Solutions and Hamiltonian Structure for Quasi-Geostrophic Flow

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Abstract of the Dissertation

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We first present that the conservation of the absolute vorticity for perfect fluids in rotating frames can be obtained by using the Lie-Poisson equation. Then we show the existence and uniqueness of the solution for the quasi-geostrophic flow which is given by the equation $\partial_t \Delta_S \, p + J[p, \Delta_S \, p + \beta y] = 0$. Finally, we put quasi-geostrophic flow into a Hamiltonian formulation to get conserved quantities.

TO MY PARENTS

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Chapter 1

Introduction

The purpose of this dissertation is to study the quasi-geostrophic flow on a rotating sphere which models the oceanic motion. The main results are to show that there exists a unique solution to the initial value problem defining this flow and to give a Hamiltonian formulation for it.

Chapter 2 is devoted to the geometric description of the fluids. In section 2.1, we give the Lagrangian description. Arnold [1] discovered that the motion of a perfect fluid on a region $M \subset \mathbf{R}^n$ can be understood as a geodesic in an infinite dimensional manifold which is $\mathcal{D}_\mu$ the group of volume-preserving diffeomorphisms of M. For M compact, Ebin and Marsden [3] studied the functional analytic detail of the manifold and group structure. They also showed the existence and uniqueness of solutions. Section 2.2 gives the equations of motion in both inertial and rotating frames.

In section 2.3, we give the Hamiltonian description which is based on the dual of the Lie algebra $\mathcal{G}^*$ of this group $\mathcal{D}_\mu$. For M simply connected, $\mathcal{G}^*$ is identified with the space of vorticities. The Poisson structure on $\mathcal{G}^*$ together with the Hamiltonian function yields the Lie-Poisson equation which is equivalent to the traditional vorticity equation. The conservation of vorticity follows from the fact that the solution of the Lie-Poisson equation is transported along the fluid motion.

When the frames rotate with possibly variable angular velocity, we show the absolute vorticity is conserved (Theorem 1). The Coriolis force changes the Poisson structure as can be seen through the momentum shift.

Chapter 3 is for the quasi-geostrophic flow. Section 3.1 gives the fundamental assumptions and to give the equation controlling the quasi-geostrophic flow. Geostrophic means that the motions are significantly influenced by the rotation of the earth. We also include the gravity as an external force. For a fluid particle moving with speed U to travel the distance L, for rotation to be important, we assume that $L/U \gg \Omega^{-1}$ where $\Omega = 7.3 \times 10^{-5}s^{-1}$ is the speed of the angular velocity of the rotation. For instance, the Gulf Stream has $L = O(100km)$ and $U = O(100cm/s)$. Under the condition $O(\frac{L}{r}) = O(\frac{U}{2L\Omega}) \ll 1$ where r is the radius of the earth, we derive the equation for the quasi-geostrophic flow (3.18) which is now an equation for the pressure function.

Section 3.2 is to show that the equation in section 3.1 has a unique solution in H^s for all $s \geq 5$ with the smooth initial condition p_o satisfying $\frac{\partial p_o}{\partial \nu}$=constant on the boundary (Theorem 2). The key ingredient in the proof is the contraction mapping theorem.

In section 3.3, we use the Poisson bracket in rotating frames to give a Hamiltonian formulation for the quasi-geostrophic flow (Theorem 3). The Hamiltonian function is given by the energy function for this flow. Analogous to the conservation of vorticity described in Theorem 1, the conserved quantity for the quasi-geostrophic flow can be found. Because of the two dimensional character of this flow, it actually has infinitely many conserved quantities.

Chapter 2

The Manifold Structure and Fluid Equations

2.1 Diffeomorphism Groups

Let M be a region in $\mathbf{R}^n$ filled with fluid. The configuration space of a perfect (incompressible, inviscid, homogeneous) fluid on M is the group of volume preserving diffeomorphisms of M. This was first discussed by Arnold [1]. The motion of the fluid can be described by a curve $\eta(t)$ in the group, with $\eta(0) = identity$. Then $\eta(t)(x)$ is the position of a fluid particle at time t, which is at x when $t = 0$. In traditional fluid mechanics, x is called the Lagrange coordinate (or body coordinate) and $\eta(t)(x)$ is called the Euler coordinate (or space coordinate) of the particle.

The manifold structure of the group of diffeomorphisms of M is shown by Ebin and Marsden [3]. Here we present some of the results. In the following, M and N are compact Riemannian manifolds without boundary. All the results can be generalized to $\partial M \neq \emptyset$.

Let $E \to M$ be a vector bundle over M. For s an positive integer, we get a Banach space $H_p^s(E)$ of sections of E whose derivatives up to order s are L^p in local charts. We have

Sobolev Embedding Theorem *For* $k \geq 0, dimM = n, s > \frac{n}{p} + k$, *where* s, p *as above, then the inclusion* $H_p^s(E) \subset C^k(E)$ *is continuous.*

Define $H_p^s(M, N)$ to be all maps from M to N such that, using local coordinates, the derivatives up to order s are L^p. Then

Proposition 1 $H_p^s(M, N)$ *is a Banach manifold.*

Proof: Given any Riemannian metric on N and its associated exponential map $exp : TN \to N$, there is a natural way to construct local charts for $H_p^s(M, N)$. For $f \in H_p^s(M, N)$, $\overline{T}_f H_p^s(M, N) = \{g \in H_p^s(M, TN) | \pi \circ g = f\}$ is a Banach space, where $\pi : TN \to N$ is the canonical projection. Define

$$\begin{aligned} \Phi : \overline{T}_f H_p^s(M, N) &\to H_p^s(M, N) \\ g &\mapsto exp \circ g. \end{aligned}$$

Let U be the neighborhood of zero in $\overline{T}_f H_p^s(M,N)$, we need to show Φ is indeed a local chart on U.

First of all, Φ is one-to-one by choosing U such that for all $g \in U, |g(x)| <$ the injectivity radius of exp at $f(x)$. Next, we show $\{\Phi, U\}$ is compatible. Suppose $\Phi : U \to H_p^s(M,N)$ is a neighborhood of f and $\Psi : V \to H_p^s(M,N)$ is a neighborhood of g, we need to show $\Psi^{-1} \circ \Phi$ is a diffeomorphism on $\Psi^{-1}(\Phi(U) \cap \Psi(V))$.

Define $Exp : TN \to N \times N$ by $(p,v) \mapsto (p, exp_p(v))$, where $exp_p : T_pN \to N$. Then we have $D_0 exp_p : T_0T_pN \to T_pN$ and $D_{(p,0)}Exp : T_{(p,0)}TN \to T_pN \times T_pN$ are identity maps. Hence, Exp is a local diffeomorphism near the zero section of TN.

Let $h = \Phi(a) = \Psi(b)$, where $a \in U, b \in V$. Since $h(x) = exp(a(x)) = exp(b(x))$, we have $Exp(a(x)) = (f(x), h(x))$ and $Exp(b(x)) = (g(x), h(x))$. Therefore,

$$\begin{aligned} b(x) &= \Psi^{-1}(\Phi(a(x))) \\ &= Exp^{-1}(g(x), h(x)) \\ &= Exp^{-1}(g(x), \Phi(a(x))). \end{aligned}$$

Hence $D(\Psi^{-1} \circ \Phi)(a) \cdot w = DExp^{-1}(g, \Phi(a)) \cdot D\Phi(a) \cdot w$. This proves the compatibility. $\square$

Definition 1 $\mathcal{D}_p^s = \{\eta \in H_p^s(M,M) | \eta^{-1} \in H_p^s(M,M)\}$.

For $s > \frac{n}{p} + 1$, $\mathcal{D}_p^s$ is an open submanifold of $H_p^s(M,M)$ and it is a topological group. The right multiplication

$$\begin{aligned} R_\eta : \mathcal{D}_p^s &\rightarrow \mathcal{D}_p^s \\ \xi &\mapsto \xi \circ \eta \end{aligned}$$

is C^∞ for $\eta \in \mathcal{D}_p^s$. The left multiplication

$$\begin{aligned} L_\eta : \mathcal{D}_p^s &\rightarrow \mathcal{D}_p^s \\ \xi &\mapsto \eta \circ \xi \end{aligned}$$

is C^l if $\eta \in \mathcal{D}_p^{s+l}$. The inverse map $\mathcal{D}_p^s \rightarrow \mathcal{D}_p^s$, $\eta \mapsto \eta^{-1}$ is continuous.

The Lie algebra $T_{id}\mathcal{D}_p^s = H_p^s(M, TM)$ is the space of H_p^s vector fields on M. Let $\mathcal{D}_p = \bigcap_{s > \frac{n}{p}} \mathcal{D}_p^s$ using the topology as the limit of topology of $\mathcal{D}_p^s$. $\mathcal{D}_p$ is called the ILB (inverse limit Banach) Lie group. See Omori [8]. When $p = 2, \mathcal{D} = \mathcal{D}_2$ is called ILH (inverse limit Hilbert) Lie group. For rest of the discussion, we assume $p = 2$.

Hodge Decomposition *Let $s \geq 0$, $\bigwedge^k$ denote the bundle of k-forms on M. Then*

$$H^s(\wedge^k) = d(H^{s+1}(\wedge^{k-1})) \oplus \delta(H^{s+1}(\wedge^{k+1})) \oplus Ker\Delta$$

where d is the exterior derivative with δ its adjoint, Δ is the Laplacian operator and $\oplus$ means an orthogonal direct sum with respect to the H^0 metric, i.e.

$$(\alpha, \beta)_0 = \int_M \alpha \wedge *\beta.$$

Also, $Ker\Delta$ is finite dimensional and may be identified with the k^{th} cohomology group of M.

Definition 2 $\mathcal{D}^s_\mu = \{\eta \in \mathcal{D}^s | \eta^*(\mu) = \mu\}$, *where* μ *is the volume form on* M.

Proposition 2 *For* $s > \frac{n}{2} + 1, \mathcal{D}^s_\mu$ *is a closed submanifold of* $\mathcal{D}^s$.

Proof: Define $\Phi : \mathcal{D}^{s+1} \to [\mu]^s$ by $\eta \mapsto \eta^*(\mu)$, where $[\mu]^s = \{\mu + df | \forall f \in H^{s+1}(\bigwedge^{n-1})\}$. Hodge decomposition implies $[\mu]^s$ is a closed affine subspace of $H^s(\bigwedge^n)$. We will show Φ is a submersion. For $s > \frac{n}{2}$, we have

$$\begin{aligned} T_\eta \Phi : T_\eta \mathcal{D}^{s+1} &\to H^s(\textstyle\bigwedge^n) \\ v &\mapsto \eta^*(\mathcal{L}_{v \circ \eta^{-1}} \mu), \end{aligned}$$

where $\mathcal{L}$ is the Lie derivative. So $T_{id}\Phi(v) = \mathcal{L}_v \mu = d\iota_v \mu$. The map $v \mapsto \iota_v \mu$ is an isomorphism since μ is nondegenerate. This shows $T_{id}\Phi(v)$ is surjective since the tangent space to $[\mu]^s$ at any point is $d(H^{s+1}(\bigwedge^{n-1}))$. Also, $T_\eta \Phi$ is surjective since R_η and η^* are isomorphisms. Therefore $\mathcal{D}^s_\mu = \Phi^{-1}(\mu)$ is a submanifold. □

$\mathcal{D}^s_\mu$ is a closed ILH subgroup of $\mathcal{D}^s$ and its Lie algebra consists of divergence free vector fields X since $\mathcal{L}_X\mu = (divX)\mu = 0$. Hodge decomposition also determines that $T_{id}\mathcal{D}^s = T_{id}\mathcal{D}^s_\mu \oplus \mathbf{grad}\ H^{s+1}(\Lambda^0)$.

2.2 Equation of Motion

We will derive the Euler equation for the perfect flow on a compact manifold.

Definition 3 *A weak Riemannian structure on a Banach manifold $\mathcal{E}$ is a smooth section of the bundle of continuous, positive definite, symmetric bilinear forms on $\mathcal{E}$.*

Note: Weak structure means that it may not define the topology on the tangent spaces $T_\zeta\mathcal{E}$ to $\mathcal{E}$ at $\zeta \in \mathcal{E}$.

Definition 4 *For $s > \frac{n}{2}+1$, define a bilinear form on $T_\eta\mathcal{D}^s$ by*

$$\ll V, W \gg = \int_M < V(x), W(x) >_{\eta(x)} \mu(x),$$

where $<,>_x$ is the Riemannian metric on M, and μ is the volume form induced by the metric on M.

Proposition 3 *The action of $\mathcal{D}^s_\mu$ on $\mathcal{D}^s$ by right multiplication preserves the Riemannian structure.*

Proof: Given $X, Y \in T_\zeta\mathcal{D}^s, \eta \in \mathcal{D}^s_\mu$,

$$\ll X \circ \eta, Y \circ \eta \gg \ = \ \int_M < X \circ \eta(x), Y \circ \eta(x) >_{\zeta\circ\eta(x)} \mu(x)$$

$$= \int_M < X(x), Y(x) >_{\zeta(x)} \mu(x)$$

$$= \ll X, Y \gg .$$

The second equality is by change of variables and $\eta^*(\mu) = \mu$. □

Restricting $\ll, \gg$ to $\mathcal{D}^s_\mu$ gives a weak Riemannian metric on $\mathcal{D}^s_\mu$.

Now we will use this metric to derive the equation of a perfect flow on M. We have the variational principle,

$$\partial_s|_{s=0} \int_0^T \mathbf{L} dt = 0,$$

where $\mathbf{L}$ is the Lagrangian of the dynamical system. In our case, the Lagrangian is the kinetic energy defined by the Riemannian metric (assume the density is 1). So

$$\mathbf{L} : T\mathcal{D}^s_\mu \rightarrow R$$

$$\mathbf{L}(\eta(t), \dot{\eta}(t)) = \frac{1}{2} \int_M < \dot{\eta}(t)(x), \dot{\eta}(t)(x) >_{\eta(t)(x)} dx.$$

The variation of η is $\eta_s(t) = \eta(t) + sw(t)$, where $div(w(t) \circ \eta(t)^{-1}) = 0$ and $w(T) = w(0) = 0$. The variational principle implies

$$\begin{aligned}
& \partial_s|_{s=0} \int_0^T \frac{1}{2} \int_M < \dot{\eta}_s(t)(x), \dot{\eta}_s(t)(x) >_{\eta_s(x)} dxdt \\
= \; & \partial_s|_{s=0} \int_0^T \frac{1}{2} \int_M < \dot{\eta}(t)(x) + s\dot{w}(t)(x), \dot{\eta}(t)(x) + s\dot{w}(t)(x) >_{\eta_s(x)} dxdt \\
= \; & \int_0^T \int_M < \dot{\eta}(t)(x), \dot{w}(t)(x) >_{\eta(t)(x)} + \frac{1}{2} D_{w(t)(x)} < \dot{\eta}(t)(x), \dot{\eta}(t)(x) >_{\eta(t)(x)} dxdt
\end{aligned}$$

$$= -\int_M \int_0^T < \ddot{\eta}(t)(x) + \overline{\Gamma}\dot{\eta}(t)(x)\dot{\eta}(t)(x), w(t)(x) >_{\eta(t)(x)} dtdx$$

where $\overline{\Gamma}$ is the Christoffel symbol of the metric on M.

$$= -\int_M \int_0^T < (\ddot{\eta}(t) + \overline{\Gamma}\dot{\eta}(t)\dot{\eta}(t)) \circ \eta(t)^{-1}(x), w(t) \circ \eta(t)^{-1}(x) >_x dtdx.$$

Therefore, $(\ddot{\eta}(t) + \Gamma\dot{\eta}(t)\dot{\eta}(t)) \circ \eta(t)^{-1} = -grad\ p$, where p is a function on M by the Hodge decomposition and Γ is the Christoffel symbol of the connection ∇ on $\mathcal{D}_\mu$ induced by the metric on M.

Lemma 1 $(\ddot{\eta} + \Gamma\dot{\eta}\dot{\eta}) \circ \eta^{-1} = \partial_t V + (V \cdot \nabla)V$ *where* $\dot{\eta}(t)(x) = V(\eta(t)(x))$.

Proof: We have $\dot{\eta}(t) = V(t) \circ \eta(t)$. Thus

$$\begin{aligned} (\ddot{\eta} + \Gamma\dot{\eta}\dot{\eta}) \circ \eta^{-1} &= \ddot{\eta} \circ \eta^{-1} + \Gamma V V \\ &= \partial_t V + DV \cdot V + \Gamma V V \\ &= \partial_t V + (V \cdot \nabla)V. \end{aligned}$$

□

We now have the Euler equations for a perfect fluid,

$$\begin{cases} \partial_t V + (V \cdot \nabla)V &= -grad\ p \\ div\ V &= 0. \end{cases}$$

where p represents the pressure function on M. Arnold [1] explained that every geodesic on $\mathcal{D}^s_\mu$ yields a solution of the Euler equations. It is shown by

Ebin and Marsden [3] that the Euler equation may be transformed to a spray on the group $\mathcal{D}^s_\mu$. Thus the existence and uniqueness for short time solutions to the Euler equations follow from showing the spray is smooth.

In the following, we will ignore the analytical treatment of the infinite dimensional manifolds involved. We derive the Euler equation on M in rotating frames.

Let $\mathbf{SO}(3)$ act on $M \subseteq \mathbf{R}^3$ such that $\forall g \in \mathbf{SO}(3), g(M) \subseteq M$ and m_t be a curve in $\mathbf{SO}(3)$ generating a vector field Z_t.

$$Z_t(m_t(x)) = \frac{d}{dt} m_t(x).$$

Let $\mathbf{SO}(3)$ act on $\mathcal{D}_\mu$ by left composition,

$$\mathbf{SO}(3) \times \mathcal{D}_\mu \longrightarrow \mathcal{D}_\mu, \ (m, \eta_R(t)(x)) \mapsto m(\eta_R(t)(x)).$$

On rotating frames, the position of the fluid particle x at time t is $\eta_R(t)(x)$ which has the true position (w.r.t. inertial frames) $m(\eta_R(t)(x))$. Therefore, the velocity vector of the fluid particle in inertial frames is

$$[m_t(\eta_R(t)(x))]^{\cdot} = T_{\eta_R(t)(x)} m_t \cdot \dot{\eta}_R(t)(x) + Z_t(m_t(\eta_R(t)(x))).$$

The Lagrangian function w.r.t. rotating frames is

$$\begin{aligned} & \mathbf{L}_R(\eta_R(t), \dot{\eta}_R(t)) \\ = \ & \frac{1}{2} \int_M |(m_t(\eta_R(t)(x)))^{\cdot}|^2 dx \end{aligned}$$

$$= \frac{1}{2}\int_M |T_{\eta_R(t)(x)}m_t \cdot \dot\eta_R(t)(x) + Z_t(m_t(\eta_R(t)(x)))|^2 dx$$
$$= \frac{1}{2}\int_M |\dot\eta_R(t)(x) + T_{\eta_R(t)(x)}m_t^{-1} \cdot Z_t(m_t(\eta_R(t)(x)))|^2 dx$$
$$= \frac{1}{2}\int_M |\dot\eta_R(t)(x) + m_t^* Z_t(\eta_R(t)(x))|^2 dx$$

where $|x|^2 =< x, x >$.

For $M = \mathbf{R}^3$, we write $m_t^* Z_t(\eta_R(t)(x)) = \Omega(t) \times \eta_R(t)(x)$ where $\Omega(t)$ is the angular velocity of the rotation. Then

$$\mathbf{L}_R = \frac{1}{2}\int_M |\dot\eta_R(t)(x) + \Omega(t) \times \eta_R(t)(x)|^2 dx.$$

The variation of η_R is $\eta_R(t,s) = \eta_R(t) + sw(t)$, where $div(w(t) \circ \eta_R(t)^{-1}) = 0$ and $w(T) = w(0) = 0$. The variational principle gives

$$\partial_s|_{s=0} \int_0^T \frac{1}{2}\int_M |\dot\eta_R(t,s)(x) + \Omega(t) \times \eta_R(t)(x)|^2 dx dt$$
$$= -\int_M \int_0^T < \ddot\eta_R(t)(x) + \dot\Omega(t) \times \eta_R(t)(x) + 2\Omega(t) \times \dot\eta_R(t)(x) +$$
$$\Omega(t) \times (\Omega(t) \times \eta_R(t)(x)), w(t)(x) > dt dx$$

Since $div(w(t) \circ \eta_R(t)^{-1}) = 0$, we have

$$\ddot\eta_R \circ {\eta_R}^{-1}(x) + 2\Omega \times (\dot\eta_R \circ {\eta_R}^{-1})(x) + \dot\Omega \times x + \Omega \times (\Omega \times x) = -grad\ p(t,x),$$

for some function p on M which is called the pressure. Let $V_R = \dot\eta_R \circ \eta_R^{-1}$, then

$$\partial_t V_R(x) + (V_R \cdot \nabla) V_R(x) + 2\Omega \times V_R(x) + \dot\Omega \times x + \Omega \times (\Omega \times x) = -grad\ p(t,x).$$

Note: $2\Omega(t) \times V_R$ is called the Coriolis force. It is orthogonal to $\Omega(t)$ and V_R. $\dot{\Omega}(t) \times x$ is called the Euler force; it is due to the non-uniformity of the rotation and $\Omega(t) \times (\Omega(t) \times x)$ is called the centrifugal force; it lies in the plane spanned by $\Omega(t)$ and x. see Batchelor [2].

We notice that $\Omega(t) \times (\Omega(t) \times x) = \frac{1}{2} grad\ |\Omega(t) \times x|^2$. $\frac{1}{2}|\Omega(t) \times x|^2$ is called the centrifugal potential. We now can write the Euler equations for a perfect fluid on $\mathbf{R}^3$ in rotating frames as follows,

$$\begin{cases} \partial_t V_R(x) + (V_R \cdot \nabla) V_R(x) + 2\Omega \times V_R(x) + \dot{\Omega} \times x &= -grad\ (p(t,x) + \frac{1}{2}|\Omega \times x|^2) \\ div\ V_R(x) &= 0. \end{cases}$$

2.3 The Poisson Structure

2.3.1 The Lie-Poisson Equation

The previous section gives the Lagrangian formulation of the fluid motion. Now we will give the Hamiltonian formulation following Marsden and Weinstein [7].

Definition 5 *A Poisson manifold P is a manifold together with a Lie algebra structure $\{\ ,\ \}$ on the space $C^\infty(P)$ of smooth real-valued functions on P such that $\{f, g\}$ is a derivation in each argument.*

Let G be a Lie group with its Lie algebra $\mathcal{G}$. The dual space $\mathcal{G}^*$ carries a Poisson structure as follows: For $u \in \mathcal{G}^*$, and $F, G \in C^\infty(\mathcal{G}^*)$, we define

$$\{F, G\}(u) = u \cdot [\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u}],$$

where $\cdot$ is the pairing between $\mathcal{G}$ and $\mathcal{G}^*$ while $[\ ,\ ]$ is the standard (right) Lie bracket on $\mathcal{G}$ and $\frac{\delta F}{\delta u} \in \mathcal{G}$ is defined for $v \in \mathcal{G}^*$,

$$\frac{d}{dt} F(u + tv)|_{t=0} = v \cdot \frac{\delta F}{\delta u}.$$

Since this formula for the bracket on $\mathcal{G}^*$ is due to Lie [5], we will call this the Lie-Poisson bracket. The bracket $\{\ ,\ \}$ is the one induced on $\mathcal{G}^*$ by identifying $C^\infty(\mathcal{G}^*)$ with the right invariant functions on T^*G.

For P a Poisson manifold, the Hamiltonian system on P corresponding to a function $\mathbf{H} : P \to R$ is that given by the real-valued function on P evolving by the rule $\dot{F} = \{F, \mathbf{H}\}$.

For the case of a perfect fluid on M, the configuration space is $\mathcal{D}_\mu$ and the phase space is $T^*\mathcal{D}_\mu$. The Lie algebra $\mathcal{X}_\mu = T_{id}\mathcal{D}_\mu = \{v : M \to TM | div(v) = 0\}$. For $v \in \mathcal{X}_\mu$, we associate a 1-form $v^\flat$ by the L^2 pairing. i.e.

$$v^\flat(\cdot) = \int_M (v, \cdot)\mu.$$

The dual of Lie algebra $\mathcal{X}_\mu^*$ can be identified as the space of 1-forms modulo exact 1-forms since $\int_M df \cdot v = 0$ if $div(v) = 0$. i.e. $\mathcal{X}_\mu^* = \{[\rho] |\ [\rho] = \rho + df, f \in C^\infty(M), \rho \in \Lambda^1(M)\}$

The Hamiltonian function $\mathbf{H} : \mathcal{X}_\mu^* \to R$ is given by

$$\mathbf{H}([\rho]) = \frac{1}{2}\|[\rho]\|^2,$$

where the metric $\| \cdot \| = \ll, \gg_{\mathcal{G}^*}$ is induced by the metric on $\mathcal{G}$.

In traditional fluid mechanics, for v the velocity vector and $\hat{\omega} = \nabla \times v$ the vorticity, the Euler equation implies the vorticity equation:

$$\partial_t \hat{\omega} + v \cdot \nabla \hat{\omega} - \hat{\omega} \cdot \nabla v = 0.$$

Proposition 4 *The Lie-Poisson equation on $\mathcal{X}_\mu^*$ is equivalent to the vorticity equation.*

Proof: For $\dfrac{\delta \mathbf{H}}{\delta[\rho]} = v$,

$$\begin{aligned} \{F, \mathbf{H}\}([\rho]) &= [\rho] \cdot [\frac{\delta F}{\delta[\rho]}, \frac{\delta \mathbf{H}}{\delta[\rho]}] \\ &= \int_M ([\rho], [\frac{\delta F}{\delta[\rho]}, \frac{\delta \mathbf{H}}{\delta[\rho]}])dx \\ &= \int_M ([\rho], [\frac{\delta F}{\delta[\rho]}, v])dx \\ &= \int_M ([\rho], \mathcal{L}_v \frac{\delta F}{\delta[\rho]})dx \\ &= -\int_M (\mathcal{L}_v[\rho], \frac{\delta F}{\delta[\rho]})dx \end{aligned}$$

where $(,)$ denotes vector-covector pairing.

Therefore, $\{F, \mathbf{H}\}([\rho]) = -DF[\rho] \cdot \mathcal{L}_v[\rho]$. This implies

$$\partial_t[\rho] = -\mathcal{L}_v[\rho].$$

Let 2-form $\omega = d[\rho]$ and take the exterior derivative of the above equation, we have

$$(\partial_t + \mathcal{L}_v)\omega = 0.$$

We notice $\hat{\omega}^\flat = *\omega$ and $\iota_v\omega = (\hat{\omega} \times v)^\flat$, so

$$\begin{aligned} \mathcal{L}_v\omega &= d\iota_v\omega + \iota_v d\omega \\ &= d\iota_v\omega \\ &= curl(\hat{\omega} \times v) \\ &= v \cdot \nabla\hat{\omega} - \hat{\omega} \cdot \nabla v. \end{aligned}$$

□

Note: For M simply connected, we can identify $[\rho]$ with $d[\rho]$ and $\mathcal{X}_\mu^*$ with the space of vorticities.

Corollary 1 *Conservation of vorticity.*

Proof: The solution to the Lie-Poisson equation $\partial_t\omega + \mathcal{L}_v\omega = 0$ with the initial condition $\omega(0)$ is $\omega(t) = (\eta^{-1})^*(t)\omega(0)$, where $\eta(t)$ is the flow for $v(t)$. So the vorticity is transported by the flow which is equivalent to the Kelvin's circulation theorem. It states that the velocity circulation around a closed "fluid" contour is constant in time. i.e.

$$constant = \oint v \cdot dc = \int curlv \cdot dA.$$

This implies *curl* v is constant along the fluid. □

Note: The corollary has another interesting interpretation. The adjoint action of $\mathcal{D}_\mu$ on $\mathcal{X}_\mu$ is

$$\begin{aligned} \mathcal{D}_\mu \times \mathcal{X}_\mu &\to \mathcal{X}_\mu \\ (\eta, v) &\mapsto Ad_\eta v = D\eta \circ v \circ \eta^{-1}. \end{aligned}$$

The coadjoint action of $\mathcal{D}_\mu$ on $\mathcal{X}_\mu^*$ is

$$\begin{aligned} \mathcal{D}_\mu \times \mathcal{X}_\mu^* &\to \mathcal{X}_\mu^* \\ (\eta, [\rho]) &\mapsto Ad_\eta^*[\rho] = \eta^*[\rho]. \end{aligned}$$

We verify the above formula for the coadjoint action as follows. Suppose $Ad_\eta^*[\rho] = \beta$, then

$$\begin{aligned} \beta(w) &= [\rho] \cdot Ad_\eta w \\ &= \int_M < [\rho], D\eta \circ v \circ \eta^{-1} > dx \\ &= \int_M < (D\eta)^{-1} \circ [\rho] \circ \eta, w > dx \\ &= \int_M < \eta^*[\rho], w > dx. \end{aligned}$$

The coadjoint orbit of $[\rho]$ is $\{\eta^*[\rho] | \ \eta \in \mathcal{D}_\mu\}$. So the corollary is equivalent to the fact that the vorticity stays on the coadjoint orbit.

2.3.2 The Momentum Shift

Now we will show the conservation of absolute vorticity in rotating frames and the momentum shift.

Theorem 1 *Absolute vorticity is conserved.*

Proof: Given the Lagrangian in rotating frames, we perform the Legendre transformation

$$[\rho_R] = \frac{\partial \mathbf{L}_R}{\partial \dot{\eta}_R} = (\dot{\eta}_R + \alpha)^\flat .$$

where $\alpha = m_t^* Z_t \circ \eta_R$. Since $div(v+\alpha) = 0$, so $[\rho_R]$ is identified as the 1-form corresponding to the absolute velocity vector field w.r.t. rotating frames. The Hamiltonian function w.r.t. rotating frames is

$$\begin{aligned} \mathbf{H}_R(\eta_R, [\rho_R]) &= [\rho_R] \cdot \dot{\eta}_R - \mathbf{L}_R \\ &= \ll [\rho_R], [\rho_R] - \alpha^\flat \gg - \frac{1}{2} \|[\rho_R]\|^2 \\ &= \frac{1}{2} \|[\rho_R]\|^2 - \ll [\rho_R], \alpha^\flat \gg . \end{aligned}$$

It is easy to see that $\mathbf{H}_R$ is invariant under the right composition by η_R, so we can define $\mathbf{H}_R$ on $\mathcal{G}^*$ as

$$\mathbf{H}_R([\rho_R]) = \frac{1}{2} \|[\rho_R]\|^2 - \ll [\rho_R], \alpha^\flat \gg .$$

We can now write down the Lie-Poisson equation using $\mathbf{H}_R$ as before.

For $D\mathbf{H}_R = [\rho_R] - \alpha^\flat = v_R^\flat$, we have $\dfrac{\delta \mathbf{H}_R}{\delta[\rho_R]} = v_R$. Therefore, if we write $\omega_R = d[\rho_R]$, we have $(\partial_t + \mathcal{L}_{v_R})\omega_R = 0$. This shows the absolute vorticity ω_R is conserved. □

For $M = \mathbf{R}^3$, the corresponding vorticity equation is the following:

Let $\hat{\omega}_R = \nabla \times V_R$, and $\nabla \times (V_R + \Omega(t) \times x) = \hat{\omega}_R + 2\Omega(t)$ so that

$$\partial_t(\hat{\omega}_R + 2\Omega(t)) + V_R \cdot \nabla(\hat{\omega}_R + 2\Omega(t)) - (\hat{\omega}_R + 2\Omega(t)) \cdot \nabla V_R = 0$$

$$(V) \qquad \partial_t \hat{\omega}_R(t) + V_R \cdot \nabla \hat{\omega}_R - \hat{\omega}_R \cdot \nabla V_R - 2\Omega(t) \cdot \nabla V_R + 2\dot{\Omega}(t) = 0.$$

The conservation of absolute vorticity implies that the conserved quantity is $\hat{\omega}_R + 2\Omega(t)$.

We now examine how the Lagrangian and Hamiltonian functions to be changed when the frames are changed. We have

$$\begin{aligned} \mathbf{L}(\eta, \dot{\eta}) &= \frac{1}{2}\|\dot{\eta}\|^2, \\ \mathbf{L}_R(\eta_R, \dot{\eta_R}) &= \frac{1}{2}\|\dot{\eta_R} + \alpha\|^2. \end{aligned}$$

Therefore $\mathbf{L}_R = \mathbf{L}$. Perform the Legendre transformation,

$$\begin{aligned} [\rho] &= \dot{\eta}^\flat, \\ [\rho_R] &= (\dot{\eta_R} + \alpha)^\flat. \end{aligned}$$

The corresponding Hamiltonians are

$$\begin{aligned}\mathbf{H}(\eta,[\rho]) &= \frac{1}{2}\|[\rho]\|^2,\\ \mathbf{H}_R(\eta_R,[\rho_R]) &= \frac{1}{2}\|[\rho_R]\|^2 - \ll [\rho_R],\alpha^\flat \gg .\end{aligned}$$

Hence, $\mathbf{H}_R = \mathbf{H} - \ll [\rho_R],\alpha^\flat \gg$.

This explains that the Lagrangian is invariant under coordinate change due to the rotation and the Hamiltonian is not. This is why the Hamiltonian formulation for the rotating frames has to be derived from the Lagrangian together with the Legendre transformation. The Hamiltonian is not covariant because of the Poisson structure. We will see this explicitly by performing a momentum shift.

Definition 6 *The momentum shift of* $[\rho_R]$ *is defined as* $\tilde{\rho}_R = [\rho_R] - \alpha$.

We get a new Hamiltonian as a function of $\tilde{\rho}_R$,

$$\tilde{\mathbf{H}}_R(\tilde{\rho}_R) = \frac{1}{2}(\|\tilde{\rho}_R\|^2 - \|\alpha\|^2).$$

Note that the term $\frac{1}{2}\|\alpha\|^2$ is just the centrifugal potential.

Write the Poisson structure in terms of $\tilde{\rho}_R$

$$\begin{aligned}\{F,G\}^{\sim}(\tilde{\rho}_R) &= (\tilde{\rho}_R+\alpha)\cdot[\frac{\partial F}{\partial\tilde{\rho}_R},\frac{\partial G}{\partial\tilde{\rho}_R}] &\quad (2.1)\\ &= \{F,G\}(\tilde{\rho}_R)+\alpha\cdot[\frac{\partial F}{\partial\tilde{\rho}_R},\frac{\partial G}{\partial\tilde{\rho}_R}]. &\quad (2.2)\end{aligned}$$

Note: The extra term α corresponds to the Coriolis force.

When Ω is time-independent, Hamilton's equation is

$$\dot{F}(\tilde{\rho}_R) = \{F, \mathbf{H}\}^{\sim}(\tilde{\rho}_R).$$

This corresponds to the vorticity equation (V) by letting $\hat{\omega}_R = d\tilde{\rho}_R$. In general, the momentum shift is time-dependent and the Hamilton's equation is no longer as above. We will see how the equation should be altered.

Suppose $\Phi_t : P \to P$ is a time-dependent map and

$$\frac{d}{dt}\Phi_t(p) = Y_t(\Phi_t(p)).$$

Assume for each t, Φ_t is a Poisson map, i.e.

$$\{F \circ \Phi_t, G \circ \Phi_t\} = \{F, G\} \circ \Phi_t, \ \forall F, G \in C^\infty(P).$$

Then $\dfrac{d}{dt}F(\Phi_t(p)) = \{F, \mathbf{H}\}(\Phi_t(p)) + Y_t F(\Phi_t(p))$.

In our case, $\Phi_t([\rho_R]) = [\rho_R] - \alpha$ and $Y_t = \dot{\alpha}$ is the Euler force.

Chapter 3

Quasi-Geostrophic Flow on a Rotating Sphere

3.1 Equation of the Flow

We now consider a model of a thin layer of inhomogeneous fluid on a rotating sphere which describes the oceanic motion. The equations of motion consist of conservation of mass

$$\frac{d\varrho}{dt} = 0, \nabla \cdot \vec{v} = 0, \tag{3.1}$$

and conservation of momentum with the Coriolis force and the gravity acting on the fluid

$$\frac{d\vec{v}}{dt} = -\frac{\nabla p}{\varrho} - 2\Omega \times \vec{v} - \vec{g} \tag{3.2}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla)$, $\vec{v}$ is the velocity vector, ϱ is the density function, p is the pressure function and $\vec{g}$ is the gravitational acceleration pointing downward. The well posedness of the inhomogeneous flow can be proved using the previous discussion. See Marsden [6].

The spherical coordinates of the position of fluid particles are r, φ, ϑ which are the distance from the center of sphere, the latitude and the longitude. We consider the motion on a sphere of radius r_o. We further suppose that the motion occurs in a mid-latitude region, distant from the equator, around some central latitude φ_o. Define

$$\begin{aligned} x &= \vartheta r_o \sin \varphi_o \\ y &= (\varphi - \varphi_o) r_o \\ z &= r - r_o. \end{aligned}$$

We now give a standard type of scaling analysis to describe a particular kind of motion by making assumptions on the relative size of quantities involved.

Let L be the horizontal (tangent to the sphere) scale constant, D be the vertical (normal to the sphere) scale constant, and U be the horizontal velocity scale constant.

To scale the dynamical variables $\vec{v} = (u, v, w), \varrho, p$, (Note u, v, w are the velocity in the eastward, northward and vertical direction.) we introduce the new variables,

$$u = U\tilde{u}, \quad x = L\tilde{x}, \quad f_o = 2\Omega \sin\varphi_o,$$

$$v = U\tilde{v}, \quad y = L\tilde{y}, \quad \delta = D/L,$$

$$w = \delta U\tilde{w}, \quad z = D\tilde{z}, \quad t = (L/U)\tilde{t}.$$

Then $r = r_o(1 + \delta\tilde{z}L/r_o)$.

To scale ϱ, p, we observe that if $\vec{v}$ is very small, then the pressure will be only slight disturbed from the value it would have in the absence of motion. That is, if we let $\vec{v} = 0$ in (3.2), we get

$$\frac{\partial p_s(z)}{\partial z} = -\varrho_s(z)g.$$

We define p_s,ϱ_s as the global averaging of p and ϱ at each level of z. We can now write

$$\begin{aligned} p &= p_s(z) + \bar{p}(x, y, z, t) \\ \varrho &= \varrho_s(z) + \bar{\varrho}(x, y, z, t). \end{aligned}$$

$\bar{p}, \bar{\varrho}$ are the variations of p, ϱ away from p_s, ϱ_s. We are interested in the motion such that the horizontal pressure gradient has the same order as the Coriolis force. From (3.2), we have

$$\bar{p} = O(\varrho_s f_o UL).$$

Similarly the vertical pressure gradient has the same order as $\bar{\varrho}g$. This implies that

$$\bar{\varrho} = O(\frac{\varrho_s f_o U L}{gD}).$$

Therefore,

$$\begin{aligned} p &= p_s(z) + \varrho_s f_o U L \tilde{p} \\ \varrho &= \varrho_s(z)(1 + \epsilon F \tilde{\varrho}) \end{aligned}$$

where

$$\epsilon = \frac{U}{f_o L}, F = \frac{{f_o}^2 L^2}{gD}.$$

Geostrophic Approximation: $\epsilon = O(L/r_o) \ll 1$

The setting is of particular interest in oceanography. For the eddies that have been observed in western Atlantic,

$$D = O(4km), U = O(5cm/s), L = O(100km), f_o = O(10^{-4}s^{-1}).$$

So

$$\epsilon = O(5 \times 10^{-3}), F = O(2 \times 10^{-3}), \delta = O(4 \times 10^{-2}).$$

We now write (3.2) in spherical coordinates as follows:

$$\frac{du}{dt} + \frac{uw}{r} - \frac{uv}{r}\cot\varphi - 2\Omega\cos\varphi v + 2\Omega\sin\varphi w = -\frac{1}{\varrho r\sin\varphi}\frac{\partial p}{\partial \vartheta} \quad (3.3)$$

$$\frac{dv}{dt} + \frac{vw}{r} + \frac{u^2}{r}\cot\varphi + 2\Omega\cos\varphi u = -\frac{1}{\varrho r}\frac{\partial p}{\partial \varphi} \quad (3.4)$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{r} - 2\Omega\sin\varphi u = -\frac{1}{\varrho}\frac{\partial p}{\partial r} - g \quad (3.5)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial\vartheta} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial r}.$$

We write (3.3) in terms of $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\varrho}$ as follows:

$$\begin{aligned} &\frac{U^2}{L}\frac{d\tilde{u}}{d\tilde{t}} + \frac{\delta U^2}{r}\tilde{u}\tilde{w} - \frac{U^2}{r}\tilde{u}\tilde{v}\cot\varphi - U\tilde{v}2\Omega\cos\varphi + \delta U\tilde{w}2\Omega\sin\varphi \\ = \;& -\frac{f_o U}{1+\epsilon F\tilde{\varrho}}\frac{r_o\sin\varphi_o}{r\sin\varphi}\frac{\partial\tilde{p}}{\partial\tilde{x}} \end{aligned} \tag{3.6}$$

where

$$\frac{d}{d\tilde{t}} = \frac{\partial}{\partial\tilde{t}} + \tilde{u}\frac{r_o\sin\varphi_o}{r\sin\varphi}\frac{\partial}{\partial\tilde{x}} + \tilde{v}\frac{r_o}{r}\frac{\partial}{\partial\tilde{y}} + \tilde{w}\frac{\partial}{\partial\tilde{z}}.$$

Dividing (3.6) by Uf_o, skipping the notation $\tilde{}$, we get

$$\frac{U}{f_oL}(\frac{du}{dt} + \frac{\delta L}{r}uw - \frac{L}{r}uv\cot\varphi) - v\frac{\cos\varphi}{\cos\varphi_o} + \delta w\frac{\sin\varphi}{\cos\varphi_o} = -\frac{r_o\sin\varphi_o}{r\sin\varphi}\frac{\partial p}{\partial x}\frac{1}{1+\epsilon F\varrho}. \tag{3.7}$$

Similarly, from (3.4), (3.5) we have

$$\frac{U}{f_oL}(\frac{dv}{dt} + \frac{\delta L}{r}vw + \frac{L}{r}u^2\cot\varphi) + u\frac{\cos\varphi}{\cos\varphi_o} = -\frac{r_o}{r}\frac{\partial p}{\partial y}\frac{1}{1+\epsilon F\varrho} \tag{3.8}$$

$$(1+\epsilon F\varrho)(\epsilon\delta^2\frac{dw}{dt} - \frac{\epsilon\delta}{r}(u^2+v^2) - \delta u\frac{\sin\varphi}{\cos\varphi_o}) = -\frac{1}{\varrho_s}\frac{\partial(p\varrho_s)}{\partial z} - \varrho. \tag{3.9}$$

Also from (3.1). $\nabla\cdot\vec{v} = 0$ gives

$$\frac{\partial w}{\partial z} + 2\frac{D}{r}w + \frac{r_o}{r}\frac{\partial v}{\partial y} - \frac{L}{r}v\cot\varphi + \frac{r_o}{r}\frac{\sin\varphi_o}{\sin\varphi}\frac{\partial u}{\partial x} = 0, \tag{3.10}$$

$\frac{d\varrho}{dt} = 0$ gives

$$\epsilon F\frac{d\varrho}{dt} + (1+\epsilon F\varrho)\frac{w}{\varrho_s}\frac{\partial\varrho_s}{\partial z} = 0. \tag{3.11}$$

Now u, v, w, p, ϱ satisfy (3.7), (3.8), (3.9), (3.10), (3.11). So they are functions of x, y, z, t and $\epsilon, \frac{U}{f_o L}, \frac{L}{r_o}, F, \delta$. We now expand u, v, w, p, ϱ in term of ϵ. Thus

$$\begin{pmatrix} u \\ v \\ w \\ p \\ \varrho \end{pmatrix} = \begin{pmatrix} u_o \\ v_o \\ w_o \\ p_o \\ \varrho_o \end{pmatrix} + \epsilon \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ p_1 \\ \varrho_1 \end{pmatrix} + \cdots$$

where $(u_i, v_i, w_i, p_i, \varrho_i)$ are functions of x, y, z, t. We now assume $\frac{1}{\varrho_s}\frac{\partial \varrho_s}{\partial z} = O(\epsilon)$. See Pedlosky [9]. The O(1) term of (3.7), (3.8), (3.9) give

$$\begin{aligned} v_o &= \frac{\partial p_o}{\partial x} \\ u_o &= -\frac{\partial p_o}{\partial y} \\ \varrho_o &= -\frac{\partial p_o}{\partial z}. \end{aligned}$$

The O(ϵ) term of (3.11) gives w_o=0 and the O(ϵ) term of (3.7), (3.8), (3.10) give

$$\frac{\partial u_o}{\partial t} + u_o \frac{\partial u_o}{\partial x} + v_o \frac{\partial u_o}{\partial y} - v_1 - v_o \frac{L}{r_o \epsilon} y \tan \varphi_o = -\frac{\partial p_1}{\partial x} - \frac{Ly}{\epsilon r_o} \cot \varphi_o \frac{\partial p_o}{\partial x} \quad (3.12)$$

$$\frac{\partial v_o}{\partial t} + u_o \frac{\partial v_o}{\partial x} + v_o \frac{\partial v_o}{\partial y} + u_1 + u_o \frac{L}{r_o \epsilon} y \tan \varphi_o = -\frac{\partial p_1}{\partial y} \quad (3.13)$$

$$\frac{\partial w_1}{\partial z} + \frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial x} - v_o \frac{L}{r_o \epsilon} \cot \varphi_o - \frac{L}{r_o \epsilon} y \cot \varphi_o \frac{\partial u_o}{\partial x} = 0 \quad (3.14)$$

Let $\xi_o = \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y}$. Differentiating (3.12) w.r.t. x and differentiating (3.13) w.r.t. y, we get

$$\begin{aligned} & \frac{\partial \xi_o}{\partial t} + u_o \frac{\partial \xi_o}{\partial x} + v_o \frac{\partial \xi_o}{\partial y} \\ = \quad & -v_o \frac{L}{r_o \epsilon} \tan \varphi_o + \frac{L}{r_o \epsilon} \cot \varphi_o \frac{\partial p_o}{\partial x} + \frac{Ly}{r_o \epsilon} \cot \varphi_o \frac{\partial^2 p_o}{\partial x \partial y} - \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y}. \end{aligned}$$

Using (3.14), we obtain

$$\frac{\partial \xi_o}{\partial t} + u_o \frac{\partial \xi_o}{\partial x} + v_o \frac{\partial \xi_o}{\partial y} + \beta v_o = -\frac{\partial w_1}{\partial z} \tag{3.15}$$

where $\beta = \frac{L \tan \varphi_o}{r_o \epsilon}$.

The $O(\epsilon^2)$ term in (3.11) implies

$$\begin{aligned} F\frac{d\varrho_o}{dt} + \frac{w_1}{\varrho_s}\frac{\partial \varrho_s}{\partial z} = 0 \\ \frac{d_o \varrho_o}{dt} - w_1 S = 0 \end{aligned}$$

where $\frac{d_o}{dt} = \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y}$ and $S(z) = -\frac{1}{F\varrho_s}\frac{\partial \varrho_s}{\partial z}$. Therefore

$$w_1 = \frac{1}{S}\frac{d_o \varrho_o}{dt}. \tag{3.16}$$

Now (3.15) gives

$$\left(\frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y}\right)\left(\xi_o + \beta y - \frac{\partial}{\partial z}\left(\frac{1}{S}\varrho_o\right)\right) = 0.$$

which is

$$\left(\frac{\partial}{\partial t} - \frac{\partial p_o}{\partial y}\frac{\partial}{\partial x} + \frac{\partial p_o}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 p_o}{\partial^2 x} + \frac{\partial^2 p_o}{\partial^2 y} + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial p_o}{\partial z}\right) + \beta y\right) = 0. \tag{3.17}$$

This is called the quasi-geostrophic flow. See Pedlosky [9].

We denote by Δ_S and $J[\cdot,\cdot]$ the following operators

$$\begin{aligned} \Delta_S\, f &= \partial_x{}^2 f + \partial_y{}^2 f + \partial_z(\frac{1}{S}\partial_z f) \\ J[a,b] &= \partial_x a \partial_y b - \partial_y a \partial_x b. \end{aligned}$$

Then (3.17) is equivalent to

$$\partial_t(\Delta_S\, p) + J[p, \Delta_S\, p + \beta y] = 0. \tag{3.18}$$

Although x, y are in principle simply new longitude and latitude coordinates, they are introduced so that for small L/r_o and D/r_o they will be the Cartesian coordinates. We now give the boundary condition for the (3.18). For simplicity, we will restrict the region to $M = \mathbf{R}^2/\mathbf{Z}^2 \times [0,1]$. This means that we consider solutions which are periodic in the horizontal direction. The boundary is $\partial M = \{(x,y,z) \in M |\ z = 0, z = 1\} = \Gamma_0 \cup \Gamma_1$ where $\Gamma_i = \mathbf{R}^2/\mathbf{Z}^2 \times \{i\}$. The boundary condition is $w_1 = 0$ at ∂M. It follows from (3.16) that

$$\frac{1}{S}(\frac{\partial}{\partial t} - \frac{\partial p}{\partial y}\frac{\partial}{\partial x} + \frac{\partial p}{\partial x}\frac{\partial}{\partial y})\frac{\partial p}{\partial z} = 0.$$

This implies

$$\frac{\partial p}{\partial z}(t, \zeta(t,x,y,z)) = \frac{\partial p}{\partial z}(0,x,y,z) \ \ at\ \partial M$$

where $\zeta(t,x,y,z)$ is the flow of $(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}, 0)$.

Note that if $\dfrac{\partial p(0,x,y,z)}{\partial z} = c_p = constant$ at ∂M (That means $c_p = c_i = constant$ at Γ_i), then the boundary condition is $\dfrac{\partial p}{\partial z}(t,x,y,z) = c_p$.

3.2 Solution of the Flow

In this section, we will consider the initial value problem for the equation

$$\partial_t(\Delta_S\ p) + J[p, \Delta_S\ p + \beta y] = 0,\ p(0) = p_o. \tag{3.19}$$

Theorem 2 *For T small, $S(z) > 0, p_o \in C^\infty(M), \partial_z p_o = c_p$ at ∂M and $s \geq 5$, (3.19) has a unique solution in $C^o([0,T], H^s)$ with $\int_M p = 0$.*

We want to solve (3.19) in the following way. For given functions $q(t)$, we consider the equation

$$\partial_t(\Delta_S\ p) + J[q, \Delta_S\ p + \beta y] = 0,\ p(0) = p_o. \tag{3.20}$$

If (3.20) has a solution p, we get a mapping $q \to p = \Phi(q)$. Then we look for a fixed point of Φ, which will be a solution of (3.19). We shall use the contraction mapping theorem to show the existence of the fixed point.

We first solve for ϕ^q satisfying the equation

$$\partial_t\phi^q + (v^q \cdot \nabla)\phi^q = -\beta v_2^q,\ \phi^q(0) = \phi_o. \tag{3.21}$$

where $v^q = (v_1^q, v_2^q, v_3^q) = (-\partial_y q, \partial_x q, 0), \phi_o = \Delta_S\ p_o$.

We will need the regularity theorem for H^s vector fields.

Regularity Theorem *Let M be a compact manifold (with boundry ∂M) and $s \geq \frac{n}{2} + 2$ ($n = dimM$). If V is an H^s vector field on M (parallel to ∂M), the* *flow η_t of V with η_o = identity is a C^1 curve in $\mathcal{D}^s$. If V is a divergence free vector field, then η_t is in $\mathcal{D}^s_\mu$.*

Proof: See Ebin and Marsden [3]. □

For $q \in H^s, v^q \in H^{s-1}, v^q$ is parallel to ∂M and $div\ v^q = 0$. It follows from the regularity theorem, if $s-1 \geq \frac{3}{2}+2$, then the flow $\eta_t{}^q$ of v^q is in $\mathcal{D}^{s-1}_\mu$.

In the following, we will omit the q in the superscript.

Taking the time derivative of the equation

$$\phi \circ \eta_t = \phi_o - \beta \int_0^t v_2 \circ \eta_s \; ds,$$

we get

$$\partial_t \phi \circ \eta_t + \nabla \phi \circ \eta_t \cdot \partial_t \eta_t = -\beta v_2 \circ \eta_t$$

which is equal to

$$\partial_t \phi + (v \cdot \nabla)\phi = -\beta v_2.$$

Therefore the solution to (3.21) is

$$\phi(t) = (\phi_o - \beta \int_0^t v_2 \circ \eta_s \; ds) \circ \eta_t^{-1}. \tag{3.22}$$

For $S(z) > 0$, the Neumann problem

$$\begin{aligned} \Delta_S f &= \phi \; at \; M \\ \frac{\partial f}{\partial z} &= c_i \; at \; \Gamma_i, i = 0, 1 \end{aligned}$$

has a solution if and only if $\int_M \phi = \int_{\Gamma_1} \frac{1}{S} c_1 - \int_{\Gamma_0} \frac{1}{S} c_0$. See Treves [10].

We now can write $\Phi(q) = \Delta_S^{-1}(\phi^q)$. The operator Δ_S^{-1} is well defined because

$$\int_M \phi^q = \int_M \phi_o - \beta \int_M \int_0^t v_2 \circ \eta_s \, ds = \int_M \phi_o = \int_{\Gamma_1} \frac{1}{S} c_1 - \int_{\Gamma_0} \frac{1}{S} c_0$$

and the uniqueness can be determined by requiring $\int_M \Phi(q) = 0$.

We now assume $T > 0, R > 0, \|\phi_o\|_{H^5} \ll R, \|c_p\|_{H^{\frac{7}{2}}(\partial M)} < R$ and fix $s = 5$.

Definition 7 $E = \{q \in C^o([0,T], H^s_o) | \; q(0) = p_o, \|q(t)\|_{H^s} \leq R\}$.

Proposition 5 *For T small, Φ maps E into itself.*

We begin with some estimates on η_t and η_t^{-1}.

Lemma 2 *If $f, g : [0,T] \to \mathbf{R}$ continuous and nonnegative. For $A \geq 0$,*

$$f(t) \leq A + \int_0^t f(s)g(s)ds.$$

Then $f(t) \leq A \exp^{\int_0^t g(s)ds}$.

Proof: Assume $A > 0$, let $h(t) = A + \int_0^t f(s)g(s)ds$ for $t \in [0,T]$. Then $h > 0$ and $d_t h = f(t)g(t) \leq h(t)g(t)$. Integration gives $h(t) \leq A \exp^{\int_0^t g(s)ds}$. For $A = 0$, replace A by $\epsilon > 0$ for all $\epsilon > 0$, then $f = 0$. □

Definition 8 *Let $f : [0,T] \to X$ be continuous, where X is a Banach space. Define*

$$\|f\|_{X,\infty} = sup_{t \in [0,T]} \|f(t)\|_X.$$

In the following, $v = v^q$ where $q \in E$ and $\eta = \eta^q$ is the flow of v^q. we will use $\mathcal{K}(\ )$ to denote an arbitrary constant which depends on the quantities inside the parenthesis.

Lemma 3 $\|D\eta\|_{C^1} \leq \mathcal{K}(T, \|v\|_{C^2,\infty})$.

Proof: Since $\partial_t \eta = v \circ \eta$, we obtain

$$\partial_t D\eta = Dv \circ \eta \cdot D\eta. \tag{3.23}$$

Using $D\eta = \int_0^t \partial_s D\eta \; ds + identity$, we get

$$\|D\eta\|_{C^o} \leq \int_0^t \|\partial_s D\eta\|_{C^o} ds + 1 \leq \int_0^t \|Dv\|_{C^o} \|D\eta\|_{C^o} ds + 1.$$

Applying Lemma 2, we get

$$\|D\eta\|_{C^o} \leq \exp(\|Dv\|_{C^o,\infty} T).$$

For $D^2\eta$, we have

$$\partial_t D^2\eta = D^2 v \circ \eta \cdot (D\eta)^2 + Dv \circ \eta \cdot D^2\eta. \tag{3.24}$$

(3.24) gives

$$\|D^2\eta\|_{C^o} \leq T\|D^2 v\|_{C^o,\infty} \|D\eta\|^2_{C^o,\infty} + \int_0^t \|Dv\|_{C^o} \|D^2\eta\|_{C^o}.$$

Thus from Lemma 2

$$\|D^2\eta\|_{C^o} \leq \|D^2 v\|_{C^o,\infty} \|D\eta\|^2_{C^o,\infty} \exp(\|Dv\|_{C^o,\infty} T).$$

□

Lemma 4 $\|D\eta\|_{H^3} \leq \mathcal{K}(T, \|v\|_{H^4,\infty})$.

Proof: We will estimate $\|D\eta\|_{H^o}$ and $\|D^2\eta\|_{H^o}$, the higher derivative can be followed by the same method. From (3.23), we have

$$\frac{1}{2}\partial_t\|D\eta\|^2_{H^o} = \int_M D\eta \cdot \partial_t D\eta = \int_M D\eta \cdot (Dv \circ \eta \cdot D\eta) \leq \|Dv\|_{C^o}\|D\eta\|^2_{H^o}.$$

So $\partial_t\|D\eta\|_{H^o} \leq \|Dv\|_{C^o,\infty}\|D\eta\|_{H^o}$. Therefore

$$\|D\eta\|_{H^o} \leq \exp(\|Dv\|_{C^o,\infty}T)\|D\eta_o\|_{H^o}.$$

From (3.24),

$$\begin{aligned}
\frac{1}{2}\partial_t\|D^2\eta\|^2_{H^o} &= \int_M D^2\eta \cdot \partial_t D^2\eta \\
&= \int_M D^2\eta \cdot (D^2v \circ \eta \cdot (D\eta)^2 + Dv \circ \eta \cdot D^2\eta) \\
&\leq \|D^2v\|_{H^o}\|D\eta\|^2_{C^o}\|D^2\eta\|_{H^o} + \|Dv\|_{C^o}\|D^2\eta\|^2_{H^o} \\
\partial_t\|D^2\eta\|_{H^o} &\leq \|D^2v\|_{H^o,\infty}\|D\eta\|^2_{C^o,\infty} + \|Dv\|_{C^o,\infty}\|D^2\eta\|_{H^o}.
\end{aligned}$$

So

$$\|D^2\eta\|_{H^o} \leq (\exp(\|Dv\|_{C^o,\infty}T) - 1)\frac{\|D\eta\|^2_{C^o,\infty}\|D^2v\|_{H^o,\infty}}{\|Dv\|_{C^o,\infty}}.$$

Apply Lemma 3 to the term $\|D\eta\|_{C^o,\infty}$ and using Sobolev Embedding Theorem, we can get the required estimate. □

Lemma 5 $\|D\eta^{-1}\|_{H^3} \leq \mathcal{K}(\|D\eta\|_{H^3})$.

Proof: We have $D\eta^{-1} = (D\eta)^{-1} \circ \eta^{-1}$, where $(D\eta)^{-1}$ is the inverse of the matrix $D\eta$. Because $Jacobian(\eta) = 1$, we have $D\eta^{-1} = Q(D\eta) \circ \eta^{-1}$ with Q a quadratic polynomial. This implies $\|D\eta^{-1}\|_{H^o} \leq K\|D\eta\|^2_{H^o}$ for some constant K.

The estimate for the higher derivatives can be obtained by repeating the differentiation. In general, $D^s\eta^{-1} = \sum(Q_i(D\eta)\cdot(D^m\eta)^{m_i}\cdot(D^n\eta)^{n_i})\circ\eta^{-1}$, where Q_i are polynomials and $m \cdot m_i + n \cdot n_i = s$. □

We continue to estimate the terms in (3.22). We first notice that by differentiating $\eta^{-1} \circ \eta = identity$, we have

$$\partial_t \eta^{-1} \circ \eta + D\eta^{-1} \circ \eta \cdot \partial_t \eta = 0.$$

This implies that

$$\partial_t \eta^{-1} = -(D\eta^{-1} \circ \eta \cdot \partial_t \eta) \circ \eta^{-1} = -D\eta^{-1} \cdot v. \tag{3.25}$$

Now

$$(\int_0^t v_2 \circ \eta_s \, ds) \circ \eta_t^{-1} = (\eta_t - \eta_o)_2 \circ \eta_t^{-1} = (identity - \eta_t^{-1})_2 = (\int_0^t D\eta_t^{-1} \cdot v)_2.$$

We define $f = \beta(\int_0^t v_2 \circ \eta_s \, ds) \circ \eta_t^{-1}$. It is easy to prove the following two lemmas by the product rule and the chain rule.

Lemma 6 $\|f\|_{H^3} \leq |\beta| T\mathcal{K} \, \|D\eta^{-1}\|_{H^3,\infty} \|v\|_{H^3,\infty}$.

Lemma 7 $\|\phi_o \circ \eta^{-1}\|_{H^3} \leq \|\phi_o\|_{H^5} \mathcal{K}(\|D\eta^{-1}\|_{H^2})$.

We can now complete the proof of the Proposition 5. For $q \in E$, we have

$$\Phi(q) = \Delta_S^{-1}\phi = \Delta_S^{-1}(\phi_o \circ \eta^{-1} - f).$$

By the regularity of the solution of Neumann problem, see Treves [10], we get

$$\|\Phi(q)\|_{H^5} \leq \mathcal{K}(S)(\|\phi_o \circ \eta^{-1} - f\|_{H^3} + \|c_p\|_{H^{\frac{7}{2}}(\partial M)}),$$

We can apply Lemma 4, 5, 6, 7 to estimate $\|\phi_o \circ \eta^{-1}\|_{H^3}$ and $\|f\|_{H^3}$, we obtain

$$\|\Phi(q)\|_{H^5} \leq \mathcal{K}(|\beta|, S, T, \|c_p\|_{H^{\frac{7}{2}}(\partial M)}, \|v\|_{H^4,\infty}, \|\phi_o\|_{H^5}).$$

Since $\|v\|_{H^4,\infty} \leq \|q\|_{H^5,\infty} \leq R$ and $\|\phi_o\|_{H^5} \ll R$, we have $\|\Phi(q)\|_{H^5} \leq R$ if T is sufficiently small.

Proposition 6 *$\Phi : E \to E$ is a contraction map w.r.t.* $\|\cdot\|_{H^2,\infty}$

Note that $(E\ ,\ \|\cdot\|_{H^2,\infty})$ is a complete metric space. From a lemma of Kato [4] which states that for $s' > s \geq 0$, if $\|p_n\|_{H^{s'}} \leq 1\ \forall n$ and $\|p_n - p_\infty\|_{H^s} \to 0$ as $n \to \infty$ then $\|p_\infty\|_{H^{s'}} \leq 1$.

In the following, for $i = 1, 2, q_i \in E, v_i = (-\partial_y q_i, \partial_x q_i, 0)$ and η_i are the flow of v_i and $f_i = (\int_0^t v_i \circ \eta_i\ ds)_2 \circ \eta_i^{-1}$.

We begin with some lemmas.

Lemma 8 $\|\eta_1 - \eta_2\|_{H^o} \leq \mathcal{K}(T, \|Dv_1\|_{C^o,\infty})\|v_1 - v_2\|_{H^o,\infty}$.

Proof:

$$\frac{1}{2}\partial_t\|\eta_1 - \eta_2\|_{H^o}^2 = \int_M (\eta_1 - \eta_2)\cdot\partial_t(\eta_1 - \eta_2) \le \|\eta_1 - \eta_2\|_{H^o}\|\partial_t(\eta_1 - \eta_2)\|_{H^o}. \quad (3.26)$$

Also

$$\begin{aligned}
\|\partial_t(\eta_1 - \eta_2)\|_{H^o} &= \|v_1 \circ \eta_1 - v_2 \circ \eta_2\|_{H^o} \\
&= \|v_1 \circ \eta_1 - v_1 \circ \eta_2 + (v_1 - v_2) \circ \eta_2\|_{H^o} \\
&\le \|v_1 \circ \eta_1 - v_1 \circ \eta_2\|_{H^o} + \|(v_1 - v_2) \circ \eta_2\|_{H^o} \\
&\le \|Dv_1\|_{C^o}\|\eta_1 - \eta_2\|_{H^o} + \|v_1 - v_2\|_{H^o}.
\end{aligned}$$

So, from (3.26)

$$\partial_t\|\eta_1 - \eta_2\|_{H^o} \le \|Dv_1\|_{C^o,\infty}\|\eta_1 - \eta_2\|_{H^o} + \|v_1 - v_2\|_{H^o,\infty}.$$

Therefore, by Lemma 2, we have

$$\|\eta_1 - \eta_2\|_{H^o} \le (\exp(\|Dv_1\|_{C^o,\infty}T) - 1)\frac{\|v_1 - v_2\|_{H^o,\infty}}{\|Dv_1\|_{C^o,\infty}}.$$

□

Lemma 9 $\|D\eta_1 - D\eta_2\|_{H^o} \le \mathcal{K}(T, \|Dv_1\|_{C^1,\infty}, \|Dv_2\|_{C^1,\infty})\|v_1 - v_2\|_{H^1,\infty}$

Proof: We use (3.23) to estimate $\partial_t(D\eta_1 - D\eta_2)$.

$$\begin{aligned}
&\|\partial_t(D\eta_1 - D\eta_2)\|_{H^o} \\
= \;& \|Dv_1 \circ \eta_1 \cdot D\eta_1 - Dv_2 \circ \eta_2 \cdot D\eta_2\|_{H^o} \\
\le \;& \|Dv_1 \circ \eta_1 \cdot (D\eta_1 - D\eta_2)\|_{H^o} + \|(Dv_1 \circ \eta_1 - Dv_2 \circ \eta_2) \cdot D\eta_2\|_{H^o}
\end{aligned}$$

The first term $\leq \|Dv_1\|_{C^o}\|D\eta_1 - D\eta_2\|_{H^o}$. The second term equals to

$$\begin{aligned}
&\|(Dv_1 \circ \eta_1 - Dv_1 \circ \eta_2) \cdot D\eta_2 + (Dv_1 - Dv_2) \circ \eta_2 \cdot D\eta_2\|_{H^o} \\
\leq\ & \|Dv_1\|_{C^1}\|\eta_1 - \eta_2\|_{H^o}\|D\eta_2\|_{C^o} + \|Dv_1 - Dv_2\|_{H^o}\|D\eta_2\|_{C^o}.
\end{aligned}$$

We apply Lemma 8 for $\|\eta_1 - \eta_2\|_{H^o}$ to obtain

$$\|\partial_t(D\eta_1 - D\eta_2)\|_{H^o} \leq \|Dv_1\|_{C^o,\infty}\|D\eta_1 - D\eta_2\|_{H^o} + \mathcal{K}\|v_1 - v_2\|_{H^1,\infty}.$$

where $\mathcal{K}$ depends on $\|Dv_1\|_{C^1,\infty}$, $\|Dv_2\|_{C^1,\infty}$. Thus

$$\|D\eta_1 - D\eta_2\|_{H^o} \leq (\exp(\|Dv_1\|_{C^o,\infty}T) - 1)\mathcal{K}\frac{\|v_1 - v_2\|_{H^1,\infty}}{\|Dv_1\|_{C^o,\infty}}.$$

□

Lemma 10 $\|\eta_1^{-1} - \eta_2^{-1}\|_{H^o} \leq \mathcal{K}(T, \|v_1\|_{H^4,\infty}, \|v_2\|_{H^4,\infty})\|v_1 - v_2\|_{H^1,\infty}$.

Proof: Apply (3.25) to estimate $\|\partial_t(\eta_1^{-1} - \eta_2^{-1})\|_{H^o}$ which is equal to

$$\begin{aligned}
&\|(D\eta_1)^{-1} \circ \eta_1^{-1} \cdot v_1 - (D\eta_2)^{-1} \circ \eta_2^{-1} \cdot v_2\|_{H^o} \\
=\ & \|(D\eta_1)^{-1} \circ \eta_1^{-1} \cdot (v_1 - v_2)\|_{H^o} + \|((D\eta_1)^{-1} \circ \eta_1^{-1} - (D\eta_2)^{-1} \circ \eta_2^{-1}) \cdot v_2\|_{H^o}.
\end{aligned}$$

The first term $\leq \|(D\eta_1)^{-1}\|_{C^o}\|v_1 - v_2\|_{H^o}$. The second term

$$\begin{aligned}
\leq\ & \|((D\eta_1)^{-1}\circ\eta_1^{-1} - (D\eta_1)^{-1}\circ\eta_2^{-1}) \cdot v_2\|_{H^o} + \|((D\eta_1)^{-1} - (D\eta_2)^{-1})\circ\eta_2^{-1} \cdot v_2\|_{H^o} \\
\leq\ & \|(D\eta_1)^{-1}\circ\eta_1^{-1} - (D\eta_1)^{-1}\circ\eta_2^{-1}\|_{H^o}\|v_2\|_{C^o} + \|(D\eta_1)^{-1} - (D\eta_2)^{-1}\|_{H^o}\|v_2\|_{C^o} \\
\leq\ & \|(D\eta_1)^{-1}\|_{C^1}\|v_2\|_{C^o}\|\eta_1^{-1} - \eta_2^{-1}\|_{H^o} + \|Q(D\eta_1) - Q(D\eta_2)\|_{H^o}\|v_2\|_{C^o}.
\end{aligned}$$

The second term in above $\leq \|DQ\|_{C^o}\|D\eta_1 - D\eta_2\|_{H^o}\|v_2\|_{C^o}$

We can apply the Lemma 9 to $\|D\eta_1 - D\eta_2\|_{H^o}$. We now obtain

$$\partial_t\|\eta_1^{-1} - \eta_2^{-1}\|_{H^o} \leq \mathcal{K}\|v_1 - v_2\|_{H^1} + \mathcal{K}\|\eta_1^{-1} - \eta_2^{-1}\|_{H^o}.$$

where $\mathcal{K} = \mathcal{K}(T, \|v_1\|_{H^4,\infty}, \|v_2\|_{H^4,\infty})$. We then use the usual method to get the result.

□

Lemma 11 $\|f_1 - f_2\|_{H^o} \leq |\beta|T\mathcal{K}(T, \|v_1\|_{H^4,\infty}, \|v_2\|_{H^4,\infty})\|v_1 - v_2\|_{H^1,\infty}$

Proof: We have

$$\begin{aligned}\|f_1 - f_2\|_{H^o} &= |\beta|\|(\textstyle\int_0^t v_1 \circ \eta_1\ ds)_2 \circ \eta_1^{-1} - (\int_0^t v_2 \circ \eta_2\ ds)_2 \circ \eta_2^{-1}\|_{H^o} \\ &\leq |\beta|(\|(\textstyle\int_0^t v_1 \circ \eta_1\ ds)_2 \circ \eta_1^{-1} - (\int_0^t v_1 \circ \eta_1\ ds)_2 \circ \eta_2^{-1}\|_{H^o} \\ &\quad +\|(\textstyle\int_0^t v_1 \circ \eta_1 - v_2 \circ \eta_2\ ds)_2 \circ \eta_2^{-1}\|_{H^o}).\end{aligned}$$

The first term $\leq \int_0^t \|v_1\|_{C^1}\|\eta_1^{-1} - \eta_2^{-1}\|_{H^o}$. The second term equals

$$\begin{aligned}&\|\textstyle\int_0^t(v_1 - v_2)_2 \circ \eta_1 + v_2 \circ \eta_1 - v_2 \circ \eta_2\ ds\|_{H^o} \\ \leq\ &\int_0^t \|v_1 - v_2\|_{H^o} + \|v_2\|_{C^1}\|\eta_1 - \eta_2\|_{H^o}.\end{aligned}$$

Then we apply lemma 8 and 10.

□

We now complete the proof of the Proposition 6. We have

$$\begin{aligned}&\|\Phi(q_1) - \Phi(q_2)\|_{H^2,\infty} \\ =\ &\|\Delta_S^{-1}(\phi_o \circ \eta_1^{-1} - f_1 - \phi_o \circ \eta_2^{-1} + f_2)\|_{H^2,\infty}\end{aligned}$$

$$\leq \mathcal{K}(S)(\|\phi_o \circ \eta_1^{-1} - \phi_o \circ \eta_2^{-1}\|_{H^o,\infty} + \|f_1 - f_2\|_{H^o,\infty})$$

$$\leq \mathcal{K}(S)(\|D\phi_o\|_{C^o,\infty}\|\eta_1^{-1} - \eta_2^{-1}\|_{H^o,\infty} + \|f_1 - f_2\|_{H^o,\infty}).$$

Applying lemma 10 and 11, we have

$$\begin{aligned}\|\Phi(q_1) - \Phi(q_2)\|_{H^2,\infty} &\leq \mathcal{K}\|v_1 - v_2\|_{H^1,\infty} \\ &\leq \mathcal{K}\|q_1 - q_2\|_{H^2,\infty},\end{aligned}$$

where $\mathcal{K}(|\beta|, S, T, \|\phi_o\|_{C^1,\infty}, \|v_1\|_{H^4,\infty}, \|v_2\|_{H^4,\infty})$ is less than 1 if T is sufficiently small.

It follows from the contraction mapping theorem that Φ has a fixed point in E which is automatically a unique solution of (3.19). We still need to show that the solution is in H^s for all s.

Proposition 7 *For the same T, the solution is in H^s for all $s \geq 5$.*

Proof: Let p_∞ be the fixed point of Φ. Then

$$p_\infty = \Delta_S^{-1}((\Delta_S\, p_o) \circ \eta_t^{-1} - \beta(\int_0^T v_2^{p_\infty} \circ \eta_s\, ds) \circ \eta_t^{-1}), \qquad (3.27)$$

where η_t is the flow of v^{p_∞}.

We have shown $p_\infty \in H^5$, so $v^{p_\infty} \in H^4$ and $\eta_t, \eta_t^{-1} \in H^4$. Therefore from (3.27), $\Delta_S\, p_\infty \in H^4$, this implies $p_\infty \in H^6$. We can repeat the argument to get $p_\infty \in H^s$ for all $s \geq 5$. □

3.3 Hamiltonian Formulation

We now put (3.18) in a Hamiltonian form for a Poisson bracket. We assume $p \in C^\infty(M), \partial_z p = c_p$ at $\partial M, \int_{\Gamma_i} p = 0, i = 0, 1$.

Using the Poisson structure defined in (2.1), we have for $\lambda \in C^\infty(M)$,

$$\{F, G\}_\lambda(p) = \int_M (\Delta_S\ p + \lambda) \cdot J[\Delta_S^{-1} \frac{\delta F}{\delta p}, \Delta_S^{-1} \frac{\delta G}{\delta p}]$$

for all F, G such that $\frac{\delta F}{\delta p}, \frac{\delta G}{\delta p} \in Im(\Delta_S)$.

We further define the Hamiltonian $H = \frac{1}{2} \int_M (\partial_x p)^2 + (\partial_y p)^2 + \frac{1}{S}(\partial_z p)^2$ which is the usual expression for the energy in the situation modeled by (3.18).

Theorem 3 *(3.18) is equivalent to $\dot{F} = \{F, H\}_\lambda$ when $\lambda = \beta y$.*

Proof: We first calculate $\frac{\delta H}{\delta p}$ as follows. For $h = \delta p$,

$$\begin{aligned} & \frac{d}{dt}|_{t=0} H(p + th) \\ = & \int_M \partial_x p \partial_x h + \partial_y p \partial_y h + \frac{1}{S} \partial_z p \partial_z h \\ = & -\int_M (\partial_x{}^2 p + \partial_y{}^2 p + \partial_z(\frac{1}{S}\partial_z p)) \cdot h + \sum_{i=0}^{1} (-1)^{i+1} \int_{\Gamma_i} \frac{1}{S} h \partial_z p \end{aligned}$$

Since

$$\int_{\Gamma_i} \frac{1}{S} h \partial_z p = \int_{\Gamma_i} \frac{c_i}{S} h = constant \int_{\Gamma_i} h = 0,$$

we have $\frac{\delta H}{\delta p} = -\Delta_S\ p$.

$$\{F, H\} \ = \ -\int_M (\Delta_S\ p + \lambda) \cdot J[\Delta_S^{-1} \frac{\delta F}{\delta p}, \Delta_S^{-1} \Delta_S\ p]$$

$$\begin{aligned} &= \int_M (\Delta_S\, p + \lambda) \cdot J[p, \Delta_S^{-1} \frac{\delta F}{\delta p}] \\ &= -\int_M \Delta_S^{-1} \frac{\delta F}{\delta p} \cdot J[p, \Delta_S\, p + \lambda]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \dot{F} &= DF \cdot \dot{p} = \int_M \frac{\delta F}{\delta p} \dot{p} \\ &= \int_M \Delta_S^{-1} \frac{\delta F}{\delta p} \cdot \Delta_S\, \dot{p} + \sum_{i=0}^{1} (-1)^{i+1} \int_{\Gamma_i} \dot{p} \frac{1}{S} \partial_z \Delta_S^{-1} \frac{\delta F}{\delta p} - \frac{1}{S} \partial_z \dot{p} \Delta_S^{-1} \frac{\delta F}{\delta p}. \end{aligned}$$

All the boundary integrals in above vanish except the first term. This follows from that at $\Gamma_i, \partial_z \dot{p} = \dot{c}_i = 0$ and $\int_{\Gamma_i} \dot{p} = 0$. Therefore

$$\partial_t \Delta_S\, p + J[p, \Delta_S\, p + \lambda] = 0. \tag{3.28}$$

For $\lambda = \beta y$, this is equivalent to (3.18). □

The conservation of H is now an immediate consequence of the Hamiltonian formulation.

Analogous to the well known conservation of integrals of functions of the vorticity in the two dimensional perfect flow, we have the conservation of a functional of the form

$$\int_M \psi(\Delta_S\, p + \lambda)$$

where p evolves according to the (3.28) and $\psi : \mathbf{R} \to \mathbf{R}$ is a smooth function. For instance, taking $\psi(\zeta) = \zeta^2$ gives the conservation of $\Delta_S\, p + \lambda$ in H^o norm.

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