\log k)^{2\gamma}$$

$$\leq \frac{C_2(\log(2n+1))^{2\gamma}(2n+1)^{2\alpha+2\beta+1}}{n^{2\alpha+2}}$$
$$\leq \frac{C_3(\log n)^{2\gamma} n^{2\alpha+2\beta+1}}{n^{2\alpha+2}}$$
$$= \frac{C_3(\log n)^{2\gamma}}{n^{1-2\beta}} \leq \frac{C_4}{n^r}$$

where $r > 0$ (this is possible: $0 \leq \beta < \frac{1}{2}$) and then

$$\sum_{n=1}^{\infty} \exp\left\{-\varepsilon\left(\sum_{k=2}^{\infty} a^2_{n,[\frac{k}{2}]} s_k^2\right)^{-1}\right\} \leq \sum_{n=1}^{\infty} \exp\left\{-\frac{\varepsilon n^r}{C_4}\right\} < +\infty$$

I) and II) prove that

$$\lim_{n\to\infty} \sum_{k=2}^{\infty} a_{n,\left[\frac{\lambda_k(t)}{2}\right]} s_{\lambda_k(t)} = s$$

almost surely, where (a_{nk}) is the Riesz mean (R, n^α), $\alpha \geq 0$.

Next, with t satisfying (1.4) and F_n, G_n, H_n as on p. 20, we have

$$\sum_{k=3}^{\infty}\left(a_{nk} - a_{n,\left[\frac{\lambda_k(t)}{2}\right]}\right) s_{\lambda_k(t)} = \sum_{k\in F_n} \cdots + \sum_{k\in G_n} \cdots - \sum_{k\in H_n}$$
$$= \Sigma_1 + \Sigma_2 - \Sigma_3$$

and the analogue of (2.6) is

$$\left|k^\alpha - \left[\frac{\lambda_k(t)}{2}\right]^\alpha\right| \leq C k^{\alpha-\frac{1}{2}}\sqrt{\log\log k} \quad k \geq 3\,. \tag{2.7}$$

i)

$$|\Sigma_1| \leq \sum_{k\in F_n} \left|a_{nk} - a_{n,\left[\frac{\lambda_k(t)}{2}\right]}\right| |s_{\lambda_k(t)}|$$
$$\leq \frac{C_0}{n^{\alpha+1}} \sum_{k\in F_n} \left|k^\alpha - \left[\frac{\lambda_k(t)}{2}\right]^\alpha\right| k^\beta (\log k)^\gamma$$
$$\leq \frac{C_1}{n^{\alpha+1}} \sum_{k\in F_n} \sqrt{\log\log k}\; k^{\alpha+\beta-\frac{1}{2}}(\log k)^\gamma$$

$$
\begin{aligned}
&\leq \frac{C_1(\log n)^{\gamma+\frac{1}{2}}}{n^{\alpha+1}} \sum_{k=1}^{n} k^{\alpha+\beta-\frac{1}{2}} \\
&\leq \frac{C_2(\log n)^{\gamma+\frac{1}{2}} n^{\alpha+\beta+\frac{1}{2}}}{n^{\alpha+1}} \\
&= \frac{C_2(\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0 \qquad \left(\beta < \frac{1}{2}\right)
\end{aligned}
$$

ii)

$$
\begin{aligned}
|\Sigma_2| &\leq \sum_{k \in G_n} |a_{nk}| \; |s_{\lambda_k(t)}| \\
&\leq \frac{C_1}{n^{\alpha+1}} \sum_{k \in G_n} k^{\alpha+\beta} (\log k)^{\gamma} \\
&\leq \frac{C_1 n^{\alpha+\beta} (\log n)^{\gamma}}{n^{\alpha+1}} \sum_{k \in G_n} 1 \\
&= \frac{C_1(\log n)^{\gamma}}{n^{1-\beta}} \cdot (1 + [C\sqrt{n \log\log n}]) \\
&\leq \frac{C_2(\log n)^{\gamma+\frac{1}{2}}\sqrt{n}}{n^{1-\beta}} = \frac{C_2(\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0
\end{aligned}
$$

iii)

$$
\begin{aligned}
|\Sigma_3| &\leq \sum_{k \in H_n} |a_{n,\left[\frac{\lambda_k(t)}{2}\right]}| \; |s_{\lambda_k(t)}| \\
&\leq \frac{C_1}{n^{\alpha+1}} \sum_{k \in H_n} \left[\frac{\lambda_k(t)}{2}\right]^{\alpha} k^{\beta} (\log k)^{\gamma} \\
&\leq \frac{C_1}{n^{\alpha+1}} \sum_{k \in H_n} (k + [C\sqrt{k \log\log k}])^{\alpha} k^{\beta} (\log k)^{\gamma} \\
&\leq \frac{C_2}{n^{\alpha+1}} \sum_{k \in H_n} k^{\alpha+\beta} (\log k)^{\gamma} \\
&\leq \frac{C_2}{n^{\alpha+1}} \cdot (n + [C\sqrt{n \log\log n}])^{\alpha+\beta} \\
&\qquad \cdot (\log(n + [C\sqrt{n \log\log n}])^{\gamma} \sum_{k \in H_n} 1
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C_3 n^{\alpha+\beta}(\log n)^{\gamma}}{n^{\alpha+1}} \cdot [C\sqrt{n \log\log n}] \\
&\le \frac{C_4 n^{\alpha+\beta+\frac{1}{2}}(\log n)^{\gamma+\frac{1}{2}}}{n^{\alpha+1}} \\
&= \frac{C_4(\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0
\end{aligned}
$$

This completes the proof of Theorem 2.2.

Unfortunately Theorem 2.2 does not apply to the Riesz mean $\left(R,\frac{1}{n}\right)$, that is, the logarithmic means. The problem lies in the fact that $\left(\frac{1}{2}a_{n,\left[\frac{k}{2}\right]}\right)$ is not stronger than (a_{nk}) because 1) b) of Theorem 2.1 does not hold:

$$
\begin{aligned}
\frac{1}{P_n}\sum_{k=1}^{n} P_{2k}\left|\frac{p_k}{p_{2k}} - \frac{p_k}{p_{2k+1}}\right| &\ge \frac{1}{1+\log n}\sum_{k=1}^{n}\frac{\log(2k+1)}{k} \\
&\ge \frac{(\log(n+1))^2}{2(1+\log n)}
\end{aligned}
$$

which is unbounded.

In this case we consider the continuous analogue (2.2) with $f(x) = \frac{1}{x}$. Then:

i) $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular: Since $f > 0$ and $\lim\limits_{x\to+\infty}\frac{f(x-\frac{1}{2})}{f(x)} = 1$ the regularity follows on account of the Corollary to Proposition 1.

ii) $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is stronger than (a_{nk}): The suprema on 2) b) and 2) c) in Theorem 2.1 are zero and 2) a) is satisfied because

$$
\frac{f(n)F(2n)}{f(2n)F(n)} \le \frac{2(1+\log 2n)}{\log(n+1)}
$$

which is bounded.

Theorem 2.3. If $\{s_n\}$ is $\left(R, \frac{1}{n}\right)$-summable to s, and for constants β, γ with $0 \leq \beta, \gamma < \frac{1}{2}$

$$\sup_{n\geq 2} \frac{|s_n|}{n^\beta (\log n)^\gamma} < +\infty \tag{2.8}$$

Then almost all subsequences of $\{s_n\}$ are $\left(R, \frac{1}{n}\right)$-summable to s.

Proof: We begin by verifying the Klee-Szüsz condition for $\left(\frac{1}{2} a_{n,\frac{k}{2}}\right)$

$$\begin{aligned}
\sum_{k=1}^{\infty} a^2_{n,\frac{k}{2}} s_k^2 &= \sum_{k=1}^{2n} a^2_{n,\frac{k}{2}} s_k^2 \\
&\leq \frac{C_0}{(\log(n+1))^2} \sum_{k=1}^{2n} \frac{k^{2\beta}(\log k)^{2\gamma}}{k^2} \\
&\leq \frac{C_0 (\log 2n)^{2\gamma}}{(\log(n+1))^2} \sum_{k=1}^{2n} \frac{1}{k^{2-2\beta}} \\
&\leq \frac{C_1 (\log n)^{2\gamma}}{(\log n)^2} \sum_{k=1}^{2n} \frac{1}{k^{2-2\beta}} \\
&\leq C_2 (\log n)^{2\gamma - 2}
\end{aligned}$$

because $2 - 2\beta > 1$ ($\beta < \frac{1}{2}$). Thus,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \exp\left\{ -\varepsilon \left(\sum_{k=1}^{\infty} a^2_{n,\frac{k}{2}} s_k^2 \right)^{-1} \right\} \\
&\quad \leq \sum_{n=1}^{\infty} \exp\left\{ -\varepsilon \frac{(\log n)^{2-2\gamma}}{C_2} \right\} < +\infty
\end{aligned}$$

because $2 - 2\gamma > 1$ ($\gamma < \frac{1}{2}$).

(Quick Proof: $\exp\{-\varepsilon (\log n)^r\} = \frac{1}{n^{\varepsilon(\log n)^{r-1}}}$. If $r > 1$ and $\varepsilon > 0$ is fixed, but otherwise arbitrary, $\varepsilon(\log n)^{r-1} > 2$ holds for sufficiently large n. So

$$\frac{1}{n^{\varepsilon(\log n)^{r-1}}} < \frac{1}{n^2}.$$

(We may also use Cauchy condensation principle.)

To finish we need to establish that

$$\lim_{n\to\infty}\sum_{k=3}^{\infty}(a_{nk}-a_{n,\frac{\lambda_k(t)}{2}})s_{\lambda_k(t)}=0$$

almost surely. Define, as on p. 20

$$F_n=\{k\mid 3\le k\le n\,,\ \lambda_k\le 2n\}$$

$$G_n=\{k\mid 3\le k\le n\,,\ \lambda_k>2n\}$$

$$H_n=\{k\mid k>n\,,\ \lambda_k\le 2n\}\qquad(n\ge 3)$$

By Lemma 2, Ch. I

$$k\in G_n\Rightarrow n-[C\sqrt{n\log\log n}]\le k\le n$$

$$k\in H_n\Rightarrow n+1\le k\le n+[C\sqrt{n\log\log n}]\,.$$

Again

$$\sum_{k=3}^{\infty}(a_{nk}-a_{n,\frac{\lambda_k(t)}{2}})s_{\lambda_k(t)}=\sum_{k\in F_n}\cdots+\sum_{k\in G_n}\cdots-\sum_{k\in H_n}\cdots$$
$$=\Sigma_1+\Sigma_2-\Sigma_3\,.$$

And to estimate Σ, we will need

$$\left|\frac{1}{k}-\frac{2}{\lambda_k}\right|\le\frac{C\sqrt{\log\log k}}{k^{\frac{3}{2}}}\qquad k\ge 3\,,\tag{2.9}$$

which is the analogue of (2.6) for $\alpha=1$ (and without [ ]). We omit its simple proof based on the Mean Value Theorem.

i)
$$|\Sigma_1|\le\sum_{k\in F_n}|a_{nk}-a_{n,\frac{\lambda_k(t)}{2}}|\,|s_{\lambda_k(t)}|$$
$$\le\frac{C_0}{\log(n+1)}\sum_{k\in F_n}\frac{\sqrt{\log\log k}}{k^{\frac{3}{2}}}\cdot k^{\beta}(\log k)^{\gamma}$$
$$\le\frac{C_0(\log n)^{\gamma+\frac{1}{2}}}{\log n}\sum_{k=1}^{n}\frac{1}{k^{\frac{3}{2}-\beta}}\qquad\left(\frac{3}{2}-\beta>1\right)$$
$$\le\frac{C_1}{(\log n)^{\frac{1}{2}-\gamma}}\to 0\qquad\left(\frac{1}{2}-\gamma>0\right)$$

ii)
$$
\begin{aligned}
|\Sigma_2| &\leq \sum_{k\in G_n} |a_{nk}|\ |s_{\lambda_k(t)}| \\
&\leq \frac{C_0}{\log n} \sum_{k\in G_n} \frac{k^\beta(\log k)^\gamma}{k} \\
&\leq \frac{C_0}{\log n} \cdot \frac{n^\beta(\log n)^\gamma}{n-[C\sqrt{n\log\log n}]} \cdot \sum_{k\in G_n} 1 \\
&= \frac{C_0 n^\beta(\log n)^\gamma(1+[C\sqrt{n\log\log n}])}{\log n(n-[C\sqrt{n\log\log n}])} \\
&\leq \frac{C_1 n^{\beta+\frac{1}{2}}(\log n)^{\gamma+\frac{1}{2}}}{n\log n} \\
&= \frac{C_1}{n^{\frac{1}{2}-\beta}(\log n)^{\frac{1}{2}-\gamma}} \to 0
\end{aligned}
$$

iii)
$$
\begin{aligned}
|\Sigma_3| &\leq \sum_{k\in H_n} |a_{n,\frac{\lambda_k(t)}{2}}|\ |s_{\lambda_k(t)}| \\
&\leq \frac{C_0}{\log n} \sum_{k\in H_n} \frac{k^\beta(\log k)^\gamma}{\lambda_k(t)} \\
&\leq \frac{C_0}{\log n} \sum_{k\in H_n} \frac{k^\beta(\log k)^\gamma}{k} \qquad (\lambda_k \geq k) \\
&\leq \frac{C_0}{\log n} \cdot \frac{(n+[C\sqrt{n\log\log n}])^\beta(\log(n+[C\sqrt{n\log\log n}]))^\gamma}{n} \cdot \sum_{k\in H_n} 1 \\
&= \frac{C_0(n+[C\sqrt{n\log\log n}])^\beta(\log(n+[C\sqrt{n\log\log n}]))^\gamma([C\sqrt{n\log\log n}])}{n\log n} \\
&\leq \frac{C_1 n^{\beta+\frac{1}{2}}(\log n)^{\gamma+\frac{1}{2}}}{n\log n} \\
&= \frac{C_1}{n^{\frac{1}{2}-\beta}(\log n)^{\frac{1}{2}-\gamma}} \to 0\,.
\end{aligned}
$$

This completes the proof.

Remarks: The growth condition (2.5) is motivated by the fact that if $\{s_n\}$ is $\left(R, \frac{1}{n^\alpha}\right)$-summable ($0 \leq \alpha < 1$) or (R, n^α)-summable ($\alpha \geq 0$), then $\lim_{n\to\infty} \frac{s_n}{n} = 0$, i.e., $\{s_n\}$ cannot grow faster than n. If $\{s_n\}$ is $\left(R, \frac{1}{n}\right)$-summable, then $\lim_{n\to\infty} \frac{s_n}{n \log n} = 0$ and $\{s_n\}$ cannot grow faster than $n \log n$. These growth restrictions on $\{s_n\}$ stem from a more general fact: If $\{s_n\}$ is (R, p_n)-summable, and $\lim_{n\to\infty} \frac{P_n}{P_{n+1}} = 1$ (which is equivalent to $\lim_{n\to\infty} \frac{p_n}{P_n} = 0$) then $\lim_{n\to\infty} \frac{p_n s_n}{P_n} = 0$. The proof is simple: Let

$$t_n = \frac{1}{P_n} \sum_{k=1}^{n} p_k s_k \to t \qquad \text{as } n \to \infty .$$

Then,

$$s_n = \frac{P_n t_n}{p_n} - \frac{P_{n-1} t_{n-1}}{p_n}$$

(the inverse) which implies

$$\frac{p_n s_n}{P_n} = t_n - \frac{P_{n-1}}{P_n} t_{n-1} \to 0 .$$

CHAPTER III. THE NØRLUND MEANS

Let $\{p_n\}$ be a sequence of nonnegative real numbers with $p_1 > 0$. The Nørlund mean (N, p_n) associated with this sequence is the triangular matrix

$$a_{nk} = \begin{cases} \frac{p_{n-k+1}}{P_n}, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{3.1}$$

where $P_n = \sum_{k=1}^{n} p_k$. It is known that (N, p_n) is regular iff $\lim_{n\to\infty} \frac{p_n}{P_n} = 0$ ([5]). We again consider the continuous case

$$a(n,x) = \begin{cases} \frac{f(n-x+1)}{F(n)}, & 0 \le x \le n \\ 0, & x > n \end{cases} \tag{3.2}$$

where $f:(0,+\infty) \to \mathbb{R}$ has $f(1) > 0$, $f(k) \ge 0$, $k = 2, 3, \ldots$ and $F(n) = \sum_{k=1}^{n} f(k)$. This (a_{nk}) is regular iff $\lim_{n\to\infty} \frac{f(n)}{F(n)} = 0$.

We now turn to find the inverse. Let $p(x) = \sum_{j=0}^{\infty} p_{j+1} x^j$. This power series has a reciprocal as a formal power series because $p(0) = p_1 > 0$

Let $q(x) = \frac{1}{p(x)} = \sum_{j=0}^{\infty} q_j x^j$. Therefore,

$$\begin{cases} (*) & \sum_{j=0}^{m} q_{m-j} p_{j+1} = \delta_{m,0} \\ (**) & \sum_{j=0}^{m} p_{m-j+1} q_j = \delta_{m,0} \end{cases} \quad m = 0, 1, 2, \ldots \tag{3.3}$$

where $\delta_{m,0}$ is Kronecker's Delta. Then $(a^{\#}_{nk})$, the inverse of (a_{nk}), is given by

$$a^{\#}_{nk} = \begin{cases} q_{n-k} P_k, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{3.4}$$

The verification of this is very simple:

If $k > n$, then

$$\sum_{j=1}^{\infty} a_{nj} a_{jk}^{\#} = \sum_{j=1}^{\infty} a_{nj}^{\#} a_{jk} = 0$$

by triangularity.

If $1 \leq k \leq n$, then

$$\begin{aligned}
\sum_{j=1}^{\infty} a_{nj} a_{jk}^{\#} &= \sum_{j=k}^{n} a_{nj} a_{jk}^{\#} = \sum_{j=k}^{n} \frac{p_{n-j+1}}{P_n} \cdot q_{j-k} P_k \\
&= \frac{P_k}{P_n} \sum_{j=k}^{n} p_{n-j+1} q_{j-k} = \frac{P_k}{P_n} \sum_{j=0}^{n-k} p_{(n-k)-j+1} q_j \\
&= \frac{P_k}{P_n} \cdot \delta_{n-k,0} = \delta_{n-k,0} = \delta_{nk}
\end{aligned}$$

by (4.2) $(**)$ with $m = n - k$. Similarly

$$\begin{aligned}
\sum_{j=1}^{\infty} a_{nj}^{\#} a_{jk} &= \sum_{j=k}^{n} a_{nj}^{\#} a_{jk} = \sum_{j=k}^{n} q_{n-j} P_j \cdot \frac{p_{j-k+1}}{P_j} \\
&= \sum_{j=k}^{n} q_{n-j} p_{j-k+1} = \sum_{j=0}^{n-k} q_{(n-k)-j} p_{j+1} = \delta_{nk}
\end{aligned}$$

and, without fear of confusion, $\sum_{j=0}^{\infty} q_j x^j$ will denote the reciprocal series in the continuous case as well.

Now we pause to give some examples of Nørlund means with their corresponding inverses.

Examples: 1) Fix $\alpha > 0$ and put $p_n = \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha)\Gamma(n)}$ where Γ is Euler's gamma function. Then (N, p_n) is (C, α), the Cesàro means of order α because $P_n = \sum_{k=1}^{n} p_k = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)\Gamma(n)}$ and it is regular because $\frac{p_n}{P_n} =$

$\frac{\alpha}{n+\alpha-1} \to 0$. For the inverse matrix we have

$$p(x) = \sum_{j=0}^{\infty} \frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)} x^j = \frac{1}{(1-x)^\alpha}, \qquad |x| < 1$$

so $q(x) = (1-x)^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} x^j$, $|x| < 1$, and $q_j = (-1)^j \binom{\alpha}{j} = \frac{(-1)^j \Gamma(\alpha+1)}{\Gamma(\alpha-j+1)\Gamma(j+1)}$.

2) Fix $N = 2, 3, \ldots$. Define $p_1 = p_2 = \cdots = p_N = 1$, $p_k = 0$, $k \geq N+1$. Then

$$P_n = \begin{cases} n, & 1 \leq n \leq N \\ N, & n \geq N+1 \end{cases}$$

and $\sum_{k=1}^{\infty} a_{nk} s_k = \frac{1}{N}(s_{n-N+1} + \cdots + s_n)$ for $n \geq N$.

This method is evidently regular. As to the inverse $p(x) = \sum_{j=0}^{N-1} p_{j+1} x^j = 1 + x^2 + \cdots + x^{N-1}$ and

$$q(x) = \frac{1}{p(x)} = \frac{1}{1 + x^2 + \cdots + x^{N-1}} = \frac{1-x}{1-x^N}$$
$$= (1-x) \sum_{j=0}^{\infty} x^{NJ} = \sum_{j=0}^{\infty} x^{NJ} - \sum_{j=0}^{\infty} x^{NJ+1}$$

for $|x| < 1$.

Thus

$$q_{NJ} = 1, \quad q_{NJ+1} = -1, \quad J = 0, 1, 2, \ldots$$

and $q_k = 0$ for all others so certainly $\sum_{r=0}^{N-1} q_{NJ+r} = 0$, $j = 0, 1, \ldots$ which implies $\sum_{r=0}^{N-1} q_{m+r} = 0$, $m = 0, 1, \ldots$. (Write $m = NJ + s$, $0 \leq s \leq N-1$.

Then

$$\begin{aligned}\sum_{r=0}^{N-1} q_{m+r} &= \sum_{r=0}^{N-1} q_{NJ+r+s} = \sum_{r=s}^{s+N-1} q_{NJ+r} \\ &= \sum_{r=0}^{s+N-1} q_{NJ+r} - \sum_{r=0}^{s-1} q_{NJ+r} \\ &= \sum_{r=N}^{s+N-1} q_{NJ+r} - \sum_{r=0}^{s-1} q_{NJ+r} \qquad \left(\sum_{r=0}^{N-1} q_{NJ+r} = 0\right) \\ &= \sum_{r=0}^{s-1} q_{NJ+N+r} - \sum_{r=0}^{s-1} q_{NJ+r} \\ &= \sum_{r=0}^{s-1} (q_{N(J+1)+r} - q_{NJ+r}) = 0\,.)\end{aligned}$$

And the sum of N consecutive q_j's is 0 and so it is the sum of $2N$ consecutive of them.

Now we investigate when is $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ or $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ stronger than (a_{nk}) where (a_{nk}) is a Nørlund mean. Before that, we need the following:

Lemma: 1) The radius of convergence of $P(x) = \sum_{j=0}^{\infty} P_{j+1}x^j$ is 1.

2) $p(x) = (1-x)P(x)$ so the radius of convergence of $p(x)$ is ≥ 1. It is, in general, $\frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\frac{p_{n+1}}{P_{n+1}}}}$.

$$3) \quad \left.\begin{aligned} \sum_{j=0}^{m} P_{m-j+1}q_j = 1 \\ \sum_{j=0}^{m} q_{m-j}P_{j+1} = 1 \end{aligned}\right\} \quad m = 0,1,2,\ldots$$

Proof: 1) $\frac{P_{j+2}}{P_{j+1}} = \frac{P_{j+2}}{P_{j+2}-p_{j+2}} = \frac{1}{1-\frac{p_{j+2}}{P_{j+2}}} \to 1$ as $j \to \infty$, so $\lim\limits_{j\to\infty} \sqrt[j]{P_{j+1}} = 1$.

2) If $|x| < 1$

$$\begin{aligned}(1-x)P(x) &= (1-x)\sum_{j=0}^{\infty} P_{j+1}x^j \\ &= \sum_{j=0}^{\infty} P_{j+1}x^j - \sum_{j=0}^{\infty} P_{j+1}x^{j+1} \\ &= P_1 + \sum_{j=1}^{\infty}(P_{j+1} - P_j)x^j \\ &= p_1 + \sum_{j=1}^{\infty} p_{j+1}x^j = \sum_{j=0}^{\infty} p_{j+1}x^j = p(x)\end{aligned}$$

which implies the statement about the radius of convergence. Finally

$$\begin{aligned}\limsup_{j\to\infty} \sqrt[j]{p_{j+1}} &= \limsup_{j\to\infty}\left(\sqrt[j]{P_{j+1}} \cdot \sqrt[j]{\frac{p_{j+1}}{P_{j+1}}}\right) \\ &= \limsup_{j\to\infty} \sqrt[j]{\frac{p_{j+1}}{P_{j+1}}} \quad (\leq 1)\end{aligned}$$

3) Recall that

$$q(x) = \frac{1}{p(x)} = \sum_{j=0}^{\infty} q_j x^j \,.$$

Thus, if $x \neq 0$ is sufficiently small (so that $\sum_{j=0}^{\infty} |q_j|\ |x|^j$ converges)

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m} P_{m-j+1}q_j\right)x^m &= \left(\sum_{j=0}^{\infty} P_{j+1}x^j\right)\left(\sum_{j=0}^{\infty} q_j x^j\right) \\ &= P(x)q(x) = \frac{P(x)}{p(x)} = \frac{1}{1-x} \\ &= \sum_{m=0}^{\infty} x^m \,.\end{aligned}$$

That is $\sum_{j=0}^{m} P_{m-j+1}q_j = 1$, $m = 0,1,2,\ldots$ and $\sum_{j=0}^{m} q_{m-j}P_{j+1} = 1$, $m = 0,1,2,\ldots$, as well.

Corollary: If $(a_{nk}^{\#})$ is the inverse of the Nørlund mean (a_{nk}), then $\sum_{k=1}^{n} a_{nk}^{\#} = 1$, $n = 1,2,\ldots$.

Proof:

$$\sum_{k=1}^{n} a_{nk}^{\#} = \sum_{k=1}^{n} q_{n-k} P_k$$
$$= \sum_{k=0}^{n-1} q_{(n-1)-k} P_{k+1} = 1 .$$

Let us return to the strength part. The (n,k) entry of the product $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)(a_{nk}^{\#})$ is zero when $k \geq 2n+2$. If $1 \leq k \leq 2n+1$,

$$\sum_{j=2}^{\infty} \frac{1}{2} a_{n,[\frac{j}{2}]} a_{jk}^{\#} = \frac{1}{2} \sum_{j=k\vee 2}^{2n+1} a_{n,[\frac{j}{2}]} a_{jk}^{\#}$$
$$(k \vee 2 = \max\{k,2\})$$
$$= \frac{P_k}{2P_n} \sum_{j=k\vee 2}^{2n+1} p_{n-[\frac{j}{2}]+1} q_{j-k}$$

and the product is

$$c_{nk} = \begin{cases} \frac{P_k}{2P_n} \sum\limits_{j=k\vee 2}^{2n+1} p_{n-[\frac{j}{2}]+1} q_{j-k} , & 1 \leq k \leq 2n+1 \\ 0 , & k \geq 2n+2 \end{cases} \tag{3.5}$$

Likewise, the (n,k) entry of $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)(a_{nk}^{\#})$ is zero when $k \geq 2n+1$ and if $1 \leq k \leq 2n$, we have

$$\sum_{j=1}^{2n} \frac{1}{2} a_{n,\frac{j}{2}} a_{jk}^{\#} = \frac{F(k)}{2F(n)} \sum_{j=k}^{2n} f\left(n - \frac{j}{2} + 1\right) q_{j-k}$$

and the product is

$$d_{nk} = \begin{cases} \frac{F(k)}{2F(n)} \sum\limits_{j=k}^{2n} f\left(n - \frac{j}{2} + 1\right) q_{j-k} , & 1 \leq k \leq 2n \\ 0 , & k \geq 2n+1 \end{cases} \tag{3.5'}$$

for the continuous case.

Before continuing, we need to establish the conditions for the regularity of $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$. We have

Proposition 1. If (a_{nk}) is a regular Nørlund mean given by a map as in (3.2), then $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular iff the following three conditions are satisfied:

a) $\lim\limits_{n\to\infty} \dfrac{f\left(n+\frac{1}{2}\right)}{\sum\limits_{k=1}^{n} f(k)} = 0$

b) $\sup\limits_{n\geq 1} \dfrac{\sum\limits_{k=1}^{n} \left|f\left(k+\frac{1}{2}\right)\right|}{\sum_{k=1}^{n} f(k)} < +\infty$

(Recall that $f(1) > 0\,,\ f(k) \geq 0\,,\ k = 2,3,\dots)$

c) $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=1}^{n} f\left(k+\frac{1}{2}\right)}{\sum\limits_{k=1}^{n} f(k)} = 1\,.$

If $f \geq 0$, c) implies b) and then only a) and c) need to be proved.

Proof: We have

$$\frac{1}{2}a_{n,\frac{k}{2}} = \begin{cases} \frac{f(n-\frac{k}{2}+1)}{2F(n)}\,, & 1 \leq k \leq 2n \\ 0\,, & k \geq 2n+1 \end{cases}$$

where $F(n) = \sum\limits_{k=1}^{n} f(k)$. Observe that $\lim\limits_{n\to\infty} \frac{F(n-1)}{F(n)} = 1$ because $\lim\limits_{n\to\infty} \frac{f(n)}{F(n)} = 0$ (regularity of (a_{nk})) so $\lim\limits_{n\to\infty} \frac{F(n-j)}{F(n)} = 1$ for any fixed $j = 0,1,2,\dots$

a) If $\lim\limits_{n\to\infty} \frac{1}{2}a_{n,\frac{k}{2}} = 0$ for $k = 1,2,\dots$ then, in particular, for $k=1$ so $\frac{f(n+\frac{1}{2})}{F(n)} = 2\cdot\frac{1}{2}a_{n,\frac{1}{2}} \to 0$ as $n\to\infty$.

If $\lim_{n\to\infty} \frac{f(n+\frac{1}{2})}{F(n)} = 0$, pick $k = 2j+1$, $j = 0, 1, \ldots$. Then, for large n,

$$\begin{aligned}\frac{1}{2}a_{n,\frac{k}{2}} = \frac{1}{2}a_{n,j+\frac{1}{2}} &= \frac{f\left(n - \left(j+\frac{1}{2}\right)+1\right)}{F(n)} \\ &= \frac{f\left(n-j+\frac{1}{2}\right)}{F(n)} = \frac{f\left(n-j+\frac{1}{2}\right)}{F(n-j)} \cdot \frac{F(n-j)}{F(n)} \\ &\to 0 \cdot 1 = 0\end{aligned}$$

as $n \to \infty$. Now pick $k = 2j$, $j = 1, 2, \ldots$. Then $\frac{1}{2}a_{n,\frac{k}{2}} = \frac{1}{2}a_{n,j} \to 0$ as $n \to \infty$ because (a_{nk}) is regular. This proves that a) is equivalent to $\lim_{n\to\infty} \frac{1}{2}a_{n,\frac{k}{2}} = 0$ for $k = 1, 2, \ldots$.

b) $$\begin{aligned}\sum_{k=1}^{2n} \frac{1}{2}\left|a_{n,\frac{k}{2}}\right| &= \frac{1}{2F(n)} \sum_{k=1}^{2n} \left|f\left(n - \frac{k}{2} + 1\right)\right| \\ &= \frac{1}{2F(n)} \sum_{k=1}^{n} |f(n-k+1)| + \frac{1}{2F(n)} \sum_{k=1}^{n} \left|f\left(n-k+\frac{3}{2}\right)\right| \\ &= \frac{1}{2} + \frac{1}{2F(n)} \sum_{k=1}^{n} \left|f\left(k+\frac{1}{2}\right)\right| \\ &= \frac{1}{2} + \frac{\sum_{k=1}^{n} \left|f\left(k+\frac{1}{2}\right)\right|}{2F(n)}\end{aligned}$$

so b) is equivalent to $\sup_{n\geq 1} \frac{1}{2} \sum_{k=1}^{2n} \left|a_{n,\frac{k}{2}}\right| < +\infty$.

c) As in b)

$$\sum_{k=1}^{2n} \frac{1}{2}a_{n,\frac{k}{2}} = \frac{1}{2} + \frac{\sum_{k=1}^{n} f\left(k+\frac{1}{2}\right)}{2F(n)}$$

so c) is equivalent to

$$\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{1}{2}a_{n,\frac{k}{2}} = 1.$$

Thus, a), b) and c) are equivalent to the Silverman-Toeplitz conditions for regularity.

Corollary. If $f \geq 0$, $f(k) > 0$ for $k = 1, 2, \ldots$, $\lim\limits_{n\to\infty} \frac{f(n+\frac{1}{2})}{f(n)} = 1$ and $\sum\limits_{k=1}^{\infty} f(k) = +\infty$, then $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular.

Proof: Only a) and c) need to be verified. Since the Nørlund Mean $(N, f(n))$ is assumed regular we have $\lim\limits_{n\to\infty} \frac{f(n)}{\sum_{k=1}^{n} f(k)} = 0$. then

$$\frac{f\left(n+\frac{1}{2}\right)}{\sum\limits_{k=1}^{n} f(k)} = \frac{f\left(n+\frac{1}{2}\right)}{f(n)} \cdot \frac{f(n)}{\sum\limits_{k=1}^{n} f(k)} \to 1 \cdot 0 = 0$$

as $n \to \infty$, which proves a).

As to c), Lemma 1 of Ch. II implies that $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n} f(k+\frac{1}{2})}{\sum_{k=1}^{n} f(k)} = 1$ which establishes c).

Theorem 3.1 1) (c_{nk}) is regular iff

a) For each $k = 1, 2, 3, \ldots$ the sequence $\{q_{2j-k} + q_{2j-k+1}\}_{j=1}^{\infty}$ is (N, p_n) summable to 0 (here we have defined $q_m = 0$ when $m \leq -1$). This holds in particular if $\{q_j + q_{j+1}\}_{j=1}^{\infty}$ converges to 0.

b) The following conditions are met:

i)
$$\sup_{n\geq 1} \frac{P_{2n+1}}{P_n} < +\infty$$

ii)
$$\sup_{n\geq 1} \frac{1}{P_n} \sum_{k=1}^{n} P_{2k} \left| \sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right| < +\infty$$

iii) $$\sup_{n\geq 2}\frac{1}{P_n}\sum_{k=1}^{n-1}P_{2k+1}$$
$$\left|p_{n-k+1}q_0+\sum_{j=k+1}^{n}p_{n-j+1}(q_{2j-2k-1}+q_{2j-2k})\right|<+\infty$$

2) (d_{nk}) is regular iff (we assume that $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular).

a) For each $k=1,2,\ldots$.

$$\lim_{n\to\infty}\frac{1}{F(n)}\left[\sum_{j=1}^{n}f(n-j+1)q_{2j-k}+\sum_{j=1}^{n}f\left(n-j+\frac{3}{2}\right)q_{2j-k-1}\right]=0$$

(Here $q_m=0$ if $m\leq -1$).

This holds in particular if, for each $k=1,2,\ldots$, $\{q_{2j-k}\}_{j=1}^{\infty}$ is $(N,f(n))$-summable to 0 and $\{q_{2j-k-1}\}_{j=1}^{\infty}$ is $\left(N,f\left(n+\frac{1}{2}\right)\right)$-summable to 0. It certainly holds if $\lim_{j\to\infty}q_j=0$.

b) $$\sup_{n\geq 1}\frac{1}{F(n)}\sum_{k=1}^{2n}F(k)\left|\sum_{j=k}^{2n}f\left(n-\frac{j}{2}+1\right)q_{j-k}\right|<+\infty$$

Proof: 1) First we establish

$$\sum_{k=1}^{2n+1}c_{nk}=1\,,\qquad n=1,2,\ldots$$

recall that $\sum\limits_{k=1}^{j} a_{jk}^{\#} = 1$ for $j = 1, 2, \ldots$, and so

$$\begin{aligned}\sum_{k=1}^{2n+1} c_{nk} &= \frac{1}{2}\sum_{k=1}^{2n+1}\sum_{j=k\vee 2}^{2n+1} a_{n,[\frac{j}{2}]}a_{jk}^{\#}\\ &= \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}a_{j,1}^{\#} + \frac{1}{2}\sum_{k=2}^{2n+1}\sum_{j=k}^{2n+1} a_{n,[\frac{j}{2}]}a_{jk}^{\#}\\ &= \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}a_{j,1}^{\#} + \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}\sum_{k=2}^{j} a_{jk}^{\#}\\ &= \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}\sum_{k=1}^{j} a_{jk}^{\#} = \frac{1}{2}\sum_{j=2}^{2n+1} a_{n,[\frac{j}{2}]}\\ &= \sum_{j=1}^{n} a_{nj} = \frac{1}{P_n}\sum_{j=1}^{n} p_{n-j+1} = 1\end{aligned}$$

because $P_n = \sum\limits_{j=1}^{n} p_j$.

a) $c_{n,1} = \frac{p_1}{2P_n}\sum\limits_{j=2}^{2n+1} p_{n-[\frac{j}{2}]+1}q_{j-1} = \frac{p_1}{2P_n}\sum\limits_{j=1}^{n} p_{n-j+1}(q_{2j-1}+q_{2j}) \to 0$ as $n \to \infty$ iff $\{q_{2j-1}+q_{2j}\}_{j=1}^{\infty}$ is (N, p_n)-summable to 0.

$$\begin{aligned}c_{n,2k} &= \frac{P_{2k}}{2P_n}\sum_{j=2k}^{2n+1} p_{n-[\frac{j}{2}]+1}q_{j-2k}\\ &= \frac{P_{2k}}{2P_n}\left[\sum_{j=k}^{n} p_{n-j+1}q_{2j-2k} + \sum_{j=k}^{n} p_{n-j+1}q_{2j-2k+1}\right]\\ &= \frac{P_{2k}}{2P_n}\sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k}+q_{2j-2k+1})\\ &= \frac{P_{2k}}{2P_n}\sum_{j=1}^{n} p_{n-j+1}(q_{2j-2k}+q_{2j-2k+1})\\ &\qquad (q_j = 0 \text{ if } j \le -1)\end{aligned}$$

which tends to 0 as $n \to \infty$, iff $\{q_{2j-2k}+q_{2j-2k+1}\}_{j=1}^{\infty}$ is (N, p_n)-summable to 0.

Finally $(k \geq 1)$

$$c_{n,2k+1} = \frac{p_{2k+1}}{2P_n} \sum_{j=2k+1}^{2n+1} p_{n-[\frac{j}{2}]+1} q_{j-2k+1}$$

$$= \frac{p_{2k+1}}{2P_n} \left[\sum_{j=k+1}^{n} p_{n-j+1} q_{2j-2k-1} + \sum_{j=k}^{n} p_{n-j+1} q_{2j-2k} \right]$$

$$= \frac{p_{2k+1}}{2P_n} \sum_{j=1}^{n} p_{n-j+1} (q_{2j-2k-1} + q_{2j-2k}) \to 0$$

as $n \to \infty$ iff $\{q_{2j-2k-1} + q_{2j-2k}\}_{j=1}^{\infty}$ is (N, p_n)-summable to 0.

b) Finally we need

$$\sup_{n\geq 1} \sum_{k=1}^{2n+1} |c_{nk}| < +\infty .$$

This is equivalent to: i) $\sup_{n\geq 1} |c_{n,2n+1}| < +\infty$,

ii) $\sup_{n\geq 1} \sum_{k=1}^{n} |c_{n,2k}| < +\infty$ and

iii) $\sup_{n\geq 2} \sum_{k=1}^{n-1} |c_{n,2k+1}|$

(If we merely want $\sum_{k=1}^{2n+1} |c_{nk}|$ to be bounded, we must add $\sup_{n\geq 1} |c_{n,1}| < +\infty$. But (c_{nk}) regular implies this because $c_{n,1} \to 0$.) Now

i) $|c_{n,2n+1}| = \left| \frac{P_{2n+1} p_1 q_0}{2P_n} \right| = \frac{P_{2n+1}}{2P_n}$ $(p_1 q_0 = 1)$

ii) $\sum_{k=1}^{n} |c_{n,2k}| = \frac{1}{2P_n} \sum_{k=1}^{n} P_{2k} \left| \sum_{j=k}^{n} p_{n-j+1} (q_{2j-2k} + q_{2j-2k+1}) \right|$

iii) $$\sum_{k=1}^{n-1} |c_{n,2k+1}| = \frac{1}{2P_n} \sum_{k=1}^{n-1} P_{2k+1}$$

$$\cdot \left| \sum_{j=k+1}^{n} p_{n-j+1} q_{2j-2k-1} + \sum_{j=k}^{n} p_{n-j+1} q_{2j-2k} \right|$$

$$= \frac{1}{2P_n} \sum_{k=1}^{n-1} P_{2k+1} \left| p_{n-k+1} q_0 + \sum_{j=k+1}^{n} p_{n-j+1} (q_{2j-2k-1} + q_{2j-2k}) \right|$$

This completes the proof.

2) First, $\lim_{n\to\infty} \sum_{k=1}^{2n} d_{nk} = 1$:

$$\sum_{k=1}^{2n} d_{nk} = \frac{1}{2} \sum_{k=1}^{2n} \sum_{j=k}^{2n} a_{n,\frac{j}{2}} a_{jk}^{\#}$$
$$= \frac{1}{2} \sum_{j=1}^{2n} a_{n,\frac{j}{2}} \sum_{k=1}^{j} a_{jk}^{\#}$$
$$= \frac{1}{2} \sum_{j=1}^{2n} a_{n,\frac{j}{2}} \to 1$$

as $n \to \infty$ due to the assumed regularity of $\left(\frac{1}{2} a_{n,\frac{k}{2}}\right)$ (we have again used that $\sum_{k=1}^{j} a_{jk}^{\#} = 1$, $j = 1, 2, \ldots$).

a) We have that $\lim_{n\to\infty} d_{nk} = 0$ for each $k = 1, 2, \ldots$ iff

$$\lim_{n\to\infty} \frac{1}{F(n)} \sum_{j=k}^{2n} f\left(n - \frac{j}{2} + 1\right) q_{j-k} = 0$$

for each $k = 1, 2, \ldots$. Now, since $q_j = 0$, if $j \le -1$, we get

$$\sum_{j=k}^{2n} f\left(n - \frac{j}{2} + 1\right) q_{j-k} = \sum_{j=1}^{2n} f\left(n - \frac{j}{2} + 1\right) q_{j-k}$$
$$= \sum_{j=1}^{n} f(n - j + 1) q_{2j-k}$$
$$+ \sum_{j=1}^{n} f\left(n - j + \frac{j}{2}\right) q_{2j-k-1}$$

So $\lim_{n\to\infty} d_{nk} = 0$, $k = 1, 2, \ldots$ is equivalent to

$$\lim_{n\to\infty} \frac{1}{F(n)} \left[\sum_{j=1}^{n} f(n - j + 1) q_{2j-k} + \sum_{j=1}^{n} f\left(n - j + \frac{j}{2}\right) q_{2j-k-1} \right] = 0$$

for $k = 1, 2, \ldots$.

If $\{q_{2j-k}\}_{j=1}^{\infty}$ is $(N, f(n))$-summable to 0, $k = 1, 2, \ldots$, this is

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} f(n-j+1) q_{2j-k}}{F(n)} = 0$$

and if $\{q_{2j-k-1}\}_{j=1}^{\infty}$ is $\left(N, f\left(n+\frac{1}{2}\right)\right)$-summable to 0, $k = 1, 2, \ldots$, this is

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} f\left(n-j+\frac{1}{2}+1\right) q_{2j-k-1}}{\sum_{j=1}^{n} f\left(j+\frac{1}{2}\right)} = 0\,,$$

but since $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular Proposition 1) c) yields

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f\left(k+\frac{1}{2}\right)}{\sum_{k=1}^{n} f(k)} = 1\,,$$ so the limit above is just

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{j} f\left(n-j+\frac{3}{2}\right) q_{2j-k-1}}{F(n)} = 0$$

because $F(n) = \sum_{k=1}^{n} f(k)$. If $\lim_{j\to\infty} q_j = 0$ then the two conditions above obtain since both $(N, f(n))$ and $\left(N, f\left(n+\frac{1}{2}\right)\right)$ are regular Nørlund means ($\left(N, f\left(n+\frac{1}{2}\right)\right)$ due to the regularity of $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$).

b) This is just the Silverman-Toeplitz condition

$$\sup_{n\geq 1} \sum_{k=1}^{2n} |d_{nk}| < +\infty\,.$$

Conditions i), ii), iii) and 2) b) of this theorem are difficult to verify in practice because we must have complete knowledge of the q_j's. Even in simple cases, to find these coefficients is no simple matter: If $p_k = \frac{1}{k}$, then

$$p(x) = \sum_{k=0}^{\infty} \frac{x^k}{k+1} = \frac{1}{x}\log\left(\frac{1}{1-x}\right)\,, \qquad |x| < 1$$

and we need $q(x) = \frac{1}{p(x)} = \frac{x}{\log(\frac{1}{1-x})} = \sum_{j=0}^{\infty} q_j x^j$.

Before considering our next theorem, we will need the following

Lemma 1.

a)
$$\binom{a-1}{c-1}(a-b) \geq \binom{a}{c} - \binom{b}{c} \geq \binom{b}{c-1}(a-b)$$
$$a \geq b \geq c \geq 1$$

b)
$$\sum_{j=0}^{m} \binom{\ell - j + 2m}{2m} \left(\binom{2m+1}{2j+1} - \binom{2m+1}{2j} \right) \geq m4^m$$
$$m = 1, 2, \ldots, \qquad \ell \geq m$$

c)
$$\sum_{j=0}^{m} \binom{\ell - j + 2m + 1}{2m+1} \left(\binom{2m+2}{2j+1} - \binom{2m+2}{2j} \right)$$
$$\geq (2m+1)(4^m - m - 1) + \binom{3m+1}{2m+1}$$
$$m = 0, 1, 2, \ldots, \qquad \ell \geq m$$

Proof: a) $\binom{n}{k} = \sum_{j=k-1}^{n-1} \binom{j}{k-1}$ so

$$\binom{a}{c} - \binom{b}{c} = \sum_{k=c-1}^{a-1} \binom{k}{c-1} - \sum_{k=c-1}^{b-1} \binom{k}{c-1}$$
$$= \sum_{k=b}^{a-1} \binom{k}{c-1} \geq \sum_{k=b}^{a-1} \binom{b}{c-1} = \binom{b}{c-1}(a-b)$$

because $\binom{k}{c-1}$ increases as k does, c fixed. Similarly
$\binom{a}{c} - \binom{b}{c} \leq \binom{a-1}{c-1}(a-b)$.

b) Let

$$
\begin{aligned}
S &= \sum_{j=0}^{m} \binom{\ell - j + 2m}{2m} \left(\binom{2m+1}{2j+1} - \binom{2m+1}{2j} \right) \\
&= \frac{1}{m+1} \sum_{j=0}^{m} \binom{\ell - j + 2m}{2m} \binom{2m+2}{2j+1} (m - 2j) \\
&= \frac{1}{m+1} \sum_{j=0}^{m} \binom{\ell + j + m}{2m} \binom{2m+2}{2j+1} (2j - m)
\end{aligned}
$$

(after $j \mapsto m - j$) so we can write

$$
\begin{aligned}
S &= \frac{1}{2(m+1)} \sum_{j=0}^{m} \left[\binom{\ell - m + 2m}{2m} - \binom{\ell + j + m}{2m} \right] \binom{2m+2}{2j+1} (m - 2j) \\
&= \frac{1}{2(m+1)} \sum_{j=0}^{[\frac{m}{2}]} \left[\binom{\ell - j + 2m}{2m} - \binom{\ell + j + m}{2m} \right] \binom{2m+2}{2j+1} (m - 2j) \\
&+ \frac{1}{2(m+1)} \sum_{j=[\frac{m}{2}]+1}^{m} \left[\binom{\ell + j + m}{2m} - \binom{\ell - j + 2m}{2m} \right] \binom{2m+2}{2j+1} (2j - m) \\
&= \frac{S_1}{2(m+1)} + \frac{S_2}{2(m+1)} .
\end{aligned}
$$

If $0 \leq j \leq \left[\frac{m}{2}\right]$, then $\ell - j + 2m \geq \ell + j + m$ and $m - 2j \geq 0$ so

$$
\binom{\ell - j + 2m}{2m} - \binom{\ell + j + m}{2m} \geq \binom{\ell + j + m}{2m - 1} (m - 2j) \geq 0
$$

by a), and therefore,

$$
\begin{aligned}
\frac{S_1}{2(m+1)} &= \frac{1}{2(m+1)} \sum_{j=0}^{[\frac{m}{2}]} \left[\binom{\ell + j + 2m}{2m} - \binom{\ell + j + m}{2m} \right] \\
&\quad \cdot \binom{2m+2}{2j+1} (m - 2j) \\
&\geq \frac{1}{2(m+1)} \sum_{j=0}^{[\frac{m}{2}]} \binom{\ell + j + m}{2m - 1} \binom{2m+2}{2j+1} (m - 2j)^2 \\
&\geq \frac{1}{2(m+1)} \cdot \binom{\ell + m}{2m - 1} \sum_{j=0}^{[\frac{m}{2}]} \binom{2m+2}{2j+1} (m - 2j)^2 .
\end{aligned}
$$

If $\left[\frac{m}{2}\right]+1 \le j \le m$, then $\ell+j+m \ge \ell-j+2m$ and $2j-m \ge 0$ and

$$\binom{\ell+j+m}{2m} - \binom{\ell-j+2m}{2m} \ge \binom{\ell-j+2m}{2m-1}(2j-m) \ge 0$$

by a), again, so

$$\begin{aligned}
\frac{S_2}{2(m+1)} &= \frac{1}{2(m+1} \sum_{j=[\frac{m}{2}]+1}^{m} \left[\binom{\ell+j+m}{2m} - \binom{\ell-j+2m}{2m}\right] \\
&\quad \cdot \binom{2m+2}{2j+1}(2j-m) \\
&\ge \frac{1}{2(m+1)} \sum_{j=[\frac{m}{2}]+1}^{m} \binom{\ell-j+2m}{2m-1}\binom{2m+2}{2j+1}(2j-m)^2 \\
&\ge \frac{1}{2(m+1)} \cdot \binom{\ell+m}{2m-1} \sum_{j=[\frac{m}{2}]+1}^{m} \binom{2m+2}{2j+1}(m-2j)^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
S &= \frac{1}{2(m+1)} S_1 + \frac{1}{2(m+1)} S_2 \\
&\ge \frac{1}{2(m+1)} \binom{\ell+m}{2m-1} \sum_{j=0}^{m} \binom{2m+2}{2j+1}(m-2j)^2 \\
&= \frac{1}{2(m+1)} \binom{\ell+m}{2m-1}(m+1)\cdot 4^m \\
&= \frac{1}{2}\binom{\ell+m}{2m-1} 4^m \ge \frac{1}{2}\cdot 2m \cdot 4^m = m4^m .
\end{aligned}$$

c)

$$\begin{aligned}
S &= \sum_{j=0}^{m} \binom{\ell-j+2m+1}{2m+1}\left(\binom{2m+2}{2j+1} - \binom{2m+2}{2j}\right) \\
&= \sum_{j=0}^{m} \binom{\ell-j+2m+1}{2m+1}\left(\binom{2m+1}{2j+1} - \binom{2m+1}{2j-1}\right) \\
&= \sum_{j=0}^{m} \binom{\ell-j+2m+1}{2m+1}\left(\binom{2m+1}{2j+1} - \binom{2m+1}{2j}\right) \\
&\quad + \sum_{j=0}^{m} \binom{\ell-j+2m+1}{2m+1}\left(\binom{2m+1}{2j} - \binom{2m+1}{2j-1}\right) \\
&= S_1 + S_2 .
\end{aligned}$$

Now S_1 may be treated as we did with the sum in b), obtaining $S_1 \geq \left(m+\frac{1}{2}\right)4^m$. Now

$$\begin{aligned}
S_2 &= \sum_{j=0}^{m} \binom{\ell-j+2m+1}{2m+1}\left(\binom{2m+1}{2j}-\binom{2m+1}{2j-1}\right) \\
&= \frac{1}{m+1}\sum_{j=0}^{m}\binom{\ell-j+2m+1}{2m+1}\binom{2m+2}{2j}(m-2j+1) \\
&= \binom{\ell+2m+1}{2m+1}+\frac{1}{m+1}\sum_{j=1}^{m}\binom{\ell-j+2m+1}{2m+1}\binom{2m+2}{2j}(m-2j+1) \\
&= \binom{\ell+2m+1}{2m+1}+\frac{1}{m+1}\sum_{j=1}^{m}\binom{\ell+j+m}{2m+1}\binom{2m+2}{2j}(2j-m-1)
\end{aligned}$$

(after $j \mapsto m-j+1$). So

$$\begin{aligned}
S_2 &= \binom{\ell+2m+1}{2m+1}+\frac{1}{2(m+1)}\sum_{j=1}^{m}\left[\binom{\ell-j+2m+1}{2m+1}-\binom{\ell+j+m}{2m+1}\right] \\
&\quad \cdot \binom{2m+2}{2j}(m-2j+1) \\
&= \binom{\ell+2m+1}{2m+1}+\frac{1}{2(m+1)}\sum_{j=1}^{\left[\frac{m}{2}\right]}\left[\binom{\ell-j+2m+1}{2m+1}-\binom{\ell+j+m}{2m+1}\right] \\
&\quad \cdot \binom{2m+2}{2j}(m-2j+1) \\
&\quad +\frac{1}{2(m+1)}\sum_{j=\left[\frac{m}{2}\right]+1}^{m}\left[\binom{\ell+j+m}{2m+1}-\binom{\ell-j+2m+1}{2m+1}\right] \\
&\quad \cdot \binom{2m+2}{2j}(2j-m-1) \\
&= \binom{\ell+2m+1}{2m+1}+\frac{1}{2(m+1)}\cdot S_3+\frac{1}{2(m+1)}\cdot S_4 .
\end{aligned}$$

Again $1 \leq j \leq \left[\frac{m}{2}\right]$ yields $\ell-j+2m+1 \geq \ell+j+m$ and $m-2j+1 \geq 0$ while $\left[\frac{m}{2}\right]+1 \leq j \leq m$ gives $\ell+j+m \geq \ell-j+2m+1$ and $2j-m-1 \geq 0$

therefore

$$S_3 = \sum_{j=1}^{[\frac{m}{2}]} \left[\binom{\ell - j + 2m + 1}{2m + 1} - \binom{\ell + j + m}{2m + 1} \right] \binom{2m+2}{2j} (m - 2j + 1)$$

$$\geq \sum_{j=1}^{[\frac{m}{2}]} \binom{\ell + j + m}{2m} \binom{2m+2}{2j} (m - 2j + 1)^2 \qquad \text{(by a) again)}$$

$$\geq \binom{\ell + m + 1}{2m} \sum_{j=1}^{[\frac{m}{2}]} \binom{2m+2}{2j} (m - 2j + 1)^2$$

and

$$S_4 = \sum_{j=[\frac{m}{2}]+1}^{m} \left[\binom{\ell + j + m}{2m + 1} - \binom{\ell - j + 2m + 1}{2m + 1} \right] \binom{2m+2}{2j} (2j - m - 1)$$

$$\geq \sum_{j=[\frac{m}{2}]+1}^{m} \binom{\ell - j + 2m + 1}{2m} \binom{2m+2}{2j} (2j - m - 1)^2 \qquad \text{(a) once again)}$$

$$\geq \binom{\ell + m + 1}{2m} \sum_{j=[\frac{m}{2}]+1}^{m} \binom{2m+2}{2j} (m - 2j + 1)^2 .$$

So

$$S_3 + S_4 \geq \binom{\ell + m + 1}{2m} \sum_{j=1}^{m} \binom{2m+2}{2j} (m - 2j + 1)^2$$

$$= \binom{\ell + m + 1}{2m} (m+1)(4^m - 2(m+1))$$

$$\geq (2m_1)(m+1)(4^m - 2(m+1)) .$$

Finally

$$S = S_1 + S_2 \geq \left(m + \frac{1}{2}\right) 4^m + S_2$$

$$= \left(m + \frac{1}{2}\right) 4^m + \binom{\ell + 2m + 1}{2m + 1} + \frac{S_3}{2(m+1)} + \frac{S_4}{2(m+1)}$$

$$\geq \left(m + \frac{1}{2}\right) 4^m + \binom{3m+1}{2m+1} + \left(\frac{2m+1}{2}\right) (4^m - 2(m+1))$$

$$= (2m+1)4^m - (2m+1)(m+1) + \binom{3m+1}{2m+1}$$

$$= (2m+1)(4^m - m - 1) + \binom{3m+1}{2m+1}$$

(in particular any sum in either b) or c) is ≥ 1).

Theorem 3.2. 1) $(C,1)$ satisfies Theorem 3.1 1), but not (C,α), $\alpha = 2,3,\ldots$.

2) Any method from example 2) above satisfies Theorem 3.1 1).

Proof: 1) $(C,1)$ is the Riesz mean (R,p_n) with $p_n = 1$ all n so this comes from Theorem 2.2. For (C,α), $\alpha = 2,3,\ldots$ we prove that b) ii) of Theorem 3.1 does not obtain. For that we show that

$$\left|\sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1})\right|$$

is bounded below by 1 for $n \geq \left[\frac{\alpha+1}{2}\right]$ and $1 \leq k \leq n - \left[\frac{\alpha-1}{2}\right]$. Now $p_n = \binom{n+\alpha-\alpha}{\alpha-1}$ and $q_j = (-1)^j \binom{\alpha}{j}$ so

$$\left|\sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1})\right|$$
$$= \left|\sum_{j=k}^{k+\left[\frac{\alpha-1}{2}\right]} \binom{n-j+\alpha-1}{\alpha-1}\left(\binom{\alpha}{2j-2k+1} - \binom{\alpha}{2j-2k}\right)\right|$$

(because $q_{2j-2k+1} \neq 0$, $q_{2j-2k} \neq 0$ iff $k \leq j \leq k + \left[\frac{\alpha-1}{2}\right]$ while $k + \left[\frac{\alpha-1}{2}\right] \leq n$ if $n \geq \left[\frac{\alpha+1}{2}\right]$ and $1 \leq k \leq n - \left[\frac{\alpha-1}{2}\right]$).

$$= \left|\sum_{j=0}^{\left[\frac{\alpha-1}{2}\right]} \binom{n-k-j+\alpha-1}{\alpha-1}\left(\binom{\alpha}{2j+1} - \binom{\alpha}{2j}\right)\right|$$

Now put $m = \left[\frac{\alpha-1}{2}\right]$ and $\ell = n-k$. Then

$$\alpha = \begin{cases} 2m+1, & m = 1,2,\ldots \\ 2m+2, & m = 0,1,2,\ldots \end{cases}$$

while $\ell = n-k \geq \left[\frac{\alpha-1}{2}\right] = m$, i.e., $\ell \geq m$.

i) $\alpha = 2m+1$, $m = 1, 2, \ldots$. We get

$$\begin{aligned}
&\left|\sum_{j=0}^{m} \binom{\ell - j + 2m}{2m} \left(\binom{2m+1}{2j+1} - \binom{2m+1}{2j} \right)\right| \\
&\quad = \sum_{j=0}^{m} \binom{\ell - j + 2m}{2m} \left(\binom{2m+1}{2j+1} - \binom{2m+1}{2j} \right) \\
&\quad \geq 1
\end{aligned}$$

By lemma 1 b).

ii) $\alpha = 2m+2$, $m = 0, 1, 2, \ldots$. We get

$$\begin{aligned}
&\left|\sum_{j=0}^{m} \binom{\ell - j + 2m + 1}{2m+1} \left(\binom{2m+2}{2j+1} - \binom{2m+2}{2j} \right)\right| \\
&\quad = \sum_{j=0}^{m} \binom{\ell - j + 2m + 1}{2m+1} \left(\binom{2m+2}{2j+1} - \binom{2m+2}{2j} \right) \\
&\quad \geq 1
\end{aligned}$$

by Lemma 1 c).

To finish this we have that, for certain positive constants (independent of n),

$$C_0 n^{\alpha} \leq \binom{n+\alpha-1}{\alpha} \leq C_1 n^{\alpha}$$

$n = 1, 2, \ldots$ (directly or from the results in the next chapter).

Then, if $n \geq \left[\frac{\alpha+1}{2}\right]$

$$\begin{aligned}
&\frac{1}{P_n} \sum_{k=1}^{n} P_{2k} \left| \sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right| \\
&\qquad \geq \frac{1}{P_n} \sum_{k=1}^{n-\left[\frac{\alpha-1}{2}\right]} P_{2k} \left| \sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right| \\
&\qquad = \frac{1}{P_n} \sum_{k=1}^{n-\left[\frac{\alpha-1}{2}\right]} P_{2k} \left| \sum_{j=k}^{k+\left[\frac{\alpha-1}{2}\right]} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right|
\end{aligned}$$

$$= \frac{1}{\binom{n+\alpha-1}{\alpha}} \sum_{k=1}^{n-\left[\frac{\alpha-1}{2}\right]} \binom{2k+\alpha-1}{\alpha}$$

$$\cdot \left| \sum_{j=k}^{k+\left[\frac{\alpha-1}{2}\right]} \binom{n-j+\alpha-1}{\alpha-1} \left(\binom{\alpha}{2j-2k} - \binom{\alpha}{2j-2k+1} \right) \right|$$

$$\geq \frac{1}{C_1 n^\alpha} \sum_{k=1}^{n-\left[\frac{\alpha-1}{2}\right]} C_0 2^\alpha k^\alpha$$

$$= \frac{2^\alpha C_0}{C_1 n^\alpha} \sum_{k=1}^{n-\left[\frac{\alpha-1}{2}\right]} k^\alpha \geq \frac{C_2}{n^\alpha} \left(n - \left[\frac{\alpha-1}{2} \right] \right)^{\alpha+1}$$

which is unbounded.

Therefore for the Cesàro means $(C, \alpha) = (a_{nk})$ with $\alpha = 2, 3, \ldots,$ $\left(\frac{1}{2} a_{n,\left[\frac{k}{2}\right]}\right)$ is not stronger than (a_{nk}).

2) a) is trivially satisfied.

b) $p_{n-j+1} = 1$ iff $n - N + 1 \leq j \leq n$

Consider n sufficiently big. Then, since $\{P_n\}$ is bounded, we only consider

$$\sum_{k=1}^{n} \left| \sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right|$$

$$= \sum_{k=1}^{n-N+1} \left| \sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right|$$

$$+ \sum_{k=n-N+2}^{n} \left| \sum_{j=k}^{n} p_{n-j+1}(q_{2j-2k} + q_{2j-2k+1}) \right|$$

$$= \Sigma_1 + \Sigma_2, \qquad \text{say}$$

$$\Sigma_1 = \sum_{k=1}^{n-N+1} \left| \sum_{j=n-N+1}^{n} (q_{2j-2k} + q_{2j-2k+1}) \right| = 0$$

because $\sum\limits_{j=n-N+1}^{n}(q_{2j-2k}+q_{2j-2k+1})$ is the sum of $2N$ consecutive q_j's, which is 0 (Example 2)).

$$\begin{aligned}\Sigma_2 &= \sum_{k=n-N+2}^{n}\left|\sum_{j=k}^{n}(q_{2j-2k}+q_{2j-2k+1})\right| \\ &\leq \sum_{k=n-N+2}^{n}\sum_{j=k}^{n}2 = 2\sum_{k=n-N+2}^{n}(n-k+1) \\ &\leq 2\sum_{k=n-N+2}^{n}(N-1) = 2(N-1)^2\end{aligned}$$

c) Again, we only consider

$$\begin{aligned}&\sum_{k=1}^{n-1}\left|p_{n-k+1}q_0+\sum_{j=k+1}^{n}p_{n-j+1}(q_{2j-2k-1}+q_{2j-2k})\right| \\ &= \sum_{k=1}^{n-N}\left|p_{n-k+1}q_0+\sum_{j=n-N+1}^{n}(q_{2j-2k-1}+q_{2j-2k})\right| \\ &\quad+\sum_{k=n-N+1}^{n-1}\left|p_{n-k+1}q_0+\sum_{j=k+1}^{n}(q_{2j-2k-1}+q_{2j-2k})\right| \\ &= \Sigma_1+\Sigma_2\end{aligned}$$

Σ_1 reduces to $\sum\limits_{k=1}^{n-N}p_{n-k+1}q_0=0$ because $n-k+1\geq N+1$ and $\sum\limits_{j=n-N+1}^{n}(q_{2j-2k-1}+q_{2j-2k})=0$ as before and

$$|\Sigma_2| \leq \sum_{k=n-N+1}^{n-1}\left(1+\sum_{j=k+1}^{n}2\right) \leq (N-1)+2(N-1)^2 .$$

To complete the proof we need to check that $\{q_{2j-k}+q_{2j-k+1}\}_{j=1}^{\infty}$ is (N,p_n)-summable to 0: for fixed k, choose any $j\geq N-1+\frac{k+1}{2}$. Then, if $s_j=q_{2j-k}+q_{2j-k+1}$,

$$\frac{1}{N}\sum_{r=0}^{N-1}s_{j-r}=\frac{1}{N}\sum_{r=0}^{N-1}(q_{2j-2r-k}+q_{2j-2r-k+1})=0\,,$$

being the sum of $2N$ consecutive q_j's.

Nevertheless, these methods do not satisfy the Klee-Szaïsz condition in a non-trivial way: we have, necessarily, that

$$\sum_{k=1}^{n} a_{nk}^2(s_{2k}^2 + s_{2k+1}^2) = \sum_{k=2}^{2n+1} a_{n,[\frac{k}{2}]}^2 s_k^2 \to 0$$

as $n \to \infty$. So, if $n \geq N$, $\frac{1}{N^2} \sum_{k=n-N+1}^{n} (s_{2k}^2 + s_{2k+1}^2) \to 0$ which implies both $s_{2n} \to 0$ and $s_{2n+1} \to 0$ so $s_n \to 0$. In which case the summability of subsequences is trivial.

Remarks: Here we prove a growth restriction for Nørlund means: suppose that $P_n \to +\infty$ and $\sum_{j=0}^{\infty} |q_j| < +\infty$, then $\{s_n\}$ (N, p_n)-summable implies $\lim_{n\to\infty} \frac{s_n}{P_n} = 0$.

Proof: Put $t_n = \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k+1} s_k \to t$ as $n \to \infty$. Then $s_n = \sum_{k=1}^{n} q_{n-k} P_k t_k$ so $s_n - t = \sum_{k=1}^{n} q_{n-k} P_k (t_k - t)$ because $\sum_{k=1}^{n} q_{n-k} P_k = 1$, $n = 1, 2, \ldots$. If $\varepsilon > 0$, there is k_0 such that $|t_k - t| < \varepsilon$ if $k \geq k_0 + 1$ and there is k_1 such that $\sum_{j=k_1+1}^{\infty} |q_j| < \varepsilon$. Then, if $n > k_0 + k_1$ we get first

$$\begin{aligned}\left|\sum_{k=k_0+1}^{n} q_{n-k} P_k (t_k - t)\right| &\leq \sum_{k=k_0+1}^{n} |q_{n-k}| P_k |t_k - t| \\ &\leq P_n \varepsilon \sum_{k=k_0+1}^{n} |q_{n-k}| \leq C_1 P_n \varepsilon\end{aligned}$$

and

$$\left|\sum_{k=1}^{k_0} q_{n-k}P_k(t_k-t)\right| \le \sum_{k=1}^{k_0} |q_{n-k}|P_k|t_k-t|$$

$$\le C_2P_{k_0}\sum_{k=1}^{k_0}|q_{n-k}|$$

$$= C_2P_{k_0}\sum_{j=n-k_0}^{n-1}|q_j|$$

$$\le C_2P_{k_0}\sum_{j=k_1+1}^{\infty}|q_j|$$

$$< C_2P_{k_0}\varepsilon$$

Thus $n > k_0 + k_1$ implies

$$\frac{|s_n - t|}{P_n} \le C_1\varepsilon + \frac{C_2P_{k_0}\varepsilon}{P_n} \le (C_1 + C_2)\varepsilon$$

and $\lim_{n\to\infty}\frac{s_n}{P_n} = 0$ since $P_n \to +\infty$. In the case of (C,α), we have $P_n \sim n^\alpha \to +\infty$ and $\sum_{j=0}^{\infty}|q_j| = \sum_{j=0}^{\infty}\left|\binom{\alpha}{j}\right| < +\infty$ so $\{s_n\}$ (C,α)-summable implies $\lim_{n\to\infty}\frac{s_n}{n^\alpha} = 0$.

Note that if $\sum_{j=0}^{\infty}|q_j| < +\infty$, and $\{P_n\}$ is bounded, then $\{P_n\}$ converges because it is nondecreasing. So, if $\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k+1}s_k \to t$ we must have $\lim_{n\to\infty}\frac{s_n-t}{P_n} = 0$ and $P_n \to P_\infty > 0$ so $\lim_{n\to\infty}(s_n - t) = 0$ and (N,p_n) is equivalent to convergence. Examples of this are furnished by $p_n = r^n$, $0 < r < 1$, then $P_n = \frac{r(1-r^n)}{1-r}$ which is bounded. And

$$p(x) = \sum_{j=0}^{\infty} r^{j+1}x^j = r\sum_{j=0}^{\infty}(rx)^j$$

$$= \frac{r}{1-rx}, \qquad |x| < \frac{1}{r}$$

so $q(x) = \frac{1}{r} - x$ and $q_0 = \frac{1}{r}$, $q_1 = -1$, $q_j = 0$, $j \ge 2$. And we obtain: if

$$\lim_{n\to\infty}\frac{(1-r)r^n}{1-r^n}\sum_{k=1}^{n}\frac{s_k}{r^k} = s \qquad (0 < r < 1)$$

Then $\lim_{n\to\infty} s_n = s$.

In general, (N, p_n) is equivalent to convergence iff

$$\sup_{n\geq 1} \sum_{k=1}^{n} |q_{n-k} P_k| < +\infty$$

Proof: If (N, p_n) is equivalent to convergence, then its inverse is regular (from Lemma 3, 2) of Ch. I). If the condition above holds, then since $\sum_{k=1}^{n} q_{n-k} P_k = 1$, $n = 1, 2, \ldots$ we need $\lim_{j\to\infty} q_j = 0$ in order to have a regular inverse. But, if $n \geq 1$,

$$\begin{aligned}\sum_{j=0}^{n} |q_j| &= \sum_{j=1}^{n+1} |q_{n-j+1}| \\ &= \frac{1}{P_1} \sum_{j=1}^{n+1} |q_{n-j+1}| P_1 \\ &\leq \frac{1}{P_1} \sum_{j=1}^{n+1} |q_{n-j+1} P_j| \leq \frac{C}{p_1}\end{aligned}$$

where $C = \sup_{n\geq 1} \sum_{k=1}^{n} |q_{n-k} P_k|$, i.e. $\sum_{j=0}^{\infty} |q_j| < +\infty$ which forces $\lim_{j\to\infty} q_j = 0$. (Note that (N, p_n) equivalent to convergence forces $\sum_{j=0}^{\infty} |q_j| < +\infty$.)

CHAPTER IV. THE CESÀRO MEANS

Definition: Let α be a positive integer. the Cesàro Mean of order α, (C,α), is the triangular matrix

$$a_{nk} = \begin{cases} \frac{\binom{n-k+\alpha-1}{\alpha-1}}{\binom{n+\alpha-1}{\alpha}}, & 1 \le k \le n \\ 0, & k > n \end{cases} \tag{4.1}$$

These methods are all regular (Example 1) Ch. III), and we already know that $\left(\frac{1}{2}a_{n,[\frac{k}{2}]}\right)$ is not stronger than $(a_{nk}) = (C,\alpha)$, $\alpha = 2,3,\ldots$ (Theorem 3.2).

Since the factorials can also be expressed through Euler's Gamma Function, we define

$$f_\alpha(x) = \frac{\Gamma(x+\alpha-1)}{\Gamma(\alpha)\Gamma(x)} \tag{4.2}$$

on $(0,+\infty)$ (here $\alpha \ge 1$) and then $a_\alpha\colon \{1,2,\ldots\} \times (0,+\infty) \to \mathrm{R}$ by

$$a_\alpha(n,x) = \begin{cases} \frac{f_\alpha(n-x+1)}{f_{\alpha+1}(n)}, & 0 \le x \le n \\ 0, & x > n \end{cases} \tag{4.3}$$

because

$$\sum_{k=1}^{n} f_\alpha(k) = \sum_{k=1}^{n} \frac{\Gamma(k+\alpha-1)}{\Gamma(\alpha)\Gamma(k)} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+1)\Gamma(n)} = f_{\alpha+1}(n).$$

We will proceed to prove that $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is stronger than $(a_{nk}) = (C,\alpha)$.

Lemma 1. 1) For each $\beta \ge 0$ $\lim_{x\to\infty} \frac{\Gamma(x+\beta)}{x^\beta \Gamma(x)} = 1$ and so, for any $a > 0$ and $\beta \ge 0$

$$0 < \inf_{x \ge a} \frac{\Gamma(x+\beta)}{x^\beta \Gamma(x)} \le \sup_{x \ge a} \frac{\Gamma(x+\beta)}{x^\beta \Gamma(x)} < +\infty$$

2) If $(a_{nk}) = (C,\alpha)$, $\alpha > 0$ then $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular.

Proof: 1) Stirling's Formula ([3]) yields

$$\Gamma(x) = x^{x-\frac{1}{2}}e^{-x+\mu(x)}\sqrt{2\pi}\,, \quad x > 0$$

where $\frac{1}{12x+1} < \mu(x) < \frac{1}{12x}$. And then

$$\begin{aligned}\frac{\Gamma(x+\beta)}{x^\beta\Gamma(x)} &= \frac{(x+\beta)^{x+\beta-\frac{1}{2}}e^{-x-\beta+\mu(x+\beta)}}{x^\beta x^{x-\frac{1}{2}}e^{-x+\mu(x)}} \\ &= \left(1+\frac{\beta}{x}\right)^{\beta-\frac{1}{2}}\left(1+\frac{\beta}{x}\right)^{x} e^{-\beta}e^{\mu(x+\beta)-\mu(x)}\end{aligned}$$

which tends to 1 as $x \to +\infty$ and so $\frac{\Gamma(x+\beta)}{x^\beta\Gamma(x)}$ is bounded above and away from zero on some interval $[b,+\infty)$, $b > 0$. This completes the proof because the function $\frac{\Gamma(x+\beta)}{x^\beta\Gamma(x)}$ is continuous and positive for $x > 0$.

2) By 1), $\lim\limits_{x\to+\infty}\frac{f_{\alpha+1}(x)}{x^\alpha} = \frac{1}{\Gamma(\alpha+1)}$ for $\alpha > 0$, so $\lim\limits_{x\to+\infty} f_{\alpha+1}(x) = +\infty$ and so

$$\lim_{n\to\infty}\sum_{k=1}^{n} f_\alpha(k) = \lim_{n\to\infty} f_{\alpha+1}(n) = +\infty\,.$$

Obviously $f_\alpha > 0$ on $(0,+\infty)$ and

$$\begin{aligned}\frac{f_\alpha\left(n+\frac{1}{2}\right)}{f_\alpha(n)} &= \frac{\Gamma\left(n+\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{\Gamma(\alpha)\Gamma(n)}{\Gamma(n+\alpha-1)} \\ &= \frac{\Gamma\left(n+\alpha-\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{\Gamma(n)}{\Gamma(n+\alpha-1)} \\ &= \frac{n+\alpha-1}{n+\alpha-\frac{1}{2}}\cdot\frac{\Gamma\left(n+\alpha+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{\Gamma(n)}{\Gamma(n+\alpha)} \\ &= \frac{2n+2\alpha-2}{2n+2\alpha-1}\cdot\frac{\Gamma\left(n+\alpha+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)^\alpha\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{n^\alpha\Gamma(n)}{\Gamma(n+\alpha)} \\ &\quad\cdot\frac{\left(n+\frac{1}{2}\right)^\alpha}{n^\alpha} \to 1\end{aligned}$$

as $n \to \infty$.

Corollary to Proposition 1, Ch. III implies that $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is regular where $(a_{nk}) = (C, \alpha)$, $\alpha > 0$.

Now we prove:

Theorem 4.1. If $(a_{nk}) = (C, \alpha)$, $\alpha = 1, 2, 3, \ldots$, then $\left(\frac{1}{2}a_{n,\frac{k}{2}}\right)$ is stronger than (a_{nk}).

Proof: We verify the conditions of Theorem 3.1, 2):

$q_j = (-1)^j \binom{\alpha}{j} \to 0$ as $j \to +\infty$ for any $\alpha > 0$ because $\sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| < +\infty$, $\alpha > 0$. Next if

$$d_{nk} = \frac{\Gamma(n)\Gamma(k+\alpha)}{2\Gamma(n+\alpha)\Gamma(k)} \sum_{j=k}^{2n} \frac{\Gamma\left(n - \frac{j}{2} + \alpha\right)}{\Gamma(\alpha)\Gamma\left(n - \frac{j}{2} + 1\right)} (-1)^{j-k} \binom{\alpha}{j-k}$$

$$= \frac{\binom{k+\alpha-1}{\alpha}}{2\binom{n+\alpha-1}{\alpha}} \sum_{j=k}^{2n} \frac{\Gamma\left(n - \frac{j}{2} + \alpha\right)}{\Gamma(\alpha)\Gamma\left(n - \frac{j}{2} + 1\right)} (-1)^{j-k} \binom{\alpha}{j-k},$$

$1 \le k \le 2n$ (we are using the notation of Theorem 3.1, 2) with $f = f_\alpha$, and $F = f_{\alpha+1}$) then $d_{nk} = 0$ if $1 \le k \le 2n - \alpha$, $\alpha = 1, 2, \ldots$.

iii) First, $d_{nk} = 0$ if $1 \le k \le 2n - \alpha$, $\alpha = 1, 2, \ldots$. Suppose this is true, and let us complete the proof. Then

$$d_{nk} \equiv \frac{\binom{k+\alpha-1}{\alpha}}{2\binom{n+\alpha-1}{\alpha}} \sum_{j=k}^{2n} \frac{\Gamma\left(n - \frac{j}{2} + \alpha\right)}{\Gamma(\alpha)\Gamma\left(n - \frac{j}{2} + 1\right)} (-1)^{j-k} \binom{\alpha}{j-k}$$

if $2n-\alpha+1 \le k \le 2n$, and 0, otherwise. Therefore, if $n > \frac{\alpha+1}{2}$

$$\sum_{k=1}^{2n} |d_{nk}| = \sum_{k=2n-\alpha+1}^{2n} |d_{nk}|$$
$$= \frac{1}{2\binom{n+\alpha-1}{\alpha}} \sum_{k=2n-\alpha+1}^{2n} \binom{k+\alpha-1}{\alpha}$$
$$\left| \sum_{j=k}^{2n} \frac{\Gamma\left(n-\frac{j}{2}+\alpha\right)}{\Gamma(\alpha)\Gamma\left(n-\frac{j}{2}+1\right)} (-1)^{j-k} \binom{\alpha}{j-k} \right|$$

Now, since $\alpha \ge 1$, Lemma 1 yields with C independent of n, j, and k

$$\frac{\Gamma\left(n-\frac{j}{2}+\alpha\right)}{\Gamma\left(n-\frac{j}{2}+1\right)} \le C\left(n-\frac{j}{2}\right)^{\alpha-1} \quad \text{if } k \le j \le 2n-1$$

and $\Gamma(\alpha)$ if $j=2n$ so certainly $\frac{\Gamma(n-\frac{j}{2}+\alpha)}{\Gamma(n-\frac{j}{2}+1)} \le C_1\left(n-\frac{j}{2}+1\right)^{\alpha-1}$ if $k \le j \le 2n$. The same lemma yields, with C_2, C_3 positive and independent of m

$$C_2 m^\alpha \le \binom{m+\alpha-1}{\alpha} \le C_3 m^\alpha .$$

Thus, for C_4 independent of n, j, k

$$\sum_{k=1}^{2n} |d_{nk}| \le \frac{C_4}{n^\alpha} \sum_{k=2n-\alpha+1}^{2n} k^\alpha \sum_{j=k}^{2n} \left(n-\frac{j}{2}+1\right)^{\alpha-1} \binom{\alpha}{j-k}$$
$$\le \frac{C_4}{n^\alpha} \sum_{k=2n-\alpha+1}^{2n} k^\alpha \left(n-\frac{k}{2}+1\right)^{\alpha-1} \sum_{j=0}^{2n-k} \binom{\alpha}{j}$$
$$\le \frac{C_4}{n^\alpha} \sum_{k=2n-\alpha+1}^{2n} k^\alpha \left(n-\frac{2n-\alpha+1}{2}+1\right)^{\alpha-1} \sum_{j=0}^{\alpha-1} \binom{\alpha}{j}$$
$$(2n-k \le \alpha-1)$$
$$\le \frac{C_5}{n^\alpha} \sum_{k=2n-\alpha+1}^{2n} k^\alpha \le \frac{C_5}{n^\alpha}(2n)^\alpha \alpha = C_5 2^\alpha \alpha$$

and iii) holds for $\alpha = 1, 2, 3, \ldots$.

To finish, we establish $d_{nk} = 0$ for $1 \le k \le 2n - \alpha$: define $(\Delta x)_m = x_m - x_{m+1}$. then

$$(\Delta^N x)_m = \sum_{j=0}^{N} \binom{N}{j} (-1)^j x_{m+j}$$

$m = 1, 2, \ldots,\ N = 1, 2, \ldots$, and $\Delta^N x \equiv 0$ iff $\{x_m\}$ is a polynomial of degree at most $N-1$ in m. On the other hand, $\frac{\Gamma(x+N)}{\Gamma(x)} = \prod_{r=0}^{N-1}(x+r)$, $x > 0$, $N = 1, 2, \ldots$ and, since $\binom{\alpha}{j-k} \ne 0$ only if $k \le j \le k+\alpha$, we get

$$d_{nk} = \frac{\binom{k+\alpha-1}{\alpha}}{2\binom{n+\alpha-1}{\alpha}} \sum_{j=k}^{k+\alpha} \frac{\Gamma\left(n - \frac{j}{2} + \alpha\right)}{\Gamma(\alpha)\Gamma\left(n - \frac{j}{2} + 1\right)} (-1)^{j-k} \binom{\alpha}{j-k}$$

when $1 \le k \le 2n - \alpha$. And so, if $\alpha \ge 3$

$$\begin{aligned} d_{nk} &= \frac{\binom{k+\alpha-1}{\alpha}}{2\Gamma(\alpha)\binom{n+\alpha-1}{\alpha}} \sum_{j=k}^{k+\alpha} \binom{\alpha}{j-k} (-1)^{j-k} \prod_{r=0}^{\alpha-2} \left(n - \frac{j}{2} + r\right) \\ &= \frac{\binom{k+\alpha-1}{\alpha}}{2\Gamma(\alpha)\binom{n+\alpha-1}{\alpha}} \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \prod_{r=0}^{\alpha-2} \left(n - \frac{j+k}{2} + r\right). \end{aligned}$$

Now, for fixed n and $\alpha \ge 2$, $x_m = \prod_{r=0}^{\alpha-2}\left(n - \frac{m}{2} + r\right)$ is a polynomial of degree $\alpha - 1$ in m so $(\Delta^\alpha x)_k = 0$, $k = 1, 2, \ldots$. That is

$$\sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j x_{k+j} = 0$$

or

$$\sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \prod_{r=0}^{\alpha-2} \left(n - \frac{j+k}{2} + r\right) = 0.$$

This proves that $d_{nk} = 0$, $1 \le k \le 2n - \alpha$, $\alpha = 1, 2, 3, \ldots$ (for $\alpha = 1$, it can be verified directly).

Thus, if $\alpha = 1, 2, 3, \ldots$, then $\left(\frac{1}{2} a_{n,\frac{k}{2}}\right)$ is stronger than (a_{nk}). The next step involves the Klee-Szüsz condition and the growth requirements on the sequence.

Theorem 4.2. Let α be a positive integer. If $\{s_n\}$ is (C, α)-summable to s and

$$\sup_{n \geq 2} \frac{|s_n|}{n^{\beta}(\log n)^{\gamma}} < +\infty \tag{4.4}$$

for some constants β, γ with $0 \leq \beta < \frac{1}{2}$ and $\gamma \geq 0$, then almost all subsequences of $\{s_n\}$ are (C, α)-summable to s.

Proof: To begin with,

$$0 \leq a_{n, \frac{k}{2}} \leq \frac{C}{n}, \qquad 1 \leq k \leq 2n,$$

C independent of k, n. Then

$$\begin{aligned}\sum_{k=1}^{2n} a_{n,\frac{k}{2}}^2 s_k^2 &\leq \frac{C_1}{n^2} \sum_{k=1}^{2n} k^{2\beta}(\log k)^{2\gamma} \\ &\leq \frac{C_2}{n^2} n^{2\beta+1}(\log n)^{2\gamma} = \frac{C_2(\log n)^{2\gamma}}{n^{1-2\beta}} \\ &\leq \frac{C_3}{n^r}\end{aligned}$$

some $r > 0$.

Therefore,

$$\sum_{n=1}^{\infty} \exp\left\{-\varepsilon\left(\sum_{k=1}^{2n} a_{n,\frac{k}{2}}^2 s_k^2\right)^{-1}\right\} \leq \sum_{n=1}^{\infty} \exp\left\{-\frac{\varepsilon n^r}{C_3}\right\} < +\infty.$$

So the Klee-Szüsz condition is satisfied.

Next, we complete the proof by showing that

$$\lim_{n\to\infty} \sum_{k=3}^{\infty} (a_{nk} - a_{n,\frac{\lambda_k(t)}{2}}) s_{\lambda_k(t)} = 0.$$

Define F_n, G_n and H_n as we did in Theorem 2.3. Then

$$k \in G_n \Rightarrow n - [C\sqrt{n \log\log n}] \leq k \leq n$$

$$k \in H_n \Rightarrow n + 1 \leq k \leq n + [C\sqrt{n \log\log n}]$$

and

$$\sum_{k=3}^{\infty}(a_{nk}-a_{n,\frac{\lambda_k(t)}{2}})s_{\lambda_k(t)}=\sum_{k\in F_n}\cdots+\sum_{k\in G_n}\cdots-\sum_{k\in H_n}\cdots$$
$$=\Sigma_1+\Sigma_2-\Sigma_3$$

Now,

$$a_{nk}=\frac{1}{\binom{n+\alpha-1}{\alpha}}\frac{1}{(\alpha-1)!}\prod_{j=1}^{\alpha-1}(n-k+j)\,,\qquad 1\le k\le n$$

and

$$a_{n,\frac{\lambda_k}{2}}=\frac{1}{\binom{n+\alpha-1}{\alpha}}\frac{1}{(\alpha-1)!}\prod_{j=1}^{\alpha-1}\left(n-\frac{\lambda_k}{2}+j\right)\,,\qquad 1\le\lambda_k\le 2n\,.$$

Therefore, since $(\alpha-1)!\binom{n+\alpha-1}{\alpha}\ge C_1n^{\alpha}$, C_1 independent of n, we obtain $|a_{nk}-a_{n,\frac{\lambda_k}{2}}|\le\frac{C_2}{n^{\alpha}}\left|P_{\alpha}(n-k)-P_{\alpha}\left(n-\frac{\lambda_k}{2}\right)\right|$ for $k\in F_n$ where $P_{\alpha}(t)=\prod_{j=1}^{\alpha-1}(t+j)$ (and $P_1(t)\equiv 1$). Now, on to the estimates:

$$\begin{aligned}|\Sigma_1|&\le\sum_{k\in F_n}|a_{nk}-a_{n,\frac{\lambda_k(t)}{2}}|\;|s_{\lambda_k(t)}|\\&\le\frac{C_2}{n^{\alpha}}\sum_{k\in F_n}\left|P_{\alpha}(n-k)-P_{\alpha}\left(n-\frac{\lambda_k(t)}{2}\right)\right|\;|s_{\lambda_k(t)}|\\&\le\frac{C_3}{n^{\alpha}}\sum_{k\in F_n}\left|P_{\alpha}(n-k)-P_{\alpha}\left(n-\frac{\lambda_k(t)}{2}\right)\right|k^{\beta}(\log k)^{\gamma}\\&\le\frac{C_3}{n^{\alpha}}\sum_{k\in F_n}\left|\frac{\lambda_k}{2}-k\right||P'_{\alpha}(\ell)|k^{\beta}(\log k)^{\gamma}\end{aligned}$$

where ℓ is between $n-k$ and $n-\frac{\lambda_k}{2}$.

Now, by its definition P_{α} and P'_{α} are both nondecreasing and positive on $[0,+\infty)$ (at least when $\alpha\ge 2$: $\alpha=1$ was already treated in Ch. II). Therefore, $0\le P'_{\alpha}(\ell)\le P'_{\alpha}(n)$ because $\ell\le n-k$ or $\ell\le n-\frac{\lambda_k}{2}$ and since

P'_α has degree $\alpha - 2$ we further obtain $P'_\alpha(n) \le C_0 n^{\alpha-2}$, C_0 independent of n, and thus,

$$\begin{aligned}
|\Sigma_1| &\le \frac{C_4}{n^2} \sum_{k \in F_n} \sqrt{k \log\log k}\; k^\beta (\log k)^\gamma \\
&\le \frac{C_4}{n^2} \sum_{k \in F_n} \sqrt{n \log n}\; n^\beta (\log n)^\gamma \\
&\le \frac{C_4}{n^2} \cdot n^{\frac{3}{2}+\beta} (\log n)^{\gamma+\frac{1}{2}} \\
&= \frac{C_4 (\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0
\end{aligned}$$

ii)

$$\begin{aligned}
|\Sigma_2| &\le \sum_{k \in G_n} |a_{nk}|\, |s_{\lambda_k(t)}| \\
&\le \frac{C_1}{n^\alpha} \sum_{k \in G_n} |P_\alpha(n-k)| k^\beta (\log k)^\gamma \\
&\le \frac{C_1 P_\alpha(n)}{n^\alpha} \sum_{k \in G_n} k^\beta (\log k)^\gamma \\
&\le \frac{C_2}{n} \sum_{k \in G_n} k^\beta (\log k)^\gamma \\
&\le \frac{C_2}{n} n^\beta (\log n)^\gamma \sum_{k \in G_n} 1 \\
&\le \frac{C_2 (\log n)^\gamma}{n^{1-\beta}} (1 + [C\sqrt{n \log\log n}]) \\
&\le \frac{C_3 n^{\frac{1}{2}} (\log n)^{\gamma+\frac{1}{2}}}{n^{1-\beta}} = \frac{C_3 (\log n)^{\gamma+\frac{1}{2}}}{n^{\frac{1}{2}-\beta}} \to 0
\end{aligned}$$

$$\begin{aligned}
|\Sigma_3| &\le \sum_{k \in H_n} |a_{n,\frac{\lambda_k(t)}{2}}|\, |s_{\lambda_k(t)}| \\
&\le \frac{C_1}{n^\alpha} \sum_{k \in H_n} \left| P_\alpha \left(n - \frac{\lambda_k(t)}{2} \right) \right| k^\beta (\log k)^\gamma
\end{aligned}$$

$$\leq \frac{C_2}{n} \sum_{k \in H_n} k^{\beta} (\log k)^{\gamma}$$

$$\leq \frac{C_2(n + [C\sqrt{n \log\log n}])^{\beta} (\log(n + [C\sqrt{n \log\log n}]))^{\gamma}}{n} \sum_{k \in H_n} 1$$

$$\leq \frac{C_3 n^{\beta} (\log n)^{\gamma}}{n} \sum_{k \in H_n} 1$$

$$= \frac{C_3 (\log n)^{\gamma}}{n^{1-\beta}} \cdot [C\sqrt{n \log\log n}]$$

$$\leq \frac{C_4 (\log n)^{\gamma + \frac{1}{2}}}{n^{\frac{1}{2} - \beta}} \to 0$$

This completes the proof because

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n, \frac{\lambda_k(t)}{2}} s_{\lambda_k(t)} = s$$

was already established: $\{s_n\}$ satisfies Klee-Szüsz condition for $\left(\frac{1}{2} a_{n, \frac{k}{2}}\right)$ and $\left(\frac{1}{2} a_{n, \frac{k}{2}}\right)$ is stronger than (a_{nk}).

Notes: Growth condition (3.5) is again motivated by the fact that if $\{s_n\}$ is (C, α)-summable ($\alpha > 0$), then $\lim_{n \to \infty} \frac{s_n}{n^{\alpha}} = 0$ (see previous chapter). When $\{s_n\}$ is bounded and (C, α)-summable to s, then $\{s_n\}$ is $(C, 1)$-summable to s ([5] p 108, Thm. 55 and p. 154, Thm. 92). Therefore, Theorem 3.2 is most important when $\{s_n\}$ is unbounded.

BIBLIOGRAPHY

1. R. C. Buck, A note on subsequences, Bull. American Math. Soc. 49, (1943), 898-899.

2. R. C. Buck and H. Pollard, Convergence and summability properties of subsequences, Bull. American Math. Soc. 49, (1943), 924-931.

3. Y. S. Chow and H. Teicher, Probability Theory, 2nd ed., Springer-Verlag, New York, 1988.

4. A. F. Dowidar and G. M. Petersen, Summability of subsequences, Quart. J. Math. Oxford (2), 13 (1962), 81-89.

5. G. H. Hardy, Divergent Series, Oxford, 1949.

6. F. R. Keogh and G. M. Petersen, Riesz summability of subsequences, Quart. J. Math. Oxford (2) 12 (1961), 33-44.

7. K. Klee and P. Szüsz, On summability of subsequences, Math. Z. 111 (1969), 205-213.

8. K. Knopp, Theory and Application of infinite series, 2nd ed. Blackie and Son, Glasgow, 1951; Reprinted, Dover Publications, New York, 1990.

9. A. Rényi, Foundations of Probability, Holden-Day, Inc., San Francisco, 1970.

10. P. Szüsz, On a theorem of Buck and Pollard, Z. Wahrscheinlichkeits theorie verw. Geb. 11, (1968), 39-40.

11. A. Wilansky, Summability through Functional Analysis, North-Holland Math. Studies No. 85, New York, 1984.