On Schottky Groups with Automorphisms

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Abstract of the Dissertation
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It is well known that every closed Riemann surface can be uniformized by Schottky groups. This fact, claimed by Felix Klein in 1883 [9] and proven rigorously by Koebe much later [10], is known in the literature as the retrosection theorem. Using the ideas of L. Bers on quasi-conformal mappings, it is possible to obtain an easy proof of this fact [3].

In this dissertation we are interested in a kind of retrosection theorem with automorphisms. To be more precise, we consider a closed Riemann surface S and a finite group of automorphisms H of S. We look for some Schottky group G, with region of discontinuity

Ω, having the property that every element of H can be lifted as a conformal automorphism of Ω.

In [8] L. Keen solved this problem in the case that S is hyperelliptic and H is the group generated by the hyperelliptic involution. She called such Schottky groups "Hyperelliptic Schottky groups". In [7] we solved this problem for the case of involutions. We called such groups "$\Gamma-$ Hyperelliptic Schottky groups". In the present work we are interested in the general case, that is, when S and H are arbitrary.

We obtain necessary conditions on the set of fixed points of the non-trivial elements of H to find a Schottky group G as desired. These conditions are trivially satisfied by groups acting fixed point freely and by groups isomorphic to the group with two elements.

We show that these necessary conditions turn out to be also sufficient in some cyclic cases. To be more precise, when each element of the cyclic group has the same fixed point set.

It would be interesting to know if these necessary conditions are also sufficient in the general case. This is still an open problem.

To my parents

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0.- INTRODUCTION.

In the literature there are many characterizations of closed Riemann surfaces with automorphisms, but in general they do not involve uniformization theory. In uniformization theory we begin with a surface S, a domain Ω contained in the Riemann sphere as a regular (Galois) covering space of S, and the corresponding group G of cover transformations given by fractional linear transformations, such that the natural projection map $\pi : \Omega \rightarrow \Omega/G = S$ is holomorphic. The pair (Ω, G) is called an uniformization of S.

Schottky groups are in some sense the lowest planar covering of closed Riemann surfaces. To be more precise, the uniformization (Ω, G) is called a Schottky uniformization if there is no non-trivial normal subgroup N of G such that the quotient surface Ω/N is planar. Schottky uniformizations are exactly those uniformizations (Ω, G), where G is a Schottky group and Ω is the region of discontinuity for the action of G on the Riemann sphere.

We are interested in finding uniformizations, via Schottky groups, which reflect symmetries (conformal automorphisms) of closed Riemann surfaces. To be more precise, let S be a closed Riemann surface and let H be a finite group of automorphisms of S. We look for a Schottky uniformization (Ω, G) of S, with $\pi : \Omega \rightarrow \Omega/G = S$ as the natural projection induced by G, such that for each transformation h in H there exists an automorphisms t of Ω satisfying
$h\,\pi = \pi\,t$.

In 1980 L. Keen [8] discussed this problem for a hyperelliptic Riemann surface S where H is the group generated by the hyperelliptic involution on

S. A closed Riemann surface S is called hyperelliptic if it admits a conformal involution with 2g+2 fixed points (the hyperelliptic involution), where g is the genus of S. In [6] and [7] we gave a similar discussion for closed Riemann surfaces which admit a general conformal involution. In general, if S is a closed Riemann surface of genus $g \geq 2$ and H is a finite group of conformal automorphisms of S, the problem of finding those Schottky groups which uniformize S and reflect the action of H is still open. In this dissertation we obtain necessary conditions, to be satisfied by the group H, to find a uniformization as desired. We show that in general if H is cyclic, then our conditions turn out to be sufficient.

In the first chapter, we give some basic definitions on Kleinian groups and Riemann surfaces. In the second chapter, we study the fixed points of elliptic transformations in geometrically finite Kleinian groups. We show that for groups of this kind these fixed points satisfy a nice property called the mixed elliptic fixed point property (M.E.F.P.). In the third chapter, we define conformal automorphisms on Riemann surfaces and we recall some basics from the theory of covering spaces. Next, we consider a closed Riemann surface S and a finite group H of conformal automorphisms of S. Necessary conditions on H, more precisely on the set of fixed points of the non-trivial elements of H, are obtained in order to find a Schottky group G uniformizing S, as desired. One corollary of our main result is that if H is a cyclic group, say generated by $f : S \rightarrow S$, with the property that the number of fixed points of f is odd, then we cannot lift the automorphism f to any Schottky covering of S. We consider two classical examples in genus one, corresponding to surfaces having an automorphism with fixed points of order four and six respectively. We observe that in such cases the necessary conditions do not hold. That is not a surprise , since if we look at the elementary groups which contain a Schottky group of genus one, we see that every elliptic element of order different than two has both fixed points in the limit set. In the last chapter, we prove our main result.

Let us remark that, if the order of H is four, then it can be shown that the necessary conditions are sufficient. The proof of this fact will appear elsewhere.

Chapter 1

PRELIMINARIES.

In this chapter, we outline some of the basic theory of Kleinian groups we will need in this dissertation. Detailed discussion of this material can be found in [2], [5] and [11].

$\hat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\} \cong \mathbb{CP}_1$ will denote the **Riemann sphere**. The group of **conformal automorphisms** of $\hat{\mathbb{C}}$ is the **Möbius group**, also called the **fractional linear group**, and denoted by $\mathbb{M}$. A Möbius transformation or fractional linear transformation has the form

$$g(z) = (az + b) / (cz + d),$$

with a, b, c, d $\in \mathbb{C}$, and ad$-$bc=1.

Let us remark that a fractional linear transformation, which is not the identity, has at most two fixed points. In fact, if z is a fixed point of a transformation g as above, then z must satisfy the quadratic equation

$$c z^2 + (d - a) z - b = 0.$$

We can identify the fractional linear transformation g as above with the representative in $PSL(2,\mathbb{C})$ of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Under the above identification, we obtain an isomorphism between $\mathbb{M}$ and $PSL(2,\mathbb{C})$. From now on, we will use freely this identification.

We can classify the fractional linear transformations as follows :

(i) g is **loxodromic** if it is conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

with $|\lambda| > 1$. Such a transformation has two fixed points.

(ii) If g is loxodromic, we call it **hyperbolic** if it is conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

with λ real, and $\lambda > 1$. A hyperbolic element keeps a circular disc invariant, while a loxodromic non-hyperbolic transformation does not.

(iii) g is **parabolic** if it is conjugate to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Such a transformation has only one fixed point in $\hat{\mathbb{C}}$.

(iv) g is **elliptic** if it is conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

with $|\lambda| = 1$, $\lambda^2 \neq 1$. Such a transformation has two fixed points in $\hat{\mathbb{C}}$.

Remark : Parabolic elements are the only fractional linear transformations with only one fixed point in $\hat{\mathbb{C}}$.

Let G be a subgroup of $\mathbb{M}$, we say that G is a **Kleinian group**, if the following holds for some $z \in \hat{\mathbb{C}}$:

(1) $stab_G(z) = \{g \in G : g(z) = z\}$ is finite; and

(2) there exists a neigborhood U of z, such that g(U)=U, for all $g \in stab_G(z)$, and $g(U) \bigcap U = \phi$, for all $g \in G - stab_G(z)$.

If G is a Kleinian group and z is as above, we say that **G acts discontinuously at z** and that z is a **regular point of G**. The set of points of $\hat{\mathbb{C}}$ at which G acts discontinuously is called the **regular set** or **region of discontinuity** of G. We denote this set by $\Omega(G)$ or Ω if there is no danger of confusion. By definition $\Omega(G)$ is an open set of $\hat{\mathbb{C}}$. Its complement $\Lambda(G) = \hat{\mathbb{C}} - \Omega(G)$ is called the **limit set** of G.

If G is a Kleinian group, then its limit set has the following possibilities :

(i) $\Lambda(G) = \phi$;

(ii) $\Lambda(G)$ has only one point;

(iii) $\Lambda(G)$ has only two points; or

(iv) $\Lambda(G)$ is a perfect, nowhere dense set in $\hat{\mathbb{C}}$.

A proof of the above fact can be found in chapter II in [11]. If we are in any of the cases (i), (ii) or (iii), we say that G is an **elementary group**, otherwise G is called a **non-elementary group**.

Let us observe that an elliptic transformation of infinite order is conjugate to a rotation $g(z) = e^{\pi i\theta}z$, where θ is irrational. Thus, for every z in the Riemann sphere we can find a subsequence of the sequence $\{g^n\}_{n\in\mathbb{Z}}$ such that $g^{n_i}(z)$ converges to z. Such a point z cannot be a regular point of any group G containing such a transformation. We conclude the following

Lemma : *Let G be a Kleinian group. Then any elliptic element of G has finite order.*

Since loxodromic elements and parabolic elements have necessarily infinite order, the torsion part of any Kleinian group G corresponds exactly to the set of elliptic elements of G.

Lemma : *Let g and h be linear fractional transformations. Assume that g has two fixed points and h shares exactly one fixed point with g, say p. Then the commutator of g and h, $[g,h] = ghg^{-1}h^{-1}$, is a parabolic transformation with fixed point p.*

Proof : Normalize g and h in such a way that the common fixed point p of both transformations is ∞, and the other fixed point of g is 0. Under this normalization the transformations g and h have the following form:

$$g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \; h = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

where t is different from 0, 1 and −1, and a is non-zero.

The commutator [g,h] has the form

$$[g, h] = \begin{pmatrix} 1 & a(t^2 - 1) \\ 0 & 1 \end{pmatrix}.$$

Since a is non-zero and t^2 is different from 1, the transformation [g,h] is parabolic with ∞ as fixed point.□

If G and H are groups of fractional linear transformations satisfying the conditions (1) and (2) below, we say that H is a **finite normal extension** of G.

(1) G is a normal subgroup of H;

(2) G has finite index in H.

Lemma : *If H is a finite normal extension of G, then H is a Kleinian group if and only if G is a Kleinian group.*

This is a consequence of the finiteness condition on the index of G in H. If the group G is non-elementary, it also can be shown by using the condition of normality. See [11] for a proof of this lemma.

In this dissertation, we are interested in some finite normal extensions of a particular class of Kleinian groups called Schottky groups; these groups will be defined latter.

About finite (normal) extensions H of Kleinian groups G, we have the following facts:

(i) $\Omega(H) = \Omega(G)$.

(ii) $\Lambda(H) = \Lambda(G)$.

(iii) If G has no parabolic elements, then H does not have any.

Facts (i) and (ii) are can be found in chapter II in [11]. The fact (iii) is revelant to our study, so we show it here. Assume h in H is a parabolic element, then h^n *and* h^m must be in different classes in H/G if $n \neq m$; otherwise $h^n h^{-m}$, which is not the identity (so a parabolic element), must belong to G. This contradicts the fact that G has no parabolic elements. Since different powers of h must now be in different classes, we see that G necessarily has infinite index in H.

Let G be a Kleinian group. By a **fundamental domain** for G we mean an open set ω of $\Omega(G)$ such that

(i) no two points of ω are equivalent under G,

(ii) every point of $\Omega(G)$ is G-equivalent to at least one point of the closure of ω,

(iii) the relative boundary of ω in $\Omega(G)$, $\delta\omega$, consists of piecewise analytic arcs, called sides, and

(iv) for every side c in $\delta\omega$, there exists a side c' in $\delta\omega$ and an element g $\in$ G such that g(c)=c'.

Proposition : *Every Kleinian group G has a fundamental domain.*

See [5] for a proof of this proposition.

A **Riemann surface** S is a complex manifold of dimension one. More precisely, S is a Hausdorff topological space such that for every point p in S there exists an open neighboorhod U and a homeomorphism $z : U \to \Delta$, $\Delta = \{z \in \mathbb{C} : \mid z \mid < 1\}$, with the property that if (U,z) and (V,w) are two such pairs as defined above then $wz^{-1} : z(U \bigcap V) \to w(U \bigcap V)$ and $zw^{-1} : w(U \bigcap V) \to z(U \bigcap V)$ are analytic mappings.

In this discussion, we will consider only connected Riemann surfaces. Examples of Riemann surfaces are :

(i) the Riemann sphere, $\hat{\mathbb{C}}$;

(ii) the complex plane, $\mathbb{C}$;

(iii) the unit disc, Δ;

(iv) the regular region of any Kleinian group $\Omega(G)$, and the quotient $\Omega(G)/G$ (G may have elliptic elements) are a set of Riemann surfaces;

(v) any closed orientable surface of genus g. In this case, except for g=0, we may have different Riemann surface structures on each topological type.

For g $\geq$ 1, let C_k, C'_k, $k = 1, ..., g$, be 2g Jordan curves on the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$, such that they are mutually disjoint and bound a 2g-connected domain. Call D the common exterior of all the curves, and suppose that for each k there exists a fractional linear transformation A_k with the following properties.

(i) $A_k(C_k)$=C'_k ;

(ii) A_k maps the exterior of C_k onto the interior of C'_k.

The transformations $\{A_i : i = 1, ..., g\}$ generate a subgroup G of Möbius transformations, necessarily Kleinian, and D is a fundamental domain for G, called a **standard fundamental domain** for G. This group is called a **Schottky group of genus g**. Observe that necessarily the transformations A_i are loxodromic. G is a free group on g generators and all its elements, except by the identity, are loxodromic [12]. These properties define in fact the Schottky groups of genus g, for g $\geq$ 1. For our purposses, we define the Schottky group of genus zero to be the group with the identity as its only element, that is the trivial group.

Theorem [12] : *Let G be a Kleinian group. G is a Schottky group if and only if G is purely loxodromic, finitely generated and free.*

Theorem [4] : *If G is a Schottky group, then corresponding to any set of free generators there exists a fundamental domain D, as above, whose boundary curves are identified by the given generators.*

If G is a Schottky group and A_1,..., A_g form a set of free generators, we say that $G = \langle A_1, ..., A_g \rangle$ is a **marked Schottky group**, and that the set of transformations A_1,..., A_g is a **marking** of G.

Let us remark that if G is a Schottky group of genus g, then $\Omega(G)/G$ is a closed Riemann surface of genus g. Moreover, if A_1,..., A_g form a set of free generators for G, and D is a standard fundamental domain for these generators with boundary curves C_k, C'_k, k=1,...,g, then these loops projects to a set of g disjoint homologically independent simple loops on S. Reciprocally, the retrosection theorem [3] says us that we can reverse this situation. A simple closed curve on a Riemann surface is a one to one continuous function,

$f : S^1 \to S$, from the unit circle to the Riemann surface in question.

Retrosection theorem : *Every closed Riemann surface S of genus g can be represented as $\Omega(G)/G$, G being a Schottky group of genus g with region of discontinuity $\Omega(G)$. More precisely, given a set of g disjoint, homologically independent, simple closed curves $\gamma_1,\ldots,\gamma_g$ on S, one can choose G, and g generators $A_1,\ldots, A_g$ of G, so that there is a standard fundamental domain D for G, bounded by curves C_1, $C_1'\ldots$, C_g, C_g' with $A_i(C_i) = C_i'$, such that γ_i is in the free homotopy class of the image of C_i under $\Omega(G) \to \Omega(G)/G$. The marked Schottky group $G =< A_1, \ldots, A_g >$ is determined by $(S, \gamma_1,\ldots,\gamma_g)$ except for replacing $A_1,\ldots, A_g$ by $BA_1^{n_1}B^{-1},\ldots, BA_g^{n_g}B^{-1}$, where B is a fractional linear transformation and $n_i \in \{-1, 1\}$.*

Remark : This theorem was first stated by Felix Klein in 1883 [9] and proved rigorously by Koebe [10] much later. See page 30 in [7] for a proof. Let us remark that an easy proof of this theorem can be done using Bers ideas on quasi-conformal mappings [3].

Since Schottky groups have no parabolic elements, no finite normal extension of such a group can have parabolic elements. Finite normal extensions of Schottky groups belong to a nice class of Kleinian groups called **geometrically finite** Kleinian groups.

There exist several (equivalent) definitions of geometrically finite Kleinian groups and they can be found in chapter VI in [11].

Let us remark that fractional linear transformations act naturally as isometries on the hyperbolic 3-space $\mathbb{H}^3 \cong \{(z, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$, with the hyperbolic metric $ds^2 = (| dz |^2 + dt^2)/t^2$.

If G is any Kleinian group, then we can define fundamental domain for the action of G on $\mathbb{H}^3$. For an exact definition of that, we refer to the chapter IV in [11]. If such a fundamental domain is a convex hyperbolic polyhedron, we call it a convex hyperbolic fundamental polyhedron for G. Geometrically finite Kleinian groups correspond exactly to those Kleinian groups that have a convex hyperbolic fundamental polyhedron with a finite number of sides. Geometrically finite Kleinian groups are in fact finitely generated as abstract groups as consequence of Poincare's theorem (chapter IV in [11]), but the converse is not necessarily true.

Chapter 2

ELLIPTIC ELEMENTS IN GEOMETRICALLY FINITE KLEINIAN GROUPS.

In this second chapter, we study the set of fixed points of elliptic elements of geometrically finite Kleinian groups. Reference for this part is chapter VI in [11]. Let us remark that in a Kleinian group the two fixed points of an elliptic element can be located either in the region of discontinuity or in the limit set or in both regions as can be seen from the following examples :

(1) Let G be generated by $A(z)=z+1$ and $B(z)=-z$. In this case any elliptic element of G is in fact conjugate in G to B. One of the fixed points of B, ∞, is fixed by the parabolic element A, so is a limit point. The other fixed point, 0, is a regular point of G.

(2) Let G be generated by $A(z)=2z$ and $B(z)=-z$. Here both fixed points of B are fixed by the loxodromic element B, so both are limit points of G.

(3) Let G be generated by $A(z)=z+1$, $B(z)=z/(z+1)$, and $C(z)=-z$. Both

fixed points of B are fixed by a parabolic element, so both are limit points. This case is essentially very different from the case (2).

(4) Let G be generated by $A(z)=-z$. In this case, both fixed points of A are regular points of G.

Let us remark at this point that much more complicated examples than the above ones can be constructed using the Klein-Maskit combination theorems [11].

We define a nice property for Kleinian groups relating the set of fixed points of its elliptic elements and we show that geometrically finite Kleinian groups satify such a property.

We say that a Kleinian group G satisfies the **mixed elliptic fixed point property (M.E.F.P.)** if, for any elliptic element h of G with x and y as fixed points, one of the following conditions hold:

(1) x and y are regular points of G, or

(2) there exists a loxodromic element g in G with x and y as fixed points, or

(3) there is a parabolic element p in G sharing a common fixed point with h, say x, and either y is a regular point of G, or there exists another parabolic element q in G with fixed point y.

The main result of this chapter is the following.

Theorem 1: *Let G be a geometrically finite Kleinian group. Then G satisfies the M.E.F.P. property.*

Proof:

If G is torsion free, then there is nothing to check. Let us assume G has torsion. Let $h \in G$ be any elliptic element with fixed points x and y.

case (1): x and y are regular points of G, in which case we are done.

case (2): x or y is a limit point.

Without loss of generality we can assume that y is a limit point. Let j∈G be a primitive elliptic element fixing y.

claim: (i) Either j(x)=x, or there is a parabolic element in G with y as fixed point.

(ii) If g(y)=y, g in G, then either g is conjugate to a power of j in G, or g is a loxodromic element with x and y as fixed points, or there is a parabolic element fixing y.

proof of claim: (i) If j(x)≠x, then the commutator $[j,h] = jhj^{-1}h^{-1}$ is a parabolic element in G with y as fixed point.

(ii) Let g in G be such that g(y)=y. Assume y is not a fixed point of any parabolic element of G. The only possibility for g is to be elliptic or loxodromic. By our assumption on y, we obtain that necessarily g(x)=x; otherwise [g,j] will be a parabolic element of G fixing the point y. At this point, g is either a power of j, or a loxodromic element with x and y as fixed points. This ends the proof of our claim.

From now on, we assume y is not a parabolic fixed point. Let L be the geodesic in $\mathbb{H}^3$ with x and y as end points. We have that j acts as the identity on L.

Let P be a convex fundamental polyhedron for G. Since y is a limit point, which is not a parabolic fixed point, y must be a point of approximation for G. See page 128 in [11]. This implies that y cannot be in the closure of P. See page 122 in [11].

By the observation above, we can find a sequence of points $y_n \in$L, converging to y, all of them non-equivalent points by G, and a sequence $g_n \in$G, all of them different, such that $g_n(y_n) = z_n \in \bar{P}$, where $\bar{P}$ denotes the Euclidean closure of P.

Let us consider a subsequence such that z_n converges, say to z, $g_n(y)$ converges, say to u, and $g_n(x)$ converges, say to t. In this way, the points u and t are limit points for the group G.

Since $z_n \in \bar{P}$, then z$\in \bar{P}$. We have two possibilities for z, that is, z is a regular point, or z is a parabolic fixed point. See page 128 in [11]. It is clear that the z_n are elliptic fixed points, in fact z_n=$g_n j g_n^{-1}(z_n)$. This implies that z_n must be in some edge of P. Since P has only a finite number of edges, we can assume all z_n are on the same edge of P. Let M be the geodesic in $\mathbb{H}^3$ containing this edge. In particular, z must belong to the closure of M.

claim: z is a regular point of G.

proof of claim: Since z belongs to the boundary of the polyhedron P, then either z is a regular point or z is a parabolic fixed point. If we assume z to be a parabolic fixed point, then the stabilizer of z in G, $stab_z G$, is a Euclidean group [11]. Since all points in $\mathbb{H}^3$ are regular points for the group G and z belongs to the geodesic M, then z must be one of the end points of M. The discreteness of G implies the existence of a horoball H contained in $\mathbb{H}^3$ which is precisely invariant under $stab_z G$ in G. In particular, z belongs to the boundary of H. The geodesic M must intersect H and such an intersection is an arc of geodesic with one end point being z. Since the sequence of points z_n are in M and they are converging to z, we can assume without lost of generality that all z_n are in this intersection. The linear transformations $g_n j g_n^{-1}$ have z_n as fixed points, so $g_n j g_n^{-1}(H) \bigcap H \neq \phi$. By the definition of H, we must have $g_n j g_n^{-1}(H) = H$ and $g_n j g_n^{-1}(z) = z$. The additional condition that $g_n j g_n^{-1}(z_n) = z_n$ implies that $g_n j g_n^{-1}(w) = w$, for all w in the geodesic M. In particular, $g_n(L) = M$. Let us consider the linear transformations $h_n = g_n g_{n+1}^{-1}$. These transformations have fixed points the end points of M, in particular z. The transformation h_n is not the identity on M, so this must be loxodromic. This contradicts the fact that z is a parabolic fixed point and that G is a discrete group. So we must have z as a regular point of G. This ends the proof of our claim.

Let us consider the geodesics $L_n = g_n(L)$ through z_n, and having end points $g_n(x)$ and $g_n(y)$. Since we have supposed $g_n(x)$ and $g_n(y)$ to converge to t and u, respectively, then L_n converges either to a point or to the geodesic with end points u and t. If L_n converges to a point, we necessarily have u=t=z. This is a contradiction to the fact that z is a regular point and u = t is a limit point for G.

The other possibility is that L_n converges to a geodesic γ, with end points u and t. In this case, since the end points of γ are limit points and z is known to be a regular point we must have z in $\gamma \bigcap \mathbb{H}^3$.

Any neighborhood of z contains z_n, for n sufficiently large. Since z is a regular point, there exists a neighborhood of z which is precisely invariant by the elements of G that fixes z, which is known to be finite. We can then assume without lost of generality that $g_n j g_n^{-1}$(z)=z, and $g_n j g_n^{-1} = h$. In other words, $(g_m^{-1} g_n) j (g_m^{-1} g_n)^{-1} = j$. Since $g_n j g_n^{-1}(z_n) = z_n$, $g_n j g_n^{-1}$(z)=z, and $z_n \neq$ z for all n, then $g_n j g_n^{-1}$(w) = w, for all w in γ. In particular, $g_n j g_n^{-1}$(t) = t and

$g_n j g_n^{-1}$(u) = u. It follows that $\{g_n(\mathrm{x}), g_n(\mathrm{y})\} = \{$ t, u $\}$. The facts that t $\neq$ u and g_n(x) converges to t imply that that g_n(x) = t and g_n(y) = u, for n sufficiently large. We may assume it holds for every n. The last observation implies that $g_m^{-1} g_n$(x) = x and $g_m^{-1} g_n$(y) = y, for all n, m.

The transformations $g_m^{-1} g_n$ also keep L invariant, and for $n \neq m$ this transformation cannot be the identity on L. This implies that $g_m^{-1} g_n$ is a loxodromic with x and y as fixed points. $\square$

Before we finish this chapter, let us say that we can also show that any function group F satisfies the M.E.F.P. property if F does not have a degenerate subgroup with torsion, see [11] for definitions. The proof of this fact will appear elsewhere.

Chapter 3

CONFORMAL AUTOMORPHISMS OF RIEMANN SURFACES.

In this chapter we recall some facts from covering spaces theory and we introduce some necessary notations to introduce our main problem. We obtain necessary conditions to answer this problem. At the end of this chapter we state our main theorem which says that our necessary conditions are also sufficient in some cyclic cases. The proof of this result will be given in the next chapter.

We need some definitions and notations before introducing our problem. Let S be a Riemann surface and $f : S \rightarrow S$ be a homeomorphism which is analytic (then the inverse is automatically analytic). We call f a **conformal (or holomorphic or analytic) automorphism of S.** For a continuous map $f : S \rightarrow S$ to be analytic we mean that for each point p in S there exist open neighboorhods U and V of p and f(p) respectively, f(U) $\subset$ V, and local coordinates $z : U \rightarrow \Delta$, $w : V \rightarrow \Delta$, where Δ denotes the unit disc $\{x \in \mathbb{C} : \mid x \mid < 1\}$, and such that $wfz^{-1} : \Delta \rightarrow \Delta$ is analytic in the usual sense.

Observe that composition of conformal automorphisms is again a conformal automorphism. A group of conformal automorphisms of a Riemann surface S will be a set of conformal automorphisms of S closed under the operation of composition of maps. See chapter 5 of [5] for more details in the theory of conformal automorphisms of Riemann surfaces.

In the rest of this discussion we will be interested in **closed Riemann surfaces.** A nice result due to Hurwitz says that if the genus of S is greater than or equal to two, then the total group of conformal automorphisms of S is finite of order at most $84(g-1)$. See page 242 in [5]. If the genus is either 0 or 1, then the groups of automorphisms is infinite. In fact, for the genus zero case, it is a three complex dimensional Lie group, and for the genus one case, it is a finite extension of a one complex dimensional compact abelian Lie group.

From now on, we will consider closed Riemann surfaces S of genus greater or equal to one and finite groups H of automorphisms of S. By Hurwitz' result, the condition of finiteness is only a restriction in the genus one case.

We are interested in studying the following question concerning conformal automorphisms and Schottky groups.

(Q) Given S and H as above; can we find a Schottky group G, uniformizing S, such that every element of H can be lifted to a conformal automorphism of $\Omega(G)$, the regular region of G ?

The genus zero case can be easily obtained since there is only one Schottky group of genus zero, this being just the identity group. So, in this case we are dealing only with the finite groups of automorphisms of the Riemann sphere, which clearly are finite normal extension of the identity group.

It is known that every conformal automorphism of the regular region of a Schottky group is the restriction of a fractional linear transformation, that is, an element of $\mathbb{M}$. See page 241 in [2].

Before we say anything about our problem, let us recall some basics from covering theory.

Definition : Let $\pi : \tilde{S} \to S$ be a continuous map between topological spaces. We say that π is a covering map if for each point p in S there exists an open neighboorhod U of p, such that the inverse image by π of U, π^{-1}(U), is the disjoint union of open sets V_i, where the restriction maps $\pi : V_i \to U$ are homeomorphisms. The topological space $\tilde{S}$ is called a covering space of S.

Definition : Let $\pi : \tilde{S} \to S$ be a covering. We say that this is regular if there exists a freely acting group G of homeomorphisms of $\tilde{S}$, called the covering group, satisfying the following property:

(i) if x, y are points in $\tilde{S}$, then π(x) = π(y) if and only if there exists g in G such that g(x) = y; or equivalently

(ii) $\pi g = \pi$, for all g in G, and G acts transitively on the fibers $\pi^{-1}(x)$, for all x in S.

Let us remark at this point that there exist coverings which are not regular.

Lemma : *If $\pi : \tilde{S} \to S$ is a covering, and S is a Riemann surface, then $\tilde{S}$ has a unique structure as Riemann surface that makes $\pi : \tilde{S} \to S$ an analytic map.*

Proof : Since the map π is locally a homeomorphism, we can lift the Riemann surface structure on S to $\tilde{S}$. In fact, if x is in $\tilde{S}$ and y=π(x), then we consider an open neighborhood V of x and an open neighborhood U of y, such that $\pi : V \to U$ is a homeomorphism. Since S is a Riemann surface, we have a local chart (W,z), y$\in$W$\subset$U. Let us consider for x the local chart

(π^{-1}(W)$\cap$V, zπ). Now it is easy to check that in this way $\tilde{S}$ has a Riemann surface structure making π analytic. The unicity follows from the fact that if $\tilde{S}$ has two such structures, then the identity map I of $\tilde{S}$ satisfies πI = π. Since π is locally bi-analytic, the map I turn out to be also bi-analytic. So, both structures are the same.□

Lemma :*If $\pi : \tilde{S} \to S$ is a regular analytic covering between Riemann surfaces with covering group G, then G is a group of conformal automorphisms of $\tilde{S}$.*

Proof : If g is in G, then πg = π. Since π is locally bi-analytic, g is analytic.□

Definition : Let $\pi_k : \tilde{S}_k \to S_k$ be covering maps, k=1,2. Assume $f : S_1 \to S_2$ is a homeomorphism. We say that a homeomorphism $\tilde{f} : \tilde{S}_1 \to \tilde{S}_2$ is a lifting of f if $\pi_2\tilde{f}$ = fπ_1.

Lemma :*In the above definition, if the coverings are analytic coverings between Riemann surfaces and f is an analytic homeomorphism, then any lifting h of f must be also analytic.*

Proof : Since π_2 is locally analytic and π_2h = fπ_1, the map h is also analytic.□

Lemma :*Let $\pi : \tilde{S} \to S$ be a regular covering with covering group G. Let $f : S \to S$ be a homeomorphism. Assume there exists a homeomorphism $\tilde{f} : \tilde{S} \to \tilde{S}$ which is a lifting of f. Then*

(1) Every lifting of f has the form $g\tilde{f}$, where g is in G, and every transforma-

tion of the above form is in fact a lifting of f.

(2) If y_1, y_2 are preimages of x and $f(x)$ respectively, then there exists exactly one lifting h of f such that $h(y_1) = y_2$.

(3) If f has a fixed point, then any lifting h of f with fixed points has the same order as f.

Proof :

(1) If t is another lifting of f, then $\tilde{f}^{-1}$t is a lifting of the identity map on S. So, $\tilde{f}^{-1}$t = g belongs to G. On the other hand, if g is in G, then clearly g$\tilde{f}$ is a lifting of the map f.

(2) Since $\tilde{f}(y_1)$ is also a lifting of f(x), then we can find g in G such that g$\tilde{f}(y_1)$ = y_2. The transformation h = g$\tilde{f}$ is also a lifting of f by the second part of (1). To get unicity, assume we have two liftings of f, say h and t, satisfying the hypotheses. Then h^{-1}t is a lifting of the identity, so belongs to G, and it fixes a point in $\tilde{S}$. This only can happen if h^{-1}t is the identity, or equivalently, if t = h.

(3) Let h be a lifting of f having a fixed point, say p. If we denote by q the projection of p to S, then q is a fixed point of f. In fact, since h is a lifting of f we have that q=π(p)=π(h(p)) = f(π(p))=f(q). If we denote by n the order of f, then h^n is a lifting of the identity. It follows that h^n belongs to G. The fact that G acts fixed point freely implies that h^n is the identity. If m denotes the order of h, then the last assertion implies that m divides n. In the other hand, the identity map h^m is a lifting of f^m, which implies that f^m is the identity. In this case n divides m. As consequence n = m, that is, the order of f and h are the same.□

Proposition : *Let* $\pi : \tilde{S} \to S$ *be a planar regular covering between Riemann surfaces with covering group* $G \leq \mathbf{M}$*. Assume G has no parabolic elements and it satisfies the M.E.F.P. property. If* $f : S \to S$ *is a conformal automorphism of S of finite order, say n, which can be lifted to a conformal automorphism* $h : \tilde{S} \to \tilde{S}$*, then the following hold.*

(1) If f acts fixed point freely, then either

(1.1) h is an elliptic element of order n and there exists g in G loxodromic commuting with h, or

(1.2) h is loxodromic and h^n*, the composition of h n-times, belongs to G.*

(2) If f has a fixed point x on S and y in $\tilde{S}$ *is a lifting of x, then there exists a unique lifting t of f with y as fixed point, and such a lifting is elliptic of order n.*

Proof : (1) The liftings of f can only be elliptic or loxodromic. In fact, if there exists a lifting of f which is parabolic, say r, then r^n is a lifting of the identity, so it must belong to the group G. Since r^n is also parabolic, this contradicts the fact that G has no parabolic points. If such a lifting is loxodromic we are done. Let us assume that a lifting, say t, is elliptic. Since f has no fixed points, then the fixed points of t must belong to the limit set of G. The M.E.F.P. property and the fact that G has no parabolics imply the existence of a loxodromic g in G commuting with t.

(2) If x is a fixed point of f and y is a lifting of x, then part (2) of the last lemma above implies the desired result.□

Now, let us come back to our problem. Let us assume we can answer **(Q)** affirmatively for H, that is, there exists a Schottky group G, uniformizing S, such that every element of H can be lifted to a conformal automorphism of the region of discontinuity of G. Let $\tilde{G}$ be the group generated by G and the liftings of the elements of H. Since H is finite, $\tilde{G}$ is a finite normal extension of G. It follows that $\tilde{G}$ is a geometrically finite function group. Theorem 1, in chapter 2, implies that $\tilde{G}$ satisfies the mixed elliptic fixed point property. Moreover, since G does not have parabolic elements, then neither does $\tilde{G}$. As consequence, only (1) and (2) in the definition of the M.E.F.P. property can happen for $\tilde{G}$.

Remark : The above results on covering maps gives us the following about the liftings of elements of H to $\tilde{G}$.

(1) If h is an element of H of order n acting without fixed points, then any lifting $\tilde{h}$ in $\tilde{G}$ of h must have one of the following properties:

(1.a) $\tilde{h}$ is elliptic of order n and there exists a loxodromic element g in G commuting with $\tilde{h}$; or

(1.b) $\tilde{h}$ is a loxodromic element with $\tilde{h}^n$ belonging to G-$\{id\}$.

(2) If h is an element of H of order n acting with fixed points, then we can find a lifting $\tilde{h}$ in $\tilde{G}$ of the transformation h which is elliptic of order n with both fixed points in Ω(G)= $\Omega(\tilde{G})$. Such lifting is unique if we fix a lifting of one fixed point of h as fixed point of $\tilde{h}$.

We need some definitions to write necessary conditions for (Q) to be answered affirmatively.

Definition : Let p, q$\in$S be fixed points of non-trivial elements in H. We will say that **p and q are paired, or that they form a pair (p,q)**, if there exists $\tilde{h} \in \tilde{G} - \{id\}$ of finite order with fixed points x and y projecting to p and q respectively.

Definition : For p∈S, **the stabilizer of p with respect to H** is the group H(p) = $\{ h \in H : h(p) = p \}$.

For the next definition, we need a classical result. Let h∈H and p∈S be as before such that h(p)=p. We can find a local coordinate system (U, ϕ) such that $\phi(p) = 0$ and $\phi \circ h \circ \phi^{-1}(z) = e^{i\alpha}z$, for all $z \in \phi(U)$. Moreover, we can assume ϕ(U) = Δ, where Δ denotes the unit disc in the complex plane $\mathbb{C}$.

Lemma : *The angle $\alpha = \alpha(h,p)$ is well defined up to a multiple of 2π, independent of the local coordinate and $\alpha(h^k, p) = k\alpha(h,p)$.*

Proof : We only need to check the independence from the local chart. Let (U,R) and (V,T) be local charts such that p belongs to U and V, R(p) = T(p) = 0, and R(U) = T(V) = Δ. Then $RhR^{-1}(z) = e^{i\alpha}z$ and $ThT^{-1}(w) = e^{i\theta}w$, since RhR^{-1} and ThT^{-1} are conformal automorphisms of the unit disc Δ fixing the origin (Schwarz's lemma). Let us consider t(q) = T$R^{-1}(q) = e^{i\eta}q$, then $e^{i\alpha}z = RhR^{-1}(z) = RT^{-1}ThT^{-1}TR^{-1}(z) = t^{-1}ThT^{-1}t(z) = e^{-i\eta}e^{i\theta}e^{i\eta}(z) = e^{i\theta}(z)$. This equation implies $e^{i\alpha} = e^{i\theta}$ and then $\alpha - \theta = 2K\pi$, for some K.□

Definition : (The rotation number) Let h∈H and p∈S be such that h(p)=p. We normalize α by assuming that $-\pi < \alpha \leq \pi$. We will call $\alpha = \alpha(h,p)$ **the rotation number of** h **at p.**

Definition : Let h be a conformal automorphism of a Riemann surface S. We denote by N(h) the number of fixed points of h.

The following results are obtained under the assumption that we can answer **(Q)** affirmatively.

Proposition 1 : *Let p∈S be fixed by some element h in H-{id}. Then there* *exists a unique point* $q \in S - \{p\}$ *which is paired to p. Moreover, if* $t \in S - \{p, q\}$ *is fixed by some non-trivial element of H, then t cannot be paired either to p or q. In particular, if h in H-*$\{id\}$ *has a fixed point, then it must have an even number of fixed points.*

Proof of Proposition 1 : Let h be an element of H and let p be a fixed point of h, that is, h(p)=p. Let x be a point in the regular region (Ω(G)) of G projecting to p. We can find a lifting (of the same order as h) $\tilde{h}$ in $\tilde{G}$ of h such that $\tilde{h}$(x)=x.

Let y be the other fixed point of $\tilde{h}$, then the M.E.F.P. property implies that y is a regular point for the group G, hence for the group $\tilde{G}$. If we show that y project on S to a point different from p, say q, then p and q are paired in the above sense. Assume y projects onto p, then there exists g in G satisfying g(x)=y. The commutator $[\tilde{h}, g\tilde{h}g^{-1}]$ will be parabolic if g(y)$\neq$x. Since $\tilde{G}$ has no parabolic elements, this is not the case. This implies that g(y)=x, and in particular g^2=1. The last is a contradiction to the fact that G has no elliptic elements. To prove the second statement of proposition 1, we assume t$\in S - \{p, q\}$ is fixed by some element in H-$\{id\}$ and it is paired to p. Then there exists j in $\tilde{G} - \{id\}$ of finite order with fixed points u and v such that u projects onto p and v projects onto t. Since p and q are paired, there exists $\tilde{h}$ in $\tilde{G} - \{id\}$ of finite order with fixed points x and y such that x projects to p and y projects to q. The condition that x and u project onto the same point p means that there exists g in G such that g(u)=x. Let us consider $k = gjg^{-1} \in \tilde{G}$. Then k is of finite order and non-trivial element of $\tilde{G} - \{id\}$ with fixed points x and g(v) projecting to p and t respectively. Since t is different from q, g(v) is also different from y. Now we form the commutator [k,$\tilde{h}$]= $k\tilde{h}k^{-1}\tilde{h}^{-1}$ which must be parabolic and we get a contradiction to the fact that $\tilde{G}$ has no parabolic elements. □

Proposition 2 : *Assume that p and q are paired under H. Then*

(1) H(p) = H(q), and

(2) $\alpha(h,p) = -\alpha(h,q)$*, for all* $h \in H(p)-\{1\} = H(q)-\{1\}$*, if h has order bigger than two, where 1 denotes the identity of H.*

Proof of Proposition 2 : Assume that p and q are paired under H, that is, there exists $\tilde{h}$ in $\tilde{G}$ of finite order with fixed points x and y projecting to p and q respectively.

By one of the lemmas in page 23, we have that for every t in H(p) there is one and only one transformation $\tilde{t}$ in $\tilde{G}$ which is a lifting of t and fixes x. Let z be the other fixed point of $\tilde{t}$, which is also in the regular region of G by the M.E.F.P. property.

Claim : z=y.

Proof of claim : If z≠y, then the commutator [$\tilde{t}$,$\tilde{h}$] in $\tilde{G}$ is parabolic. But $\tilde{G}$ cannot have parabolic elements, so z=y as we required.

We have shown that t also belongs to H(q), and by symmetry we get H(p)=H(q). Also we have shown that for every t in H(p)=H(q) there exists a lifting $\tilde{t}$ of finite order (the same as t) with x and y as fixed points. In particular $\alpha(\tilde{t},x)=-\alpha(\tilde{t},y)$. But $\alpha(\tilde{t},x)=\alpha(t,p)$ and $\alpha(\tilde{t},y)=\alpha(t,q)$.□

The above two propositions give us the following necessary conditions, on our group H, to find a Schottky group G as desired. **The set of fixed points of the non-trivial elements of H can be paired in the following way :**

(1) if (p,q) is such a pair, then p $\neq$ q, H(p) = H(q) and α(h,p) = $-\alpha$(h,q) for all h $\in$ H of order greater than two;

(2) if (p,q) and (r,s) are two such pairs, then either {p,q} = {r,t}, or {p,q} $\bigcap$ {r,t} = ϕ.

Proposition 3 : *Assume p and q are paired under H. If there exists h in H such that h(p)=q, then h is an involution, that is, $h^2 = 1$. In particular, if H has no elements of order 2, then any pair (p,q) will project onto two different points on the quotient Riemann surface obtained by the action of H on S.*

Proof of Proposition 3 : Let us assume there exists a pair (p,q) and an element h in H such that h(p)=q. Let $\tilde{g} \in \tilde{G}$ be an elliptic element with fixed points x and y projecting on S to p and q respectively. Since there exists h in H with h(p)=q, then there exists a unique lifting $\tilde{h} \in \tilde{G}$ such that $\tilde{h}(x) = y$. Let us consider $\tilde{g}$ and $\tilde{h} \circ \tilde{g} \circ \tilde{h}^{-1}$; both elliptic elements of the same order with y as a common fixed point. Since $\tilde{G}$ has no parabolic elements, they must also have x as a common fixed point, that is, $\tilde{h}(y) = x$. This means that $\tilde{h}^2 = 1$ and $\tilde{h} \neq id$. Since G has no elliptic elements, $\tilde{h}$ cannot belong to G. We can see that $\tilde{h}$ induces h as an involution in S and permuting the points p and q as we require.□

Remark : If H is a cyclic group, then each pair (under H) of points p and q is formed by non H-equivalent points. In fact, since p and q are paired under H, there exist an elliptic element $\tilde{h}$ in $\tilde{G}$ with fixed points x and y, such that π(x)=p and π(y)=q. Denote by h the element of H such that $\pi\tilde{h} = h\pi$. Assume there exists t in H such that t(p)=q, then we can find $\tilde{t}$ in $\tilde{G}$ which is a lifting of t, i.e, $\pi\tilde{t} = t\pi$, such that $\tilde{t}$(x)=y. Since $\tilde{G}$ has no parabolic elements, $\tilde{t}$(y)=x, so $\tilde{t}^2 = 1$ and $t^2 = 1$. The transformations $\tilde{t}$, $\tilde{h}$ satisfy the equation $\tilde{t}\tilde{h}\tilde{t} = \tilde{h}^{-1}$. In fact, normalize such that x=0 and y=∞. Under this normalization $\tilde{h}(Z) = e^{i\theta}Z$ and $\tilde{t}(Z) = R/Z$, for some non-zero complex number R. In this case, $\tilde{t}\tilde{h}\tilde{t}(Z) = e^{-i\theta}Z = \tilde{h}^{-1}(Z)$. But now $\tilde{t}$ and $\tilde{h}$ project to t and h respectively, and the above relation says that t has order two and that th has order two. Since we have assumed t to permute the two fixed points of h, then every non-trivial power of h is different from t., that is, we get the Dihedral group $\mathbb{D}_{2n}$, where n is the order of h, as a subgroup of H. In particular, H cannot be cyclic.

The next two propositions, whose proofs are not given here, will not be used in what follows, but we put them here as matter of interest. If F is a subgroup of the group H, we will denote by [H:F] the index of F in H.

Proposition 4 : *If H has no elements of order 2, then for every* $h \in H - \{id\}$ *we have*

$N(h) = 2[H :< h >]n(h)$, *where* $n(h)$ *is a non-negative integer.*

Proposition 5 : *If H has elements of order 2, then for every* $h \in H - \{id\}$ *we have*

$N(h) = [H :< h >](2n_1(h) + n_2(h))$, *where* $n_1(h)$ *and* $n_2(h)$ *are non-negative integers, and* $n_2(h)$ *denotes the number of pairs (p,q) with* $H(p) = H(q) = < h >$ *and such that there exists some* $g \in H$, $g^2 = 1$ *and g(p)=q.*

Let H be a cyclic group of order o(f), say H=$< f >$, where o(f) is the order of f. Assume that every fixed point of any non-trivial power of f is also a fixed point of f; then the necessary conditions founded above, which will be called the condition **(A)**, can be written as follow.

(A1) f has an even number of fixed points. Denote by N(f) = 2n(f), for some non-negative integer n(f), the number of fixed points of f;

(A2) The fixed points of the automorphism f can be paired in the following sense; if (p,q) is such a pair, then p $\neq$ q and α(f,p) = $-\alpha$(f,q) if o(f) $\neq$ 2; and

(A.3) If (p,q) and (r,t) are paired as in (A.2), then either {p,q} = {r,t}, or {p,q} $\bigcap$ {r,t} = ϕ.

As result of this condition we obtain the following

Corollary 2 : *Let S be a closed Riemann surface, and let $f : S \to S$ be a conformal automorphism of finite order. Assume that f has an odd number of fixed points. Then there exist no Schottky group G, uniformizing S, such that f can be lifted to a conformal automorphism of the regular region of G.*

Example : The following shows an example of a closed Riemann surface of genus three, non-hyperelliptic, with an automorphism of order three with five fixed points.

Let us consider the non-singular e irreducible quartic

$$X^4 + Y^4 + XY^3 + aX^2Y^2 + bX^3Y + Z^3X + Z^3Y = 0,$$

for suitables complex numbers a and b. This quartic is a closed Riemann surface of genus three, non-hypereliptic, admiting the automorphism of order three h induced by the linear transformation

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w \end{pmatrix},$$

where $w^2 + w + 1 = 0$. It is easy to check that this automorphism has in fact only five fixed points.

(Q1) Is condition (A) sufficient?

The following will answer **(Q1)** in the case when H is a cyclic group with the property that the set of fixed points of all non-trivial elements of H are the same.

Theorem 2 : *Let S be a closed Riemann surface of genus $g \geq 0$, and $f : S \to S$ be a conformal automorphism of finite order, say $o(f)$. Assume that every fixed point of an element f^k is also a fixed point of f, for all $k \in \{1, ..., o(f) - 1\}$. Then condition (A) is necessary and sufficient to find a Schottky group G, uniformizing S, such that the automorphism f can be lifted to a conformal automorphism of the region of discontinuity (regular region) of G, $\Omega(G)$.*

Since any involution trivially satisfies the hypotheses of theorem 2, we obtain the following.

Corollary 3 : *Let S be a closed Riemann surface of genus $g \geq 0$, and $f : S \to S$ be a conformal involution. Then there exists a Schottky group G, uniformizing S, such that f can be lifted to a conformal automorphism of* *the region of discontinuity of G, $\Omega(G)$.*

Remark : In the particular case when S is hyperelliptic and f is the hyperelliptic involution, corollary 3 was proven by L. Keen [8]; the general case first appeared in [6].

The proof of theorem 2 appears in the next chapter. Before we go on, let us look at two examples in genus one. In genus one Riemann surfaces, tori, there are only two classes of conformally different tori with an automorphism having fixed points. These tori are given by the quotient of the complex plane $\mathbb{C}$ by the group G_1, generated by the parabolic transformations A(z)=z+1 and B(z)=z+i, and the group G_2, generated by the parabolic transformations A(z)=z+1 and B(z)=z+ρ, where $\rho = 1/2 + i\sqrt{3}/2$. In the first case, $T_1 = \mathbb{C}/G_1$ has an automorphism j of order four having two fixed points, where j^2 is an involution with four fixed points. It is easy to see that the rotation number of j at both fixed points is the same; figure 3.1 shows the action of j at its fixed points when we lift it to the universal covering of T_1. In particular the necessary conditions to find a Schottky group G of genus one uniformizing T_1 such that j can be lifted to a conformal automorphism of $\Omega(G)$ are not satisfied by $H = < j > \cong \mathbb{Z}/4\mathbb{Z}$.

In the second case, $T_2 = \mathbb{C}/G_2$ has an automorphism t of order six having only one fixed point. There are three points in T_2 which are permuted between them by t and another two points which are permuted between them by t. Figure 3.2 shows the action of t when we lift to the universal covering of T_2. Again, the necessary conditions are violated by $H = < t > \cong \mathbb{Z}/6\mathbb{Z}$.

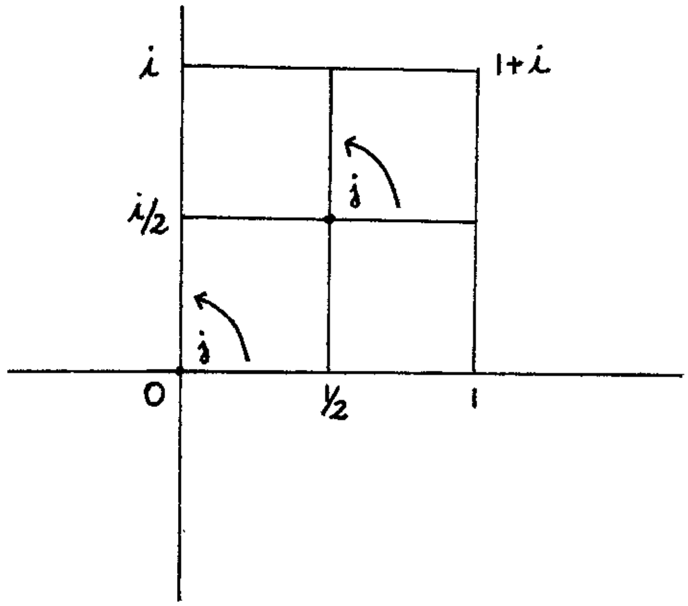

Figure 3.1: The rotation of j at the fixed points.

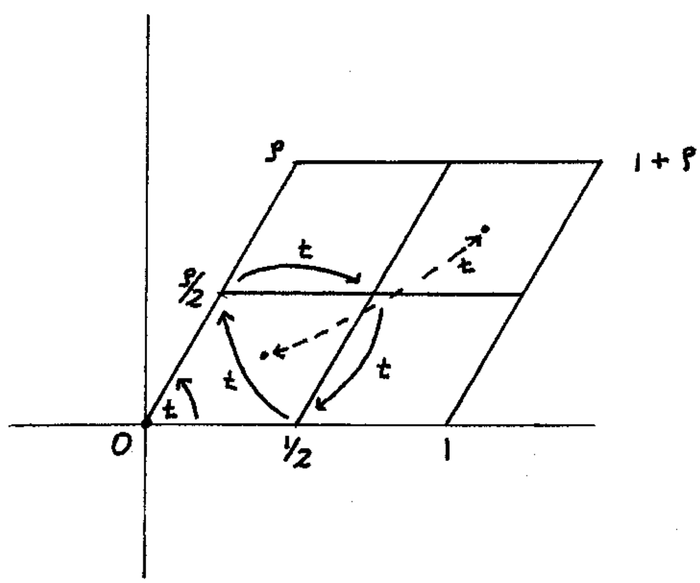

Figure 3.2: The action of t at the universal covering.

If we look at the elementary Kleinian groups with exactly two limit points [11], we can see that it is impossible to find a finite extension of a Schottky group of genus one with an elliptic transformation of order greater than two having a fixed point as regular point of a such group. Let us also remark that any torus T admits involutions with 4 fixed points. These involutions satisfy trivially the (A) condition. Since these involutions act on the fundamental group $\Pi_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$ as the transformation $(n, m) \to (-n, -m)$, it is easy to see that they lift to any covering of T. In particular, to every Schottky covering of T.

Let us remark also that if a subgroup of conformal automorphisms H of a closed Riemann surface S has order 4, and H satisfies the necessary conditions above, then we can find a Schottky group G with region of discontinuity Ω, and a finite extension $\tilde{H}$ of G where G is a normal subgroup of index four of $\tilde{H}$, and for every h in H there exists $\tilde{h} \in \tilde{H}$ with $\pi\tilde{h} = \pi h$, where $\pi : \Omega \to \Omega/G = S$ denotes the natural holomorphic covering induced by G on Ω. The proof of this fact will appear elsewhere. The general case is still an open problem.

We can also generalize this problem for more general uniformizations (Δ, F) of closed Riemann surfaces, but in this case it is easy to show that our necessary conditions are not in general sufficient. This problem in this generality will be discussed elsewhere.

Chapter 4

THE PROOF OF THE MAIN THEOREM.

In this last chapter, we prove theorem 2 as a consequence of two lemmas. Those lemmas describe the topological action of any conformal automorphism satisfying our necessary conditions.

Let S be a closed Riemann surface of genus $g \geq 0$, and let $f : S \to S$ be a finite order conformal automorphism of the surface S. Let us denote by o(f) the order of this transformation. Assume that every fixed point of the transformation f^k is also a fixed point of f, for all $k \in \{1, ..., o(f) - 1\}$. We make the following definitions.

(i) $\tilde{S}$ is the quotient Riemann surface obtained by the action of $< f >$ on S ;

(ii) N(f) is the number of fixed points of f;

(iii) γ is the genus of $\tilde{S}$; and

(iv) π:S$\rightarrow$ $\tilde{S}$ is the natural holomorphic projection induced by f on S.

In this case, we can re-write condition **(A)** as:

(A1) f has an even number of fixed points, that is, N(f) = 2n(f), for some non-negative integer n(f);

(A2) The fixed points of the automorphism f can be paired in the following sense; if (p,q) is such a pair, then p $\neq$ q and α(f,p) = $-\alpha$(f,q) if o(f) $\neq 2$; and

(A.3) If (p,q) and (r,t) are paired as in (A.2), then either {p,q} = {r,t}, or {p,q} $\bigcap$ {r,t} = ϕ.

The Riemann-Hurwitz formula [5] gives us the following relation between the genera of the surfaces S and $\tilde{S}$, the number of fixed points of f and the order of f :

$$g = o(f)\gamma + (n(f) - 1)(o(f) - 1).$$

The proof of theorem 2 will be obtained as consequence of the following two lemmas.

Lemma 1 : *Assume f is fixed point free.*

(a) For some (non-dividing) simple closed curve η on $\tilde{S}$, $\pi^{-1}(\eta)$ consists of o(f) disjoint non-dividing simple closed curves which divide S into o(f) parts.

(b) The canonical projection, π, maps each part topologically onto $\tilde{S} - \eta$ and each part is bounded by exactly two of the o(f) loops in (a).

(c) Some element g identifies the two boundary components simultaneously for each part, where $< f > = < g >$.

Observe that in the fixed point free case, the Riemann-Hurwitz formula says that

g=o(f)($\gamma - 1$)+1. The proof of this lemma, when o(f) is prime, was done by Accola in [1]. For the general case it was proved by R. Ruedy in [13]. Figure 4.1 shows lemma 1 in the case o(f)=4, γ=1 and $g = f$.

The following lemma is a generalization of lemma 2 in [13]. The proof

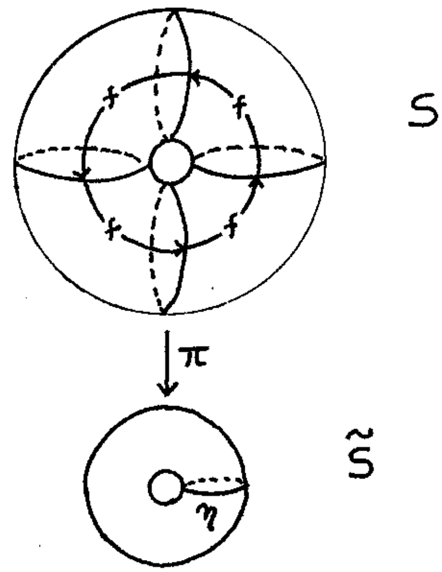

Figure 4.1: o(f)=4, γ=1 and $g = f$.

follows in the same way and it will be done after we show how to obtain the theorem 2 from the above two lemmas.

Lemma 2 : *Assume n(f)$\neq$ 0.*

(a) The branch values $p_1, ..., p_{2n(f)}$ on $\tilde{S}$, of π:S$\rightarrow$ $\tilde{S}$, can be arranged in such a way that there are n(f) disjoint simple paths η_j joining p_{2j-1} and p_{2j}, $j = 1, ..., n(f)$, whose preimages divide S into o(f) parts.

(b) π maps each part topologically onto $\tilde{S} - \bigcup\{\eta_j : j = 1, ..., n(f)\}$ and each part is bounded by exactly n(f) disjoint simple closed loops, each one containing exactly two fixed points of f.

(c) The fixed points of f divide these loops into two halves and some generator of $< f >$, say f^{k_j} (depending on $\alpha(f, p_{2j})$), maps one half of the curve onto the other half.

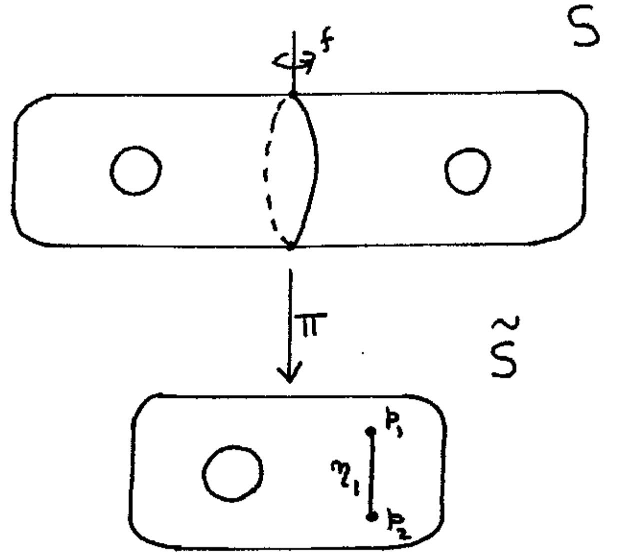

Figure 4.2: o(f)=2, γ=1 and n(f)=1.

Figure 4.2 shows lemma 2 in the case o(f)=2, γ=1 and n(f)=1. Before we prove it, let us see how we obtain theorem 2 by using the above two lemmas.

Fixed Point Free Case (**n(f)=0**).

Lemma 1 implies that S may be obtained by pasting together o(f) spheres, each with two holes and $\gamma - 1 = (g-1)/o(f)$ handles, where some non-trivial power of f, say g = f^{k_0}, permutes these surfaces with boundary in a cyclic way; mapping each one of them into an adjacent one. See figure 4.3, for the case γ=2, o(f)=4, $g = f$.

Let us consider a set of 2γ loops $\alpha_r, \beta_r,\ r = 1, ..., \gamma$, on $\tilde{S}$, satisfying:

(1) $\beta_1 = \eta$;

(2) $\alpha_i \bigcap \alpha_j = \phi \ = \ \beta_i \bigcap \beta_j,\ if\ i \neq j$;

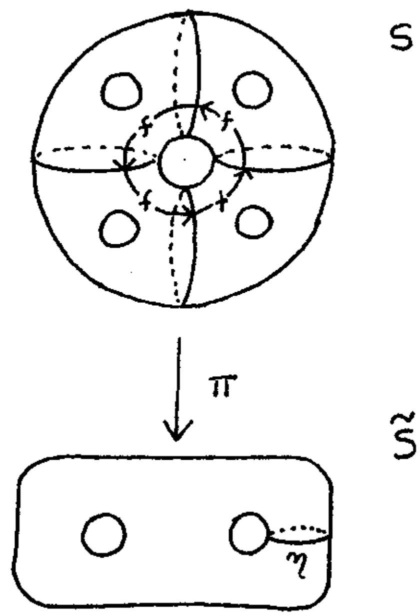

Figure 4.3: γ=2, o(f)=4, $g = f$.

(3) $\alpha_i \bigcap \beta_i$ consists of one point; and

(4) their homological classes form a canonical basis for the homology of $\tilde{S}$.

See figure 4.4 in the case γ=2, o(f)=4, $g = f$.

Let us consider the lifting under π of the loops α_r *and* β_r, $r = 1, ..., \gamma$. Let us denote by δ_1 the lifting of the loop α_1, by $\delta_{r,s}$, s=1,...,o(f), the liftings of α_r for r=2,...,γ and by $\eta_{r,s}$, s=1,...,o(f), the liftings of β_r for r=2,...,γ. See figure 4.5 in the case γ=2, o(f)=4, $g = f$.

In this way we obtain a set of simple loops on S satisfying the following conditions

(1) $\delta_1 \bigcap \delta_{r,s} = \phi$;

(2) $\delta_{r,s} \bigcap \delta_{l,t} = \phi$, if $r \neq l$, or $s \neq t$;

(3) $\eta_{r,s} \bigcap \eta_{l,t} = \phi$, if $r \neq l$, or $t \neq s$;

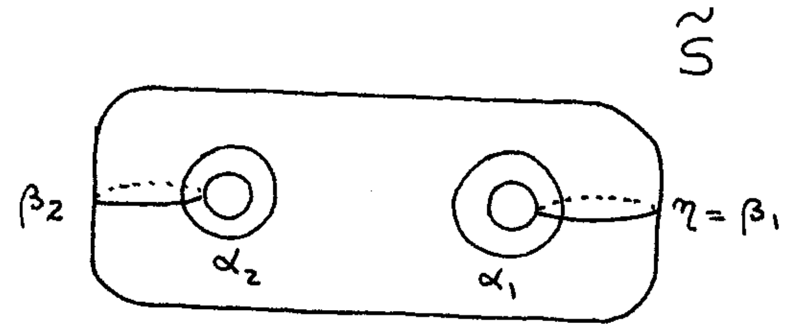

Figure 4.4: A special canonical homology basis in $\tilde{S}$ in the case γ=2, o(f)=4, $g = f$.

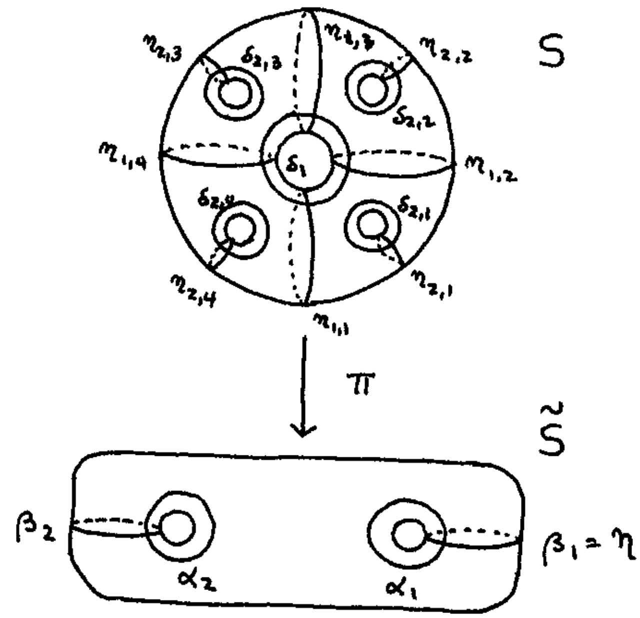

Figure 4.5: Lifting to S of a special canonical homology basis in $\tilde{S}$ in the case γ=2, o(f)=4, $g = f$.

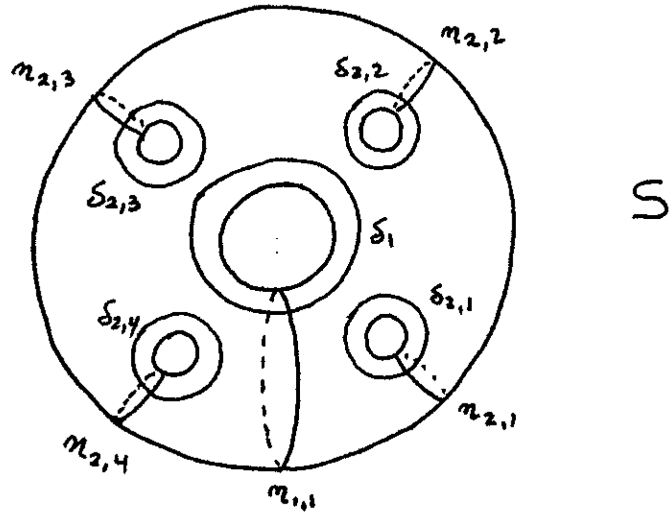

Figure 4.6: A special canonical homology basis in S in the case γ=2, o(f)=4, $g = f$.

(4) $\delta_1 \bigcap \eta_{1,s}$ is a point;

(5) $\delta_1 \bigcap \eta_{r,s} = \phi$, if r=2,...,o(f);

(6) $\delta_{r,s} \bigcap \eta_{l,t} = \phi$, if $r \neq l$, or $s \neq t$;

(7) $\delta_{r,s} \bigcap \eta_{r,s}$, is a point; and

(8) the loops δ_1, $\delta_{r,s}$, $\eta_{1,1}$, $\eta_{r,s}$, s=1,...,o(f), r=2,...,γ, form a canonical homology basis on the surface S.

See figure 4.6 in the case γ=2, o(f)=4, $g = f$.

Observe that the loop δ_1 is invariant under f, and for each r=2,...,γ, the loops $\delta_{r,s}$, s=1,...,o(f), are permuted cyclically by the automorphism f. The g (g = o(f)($\gamma - 1$) + 1) loops δ_1, $\delta_{r,s}$, s=1,...,o(f), r=2,...,γ, on S are homologically independent and, as a set of loops, it is f-invariant. The $\delta-$ loops define a

Schottky group G (up to equivalence), uniformizing the surface S, such that the conformal automorphism f can be lifted as a conformal automorphism, say F, of the region of discontinuity of G. It can be seen that F can be chosen in such a way that there exist a set of generators for G invariant under the action of F. More precisely, there exist A_1, $A_{r,t}$, $r = 2, ..., \gamma$ *and* $t = 1, ..., o(f)$, linear fractional transformations, which form a set of generators for G, such that :

(1) $F \circ A_1 \circ F^{-1} = A_1$;

(2) $F \circ A_{r,t} \circ F^{-1} = A_{r,t+1}$, $r = 2, ..., \gamma$; $t = 1, ..., o(f) - 1$; and

(3) $F \circ A_{r,o(f)-1} \circ F^{-1} = A_{r,1}$, $r = 2, ..., \gamma$.

Observe that F choose in this way must have finite order, in fact the same order as f. See figure 4.7 in the case γ=2, o(f)=4, $g = f$.

Let us remark that we can also use the $\eta-$ loops to define a Schottky group K uniformizing the surface S, such that f can be lifted as a conformal automorphism, say R, of the region of discontinuity of K. This uniformization is essentially different from the above one. In this case f has a lifting R which is a loxodromic transformation, such that $R^{o(f)}$ belongs to K. In fact, we can choose a set of generated for K, B_1, $B_{r,t}$, $r = 2, ..., \gamma$ *and* $t = 1, ..., o(f)$, such that

(1) $R \circ B_1 \circ R^{-1} = B_1$;

(2) $R^{o(f)} = B_1$;

(3) $R \circ B_{r,t} \circ R^{-1} = B_{r,t+1}$, $r = 2, ..., \gamma$; $t = 1, ..., o(f) - 1$; and

(4) $R \circ B_{r,o(f)-1} \circ R^{-1} = B_1 \circ B_{r,1} \circ B_1^{-1}$, $r = 2, ..., \gamma$.

See figure 4.8 in the case γ=2, o(f)=4, $g = f$.

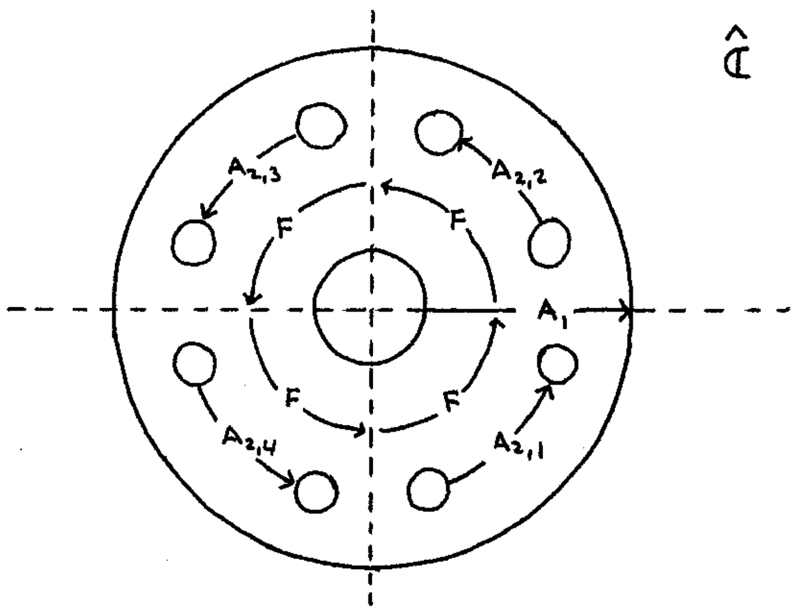

Figure 4.7: A Schottky group G uniformizing the surface S defined by the $\delta-$ loops in the case γ=2, o(f)=4, $g = f$.

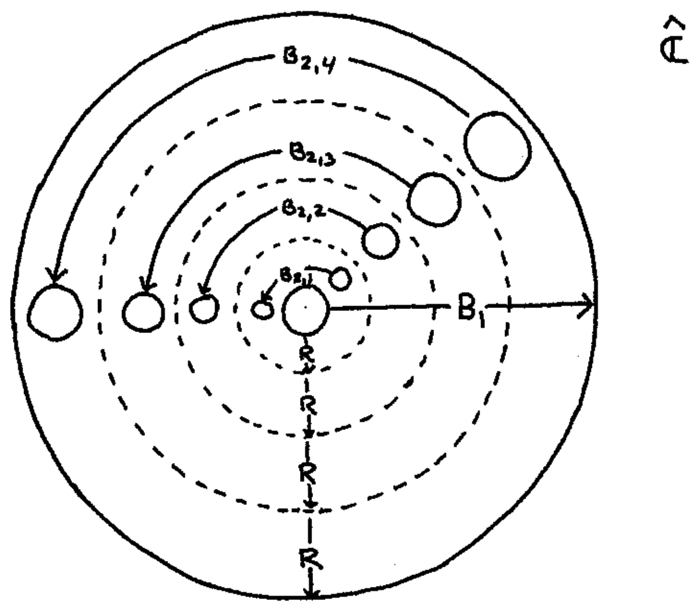

Figure 4.8: A Schottky group K uniformizing the surface S defined by the $\eta-$ loops in the case γ=2, o(f)=4, $g = f$.

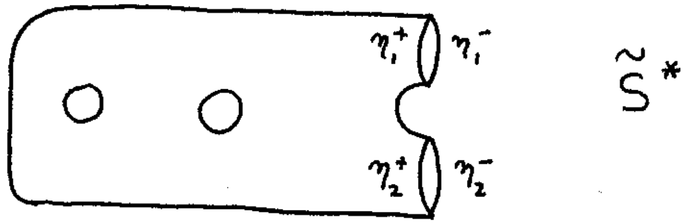

Figure 4.9: $\gamma = 2$, and n(f)=2

Fixed Point Case (n(f)$\neq$ 0).

Lemma 2 implies that S can be obtained by pasting together o(f) spheres with n(f) holes and γ handles each as follows:

let $\tilde{S}$ and η_j, j=1,...,n(f), be as in lemma 2. Cut along all the paths η_j to obtain a new surface, say $\tilde{S}^*$, with boundary $\eta_j^+ \bigcup \eta_j^-$, for j=1,..,n(f). See figure 4.9 in the case $\gamma = 2$ and n(f) = 2. Now we procede to assign to each path η_j an element σ_j in the permutation group of o(f) elements as follow. Let t_j be such that $0 <\mid \alpha(f^{t_j}, p_{2j}) \mid \leq \mid \alpha(f^k, p_{2j}) \mid$, $all\ k = 1, ..., o(f) - 1$. Then σ_j corresponds to the cyclic rotation determined by $\sigma_j(1) = t_j$. Let us consider o(f) copies of the surface $\tilde{S}^*$. Let us denote these copies by Σ_s, $s = 1, ..., o(f)$. Glue these o(f) copies in the following way:

identify each point in $\eta_j^+\ of\ \Sigma_s$ to the appropriate point in $\eta_j^-\ of\ \Sigma_{\sigma_j(s)},\ for\ all\ j\ and\ s$, so that these two identified points correspond to the same point on $\tilde{S}$. See figure 4.10 in the case that $\gamma = 0$, n(f) = 1, o(f) = 4 and σ_1 is the permutation corresponding to a rotation of angle $2\pi/4$.

The new surface obtained above gives us a topological model of both the

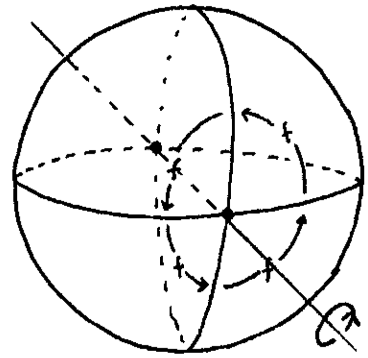

Figure 4.10: $\gamma = 0$, n(f) = 1, o(f) = 4 and σ_1 is the permutation corresponding to a rotation of angle $2\pi/4$.

surface S and the action of f on S.

To obtain a set of g disjoint homologically independent and f-invariant simple loops on S (again as a set of loops), we procede in the following way. Consider on $\tilde{S}$ a set of γ disjoint (homologically independent) simple loops, disjoint from the paths η_j, all j. Call them α_r, $r = 1, ..., \gamma$. Now, let us consider disjoint simple loops β_l, $l = 1, ..., n(f)$, also disjoint from the α-loops and such that β_l is free homotopic in $\tilde{S} - \bigcup\{\eta_j : j = 1, ..., n(f)\}$, to the boundary curve η_l. See figure 4.11, for the case $\gamma = 2$ and n(f)=2.

Let $\alpha_{r,k}$ be the liftings of α_r, and $\beta_{l,m}$ be the liftings of β_l, where k, $m = 1, ..., o(f)$; $r = 1, ..., \gamma$; $l = 1, ..., n(f)$. Then by lemma 2, up to rename, we have that

(1) $f(\alpha_{r,k}) = \alpha_{r,k+1}$, k *modulo* $o(f)$, $r = 1, ..., \gamma$,

(2) $f(\beta_{l,m}) = \beta_{l,m+1}$, m *modulo* $o(f)$, $l = 1, ..., n(f)$,

(3) $\beta_{l,o(f)}$ is homologous to $-\sum_{m=1}^{o(f)} \beta_{l,m}$.

(4) $\{\alpha_{r,k},\ \beta_{l,m}\}$, where $k = 1, ..., o(f)$; $r = 1, ..., \gamma$; $l = 1, ..., n(f) - 1$;

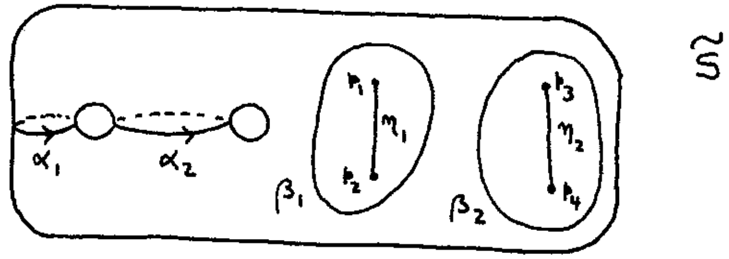

Figure 4.11: $\gamma = 2$, and n(f)=2

$m = 1, ..., o(f) - 1$, are g simple loops homologically independent on S. See figure 4.12 in the case that o(f)=γ=n(f)=2 and g=5.

The family of simple loops, given in (4), will define a Schottky group G, uniformizing S, such that f can be lifted to a conformal automorphism, F, of its region of discontinuity. As in the fixed point free case, we can find a set of generators for G, invariant under conjugation by F in the following sense. There exists linear fractional transformations $A_{r,k}$, $B_{l,m}$ $k = 1, ..., o(f)$; $r = 1, ..., \gamma$; $l = 1, ..., n(f) - 1$; $m = 1, ..., o(f) - 1$, forming a set of generators for G such that:

(1) $F \circ A_{r,k} \circ F^{-1} = A_{r,k+1}$, k *modulo* $o(f)$; $r = 1, ..., \gamma$.

(2) $F \circ B_{l,m} \circ F^{-1} = B_{l,m+1}$, $m = 1, ..., o(f) - 2$; $l = 1, ..., n(f) - 1$.

(3) $F \circ B_{l,o(f)-1} \circ F^{-1} = \prod_{n=1}^{o(f)-1} B_{l,n}^{-1}$, $l = 1, ..., n(f) - 1$.

To prove lemma 2, we will need the following.

Lemma 3 : *Let S be a closed Riemann surface, and let $f : S \to S$ be any conformal automorphism of order o(f)=pq, where p is prime and q$\geq$ 2. Assume f satisfies the condition (A) and that every fixed point of f^k is also*

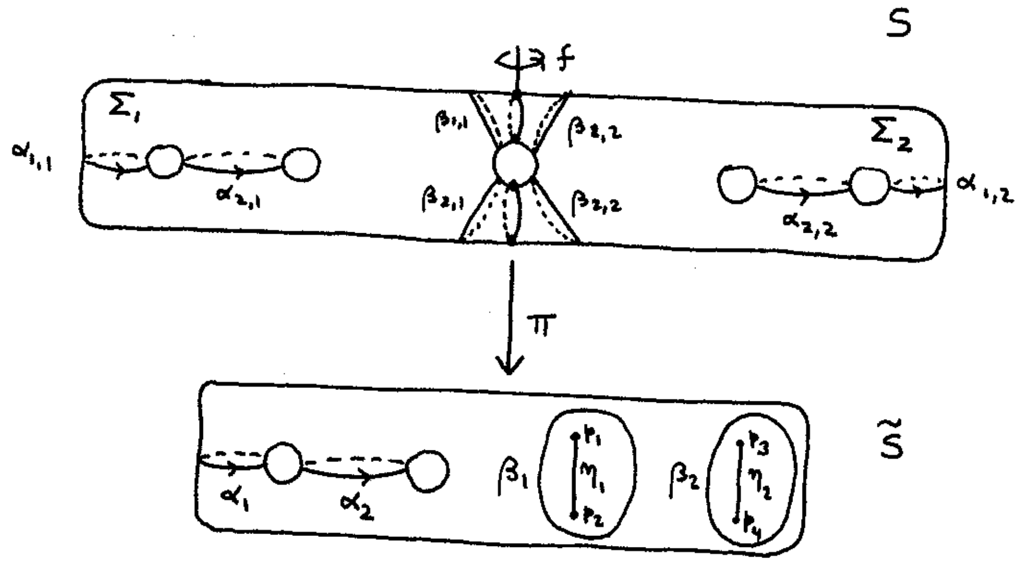

Figure 4.12: o(f)=γ=n(f)=2, g=5

fixed by f, for all $k = 1, ..., o(f) - 1$.

Let S_0*=*$S/<f^q>$ *be the quotient Riemann surface obtained by the action of* f^q *on S, and let* $\pi : S \to S_0$ *be the natural projection induced by* $<f^q>$*. Then f induces an automorphism* $g : S_0 \to S_0$*, of order* $o(g)=q$*, and* $<g> \cong <f> / <f^q>$*, such that:*

(a) $\{x \in S_0 : g(x) = x\} = \{x \in S_0 : g^k(x) = x\}$ = *the projection by* π *to* S_0 *of the set of fixed points of f, for all* $k = 1, ..., o(f) - 1$*; and*

(b) g satisfies the condition (A).

Proof :

To see the first equality in part (a), observe that the contention

$$\{x \in S_0 : g(x) = x\} \subseteq \{x \in S_0 : g^k(x) = x\}$$

is trivial. Now let us prove the other contention. If x_0 is a fixed point of the transformation g^k, then the preimage by π of x_0 is the set $\{x,\ f^q(x),\ f^{2q}(x)\ ,...,\ f^{(p-1)q}(x)\}$, where x is a point in S such that $\pi(x) = x_0$. If x is a fixed point of f, then x_0 is a fixed point of g. If we assume x not a fixed point of f, then x cannot be a fixed point of any non-trivial power of f by hypotheses. In any case $f^k(\mathrm{x}) \in \pi^{-1}(x_0)$, that is, $f^k(\mathrm{x}) = f^{sq}(\mathrm{x})$ for some s, $1 \leq s \leq p-1$. This means that $f^{sq-k}(\mathrm{x}) = \mathrm{x}$. The above observation implies that $sq-k$ is congruent to 0 modulo pq. But $0 < 1 < q-k < sq-k < pq-1 < pq$ implies that the last congruence is impossible. We have shown that every fixed point of a non-trivial power of g is also a fixed point of the transformation g. To show the other equality, let us observe that the fixed points of f project on S_0 to fixed points of g. We only need to show that they are in fact all of them. Let $x_0 \in S_0$ be fixed by g, and let x∈S be such that $\pi(x) = x_0$. The preimage by π of x_0 is the set $\{x,\ f^q(x),\ f^{2q}(x)\ ,...,\ f^{(p-1)q}(x)\}$. If f(x)=x, we are done. Let us suppose f(x)≠x. Since we are supposing g(x_0)=x_0, we must have $f(x) \in \{x,\ f^q(x),\ f^{2q}(x)\ ,...,\ f^{(p-1)q}(x)\}$. In this case, we obtain that $f(x) = f^{kq}(x)$, some $1 \leq k \leq p-1$, but this is equivalent to $x = f^{kq-1}(x)$. Since, we have assumed the fixed points of any power of f to be also fixed points of f, and we are supposing f(x)≠x, we must have $kq = 1$, which is a contradiction. We observe that if x and y are paired by f, then $\pi(x)$ and $\pi(y)$ are also paired by g. Part (b) follows from the above.□

Proof of lemma 2 : Let $p_0 \in \tilde{S}$ be a point which does not belong to the set X, where X denotes the set of points on $\tilde{S}$ corresponding to the projections, via π, of the set of fixed points of f on S. On $\tilde{S}$ there exists a family of (oriented) simple loops through p_0, which have no other points in common, disjoint from the set X and whose homotopy classes define a canonical basis for homology on the surface $\tilde{S}$. The complement of these curves is simply connected. Next, we show that we can modify this system of loops, without destroying any of the properties above, such that the following hold:

Claim : *Each of the above loops lifts to a loop on S.*

Proof of claim : To do it, we use induction on the decomposition of the integer o(f) in prime factors. Let us write the order of f as o(f)$=\prod_{i=1}^{N} p_i$, where p_i is prime, and N$\geq$ 1.

(1) case N=1.

In this case o(f)=p, where p is prime. Let us consider α_r, β_r, $r = 1, ..., \gamma$, simple loops as above. We have the following possibilities, for $r \in \{1, ..., \gamma\}$.

(a) α_r and β_r both lift to loops.

(b) α_r lifts to a loop and β_r lifts to a path.

(c) α_r lifts to a path and β_r lifts to a loop.

(d) α_r and β_r both lift to paths.

In case (a), we are done. In case (d), we can replace our loops to satisfy the case (b) as follow. If $\tilde{p_0}$ $\in$S is a lifting of p_0, then α_r and β_r lift to paths starting at $\tilde{p_0}$ and ending at $f^{k_1}(\tilde{p_0})$ and at $f^{k_2}(\tilde{p_0})$ respectively. Since o(f) is prime, we can find t such that β_r^t lifts to a path starting at $\tilde{p_0}$ and ending at $f^{k_1}(\tilde{p_0})$. Let us replace our loops α_r and β_r by simple loops α_r^* and β_r^* respectively, where α_r^* is homotopic to $\alpha_r\beta_r^{-t}$ and β_r^* is homotopic to β_r. In case (c), we replace our loops α_r and β_r by simple loops α_r^* and β_r^* respectively, where α_r^* is homotopic to β_r^{-1} and β_r^* is homotopic to α_r.

As observed above , we only need to solve the case (b). Let us consider α and β as in (b), that is, α and β are (oriented) simple loops through p_0, α lifts to a loop and β lifts to a path. Fix any s, $1 \leq s \leq o(f)$, and let $\tilde{p_0}$ be as before. Then, β lifts to a path starting at $\tilde{p_0}$ and ending at $f^k(\tilde{p_0})$, some $1 \leq k \leq o(f)$. Since o(f) is prime, there exists t such that β^t lifts from $\tilde{p_0}$ and ends at $f^s(\tilde{p_0})$. We replace our loops α and β by simple loops homotopic to $\alpha\beta^t$ and β respectively. We can assume, as above, α lifts to a path starting at $\tilde{p_0}$ ending at $f^s(\tilde{p_0})$ and β lifts to a path starting at $\tilde{p_0}$ ending at $f^k(\tilde{p_0})$. We replace α and β again by simple loops homotopic to α and $\beta\alpha^{-R}$, where R is such that α^R lifts from $\tilde{p_0}$ and ends at $f^k(\tilde{p_0})$ (o(f) is prime). Now, we can suppose α lifts to a path starting at $\tilde{p_0}$ and ending at $f^s(\tilde{p_0})$ and β lifts to a loop. Change α and β to β^{-1} and α respectively. At this point we can assume α lifts to a loop and β lifts to a path from $\tilde{p_0}$ and ending at $f^s(\tilde{p_0})$.

Let us suppose, we have α_i, β_i, α_j, β_j, $i \neq j$, such that α_i and α_j both lift to loops and β_i and β_j both lift to paths starting at the same point and ending at the same point. Then we change them by simple loops homotopic to $\alpha_i\alpha_j$, β_i, α_j and $\beta_i\beta_j^{-1}$ respectively, see figure 4.13.

Now, we can assume all loops α_r, β_s, $r = 1, ..., \gamma$; $s = 2, ..., \gamma$, lift to loops on S.

Let p$\in \tilde{S}$ be the projection, by π, of some of the fixed points of f. Let η be a (oriented) simple loop on $\tilde{S}$ through p_0, disjoint from all other loops with the exception of p_0, freely homotopic in $\tilde{S}-$X to a small simple loop around p. We can choose the integer s, $1 \leq s \leq o(f)$, in the above argument in such a way that $\beta_1\eta$ (*or* $\beta_1\eta^{-1}$) is homotopic to a simple loop on $\tilde{S}$, say δ, which lifts

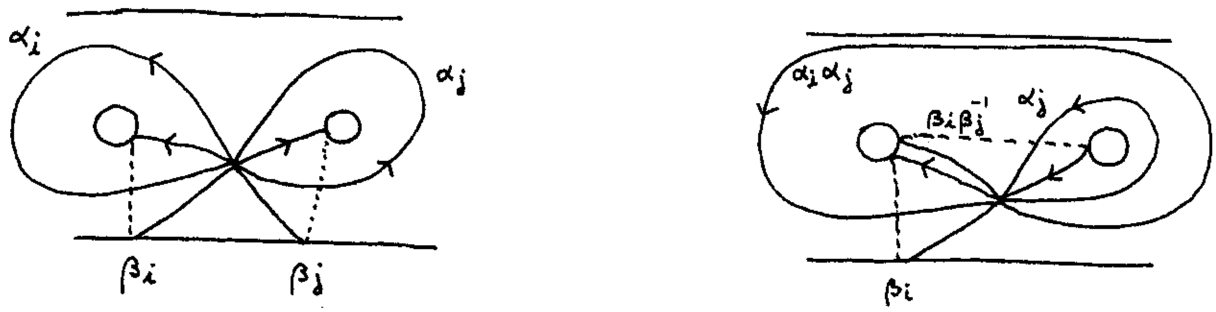

Figure 4.13:

to a loop on S. Replace β_1 by any simple loop homotopic to δ. Now, all loops α and β will lift to loops on S as required, and we are done in the case N=1.

(2) Assume we can do it for N $-$ 1 and let us prove we can do it for N. Let f be a conformal automorphism of S, satisfying (A1), (A2), (A3) and such that the fixed points of any non-trivial power of f are also fixed points of f. Let o(f) be of the form: o(f)=$\prod_{n=1}^{N} p_n$. Let q=$\prod_{n=1}^{N-1} p_n$ and p=p_N, then o(f)=pq, where p is prime and q$\geq$2. Consider the cyclic subgroup generated by f^q, $< f^q >$, and let S_1 be the quotient Riemann surface obtained by the action of $< f^q >$ on S. Let $\pi_1 : S \to S_1$ be the natural projection induced by f^q. By lemma 3, f induces an automorphism g, of order o(g)=q, satisfiying the conditions of lemma 2. Let S_2 be the quotient Riemann surface obtained by the action of g on S_1. Let $\pi_2 : S_1 \to S_2$ be the natural projection given by such action. Then we have that S_2 =S/$< f >$=$\tilde{S}$ and $\pi = \pi_2 \circ \pi_1$. See figure 4.14.

Let g_i be the genus of S_i, $i = 1, 2$. By induction hypotheses, we can find a set of simple closed curves α_r, β_r, $r = 1, ..., \gamma$, through $p_0 \in S_2$−X, which

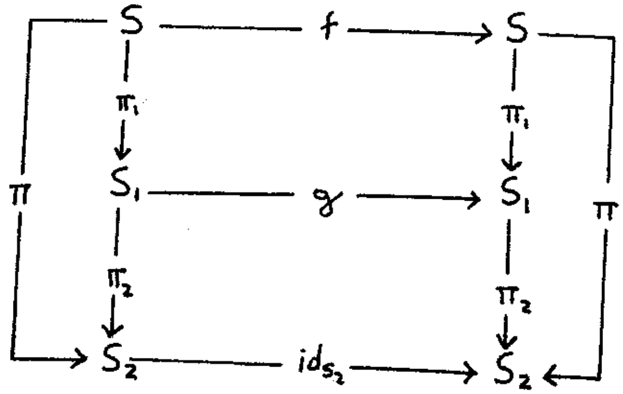

Figure 4.14:

have no other points in common, whose homology classes form a canonical basis for the homology of S_2 and such that each of them lift to a loop on S_1. Let $\alpha_{r,k}$ be the liftings of α_r and let $\beta_{l,m}$ be the liftings of β_l, on S_1, such that $\alpha_{r,k}$ and $\beta_{r,k}$ have non zero intersection number. Let us orient them such that this intersection number is +1. We have the following possibilities for fixed r:

(a) $\alpha_{r,k}$ and $\beta_{r,k}$ both lift to loops on S.

(b) $\alpha_{r,k}$ lift to a loop on S and $\beta_{r,k}$ lift to a path on S.

(c) $\alpha_{r,k}$ and $\beta_{r,k}$ both lift to paths on S.

In case (a), we are done. In case (b), we procede to change β_r without changing any of the previous properties except that $\beta_{r,k}$ will lift to a loop on S. Consider a small loop σ around $p_1 \in$X and a simple path δ connecting p_0 to σ, disjoint from all other curves with the exception of p_0. Take the loop $\beta_r(\delta\sigma\delta^{-1}\alpha_r^{-1})^s$, see figure 4.15 for the case s = 2. This loop will lift to a loop on S for some s=s_0. We replace β_r by a simple loop homotopic to $\beta_r(\delta\sigma\delta^{-1}\alpha_r^{-1})^{s_0}$. In case (c), we consider the loops α_r and $\alpha_r^t\beta_r$. Then, for some t=t_0, $\alpha_r^{t_0}\beta_r$ will lift to

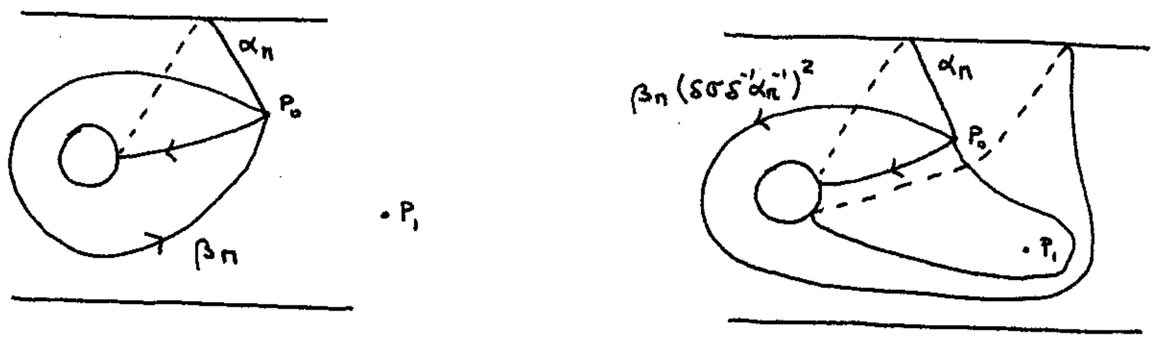

Figure 4.15: The loop $\beta_r(\delta\sigma\delta^{-1}\alpha_r^{-1})^2$

a loop on S, and we are in case (b).

This finishes the argument, and we have a set of α and β loops as required in our claim.

Now we finish the proof of lemma 2. Let Γ be a set of loops as in the claim above. We label the branch points as $p_r,\ q_r,\ r = 1, ..., n(f)$ in such way that p_r and q_r are the projections of paired fixed points. Choose disjoint simple arcs, $\delta_r : [-1, 1] \to (\tilde{S} - \Gamma) \bigcup \{p_0\}$, with the property that $\delta_r(-1) = p_0$, $\delta_r(0) = p_r$ *and* $\delta_r(1) = q_r$, $r = 1, ..., n(f)$. Since the complement of $\bigcup_{r=1}^{n(f)} \delta_r \bigcup \Gamma$ is simply connected, we can find on it o(f) continuous branches of π^{-1}. These branches have a continuous extension to Γ since the liftings of all the loops in Γ are simple loops. We also can extend them continuously to $\delta_r \mid [-1, 0]$ (the restriction of δ_r to [-1,0]), but we cannot extend them, continuosly, to $\delta_r \mid [0, 1]$. If π_1 is a branch of π^{-1}, then $\pi_1(\tilde{S} - \bigcup \delta_r \mid [0, 1])$ is a fundamental domain, for the action of the cyclic group generated by f, bounded by two images of $\pi_1(\bigcup \delta_r \mid [0,1])$. Since $\mid \alpha(f, p_r) \mid = 2\pi k_r / o(f)$, some $k_r = 1, ..., o(f) - 1$, we can find $t_r \in \{1, ..., o(f) - 1\}$, such that $\mid \alpha(f^{t_r}, p_r) \mid = 2\pi / o(f)$. It follows that f^{t_r} maps one of these two images of $\pi_1(\delta_r \mid [0,1])$ onto the other, for all $r = 1, ..., n(f)$. This ends the proof of lemma 2.□

We end this work with the following conjecture.

Conjecture. *Let S be a closed Riemann surface of genus $g \geq 0$, and $f : S \to S$ be a conformal automorphism of finite order, say $o(f)$. Then the condition (A) is necessary and sufficient to find a Schottky group G, uniformizing S, such that f can be lifted to a conformal automorphism of the region of discontinuity of G, $\Omega(G)$.*

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