Nonnegative Ricci curvature near infinity and the geometry of ends

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Abstract of the Dissertation

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We study the geometric growth of ends of a complete noncompact Riemannian manifold M^n whose Ricci curvature is nonnegative outside a compact set B, i.e. $Ric_{M-B} \geq 0$. Our main result is the establishment of the following ball covering property: for a fixed point p_0 and $0 < \mu < 1$, $B_{p_0}(r)$ can be covered by a bounded number of balls of radius μr; this bound, called the packing number, is independent of r. As consequences, M has a bounded number of ends, the diameter growth of ends is at most linear, and the topological growth of M is bounded by a polynomial of degree n. We also discuss the application of the ball covering property in the study of harmonic functions.

Dedicated to my parents in China

To my wife Ching and daughter Karen

Contents

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Chapter 1. Introduction

In this dissertation we study the geometry of ends of complete, noncompact Riemannian manifolds with nonnegative Ricci curvature outside a compact set, a problem that has received increasing attention in recent years.

On a complete open Riemannian manifold with nonnegative Ricci curvature, the volume comparison of Bishop and Gromov ([BC] and [GLP]) is a very powerful tool. It can be used to show that the geometric growth of the ends of the manifold is well controlled. Namely:

(a) The volume of the geodesic ball $B_p(r)$ is bounded above by $C \cdot r^n$, where n is the dimension of the manifold and C is the volume of the unit ball in $\mathbb{R}^n$; (Bishop)

(b) The volume of $B_p(r)$ grows at least linearly; (Calabi, Yau)

(c) There are a bounded number of ends; (Cheeger-Gromoll's splitting theorem implies that there are at most two ends)

(d) Every end has at most linear diameter growth ([AG]);

(e) There is a *Ball Covering Property*, a simplified form of which is ([LT]): for a fixed point p and any large r, there are a bounded number of points $p_1, ..., p_k \in \partial B_p(2r)$, where the number k is independent of r, such that

$$\partial B_p(2r) \subset \bigcup_{j=1}^{k} B_{p_j}(r).$$

Here (e) can be easily derived from the volume comparison, as demonstrated in the next chapter. On the other hand, the ball covering property implies (c) and (d) without any curvature conditions. (We will show this in chapter 4.)

When the global condition $Ric_M \geq 0$ is relaxed to $Ric_{M-B} \geq 0$, where B is a compact set, the same argument that was used to show Bishop-Gromov's volume comparison can also be used to show that (a) and (b) remain true, although the constant C now depends on the manifold; see for example [LT2]. However, as we shall see in the next chapter, the relative volume comparison becomes so weak that it does not yield the ball covering property as before. Thus, it is important to establish the ball covering property, so that we still have control over the geometric growth of the ends.

Our main result is the proof of the ball covering property in its general form for complete Riemannian manifolds with nonnegative Ricci curvature outside a compact set.

Theorem 1 *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set B. Assume that $Ric_B \geq (n-1)H$ and $B \subset B_{p_0}(D_0)$. Then for any $\mu > 0$, there exists $N = N(n, HD_0^2, \mu) > 0$ such that for any $r > 0$, and for any subset S satisfying $S \subset \overline{B}_{p_0}(r)$, we can find $p_1, ..., p_k \in S$, $k \leq N$, with $\bigcup_{j=1}^k B_{p_j}(\mu \cdot r) \supset S$.*

The topological constraints on a manifold which admits a metric with nonnegative Ricci curvature are much weaker than that to admit a metric with nonnegative sectional curvature. In [CG1], Cheeger and Gromoll proved that a complete Riemannian manifold M with nonnegative curvature is diffeomorphic to the normal bundle of its soul, a compact totally geodesic submanifold. M. Gromov showed that the sum of betti numbers of such a manifold has an *a priori* upper bound. One might be tempted to try to extend these results to the case of nonnegative Ricci curvature. However, the examples of Sha-Yang [SH1, 2] and Anderson-Kronheimer-LeBrun [AKL] show that neither Cheeger-Gromoll's Soul Theorem nor Gromov's bound on betti numbers holds in this case. Actually a complete manifold of nonnegative Ricci curvature can have infinite topological type and infinite betti numbers.

Nevertheless, the existance of a metric of nonnegative Ricci curvature does put some constraints on the topology of the manifold. J. Milnor ([Mi]) proved that if a manifold M^n admits a complete metric with nonnegative Ricci curvature, then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree n. For the betti numbers, Z. Shen ([Sh2]) proved that if M has nonnegative Ricci curvature and has a lower bound on the sectional curvature, say $K \geq -1$, then the growth of betti numbers is bounded by a polynomial of degree n, i.e.,

$$\sum_{0 \leq i \leq n} b_i(p,r) \leq C(n)(1+r)^n, \qquad r > 0.$$

where $b_i(p,r)$ denotes the rank of $i_* : H_i(B_p(r), \mathbb{R}) \rightarrow H_i(M, \mathbb{R})$. This can be viewed as giving a certain amount of control over the topological growth of M.

When Ricci curvature is nonnegative outside a compact set B, the result of Shen can also be generalized. We have:

Theorem 2 *Assume that M is a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set B and sectional curvature $K_M \geq -1$. For fixed $p_0 \in M$, assume that $B \subset B_{p_0}(D_0)$. Then there is a constant $C(n, D_0)$ such that*

$$\sum_{0 \leq i \leq n} b_i(p_0, r) \leq C(n, D_0)(1+r)^n, \qquad r > 0.$$

The analytical motivation of our work is the study of the space of harmonic functions. S.T. Yau proved in [Y1] that a positive harmonic function on a complete manifold with nonnegative Ricci curvature must be a constant. If M has nonnegative Ricci curvature outside a compact set, H. Donnelly ([Do]) showed that the space of bounded harmonic functions has finite dimension. In their paper [LT1], P. Li and L. Tam gave a complete description of the space of positive harmonic functions on a complete noncompact Riemannian manifold with nonnegative *sectional curvature* outside a compact set. The ball covering property, which they showed is true in their case, plays a crucial role. They conjectured that their description of the space of positive harmonic functions should continue to hold if one only assumes that the

Ricci curvature is nonnegative outside a compact set. One of the difficulties was, as they pointed out, that the ball covering property was not proved in this case.

Although we still can not prove their conjecture, our Theorem 1 combined with the Harnack inequality ([LY]) yields the following:

Theorem 3 *Let M be a complete noncompact manifold with nonnegative Ricci curvature outside a compact set. Assume that $f(x)$ is a positive harmonic function defined on an end E. Then either $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = a$ for some constant a,* provided that *for any $r > 0$, there exists $R > r$, such that the unbounded component of $E - B_{p_0}(R)$ has a connected boundary.*

Let us conclude this chapter by posing some problems.

1. Cheeger and Gromoll ([CG2]) proved that if a complete Riemannian manifold M with nonnegative Ricci curvature contains a line γ, then M splits isometrically as a product $M^{n-1} \times \gamma$. One naturally asks if Ricci curvature is nonnegative outside a compact set B, and if γ is a line that does not pass B, then is there a tubular neighborhood $T_\delta(\gamma)$ which splits isometrically as a product $\gamma \times U^{n-1}$? The author discussed this with G. Galloway and we are able to give an affirmative answer to this question. The details will appear elsewhere. This leads us to ask whether the following stronger version of the local splitting is true or not: if $\gamma \subset M$ is a line and

in some ϵ-tubular neighborhood $T_\epsilon(\gamma)$, the Ricci curvature is nonnegative, then can $T_\epsilon(\gamma)$ split isometrically as a product?

2. Can we generalize the ball covering property to the case in which the Ricci curvature is "asymptotically nonnegative"? This concept is defined as follows. Let $\lambda(r)$ be a non-increasing function such that

$$(I_k) \qquad \int_0^\infty r^k \lambda(r)\, dr < \infty.$$

We say that the lower bound of Ricci curvature of M satisfies decay condition I_k if for some $p \in M$ there is a $\lambda(r) = \lambda_p(r)$ satisfying (I_k) and the Ricci curvature at any x is bounded below by $-\lambda(d(p,x))$. When $k = 1$, we say that the Riemannian manifold M has asymptotically nonnegative Ricci curvature. There are two indications that the ball covering property may still hold in this case: a) in the case that sectional curvature is asymptotically nonnegative, U. Abresch ([A1, 2]) proved that the ball covering property is true and that it is false for any weaker decay conditions on sectional curvature; b) P. Li and L. Tam ([LT2]) showed that if the lower bound of the Ricci curvature of M satisfies decay condition I_{n-1}, then M has a bounded number of ends. Note that this condition on the Ricci curvature is stronger than asymptotic nonnegativity when $n > 2$.

3. Let M^3 be a three dimensional complete open Riemannian manifold whose Ricci curvature is nonnegative outside a compact set. If for a fixed $p \in M$, there a constant C such that for every end E, $Vol(E \bigcap B_p(r)) \geq C\, r^3$, then does M have finite

topological type? That is, is M topologically the interior of a compact manifold with boundary? When M has nonnegative Ricci curvature globally, S. Zhu ([Z]) proved that M is actually contractible. Our conjecture seems to be a natural generalization of Zhu's result.

Chapter 2. Volume Comparison

Let M be an n-dimensional Riemannian manifold. Let $B_p(r)$ denote the geodesic ball of radius r at $p \in M$. Put $A_p(R,r) := B_p(R) - \overline{B_p(r)}, (R > r > 0)$, $V_p(r) := vol(B_p(r))$, and $V_p(R,r) := vol(A_p(R,r))$. We use $V^H(r)$ to denote the volume of a ball of radius r in the space form of constant curvature H of the same dimension. $V^H(R,r) := V^H(R) - V^H(r)$. Then we have the well known relative volume comparisons (see [GLP], also see [GHL]).

Lemma 1 *If on $B_p(R)$, $Ric \geq (n-1)H$, and $0 < r < R$, then*

$$V_p(R)/V_p(r) \leq V^H(R)/V^H(r), \tag{1}$$

$$V_p(R,r)/V_p(R) \leq V^H(R,r)/V^H(R), \quad \textit{and} \tag{2}$$

$$V_p(R,r)/V_p(r) \leq V^H(R,r)/V^H(r). \tag{3}$$

Let us see how easily the ball covering property stated in Theorem 1 is induced from the above lemma when M has nonnegative Ricci curvature globally.

For any $r > 0$ and any $S \subset \overline{B}_{p_0}(r)$, take a maximal set of points $\{p_1, ..., p_m\} \subset S$ such that $dist(p_i, p_j) \geq \mu r$, $i \neq j$. Then

$$\bigcup B_{p_i}(\mu r) \supset S;$$

$$B_{p_i}((\mu/2)r) \bigcap B_{p_j}((\mu/2)r) = \emptyset, \qquad i \neq j.$$

Suppose $B_{p_s}((\mu/2)r)$ has the smallest volume among all $B_{p_i}((\mu/2)r)$. Since

$$\bigcup_{j=1}^{m} B_{p_j}((\mu/2)r) \subset \; B_{p_s}((2+\mu)r) \,,$$

Lemma 1 (with $H = 0$) implies that

$$m \leq V_{p_s}((2+\mu)r)/V_{p_s}((\mu/2)r) \leq V^0((2+\mu)r)/V^0((\mu/2)r) = (2+\mu)^n 2^n (\mu)^{-n}.$$

When $Ric_{M-B} \geq 0$, this argument fails because $B_{p_s}((2+\mu)r) \supset B$ for r sufficiently large.

There is also a version of the relative volume comparison for star-shaped sets, which is our main tool in the proof of Theorem 1. A star-shaped set S_p at p is a set containing p such that whenever $x \in S_p$ is not on the cut-locus of p, any point on the minimal geodesic joining p and x is also in S_p. We then have ([CGT] §4, Remark 4.1)

Lemma 2 *Let S_p be a star-shaped set and $R > r > 0$. Suppose $Vol(S_p \bigcap B_p(r)) > 0$. If*

$$Ric|(S_p \bigcap B_p(R)) \geq (n-1)H \,, \tag{4}$$

then

$$Vol(S_p \bigcap B_p(R))/\; Vol(S_p \bigcap B_p(r)) \;\leq V^H(R)/\; V^H(r) \,. \tag{5}$$

Remark 1. The above form of comparison first appears in [CGT] without explicitly mentioning the Ricci curvature condition (4) as we did here. To understand why the proof of Lemma 1 (as given in [GLP] or [GHL] with more details) also implies Lemma 2, it suffices to point out that the proof does not need M being complete; it only needs M being a star-shaped set at p. (Note that for any $R > 0$, $S_p \bigcap B_p(R)$ is a star-shaped set at p.)

Remark 2. If M is a complete Riemannian manifold which has nonnegative Ricci curvature outside a compact set, or more generally, if it has asymptotically nonnegative Ricci curvature as defined in Chapter 1, it can be shown that for any $q \in M$, we have some relative volume comparison of the form

$$V_q(R)/V_q(r) \leq C \cdot R^n/r^n , \qquad 0 < r < R .$$

However the constant C now depends on the integral $\int_0^\infty r\lambda_p(r)\, dr$, on $dist(p,q)$, as well as on the numerical order among R, r and $dist(p,q)$. See for example [Ca] or [LT2] for more details. Hence the proof of the ball covering property in the case of nonnegative Ricci curvature outside a compact set can not be carried out as in the above case where the Ricci curvature is globally nonnegative.

Chapter 3. Proof of the Theorems

We assume that all geodesics are parametrized by arc length.

Proof of Theorem 1.

If $H \geq 0$, i.e. $Ric_M \geq 0$, the Theorem is already known. So we assume that $H < 0$. If we multiply the metric on M by $\sqrt{-H}$, then for the new metric, $Ric \geq -1$, and $B \subset B_{p_0}(\sqrt{-H} \cdot D_0)$. This normalization does not affect the validity of the ball covering property. Thus we put $D = \sqrt{-H} \cdot D_0$ and work with the rescaled metric.

Let $\mu > 0$ be given. We may assume that $\mu \leq 2$. Otherwise the theorem is trivial. We divide S into the union of S_1 and S_2, where

$$S_1 = S \bigcap B_{p_0}(\mu r/2), \qquad S_2 = S - S_1.$$

If S_1 is not empty, it can be covered by just one $B_p(\mu r)$ with p in S_1. So we only have to estimate the covering number for S_2, which is contained in $\overline{B}_{p_0}(r) - B_{p_0}(\mu r/2)$. Also, it suffices to count the number of balls of radius $\mu r/4$ needed to cover S_2. Let us denote $t := \mu/4$.

First, we assume that $t \cdot r > 2D$. Note that for any $q \in S_2$, $B_q(tr) \bigcap \overline{B}_{p_0}(2D) = \emptyset$.

Divide $\partial B_{p_0}(2D)$ into a bounded number of subsets $\{U_1, ..., U_m\}$ such that $\forall x, y \in U_a$, $d_M(x, y) \leq 2D$. This can be done as follows. Take a maximal set of points $\{q_1, ..., q_m\} \subset \partial B_{p_0}(2D)$ such that $dist(q_a, q_b) \geq D, \qquad a \neq b$. Then

$$\bigcup B_{q_a}(D) \supset \partial B_{p_0}(2D);$$

$$B_{q_a}(D/2) \bigcap B_{q_b}(D/2) = \emptyset, \qquad a \neq b.$$

Suppose $B_{q_s}(D/2)$ has the smallest volume among all $B_{q_a}(D/2)$. Since $\bigcup_{a=1}^{m} B_{q_a}(D/2) \subset B_{q_s}(5D)$, relative volume comparison (Lemma 1) implies that

$$m \leq V_{q_s}(5D)/V_{q_s}(D/2) \leq V^{-1}(5D)/V^{-1}(D/2). \tag{1}$$

Note that the right hand side of (1) depends only on n and D. We define

$$U_1 = B_{q_1}(D) \bigcap \partial B_{p_0}(2D),$$

$$U_b = B_{q_b}(D) \bigcap \partial B_{p_0}(2D) - \bigcup_{a=1}^{b-1} B_{q_a}(D), \qquad b = 2, ..., m.$$

Denote

$$M_r := \{x : \exists \text{ a minimal geodesic } \gamma : [0, A] \to M, \text{ passing through } x$$

$$\text{with } \gamma(0) = p_0 \text{ and } A \geq r\},$$

i.e. M_r consists of all minimal geodesics emanating from p_0 that are no shorter than r. Hence $M_r \supset M - B_{p_0}(r)$, and M_r is star-shaped at p_0.

We now divide M_{2D} into m cones K_a by:

$$K_a := \{x : \exists \text{ a minimal geodesic } \gamma : [0, A] \to M, \text{ passing through } x$$

$$\text{with } \gamma(0) = p_0 \text{ , } A \geq 2D, \text{ and } \gamma(2D) \in U_a\},$$

i.e. K_a consists of all minimal geodesics emanating from p_0 that intersect U_a. Note that by the triangle inequality, if

$$d(x_i, p_0) > 2D, \qquad x_i \in K_a, i = 1, 2,$$

then any minimal geodesic connecting x_1 and x_2 will not pass through $B_{p_0}(D)$. Indeed, let γ_i be a minimal geodesic from p_0 to x_i with $\gamma_i(2D) \in U_a$, $i = 1, 2$. Then the broken geodesic from x_1 to $\gamma_1(2D)$ to $\gamma_2(2D)$ to x_2 has length $\leq d(x_1, p_0) + d(x_2, p_0) - 2D$. On the other hand, if a minimal geodesic connecting x_1 and x_2 intersects $B_{p_0}(D)$, then it would have a length $> d(x_1, p_0) + d(x_2, p_0) - 2D$, which is a contradiction.

Now we estimate the covering number N.

On S_2, take a maximal set of points $\{p_1, ..., p_k\}$ such that $dist(p_i, p_j) \geq tr, \quad i \neq j$. Then

$$\bigcup B_{p_i}(tr) \supset S_2; \tag{2}$$

$$B_{p_i}(tr/2) \bigcap B_{p_j}(tr/2) = \emptyset, \quad i \neq j. \tag{3}$$

To estimate the bound on k, we divide the balls $B_{p_i}(tr/2)$ into m families as follows: for each ball $B_{p_i}(tr/2)$, look at $Vol(B_{p_i}(tr/2) \bigcap K_a)$, $a = 1, ..., m$. Fix an a_i such that $Vol(B_{p_i}(tr/2) \bigcap K_{a_i})$ is maximal. Then

$$Vol(B_{p_i}(tr/2) \bigcap K_{a_i}) \geq \frac{1}{m} Vol(B_{p_i}(tr/2)). \tag{4}$$

We denote

$$B_{p_i}(tr/2) \bigcap K_{a_i} = B_{p_i}^{L, a_i},$$

or simply $B_{p_i}^L$, and place the ball $B_{p_i}(tr/2)$ in the $a_i - th$ family. Here the superscript L stands for *largest* (in volume of intersection).

Now we estimate the number of balls in a cone K_a. Suppose B_p^L has the smallest volume among all $B_{p_i}^{L,a}$ in this cone. Since the center p may not be in K_a, we need the following:

Lemma 1 *Assume that* $Ric|B_p(R) \geq 0$. *There exists a* δ *with* $0 < \delta < 1$, *depending only on* n *and* m, *such that whenever a subset* $W \subset B_p(R)$ *has*

$$Vol(W) \geq 1/m \cdot Vol(B_p(R)) \ ,$$

then there exists $q \in W$ *such that* $dist(q,p) \leq \delta R$. *Hence* $B_q((1-\delta)R) \subset B_p(R)$.

Proof. If for any $q \in W$, $dist(p,q) > \delta R$, then $W \subset A_p(R, \delta R)$. By Lemma 1,

$$V_p(R, \delta R)/V_p(R) \leq \frac{R^n - \delta^n R^n}{R^n} = 1 - \delta^n.$$

So $1 - \delta^n \geq 1/m$, i.e. $\delta \leq (1 - 1/m)^{1/n}$. Thus we can take

$$\delta = (1 - 1/(2m))^{1/n}. \tag{5}$$

This finishes the proof of Lemma 1.

Now we continue the proof of the theorem. By the above lemma and (4), there is a point $q \in B_p^L$ such that

$$B_q((1-\delta)tr/2) \subset B_p(tr/2). \tag{6}$$

We construct a star-shaped set W_q at q as follows. $y \in W_q$ if and only if there is a point x belonging to either $B_q((1-\delta)tr/2)$ or one of $B_{p_i}^{L,a}$ in the cone K_a and there is a minimal geodesic γ connecting q and x which passes y. Note that

(a) This geodesic γ will not pass through $B_{p_0}(D)$. To see this note that if x is in one of $B_{p_i}^{L,a}(tr/2)$, then both x and q are in K_a, the claim follows from the remark following the definition of K_a; if x is in $B_q((1-\delta)tr/2)$ then the claim is immediate by the triangle inequality.

(b) The length of γ is not bigger than $tr/2 + r + r + tr/2 = (2+t)r$.

(c) $B_q((1-\delta)tr/2) \subset W_q \subset M - B_{p_0}(D)$.

Hence we can apply Lemma 2 of Chapter 2. We have

$$Vol(W_q)/V_q((1-\delta)tr/2) \leq V^0((2+t)r)/V^0((1-\delta)tr/2) = 2^n(2+t)^n t^{-n}(1-\delta)^{-n}. \quad (7)$$

On the other hand let N_a be the number of balls in the a-th family. By (4) and (6),

$$Vol(W_q)/V_q((1-\delta)tr/2) \geq \sum_{B_{p_i}^L \in a-th\ family} Vol(B_{p_i}^L)/V_p(tr/2) \geq N_a/m\ . \quad (8)$$

So (7) and (8) imply $N_a \leq m \cdot 2^n(2+t)^n t^{-n}(1-\delta)^{-n}$. Adding up the contributions from all m families, we have

$$k \leq m^2 2^n (2+t)^n t^{-n} (1-\delta)^{-n}\ , \quad (9)$$

where δ is defined by (5). Since m depends only on n and D, the right hand side of (9) is a function of n, D, and μ. (Recall that $t = \mu/4$).

If $tr \leq 2D$, that is $\mu r \leq 8D$, as in the proof of (1), we can bound k by

$$\max_{0<\mu r \leq 8D} V^{-1}((2+\mu)r)/V^{-1}(\mu r/2),$$

which again depends only on μ, n and D and not on r. Q.E.D.

Proof of Theorem 2.

When $B = \emptyset$, this theorem is proved by Z. Shen in [Sh2]. We follow his line of argument.

For $X \subset Y$, let $b_i(X, Y)$ denotes the rank of $i_* : H_i(X, \mathbb{R}) \to H_i(Y, \mathbb{R})$.

Let B be a ball in M with radius r and for any $\rho > 0$, let ρB denote the concentric ball of radius ρr. The following two lemmas are due to M. Gromov (*c.f.* [G], [A2]). We state the versions adopted in [Sh2].

Lemma 2 *Let M^n be a complete manifold with sectional curvature $K_M \geq -1$. Then there is a constant $C(n) > 1$ such that for any $0 < \epsilon \leq 1$ and any bounded subset $X \subset M$,*

$$\sum_{0 \leq i \leq n} b_i(X, U_\epsilon X) \leq (1 + diam_M(X)/\epsilon)^n \cdot C(n)^{1+diam_M(X)}, \tag{10}$$

where $U_\epsilon X$ denotes the ϵ-neighborhood of X in M.

Lemma 3 *Let M^n be a complete manifold and $p \in M$. For any fixed numbers $r > 0$ and $0 < r_0 \leq 7^{-n-1}$, let $B_j^0 = B_{p_j}(r_0)$, $j = 1, ..., N$, be a ball covering of $B_p(r)$ with $p_j \in B_p(r)$. For each j let $B_j^k = 7^k B_j^0$, $k = 0, 1, ..., n+1$. Then*

$$\begin{aligned} \textstyle\sum_{0 \leq i \leq n} \quad & b_i(B_p(r), B_p(r+1)) \leq \\ \leq \quad & (e-1)Nt^n \sup\{\sum_{0 \leq i \leq n} b_i(B_j^k, 5B_j^k);\ 0 \leq k \leq n,\ 1 \leq j \leq N\}, \end{aligned} \tag{11}$$

where t is the smallest number such that each ball B_j^n intersects at most t other balls B_i^n.

We observe the following fact which will be used repeatedly in our proof.

Lemma 4 *Suppose $A \subset B \subset C \subset D$. From the sequence of compositions*

$$H_i(A,\mathbb{R}) \to H_i(B,\mathbb{R}) \to H_i(C,\mathbb{R}) \to H_i(D,\mathbb{R})$$

we see that $b_i(A,D) \leq b_i(B,C)$.

Now we prove Theorem 2 as follows. By Lemma 2 and Lemma 4, $\exists C_1(n)$ depending only on n such that for all balls B with radii $r \leq 1$,

$$\sum_{0\leq i\leq n} b_i(B,5B) \leq C_1(n). \tag{12}$$

Thus by Lemma 3 and Lemma 4,

$$\begin{aligned} \sum_{0\leq i\leq n} b_i(B_p(r),M) &\leq \sum_{0\leq i\leq n} b_i(B_p(r),B_p(r+1)) \\ &\leq (e-1)Nt^nC_1(n). \end{aligned} \tag{13}$$

where N and t are defined in Lemma 3. Hence we are left to show that we can find $\{B_{p_j}(r_0),\ j=1,...,N,\ \ r_0 = 7^{-n-1}\}$, a covering of $B_p(r)$ with $p_j \in B_p(r)$ such that t is bounded above by some $T(n,D_0)$ independent of r, and N is bounded by some $C_2(n,D_0)\cdot(1+r)^n$.

In the proof of Theorem 1, replace μr by r_0. Set $S=\overline{B}_p(r)$, $S_1 = B_p(2D_0)$, and $S_2 = S - S_1$. Let $\{B_{p_j}(r_0/2),\ j=1,...,N\ p_j \in B_p(r)\}$ be a maximal set of disjoint balls covering S. The same argument used there shows that N is bounded by

$$m^2 2^n (2 + \frac{r_0}{4r})^n (\frac{r_0}{4r})^{-n} (1-\delta)^{-n} + \frac{V^{-1}(4D_0 + r_0)}{V^{-1}(0.5r_0)}$$
$$\leq C_2(n, D_0) \cdot r^n, \tag{14}$$

for r sufficiently large, where m is bounded by (1) and δ is defined in (5).

As for t, suppose that a B_i^n intersects t other balls B_j^n, where $B_j^n = 7^n B_{p_j}(r_0)$ as defined in Lemma 3. If $B_{p_s}(r_0/2)$ has the smallest volume among all the $t+1$ balls $B_{p_j}(r_0/2)$, then $B_{p_s}(6/7)$ contains all other t $B_{p_j}(r_0/2)$ (recall that $r_0 = 7^{-n-1}$). The argument as in Chapter 2 (with $H = -1$) implies that

$$t + 1 \leq \frac{V^{-1}(6/7)}{V^{-1}(r_0/2)} = \frac{V^{-1}(6/7)}{V^{-1}(7^{-n-1}/2)}. \tag{15}$$

Q.E.D.

We will prove Theorem 3 in the next chapter.

Chapter 4. Applications

Let M be a complete open Riemannian manifold. The number of unbounded components of $M - B_{p_0}(r)$ is a nondecreasing function of r. If this number has an upper bound, we say that M has *finitely many ends* and call the least upper bound *the number of ends of* M. This number is independent of p_0.

Our first application of the ball covering property is to give a very simple proof of the following result, due to Cai ([Ca]) and Li-Tam ([LT2]) independently, that a complete manifold with nonnegative Ricci curvature outside a compact set can only have a bounded number of ends.

Corollary 1 *Under the same assumption as in the theorem, the number of ends is finite and bounded by some* $N_1 = N_1(n, D_0^2 H)$.

Proof. We let $N_1 = N(n, D_0^2 H, 1/2)$ as in our Theorem 1. If the above claim is not true, take r large enough so that in $M - B_{p_0}(r)$, there are more than N_1 unbounded components E_i. It is apparent that balls of radius r with centers in different components $E_i \bigcap \partial B_{p_0}(2r)$ do not intersect. Thus we need more than N_1 balls of radius r to cover $\partial B_{p_0}(2r)$. This contradicts the theorem. Q.E.D.

Note. In the original version of our paper [L] (submitted for publication in May 1990), we assumed an arbitrary lower bound on the sectional curvature in addition to the assumption that $Ric_{M-B} \geq 0$ and proved the ball covering property. As a corollary M has a bounded number of ends in this case. At the AMS Summer Institute on Differential Geometry held in U.C.L.A., July 1990, P. Li announced that he and L. Tam had proved that a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set has finitely many ends without assuming a lower bound on the sectional curvature ([LT2]). Their approach is analytic in nature. At about the same time, M. Cai ([Ca]) independently proved this same result by purely geometric means. (Ball covering is not treated in either of the aforementioned works.)

There are several variations of the definition of "diameter growth". See for example, [AG], [C], [Sh1] and [Sh2]. In [AG] it is proved that if M^n has asymptotically nonnegative Ricci curvature and a lower bound on the sectional curvature, and if the diameter growth of M is $o(r^{1/n})$, then M has finite topological type. For this purpose, all definitions mentioned above are equivalent. However, the diameter growth defined in [C] and [Sh2] are always bounded above by $2r$. This is not what we are interested in. Our concern is to know how fast the intrinsic diameter of the geodesic sphere grows. For example, in the space form M^{-1} of constant curvature -1, the intrinsic diameter of $\partial B_0(r)$ grows exponentially ($4\sinh^2(r/2)$ to be exact), whereas in $\mathbb{R}^n$, it is linear ($= \pi r$).

Since in general, $\partial B_p(r)$ is not a submanifold, we measure its diameter in an annulus of size proportional to r. The following definition is adapted from that in [Sh1]. The definition given here is stronger (in the sense that the diameter is larger) than any one of the four definitions mentioned above.

Definition 1 Let M be a complete noncompact Riemannian manifold and $p \in M$

a fixed point. For any connected component Σ of the annulus $A_p(2r,\ (3/4)r)$, and any two points $x, y \in \Sigma \bigcap \partial B_p(r)$, let $d_r(x,y) = \inf Length(\phi)$, where the infimum is taken over all piecewise smooth curves ϕ from x to y in $M - \overline{B}_p(1/2\ r)$. Set $diam(\Sigma \bigcap \partial B_p(r)) = \sup_{x,y} d_r(x,y)$, where $x, y \in \Sigma \bigcap \partial B_p(r)$. The diameter of ends at r from p is defined to be

$$Diam_p(r) := \sup diam(\Sigma \bigcap \partial B_p(r))\ ,$$

where the supremum is taken over all connected components Σ of $A_p(2\ r,\ 3/4\ r)$.[1]

Corollary 2 *If a complete manifold M has nonnegative Ricci curvature outside a compact set, then the diameter growth of ends is at most linear.*

Proof. Let B, H, D_0 as in Theorem 1. With notations as in the above definition, apply Theorem 1 to $S = \Sigma \subset \overline{B}_p(2r), \mu = 1/8$. Cover Σ by no more than $N = N(n, HD_0^2, 1/8)$ balls $B_i = B_{q_i}(r/4),\ i \leq N$. For any two points $x, y \in \Sigma \bigcap \partial B_p(r)$, since Σ is connected, we can find a subsequence of balls $B_{i_1}, ..., B_{i_k},\ k \leq N$ such that $x \in B_{i_1},\ B_{i_j} \bigcap B_{i_{j+1}} \neq \emptyset,\ y \in B_{i_k}$. Fix a $z_j \in B_{i_j} \bigcap B_{i_{j+1}}$. By connecting $x,\ q_{i_1},\ z_1,\ q_{i_2},\ z_2, ..., z_{k-1},\ q_{i_k}$ and y consecutively, we obtain a curve γ which is in $M - \overline{B}_p(1/2\ r)$ and which has length $\leq 2k \cdot r/4 \leq (N/2)r$. Q.E.D.

An immediate consequence of Theorem 1 in the study of harmonic functions is the following Proposition, which contains Theorem 3.

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[1]In [Sh1], as well as in [AG], [C] and [Sh2], only *unbounded* connected components are considered.

Proposition 1 *Assume that $Ric_{M-B} \geq 0$, where $B \subset B_{p_0}(D)$ is a compact subset of M. Let E be an unbounded component of $M - B_{p_0}(r_0)$. Assume that $f(x)$ is a positive harmonic function defined on E. Then either $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = a$ for some constant a,* provided that one of the following three conditions on E holds:

(a) There is an $R_0 > 0$, such that $\pi_1(E - B_{p_0}(R_0)) = 0$;

(b) $\forall r > 0$, $\exists R > r$, such that the unbounded component of $E - B_{p_0}(R)$ has a connected boundary;

(c) There is a sequence of connected $n-1$ dimensional compact submanifolds $S_j \subset E$, such that (i) S_j and S_{j+1} bound a connected compact region D_j for $j = 1, 2, 3, \ldots$; (ii) for some large r, $\bigcup D_j \supset E - B_p(r)$; (iii) $dist(p_0, S_j) \to \infty$; and (iv) for some $C > 0$,

$$\max_{x\in S_j} dist(p_0, x) \;/\; \min_{x\in S_j} dist(p_0, x) \quad \leq C.$$

Proof. We follow the same line of argument as in the proof of Theorem 3.2 and 3.3 in [LT1].

Since (a) implies (b) (see [AG] Proposition 4.3 and its proof) which in turn implies (c), let us assume that the end E satisfies condition (c). The proposition is a consequence of the following two lemmas.

Lemma 1 (Harnack inequality. See [Y], [LY], also see [LT1] Lemma 3.2) *There is a constant C_1, such that if $f(x)$ is a positive harmonic function on E and $Ric_{B_q(2r)} \geq 0$, where $B_q(2r) \subset E$, then*

$$|\nabla f(x)|/f(x) \leq C_1/r$$

for any $x \in B_q(r)$.

Lemma 2 *There is a constant* $C_2 > 0$ *independent of* j *such that for any harmonic function* $f(x)$ *defined on* E *and any large* j, *we have*

$$\max_{x\in S_j} f(x) \leq C_2 \min_{x\in S_j} f(x).$$

Proof. Choose $y, z \in S_j$ with

$$f(y) = \max_{x\in S_j} f, \quad f(z) = \min_{x\in S_j} f.$$

Let γ be a curve in S_j connecting y and z. Denote $r_j = dist(p_0, S_j)$. Theorem 1 implies that there are a bounded number of points $y = y_1, ..., y_k = z$, $k \leq C_3$, on the curve γ such that

$$\gamma \subset \bigcup_i B_{y_i}(1/4 \cdot r_j).$$

Now the Harnack inequality implies that

$$f(y)/f(z) \leq C_1^{2k} \leq C_1^{2C_3}.$$

This proves Lemma 2.

Now, Proposition 1 follows from the above Lemma. The argument is as in the proof Theorem 3.3 of [LT1]. Namely, suppose that $f(x)$ is not bounded on E. We are going to show that

$$\lim_{x\to\infty} f(x) = \infty.$$

Since $f(x)$ is not bounded, there is a sequence of points x_j, $\lim dist(x_j, p_0) = \infty$, such that $\lim f(x_j) = \infty$. Suppose $x_j \in D_{i_j}$, then

$$\max_{x \in D_{i_j}} f(x) \geq f(x_j).$$

By the maximum principle the maximum is either on S_{i_j}, in which case we denote $\Sigma_j = S_{i_j}$, or on S_{i_j+1}, in which case we denote $\Sigma_j = S_{i_j+1}$. We write U_j for the domain bounded by Σ_j and Σ_{j+1}. Then we have

$$\min_{x \in \Sigma_j} f(x) \geq 1/C_2 \max_{x \in \Sigma_j} f(x) \geq 1/C_2 f(x_j).$$

Thus by the maximum principle $\min_{x \in U_j} f(x) \geq 1/C_2 \min\{f(x_j), f(x_{j+1})\}$. Therefore $\lim_{x \to \infty} f(x) = \infty$.

Suppose that $f(x)$ is bounded on E. Let

$$A = \lim_{r \to \infty} (\inf_{E - B_{p_0}(r)} f).$$

$0 \leq A < \infty$. For any $\epsilon > 0$, there exists a sequence of points x_j, $\lim dist(x_j, p_0) = \infty$, such that $A - \epsilon < f(x_j) < A + \epsilon$. Suppose $x_j \in D_{i_j}$. By the maximum principle $\min_{D_{i_j}} f$ is either on S_{i_j}, in which case we denote $\Sigma_j = S_{i_j}$, or on S_{i_j+1}, in which case we denote $\Sigma_j = S_{i_j+1}$. We write U_j for the domain bounded by Σ_j and Σ_{j+1}. Also, there is a j_0, such that

$$\inf_{\bigcup_{i>j} U_i} f > A - \epsilon$$

for any $j \geq j_0$.

Let $g = f - (A - \epsilon)$, which is a positive harmonic function on $\bigcup_{i>j_0} U_i$. Lemma 2 implies that for sufficiently large j,

$$\max_{x\in\Sigma_j} g(x) \leq C_2 \min_{x\in\Sigma_j} g(x).$$

Hence

$$\max_{x\in\Sigma_j} g(x) \leq C_2\{\min_{x\in\Sigma_j} f - (A-\epsilon)\} \leq 2C_2\epsilon.$$

$$\max_{x\in\Sigma_j} f \leq (2C_2 - 1)\epsilon + A.$$

Since ϵ is arbitrary, by the maximum principle again we have

$$\lim_{j\to\infty}\Big(\sup_{\bigcup_{i>j} U_i} f\Big) \leq A.$$

This shows that $\lim_{x\to\infty} f = A$. Q.E.D.

Remark. It is proved in [AG] that if $Ric_M \geq 0$ and if E is an end of M, then any unbounded component of $E - B_{p_0}(R)$ has a connected boundary. Intuitively, we say that E "has no holes". Since their proof uses the splitting theorem on the universal covering of M, it does not apply to the case of nonnegative Ricci curvature outside a compact set. Thus it remains an interesting question whether an end E "has holes" in our case. If the answer is no, then all three extra conditions in our proposition can be dropped automatically.

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