The Euler-Chow series for Toric Varieties

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Abstract of the Dissertation
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Let $X$ be a smooth $n$-dimensional projective variety. Consider the space $C_\lambda$ of all $p$-dimensional effective cycles on $X$ with a given homology class $\lambda$. We define the Euler-Chow series $E$ of $X$ as the element $\sum_{\lambda \in H_{2p}(X)} \chi(C_\lambda) \lambda \in \mathbb{Z}^H$. If we choose a basis $A$ for $H_{2p}(X, \mathbb{Z})$, $E$ can be written as $E = \sum_{\lambda \in \mathbb{Z}^n} \chi(C_\lambda) t^\lambda$ where $m$ is the rank of $H_{2p}(X)$. We introduce a concept of rationality in $\mathbb{Z}^H$ and prove that this does not depend of the basis $A$. It is shown that if $X$ is a smooth projective toric variety the Euler-Chow series is rational and an explicit computation is given in terms of an arbitrary basis $A$. In particular, we compute the euler characteristic.
of $C_\lambda$, for all $\lambda$ in $H_2(X, \mathbb{Z})$. The idea behind the proof is to define the Equivariant Euler-Chow series $E_T$ for $X$, using equivariant cohomology instead of singular cohomology. Making use of properties of toric varieties, it is not complicate to compute explicity $E_T$. Then, there is a map $I$ from $\mathbb{Z}^H$ to $\mathbb{Z}^{H_T}$, induced by the projection map $\pi$ from $H_T^{2p}(X, \mathbb{Z})$ to $H^{2p}(X, \mathbb{Z})$ which sends $E_T$ to $E$. The computation for $E$ follows since $E_T$ and $I$ are known.
Para Alicia, mi inseparable media naranja.

Para mi madre Arcelia.

A la memoria de Guillermo Torres.
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Chapter 1

Introduction

Algebraic geometers have long been interested in the space of all effective algebraic cycles on a given projective algebraic variety. It is well known that each connected component of this space is itself a projective algebraic set which is not in general irreducible. Many years have been devoted to understanding the structure of this space. However, there are still many important and fundamental questions which have not been answered. For instance, we do not know the Betti numbers and in general the Euler characteristic of these algebraic sets. In order to state our results we need to give a more precise meaning to “the space of all effective algebraic cycles”.

Let $X$ be a fixed $n$-dimensional variety contained in $\mathbb{P}^N$. For each pair of integers $p$ and $d$, with $d \geq 1$ and $0 \leq p < n$, let $\mathcal{C}_{p,d}(X)$ be the set of all finite formal sums

$$c = \sum n_\alpha V_\alpha$$

where $n_\alpha$ is a positive integer, $V_\alpha$ is an irreducible algebraic $p$-dimensional
variety of $X$ and

$$\text{deg}(c) \overset{df}{=} \sum n_\alpha \text{deg}(V_\alpha) = d$$

where $\text{deg}(V_\alpha)$ is the homological degree of $V_\alpha$ as a subvariety of $\mathbb{P}^N$. Each space $C_{p,d}(X)$ has the structure of a projective algebraic set and is called a Chow variety. The inclusion map $i : X \hookrightarrow \mathbb{P}^N$ induces a map $i_* : H_{2p}(X,\mathbb{Z}) \to H_{2p}(\mathbb{P}^N,\mathbb{Z})$. For each homology class $\lambda$ in $H_{2p}(X,\mathbb{Z})$ with $i_* \lambda = d[\mathbb{P}^p]$, the space $C_{p,d}(X)$ of all elements of $C_{p,d}(X)$ with homology class $\lambda$, is a subvariety of $C_{p,d}(X)$. We are interested in computing the Euler characteristic of $C_{p,d}(X)$, for all $\lambda$, and in looking for relations among them. The case of $X = \mathbb{P}^N$ was worked out in a paper by Blaine Lawson and Stephen S. Yau. They prove the following equality

$$\sum_{d=0}^{\infty} \chi(C_{p,d}(X)) t^d = \left( \frac{1}{1-t} \right)^{\binom{n+1}{p+1}}$$

We would like to state the problem for an arbitrary algebraic projective variety, and solve it for the case of a smooth projective toric variety. The series that we obtain is not in general a formal power series. We have to clarify, and give to it, the meaning of being rational. Homology and cohomology are considered without torsion.

**Definition 1.0.1.** For each nonnegative integer $p$, the Euler-Chow series $E^p$ of $X$ is the element

$$E^p = \sum_\lambda \chi(C_{p,d}(X)) \lambda \in \mathbb{Z}^{H_{2p}(X,\mathbb{Z})},$$

where $\mathbb{Z}^{H_{2p}(X,\mathbb{Z})}$ is the set of all functions from $H_{2p}(X,\mathbb{Z})$ to $\mathbb{Z}$. 
If we fix a basis \( \{x_1, \ldots, x_m\} \) for \( H_{2p}(X, \mathcal{I}) \) and if \( \lambda \) is written in terms of the basis as \( \lambda = \sum_{i=1}^{m} \alpha_i x_i \), then the Euler-Chow series takes the form

\[
E^p = \sum_{\lambda} \chi(C_\lambda) t^{\alpha(\lambda)} \in \mathbb{Z}^m.
\]

where \( t^{\alpha(\lambda)} = t_1^{\alpha_1} \cdots t_m^{\alpha_m} \). Since we can have negative powers, \( E^p \) is not necessary a formal power series, so it does not make sense in general to ask if \( E^p \) is rational. In the case of \( \mathbb{P}^N \), the series runs over positive numbers, so it is itself a formal power series, and it is rational. However, we will be able to define a similar concept for \( E^p \). In order to do this, we observe that there is a map \( p \), from a subgroup of \( \mathbb{Z}[[t_1, \ldots, t_m, t_1^*, \ldots, t_m^*]] \) to \( \mathbb{Z}^m \), which sends the series \( S_\alpha = \sum_{\alpha \in \mathbb{N}^m} n_{\alpha} t_1^{\alpha_1} \cdots t_m^{\alpha_m} (t_1^{\bar{\alpha}_1} \cdots t_m^{\bar{\alpha}_m}) \) to \( S_\beta = \sum_{\beta \in \mathbb{Z}^m} n_{\beta} t_1^{\beta_1} \cdots t_m^{\beta_m} \) where \( \beta_i = \alpha_i - \bar{\alpha}_i \), and the sum \( n_{\beta} = \sum_{\alpha} n_{\alpha} \) is taken over the \( \alpha \)'s with the property \( \beta_i = \alpha_i - \bar{\alpha}_i \). [We have that \( S_\alpha \) is in the domain of \( p \) if, for all \( i \), \( \beta_i = \alpha_i - \bar{\alpha}_i \) with \( n_{\alpha} \neq 0 \), has a finite number of solutions on \( \mathbb{P}^1 \).] We arrive to the following definition.

**Definition 1.0.2.** An element \( E \) of \( \mathbb{Z}^{H_{2p}(X, \mathcal{I})} \) is called rational if there is a rational function \( S_\alpha \) in the domain of \( p \) such that \( p(S_\alpha) = E \).

Here, we are just associating to the Euler-Chow series a rational function in the variables \( \{t_1, \ldots, t_m\} \). We show that this definition does not depend on the basis, in other words, rational is an intrinsic concept.

**Theorem 1.0.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two integral basis of \( H_{2p}(X, \mathcal{I}) \). An element \( E \) of \( \mathbb{Z}^{H_{2p}(X, \mathcal{I})} \) is rational with respect to the basis \( \mathcal{A} \) if and only if it is with respect to the basis \( \mathcal{B} \).
We recall that a Toric Variety $X$, is a normal variety that contains a torus $T$ as an open dense set, and the action of the torus on itself can be extended to the whole variety. In order to study the Euler-Chow series for a smooth projective toric variety we need to define the equivariant Euler-Chow series. From now on, toric variety means smooth toric variety.

**Definition 1.0.3.** Let $X$ be a toric variety, and let $H^{2p}_T (X, \mathbb{Z})$ be the equivariant cohomology of $X$. The **equivariant Euler-Chow series** of $X$ is the element

$$E^p_T = \sum_{\xi} \chi \left( (C^T)_\xi \right) \xi \in \mathcal{Z}^{2p}_T (X, \mathbb{Z})$$

where $\xi \in H^{2p}_T (X, \mathbb{Z})$ and $(C^T)_\xi$ is the space of all invariant effective cycles on $X$, with equivariant cohomology class $\xi$.

Consider a basis $\mathcal{A}$ for $H^{2p}_T (X, \mathbb{Z})$ which contains as elements, the cohomology class of the closure of the orbits under the action of the torus $T$ on $X$. The equivariant Euler-Chow series is a rational function, and its expression with respect to the basis $\mathcal{A}$ is

$$E^p_T = \prod_{i=1}^N \left( \frac{1}{1 - s_i} \right)$$

where $N$ is the number of orbits of dimension $p$. The map $\pi$, from the equivariant cohomology to the singular cohomology, induces a map $I$ from a subset $\mathcal{D}$, containing $E^p_T$, of $\mathcal{Z}^{2p}_T (X, \mathbb{Z})$ to $\mathcal{Z}^{2p} (X, \mathbb{Z})$. From the definition of $I$, it easy to see that this map sends the equivariant Euler-Chow series $E^p_T$ to the Euler-Chow series $E^p$. All these results, together with the important fact that $C_\lambda$ is compact, allow us to prove the following theorem.
Theorem 1.0.2. For any nonsingular projective toric variety \( X \) over \( \mathbb{C} \), the Euler-Chow series is rational. In fact, if \((\mathcal{O}_1, \ldots, \mathcal{O}_N)\) denotes all the orbits of codimension \( 2p \), and if \( \mathcal{B} = \{x_1, \ldots, x_m\} \) is a basis for \( H^{2p}(X, \mathbb{Z}) \). Then

\[
E^p = \prod_{i=1}^{N} \left( \frac{1}{1 - \prod_{j=1}^{m} t_j^{a_{ij}}} \right)
\]

where \([\overline{O}_i] = \sum_{j=1}^{m} a_{ji} x_j\) is the expression, in terms of the basis \( \mathcal{B} \), of the cohomology class of the closure \( \overline{O}_i \) of the orbit \( O_i \).

An important remark that should be made here is that the matrix \( A = (a_{ji}) \) is directly computable in terms of the fan of \( X \). This allows us to compute easily calculate many examples. We recover the results of B. Lawson and S.S. Yau. Chapter 2 contains the definition of the Euler-Chow series, the definition of rationality for an element in \( \mathbb{Z}^m \), and the proof that this definition does not depends on the basis.

Chapter 3 contains a review of Toric varieties. The most important result that will be used later is the last theorem of the chapter. It is characterized by the cohomology ring of a smooth projective toric variety.

Chapter 4 contains the main results of this work we find an explicit expression for the equivariant Euler-Chow which we use to compute the Euler-Chow for a smooth projective toric variety. The appendix contains a letter from Emili Bifet. He suggests how to reformulate, in an intrinsic way, the results contained in this thesis. I consider the approach suggested for him very elegant. Unfortunaly, this letter was written one week before the defense, and it was very hard to rewrite everything in the terms suggested by him.
Chapter 2

The Euler-Chow Series and the Space of Cycles

2.1 Definitions and Properties

In this chapter we define the Euler-Chow series and state the problem in which we are interested. At the end we discuss a few examples. All the definitions and results are restricted to the complex numbers, unless otherwise stated.

Throughout this chapter all homology and cohomology are considered modulo torsion.

A subset $X$ of $\mathbb{P}^N$ is an algebraic set if there exists a finite number of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_N]$ such that $X$ is the set of common zeros of these polynomials. $X$ is always provided with an embedding $j : X \to \mathbb{P}^N$.

$X$ is called irreducible if it is not a non-trivial finite union of algebraic sets, no one of which contained in another. We define the Zariski topology on
\( \mathbb{P}^N \) by taking the open sets to be the complements of algebraic sets.

A **projective algebraic variety** is an irreducible algebraic set of \( \mathbb{P}^N \) where its dimension is defined as that of a topological space. A subvariety \( Y \) of \( X \) is a subset of \( X \) which is itself a projective algebraic variety. Let \( (X, j) \) be an algebraic subset. The degree, \( \deg(V) \), of an irreducible subvariety \( V \subseteq \mathbb{P}^n \) of dimension \( p \) is the number of points in the intersection of \( V \) with a generic \((N-p)\)-dimensional linear subspace of \( \mathbb{P}^N \), and it is the same as the degree of its fundamental class in \( H_{2p}(\mathbb{P}^N, \mathbb{Z}) = \mathbb{Z} \) (see [GH78]).

**Definition 2.1.1** An effective \( p \)-cycle \( c \) on \( X \) is a finite (formal) sum \( c = \sum n_s V_s \) where each \( n_s \) is a nonnegative integer and each \( V_s \) is an irreducible \( p \)-dimensional subvariety of \( X \). We define the degree of any effective cycle \( c = \sum n_s V_s \) as \( \deg(c) = \sum n_s \deg(V_s) \). From now on, we shall use the term cycle for effective cycle.

For any algebraic set \( j : X \hookrightarrow \mathbb{P}^N \) and integers \( p \geq 0, d \geq 0 \), we denote by \( C_{p,d}(X) \) the space of all cycles of dimension \( p \) and degree \( d \) on \( X \). We set \( C_{p,0}(X) = \{ \emptyset \} \) by convention. It is well known that \( C_{p,d}(X) \) is a projective algebraic set of \( \mathbb{P}^{N(p,d)} \) and that \( C_{p,d}(X) \) is an algebraic subset of \( C_{p,d}(\mathbb{P}^N) \) (see [Sam55], [Sha74]). Let \( \lambda \) be an element in \( H_{2p}(X, \mathbb{Z}) \), and denote by \( C_\lambda(X) \) the set of all cycles on \( X \) whose homology class is \( \lambda \). Note that \( C_\lambda(X) \) is contained in \( C_{p,d}(X) \) by the inclusion \( j : X \hookrightarrow \mathbb{P}^N \) where \( j_* \lambda = d[\mathbb{P}^p] \). Since any algebraic set is the finite union of irreducible components (see [Har77]), we have \( C_{p,d}(X) = \bigcup_{i=1}^M C_{p,d}^i(X) \), where \( C_{p,d}^i(X) \) are its irreducible components. Suppose \( C_\lambda(X) \cap C_{p,d}^e(X) \neq \emptyset \). Since any two cycles in \( C_{p,d}^e \) are algebraically equivalent, (}
see [Ful84]) they represent the same element of homology, so \( \mathcal{C}_{p,d}^{\lambda}(X) \subset \mathcal{C}_\lambda(X) \) for some \( \lambda \). Therefore, \( \mathcal{C}_\lambda(X) = \bigcup_{j=1}^l \mathcal{C}_{p,d}^{i_j}(X) \) where \( \{i_1, \ldots, i_l\} \) is a subset of \( \{1, \ldots, m\} \) with \( \mathcal{C}_\lambda(X) \cap \mathcal{C}_p^{i_0}(X) \neq \emptyset \). We have proved the following lemma:

**Lemma 2.1.1** Let \( \lambda \) be an element of \( H_{2p}(X, \mathbb{Z}) \), then \( \mathcal{C}_\lambda(X) \) is a projective algebraic set.

As a consequence of this lemma we have that \( \mathcal{C}_\lambda(X) \) has a finite triangulation (see for example [Hir75]).

### 2.2 The Euler-Chow Series

In this section we define the Euler-Chow series of \( X \) in dimension \( p \), and discuss some of its properties.

Let \( H \) be a free abelian group. We denote by \( \mathbb{Z}^H \) the set of all functions from \( H \) to \( \mathbb{Z} \). We shall write the elements of \( \mathbb{Z}^H \) as \( \sum_{h \in H} m(h) h \) where \( m(h) \in \mathbb{Z} \) for each \( h \) in \( H \). We are ready for the following definition.

**Definition 2.2.1** Let \( p \) be a fixed nonnegative integer. The Euler-Chow series of \( X \) in dimension \( p \) is the element

\[
E^p = \sum_{\lambda} \chi(C_\lambda) \lambda \in \mathbb{Z}^{H_{2p}(X, \mathbb{Z})},
\]

where \( C_\lambda(X) \) is the space of all effective cycles on \( X \) with homology class \( \lambda \) and where \( \chi(C_\lambda(X)) \) is the Euler characteristic of \( C_\lambda \). If \( C_\lambda \) is the empty set, by convention its Euler characteristic is zero.
Let \( P = \mathbb{Z}[t_1, \ldots, t_r] \) be the ring of polynomials with coefficients in \( \mathbb{Z} \). Let \( S \) be the multiplicative set of all polynomials of \( P \) whose constant term is \( \pm 1 \). We denote by \( P_S \) the localisation of \( P \) with respect \( S \). There is an injective morphism

\[
\mathbb{Z}[t_1, \ldots, t_r]_S \xrightarrow{\phi} \mathbb{Z}[[t_1, \ldots, t_r]]
\]

from the localisation \( P_S \) to the ring of formal power series in the same variables, given by the universal property of \( P_S \). We call an element of \( \mathbb{Z}[[t_1, \ldots, t_r]] \) a rational function if it is in the image of the morphism \( \phi \).

Let \( H \) be a free abelian group and \( m \) its rank. Let \( \mathcal{A} \) be an integral basis of \( H \). This gives us an isomorphism

\[
F : \mathbb{Z}^m \rightarrow H
\]

and an isomorphism of groups

\[
\Psi_\mathcal{A} : \mathbb{Z}^H \rightarrow \mathbb{Z}^{\mathbb{Z}^m}.
\]

We shall express any element of \( a \in \mathbb{Z}^{\mathbb{Z}^m} \) in exponential notation as

\[
a = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha
\]

where \( a_\alpha = a(\alpha) \) and \( t^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m} \) for \( \alpha = (\alpha_1, \ldots, \alpha_m) \). Observe that the elements of \( \mathbb{Z}^{\mathbb{Z}^m} \) are not in general formal power series, therefore we can not say, or ask, if they are rational functions. However, we will give a definition of rational, and will prove it is an intrinsic concept.

Let \( t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m \) be variables and denote by \( \mathcal{R} \) the ring of formal power series \( \mathbb{Z}[[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]] \) in these variables. We write each element
\[ \sigma = \sum_{(\beta, \bar{\beta}) \in \mathbb{N}^{2m}} n_{\beta \bar{\beta}} t^\beta \bar{t}^{\bar{\beta}}, \]

where

\[(\beta, \bar{\beta}) = (\beta_1, \ldots, \beta_m, \bar{\beta}_1, \ldots, \bar{\beta}_m) \text{ and } t^\beta = t_{\beta_1} \cdots t_{\beta_m}, \quad \bar{t}^{\bar{\beta}} = \bar{t}_{\bar{\beta}_1} \cdots \bar{t}_{\bar{\beta}_m}.\]

Let \(\sigma = \sum_{\beta \bar{\beta}} n_{\beta \bar{\beta}} t^\beta \bar{t}^{\bar{\beta}}\) and \(\alpha = (\alpha_1, \ldots, \alpha_m)\) be elements in \(\mathcal{R}\) and \(\mathbb{N}^m\), respectively. We write

\[ N_{\sigma, \alpha} = \text{card} \{(\beta, \bar{\beta}) \in \mathbb{N}^{2m} | n_{\beta \bar{\beta}} \neq 0 \text{ and } \alpha = \beta - \bar{\beta}\}. \]

Let \(\mathcal{D}\) be the set of all elements \(\sigma \in \mathcal{R}\) with the property that the number \(N_{\sigma, \alpha}\) is finite for all \(\alpha\) in \(\mathbb{Z}^m\). We have a map

\[ p : \mathcal{D} \subset \mathcal{R} \rightarrow \mathbb{Z}^m \]

defined by

\[ \sum_{\beta \bar{\beta} \in \mathbb{N}^{2m}} n_{\beta \bar{\beta}} t^\beta \bar{t}^{\bar{\beta}} \xmapsto{p} \sum_{\alpha} v_\alpha t^\alpha \quad (2.1) \]

where

\[ v_\alpha = \sum_{\beta - \bar{\beta} = \alpha} n_{\beta \bar{\beta}}. \]

We arrive at the following definition.

**Definition 2.2.2** Let \(\mathcal{A}\) be a basis for \(H\). We say that \(E\) in \(\mathbb{Z}^H\) is \(\mathcal{A}\)-rational if there is a rational function \(\sigma \in \mathcal{D}\), such that \(p(\sigma) = \Psi_\mathcal{A}(E)\).

Observe that in this definition \(\Psi_\mathcal{A} : \mathbb{Z}^H \rightarrow \mathbb{Z}^m\) is the isomorphism defined above in terms of the basis \(\mathcal{A}\). The next theorem shows that this definition is intrinsic, in other words, it does not depend on the choice of basis.
Theorem 2.2.1 Let $A$ and $B$ be two integral bases for $H$. An element $E$ of $Z^H$ is $A$-rational if and only if it is $B$-rational.

Before we go on with the proof of the theorem, we give the following definition as a consequence of the theorem.

Definition 2.2.3 An element $E$ of $Z^H$ is rational if it is $A$-rational for some basis $A$ of $H$.

Proof of the theorem

The heart of the argument is to show that the following diagram commutes

$$
\begin{array}{c}
\mathcal{R} \circ \mathcal{D} \xrightarrow{P} Z^m \\
\downarrow \Phi \quad \downarrow \Phi \\
\mathcal{R} \circ \mathcal{D} \xrightarrow{P} Z^m,
\end{array}
$$

where $\Phi$ and $\Phi$ are maps induced by the change of bases. We start with a description of these two maps. For any $a \in Z$, let

$$
a^+ = \max \{a, 0\} \quad \text{and} \quad a^- = \max \{-a, 0\} \quad \text{and} \quad a = a^+ - a^-.
$$

Let $A = (a_{ij})$ be the matrix representing the change of bases. Then $A = A^+ - A^-$ where $A^+ = (a^+_{ij})$ and $A^- = (a^-_{ij})$. The isomorphism $A$ induces two maps. The first map is the isomorphism of groups

$$
\Phi : Z^m \rightarrow Z^m
$$
given by

\[
\Phi \left( \sum_\alpha n_\alpha t^\alpha \right) = \sum_\alpha n_\alpha t^{A(\alpha)} = \sum_\gamma n_{A^{-1}(\gamma)} t^\gamma .
\]

The second one is the isomorphism

\[
\Phi : \mathcal{R} \longrightarrow \mathcal{R}
\]

defined at each generator \( t_i \) and \( \bar{t}_i \) by

\[
\Phi (t_i) = \prod_{j=1}^m t_j^{a_j i} \bar{t}_j^{\bar{a}_j i} \quad \text{and} \quad \Phi (\bar{t}_i) = \prod_{j=1}^m t_j^{a_j i} \bar{t}_j^{\bar{a}_j i} .
\]

To see that \( \Phi \) is an isomorphism one checks that for any two basis \( \mathcal{A} \) and \( \mathcal{B} \), we have \( \Phi_A \circ \Phi_B = \Phi_{\mathcal{A}} \circ \Phi_{\mathcal{B}} \). Observe that By the universal property of localisation, the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]_S & \overset{\Phi}{\longrightarrow} & \mathcal{R} = \mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m] \\
\downarrow \Phi & & \downarrow \bar{\Phi} \\
\mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]_S & \overset{\Phi}{\longrightarrow} & \mathcal{R} = \mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m] ,
\end{array}
\]

where \( \mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]_S \) is the localisation of the ring of polynomials \( \mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m] \) with respect to the multiplicative set \( S \) which consists of all polynomials with constant term 1 or -1. Consequently \( \Phi \) sends rational functions into rational functions.

Next, we would like to prove that \( \Phi(D) \subset D \). Let \( \sigma \in D \) be an element in the domain of \( \Phi \). We write \( \sigma \) as \( \sigma = \sum_{\beta} n_{\beta \bar{\alpha}} t^\beta \bar{t}^{\bar{\alpha}} \). By the definition of \( D \), for each \( \alpha \) the number of solutions \( N_{\sigma, \alpha} \) of

\[
\alpha = \beta - \bar{\beta} \quad \text{with} \quad n_{\beta \bar{\alpha}} \neq 0 \quad (2.2)
\]
is finite. From the definition of $\overline{\Phi}$ we obtain

$$
\overline{\Phi}(\sigma) = \overline{\Phi} \left( \sum_{\beta \overline{\beta}} n_{\beta \overline{\beta}} \ell^{\beta} \overline{\ell}^{\overline{\beta}} \right) = \sum_{\beta \overline{\beta}} n_{\beta \overline{\beta}} \ell^{(A^+\beta + A^-\overline{\beta})} \overline{\ell}^{(A^-\beta + A^+\overline{\beta})}.
$$

Observe that the system

$$
\delta = A^+\beta + A^-\overline{\beta} \quad \text{and} \quad \overline{\delta} = A^-\beta + A^+\overline{\beta} \quad \text{with} \quad (\beta, \overline{\beta}) \in \mathbb{N}^{2m} \quad (2.3)
$$

has a unique solution, since $\overline{\Phi}$ is an isomorphism. We write

$$
\overline{\Phi}(\sigma) = \overline{\Phi} \left( \sum_{\beta \overline{\beta}} n_{\beta \overline{\beta}} \ell^{\beta} \overline{\ell}^{\overline{\beta}} \right) = \\
= \sum_{\beta \overline{\beta}} n_{\beta \overline{\beta}} \ell^{(A^+\beta + A^-\overline{\beta})} \overline{\ell}^{(A^-\beta + A^+\overline{\beta})} = \sum_{\delta \overline{\delta}} u_{\delta \overline{\delta}} \ell^{\delta} \overline{\ell}^{\overline{\delta}} = \overline{\sigma}.
$$

Where

$$
u_{\delta \overline{\delta}} = n_{\beta \overline{\beta}} \quad \text{with} \quad (\delta, \overline{\delta}) = (A^+\beta + A^-\overline{\beta}, A^-\beta + A^+\overline{\beta})
$$

We now prove that $\overline{\sigma} \in \mathcal{D}$. In order to do that, we have to show that for any $\gamma \in \mathbb{Z}^m$, there is a finite number of solutions on $(\delta, \overline{\delta})$ to the system

$$
\gamma = \delta - \overline{\delta} \quad \text{with} \quad u_{\delta \overline{\delta}} \neq 0. \quad (2.4)
$$

However, any solution $(\delta, \overline{\delta})$ to this system, we have

$$
\gamma = \delta - \overline{\delta} = A^+\beta + A^-\overline{\beta} - A^-\beta - A^+\overline{\beta} = A(\beta - \overline{\beta}) \quad \text{with} \quad n_{\beta \overline{\beta}} \neq 0.
$$

This is equivalent to the system

$$
A^{-1}(\gamma) = A^{-1}A(\beta - \overline{\beta}) = \beta - \overline{\beta} \quad \text{with} \quad n_{\beta \overline{\beta}} \neq 0. \quad (2.5)
$$
However, this is the system 2.2, which has a finite number \( N_{A^{-1}(\sigma), \sigma} \) of solutions. Then 2.4 has a finite number of solutions. Therefore, \( \Phi(\sigma) = \sigma \in \mathcal{D} \) and

\[
p(\Phi(\sigma)) = p \left( \sum_{\delta} u_{\delta} t^{\delta} \right) = \sum_{\gamma \in \mathcal{I}} w_{\gamma} t^{\gamma}
\]

(2.6)

where

\[
w_{\gamma} = \sum_{\beta - \beta = A^{-1}(\gamma)} n_{\beta \beta}.
\]

Next, from the definition of \( p \) (see 2.1), we have

\[
p(\sigma) = \sum_{\alpha \in \mathcal{I}} v_{\alpha} t^{\alpha}, \quad \text{where} \quad v_{\alpha} = \sum_{\alpha = \beta - \beta} n_{\beta \beta}.
\]

Then,

\[
\Phi(p(\sigma)) = \Phi \left( \sum_{\alpha \in \mathcal{I}} v_{\alpha} t^{\alpha} \right) = \sum_{\gamma \in \mathcal{I}} N_{A^{-1}(\gamma)} t^{\alpha}
\]

(2.7)

with

\[
v_{A^{-1}(\gamma)} = \sum_{A^{-1}(\gamma) = \beta - \beta} n_{\beta \beta}.
\]

Finally, comparing 2.6 and 2.7 we obtain

\[
p(\Phi(\sigma)) = \Phi(p(\sigma)).
\]

2.3 Examples

I) The projective space \( P^n(C) \).

Let \([x_0 : \ldots : x_{n+1}]\) be homogenous coordinates for \( P^n(C) \).
For each $(p+1)$-tuple of integers $\alpha = (\alpha_0, \ldots, \alpha_p)$ with $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_p \leq n$, consider the coordinate $(p+1)$-plane

$$C^{p+1}_\alpha = \{ z \in C^{n+1} | z_i = 0 \text{ if } i \neq \alpha_i \text{ for some } j \} \subset C^{n+1}$$

and denote by

$$P^p_\alpha \subset P^n$$

the corresponding projective $p$-plane. It is not hard to see that any $p$-dimensional invariant cycle on $P^n$ has the form

$$V = \sum \eta_\alpha P^p_\alpha \text{ for nonegative integers } \eta_\alpha.$$

In [LY87] it is proved that

$$\chi(C_{p,d}) = \text{card } \{(m_1, \ldots, m_\nu) \in \mathbb{Z}^\nu : m_j \geq 0 \text{ and } \sum m_j = d\} = \binom{\nu + d - 1}{d},$$

where $\nu = \binom{n+1}{p+1}$. Therefore,

$$E^p = \sum_{d=0}^{\infty} \chi(C_{p,d}) t^d = \left(\frac{1}{1 - t}\right)^{\binom{n+1}{p+1}}.$$

II) $P^n \times P^m$.

Using the same method as in the last example, and the canonical decomposition

$$H_{2p}(X, Z) = \bigoplus_{k+l=p} H_{2k}(P^n) \otimes H_{2l}(P^m)$$

we obtain, for any integer $p$, and $0 \leq p \leq n + m$

$$E^p = \prod_{k+l} (1 - t_{k,l})^{-\binom{n+1}{k+1}\binom{m+1}{l+1}}.$$

III) Zero cycles.
Let $X$ be a projective algebraic variety. We denote by $SP^d(X)$ the $d$-fold symmetric product on $X$. Let $b_0, b_1, \ldots$ be the Betti numbers of $X$. MacDonalld proves (see [Mac62]) that the $k^{th}$ Betti number of $SP^d(X)$ is the coefficient of $x^k t^d$ in the power series expansion of

$$\frac{(1 + xt)^{b_1} (1 + x^2 t)^{b_2} \cdots}{(1 - t)^{b_0} (1 - x^2 t)^{b_2} (1 - x^4 t)^{b_4} \cdots},$$

and therefore the Euler-Poincaré characteristic of $SP^d(X)$ is the coefficient of $t^n$. Finally we have

$$E^p = \sum_{d=0}^{\infty} \chi(SP^d(X)) t^d = \frac{1}{(1 - t)^{\chi(X)}}.$$
Chapter 3

Toric Varieties

3.1 Strongly Convex Rational Polyhedral Cones

In this section we state some definitions and theorems without proof about convex polyhedral cones, we refer the reader to ([Ful89], [Oda88], [Oda91]). We start with the following definition,

Definition 3.1.1 A subset $\sigma \subseteq V$ of the vector space $V$ is called a convex polyhedral cone if

$$\sigma = \mathbb{R}_{\geq 0} V_1 + \cdots + \mathbb{R}_{\geq 0} V_s = \{r_1 V_1 + \cdots + r_s V_s | r_i \in \mathbb{R}, r_i \geq 0, V_i \in V\}$$

$\{V_i\}$ (sometimes $\mathbb{R}_{\geq 0} V_i$) are called generators of $\sigma$.

The dimension of $\sigma$ is the dimension of the subspace $\mathbb{R}\sigma = \sigma + (-\sigma)$ of $V$.

The first Lemma that we will need later is due to Carathéodory:

Lemma 3.1.1 (Carathéodory)

Let $\sigma$ be a convex Polyhedral cone with generators $V_1, \ldots, V_s$ the any nonzero...
vector \( V \) in \( \sigma \) can be written as
\[
V = \sum_{i=1}^{l} r_i V_i, \quad \text{with} \quad l \leq \dim \sigma \quad \text{and} \quad \{V_i\} \quad \text{is a subset of linearly independent vectors of} \quad \{V_1, \ldots, V_l\}. 
\]

Let \( V^* \) be the dual space with dual pairing denoted by
\[
\langle , \rangle : V^* \times V \rightarrow \mathbb{R}
\]
The dual cone \( \sigma^* \) of \( \sigma \) is the subset of \( V^* \) defined by
\[
\sigma^* = \{u \in V^* | \langle u, v \rangle \geq 0 \quad \forall v \in \sigma\}
\]
and it is easy to see that \( \sigma^* \) is also a convex polyhedral cone.

A cone \( \sigma \) is called **Strictly Convex** if it contains no vector subspace but \( \{0\} \), in other words, if \( \sigma + \{-\sigma\} = \{0\} \). In fact, \( \sigma \) is strictly convex if and only if \( \sigma^* \) spans \( V^* \). For any cone \( \sigma \), the cone
\[
\sigma^\perp = \{u \in V^* | \langle u, v \rangle = 0 \quad \forall v \in \sigma\} = \sigma^* \cap (-\sigma)^*
\]
is a vector space.

**Definition 3.1.2** A subset \( \tau \) of \( \sigma \) is called a **face** if there exists \( u_0 \in V^* \) such that
\[
\tau = \sigma \cap u_0^\perp.
\]
\( \tau \) itself is a convex polyhedral cone.

**Definition 3.1.3** \( \sigma \) is simplicial if it is spanned by linearly independent generators.
Let $\mathbb{N}$ be a lattice, there is $r$ such that $\mathbb{N} \cong \mathbb{Z}^r$. Let $M = \text{Hom}(\mathbb{N}, \mathbb{Z})$ be the dual lattice with pairing

$$<,>: M \times N \rightarrow \mathbb{Z}$$

Let $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$ (resp. $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$) be the vector space over $\mathbb{R}$ generated by $N$ (resp. $M$).

**Definition 3.1.4** A strictly convex polyhedral cone is rational if it has generators $V_1, \ldots, V_s$ with each $V_i$ in $N$.

Everything said up to this point preserves rationality. From now on, a cone $\sigma$ means a strictly convex polyhedral cone. The following proposition is one of the main keys to construct toric varieties.

**Proposition 3.1.1** Let $\sigma$ be a cone in $N_\mathbb{R}$. Then

1. $S_\sigma := M \cap \sigma^* = \{ m \in M \mid <m, v> \geq 0 \forall v \in \sigma \}$ is an additive subsemigroup, containing 0, of $M$.

2. $S_\sigma$ is finitely generated as an additive semigroup, in other words, there exist $m_1, \ldots, m_p \in S_\sigma$ such that $S_\sigma = \mathbb{Z}_{\geq 0}m_1 + \cdots + \mathbb{Z}_{\geq 0}m_p$.

3. $S_\sigma$ generates $M$ as a group, i.e. $S_\sigma + (-S_\sigma) = M$.

4. $S_\sigma$ is saturated, i.e. $cm \in S_\sigma; m \in M, c \in \mathbb{Z}_{\geq 0} \Rightarrow m \in S_\sigma$.

Conversely, for any additive semigroup $S$ of $M$ satisfying the properties (1) through (4), there exists a unique strongly convex rational polyhedral cone $\sigma$ in $N_\mathbb{R}$ such that $S = S_\sigma$.

We are going to prove only (2), which is known as Gordan's lemma.

**Proof of (2):** Because of Carathéodory's lemma, we can assume the cone $\sigma^*$
is generated by linearly independent vectors, \( u_i \in \sigma^* \cap M \). Now

\[
K = \{ \sum r_i u_i \mid 0 \leq r_i \leq 1 \} \cap M
\]

is compact and discrete, so finite, and since any \( u \in \sigma^* \cap M \) has the form

\[
\sum (a_i + r_i) u_i
\]

for \( a_i \in \mathbb{Z} \) and \( r_i \in [0,1] \), then \( K \) generates \( \sigma^* \cap M \) as semigroup.

The next lemma is the last one in this introductory section.

**Lemma 3.1.2** If \( \tau \) is a face of \( \sigma \) and if \( S_\sigma = \sigma^* \cap M \), the there is a \( u \in \sigma^* \cap M \) with \( \tau = \sigma \cap \tau^\perp \) and

\[
\tau^* \cap M = S_\sigma + \mathbb{Z}_{\geq 0}(-u).
\]

For a proof, see ([Oda88], chap1.).

### 3.2 Affine Toric Varieties

Let \( \mathbb{C}[S] \) be the commutative \( \mathbb{C} \)-algebra determined by a semigroup \( S \).

As a vector space, \( \mathbb{C}[S] \) has a basis \( \{ x^s \mid s \in S \} \) and multiplication given by \( x^s x^{s'} = x^{s+s'} \). If \( \{ s_i \} \) is a set of generators for \( S \), then \( \{ x^{s_i} \} \) is a set of generators for \( \mathbb{C}[S] \). Now, consider \( \text{Spec} (\mathbb{C}[S]) \), here, when we speak of a point \( p \) of \( \text{Spec} (\mathbb{C}[S]) \) we understand that \( p \) is a closed point. We recall that any homomorphism \( A \to B \) of \( \mathbb{C} \)-algebras determines a morphism \( \text{Spec} (B) \to \text{Spec} (A) \) of varieties, in particular we have that closed points correspond to \( \mathbb{C} \)-algebra homomorphisms from \( A \) to \( \mathbb{C} \). And if \( X = \text{Spec} (A) \), for each element \( x \in \text{Spec} (A) \), \( x \neq 0 \), the principal open subset \( X_x = \text{Spec} (A_x) \subset X = \text{Spec} (A) \) correspond to the localisation homomorphism \( A \to A_x \).

An important example is the torus:
$S = M$ is a lattice. Choosing an isomorphism $M \cong \mathbb{Z}^n$ determines $2n$ semigroup generators
\[-1,0,\ldots,0), (0, -1, \ldots, 0), \ldots, (0, \ldots, 0, -1)\].

Letting $X_1 = \chi^{(1,0,\ldots,0)}$, so $X_1^{-1} = \chi^{(-1,0,\ldots,0)}$, and similarly for other basis elements, we have
\[C[M] = C[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]\]
\[C[X_1, \ldots, X_n]_{X_1, \ldots, X_n}\]
the ring of Laurent polynomials in $n$ variables. So
\[\text{Spec } (C[M]) = C^* \times \cdots \times C^* = (C^n = T)\]
is an affine algebraic torus. Any point of $T$ can be identify with a homomorphism from $M$ to $C^*$ as a group, i.e. $T^n \cong \text{Hom}_{gr}(M, C^*) = N \otimes \mathbb{Z} C^*$. We denote this group by $T_N$. Furthermore, for $S$ a semigroup, the points of $\text{Spec } (C[S])$ correspond to semigroup homomorphisms from $S$ to $C$:
\[\text{Spec } (C[S]) = \text{Hom}_{sg}(S, C)\].

**Definition 3.2.1** Let $\sigma$ be a rational strictly convex polyhedral cone in a lattice $N$. Let $S_\sigma$ be the semigroup $\sigma^* \cap M$ and let $A_\sigma := C[S_\sigma]$. Then
\[U_\sigma = \text{Spec}(A_\sigma) = \text{Hom}_{sg}(S_\sigma, C)\]
is called the Affine Toric Variety determined by $\sigma$. 
Example

\( N = \mathbb{Z}^n, \ \{ e_1, \ldots, e_n \} \) a basis for \( N \).

\[ \sigma = R_{\geq 0} e_1 + \cdots + R_{\geq 0} e_k, \ k \leq n \]

and

\[ \sigma^* \cap M = \mathbb{Z}_{\geq 0} e_1^* + \cdots + \mathbb{Z}_{\geq 0} e_k^* + \mathbb{Z}_{\geq 0} e_{k+1}^* + \cdots + \mathbb{Z}_{\geq 0} e_n^*. \]

\( A_\sigma = \mathbb{C}[x_1, \ldots, x_k, x_{k+1}, x_{k+1}^{-1}, \ldots, x_n, x_n^{-1}] \) so

\[ U_\sigma = \mathbb{C} \times \cdots \times \mathbb{C}^* \times \mathbb{C}^* = \mathbb{C}^k \times \left( \mathbb{C}^* \right)^{n-k} \]

and \( U_\sigma \) is nonsingular. Let \( \tau \) be a face of \( \sigma \), then from the previous section we have that there exists \( u \in \sigma^* \cap M \) such that \( \tau = \sigma \cap u^\perp \) and \( \tau^* \cap M = \sigma^* \cap u + \mathbb{Z}_{\geq 0} (-u) \). Then each element of \( \mathbb{C}[r^* \cap M] \) has the form \( \chi^{w - pu} = \chi^w / (\chi^u)^p \) for \( w \in \sigma^* \cap M \). Therefore \( A_\tau = (A_\sigma)^\chi^u \) and we have that \( U_\tau = \text{Spec}(A_\tau) \subset \text{Spec}(A_\sigma) = U_\sigma \) is a rational open subset.

The process of associating the toric variety \( U_\sigma \) with the pair \((N, \sigma)\) is a contravariant functor since, if \( \varphi : N' \to N \) is a homomorphism of lattices that maps \( \sigma' \subset N' \) to \( \sigma \subset N \) then dual \( \varphi^* : M \to M' \) maps \( \sigma^* \) to \( \sigma'^* \), determining a homomorphism \( A_\sigma \to A_{\sigma'} \) and hence \( U_{\sigma'} \to U_\sigma \).

Finally, it is easy to see that we have an action of \( T_N \) over \( U_\sigma \) which extends the action of \( T_N \) on itself, given by multiplication. More precisely, a point \( t \in T_N \) is represented by a map \( M \to \mathbb{C}^* \) of groups, and \( \chi \in U_\sigma \) by a map \( S_\sigma \to \mathbb{C} \) of semigroups. The product \( t \cdot \chi \) is the map of semigroups \( S_\sigma \to \mathbb{C} \) given by \( s \mapsto t(s) \chi(s) \) and the dual map on algebras \( \mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M] \) is given by \( \chi^s \mapsto \chi^s \otimes \chi^s \) for \( s \in S_\sigma \).

Now let us go to see some basic properties of affine varieties.
Proposition 3.2.1 An affine toric variety $U_{\sigma}$ is nonsingular if and only if $\sigma$ is generated by part of a basis for the lattice $N$, in which case

$$U_{\sigma} \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, \ k = \dim(\sigma).$$

Proof

$\Leftarrow$ See example above.

$\Rightarrow$ We can assume $k=n$. Let $\mathcal{M}$ be the maximal ideal in $A_{\sigma}$ generated by all $\chi^u$ for $u \in S_{\sigma} - \{0\}$. Then $\mathcal{M}/\mathcal{M}^2$ is generated by all $\chi^u$ such that $u$ in $S_{\sigma} - \{0\}$ and $u$ is not the sum of two elements in $S_{\sigma} - \{0\}$. Let $P_M$ be the point in $U_{\sigma}$ that correspond to $\mathcal{M}$, then if $U_{\sigma}$ is not singular at $P_M$, the cotangent space $\mathcal{M}/\mathcal{M}^2$ is n-dimensional, then $\sigma^*$ have n edges and the minimal generators along these generate $S_{\sigma}$, but $S_{\sigma}$ generates $M$ as a group so the dual $\sigma$ is generated by a part of a basis for $N$, therefore $U_{\sigma} \cong \mathbb{C}^n$.

Proposition 3.2.2 $U_{\sigma}$ is normal.

Proof

Let $A_{\sigma} = \mathbb{C}[\sigma^* \cap M]$, we have to show that $A_{\sigma}$ is integrally closed. Let $\tau_1, \ldots, \tau_n$ be the faces of codimension 1 of $\sigma$. Then we have that $\Sigma^* = \cap_{i=1}^n \tau_i^*$. Therefore $A_{\sigma} = \cap A_{\tau_i}$ is isomorphic to $\mathbb{C}[x_1, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ which is integrally closed. So is $A_{\tau_i}$.

3.3 Toric Varieties and their Orbits

Definition 3.3.1 A Fan $\Delta$ is a set of cones $\sigma$ in $N$ such that
1) if \( \tau \) is a face of \( \sigma \in \Delta \) then \( \tau \in \Delta \),

2) if \( \sigma, \sigma' \in \Delta \) then \( \sigma \cap \sigma' \) is a face of both, \( \sigma \) and \( \sigma' \).

We are going to consider only finite fans. For each \( \sigma \in \Delta \) consider its respective \( U_\sigma \). We can glue all the \( U_\sigma \) in the following way: take \( \sigma \) and \( \tau \) in \( \Delta \), then \( \sigma \cap \tau \) is a face of both \( \sigma \) and \( \tau \). Then \( U_{\sigma \cap \tau} \) is identified as an open subvariety of \( U_\sigma \) and of \( U_\tau \). We glue \( U_\sigma \) and \( U_\tau \) by identifying this common subvariety \( U_{\sigma \cap \tau} \). Furthermore, it is proven easily that the action of \( T_N \) is compatible.

Now, suppose \( \varphi : N' \to N \) is a homomorphism of lattices and that \( \Delta \subset N \), \( \Delta' \subset N' \) are two fans in \( N \) and \( N' \) respectively. Assume that for each cone \( \sigma' \) in \( \Delta' \) there exists a cone \( \sigma \) in \( \Delta \) such that \( \varphi(\sigma') \subset \sigma \), then the dual

\[ \mathcal{L} - \text{homomorphism } \varphi^* : M' \to M \]

induces a semigroup homomorphism

\[ \varphi^* : M' \cap (\sigma')^* \to M \cap \sigma^* \]

which induces an equivariant morphism \( \varphi_* : U_\sigma \to U'_{\sigma'} \). We can glue equivariant morphisms of this form for toric varieties and obtain a general equivariant morphism. In other words:

**Theorem 3.3.1** Via a covariant functor

\[ (N, \Delta) \mapsto X(\Delta) \text{and } \varphi \mapsto \varphi_* \]

the category of fans together with maps of fans is equivalent to the category of toric varieties over \( \mathbb{C} \) together with equivariant morphisms.
The main reason for introducing fans is that the algebro-geometric properties of toric varieties can be interpreted (in a combinatorial way) using the fan associated to them. The next two proposition are examples of this fact.

**Proposition 3.3.1** The toric variety $X(\Delta)$ is nonsingular iff each $\sigma \in \Delta$ is generated by elements of a basis of $N$.

We have already given a proof of this result for the affine case.

**Proposition 3.3.2** $X(\Delta)$ is compact iff $\Delta$ is finite and complete, where complete means that $N_{\mathbb{R}} = \cup_{\sigma \in \Delta} \sigma$.

For a complete proof of these two facts see [Oda88], [Ful89].

**ORBITS:**

We would like to describe the orbits of $X(\Delta)$ under the action of $T_N$. For each $\tau \in \Delta$, let $N_\tau$ be the lattice generated by $\tau \cap N$.

Consider $N(\tau) := N/N_\tau$ and its dual $M(\tau) = \tau^\perp \cap M$. Define

$$O_\tau := T_{N_\tau} = \text{Spec} (\mathbb{C}[M(\tau)]) = \text{Hom} (M(\tau), \mathbb{C}^*) = N(\tau) \otimes \mathbb{C}^*.$$

This is the torus associated to the lattice $N_\tau$. Its dimension is $n-k$, where $n=\dim \Delta$ and $k=\dim \tau$. Observe that $T_N$ acts transitively via the projection $T_N \to T_{N_\tau}$, i.e. $O_\tau$ is an orbit.

Let us describe the closure $V(\tau)$ of $O_\tau$. Consider the following set $A := \{\sigma \in \Delta | \tau < \sigma\}$. For any $\sigma \in A$ we denote the image of $\sigma$ in $N_\tau$ by $\overline{\sigma}$. The set $\{\overline{\sigma} | \tau < \sigma\}$ forms a fan $st(\tau)$ in $N_\tau$. Set $V(\tau) = X(st(\tau))$; here $O_\tau$ corresponds
to the cone \( \{0\} = \tau \) in \( N_\tau \). This toric variety \( V(\tau) \) is covered by \( \{U_\sigma(\tau)\}_{\sigma \in \Delta, \tau < \sigma} \) and

\[
U_\sigma(\tau) = \operatorname{Spec}(C[\mathcal{F}^* \cap M(\tau)]) = \operatorname{Spec}(C[\sigma^* \cap \tau^\perp \cap M]).
\]

We have the projection map from \( \sigma^* \cap M \) to \( C[\sigma^* \cap \tau^\perp \cap M] \), both considered as semigroups, the latter with the multiplication operation, this map sends \( \sigma^* \cap \tau^\perp \cap M \) to the identity and the complement goes to zero. This induces a surjective map

\[
C[\sigma^* \cap M] \to C[\sigma^* \cap \tau^\perp \cap M]
\]

and this gives us a map

\[
U_\sigma(\tau) = \operatorname{Spec}(C[\sigma^* \cap \tau^\perp \cap M]) \leftarrow \operatorname{Spec}(C[\sigma^* \cap M]) = U_\sigma.
\]

In other words, we have a closed embedding from \( U_\sigma(\tau) \) to \( U_\sigma \), for each \( \sigma > \tau \).

It is easy to see that these maps are compatible on \( \{U_\sigma(\tau)\}_{\sigma \in \Delta, \tau \in \sigma} \) then we have a closed embedding \( V(\tau) \leftarrow X(\Delta) \).

Finally, we have an ordering reversing correspondende from cones \( \{\tau\} \) in \( \Delta \) to closure of orbits \( V(\tau) \) in \( \Delta \).

To finish this section, we state the relations among orbits \( O_\tau \), orbit closures \( V(\tau) \), and the affine open sets \( U_\sigma \).

**Proposition 3.3.3** Let \( U_\sigma V(\tau) \) and \( O_\tau \) as before, then

I) \( U_\sigma = \bigcup_{\tau < \sigma} U_\tau \)

II) \( V(\tau) = \bigcup_{\gamma > \tau} V_\gamma \)

III) \( O_\tau = V(\tau) - \bigcup_{\gamma > \tau} V(\gamma) \)
In particular, it says that $X(\Delta)$ is a disjoint union of the $O_\tau$, which are the orbits of $T_n$ and $O_{\sigma} \subset O = V(\sigma)$ if and only if $\sigma$ is a face of $\tau$.

sketch of the proof

Take $\varphi \in U_\sigma$, in other words, $\varphi \in Hom_{sg}(s_\sigma, C)$. Using lemma 3.1.2, and the fact that two elements of $\sigma^*$ can not be in $\varphi^{-1}(C^*)$ unless both are in $\varphi^{-1}(C^*)$ we can prove that the set $\{m \in s_\tau | \varphi^{-1}(m) \in \varphi^{-1}(C^*)\}$ is of the form $M \cap \sigma^* \cap \tau^*$, for a unique face $\tau$ of $\sigma$. In other words, for each $\varphi \in U_\sigma$, $\varphi^{-1}(C^*) = M \cap \sigma^* \cap \tau^*$ and this means that $\varphi$ corresponds to a point of $O_\tau$. This prove 1. For 3, if we go to $N(\tau)$, i.e. in $V(\tau)$ we may assume $\tau = 0$, we have $T_n = X(\Delta) - \cup_{\sigma \neq [0]} V(\sigma)$, now, intersecting with $U(\sigma)$ three follows from one, and two follows from 3 using induction on dimension.

3.4 Cohomology

Let $A^p(X(\Delta))$ be the group of the formal finite $\mathbb{Z}$-linear combinations of closed irreducible subvarieties of $X(\Delta)$ of codimension $p$, modulo rational equivalence. The intersection of cycles induces a product

$$A^p(X(\Delta)) \times A^q(X(\Delta)) \rightarrow A^{p+q}(X(\Delta))$$

so

$$A^r(X(\Delta)) = \oplus_{i=0}^{r} A^i(X(\Delta))$$

is a graded ring of $X(\Delta)$.

For each closed irreducible subvariety we can associate to it its fundamental
cohomology class and this map induces an additive homomorphism

\[ A^p(X(\Delta)) \longrightarrow H^{2p}(X(\Delta), \mathcal{I}) \]

which doubles the degrees.

Let \( \Delta(J) \) be the set of \( J \)-dimensional cones in \( \Delta \). For each \( \sigma \in \Delta(j) \) let us denote by \( v(\sigma) \in A^j(X) \) the rational equivalence class of the irreducible cycle \( V(\sigma) \) on \( X \) of codimension \( j \). For distinct \( D_1, \ldots, D_s \) in \( \Delta(1) \) we have

\[
v(D_1) \cdot \cdots \cdot v(D_s) = \begin{cases} 
D_1 + \cdots + D_s & \text{if } D_1 + \cdots + D_s \in \Delta \\
0 & \text{otherwise}
\end{cases}
\]

Furthermore, each \( m \in M \) determines a divisor that corresponds to the rational function \( \chi_m \) which is rationally equivalent to zero

\[
\sum_{i=1}^{r} <m, m(D_i) > v(D_i) = 0 \quad \text{for all } m \in M
\]

Here, \( r \) is the number of invariant divisors in \( X(\Delta) \) or equivalently, the cardinality of \( \Delta(1) \). For each \( D_i \in \Delta(1) \), we introduce the variable \( t_i \) and consider the polynomial ring \( S = \mathbb{Z}[t_1, \ldots, t_r] \) which is the symmetric algebra associated to the \( \mathbb{Z} \)-module \( T_N Div(X(\Delta)) \) of \( T_N \)-invariant divisors on \( X((\Delta)) \).

Let \( J \) be the ideal in \( S \) generated by

\[
\{ t_{i_1} \cdots t_{i_s} \mid t_{i_l} \neq t_{i_m} \text{ if } l \neq m \text{ and } D_1 + \cdots + D_s \not\in \Delta \}
\]

and let \( I \) be the ideal in \( S \) generated by

\[
\{ \sum_{j=1}^{r} <m, m(D_j) > v(D_i) \mid m \in M \}
\]
we have the following homomorphisms of graded rings

\[ S/(I + J) \longrightarrow A^\ast(X(\Delta)) \longrightarrow H^\ast(X(\Delta), \mathbb{Z}) \]

The main theorem of this section is due to Jurkiewicz and Danilov:

**Theorem 3.4.1** Jurkiewicz-Danilov's theorem.

Let \( X(\Delta) \) be an \( n \)-dimensional compact toric variety then,

I) If \( X(\Delta) \) is nonsingular, the homomorphisms of graded rings described above are in fact isomorphisms

\[ S/(I + J) \xrightarrow{\sim} A^\ast(X(\Delta)) \xrightarrow{\sim} H^\ast(X(\Delta), \mathbb{Z}). \]

II) If \( \Delta \) is simplicial, we can define the chow ring with coefficients in \( \mathbb{Q} \) as,

\( A^\ast(X(\Delta)) \otimes_{\mathbb{Z}} \mathbb{Q} \) and again we get isomorphisms of \( \mathbb{Q} \)-algebras

\[ S/(I + J) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} A^\ast(X(\Delta)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H^\ast(X(\Delta), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \]

and again we refer to the reader to [Dan78] for the proof.
Chapter 4

The Euler-Chow Series for Toric Varieties

4.1 General properties

Through this chapter toric variety means smooth projective toric variety, unless otherwise stated. Poincaré duality allow us to express the Euler-Chow series in terms of the cohomology groups of the variety, i.e., we can write

\[ E = \sum_{\lambda} \chi(C_\lambda) \lambda \quad \lambda \in H^{2p}(X, \mathbb{Z}) \]

where \( C_\lambda \) is the space of all effective cycles whose homology class is the Poincaré dual of \( \lambda \).

Let \( X(\Delta) \) be the toric variety associated to the fan \( \Delta \). Let \( m \) be the rank of \( H^{2p}(X, \mathbb{Z}) \). We know that

\[ m = \sum_{i=1}^{n} (-1)^{i-p} (\binom{p}{i}) d_{n-i} \]

where \( d_k \) is the number of cones of dimension \( k \) in \( \Delta \) and \( n \) is the complex dimension of \( X(\Delta) \). The following lemma describes the invariant effective cycles of \( X \).
Lemma 4.1.1 Let $X(\Delta)$ be a nonsingular projective toric variety. Then any irreducible subvariety $V$ of $X(\Delta)$ which is invariant under the torus action is the closure of an orbit. Therefore, any invariant cycle has the form $c = \sum n_i \overline{O}_i$ where each $n_i$ is a nonnegative integer and each $\overline{O}_i$ is the closure of the orbit $O_i$.

Proof Since the variety is compact, the fan $\Delta$ is finite. Hence there are a finite number of cones and, therefore a finite number of orbits. Let $V$ be an invariant irreducible subvariety of $X(\Delta)$, then $V$ is the closure of the union of orbits. Since there are a finite number of them we must have that

$$V = \overline{O}_1 \cup \overline{O}_2 \cup \cdots \cup \overline{O}_N$$

where $\overline{O}_i$ is the closure of an orbit. Finally since $V$ is irreducible, there must be $i_0$ such that $V = \overline{O}_{i_0}$.

Let $C^T_{\lambda}$ and $C_{\lambda}$ be the spaces of all effective invariant cycles and effective cycles respectively, with homology class $\lambda$. It is proved in [LY87] that

$$\chi\left(C^T_{\lambda}\right) = \chi\left(C_{\lambda}\right).$$  \hspace{1cm} (4.1)

We end this section with the following important lemma,

Lemma 4.1.2 Let $\lambda$ be an element in $H^{2p}(X, \mathbb{Z})$. Then $C_{\lambda}$ is a finite set.

Proof. Since any invariant effective cycle $c$ in $C^T_{\lambda}$ has the form $c = \sum_{i=1}^{N} \beta_i \overline{O}_i$ with $\beta_i \in \mathbb{N}$, we obtain that $C^T_{\lambda}$ has a countable number of elements. By Lemma 2.1.1, $C_{\lambda}$ is a projective algebraic variety, and since $C^T_{\lambda}$ is Zariski closed (see [Hum85], pag.59), we have that $C^T_{\lambda}$ is finite.
4.2 The equivariant Euler-Chow series

In this section we introduce the equivariant cohomology and the equivariant Euler-Chow series. We show it is rational, and find a rational function associated to it. We should start with the definition of equivariant cohomology. We refer the reader to ([AB84], [Bif92]). For any topological group $G$ the classifying space $BG$ is the base of a certain principal $G$-bundle $EG \to BG$ whose total space is contractible. If $G$ acts on a space $X$, we define $X_G = EG \times_G X$ as the associated bundle over $BG$ with fiber $X$. The group $G$ acts on $EG$ from the right, on $X$ from the left, and $X_G = EG \times_G X$ is constructed by identifying $(eg, x)$ with $(e, gx)$, for all $g \in G$. The equivariant cohomology of $X$ is defined as

$$H^n_G(X) = H^*(X_G).$$

Observe that the inclusion $X \hookrightarrow X_G$ gives an homomorphism

$$\pi : H^n_G(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$$

(4.2)

Let $\mathcal{O}$ be an irreducible invariant cycle in a smooth toric variety. Since $\mathcal{O} \subset X$ is smooth, we have an equivariant Thom-Gysin sequence

$$H^{i-2\text{codim}\mathcal{O}}_G(\mathcal{O}) \to H^i_T(X) \to H^i_T(X - \mathcal{O}) \to$$

and we define $[\mathcal{O}]_T$ as the image of 1 under

$$H^0_T(\mathcal{O}) \to H^{2\text{codim}\mathcal{O}}(X).$$

Let $\{D_1, \ldots, D_K\}$ be the set of $T$-invariant divisors on $X$. To each $D_i$ we associate the variable $t_i$ in the polynomial ring $\mathbb{Z}[t_1, \ldots, t_K]$. Let $I$ be the
ideal generated by the (square free) monomials \( \{ t_{i_1}, \ldots, t_{i_m} | i_m \neq i_n, m \neq n \) and \( D_i + \ldots + D_i \notin \Delta \). 

It is proved in [BDP90] that

\[
\mathbb{Z}[t_1, \ldots, t_K]/\mathcal{I} \cong H_T(X, \mathbb{Z})
\]  

(4.3)

The arguments given there also prove the following.

**Proposition 4.2.1** For any \( T \)-orbit \( \mathcal{O} \) in \( X \), one has

\[
[\mathcal{O}]_T = \prod_{\mathcal{O} \subset D_i} [D_i]_T.
\]

Furthermore if \( \mathcal{O} \) and \( \mathcal{O}' \) are distinct orbits, then

\[
[\mathcal{O}]_T \neq [\mathcal{O}']_T.
\]

It is natural to define the cohomology class for any effective invariant cycle \( V = \sum m_i \mathcal{O}_i \) as \( [V]_T = \sum_i m_i [\mathcal{O}_i]_T \). Where \( \mathcal{O}_i \neq \mathcal{O}_j \) if \( i \neq j \). We are ready for the following definition:

**Definition 4.2.1** Let \( X \) be a smooth projective toric variety, and let \( H_T^{2p}(X, \mathbb{Z}) \) be the equivariant cohomology of \( X \). Let us denote by \( C^T_\mathcal{X} \) the space of all invariant effective cycles on \( X \) whose equivariant cohomology class is \( \mathcal{X} \). The **equivariant Euler-Chow series of \( X \)** is the element

\[
E_T = \sum_\mathcal{X} \chi (C^T_\mathcal{X}) \cdot \mathcal{X} \in \mathbb{Z}^{H_T^{2p}(X, \mathbb{Z})}
\]

where the sum is over \( \mathcal{X} \in H_T^{2p}(X, \mathbb{Z}) \).

Next, we compute \( E_T \) for a certain basis.
Theorem 4.2.1 Let $X$ be a smooth projective toric variety of dimension $n$. Let $\{\mathcal{O}_1, \ldots, \mathcal{O}_N\}$ be the set of all distinct orbits on $X$ of codimension $p$. Let $B = \{y_1, \ldots, y_r\}$ be an integral basis for $H_T^{2p}(X,\mathbb{Z})$ such that $y_i = [\mathcal{O}_i]_T$ for $i = 1, 2, \ldots, N$. Then with respect to this basis

$$E_T = \prod_{i=1}^{N} \frac{1}{1-s_i}.$$ 

**Proof**

Two invariant cycles $V_1 = \sum m_i \mathcal{O}_i$ and $V_2 = \sum n_i \mathcal{O}_i$ represent the same equivariant cohomology class if and only if $m_i$ is equal to $n_i$. In other words

$$\chi(C^T_\xi) = \begin{cases} 1 & \text{if } \xi = \sum m_i [\mathcal{O}_i]_T \text{ where } m_i \geq 0 \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$ 

and so

$$E_T = \sum_{m \in \mathbb{N}} s_1^{m_1} \cdots s_N^{m_N} = \prod_{i=1}^{N} \frac{1}{1-s_i}.$$ 

4.3 The Euler-Chow series

In this section we prove that the Euler-Chow series is rational. In order to do that, we need to define a map $I$ from $D_0 \subset H_T^{2p}(X,\mathbb{Z})$ to $H^{2p}(X,\mathbb{Z})$. Let us begin by giving a description of the domain of $I$. Let $D_0$ be the subgroup of $H^{2p}(X,\mathbb{Z})$ whose elements $\sum n_\xi \cdot \xi$ have the property that for any $\lambda \in H^{2p}(X,\mathbb{Z})$, there are only a finite number of $\xi$'s with $n_\xi \neq 0$, and $\pi(\xi) = \lambda,$
where $\pi$ is the homomorphism defined in equation 4.2. Now, we can define
the map $I$ as follows.

**Definition 4.3.1** Let

$$I : \mathcal{D}_0 \subseteq \mathbb{Z}_T (X, \mathcal{X}) \rightarrow \mathbb{Z}_T (X, \mathcal{X})$$

be the map defined by

$$I \left( \sum_\xi n_\xi \xi \right) = \sum_\lambda \left( \sum_{\pi(\xi) = \lambda} n_\xi \right) \lambda$$

The first interesting property of $I$ is given in the following proposition.

**Proposition 4.3.1** Let $E_T$ and $E$ be the Euler-Chow series for the equivariant
cohomology and for the ordinary cohomology respectively. Then

$$E_T \in \mathcal{D}_0 \quad \text{and} \quad I(E_T) = E.$$

**Proof.** We start proving that $E_T \in \mathcal{D}_0$. In the proof of 4.2.1, we found
that $\chi(C^T_\xi) \neq 0$ if $C^T_\xi$ is the set which contains the unique element $\xi = \sum_{i=1}^N n_i [O_i]_T$. Since $\pi((O_i)_T) = [O_i]$ we obtain that

$$\pi(\xi) = \sum_{i=1}^N n_i [O_i].$$

By Lemma 4.1.2 we know that $C^T_\xi$ is finite. Therefore, there is a finite number
of $\xi$ such that $\pi(\xi) = \lambda$. The second part of the theorem is proved by the
following equalities,

$$I \left( \sum_\xi \chi(C^T_\xi) \xi \right) = \sum_\lambda \left[ \sum_{\pi(\xi) = \lambda} \chi(C^T_\xi) \right] \lambda = \sum_\lambda \chi(C^T_\lambda) \lambda = \sum_\lambda \chi(C_\lambda) \lambda$$

We are ready for the main theorem of this chapter.
Theorem 4.3.1. For any nonsingular projective toric variety $X$ over $\mathbb{C}$, the Euler-Chow series is rational. In fact, if $(\mathcal{O}_1, \ldots, \mathcal{O}_N)$ denote all the orbits of codimension $p$, and if $A = \{x_1, \ldots, x_m\}$ is a basis for $H^{2p}(X, \mathbb{Z})$. Then

$$E^p = \prod_{i=1}^{N} \left( \frac{1}{1 - \prod_{j=1}^{m} t_{ij}} \right)$$

where $[\mathcal{O}_i] = \sum_{j=1}^{m} a_{ji} x_j$ is the expression, in terms of the basis $A$, of the cohomology class of the closure $\overline{\mathcal{O}}_i$ of the orbit $\mathcal{O}_i$.

Proof.

We start by choosing a basis $B$ for $H^{2p}_T(X)$ and by expressing the map $I$ in terms of the bases $A$ and $B$. Let $B = \{y_1, \ldots, y_r\}$ be an integral basis for $H^{2p}_T(X, \mathbb{Z})$, such that $y_i = [\mathcal{O}_i]_T$, for $i = 1, \ldots, N$. It was proved in theorem 4.2.1 that

$$E_T = \prod_{i=1}^{N} \frac{1}{1 - s_i} = \sum_{i=1}^{N} s_{i_1} \cdots s_{i_N}.$$

Let

$$I_{AB} : \mathbb{Z}^r \to \mathbb{Z}^m$$

be the map $I$ with respect to the bases $A$ and $B$. Consider the following diagram

\[
\begin{array}{ccc}
\mathbb{Z}[[s_1, \ldots, s_N]] & \xrightarrow{\Phi} & \mathbb{Z}^r \\
\downarrow \Phi & & \downarrow I_{AB} \\
\mathbb{Z}[[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]] & \supset \mathcal{D} & \xrightarrow{D} & \mathbb{Z}^m
\end{array}
\]
where \( \Phi \) is defined on each generator \( s_i \) as

\[
\Phi(s_i) = \prod_{j=1}^{m} t_j^{a_{ji}^+} \tau_j^{a_{ji}^-}.
\]

We recall that for any integer number \( a \) we defined \( a^+ = \max \{a, 0\} \) and \( a^- = \max \{-a, 0\} \). Before we prove that the diagram commutes, we take a closer look at the number of effective invariant cycles in a given cohomology class. Let \( \lambda \in H^{2p}(X) \), and denote by \( \gamma = (\gamma_1, \ldots, \gamma_m) \) the coordinates of \( \lambda \) in terms of the basis \( A \). Let \( c = \sum_{i=1}^{N} \beta_i \mathcal{O}_i \) be an invariant effective cycle with cohomology class \([c] = \lambda\). Then,

\[
[c] = \sum_{i}^{N} \beta_i [\mathcal{O}_i] = \sum_{i}^{N} \beta_i \left( \sum_{j=1}^{m} a_{ji} x_j \right) = \sum_{j=1}^{m} \gamma_j x_j = \lambda.
\]

Therefore, the number \( \chi(C_{\lambda}) \) of invariant cycles with cohomology class \( \lambda \) is equal to the number \( N_\lambda \) of solutions to the system

\[
A \beta = \gamma \quad \text{for} \quad \beta \in \mathbb{N}^N. \tag{4.4}
\]

But it was proved in Lemma 4.1.2 that this number is finite. Therefore, the equality \( I(E_T) = E \) proved in the last proposition becomes

\[
I_{A\beta}(E_T) = I_{A\beta} \left( \sum_{\beta \in \mathbb{N}^N} s^\beta \right) = \sum_{\gamma \in \mathbb{Z}^m} N_\gamma t^\gamma = E.
\]

Next, we show that \( \Phi(\prod_{i=1}^{N} \frac{1}{1-s_i}) \in \mathcal{D} \). We have
\[ \Phi \left( \prod_{i=1}^{N} \frac{1}{1 - s_i} \right) = \prod_{i=1}^{N} \left( \frac{1}{1 - \prod_{j=1}^{m} t_j^{a_i^j m_j^i - n_j^i}} \right) \]
\[ = \sum_{\beta \in \mathbb{N}^N} \left( \prod_{j=1}^{m} t_j^{a_j^1 \beta_1} \cdots \prod_{j=1}^{m} t_j^{a_j^m \beta_m} \right) \beta_N \]
\[ = \sum_{\beta \in \mathbb{N}^N} t_1^{\sum_{i=1}^{m} a_1^i \beta_i} \cdots t_m^{\sum_{i=1}^{m} a_m^i \beta_i} \]
\[ = \sum_{\beta \in \mathbb{N}^N} t^{A^\beta} \tau^{-A^\beta}, \]

and we know that for each \( \gamma \in \mathbb{Z}^m \) the number of solutions to

\[ \gamma = A^+ \beta - A^- \beta = A \beta \quad \text{for} \quad \beta \in \mathbb{N}^N \]

is finite, and equal to \( N_\gamma \). Therefore, \( \Phi \left( \prod_{i=1}^{N} \frac{1}{1 - s_i} \right) = \sum_{\beta} t^{A^\beta} \tau^{-A^\beta} \in \mathcal{D} \).

Furthermore, we have that

\[ p \left( \Phi \left( \sum_{\beta \in \mathbb{N}^N} s^\beta \right) \right) = p \left( \Phi \left( \prod_{i=1}^{N} \frac{1}{1 - s_i} \right) \right) \]
\[ = p \left( \sum_{\beta} t^{A^\beta} \tau^{-A^\beta} \right) \]
\[ = \sum_{\beta} t^{A^\beta} \]
\[ = \sum_{\gamma} N_\gamma t^\gamma = E. \]

Finally,

\[ E = \sum_{\gamma} N_\gamma t^\gamma = \sum_{\gamma} N_\gamma t^{\gamma_1} \cdots t^{\gamma_m} \]
= \sum_{\beta \in \mathbb{N}^N} t_1^{a_{i1}\beta_1} \cdots t_m^{a_{mi}\beta_m} \\
= \sum_{\beta \in \mathbb{N}^N} \left( \prod_{j=1}^m t_j^{a_{ji}} \right) \beta_1 \cdots \left( \prod_{j=1}^m t_j^{a_{ji}} \right) \beta_N \\
= \prod_{i=1}^N \left( \frac{1}{1 - \prod_{j=1}^n t_j^{a_{ji}}} \right).

4.4 Examples

I) The projective space \( \mathbb{P}^n \)

Let \( X = \mathbb{P}^n \) be the complex projective space of dimension \( n \). Let \( \{e_1, \ldots, e_n\} \) be the standard basis for \( \mathbb{R}^n \). Consider \( A = \{e_1, \ldots, e_{n+1}\} \) a set of generators of the fan \( \Delta \) where \( e_{n+1} = -\sum_{i=1}^n e_i \). We have the following equality

\[ H^*(X, \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_{n+1}] / I \]

where \( I \) is the ideal generated by

\[ i) \quad t_1 \cdots t_{n+1} \]

and

\[ ii) \quad \sum_{j=1}^{n+1} e_i^*(e_j) t_j \quad \text{for} \quad i + 1, \ldots, n + 1, \]

where \( e_i^* \in (\mathbb{R})^* \) is the element dual to \( e_i \).

However \( ii) \) says that \( t_i \sim t_j \) for all \( i, j = 1, \ldots, n + 1 \). Therefore

\[ H^*(X, \mathbb{Z}) = \mathbb{Z}[t] / t^{n+1}. \]
Consequently any two cones of dimension $p$ represent the same element in cohomology, and Theorem 4.3.1 implies, that

$$\prod_{i=1}^{n+1} \left( \frac{1}{1-t} \right) = \left( \frac{1}{1-t} \right)^{(n+1)} = E^p.$$  

where, as always, $\alpha \in H^{2p}(X, \mathbb{Z})$. Note that the power $^{(n+1)}p$ is different from the power $^{(n+1)}p+1$ obtained in Example I, chapter II. This is because we work with cohomology while in chapter II we worked with homology. If we would consider cones of dimension $n - p$ instead of cones of dimension $p$, then we would get

$$\left( \frac{1}{1-t} \right)^{(n+1)} = \left( \frac{1}{1-t} \right)^{(n+1)} = E^p$$

since $^{(n+1)}n = ^{(n+1)}n+1$, we obtain the same answer as in Chapter II.

II) $P^n \times P^m$

Recall that $X(\Delta') \cong X(\Delta) \times X(\Delta')$. Using the same notation as that in I) we have that a set of generators of $\Delta \times \Delta'$ is given by

$$\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}, e_{n+m+1}, e_{n+m+2}\}$$

with $e_{n+m+1} = -\sum_{i=1}^n e_i$ and $e_{n+m+2} = -\sum_{i=n+1}^{n+m} e_i$. and $\{e_1, \ldots, e_{n+m}\}$ is a basis for $P^n \times P^m$. Then

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_{n+m+2}]/I$$

where $I$ is the ideal generated by

i) $\{t_1 \cdots t_n t_{n+m+1}, t_{n+m+1} \cdots t_{n+m+2}, \prod_{i=1}^{n+m+2} t_i\}$
and

\[ \sum_{j=1}^{n+m+2} c_i(e_j) t_j \quad i = 1, \ldots, n+m. \]

From ii) we obtain,

\[ t_i \sim t_{n+m+1} \quad \text{if} \quad 1 \leq i \leq n \]

\[ t_j \sim t_{n+m+2} \quad \text{if} \quad n+1 \leq j \leq n+m \]

The number of cones of dimension \( p \) is equal to \( \sum_{k+i=p} \binom{n+1}{k} \binom{m+1}{i} \).

Denote by \( t_{J_1} = t_{i_1} \cdots t_{i_k} t_{j_1} \cdots t_{j_l} \). Then

\[ \prod_{k+i=p} \left( \frac{1}{1-t_{k,j}} \right) \binom{n+1}{k} \binom{m+1}{i} = E^p \]

where \( t_{k,j} = t_{n+m+1}^{k} t_{n+m+2}^{l} \).

III) Blow up of \( \mathbb{P}^n \)

Let \( \Delta \) be the fan associated to \( \mathbb{P}^n \). We know by example 1 that \( \Delta \) is generated by \( \{ e_1, \ldots, e_{n+1} \} \) where \( e_{n+1} = -\sum_{i=1}^{n} e_i \) and \( \{ e_1, \ldots, e_n \} \) is the canonical basis for \( \mathbb{R}^n \). The fan \( \bar{\Delta} \) associated to the blow up \( \mathbb{P}^n \) of the projective space at the fixed point given by the cone \( \mathbb{R}^+ e_2 + \cdots + \mathbb{R}^+ e_{n+1} \) is generated by \( \{ e_1, \ldots, e_{n+1}, e_{n+2} \} \) where \( e_{n+2} = -e_1 \). Denote by \( D_i \) the 1-dimensional cone \( \mathbb{R}^+ e_i \) and by \( s_i \) its class in cohomology where

\[ H^*(X, \mathbb{Z}) = \mathbb{Z}[s_1, \ldots, s_{n+2}] / I \]

and \( I \) is the ideal generated by

i) \( \{ s_{i_1} \cdots s_{i_k} : D_{i_1} \cdots D_{i_k} \text{ is not in } \bar{\Delta} \} \)

and

\[ \sum_{j=1}^{n+2} c_i(e_j) s_j \quad i = 1, \ldots, n. \]
However \( ii \) is equivalent to

\[
\begin{align*}
&\mathrm{ii)} \ s_2 \sim \cdots \sim s_{n+1} \quad \text{and} \quad s_1 \sim s_{n+1} + s_{n+2} \end{align*}
\]

Note that a p-dimensional cone cannot contain \( D_{n+2} \) and \( D_1 \). The reason is that \( D_{n+2} \) is generated by \(-e_1\) and \( D_1 \) by \( e_1 \), but by definition, a cone does not contain a subspace of dimension greater than 0. We would like to find a basis for \( H^*(\bar{\mathbb{P}}^n) \) and write any monomial of degree \( p \) in terms of it. Consider the monomial \( s_{i_1} \cdots s_{i_p} \). There are three possible situations:

1) \( s_{i_j} \) is different from both \( s_{n+2} \) and \( s_{n+1} \). In this situation we have from \( \mathrm{ii} \) that \( s_{i_1} \cdots s_{i_p} = s_{n+1}^p \).

2) \( s_{n+2} \) is equal to \( s_{i_j} \) for some \( j, 1, \ldots, p \). Then from \( \mathrm{ii} \) we obtain that \( s_{i_1} \cdots s_{i_p} = s_{n+1}^{p-1} s_{n+2} \).

3) \( s_1 \) is equal to \( s_{i_j} \) for some \( j, 1, \ldots, p \). Then from \( \mathrm{ii} \) we obtain \( s_{i_1} \cdots s_{i_p} = (s_{n+1} + s_{n+2}) s_{n+1}^{p-1} = s_{n+1}^p + s_{n+2} s_{n+1}^{p-1} \) which is the sum of 1) and 2).

We conclude that \( s_{n+1}^p \) and \( s_{n+2} s_{n+1}^{p-1} \) form a basis for \( H^{2p} \) if \( p \leq n \). If \( p = n \) then \( s_{n+1}^p = 0 \) and the only generator is \( s_{n+2} s_{n+1}^{p-1} \). Let us call \( s_{n+1} \) by \( t_1 \) and \( s_{n+2} s_{n+1}^{p-1} \) by \( t_2 \). The Euler-Chow series for \( \bar{\mathbb{P}}^n \) is:

\[
E^p = \left( \frac{1}{1-t_1} \right)^{\binom{p}{2}} \left( \frac{1}{1-t_1 t_2} \right)^{\binom{p-1}{p}} \left( \frac{1}{1-t_2} \right)^{\binom{p}{p}} \quad \text{if} \quad p \leq n.
\]

\[
E^n = \left( \frac{1}{1-t_2} \right)^{\binom{n+2}{p}} \quad \text{if} \quad p = n.
\]

**IV) Hirzebruch surfaces**

A set of generators for the fan \( \Delta \), in \( \mathbb{R}^2 \), that represents the Hirzebruch surface \( X(\Delta) \) is given by, \( \{e_1, \ldots, e_4\} \) with \( \{e_1, e_2\} \) the standard basis for \( \mathbb{R}^2 \), and \( e_3 = -e_1 + ae_2, \ a > 1 \) and \( e_4 = -e_2 \).
With the same notation as in last examples, we have,

\[ H^*(x(\Delta)) = \mathbb{Z}[t_1, \ldots, t_4] / I \]

where I is generated by

1) \{t_1t_3, t_2t_4, t_1t_2t_3, t_1t_2t_3t_4 : i_j \neq i_{j'} \text{ for } j \neq j' \}

and

2) \{t_1 - t_3, t_2 + at_3 - t_4 \}

from ii) we have the following conditions for the \( t_i \)'s in \( H^*(X) \)

\[ (*) \quad t_1 \sim t_3 \text{ and } t_2 \sim t_4 - at_3. \]

A basis for \( H^*(x(\Delta)) \) is given by \{\{0\}, t_3, t_4, t_4t_1\} (see [Dan78]). The Euler-Chow series for each dimension is:

1) Codimension 2: There are four orbits (four cones of dimension 2), and all of them are equivalent in homology. From Theorem 4.3.1 we obtain

\[ E^2 = \left( \frac{1}{1-t} \right)^4 \]

2) Codimension 1: Again, there are four orbits (four cones of dimension 1), and the relation among them, in homology is given by \( (*) \). From Theorem 4.3.1 we obtain

\[ E^1 = \left( \frac{1}{1-t_3} \right)^2 \left( \frac{1}{1-t_4} \right) \left( \frac{1}{1-t_5 t_4} \right). \]

3) Codimension 0: The only orbit is the torus itself, from Theorem 4.3.1 we have

\[ E^0 = \left( \frac{1}{1-t} \right). \]
Appendix A

Letter from Emili Bifet

As I mentioned before, Emili’s letter gives a sketch of how to reformulate, in intrinsic form, the work done in this thesis. I received it one week before the defense, so it was hard to rewrite everything in the terms he had suggested. Since Emili’s approach is very elegant, I decided it would be worth to add the letter as an appendix.

August 1, 1992

Dear Blaine and Javier,

Here follows what I hope is an intrinsic reformulation of Javier’s results (I have not checked everything in full detail, but it seems to work.)

1. $H$ (resp. $H_T$) denotes the $2p$ cohomology (resp. equivariant cohomology) group.
2. $C \subset H$ (resp. $C_T \subset H_T$) denotes the sub-monoid of cohomology (resp. equivariant cohomology) classes of effective (resp. invariant effective) $p$-cycles.

3. $\pi : H_T \to H$ denotes the standard surjection.

4. $\pi : C_T \to C$ is surjective with finite fibres (the Chow variety is projective and there are countably many fixed points.)

5. $+: C_T \times C_T \to C_T$ has finite fibres ($C_T \simeq N^N$, where $N$ is the number of $p$-orbits.)

6. The following diagram commutes:

\[
\begin{array}{c}
\pi \times \pi \\
\downarrow \quad \downarrow \pi \\
C \times C &\to & C \\
\end{array}
\]

$+$

$(\pi$ is a homomorphism.)

7. $+: C \times C \to C$ has finite fibres (follows from 4, 5 and 6.)

8. $\mathcal{A} \subset \mathcal{Z}^H$ (resp. $\mathcal{A}_T \subset \mathcal{Z}^{HT}$) denotes the functions with support in $C$ (resp. $C_T$.)

\[
\begin{array}{c}
\end{array}
\]
9. \( \mathcal{A} \) (resp. \( \mathcal{A}_T \)) is a commutative ring (from 7 and 5.)

10. \( I : \mathcal{A}_T \to \mathcal{A} \) (defined by \( I(u)(\lambda) = \sum_{\xi \in \pi^{-1}(\lambda)} u(\xi) \)) is a ring homomorphism.

11. \( E \) (resp. \( E_T \)) denotes the Euler-Chow (resp. equivariant Euler-Chow) invariant.

12. \( E_T \in \mathcal{A}_T \) and \( I(E_T) = E \).

13. Define for each \( p \)-orbit \( \mathcal{O}_i \) an element \( f_i \in \mathcal{A} \) (resp. \( f_i^T \in \mathcal{A}_T \)) by:

\[
 f_i(\lambda) = \begin{cases} 
 1 & \text{if } \lambda = n \cdot [\mathcal{O}_i], \ n \geq 0 \\
 0 & \text{otherwise.}
\end{cases}
\]

(resp.

\[
 f_i^T(\xi) = \begin{cases} 
 1 & \text{if } \xi = n \cdot [\mathcal{O}_i]_T, \ n \geq 0 \\
 0 & \text{otherwise.}
\end{cases}
\]

We also denote \( e_\lambda \) (resp. \( e_\xi \)) the characteristic function of \( \{\lambda\} \) (resp. \( \{\xi\} \)). Note in particular that \( 1 = e_0 \).

14. \( I(f_i^T) = f_i \) and \( I(e_\xi) = e_{\pi(\xi)} \).

15. \( \pi[\mathcal{O}_i]_T = [\mathcal{O}_i] \).

16. \( E_T = \prod_{1 \leq i \leq N} f_i^T \) and \( (1 - e_{[\mathcal{O}_i]_T}) \cdot f_i^T = 1 \).

17. \( E = \prod_{1 \leq i \leq N} f_i \) and \( (1 - e_{[\mathcal{O}_i]}) \cdot f_i = 1 \) (apply \( I \) to 16.)
18. Finally:

\[ E = \frac{1}{\prod_{1 \leq i \leq N} (1 - e^{[\beta_i]})}. \]

19. The ring \( \mathcal{A} \) contains the monoid-ring \( \mathbb{Z}[C] \) (\( = \) functions with finite support.) It is perhaps natural to say that an element of \( \mathcal{A} \) is "rational" provided it is the quotient of two elements of \( \mathbb{Z}[C] \). If one wants to consider bases, one will perhaps have to use Gordan's lemma and ...

20. It is tempting to speculate about the general case (with \( H \) being the homology group and \( C \) the sub-monoid of homology classes of effective cycles.) When is \( E \in \mathcal{A} \) rational in the above sense? Well ...

Sincerely,

Emili
Bibliography


Bibliography


