# On a Class of 4-dimensional Minimum Energy Metrics and Hyperbolic Geometry

A Dissertation Presented

by

Jongsu Kim

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

December 1991

State University of New York
at Stony Brook

The Graduate School

Jongsu Kim

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Graduate School

# Abstract of the Dissertation

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A general problem in Differential Geometry is that how we can understand the topology of a 4-manifold in terms of 'optimal', 'canonical' or 'best' metrics. It is not easy to define what these are, but such 'best' metrics should be the ones easily perceivable to us, for instance it should have much symmetry or satisfies meaningful equations.

We consider minimum energy or critical metrics of two natural energy functionals in a natural moduli space of metrics; a scale invariant energy $\int_{\mathrm{M}} |R_g|^2 dvol_g$ and the conformally invariant energy $\int_{\mathrm{M}} |W_g|^2 dvol_g$, where $R$ is the curvature tensor and $W$ is the

conformal Weyl tensor of a metric $g$. We hope that these critical metrics provides many 'best' metrics. Recent major developments in the study of the critical metrics motivated the works in this dissertation.

We start by discussing the Kähler critical metrics of two functionals. We then focus on 'self-dual' metrics which give minimum energies of the above functionals. We construct explicit hyperbolic ansatz self-dual metrics with semi-free conformal $S^1$ action on connected sums of some conformally flat metrics with some number of $\mathbb{CP}^2$'s. The whole process depends on a study of hyperbolic geometry and Kleinian groups.

We observe that, from the explicitness of the metrics, we can read off the scalar curvature behavior without difficulty. Our construction includes all nonnegative scalar curved self-dual metrics with semi-free action. We also discuss a problem concerning scalar curvatures on the moduli space of self-dual conformal structures.

We also discuss examples in the context of minimum energy metrics of the above functionals.

**To Kyeonghi and my parents.**

# Contents

## Acknowledgements

I would like to thank, first and foremost, my thesis advisor Professor Claude LeBrun for his inspiring guidance and encouragements without which this dissertation could never have been completed.

I am very grateful to Professor Bernard Maskit, Professor Michael T. Anderson and Dr. Rubén A. Hidalgo for helpful conversations and suggestions during the preparation of this dissertation.

I also wish to express deep gratitute to Professor Dong-Pyo Chi in Seoul National University, who introduced me to Differential Geometry.

I am also obliged to thank all my friends at Stony Brook who brought joy to my life and helped me all the way. In particular, I would like to thank Christian Bäer, Gregorio P. Bessa, Adam Harris, Seungsu Hwang, Byung Bae Kim, Myong-Hi Kim, Jaeman Kim, Francois Lamongtagne, Andrew P. Mchugh, Sunil Nair, Zhongmin Shen, Santiago Simanca, Myong Hee Sung, Edward Taylor, Lisa Traynor, Wen-Xiang Wang, Seung-Mok Yi, Gabjin Yun, Jian Zhou, Shunhui Zhu.

My sincere thanks also go to to the faculty, staff and all the other members in the Mathematics Department of the State University of New York at Stony

Brook.

I would like to thank Joann Debis for helping me to prepare for completion of thesis and finally, I thank my wife, Kyeonghi, for her heartful support.

# Chapter 1

# Introduction

## 1.1 Motivation

The goal of this dissertation is to study some special classes of Riemannian metrics which have been considerably explored recently. In Riemannian geometry, it has been an interesting question to find 'optimal' or 'best' metrics on smooth manifolds. Although it is not clear how to define such metrics, they should be the ones we may get familiar with easily, so that they provide a 'language', in terms of which we understand the manifolds topologically or geometrically. This idea has been successful in lower dimensions.

For convenience, the author would like to use the term *'best'* metrics in this dissertation.

Constant curvature metrics can certainly be called 'best' metrics.

In dimension two, any metric on a closed oriented surface can be conformally deformed to another metric of constant curvature by Uniformization theory for Riemann surfaces.

In dimension three, it gets more complicated. The existence of a constant curvature metric on a simply connected n-manifold with $n \geq 3$ implies that the manifold is isometric to either a sphere $S^n$, an Euclidean space $\mathbb{R}^n$ or a hyperbolic space $\mathbb{H}^n$ so that it is diffeomorphic to $S^n$ or $\mathbb{R}^n$. So not every 3-manifold admit an Einstein metric, e.g. $S^2 \times S^1$ doesn't admit one because its universal covering space is not diffeomorphic to a sphere or an Euclidean space.

However, the Thurston geometrization conjecture [35] has been highly successful with many positive results already. The conjecture says that there is a canonical decomposition of the interior of every compact 3-manifold and that each manifold piece in the decomposition has one of *only eight* special geometries.

In the next dimension *four*, it may be hard to get such a strong geometrization program as Thurston's, and this remains as largely unexplored. However at least we may well hope to get much information on the the topology of 4-manifolds by studying 'best' metrics or special geometries.

So far, several categories of 'best' metrics, in a sense, have been found. Among them, let's consider the Einstein metrics briefly. Here $(M, g)$ is Einstein if $g$ has constant ricci curvature i.e. the ricci tensor $r$ such that $r = \lambda g$ for a constant $\lambda$. Einstein metric may certainly be called a 'best' metric, and there has been a huge literature [4]. We know existence of some Einstein metrics as well as some obstruction theories to existence, including the Hitchin-Thorpe Inequality [4];

$2\times(\textit{Euler characteristic}) \geq 3|\textit{signature}|$.

For example, $m\mathbb{CP}^2$ the connected sum of m copies of $\mathbb{CP}^2$, the complex projective plane for $m \geq 5$ can't admit Einstein metrics.

We also have the moduli space theory by Koiso and other people [4]. Yet, in a sense, a number of basic questions have not been answered, for instance are there many manifolds admitting an Einstein metric?

The technical difficulty involved in the study of Einstein metrics and the obstruction above are probably telling us that Einstein metrics are too restrictive to cover many topological manifolds. It is a very interesting question whether every manifold has a 'best' metric in a sense.

In this dissertation we are going to study minimal or critical energy metrics of energy functionals on the hope that these may become a better 'language' to understand 4-dimensional manifolds. A number of recent major developments motivated this study. Noting that Einstein metrics are the critical points of the total scalar curvature functional, we define other natural Riemannian functionals and study their critical points. Consider two functionals;

$\mathcal{R}(g) = \int_{\mathrm{M}} |R_g|^2 dvol_g$ and $\mathcal{W}(g) = \int_{\mathrm{M}} |W_g|^2 dvol_g$ on $M$.

Then Einstein metrics again are critical points of each functional. We will see a strong tie of these functionals with the topology of compact oriented 4-manifolds in the next section. Then the critical points or, more strongly, the extremal metrics of these functionals are interesting, because they minimize natural energies. Large classes of known interesting riemannian metrics are the critical points of the functionals. More interestingly, we may hope that there exists a critical or extremal metric on *every* manifold. There arise naturally *(anti)-self-dual* metrics which give critical or extremal points of the above functionals.

The critical or minimal points of $\mathcal{W}(g) = \int_M |W_g|^2 dvol_g$ is particularly analogous to the Yang-Mills or (anti)-self-dual, respectively, connections in gauge theory.

We would like to describe the developments after a short break in the next section to explain several basic notions.

## 1.2 Basic notions

For a Riemannian manifold $(M, g)$, the curvature tensor $R$ of the Riemannian metric $g$ satisfies $g(R(x,y)z,w) = g(R(z,w)x,y)$ for tangent vectors $x, y, z, w$ at each point in $M$. So $R$ can be considered as an element of $S^2(\Lambda^2 M)$, where $S^2(\cdot)$ is the symmetric tensor product and $\Lambda^2 M$ is the space of 2-forms in $M$ or as a self adjoint endomorphism of $\Lambda^2 M$.

In 4-dimensional oriented Riemannian manifold, the Hodge star operator $*$ on $\Lambda^2 M$ has the property that $*^2 =$ identity and so $\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M$, here $\Lambda^2_+ M$ is the $+1$ eigenspace of $*$ called the space of self-dual 2-forms and

$\Lambda^2_- M$ is the $-1$ eigenspace of $*$ called that of anti-self-dual 2-forms.

For the next description of the curvature tensor, refer to [4]. We may express $R \in S^2(\Lambda^2_+ M \oplus \Lambda^2_- M)$ as a matrix;

$$\begin{pmatrix} \frac{s}{12} + W^+ & Z \\ Z^T & \frac{s}{12} + W^- \end{pmatrix}$$

Also we may write $R$ as follows;

$$R = \frac{1}{12}s \cdot Id + \frac{1}{2}(r - \frac{s}{4}g) \diamond g + W^+ + W^-, \tag{1.1}$$

where s is the scalar curvature, r is the ricci curvature, $Z = r - \frac{s}{4}g$ is the trace free Ricci tensor, $h \diamond k$ is the Kulkarni-Nomizu product of symmetric 2-tensors h and k defined by;

$$(h \diamond k)(x, y, z, t) = h(x, z)k(y, t) + h(y, t)k(x, z) - h(x, t)k(y, z) - h(y, z)k(x, t),$$

and $W^+$ and $W^-$ are called respectively the self-dual and anti-self-dual Weyl curvature tensor.

$W = W^+ + W^-$, called the Weyl curvature tensor of the metric, is the conformally invariant part of the full curvature tensor in a natural sense. In fact, as a (3,1) tensor, $W_1 = W$ for a new conformal metric $g_1 = f^2 g$ and it is true that $W = 0$ if and only if it is a locally conformally flat metric i.e. for each point $p$ of M, there exist a neighborhood $U_p$ and a positive function $f$ on $U_p$such that $f^2 g$ is flat on $U_p$.

**Definition:** A Riemannian metric $g$ on an oriented manifold is called self-dual and anti-self-dual if and only if $W^- = 0$ and $W^+ = 0$ respectively. $g$ is called half conformally flat if it is either self-dual or anti self-dual.

Basic examples are $S^4$ with the standard metric and $\mathbb{CP}^2$ with Fubini-Study metric.

Note that (anti)-self-duality is a conformally invariant property, and that if we switch the orientation of the manifold, then self-dual metric becomes anti-self-dual and vice versa.

In dimension four, $\mathcal{R}(g) = \int_{\mathrm{M}} |R_g|^2 dvol_g$ is a scale invariant energy functional of a metric g in the sense that $\mathcal{R}(g) = \mathcal{R}(cg)$ with $c$ any positive number, and $\mathcal{W}(g) = \int_{\mathrm{M}} |W_g|^2 dvol_g$ is a conformally invariant energy functional in the sense that $\mathcal{W}(g) = \mathcal{W}(fg)$ with $f$ any positive function defined on M. Both of them has at least one strong connection with the topology of a 4-dimensional oriented compact manifold via the Generalized Gauss-Bonnet formula for the Euler characteristic;

$$\chi(M) = \frac{1}{8\pi^2}\int_M (|W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{1}{6}|Z|^2) dvol_g \tag{1.2}$$

and the formula for the signature;

$$\tau(M) = \frac{1}{12\pi^2}\int_M (|W_+|^2 - |W_-|^2) dvol_g = b_+ - b_-, \tag{1.3}$$

where $b_+$ is the dimension of the space of self-dual harmonic 2-forms and $b_-$ is that of anti-self-dual harmonic 2-forms. In the above notation,

$$\mathcal{R}(g) = \int_M (|W_+|^2 + |W_-|^2 + \frac{s^2}{24} + \frac{1}{6}|Z|^2) dvol_g. \tag{1.4}$$

$$\mathcal{W}(g) = \int_{M}(|W_+|^2 + |W_-|^2)dvol_g. \tag{1.5}$$

## 1.3 Critical metrics

Let $\mathcal{F}$ be one of the $\mathcal{R}$ and $\mathcal{W}$ functionals. We may define its gradient at a metric $g$ to be the element grad$\mathcal{F}_g$ in $S^2M$ such that

$$\frac{d}{dt}|_{t=0}\mathcal{F}(g+th) =< \text{grad}\mathcal{F}_g, h >_g$$

for every $h$ in $S^2M$. We can compute the gradient in each functional [4];

$$\text{grad}\mathcal{R}_g = \delta^D\delta W + \frac{s}{6}Z + \overset{\circ}{W} r. \tag{1.6}$$

$$\text{grad}\mathcal{W}_g = -2\delta^D\delta W - 2\overset{\circ}{W} r. \tag{1.7}$$

Here $\delta$ is the divergence of a (4,0) tensor and $\delta^D$ is the formal adjoint of $d^D$ which takes the exterior differential of a symmetric 2-tensor viewed as a one form with values in the tangent bundle,

$$d_g h(x,y,z) = D_x h(y,z) - D_y h(x,z).$$

And $\overset{\circ}{W} r_{ij} = g^{kl}R_{kijl}$. The grad$\mathcal{W}$ is called the Bach tensor. The following is easy to observe.

**Remark 1.3.1:**

i) Einstein metrics, i.e. $Z = 0$, are absolute minima of $\mathcal{R}$ from (1.2) and (1.4).

ii) Half conformally flat and zero scalar curved metrics are also absolute minima of $\mathcal{R}$ from (1.2), (1.3) and (1.4).

iii) Half conformally flat metrics are absolute minima of $\mathcal{W}$ from (1.3) and (1.5).

iv) Next, any metrics locally conformal to an Einstein metric are smooth critical points of the $\mathcal{W}$-functional. In fact, for an Einstein metric, by the differential Bianchi identity

$$\delta W = -\frac{1}{2}d^D(r - \frac{sg}{6}) = 0$$

so we get zero Bach tensor, which is a conformally invariant condition.

(1.2) and (1.3) give the Hitchin-Thorpe inequality $2\chi \geq 3|\tau|$ for Einstein metrics, which is the only known obstruction to the existence of an Einstein metric. (1.3) gives $\tau \geq 0$ for self-dual metrics and $\tau \leq 0$ for anti-self-dual metrics. From (1.1) and (1.2) again, if $s = 0$ and half conformally flat, then $2\chi \leq 3|\tau|$.

In the light of these inequalities, we may ask the following natural questions.

**Quetions:**

**A**. Does every smooth (oriented) 4-manifold admit Einstein or half conformally flat zero scalar curvature metrics?

**B**. Does every smooth (oriented) 4-manifold admit half conformally flat metrics?

**C**. Does every smooth (oriented) 4-manifold admit $\mathcal{R}$-critical metrics?

**D**. Does every smooth (oriented) 4-manifold admit $\mathcal{W}$-critical metrics?

We may find above questions all the more interesting if we compare them to the corresponding ones in Yang-Mills theory. The conformally invariant $\mathcal{W}$-energy critical and extremal metrics are analogous to Yang-Mills connections and instantons respectively.

While the Hitchin-Thorpe inequality is not known to be the necessary and sufficient condition to the existence of Einstein metrics, zero scalar half conformally flat metrics are known to have more restriction [23]. On question **A**, it is known that not every 4-manifold has Einstein or half conformally flat zero scalar curvature metrics.

Concerning Question **B**, not every four manifold admits such metrics. For example, the product of two spheres $S^2 \times S^2$ or the connected sum $\mathbb{P}^2 \sharp \overline{\mathbb{P}^2}$ of the complex projective plane $\mathbb{P}^2$ and the complex projective plane with reverse orientation $\overline{\mathbb{P}^2}$ both have signature zero, so if they admit self-dual or anti-self-dual metrics, then $W = 0$. But by Kuiper theorem which states that any simply connected $W = 0$ compact manifold is conformally equivalent (so diffeomorphic) to sphere, so neither $S^2 \times S^2$ nor $\mathbb{P}^2 \sharp \overline{\mathbb{P}^2}$ admit such absolute minimum $\mathcal{W}$-energy.

Question **C** is very interesting but little is known.

Question **D** is rather a conjecture now. There have been considerable interests on this conjecture recently and a partial positive result has been announced [24].

In the course of approaching the above questions, we have some understanding in the Kähler manifold category.

First, Kähler $\mathcal{W}$-energy extremal metrics are characterized. The following has been proved by Flaherty, for the first time, in physics literature and Gauduchon [13] in mathematics literature, and reproved by others in different arguments [8, 16, 20].

**Theorem 1.3.1** *A Kähler metric on a complex surface has zero scalar curvature if and only if it is anti-self-dual with respect to the complex orientation.*

**Sketch of proof:** For one approach we may refer to the Derdzińsky paper [8] to derive the key formula $W_+ = sA$ for a Kähler metric on a complex surface where $A$ is a paralell tensor. Then the theorem follows. Computation is done in the terminology of the local basis of self-dual and anti-self-dual 2-forms. **q.e.d.**

**Theorem 1.3.2** ([8]) *Any compact self-dual Kähler manifold of real four dimension is locally symmetric.*

**Sketch of proof:** From $W_+ = sA$, $W = W_+ = sA$ is parallel if $s$ is a constant. So $\nabla W = 0$. $ds = 0$ implies $\delta r = 0$. Set a tensor $T_{ij} = r_{ik}\omega_j^k$. Since g is Kähler, $\nabla_l T_{ij} = (\nabla_l r_{iq})\omega_j^q = 0$ We get $\nabla r = 0$. So, $\nabla R = 0$ i.e. $g$ is locally symmetric. If s is not a constant, then $\nabla s$ is a nontrivial holomorphic vector field with zeros on $M$ and gives a strong restriction on the possible compex surface type for $M$. Moreover the signature $\tau > 0$ implies that $M$ is biholomorphic to $\mathbf{P}^2$ and $g$ is Einstein and so scalar curvature is constant. So, the scalar curvature is always a constant. This is a contradiction.

**q.e.d.**

Next, Kähler $\mathcal{W}$-energy critical metrics are characterized.

**Proposition 1.3.1** ([8]) *A Kähler manifold $(M, g, \omega)$ in real dimension four is $\mathcal{W}$-critical if and only if $s^{-2}g$, defined where $s \neq 0$, is Einstein.*

We note that Kähler $\mathcal{R}$-critical metrics can be characterized, based on Derdziński [8] argument.

**Proposition 1.3.2** *A Kähler manifold $(M, g, \omega)$ in real dimension four is $\mathcal{R}$-critical if and only if it is Kähler Einstein or Kähler zero scalar curvature.*

**Proof:** Any Kähler $\mathcal{R}$-critical metric in real dimension four has constant scalar curvature and satisfies the following Euler Lagrange equation of the $\mathcal{R}$ functional;

$$\delta^D \delta W + \frac{s}{6} Z + \overset{\circ}{W} r = 0.$$

Let's assume that $s$ is not zero. By Bianchi identity, $\delta W = -\frac{1}{2} d^D r$. The argument in the proof of Theorem 1.3.2 shows $\nabla r = 0$, so $\delta W = 0$. By writing formula (1) for $W$ in local coordinates, we can compute

$$\begin{aligned} & 6w_{ik}(\delta^D \delta W + \frac{s}{6}Z + \overset{\circ}{W}\, r)_{kj} = 0 \\ = \;& 6w_{ik}(\frac{s}{6}Z + \overset{\circ}{W}\, r)_{kj} \\ = \;& sw_{ik}Z^k_j + w_{ik}(4sr^k_j - 12r^{kp}r_{pj} + (3 \mid r \mid^2 - s^2)\delta^k_j). \end{aligned}$$

By direct matrix computation,

$$4sr^k_j - 12r^{kp}r_{pj} + (3 \mid r \mid^2 - s^2)\delta^k_j = -2s(r - \frac{s}{4}g^k_j = -2sZ).$$

Now we get $0 = -2sZw + swZ = -swZ$. Since s is a nonzero constant, we get $Z = 0$, i.e. g is Einstein.

Converse is a direct consequence of theorem 1.3.1, remark 1.3.1 and the fact that if $s = 0$, $\mathcal{W}$-critical condition is equivalent to $\mathcal{R}$-critical from (5) and (6).

**q.e.d.**

In an effort to generalize above, we may ask ;

**Questions:**

i) Can we classify Hermitian $\mathcal{R}$-critical or $\mathcal{W}$-critical metrics?

ii) Does there exist a half conformally flat scalar curvature zero which is not conformal to Kähler metrics?

## 1.4 Self-dual metrics

Above theorem 1.3.1 and theorem 1.3.2 provide some (anti)-self-dual metrics examples; there exist Calabi-Yau Kähler ricci flat metrics and also Yau classified Kähler scalar curvature zero metrics [37]. C. Boyer [6] generalized these results to hermitian case and C. LeBrun [20] classified the homeomorphism types of simply connected nonnegatively scalar curved self-dual metrics.

So far at this point, we have only a few known restricted classes of examples, so it has been an extremely interesting question to construct more examples, in paticular ones which are not conformal to symmetric spaces and zero-scalar-Kähler metrics.

One interesting result, which motivated many important subsequent results, was established by Y.S. Poon. Before we state this result, we would like to explain the interaction between so called the twistor theory in three dimensional complex manifolds and the self-dual gravity.

**Detour through Twistor Theory [28]:**

Roger Penrose has constructed the twistor space. It is defined to be $Z = S(\Lambda^- M)$, the unit sphere bundle of anti-self-dual two forms on $M$, with a naturally defined almost complex structure. This is a fiber bundle over $M$ with two dimensional sphere as fiber. The almost complex structure can be described as follows. A unit anti-self-dual 2-form $\omega$ at $x \in M$ becomes a complex structure $J_{\omega(x)}$ on the tangent space $T_xM$ via $J_\omega(v) = \sum \omega(v, e_i)e_i$ where $v \in T_xM$ and $e_i$, $i = 1, 2, 3, 4$ is an orthonormal basis of $T_xM$. Let $\pi$

be the projection map $Z \longrightarrow M$. We have the splitting of the tangent bundle $TZ \cong TF \oplus \pi^*TM$ using the Levi-Civita connection on $M$, here $TF$ is the tangent bundle along the fibre. At a point $z = \omega = J_{\omega(x)} \in Z$, define a complex structure on $T_zZ$ by taking $J_x$ on $\pi^*TM$ and the standard complex structure on $TF$. From this description, it is easy to see that the almost complex structure is conformally invariant in the sense that $(M, g)$ and $(M, f^2g)$ with $f$ a positive function both give rise to an isomorphic almost complex structure.

Then, a fundamental theorem by Penrose and Atiyah-Hitchin-Singer [3] states that for an oriented Riemannian manifold $(M, g)$, the almost complex structure on the associated twistor space $Z_M$ is integrable if and only if $(M, g)$ is self-dual, i.e. $W_- = 0$. Moreover the Penrose Transform further gives one-to-one correspondence between holomorphic objects on $Z$ and the solutions of interesting field equations on $M$. So the study of self-dual geometry can be translated into the terms of 3-dimensional complex geometry.

One application of this theory is that holomorphic deformations of twistor spaces may produce self-dual spaces.

**Notation:** For a natural number $n$ and a manifold $M$, $nM$ would mean the connected sum $M \sharp M ... \sharp M$ of $n$ copies of $M$.

$\mathbb{P}^2$ with the Fubini-Study metric is, in a sense, the simplest closed manifold with a nonflat self-dual metric which is not conformally flat. Naturally we can ask whether $2\mathbb{P}^2$ can have self-dual metrics. This has been the first step of the program to construct new self-dual metrics. Using the twistor correspondence,

**Theorem 1.4.1 (Poon [29])** *There exist one dimensional family of self-dual conformal metrics on $\mathbb{P}^2 \sharp \mathbb{P}^2$. They admit positive scalar curvaturse.*

By R. Schoen [30], each conformal structure on a compact manifold has a constant scalar curvature with unique sign. Poon's result naturally led to the question on $n\mathbb{P}^2$. Again using holomorphic deformations,

**Theorem 1.4.2 (S. Donaldson, R. Friedman [9])** *There exist self-dual conformal metrics on $n\mathbb{P}^2$ for every natural number $n$ and $N\overline{K_3} \sharp n\mathbb{P}^2$ for $N \geq 0$ and $n \geq 2N+1$.*

Independently of above, using a completely different argument of nonlinear elliptic PDE estimates, Andre Floer proved;

**Theorem 1.4.3 (A. Floer [11])** *There exist self-dual conformal metrics on $n\mathbb{P}^2$ for every natural number $n$.*

Floer and Donaldson theorems show the existence of half conformally flat metrics in interesting ways, but it's hard, by its nature, to read out how the Riemannian metrics behave. If we hope to analyze and classify the $\mathcal{W}$-energy extrema or $\mathcal{W}$-critical metrics in the Riemannian geometry context, we need more detailed information on the metrics. As we get more and more constructions of self-dual metrics, it has become an interesting question to classify them in certain ways. For instance, some prototype questions might be as follows;

1). Classify self-dual or $\mathcal{W}$-critical metrics with positive or nonnegative sectional curvature.

2). Classify self-dual or $\mathcal{W}$-critical metrics with positive or nonnegative ricci curvature.

3). Classify self-dual or $\mathcal{W}$-critical metrics with positive or nonnegative scalar curvature.

4). Does there exist a negative scalar curved self-dual metric on $\mathbb{P}^2$?

Self-dual metrics are conformally best metrics in some sense, so it'll be interesting to find any topological finiteness or other topological implications. We have partial results on ricci curvature and scalar curvature cases;

**Theorem 1.4.4 (LeBrun)** *Let $M$ be a connected simply connected compact self-dual 4-manifold with nonnegative scalar curvature. Then $M$ is one of the following*

*i) conformally isometric to $S^4$*

*ii) conformally isometric to a ricci-flat K3 surface with reverse orientation.*

*iii) homeomorphic to $m\mathbb{P}^2$ for $m > 0$*

*iv) diffeomorphic to $m\mathbb{P}^2 \sharp \mathbb{P}^2$ for $m > 9$*

Let $\lambda_1 \leq ... \leq \lambda_4$ be the eigenvalues of the ricci tensor. In the statement of next theorem, the pinching condition below is equivalent to the nonnegativity of the so called *ricci operator* [12]. Here nonnegative ricci operator implies nonnegative ricci curvature.

**Theorem 1.4.5 (Gauduchon [13])** *Let $(M, g)$ be a compact self-dual 4-manifold of positive scalar curvature with the following pinching condition*

$$\lambda_3 + \lambda_4 \leq 2(\lambda_1 + \lambda_2).$$

*then,*

*either i) $(M, g)$ is locally isometric to the product $S^1 \times S^3$,*

*or ii) $M$ is simply connected, homeomorphic to $m\mathbb{P}^2$ with $0 \leq m \leq 3$.*

It is necessary to have much information involving the curvature of self-dual metrics. In this respect, the next result is particularly interesting.

**Theorem 1.4.6 (Le Brun [21, 22])** *There exist explicit* hyperbolic ansatz *self-dual conformal metrics on $n\mathbb{P}^2$, $l(S^3 \times S^1)\sharp n\mathbb{P}^2$ for every natural number $l$ and $n$ , and $S^2 \times S_g \sharp n\mathbb{P}^2$ for natural number $n \geq 2$.*

In the above, hyperbolic ansatz means the hyprebolic analogues of the Gibbons-Hawking metrics. The metric is described as $g = f^2(Vh + V^{-1}w^2)$ where h is a hyperbolic metric on (possibly incomplete) manifold N, $V$ a positive function on $N$, $w$ a connection on a $S^1$ bundle over $N$.

One of the main works in this dissertation is concerned with constructing such hyperbolic ansatz metrics, generalizing theorem 1.4.8. That can be described as below.

R. Schoen and S.T. Yau studied a large class of conformally flat manifolds of general dimension, whose developing maps are injective (conformal immersion). For example, any complete conformally flat manifold with nonnegative scalar curvature has an injective developing map. These manifolds are quotients of domains in $S^n$ by Kleinian groups.

Meanwhile, there is an explicit description of 4-dimensional conformally flat manifolds with $S^1$ conformal action on quotients of domains in $S^4$ by Kleinian groups by Peter Braam; these conformally flat manifolds are the *conformal* compactification $\mathbf{X}$ of $S^1 \times N$, where $N$ is a noncompact hyperbolic 3-manifold. For details, refer to section 3.2.

Note that by the Thurston Uniformization Theorem, *most* irreducible homotopically atoroidal compact 3-manifolds $\overline{N}$ with nonempty boundary $\partial N$ have such hyperbolic structures in the interior. This means that the conformally compactified manifolds constitute a large class of manifolds.

In the following we again let $X$ be the conformal compactification of $S^1 \times N$, such that $\Gamma$ is a no cusp, geometrically finite Kleinian group and $N = \mathcal{H}^3/\Gamma$ is a noncompact hyperbolic 3-manifold, which is topologically the interior of a compact 3-manifold $\overline{N}$ with $\partial N$ being the union of compact riemann surfaces (let's call these *boundary surfaces*).

Then we can show;

**Theorem 1.4.7** *There exist explicit hyperbolic ansatz self dual metrics on $X \sharp m\mathbb{P}^2$ for all sufficiently large natural number $m$, if there is at most one torsion free element which cannot be generated by boundary surfaces in $H_2(N, Z)$.*

The sign of scalar curvature has been a useful tool to study the conformally flat metrics [32], which is related to the Hausdorff dimension (to be defined in section 2.1) of a natural geometrical set. In our case we can get such relation by exploiting the explicitness of the metric and also computing the curvature tensor; the relation between scalar curvature on constructed manifolds above and Hausdorff dimension of the limit set of the Kleinian group is as follows:

**Theorem 1.4.8** *Suppose that* $\Gamma$ *is any group as in Theorem 1.4.7. If the sign of 1 - Hausdorff dimension*($\Lambda(\Gamma)$) *is* $+$, *0 or* $-$,

*then there exists a representative metric of scalar curvature positive, zero or negative respectively, in the self dual conformal structure on* $X\sharp m\mathbb{P}^2$, *constructed in theorem 1.4.7.*

Theorem 1.4.7 and 1.4.8 together construct a number of explicit self-dual metrics and describe their scalar curvatures. This construction exhausts all self-dual metrics of nonnegative scalar curvature with semi-free circle action. A Semi-free action means its isotropy group is either the whole group or the identity subgroup.

There has been a conjecture [18] that if the signature $\tau(X) \neq 0$, for $X$ a self-dual manifold, then the Yamabe invariant doesn't change sign on smooth points of any connected component of *the moduli space of self-dual conformal structures with trivial conformal isometry group.*

However, from theorem 1.4.8, we can prove the following;

**Theorem 1.4.9** *There exists a continuous family of self-dual metrics on a connected component of the moduli space of self-dual conformal structures on* $l(S^3 \times S^1)\sharp m\mathbb{P}^2$ *for some* $l$ *and arbitrary* $m \geq 1$*, but which change the sign of the scalar curvatures.*

Klein-Maskit combination theorem of Kleinian group theory is a useful tool to get specific examples. We explain and discuss examples in section 4.1.

## 1.5 More recent developments and questions

For the rest of this chapter, we would like to describe other developments and also discuss problems.

Motivated by above existence theorems of Poon-Donaldson-Friedman-Floer and also by the Taubes existence theorem of self-dual connections in Yang-Mills theory [33], there has been an existence question of (anti)-self-dual metrics in general context.

In fact, the following has been a conjecture for a while.

**Theorem 1.5.1 (Taubes [34])** *For any smooth, compact, oriented 4-manifold* $M$*,* $M\sharp n\mathbb{P}^2$ *for sufficiently large* $n$*, admits a self-dual metric.*

It was solved by deep elliptic estimates generalizing Floer techinques.

**Remark:** This has a corollary that every finitely presentable group is the fundamental group of a complex 3-manifold by the twistor theory.

The following theorems give sufficient conditions for a self-dual manifold to arise by the hyperbolic ansatz.

**Theorem 1.5.2 (LeBrun [23])** *Let $(M, g)$ be a compact half conformally flat 4-manifold with a semi-free conformal $S^1$ action. Suppose one of the following holds;*

*i) $M$ has positive definite intersection form.*

*ii) $(M, g)$ is not conformally flat and has non-negative scalar curvature.*

*Then, (M,g) arises via the hyperbolic ansatz described above, with a hyperbolic manifold $N$ and a finite collection of points $q_1, ...., q_n$ in $N$.*

This theorem covers only a part of hyperbolic ansatz constructions; e.g. the metrics on $2(S^2 \times S_g)\sharp 3\mathbb{P}^2$ do not belong to i) or ii) of the above theorem.

**Open Problems:**

There are interesting questions in the field. Let's list a few;

**Question 1)** Can we find a topological invariant $m_0$ such that $M\sharp m\mathbf{P}^2$ for any $m \geq m_0$ admits self-dual metrics?

**Question 2)** Does there exist positive ricci curved self-dual metrics on $2\mathbb{P}^2$? The author conjectures this is true.

If yes, how about on $m\mathbb{P}^2$ for any $m > 0$? Confer to theorem 1.4.7.

D.G. Yang and J.P. Sha [36] constructed positive Ricci metrics on $m\mathbb{P}^2$ for any $m > 0$. Question 2) tests whether self-dual conformal structures can hold as much positivity of ricci curvature as any other conformal structures.

**Question 3)** Construct $\mathcal{R}$-critical and $\mathcal{W}$-critical metrics and classify them.

**Question 4)** Comparing with a convergence of Einstein metric case [1], we can ask the following; if we have a sequence of self-dual metrics $g_i$ of constant scalar curvature under some bound on geometries, e.g. volume or scalar curvature on $M$, does a subsequence converge in some category?

In the rest of the dissertation, we give the proofs of the theorem 1.4.7, 1.4.8 and 1.4.9, and give more details of examples.

When we prove theorem 1.4.7, we have to face the problem of constructing nontrivial $S^1$ bundles over $\overline{N} - \{a\ finite\ number\ of\ points\}$.

This reduces the main problem to veryfing the integrality of a certain second cohomology class, which is done in Chapter 2.

Chapter 3 describes how to construct the manidfolds and metrics.

Chapter 4 discusses the scalar curvature of the self-dual metrics and proves theorem 1.4.8 and 1.4.9.

# Chapter 2

# Hyperbolic Geometry

## 2.1 Hyperbolic Geometry

We will denote $\Gamma$ to be a discrete subgroup of the isometry group $\mathbf{PSL(2,\mathbb{C})} = \mathbf{SL(2,\mathbb{C})}/\{\pm 1\}$ of the hyperbolic space $\mathcal{H}^3$. There are two standard models for $\mathcal{H}^3$.

The first one is the unit ball $\mathbb{B}^3$ centered at the origin in $\mathbb{R}^3$ equipped with the metric $g = \frac{4}{(1-||x||^2)^2}(dx_1^2 + dx_2^2 + dx_3^2)$ with $||x||^2 = x_1^2 + x_2^2 + x_3^2$.

The second one is the upper half space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3)|x_3 > 0\}$ in $\mathbb{R}^3$ with the metric $g = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)$.

We are going to to explain several terms now. Readers may refer to [26, 31] as good references. A discrete subgroup $\Gamma$ of $\mathbf{PSL(2,\mathbb{C})}$ extends to act on $\partial\mathcal{H}^3 \approx S^2$ by conformal transformations. The *limit set* of $\Gamma$, $\Lambda(\Gamma)$, is the set of all points $x \in \partial\mathcal{H}^3 = S^2$ such that there exist a sequence $\{r_j\} \subset \Gamma$ and a point $y \in \mathcal{H}^3$ with $r_j \cdot y \longrightarrow x$. The *region of discontinuity* $\Omega(\Gamma)$ is

$S^2 - \Lambda(\Gamma)$, where $\Gamma$ acts properly discontinuously. $\Gamma$ is *Kleinian* if $\Gamma$ acts properly discontinuously on a nonempty open set $\Omega \subset S^2$. $\Gamma$ is *geometrically finite* if there is a finitely sided fundamental polyhedron for $\Gamma$ action on $\mathcal{H}^3$. $\Gamma$ has a *cusp* if there is a *parabolic* element i.e. an element having only one fixed point in $S^2$. A *loxodromic* element is an element having exactly two fixed points in $\mathcal{H}^3 \cup S^2$. The *convex hull* of a set $\Lambda$ in $\partial\mathcal{H}^3 \approx S^2$ is the intersection of all the closed half spaces in $\mathcal{H}^3$ whose boundary in $S^2$ contains $\Lambda$. $\Gamma$ is *convex cocompact* if the action of $\Gamma$ on the convex hull $C(\Lambda(\Gamma))$ has a compact fundamental domain.

We will be mainly interested in hyperbolic quotient manifolds. The following description of noncompact hyperbolic manifolds is due to [7, 27]. Let $\overline{N}$ be an oriented, irreducible, homotopically atoroidal, compact 3-manifold with its interior $N$ and nonempty boundary $\partial N$. Homotopically atoroidal means that every map from $T^2$ to $\overline{N}$ has a nontrivial kernel on the level of fundamental groups. For simplicity, to avoid cusps, assume that either $\partial N$ has no torus component or $\overline{N} = \overline{D^2} \times S^1$. Then by the Thurston Uniformization Theorem, $N = \overline{N} - \partial N$ has a complete, geometrically finite, hyperbolic structure. This means that $N$ can be realized as a quotient of $\mathcal{H}^3$ by a purely loxodromic, geometrically finite, Kleinian group $\Gamma$ without cusp. Conversely, if $\Gamma$ is such group, then $N = \mathcal{H}^3/\Gamma$ is the interior of a compact, smooth manifold $\overline{N} = (\overline{\mathcal{H}}^3 - \Lambda(\Gamma))/\Gamma$ which has its boundary $\partial N = \Omega(\Gamma)/\Gamma$ being the union of a finite number of compact Riemann surfaces without boundary.

Also, we would like to introduce some results on Hausdorff dimension of the limit set of hyperbolic actions.

For an infinite discrete group $\Gamma$ of hyperbolic motions in $\mathcal{H}^3$, a positive real number s and $x, y \in \mathcal{H}^3$, define

$$g_s(x,y) = \sum_{\alpha\in\Gamma} \frac{1}{e^{s\rho(x,\alpha y)}},$$

here $\rho(x, \alpha y)$ is the hyperbolic distance between x and $\alpha y$.

Let $s_k$ be the number of orbit points in $\overline{B_x(k+\frac{1}{2})}/\overline{B_x(k-\frac{1}{2})}$ $k \geq 1$, where $B_x(r)$ is the ball of radius r with center at x. Then if we define

$$\delta = \overline{\lim_{k\to\infty}} \frac{1}{k} \log s_k,$$

the above series $g_s$ converges for $s > \delta$ and diverges for $s < \delta$. We can see easily that $\delta$ depends only on $\Gamma$. This $\delta(\Gamma)$ is called the critical exponent of the group $\Gamma$.

For a set $X$ in a metric space, define the $r^\delta$-Hausdorf measure of **X** to be

$$H_\delta(x) = \liminf_{\epsilon\to 0} \inf_{\mathcal{C}} \sum_{|r_i|<\epsilon} r_i^\delta$$

where infimum is over all coverings $\mathcal{C}$ of $X$ by countably many small balls of radii $r_i < \epsilon$.

The Hausdorff dimension of $X$, denoted by $D(X)$, is defined to be the number $\delta_0$ such that $H_\delta(X) = \infty$ for $\delta < \delta_0$ and $H_\delta(X) = 0$ for $\delta > \delta_0$. Hausdorff dimension is a natural geometric criterion to analyze the discrete groups.

Then, we have the theorems of D. Sullivan [31];

**Theorem 2.1.1** *For a convex cocompact group* $\Gamma$,

$$D(\Lambda(\Gamma)) = \delta(\Gamma).$$

**Theorem 2.1.2** *For a convex cocompact group, there are constants* $c$ *and* $C$ *so that* $n_r$*, the number of orbit points in the ball of radius* $r$*, satisfies* $ce^{\delta r} \leq n_r \leq Ce^{\delta r}$.

This estimate will be essential in next sections. For definition of groups below, confer [26].

**Theorem 2.1.3** *For Kleinian convex cocompact groups,* $1 < D(\Lambda(\Gamma)) < 2$ *except that* $\Gamma$ *is Schottky, Fuchsian (or extended Fuchsian).*

**Remark 2.1.1**: The convex cocompact condition is equivalent to the condition that the fundamental domain has finitely many sides and doesn't meet the limit set. Therefore any Kleinian, geometrically finite, without cusp group is convex cocompact. Note that for Schottky group, $0 \leq D < 2$ [5] and for (extended) Fuchsian group, $0 \leq D \leq 1$.

## 2.2 Green's Functions on Hyperbolic 3-space

In this section we are concerned with establishing a Green's funtion on a hyperbolic manifold. We will see that Green's functions can be constructed in rather general context, due to D. Sullivan's work on discrete hyperbolic motions.

Let $p_0$ be a fixed point in $\mathcal{H}^3$ and $\Gamma p_0$ be the orbit of $p_0$ with respect to $\Gamma$ action. Note that $G_\alpha(p) = \frac{1}{e^{2\rho_\alpha(p)}-1}$ with $\rho_\alpha(p) = distance(\alpha p_0, p)$ satisfies

$$\Delta_{\mathcal{H}^3} G_\alpha(p) = -\frac{1}{2}\delta_{\alpha p_0}$$

i.e. $G_\alpha$ is a positive Green function. Then,

**Lemma 2.2.1** *Let $\Gamma$ be a Kleinian , geometrically finite group without cusp. Then*

$$\mathbf{V}(p) = 1 + \sum_{\alpha\in\Gamma} G_\alpha(p)$$

*is a smooth function on $\mathcal{H}^3 - \{\Gamma p_0\}$.*

**Proof of Lemma 2.2.1**: First we want to show that $\mathbf{V} - 1$ is a continuous function. For a point $p \in \mathcal{H}^3 - \{\Gamma p_0\}$, set $B_p^\epsilon$ be the open ball of radius $\epsilon$ (may be assumed to be small) centered at p in $\mathcal{H}^3 - \{\Gamma p_0\}$ with its closure also in $\mathcal{H}^3 - \{\Gamma p_0\}$. By Remark 2.1.1, $\Gamma$ is convex cocompact and so by the theorems of section 2.1, the critical exponent satisfies that $\delta(\Gamma) = D(\Lambda(\Gamma)) < 2$. Let S be the finite set $\{\alpha \in \Gamma \mid dist_{\mathcal{H}^3}(\alpha p_0, y) < 1 \text{ for some } y \in B_p^\epsilon\}$. Then on $B_p^\epsilon$,

$$\sum_{\alpha\in\Gamma} G_\alpha \leq \sum_{\alpha\in S} \frac{1}{e^{2\rho_\alpha(p)}-1} + \sum_{\alpha\in\Gamma-S} \frac{2}{e^{2\rho_\alpha}} \leq \sum_{\alpha\in S} \frac{1}{e^{2\rho_\alpha(p)}-1} + \sum_{r=2}^{\infty} \frac{2n_{r+1}}{e^{2(r-1)}}$$

$$\leq \sum_{\alpha\in S} \frac{1}{e^{2\rho_\alpha(p)}-1} + \sum_{r=2}^{\infty} \frac{2Ce^{\delta(\Gamma)(r+1)}}{e^{2(r-1)}},$$

here $n_{r+1}$ is the number of orbit points of $p_0$ in $B_p^{r+1}$. RHS uniformly converges on $B_p^\epsilon$, so $\sum_{\alpha\in\Gamma} G_\alpha$ is a continuous function on $B_p^\epsilon$, and so on $\mathcal{H}^3 - \{\Gamma p_0\}$. Now $\mathbf{V}$ is a smooth function on $\mathcal{H}^3 - \{\Gamma p_0\}$ by elliptic regularity. Furthermore it is not hard to show that $\mathbf{V} - 1$ is a distribution on $\mathcal{H}^3$ i.e. is a continuous linear functional: $\mathcal{D} \to \mathbb{R}$ where $\mathcal{D}$ is the space of smooth functions with compact support. Then $\Delta(\mathbf{V} - 1) = -2\pi \sum_\alpha \delta_\alpha$. **q.e.d.**

Now we are going to show that this smooth function extends smoothly to the boundary. Let's take the upper half space model for the hyperbolic space, i.e. $\mathbb{R}^3_+ = \{(x,y,z) \mid z > 0\}$ with the metric $h = \frac{dx^2+dy^2+dz^2}{z^2}$ and consider $\mathbf{V}$ as a function defined on $\mathbb{R}^3_+ - \{\Gamma p_0\}$. Here $\partial\mathcal{H}^3 \approx \{z = 0\} \cup \{\infty\}$. $\Lambda(\Gamma)$ is empty only if $\Gamma$ is the trivial group $< e >$. So we may assume that $\Lambda(\Gamma)$ is nonempty and that, by rotation, $\infty \in \Lambda(\Gamma)$.

**Proposition 2.2.1** $\mathbf{V}$ *is smooth up to the boundary* $\{z = 0\} - \Lambda(\Gamma)$ *i.e.* $\partial\mathcal{H}^3 - \Lambda(\Gamma)$.

Let's prove the following technical lemma first:

**Lemma 2.2.2** *Let* $\Gamma$ *be as in Lemma 2.2.1.*

*Fix a point* $p_0$ *in* $\mathcal{H}^3$. *For small* $\epsilon$, *let* $\mathcal{B}_\epsilon$ *be an Euclidean closed half ball of radius* $\epsilon$ *in* $\mathcal{H}^3 \cup \{\partial\mathcal{H}^3 - \Lambda(\Gamma)\}$, *centered at a fixed point* $(a, b, 0) \in \partial\mathcal{H}^3 - \Lambda(\Gamma)$ *such that* $\mathcal{B}_{10\epsilon} \cap \Lambda(\Gamma) = \phi$. *We may assume that* $\mathcal{B}_{10\epsilon}$ *doesn't contain any point in the orbit* $\Gamma p_0$. *Set* $x_\alpha = dist_{\mathcal{H}^3}(\alpha p_0, (a, b, 1))$. *Then ,*

$$\ln(e^{\rho_\alpha} \cdot z) - (\frac{n}{n+1})x_\alpha \geq C(n, \epsilon) \tag{2.1}$$

*on $\mathcal{B}_\epsilon$ for any element $\alpha$ of $\Gamma$. Here $n$ is any natural number such that $1 > \frac{n}{n+1} > \frac{D(\Lambda(\Gamma))}{2}$ and $C(n,\epsilon)$ is a real number depending only on $n$ and $\epsilon$.*

**Proof of Lemma 2.2.2:** Set $\alpha p_0 = (a_\alpha, b_\alpha, c_\alpha)$. Recall that the hyperbolic distance from $\alpha p_0$ to $(x,y,z)$ is given by;

$$\begin{aligned}\rho_\alpha &= dist_{\mathcal{H}_3}(\alpha p_0,(x,y,z)) \\ &= \cosh^{-1}[\frac{(x-a_\alpha)^2+(y-b_\alpha)^2+z^2+c_\alpha^2}{2zc_\alpha}].\end{aligned}$$

For convenience, set

$$r_\alpha((x,y,z)) = (x-a_\alpha)^2+(y-b_\alpha)^2+z^2+c_\alpha^2.$$

Then we have

$$\frac{r_\alpha}{2zc_\alpha} \le e^{\rho_\alpha} \le \frac{r_\alpha}{zc_\alpha}$$

everywhere for every $\alpha \in \Gamma$. Now for $(x,y,z) \in \mathcal{B}_\epsilon$, and setting $A = \ln(e^{\rho_\alpha}\cdot z) - \frac{n}{n+1}x_\alpha$,

$$\begin{aligned}A &\ge \ln\frac{r_\alpha}{c_\alpha} - \frac{n}{n+1}x_\alpha - \ln 2 \\ &\ge \ln\frac{r_\alpha}{c_\alpha} - \frac{n}{n+1}\{dist_{\mathcal{H}^3}(\alpha p_0,(x,y,1)) + dist_{\mathcal{H}^3}((a,b,1),(x,y,1))\} \\ &\quad -\ln 2 \\ &\ge \ln\frac{r_\alpha}{c_\alpha} - \frac{n}{n+1}\ln\{\frac{(x-a_\alpha)^2+(y-b_\alpha)^2+1+c_\alpha^2}{c_\alpha}\} \\ &\quad -\frac{n}{n+1}\cosh^{-1}(\frac{1+\epsilon^2}{2}) - \ln 2 \\ &\ge \frac{1}{n+1}\ln[\frac{B_\alpha^{n+1}}{(B_\alpha+1)^n c_\alpha}] - \frac{n}{n+1}\cosh^{-1}(\frac{1+\epsilon^2}{2}) - \ln 2.\end{aligned}$$

In the above,

$$B_\alpha = (x-a_\alpha)^2+(y-b_\alpha)^2+c_\alpha^2,$$

which is the square of the Euclidean distance from (x,y,0) to $\alpha p_0$. $B_\alpha \geq \frac{1}{M}$ on $\mathcal{B}_\epsilon$ for some positive large number M independent of $\alpha$ by assumption. So,

$$\frac{B_\alpha^{n+1}}{(B_\alpha+1)^n c_\alpha} \geq \frac{1}{(M+1)^n} \cdot \frac{1}{\sqrt{M}}.$$

Now

$$\begin{aligned} A &\geq \frac{1}{(n+1)} \ln\{\frac{1}{(M+1)^n\sqrt{M}}\} - \frac{n}{n+1}\cosh^{-1}(\frac{2+\epsilon^2}{2}) - \ln 2 \\ &= C \end{aligned}$$

on $\mathcal{B}_\epsilon$ for all $\alpha \in \Gamma$.

**q.e.d.**

**Proof of Proposition 2.2.1:** First exponentiating (2.1) and then squaring and taking reciprocals, we get

$$\sum_{\alpha\in\Gamma} \frac{1}{z^2 e^{2\rho_\alpha}} \leq \sum_{\alpha\in\Gamma} \frac{1}{e^{2C} e^{2(\frac{n}{n+1})x_\alpha}},$$

on $\mathcal{B}_\epsilon$ centered at $(a, b, 0)$.

For convenience, set $\acute{\delta} = \frac{n}{n+1}$. We introduce $m_r$ to be the number of orbit points in the ball of radius r, centered at $(a, b, 1)$. Then by theorem 2.1.2,

$$\sum_{\alpha\in\Gamma} \frac{1}{e^{2\acute{\delta} x_\alpha}} \leq \sum_{r=2} \frac{m_r}{e^{2\acute{\delta}(r-1)}} + \sum_{\alpha\in\Gamma_0} \frac{1}{e^{2\acute{\delta} x_\alpha}} \leq \sum_{r=2} \frac{C e^{\delta r}}{e^{2\acute{\delta}(r-1)}} + \sum_{\alpha\in\Gamma_0} \frac{1}{e^{2\acute{\delta} x_\alpha}}$$

where $\Gamma_0$ is a finite set; $\Gamma = \{\alpha \in \Gamma \mid x_\alpha = dist_{\mathcal{H}^3}(\alpha p_0, (a, b, 1)) \leq 1\}$. Since $2\acute{\delta} = 2\frac{n}{n+1} > \delta = D(\Lambda(\Gamma))$, RHS converges for any such n.

So this implies that $\frac{1}{z^2}\sum_{\alpha\in\Gamma}\frac{1}{e^{2\rho_\alpha}-1}$ also uniformly converges on $\mathcal{B}_\epsilon$. By this local uniform convergence, $\frac{1}{z^2}\sum_{\alpha\in\Gamma}\frac{1}{e^{2\rho_\alpha}-1}$ is a continuous function , say $f$, on $\mathcal{H}^3 \cup \{\partial\mathcal{H}^3 - \Lambda(\Gamma)\}$. Now

$$\mathbf{V} = 1 + z^2 f = 1 + \sum_{\alpha\in\Gamma} \frac{1}{e^{2\rho_\alpha} - 1}$$

satisfies $\Delta\mathbf{V} = 0$ and identically equal to 1 along $\{z = 0\} - \Lambda(\Gamma)$. So by elliptic regularity for degenerate ellitic differential operators of characteristic type [15, 22], $\mathbf{V}$ and $f$ is smooth up to $\{z = 0\} - \Lambda(\Gamma)$. This finishes the proof of Proposition 2.2.1.

## 2.3 Construction of principal circle bundles

Suppose $\Gamma$ is a geometrically finite, Kleinian group without cusp such that $\mathcal{H}^3/\Gamma$ is a noncompact hyperbolic manifold. Set $N_0 = \mathcal{H}^3/\Gamma$, then as described in the beginning of section 2.1, $N_0$ is the topological interior of a compact smooth manifold $\overline{N_0}$ with the boundary $\partial N_0$ consisting of a finite number of closed Riemann surfaces.

Let's pick out a finite number of points $p_1, p_2, ..., p_l$ from $\mathcal{H}^3$ such that $\pi(p_i)$ are all distinct for the projection $\pi : \mathcal{H}^3 \to \mathcal{H}^3/\Gamma$. We want to construct a $S^1$ principal fiber bundle over $N = (\mathcal{H}^3 - \{\Gamma p_j\})/\Gamma$. Let

$$\mathbf{V} = 1 + \sum_{j=1}^{l} G_{\pi(p_j)} = 1 + \sum_{j=1}^{l} \sum_{\alpha\in\Gamma} G_{\alpha p_j}$$

on M, where $G_p = \frac{1}{e^{2\rho_\alpha}-1}$ is the Green's function in section 2.2. $\mathbf{V}$ is a smooth function on $N$ by Lemma 2.2.1. We are going to show that the $\frac{1}{2\pi} * d\mathbf{V}$ is an integral class as an element of $H^2(N, \mathbb{R})$ so that there exists a $S^1$ principal bundle over $N$ with connection $\omega$ such that $d\omega = \frac{1}{2\pi} * d\mathbf{V}$ by Chern-Weyl theorem. Let's restrict on $\Gamma$ such that $N_0 = \mathcal{H}^3/\Gamma$ has either only torsion elements or those elements generated by boundary surfaces in $H_2(N_0, \mathbf{Z}) \approx$

$H_2(\overline{N_0}, \mathbf{Z})$. Here a boundary surface means an element of $H_2(N_0, Z)$ isotopic to a compact Riemannian surface in $\partial\overline{N_0}$.

We will prove the integerality of the 2-form above for such class of hyperbolic 3-manifolds. After this is done, we will extend the integrality to the case where there *is* a torsion free element which can't be generated by boundary surfaces.

**Proposition 2.3.1** *Assume that $H_2(N_0, Z)$ has either only torsion elements or those elements generated by boundary surfaces. If $N_0$ has k boundary surfaces, then for some choice of l points with $l \geq k$, $p_1, p_2, ..., p_l$ in $\mathcal{H}^3$, $[\frac{1}{2\pi} * d\mathbf{V}]$ is an integral class in $H^2(N, \mathbf{R})$.*

We are going to split the proof of proposition 2.3.1 into two lemmas below. What we have to show is that for a nontrivial homology class $[S] \in H_2(N, \mathbf{Z})$ with its representative compact surface S,

$$\int_S \frac{1}{2\pi} * d\mathbf{V} \in \mathbf{Z}.$$

If $[S]$ is a torsion element, then there is nothing to do. If $[S]$ is represented by small sphere centered at some $p_i$, then we will see easily below that the integration value is an integer. Now assume that S is homologous to a boundary surface $\tilde{S} \subset \partial N$. By tubular neigborhood argument we have a diffeomorphism between a neighborhood $U$ of $\tilde{S}$ and $\tilde{S} \times [0, 1]$, with $S = \tilde{S} \times \{1\}$ and $\tilde{S} = \tilde{S} \times \{0\}$

In section 2.2, we proved that for $p_j \in \mathcal{H}^3$, $G_{\pi_{p_j}}$ is smooth up to the boundary with $G_{\pi_{p_j}} = z^2 f_{p_j}$ with $f_{p_j}$ smooth. Also easily we can see, by writing down terms, that $*dG_{\pi_{p_j}} = *d(z^2 f_{p_j})$ can be extended to be a smooth 2-form up to the boundary. So,

$$\begin{aligned}\int_S \frac{1}{2\pi} * d\mathbf{V} &= \sum_j \int_{\tilde{S}} \frac{1}{2\pi} * dG_{\pi_{p_j}} = \sum_j \int_{\tilde{S}} \frac{1}{2\pi} * d(\sum_{\alpha\in\Gamma} G_{\alpha p_j}) \\ &= \sum_j \sum_{\alpha\in\Gamma} \frac{1}{2\pi} \int_{\tilde{S}} *dG_{\alpha p_j} = \sum_j \frac{1}{2\pi} \sum_{\alpha\in\Gamma} \int_{\tilde{S}\circ\alpha^{-1}} *dG_{p_j} \\ &= \sum_j \frac{1}{2\pi} \int_{\cup_{\alpha\in\Gamma}\tilde{S}\circ\alpha^{-1}} *dG_{p_j}.\end{aligned}$$

Let's consider $l = k$ case first. Let $S_1, S_2, ..., S_k$ be the k boundary surfaces. Then $\partial\mathcal{H}^3 - \Lambda(\Gamma)$ is partitioned into k open sets $\Omega_1, \Omega_2, ..., \Omega_k$ with $S_i = \Omega_i/\Gamma$.

We define a map $\Phi : (\mathcal{H}^3)^k \longrightarrow \mathbb{R}^k$ by

$$\Phi(p_1, p_2, .., p_k) = (x_1, x_2, .., x_k)$$

with $x_i = \sum_{j=1}^k \frac{1}{2\pi} \int_{\Omega_i} *dG_{p_j}$.

Note that

i) Under the assumption on $\Gamma$, Lebesgue measure of $\Lambda(\Gamma)$ in $S^2 \approx \partial\mathcal{H}^3$ is zero [26, 31].

ii) For $\Omega \subset \partial\mathcal{H}^3 - \Lambda(\Gamma)$, $\int_\Omega *dG_p$ is equal to $(\frac{-1}{2})\cdot$ {Lebesgue measure (with respect to the unit tangent sphere at p) of the set of oriented hyperbolic geodesics through p which hit $\Omega$ at infinity}, cf [22].

iii) There is a consevation property from i) and ii) that for each p in $\mathcal{H}^3$,

$$\sum_{i=1}^k \frac{1}{2\pi} \int_{\Omega_i} *dG_p = -1.$$

These ii) and iii) will be exploited fully later.

$\Phi$ actually maps into a $(k-1)$ dimensional simplex

$$\Delta^{k-1} = \{(x_1, x_2, .., x_k) \in \mathbf{R}^k \mid x_i \leq 0, \sum_{i=1}^{k} x_i = -k\}$$

We claim that $\Phi$ is a smooth map. We have to show that for each i, $\frac{1}{2\pi}\int_{\Omega_i} *dG_p$ is smooth as a function of p in $\mathcal{H}^3$. If we set $G_p = z^2 f_p$ as before, then $\int_{\Omega_i} *dG_p = \int_{\Omega_i} 2f_p dxdy$. $k = 1$ is a trivial case. For $k \geq 2$, we may assume that $\Omega$ is bounded in the upper half space model with its coordinate $(x, y, z)$ by the conservation property iii). Recall that $f_p(q) = f(p, q)$, which is smooth on $\{\mathcal{H}^3 \times \mathcal{H}^3 - diagonal\}$, can be extended smoothly upto $\{\mathcal{H}^3 \times (\partial\mathcal{H}^3 - \Lambda(\Gamma))\}$. Therefore, $\int_\Omega 2f_p dxdy$ is a smooth function of p in $\mathcal{H}^3$.

Now we need more subtle result;

**Lemma 2.3.1** *$\Phi$ can be extended smoothly to*

$$\{\mathcal{H}^3 \cup (\partial\mathcal{H}^3 - \Lambda(\Gamma))\}^k \longrightarrow \Delta^{k-1}$$

**Proof of Lemma 2.3.1**: From above argument and symmetry of $G(p,q) = G_p(q)$, we deduce that $G(p,q)$ is smooth on $\{\mathcal{H}^3 \times \mathcal{H}^3 - diagonal\} \cup \{(\partial\mathcal{H}^3 - \Lambda(\Gamma)) \times \mathcal{H}^3\} \cup \{\mathcal{H}^3 \times (\partial\mathcal{H}^3 - \Lambda(\Gamma))\}$. The case $k = 1$ is trivial so assume $k \geq 2$. ii) implies that $\Phi$ can be extended continuously up to the boundary $\cup_{i=1}^k \Omega_i = \partial\mathcal{H}^3 - \Lambda(\Gamma)$.

$\frac{1}{2\pi}\int_{\Omega_j} *dG_p$ equals zero if $p \in \{\cup_{i=1}^k \Omega_i\} - \Omega_j$, and $-1$ if $p \in \Omega_j$ by conservation property iii). It will be enough to show only that $\int_{\Omega_j} *dG_p$ is smooth as a function of $p$ upto $\Omega_i$ with $i \neq j$. Let's set $j = 2$ and $i = 1$. We are going to use again the upper half space model with (x,y,z) coordinates with $z > 0$, for hyperbolic space, and may assume that $\Omega_2$ is bounded in $z = 0$, because by symmetricity we may assume that $\infty \in \Omega_1$.

Set $F(p) = \int_{\Omega_2} *dG_p = \int_{\Omega_2} 2f_p(\cdot)dxdy$ for $p \in \mathcal{H}^3$ (Recall that $G_p = z^2 f_p$ ). We claim that $f_p(\cdot) = f(p, \cdot)$ is smooth on $(\mathcal{H}^3 \cup \Omega_1) \times (\mathcal{H}^3 \cup \Omega_2)$. Set $p = (x, y, z)$. Here $\rho = dist(p, q = (x_1, y_1, z_1))$. Then we can compute, from the hyperbolic distance formula,

$$\begin{aligned} G_p(x,y,z) &= \frac{1}{e^{2\rho}-1} = z^2 f_p \\ &= \frac{1}{2}\{\frac{(x-x_1)^2+(y-y_1)^2+z^2+z_1^2}{\sqrt{((x-x_1)^2+(y-y_1)^2+z^2+z_1^2)^2-4z^2z_1^2}} - 1\} \end{aligned}$$

Set $L = (x-x_1)^2 + (y-y_1)^2 + z^2 + z_1^2$. Then

$$f_p = \frac{2z^2}{\sqrt{L^2 - 4z^2z_1^2} \cdot \{L + \sqrt{L^2 - 4z^2z_1^2}\}} \tag{2.2}$$

Denomenator equals zero only when $(x, y, z) = (x_1, y_1, z_1)$.

So $f_p(x_1, y_1, z_1) = f(x, y, z, x_1, y_1, z_1)$ is smooth on $(\mathcal{H}^3 \cup \Omega_1) \times (\mathcal{H}^3 \cup \Omega_2)$. So the claim is shown and this means that $F(p)$ is smooth on $\mathcal{H}^3 \cup \Omega_1$. Therefore $\Phi$ can be extended smoothly.

**q.e.d.**

Note that $\Phi(p_{1_o}, p_{2_o}, .., p_{k_o}) = (-1, -1, .., -1)$ if $p_{i_o} \in \Omega_i$. Now consider curves $C_i(t) : [0, \epsilon_i] \longrightarrow \mathcal{H}^3 \cup \Omega_i$ , with $\epsilon_i$ small, starting at $p_{i_o}$ with $C_i'(0) = \frac{dC_i}{dt}(0)$ being an inward normal vector. Set

$$F_{ij}(t) = \int_{\Omega_i} *dG_{C_j(t)},$$

then $F_{ij}(0) = -\delta_{ij}$. Let's observe the following property that $F_{ij}(t)$ is decreasing for $i \neq j$ and increasing for $i = j$ as t increases near zero, because of the conservation property and nonpositivity of integrals. So we may hope to get some configurations of points in neighborhoods of the points $p_{1_o}, p_{2_o}, .., p_{k_o}$ such that the configurations give an integral values for $\Phi$. Our strategy is as follows. We want to get k points $p_i \in \mathcal{H}^3$ $i = 1, 2, .., k$ with each $p_i$ close to $\Omega_i$ such that $\Phi(p_1, p_2, .., p_k) = (-1, -1, .., -1)$. Note that this choice of $\{p_i\}$ gives the integrality of $\frac{1}{2\pi} * d\mathbf{V}$.

Considering the above increasing (or decreasing) property, it will be natural to apply the maximal rank theorem to $\Phi$ at $(p_{1_o}, p_{2_o}, .., p_{k_o})$. But we have to overcome some technical difficulties.

So, for $\Phi : \{\mathcal{H}^3 \cup (\partial\mathcal{H}^3 - \Lambda(\Gamma))\}^k \longrightarrow \Delta^{k-1}$, set $p_{i_0} = (a_i, b_i, 0)$ and $\tilde{C}_i(t) = (p_{1_o}, p_{2_o}, .., C_i(t), ., p_{k_o})$ with $C_i$ is in i-th entry. And set $v_i = \Phi_*(\tilde{C}_i'(0))$.

If a set of vectors $\{w_1, w_2, .., w_s\}$ in a vector space W spans W as linear combinations of $w_i$ with positive (nonnegative) coefficients, then we shall simply say that the vectors positively ( nonnegatively, respectively) span W.

Define $C_i(t) = (a_i, b_i, t)$ with $t = z^2$, where $(x, y, z)$ are the upper half space coordinates. And define $\tilde{C}_i$ as above. Then

$$\Phi((p_{1_o}, p_{2_o}, .., C_i(t), ., p_{k_o})) = (\phi_1, \phi_2, .., \phi_k)$$

with

$$\phi_j(t) = \frac{1}{2\pi} \sum_{s \neq i} \int_{\Omega_j} *dG_{p_{s_0}} + \frac{1}{2\pi} \int_{\Omega_j} *dG_{C_i(t)}.$$

So,

$$v_i = \Phi_*(\tilde{C}_i'(0)) = (\phi_1'(0), \phi_2'(0), .., \phi_k'(0)),$$

with

$$2\pi\phi_j'(0) = \frac{d}{dt}|_{t=0} \int_{\Omega_j} 2f_{C_i}(\cdot)dxdy,$$

If we set $R = (x - a_i)^2 + (y - b_i)^2 + z^2$, then from the formula (2.3), (for $j \neq i$) we can compute

$$\begin{aligned} 2\pi\phi_j'(0) &= \int_{\Omega_j} \frac{1}{R^2} dxdy \\ &= \int_{\Omega_j} \frac{1}{\{(x-a_i)^2 + (y-b_i)^2\}^2} dxdy < 0, \end{aligned}$$

here above integration is negative because $\Omega_j$ has the opposite orientation to $(dx, dy)$. So the entries of $v_i = \Phi_*(\tilde{C}_i'(0)) = (v_{i1}, v_{i2}, .., v_{ik})$ has sign $v_{ii} > 0$ and $v_{ij} < 0$ for $i \neq j$, and $\sum_{j=1}^k v_{ij} = 0$.

These $v_i$ have the property as follows:

**Lemma 2.3.2** *$v_1, v_2, .., v_k$ nonnegatively spans $T_{(-1,-1,.,-1)}\Delta^{k-1}$ ( let's denote this simply by $T$ ) and there exist positive numbers $c_1, c_2, ., c_k$ such that*

$$\sum_{i=1}^k c_i v_i = 0.$$

**Proof of Lemma 2.3.2**: First we will see that $v_1, v_2, .., v_{k-1}$ are linearly independent. Using the entry matrix for $v_1, v_2, .., v_{k-1}$;

$$\begin{pmatrix} v_{11} & v_{12} & .... & v_{1k} \\ v_{21} & v_{22} & .... & v_{2k} \\ .... & ... & & .. \\ v_{k-1,1} & v_{k-1,2} & .... & v_{k-1,k} \end{pmatrix},$$

it is an easy exercise to see that we can apply row reduction (k-2) times to remove entries $v_{ij}$ with $i > j$ until we get the matrix of the sign

$$\begin{pmatrix} + & - & - & - & .... & - & - \\ 0 & + & - & - & .... & - & - \\ 0 & 0 & + & - & .... & - & - \\ 0 & 0 & 0 & + & .... & - & - \\ . & . & . & . & .... & . & - \\ 0 & 0 & 0 & 0 & .... & + & - \end{pmatrix}.$$

So $v_1, v_2, .., v_{k-1}$ are linearly independent and their nonnegative linear combinations form a (k-1) dimensional convex cone. And $v_k$ is clearly not in the cone because $v_{kk} > 0$. If $v_1, v_2, .., v_k$ does not nonnegatively span whole T which is to be identified with $\{x_1 + x_2 + .. + x_k = 0\}$ , then there exists (k-2) dimensional subspace W in T such that $v_1, v_2, .., v_k$ are all on one component of $T - W$ and so, ther exist a nonzero vector $v$ in T such that $v \cdot \alpha_i \geq 0$. Setting $v = (a_1, a_2, .., a_k)$ with $\sum_{i=1}^{k} a_i = 0$, we have, for each i,

$$\begin{aligned} v_i \cdot v &= \sum_{j=1}^{k} v_{ij} a_j = \sum_{j \neq i} v_{ij} a_j + v_{ii} a_i \\ &= \sum_{j \neq i} v_{ij} a_j - \sum_{j \neq i} v_{ij} a_i = \sum_{j \neq i} (v_{ij} a_j - v_{ij} a_i) \\ &= \sum_{j \neq i} v_{ij} (a_j - a_i) \geq 0 \end{aligned}$$

Since $v_{ij} < 0$ for $i \neq j$, this implies that $v$ is zero vector. This gives a contradiction. So, $v_1, v_2, .., v_k$ nonnegatively span T and the last statement of the lemma is also true because $-v_k$ is clearly in the cone from above argument. This finshes the proof of Lemma 2.3.2.

**Proof of Proposition 2.3.1**: Now we apply the maximal rank theorem and get a continuous curve $\gamma(t)$ in $\{\mathcal{H}^3 \cup (\partial\mathcal{H}^3 - \Lambda(\Gamma))\}^k$ such that $\gamma(t) \in (\mathcal{H}^3)^k$ and $\Phi = (-1, -1, ., -1)$ on $\gamma$. This proves the case $l = k$. We have to prove $l > k$ case. We define the map $\Phi$ similarly to (2.2);

$$\Phi : \{\mathcal{H}^3 \cup (\partial\mathcal{H}^3 - \Lambda(\Gamma))\}^l \longrightarrow \{(x_1, .., x_k) \in R^k \mid x_i \leq 0, \sum_{i=1}^{k} x_i = -l\}. \quad (2.3)$$

Lemma 2.3.1 can be carried out in the same way, so we only have to prove Lemma 2.3.2. Given $v_1, .., v_{k-1}, v_k, v_{k+1}, ., v_l$ in T, from $v_i = \Phi_*(\tilde{C}_i'(0))$, then reorder them $v_{11}, v_{12}, .., v_{1l_1}, v_{21}, v_{22}, .., v_{2l_2}, ....., v_{k1}, v_{k2}, .., v_{kl_k}$ , so that $v_{i\cdot}$ is induced from the curve starting off $p_i \in \Omega_i$ as in the above. Next set $w_i =$

$\sum_s v_{is} a_{is}$, here $a_{is}$ with $1 \leq i \leq k, 1 \leq s \leq l_i$ are any positive numbers. Then we can easily check that the vectors $w_i$ has the property of Lemma 2.3.2. So we conclude that there exist $a_1, a_2, .., a_l > 0$ such that $\sum_{i=1}^{l} a_i v_i = 0$. By the same argument as above, there exist at least a curve satisfying $\Phi = (-l_1, -l_2, ., -l_k)$ on it.

Therefore, for $l \geq k$, we get configurations of points $p_1, p_2, ..., p_l$ in $\mathcal{H}^3$ such that $[\frac{1}{2\pi} * d\mathbf{V}]$ is an integral class in $H^2(M, \mathbf{R})$.

Finally, note that we can get the integrality of our cohomolgy element in the presence of any extra *torsion* elements. **q.e.d.**

As a consequence of Proposition 2.3.1, there exists a $S^1$ principal fiber bundle P over $(\mathcal{H}^3 - \{p_j \Gamma\}_{j=1,2,..,l})/\Gamma$ with a connection form $\omega$ such that $d\omega = *d\mathbf{V}$.

The rest of this section is devoted to the discussion on how to get the integrality in the presence of torsion free *interior* homology elements i.e. the elements which is not in the span of the homology elements represented by boundary surfaces. In some sense this is similar to the previous case, but since we can see a different phenomena here and also we are aiming to get integrality over general noncompact hyperbolic 3-manifolds eventually, we are going to spare a space for discussing it.

Take a torsion free generator $[S]$ of $H_2(N_0, Z)$ with a compact surface representative $S$. Again in this section we assume that $N_0 = \mathcal{H}^3/\Gamma$ has k boundary surfaces $S^1, S^2, .., S^k$ as before. Assume that the torsion free part of

$H_2(N_0, Z)$ is generated by $[S]$ only. In addition to $Phi$ in (2.4), we also define

$$\begin{aligned}\Phi_S(p_1, p_2, .., p_l) &= \frac{1}{2\pi}\int_S *d\mathbf{V} \\ &= \frac{1}{2\pi}\int_{\Gamma\cdot S}(*dG_{p_1} + ... + *dG_{p_l}).\end{aligned}$$

$\Phi_S$ is then defined on $\{(\mathcal{H}^3 - \Gamma\cdot S)\cup\Omega(\Gamma)\}^l$. Since $\Phi$ in (2.4) has the maximal rank along $U_1 \times U_2 \times .. \times U_l$, where $U_i$ is an open set in some $\Omega_{j(i)}$, $\mathcal{U} \cap \Phi^{-1}(-l_1, -l_2, .., -l_k)$ with natural numbers $l_1 + l_2 + .. + l_k = l$ is (3l - k + 1)-dimensional submanifold of $\mathcal{U}$ which is a neighborhood of $U_1 \times U_2 \times .. \times U_l$ in $\{\mathcal{H}^3 \cup \Omega(\Gamma)\}^l$. For convenience, we may suppose $\mathcal{U}$ to be $V_1 \times V_2 \times ... \times V_l$ where $V_i$ is diffeomorphic to $U_i \times [0, 1)$.

Note that we may consider $\Phi_S$ to be smooth on $\mathcal{U}$, because $S$ is at least smooth almost everywhere, the integand $*dG_{p_i}$ is smooth on the appropriate domain and in upper half space coordinates $\Gamma \cdot S$ can be taken to be sitting in compact neighborhood. Define $\mathcal{V} = \mathcal{U} \cap \Phi^{-1}(-l_1, -l_2, .., -l_k) - \{(p_1, p_2, ..p_l) \mid$ some $p_i \in U_i\}$. Let's restrict $\Phi_S$ on $\mathcal{V}$ : $\Phi_S : \mathcal{V} \longrightarrow \mathbf{R}^+$ . Since $\Phi_S$ is a priori a nonconstant map, $d\Phi_S \neq 0$ at some point $p \in \mathcal{V}$ (actually we could just compute it as in the proof of proposition 2.3.1). So $d\Phi_S \neq 0$ on a neighborhood $\mathcal{U}_p$ of $p$ in $\mathcal{V}$, and so for some rational number $\frac{n}{m}$ $\Phi_S^{-1}(\frac{n}{m})$ is a (3l - k)-dimensional submanifold of $\mathcal{U}_p$. From this we can deduce that we can choose as many points as possible (here we need any multiple of m points) from $\Phi_S^{-1}(\frac{n}{m})$. The only thing we have to worry about is two points $\alpha_i = (p_{i_1}, p_{i_2}, .., p_{i_l})$ i=1,2 with $p_{1_s} = p_{2_s}$ for some s in $\{1, 2, .., l\}$, but when $\alpha_1$ is chosen first, this does not give restriction to the choice of next one $\alpha_2$ because first choice doesn't reduce the dimension of the data space.

Now we strengthened the proposition 2.3.1. Recall the notation $N_0 = \mathcal{H}^3/\Gamma$;

**Proposition 2.3.2** *Assume that $H_2(\overline{N_0}, Z)$ has at most one torsion free element which can not be generated by boundary surfaces. Then there exist a natural number $m_0$ such that for any $m \geq m_0$ there exist m-points $p_1, p_2, ..., p_m$ in $\mathcal{H}^3$, for which $[\frac{1}{2\pi} * d\mathbf{V}]$ is an integral class in $H^2(N, \mathbf{R})$.*

So there exists a $S^1$ bundle over $N$.

# Chapter 3

# Construction of Compact Self-Dual Manifolds

## 3.1 Hyperbolic ansatz metrics

In [14], Gibbons and Hawking found the ansatz for complete noncompact (strongly) self-dual metrics (i.e. Kähler ricci flat metrics) with circle actions which were later called $A_k$ gravitational instantons. In short, it generates such metrics in terms of solutions of Laplacian's on $\mathbb{R}^3$, the Euclidean space. In [21], Le Brun developed *hyperbolic* ansatz for Kähler scalar flat metrics with circle actions. One strong point of this hyperbolic ansatz metrics is that it is conformally compactifiable so that they produce *compact* self-dual metrics.

The following describes the basic structure of the hyerbolic ansatz self-dual metrics. For convenience, we are going to use the upper half space model with $(x, y, z)$-coordinates for $\mathcal{H}^3$ as in section 2.1. Recall that Kähler metrics of zero scalar are conformally anti-self-dual by theorem 1.2.1.

**Proposition 3.1.1 (C. LeBrun [21])**

*Let* $\mathbf{V}$ *be any positive solution of the Laplace-Beltrami equation*

$$\Delta\mathbf{V} = 0$$

*on a region* $\mathcal{V} \subset \mathcal{H}^3$ *of hyperbolic 3-space, and assume that the cohomology class of* $\frac{1}{2\pi} \star d\mathbf{V}$ *is integral, where* $\star$ *is the Hodge star operator of* $\mathcal{H}^3$*. Then if* $\omega$ *is a connection form for a circle bundle whose curvature is* $\star d\mathbf{V}$*, then*

$$[g] = [\mathbf{V}h + \mathbf{V}^{-1}\omega^2]$$

*is half conformally flat. Moreover, the metric*

$$g = z^2(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$$

*is Kähler, with scalar curvature zero.*

For reader's convenience, we are going to sketch the proof in [21].

**Sketch of Proof:**

First change coordinates by $2q = z^2$ to get; $h = \frac{dx^2+dy^2}{2z} + \frac{dz^2}{4z^2}$ for the upper half space $\{(x, y, q)|q > 0\}$ in $\mathbb{R}^3$. For convenience, set $V = 2qw$. Then applying the Laplacian of $h$,

$$\Delta V = 4q^2(\omega_{xx} + \omega_{yy} + V_{qq}) = 0.$$

Set $g = V(dx^2 + dy^2) + w \cdot dq^2 + \frac{1}{w} \cdot \omega^2$. Define an almost complex structure J on $M$ by $dq \to \frac{1}{w} \cdot \omega$, $dx \to dy$. Then using the relation

$$*dV = d\omega = w_x dy \wedge dq + w_y dq \wedge dx + V_q dx \wedge dy,$$

we can show that $wdq + i\omega$ and $dx + idy$ form a closed differential ideal. So J is integrable and $g$ is hermitian w.r.t. J. The associated 2-form of $g$ is $\Omega = dq \wedge \omega + V dx \wedge dy$. Then it is easy to see that $d\Omega = 0$, so $g$ is a Kähler metric. We may choose a local holomorphic basis of (1,0)-forms explicilty including $dx + idy$ in which we compute the ricci form $\rho$ and verify that the scalar curvature is zero. **q.e.d.**

## 3.2 Construction of compact self dual manifolds

In section 2.3, we constructed a $S^1$ principal fiber bundle

$$\pi : P \longrightarrow (\mathcal{H}^3 - \cup_{j=1,2,..,l}\{p_j\Gamma\})/\Gamma$$

with a connection form $\omega$ such that $d\omega = *d\mathbf{V}$. We aim to get a compact (anti)-self-dual metric on the *conformal* compactification of P as a representative of the conformal structure $[\mathbf{V}h + \mathbf{V}^{-1}\omega^2]$ on P. We will be able to get a conformal factor function $f$ of the form $\frac{z^2}{f_o^2}$, where $f_o$ is a positive function on $\mathcal{H}^3 \cup \{\partial\mathcal{H}^3 - \Lambda(\Gamma)\}$, such that $f(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$ is a well defined smooth metric on the compactification of P.

Next argument follows LeBrun [21, 22]. We can get a smooth compactification of P by adding a finite number of points and boundary surfaces $\{S_i\}_{i=1,..,k}$ which are to be identified with the ideal surfaces at infinity

$\{\partial\mathcal{H}^3 - \Lambda(\Gamma)\}/\Gamma$, as follows. Note that the case where we add *zero* point is a special case and this is described in [28].

For each point $p \in \cup_{j=1,2,..,l}\{[\Gamma p_j]\} \subset \mathcal{H}^3/\Gamma$, choose a small sphere $S$ around $p$. Then $\int_S \frac{*d\mathbf{V}}{2\pi} = -1$. Then $\pi^{-1}(B-p)$ is diffeomorphic to $\mathbb{R}^4 - \{origin\} \approx S^3 \times \mathbb{R}^+$ for a sufficiently small open ball $B$ in $\mathcal{H}^3/\Gamma$ around the point $p$.

In section 2.3, we showed that $*d\mathbf{V}$ can be extended smoothly and vanishes along the boundary $\{\partial\mathcal{H}^3 - \Lambda(\Gamma)\}/\Gamma$. Therefore near the boundary, $\pi$ is diffeomorphically conjugate to the map from $\mathbb{R}^4 = \{(x,y,r,\theta)\}$ where $(r,\theta)$ is considered to be the polar coordinates. to $\{(x,y,r)|r > 0\} \subset \mathbb{R}^3$ such that $(x,y,r,\theta) \mapsto (x,y,r)$.

So we can get a smooth extension map of $\pi$ (also denote by $\pi$) :

$$M = P \cup \{[\Gamma p_j]\}_j \cup \{S_i\}_i \longrightarrow (\{\mathcal{H}^3 - \{[\Gamma p_j]\}_j\}/\Gamma) \cup \{[\Gamma p_j]\}_j \cup \{\partial\mathcal{H}^3 - \Lambda(\Gamma)/\Gamma\}.$$

The $S^1$ action on P extends to M such that $\{[\Gamma p_j]\}_j$ and $\{S_i\}_i$ become the fixed poins of the extended $S^1$ action. This comactification is in fact a *conformal* compactification, namely that a metric representative of the above conformal structure extends over M. In fact, for given $\Gamma$ and $p_j \in \mathcal{H}^3$, $g_o = z^2(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$ gives a metric on $\tilde{P} \cup \{\widehat{\alpha p_j}\}_{\alpha\in\Gamma}$, where $\tilde{P}$ is the universal covering of P and $\widehat{\alpha p_j}$ is the one point inverse image of $\alpha p_j$ under the obvious projection map, and $g_o$ is asymptotically flat near $\partial\mathcal{H}^3 - \Lambda(\Gamma)$ and can be extended conformally over $\tilde{M} = \tilde{P} \cup \{\alpha\hat{p}_j\}_{\alpha\in\Gamma} \cup \partial\mathcal{H}^3 - \Lambda(\Gamma)$. See [21] for details.

$M = \tilde{M}/\Gamma$ is now a compact manifold equipped with the local conformal structure induced from $g_o$. So using partitions of unity over M, we get a global metric g on M which is pointwise conformal to the metric $\mathbf{V}h + \mathbf{V}^{-1}\omega^2$ defined on the quotient $P = \tilde{P}/\Gamma$. Note that this metric g should be of the form $\frac{z^2}{f_o{}^2}(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$ for some positive function $f_o$ such that $\frac{z}{f_o}$ is invariant under $\Gamma$. Therefore we completed the construction of compact (anti)-self-dual metrics.

Now let's analyze the topology of $M$. Let $X$ be the conformal compactification of $S^1 \times N$.

Consider $\pi^{-1}(\gamma)$ of a curve $\gamma$ from a point $[\Gamma p_j]$ to a boundary surface in $\{\partial\mathcal{H}^3 - \Lambda(\Gamma)\}/\Gamma$ such that $\pi^{-1}(\gamma)$ is topologically $S^2$. We can show that $\pi^{-1}(\gamma)$ has self intersection number $-1$, by considering the intersection number of $\pi^{-1}(\gamma)$ with $\pi^{-1}(\tilde{\gamma})$, where $\tilde{\gamma}$ is a nearby arc from $[\Gamma p_j]$ to the same boundary surface which intersects $\gamma$ only at the point $[\Gamma p_j]$.

If we reverse the orientation, then the self-intersection number of $\pi^{-1}(\gamma)$ is 1 and a neighborhood of $\pi^{-1}(\gamma)$ is diffeomorphic to $\mathbb{CP}^2 - ball$. Now with this reversed orientation we get self-dual metrics on a connected sum $M \approx X\sharp m\mathbb{CP}^2$.

Now we can summarize above;

**Theorem 3.2.1** *Suppose that $\Gamma$ is a no cusp, geometrically finite, Kleinian group such that $N = \mathcal{H}^3/\Gamma$ is a noncompact hyperbolic 3-manifold. If there is at most one torsion free element which cannot be generated by boundary surfaces in $H_2(N, Z)$, then, there exist (anti)-self-dual metrics on $\mathbf{X}\sharp m\mathbb{CP}^2$ for all sufficiently large natural number $m$.*

It will be interesting to find any topological meaning of the smallest number $m_0$ such that $\mathbf{X}\sharp m\mathbb{CP}^2$ admits self-dual metrics for any natural number $m \geq m_0$ or to find any topological invariant $m_0$ with the same property. For this purpose, we state the following which is a consequence of proposition 2.3.1.

**Proposition 3.2.1** *Let $\Gamma$ be as in theorem 3.2.1. Moreover, if there is no torsion free element in $H_2(\overline{N}, Z)$ except the ones generated by boundary surfaces, then for any natural number m bigger than or equal to the number of boundary surfaces in $N$, $\mathbf{M}_0\sharp m\mathbb{CP}^2$ admits self dual metrics.*

Author believes that theorem 3.2.1 should be true in the general case of arbitrary number of interior second homology elements. Above argument produces an interesting picture, namely that it shows that in order to get self dual metrics on connected sum of a conformally flat metric with sufficiently many $\mathbb{CP}^2$'s, how we should increase the number of attaching $\mathbb{CP}^2$'s to overcome nontrivial topologies.

# Chapter 4

# Geometry of the Self-Dual Manifolds and Examples

## 4.1 Geometry of the constructed manifolds

Recall that any compact riemannian manifold admits, by conformal deformation, a metric of constant scalar curvature [30] and the sign of its scalar curvature is uniquely determined by the conformal structure. We call the metric is of type +, 0, − if it admits a conformal metric of scalar curvature positive, zero, negative respectively. In [32], Schoen and Yau showed that for a quotient manifold of $S^4 - \Lambda$ by conformal transformations, it is of type +, 0, − if and only if $1 - D(\Lambda(\Gamma))$ is +, 0, − respectively. Since our construction is based on such quotient manifolds description, we may hope to get analogous statement for our self dual metrics. Indeed, by curvature computation and some argument we could get;

**Theorem 4.1.1** *Suppose that* $\Gamma$ *is any group as in Theorem 3.2.1.*

*If* $1 - D(\Lambda(\Gamma))$ *is* $+$, *0*, $-$, *respectively,*

*then there exists a representative metric of scalar curvature positive, zero, negative, respectively, in the self dual conformal structure on* $\mathrm{M}_0\sharp m\mathbb{CP}^2$ *in theorem 3.2.1.*

**Remark 4.1.1:** If we succeed to get self dual metrics on connected sum of conformal compactification of $\mathcal{H}^3/\Gamma \times S^1$ for *general* $\Gamma$ with many $\mathbb{CP}^2$'s by similar argument as shown in Theorem 3.2.1, then theorem 4.1.1 is also true for this general case.

**Proof:** Here we consider the self-dual metrics $g$ of the form $g = F^2(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$ as constructed in section 2.3 on the underlying differentiable manifold for fixed $\Gamma$. Let's use $(x, y, z)$ coordinates in the upper half space model again. If we compute the scalar curvature of the above metric in some routine way, we get ;

$$R(g) = \frac{\mathbf{V}^{-1}}{F^3}\{F - zF_z - z^2(F_{xx} + F_{yy} + F_{zz})\}. \tag{4.1}$$

Note that the sign of scalar curvature depends only on F and its derivatives. Conformally flat case is a special case of these, where $\mathbf{V} = 1$.

Suppose that $D(\Lambda(\Gamma)) < 1$. The other cases that $D(\Lambda(\Gamma)) > 1$ or $D(\Lambda(\Gamma)) = 1$ will be handled in similar way. We have to show that there exists a function F on M such that $R(g)$ is positive, for the metric $g = F^2(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$ on M. First note that Schoen-Yau [32] result says that if $\mathbf{V} = 1$, we can choose a smooth function, say $F_1$ positive except along $z = 0$, such that (4.1) is positive. The important thing is that we can choose $F_1$ as

$S^1$-invariant. In fact, set $g_1 = F_1^2(h + dt^2)$, then from the formula involving conformal Laplacian, we get

$\frac{1}{6}F_1^3 \cdot R(g_1) = (d^*dF_1 - \frac{1}{6}F_1) > 0$. except along $z = 0$. Set $\tilde{F} = \frac{1}{2\pi}\int_{S^1} F_1(x, y, z, t)dt$ the average function over $S^1$ fibers which is $S^1$ invariant. If we define $g_2 = \tilde{F}^2(h + dt^2)$, then

$$\begin{aligned} \frac{1}{6}\tilde{F}^3 \cdot R(g_2) &= d^*d\tilde{F} - \frac{1}{6}\tilde{F} \\ &= \frac{1}{2\pi}\int_{S^1}(d^*dF_1 - \frac{1}{6}F_1)dt \\ &= \frac{1}{2\pi}\int_{S^1}\frac{1}{6}F_1^3 \cdot R(g_1)dt > 0, \end{aligned}$$

except along $z = 0$. So $R(g_2)$ is of type $+$. This $\tilde{F}$ is the function we were looking for; $g = \tilde{F}^2(\mathbf{V}h + \mathbf{V}^{-1}\omega^2)$ is a well defined metric on the manifold by the nature of the construction, namely that, g can be extended smoothly to $\{[\Gamma p_j]\}_j$ and also to the boundary surfaces because of the $S^1$ invariance. **q.e.d.**

Now we would like to describe specific examples and discuss a theorem on the scalar curvature and the moduli space of self-dual conformal structures. Let's start with the basic example.

**Example 4.1.1:** $S^4$ with standard metric : No points taken out and $\Gamma = \{e\}$. Then, the fixed surface is $S^2 \approx \partial\mathcal{H}^3$ . We start with

$$\pi : P = \mathcal{H}^3 \times S^1 \longrightarrow \mathcal{H}^3$$

which is the trivial $S^1$ bundle, and extend to

$$\pi : \{\mathcal{H}^3 \times S^1\} \cup S^2 = \{S^4 - S^2\} \cup S^2 = S^4 \longrightarrow \mathcal{H}^3 \cup \partial\mathcal{H}^3$$

And this can be understood at metric level; with $f_o = 1, \mathbf{V} = 1$,

$$\begin{aligned} g &= \frac{z^2}{f_o^2}(\mathbf{V}h + \mathbf{V}^{-1}\omega^2) \\ &= z^2(\frac{dx^2 + dy^2 + dz^2}{z^2} + dt^2) \\ &= dx^2 + dy^2 + dz^2 + z^2 dt^2. \end{aligned}$$

Considering $(z, t)$ as polar coordinates, g is the Euclidean metric and so conformally compactifiable to $S^4$, using the conformal factor function arising from the stereographic projection, if we add the boundary surface, $S^2$ in this case, at infinity.

**Example 4.1.2:** Conformally flat metrics [7]: No points taken out and $\Gamma$ general ( i.e. $\mathcal{H}^3/\Gamma$ is a noncompact hyperbolic 3 manifold with a no cusp, geometrically finite Kleinian group). The fixed surfaces are $(\partial\mathcal{H}^3 - \Lambda(\Gamma))/\Gamma$. In this case, we don't need to worry about integrality because we can take trivial $S^1$ bundle. Peter Braam described conformally flat metrics on $M_\Gamma$, the compactification of these trivial bundle.

**Example 4.1.3:** Self-dual metrics on $l\mathbb{CP}^2$ [21] : The fixed surface is $S^2 = \partial\mathcal{H}^3$. Integrality is trivial in this case because $\Gamma = \{e\}$. We take out $l$ points $p_1, ..., p_l$ from $\mathcal{H}^3$. This is an interesting family of self dual metrics, cf Chapter 1.

**Example 4.1.4:** Kähler metrics metrics with zero scalar curvature (so anti-self-dual)on $S^2 \times S_g \sharp l\overline{\mathbb{CP}^2}$ with $l \geq 2$ [22]. Here $S^2 \times S_g$ may be replaced by some ruled surfaces over $S_g$. Here we take a fuchsian group $\Gamma$ such that $\mathcal{H}^2/\Gamma$

is a compact Rieman surface of genus g $\geq 2$. Some specific conformal factor function and complex structure are described to construct the first explicit nontrivial compact scalar zero Kähler metrics on blown ups of ruled surfaces.

Before next exampless, we need to introduce the Klein-Maskit combination theorems of Kleinian group theory and the boundary connected sum construction of new examples of hyperbolic 3-manifolds. [7, 26, 27]

Definition: Assume that $\Gamma_0, \Gamma_1$ are Kleinian groups such that $N_i = \mathcal{H}^3/\Gamma_i$ are noncompact hyperbolic 3 manifolds with nonempty boundaries as in section 2.1. For a pair of points $x_i \in \partial\overline{N}_i$, take open half ball $K_i$ centered at $x_i$. Then *the boundary connected sum* of $N_1$ and $N_2$, $N_1 \natural N_2$ is defined to be

$$\{\mathcal{H}^3/\Gamma_1 - K_1\} \cup_\rho \{\mathcal{H}^3/\Gamma_2 - K_2\}$$

wher $\rho$ is an isometry : $\partial K_1 \longrightarrow \partial K_2$. The first combination theorem says that under same hypothesis as above, the boundary connected sum, say N, is isometric to $\mathcal{H}^3/\Gamma$ with $\Gamma$ being a Kleinian group which, in $PSL(2, \mathbb{C})$, is the free product of $\Gamma_1$ and $g\Gamma_2 g^{-1}$ for some $g \in PSL(2, \mathbb{C})$.

In conformally flat metrics case, P. Braam observed:

**Proposition 4.1.1** *If $\Gamma$ is the Kleinian group corresponding to a boundary connected sum of $\mathcal{H}^3/\Gamma_1$ and $\mathcal{H}^3/\Gamma_2$, then $\Gamma$ is a Kleinian group such that $M_\Gamma$ (cf; Example 4.1.2) is the $S^1$-equivariant conformal connected sum of $M_{\Gamma_1}$ and $M_{\Gamma_2}$ at points in the fixed surfaces.*

This is true because $\pi^{-1}(\bar{K}_j)$ for the map

$$\pi : P \cup \{S_j\} \longrightarrow \mathcal{H}^3/\Gamma_i \cup \{(\partial\mathcal{H}^3 - \Lambda(\Gamma))/\Gamma_i\}$$

are balls around the points $x_j$ in the fixed surfaces and we are identifying $\pi^{-1}(K_1)$ and $\pi^{-1}(K_2)$ by $S^1$ equivariant conformal maps.

**Remark 4.1.2:** If the conformally flat manifolds $M_{\Gamma_1}$ and $M_{\Gamma_2}$ in Proposition 4.1.1 is replaced by self dual metrics in our construction , then the Proposition is not true because these two self dual metrics on $M_{\Gamma_i}$ are not conformally equivalent with different potential functions $\mathbf{V}_{\Gamma_i}$.

Now we are going to use Proposition 4.1.1. The boundary connected sum of two manifolds $N_i = \mathcal{H}^3/\Gamma_i$ with $k_i$ (i = 1,2) boundaries cannot create extra $H_2$ interior homology elements and the number of boundary surfaces for $N_1 \natural N_2$ is $k_1 + k_2 - 1$. By Proposition 4.1.1 and Theorem 3.2.1, we get self dual metrics on connected sum of $M_{\Gamma_1} \sharp M_{\Gamma_2}$ with $\mathbb{CP}^2$'s if each $M_{\Gamma_i} \sharp \mathbb{CP}^2$ admits self dual metric by the theorem 3.2.1. This helps to get explicit self dual metrics on a number of topological manifolds. We can take for instance;

**Example 4.1.5:** Self dual metrics on $k(S^3 \times S^1) \sharp m(S^2 \times S_g) \sharp l\mathbb{CP}^2$ for $k \geq 0$ and $l \geq m+1$.

Here are some remarks on the geometry of Example 4.1.5.

**Remark 4.1.3:** The self-dual metrics on $l(S^3 \times S^1) \sharp m\mathbb{CP}^2$ form a larger class than those in theorem 1.4.6 and have any sign of metrics from theorem 4.1.1 and the fact that there exist Schottky groups of any Hausdorff dimension between 0 and 2 [7, 32]. Particularly, there exist self-dual, zero scalar curvature ($\mathcal{R}$-minimum) non-hermitian metrics.

**Remark 4.1.4:** $k(S^3 \times S^1)\sharp m(S^2 \times S_g)\sharp l\mathbb{CP}^2$ for $m \geq l+1$ have self-dual metrics with negative scalar curvature if either $m \geq 2$ or $k, m \geq 1$.

**Remark 4.1.5:** By the theorem 1.5.2, and [7], the construction in this thesis exhausts all self-dual metrics of nonnegative scalar curvature witha semi-free circle action. A Semi-free action means its isotropy group is either the whole group or the identity subgroup.

**Example 4.1.6** Given a hyperbolic 3-manifold as in theorem 3.2.1, we can show that $\tilde{N} = N\sharp(S^3/\Gamma)$, where $\Gamma$ is a finite group acting freely and properly discontinously on $S^3$, also satisfies the conditions of the theorem 3.2.1. Let's say $\tilde{N} = \mathcal{H}^3/\tilde{\Gamma}$. There exist self-dual metrics on $M_{\tilde{\Gamma}}\sharp m\mathbb{CP}^2$.

A. King and Kotschick [18] studied the moduli space $\mathcal{M}^*$ of self-dual conformal structures with trivial conformal isometry groups and

conjectured that if the signature $\tau(X) \neq 0$, for $X$ a self-dual manifold, then the Yamabe invariant doesn't change sign on smooth points of any connected component of $\mathcal{M}^*$.

However from theorem 4.1.1, we can prove;

**Theorem 4.1.2** *There exists a continuous family of self-dual metrics on a connected component of the moduli space of self-dual structures on* $l(S^3 \times S^1)\sharp m\mathbf{CP}^2$ *for any* $m \geq 1$*, and for some* $l \geq 2$*, which changes the sign of the scalar curvature.*

**Proof 4.1.2:** We are going to use the notations in [18]. We basically follows that of theorem 3.9 in the paper. We now have, from Remark 4.1.3, self-dual metrics of scalar curvature zero on $l(S^3 \times S^1)\sharp m\mathbf{CP}^2$ for any $m \geq 1$, and for some $l \geq 2$. Let's denote any such metric by $g$. Then since $g$ can't be ricci flat by Hithin-Thorpe inequality, this forces $dimH^2_g = dim\bar{H}^2_g$. The rest now follows in similar way to [18]. So we get a 1-parameter family of self-dual metrics with changing sign of the scalar curvatures. **q.e.d**

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