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Abstract of the Dissertation

Clifford Cohomology and Kähler Geometry

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In this thesis we use Clifford cohomology with coefficients in holomorphic vector bundles to prove a wide range of vanishing theorems for compact Kähler manifolds. These include vanishing theorems on holomorphic sections, vanishing theorems of higer dimensional cohomology groups for "semi–negative" line bundles and vanishing theorems for fractional powers of the canonical line bundle. In particular we apply the Clifford cohomology theory to hypersurfaces of complex projective space to study the differential geometry and complex structure of these manifolds.

To My Parents

To My Husband and Son

Contents

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Chapter 1

1.1 Introduction

In recent years mathematicians have found many applications of spin geometry and Clifford algebra structures to various aspects of geometry, topology and analysis. One basic case of this is the paper of M. L. Michelsohn [8], who applied these ideas to the study of complex Kähler geometry. In her paper she replaced the bundle of complex exterior forms on a manifold X with the complexified Clifford bundle $\mathbf{Cl(X)} \equiv Cl(X) \otimes_R \mathbf{C}$. She showed that this admits a pretty decomposition

$$\mathbf{Cl(X)} \equiv \bigoplus_{|p|+|q|\leq n} \mathbf{Cl}^{p,q}(\mathbf{X})$$

where the n is the complex dimension of X and that there is a first order operator $\mathcal{D}$ with the property that $\mathcal{D}$: $\Gamma\mathbf{Cl}^{p,q} \longrightarrow \Gamma\mathbf{Cl}^{p+1,q+1}$ and $\mathcal{D}^2 = 0$. In fact, $\mathcal{D}^2 = 0$ is a sufficient condition for X being Kähler. Each complex

$$\cdots \xrightarrow{\mathcal{D}} \Gamma\mathbf{Cl}^{p-1,q-1} \xrightarrow{\mathcal{D}} \Gamma\mathbf{Cl}^{p,q} \xrightarrow{\mathcal{D}} \Gamma\mathbf{Cl}^{p+1,q+1} \rightarrow \cdots$$

is elliptic and therefore one finds that X has finite dimensional cohomology groups $H_{cl}^{p,q}(X)$, when X is compact. The construction carries over to $\mathbf{Cl(X)} \otimes \mathbf{W}$, where $\mathbf{W}$ is any holomorphic vector bundle over X. Thus one obtains cohomology groups $H_{cl}^{p,q}(X, W)$. Note that the (p,q) decomposition of $\mathbf{Cl(X)}$ does not directly correspond to the Dolbeault decomposition (r,s). However there is an isomorphism between the Clifford cohomology and Dolbeault cohomology groups:

$$H_{cl}^{r-s,n-r-s}(X, W) \simeq H_{Dol}^{s}(X, \theta^{r}(W^{*}))$$

Therefore the Clifford cohomology groups are independent of the Kähler metric on X.

In this thesis, we shall extensively apply Clifford cohomology theory to investigate the relationship between the differential geometry and the complex structure of Kähler manifolds. The Clifford cohomology enters into Kähler geometry in a natural and very interesting way. Indeed it provides an alternate calculus for studying the properties of Kähler manifolds.

We shall recapture a number of classical results. These include vanishing theorems on holomorphic sections and vanishing theorems of Kodaira, Nakano, Vesentini, Girbau and Gigante, i.e., vanishing of higher dimensional cohomology groups for "semi–negative" line bundles. However we shall show how the Clifford bundle approach greatly simplifies the proofs and gives a transparent unification of all the fundamental elliptic complexes in Kähler geometry. Everything will follow from a single formula which applies to any holomorphic vector bundle.

We shall also recover most of the results of Michelsohn [8]. We use consistently the Clifford cohomology groups with coefficients in a holomorphic vector bundle. We also make use of the observation that the subspace $\mathbf{Cl}_n^{p,q}$ with $|p| + |q| = n$ has a basis in which each element has a "nice" algebraic property. This enables us to relax the curvature condition in Theorem 7.15 [8] to a weaker one.

Finally, we shall apply the Clifford cohomology to study the differential geometry, complex structure and topological properties of complex hypersurfaces of complex projective space $\mathbf{CP}^{n+1}$. One of the main theorems is

Theorem 3.4.3: Let M be a complex hypersurface imbedded in $\mathbf{CP}^{n+1}$ with the induced metric and assume $c_1(M)$ is even. If the eigenvalues of the second fundamental form of M satisfy

$$\lambda_j^2 \leq \frac{n^2 - n + 6}{4(5n - 1)}$$

for all j, then for $n \geq 5$, degree $(M) < [\frac{n+2}{3}]$.

In particular, we have the following rigidity results.

Corollary 3.4.5: Suppose that M is as in Theorem 3.4.3 . If the eigenvalues of the second fundamental form of M satisfy

$$\lambda_j^2 \leq \frac{13}{48} \qquad \text{when } n = 5$$

$$\text{or} \quad \lambda_j^2 \leq \frac{6}{17} \qquad \text{when } n = 7$$

for all j, then M has degree 1, i.e. M is the complex projective space $\mathbf{CP}^n$.

Corollary 3.4.6: Suppose that M as in Theorem 3.4.3 . If the eigenvalues of the second fundamental form of M satisfy

$$\begin{aligned} & \lambda_j^2 \leq \frac{9}{29} && \text{when } n = 6 \\ \text{or} \quad & \lambda_j^2 \leq \frac{31}{78} && \text{when } n = 8 \\ \text{or} \quad & \lambda_j^2 \leq \frac{24}{49} && \text{when } n = 10 \end{aligned}$$

for all j, then M is an algebraic manifold with degree 2, i.e. M is the complex hypersphere $Q_n(\mathbf{C})$.

The dissertation is organized as follows. In Chapter 1 we review the Clifford cohomology theory on Kähler manifolds. In Chapter 2 we first give a specific description of the subspaces of $\mathbf{Cl}_n^{p,q}$ with $|p| + |q| = n$, then we establish a series of vanishing theorems for Clifford cohomology groups. In the last Chapter we apply the Clifford cohomology to spin complex hypersurfaces of $\mathbf{CP}^{n+1}$ to study the differential geometry of these submanifolds.

1.2 Basic concepts

1.2.1 Clifford algebra

We begin by recalling the concept of a Clifford algebra. For more details the reader is referred to the paper [1].

Let V be a vector space with a quadratic form Q. Consider the tensor algebra

$$T(V) = \sum_{r=0}^{\infty} \otimes^r V$$

and let I denote the ideal in $T(V)$ generated by elements of the form $v \otimes v + Q(v) \cdot 1$ for $v \in V$. Then the quotient $Cl_Q(V) = T(V)/I$ is defined to be the Clifford algebra of V with quadratic form Q. For this paper, we consider $V = \mathbf{R}^n$, $Q(v) = < v, v >$ where $< \cdot, \cdot >$ is the standard inner product on $\mathbf{R}^n$. Let $e_1, \cdots, e_n$ be any orthonormal basis for $\mathbf{R}^n$. Then $Cl_Q(V)$ is generated as an algebra with unit by $e_1, \cdots, e_n$ subject to the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$$

for $1 \le i, j \le n$. We denote this algebra by Cl_n. These algebras satisfy the periodicity relation $Cl_{n+8} = Cl_n \otimes_R Cl_8$. We also consider the complexification of the algebras Cl_{2n} i.e. $\mathbf{Cl}_n = Cl_{2n} \otimes_R \mathbf{C}$ which satisfy $\mathbf{Cl}_{n+2} = \mathbf{Cl}_n \otimes_{\mathbf{C}} \mathbf{Cl}_2$. The first eight are given by the following table.

n	1	2	3	4	5	6	7	8
Cl_n	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H} \oplus \mathbf{H}$	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$\mathbf{R}(8) \oplus \mathbf{R}(8)$	$\mathbf{R}(16)$
$\mathbf{Cl}_n$		$\mathbf{Cl}(2)$		$\mathbf{C}(4)$		$\mathbf{C}(8)$		$\mathbf{C}(16)$

1.2.2 Clifford cohomology groups $H_{cl}^{p,q}(X)$

Our exposition here follows closely the paper by Michelsohn [8].

a. (p,q) decomposition of Cl(X)

Let X be a compact Kähler manifold of complex dimension n. Consider the tangent bundle $T(X)$ of X as a real 2n-dimensional vector bundle.

We then form the bundle $Cl_{2n}(X)$ of real Clifford algebras and take its complexification $\mathbf{Cl}_n(X) = Cl_{2n}(X) \otimes_R \mathbf{C}$

At each point $x \in X$, the linear map $\mathcal{J}_o : T_x(X) \longrightarrow T_x(X)$, which is the complex structure of X and is parallel in the canonical Riemannian connection, extends naturally to $Cl_{2n}(X)_x$ and therefore to $\mathbf{Cl}_n(X)_x$, as a derivation. For any $x \in X$, we can choose an orthonormal basis of the form: $e_1, \mathcal{J}_o e_1, \cdots e_n, \mathcal{J}_o e_n$. In terms of these we define a new basis $\epsilon_1, \cdots, \epsilon_n, \bar{\epsilon}_1, \cdots, \bar{\epsilon}_n$ of $T_x(X) \otimes_R \mathbf{C}$ by setting

$$\begin{aligned} \epsilon_j &= \frac{1}{2}(e_j - i\mathcal{J}_o e_j) \\ \bar{\epsilon}_j &= \frac{1}{2}(e_j + i\mathcal{J}_o e_j) \end{aligned} \tag{1.1}$$

for $j = 1, \cdots n$. These elements have the property that $\mathcal{J}_o(\epsilon_j) = i\epsilon_j$, $\mathcal{J}_o(\bar{\epsilon}_j) = -i\bar{\epsilon}_j$. For convenience we set

$$\mathcal{J} = \frac{1}{i}\mathcal{J}_o$$

then $\mathcal{J}(\epsilon_j) = \epsilon_j$, $\mathcal{J}_o(\bar{\epsilon}_j) = -\bar{\epsilon}_j$. In the algebra $\mathbf{Cl}_n$, all pairs of elements from (1.1) anti-commute except for those of type $\epsilon_j, \bar{\epsilon}_j$ which satisfy the relations

$$\epsilon_j \cdot \bar{\epsilon}_j + \bar{\epsilon}_j \cdot \epsilon_j = -1$$

In particular

$$\epsilon_j \cdot \epsilon_j = \bar{\epsilon}_j \cdot \bar{\epsilon}_j = 0$$

where $\cdot$ means Clifford multiplication. Therefore the elements of the form $\epsilon_I \bar{\epsilon}_J = \epsilon_{i_1} \cdots \epsilon_{i_r} \bar{\epsilon}_{j_1} \cdots \bar{\epsilon}_{j_s}$, where I, J range over all strictly ascending multi-indices from $\{1, \cdots n\}$, form an additive basis for $\mathbf{Cl}_n(X)$. By the derivation property of $\mathcal{J}$

$$\mathcal{J}(\epsilon_I\bar{\epsilon}_J) = (|I| - |J|)\epsilon_I\bar{\epsilon}_J = (r - s)\epsilon_I\bar{\epsilon}_J$$

where $|I|, |J|$ denote the lengths of the multi-indices I and J.

Now we can define three natural operators $\mathcal{L}$ $\bar{\mathcal{L}}$ and $\mathcal{H}$ on the bundle $\mathbf{Cl}_n(X)$ as follows. For any $\varphi \in \mathbf{Cl}(X)$,

$$\mathcal{L}(\varphi) = -\sum_{j=1}^{n} \epsilon_j \cdot \varphi \cdot \bar{\epsilon}_j$$

$$\bar{\mathcal{L}}(\varphi) = -\sum_{j=1}^{n} \bar{\epsilon}_j \cdot \varphi \cdot \epsilon_j$$

$$\mathcal{H} = [\mathcal{L}, \bar{\mathcal{L}}]$$

These operators are independent of the choice of basis $\epsilon_1, \cdots \epsilon_n, \bar{\epsilon}_1, \cdots \bar{\epsilon}_n$. The operators $\mathcal{L}$, $\bar{\mathcal{L}}$, $\mathcal{H}$ define a representation of $\mathfrak{Sl}(2, \mathbf{C})$ on $\mathbf{Cl}_n(X)$. Therefore we can define the subspaces

$$\mathbf{Cl}_n^{p,q}(X) = \{\varphi \in \mathbf{Cl}_n(X) \ \mid \ \mathcal{H}\varphi = q\varphi, \mathcal{J}\varphi = p\varphi\}$$

We get a decomposition

$$\mathbf{Cl}_n(X) = \bigoplus_{p,q} \mathbf{Cl}_n^{p,q}(X)$$

For a given p, $-n \leq p \leq n$, only certain of the bundles $\mathbf{Cl}^{p,q}(X)$ can be non-zero. In fact $\mathbf{Cl}^{p,q}(X)$ is non-zero only if $|p| + |q| \leq n$ and $p + q + n = 0 (\text{mod } 2)$. Hence the bundles appear only for values of (p, q) marked in the "diamond" pictured on the page 13.

b. Differential operators $\mathcal{D}, \bar{\mathcal{D}}$

We now define differential operators on sections of the Clifford bundle as follows

$$\mathcal{D} = \sum_{j=1}^{n} \epsilon_j \cdot \nabla_{\bar{\epsilon}_j}$$
$$\bar{\mathcal{D}} = \sum_{j=1}^{n} \bar{\epsilon}_j \cdot \nabla_{\epsilon_j}$$

where ∇ denotes the Riemannian connection. These operators are independent of the choice of the ϵ_j's. The following theorem defines the Clifford cohomology groups on X.

Theorem [8]: The operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoints of one another. They satisfy $\mathcal{D}^2 = 0 = \bar{\mathcal{D}}^2$. Furthermore, the complex $\xrightarrow{\mathcal{D}} \Gamma\mathbf{Cl}^{p-1,q-1}(X) \xrightarrow{\mathcal{D}} \Gamma\mathbf{Cl}^{p,q}(X) \xrightarrow{\mathcal{D}} \Gamma\mathbf{Cl}^{p+1,q+1}(X) \rightarrow$ is elliptic. Hence there are defined finite dimensional Clifford cohomology groups

$$H_{cl}^{p,q}(X) = ker\mathcal{D}/Im\mathcal{D} \cap \Gamma\mathbf{Cl}^{p,q}(X)$$

Remark: More generally, if W is a holomorphic hermitian bundle over X, we introduce the canonical hermitian connection on W, the tensor product connection on $\mathbf{Cl}(X, W)$, and then define $\mathcal{D}$ and $\bar{\mathcal{D}}$ as above on $\Gamma\mathbf{Cl}(X, W)$. In this case the theorem still holds. Thus we have Clifford cohomology groups with coefficients in W.

$$H_{cl}^{p,q}(X, W) = ker\mathcal{D}/Im(\mathcal{D}) \cap \Gamma\mathbf{Cl}^{p,q}(X, W)$$

which satisfy a fundamental duality theorem.

c. Hodge theory for the Clifford cohomology

From the standard elliptic theory, we introduce the Laplacian

$$\Delta = \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}$$

and consider the space of harmonic (p,q)-elements.

$$H^{p,q}(X,W) = ker(\Delta) \cap \Gamma\mathbf{Cl}^{p,q}(X,W)$$

Then there is an orthogonal decomposition

$$\Gamma\mathbf{Cl}^{p,q}(X,W) = H^{p,q}(X,W) \oplus Im\mathcal{D} \oplus Im\bar{\mathcal{D}}$$

which holds also for the space of L^2 sections. This gives an isomorphism

$$H^{p,q}_{cl}(X,W) \approx H^{p,q}(X,W)$$

Therefore for any $[\varphi] \in H^{p,q}_{cl}(X,W)$, there is a harmonic representative $\tilde{\varphi}$ i.e. $\Delta\tilde{\varphi} = 0$ and $\tilde{\varphi} \in [\varphi]$. With the construction we have, we can prove the vanishing of certain Clifford cohomology groups under some curvature assumptions by using the Bochner technique [5].

Fix $x \in X$ and choose local frames $\epsilon_1, \cdots, \epsilon_n, \bar{\epsilon}_1, \cdots, \bar{\epsilon}_n$ as before such that $(\nabla\epsilon_j)_x = (\nabla\bar{\epsilon}_j)_x = 0$ i.e. choose local normal coordinates at x.

Consider that

$$\begin{aligned}\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} &= \sum_{j,k}\{\epsilon_j \cdot \bar{\epsilon}_k \cdot \nabla_{\bar{\epsilon}_j,\epsilon_k} + \bar{\epsilon}_k \cdot \epsilon_j \cdot \nabla_{\epsilon_k,\bar{\epsilon}_j}\} \\ &= \sum_{j,k}\{\epsilon_j \cdot \bar{\epsilon}_k \cdot \nabla_{\bar{\epsilon}_j,\epsilon_k} - (\epsilon_j\bar{\epsilon}_k + \delta_{j,k})\nabla_{\epsilon_k,\bar{\epsilon}_j}\} \\ &= \sum_{j,k}\{\epsilon_j \cdot \bar{\epsilon}_k \cdot R_{\bar{\epsilon}_j,\epsilon_k} - \sum_j \nabla_{\epsilon_j,\bar{\epsilon}_j}\}\end{aligned}$$

where $\sum_{j,k} = \sum_{j,k=1}^{n}$, $R_{V,W} \equiv \nabla_{V,W} - \nabla_{W,V}$ is the curvature of the connection and $\nabla_{V,W} \equiv \nabla_V \nabla_W - \nabla_{\nabla_V^W}$ is the invariant second covariant derivative. We define operators on $\Gamma\mathbf{Cl}(X, W)$ by

$$\nabla^*\nabla = -\sum_j \nabla_{\epsilon_j, \bar{\epsilon}_j}$$
$$\bar{\nabla}^*\bar{\nabla} = -\sum_j \nabla_{\bar{\epsilon}_j, \epsilon_j}$$
$$R = \sum_{j,k} \epsilon_j \cdot \bar{\epsilon}_k \cdot R_{\bar{\epsilon}_j, \epsilon_k}$$
$$\bar{R} = \sum_{j,k} \bar{\epsilon}_j \cdot \epsilon_k \cdot R_{\epsilon_j, \bar{\epsilon}_k}$$

We have the following Weitzenböck type formula.

Proposition [8]:

$$\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \nabla^*\nabla + R = \bar{\nabla}^*\bar{\nabla} + \bar{R}$$

The importance of the formula derives from the following.

Proposition [8]: The operators $\nabla^*\nabla$ and $\bar{\nabla}^*\bar{\nabla}$ are non-negative, elliptic, formally self-adjoint differential operators. The zero-order operators R and $\bar{R}$ are self-adjoint.

Remark: Since the connection on $\Gamma\mathbf{Cl}(X)$, defined by the connection on the tangent bundle $T(X)$, acts as a derivation, i.e.

$$\nabla(U \cdot V) = (\nabla^T U) \cdot V + U \cdot (\nabla^T V)$$

where ∇^T denotes the connection on $T(X)$. So does the curvature tensor, i.e.

$$R(U \cdot V) = (R^T U) \cdot V + U \cdot (R^T V)$$

where R^T denotes the curvature tensor on $T(X)$. Similarly, the connection on $\Gamma\mathbf{Cl}(X, W)$ acts as a derivation, i.e.

$$\nabla(\sigma \otimes w) = \nabla^{cl}\sigma \otimes w + \sigma \otimes \nabla^W w$$

where ∇^{cl}, ∇^W denote the connections on $\mathbf{Cl}(X)$ and W respectively. So does the curvature tensor on $\Gamma\mathbf{Cl}(X, W)$, i.e.

$$R(\sigma \otimes w) = R^{cl}\sigma \otimes w + \sigma \otimes R^W w$$

where R^{cl} and R^W denote the curvature tensors of $\mathbf{Cl}(X)$ and W respectively.

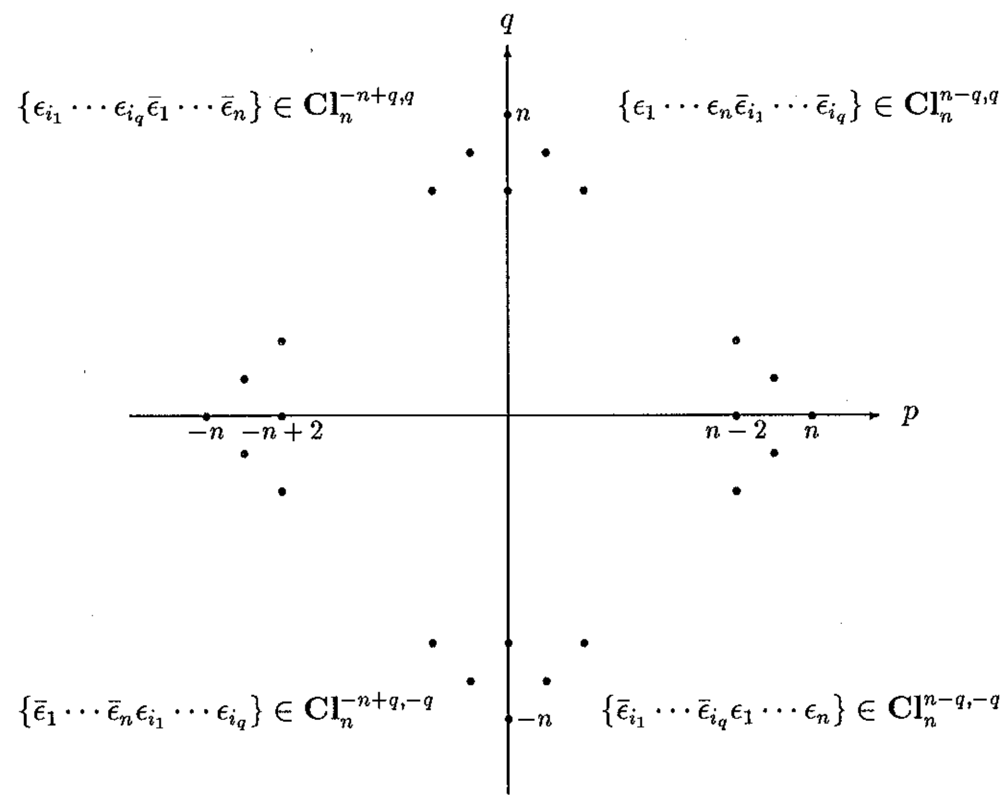
Chapter 2

In this chapter we shall show the Clifford bundle intimately reflects the properties of the Kähler manifold. Because the rich algebraic structure which is presented using Clifford multiplication on $\mathbf{Cl}(X)$ depends on the Riemannian metric, this bundle carries more information than $\bigwedge^*(X)$ for studying the Riemannian structure of the manifolds. The Clifford formalism is extremely effective in relating the differential geometry and the complex structure of a Kähler manifold. One sees precisely the role of Ricci tensor in the formulas for $\mathbf{Cl}_n$. One sees that it is the holomorphic square root of the canonical line bundle on X which reveals the profound relations between scalar curvature and $\hat{A}$ -genus of X. One also sees that the Clifford formalism indeed offers unified, systematic and comprehensive proofs for a wide variety of vanishing theorems.

2.1 The subspaces $\mathbf{Cl}_n^{p,q}(X)$ with $|p| + |q| = n$

First let us give a specific description for the subspaces $\mathbf{Cl}_n^{p,q}(X)$ of $\mathbf{Cl}_n$ with $|p| + |q| = n$.

Proposition 2.1.1 *In the local coordinates at* x *chosen as before, the elements* $\{\epsilon_1 \cdots \epsilon_n \, \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q}\}$*, where* $i_1 < \cdots < i_q$, $i_j \in \{1, \cdots n\}, 1 \leq j \leq q$*, form a basis for* $\mathbf{Cl}^{n-q,q}_{n,x}(X)$*. In fact we have the following "diamond" [8].*



Proof: Let P_n be the set of primitive elements in $\mathbf{Cl}_n$. From the proposition 2.10 [8], we know that:

$$P_n = \bigoplus_{|r|+s \leq n} P^{r,s}_n \qquad s > 0$$

$$P_n = \{\epsilon_n \cdot P_{n-1}\} \oplus \{P_{n-1} \cdot \bar{\epsilon}_n\} \oplus \{\epsilon_n P_{n-1} \bar{\epsilon}_n\}$$
$$\oplus \{\bar{\mathcal{L}}_n(\epsilon_n P_{n-1} \bar{\epsilon}_n) - z \bar{\epsilon}_n \epsilon_n P_{n-1}\} \tag{2.1}$$

where $z = s + 1$. If $\varphi \in P_{n-1}^{r,s}$, then

$$\begin{aligned}
&\epsilon_n \cdot \varphi \in P_n^{r+1,s} \\
&\varphi \cdot \bar{\epsilon}_n \in P_n^{r-1,s} \\
&\epsilon_n \varphi \bar{\epsilon}_n \in P_n^{r,s+1} \\
&\bar{\mathcal{L}}_n(\epsilon_n \cdot \varphi \cdot \bar{\epsilon}_n) - z\bar{\epsilon}_n \epsilon_n \varphi \in P_n^{r,s-1} \qquad (2.2)
\end{aligned}$$

More precisely, from (2.1) (2.2) above we have:

$$\begin{aligned}
P_n^{p,q} = \{\epsilon_n \cdot P_{n-1}^{p-1,q}\} \oplus \{P_{n-1}^{p+1,q} \cdot \bar{\epsilon}_n\} \oplus \{\epsilon_n \cdot P_{n-1}^{p,q-1} \cdot \bar{\epsilon}_n\} \\
\oplus \{\bar{\mathcal{L}}_n(\epsilon_n \cdot P_{n-1}^{p,q+1} \cdot \bar{\epsilon}_n) - (q+2)\bar{\epsilon}_n \cdot \epsilon_n \cdot P_{n-1}^{p,q+1}\}
\end{aligned}$$

In particular, if $p = n - q$, *i.e.* , $p + q = n$

$$\begin{aligned}
P_n^{n-q,q} = \{\epsilon_n \cdot P_{n-1}^{n-q-1,q}\} \oplus \{P_{n-1}^{n+1-q,q} \cdot \bar{\epsilon}_n\} \oplus \{\epsilon_n \cdot P_{n-1}^{n-q,q-1} \cdot \bar{\epsilon}_n\} \\
\oplus \{\bar{\mathcal{L}}_n(\epsilon_n \cdot P_{n-1}^{n-q,q+1} \cdot \bar{\epsilon}_n) - (q+2)\bar{\epsilon}_n \cdot \epsilon_n \cdot P_{n-1}^{n-q,q+1}\}
\end{aligned}$$

But

$$\begin{aligned}
P_{n-1}^{n+1-q,q} = \phi, \quad \text{since} \quad n + 1 - q + q = n + 1 > n - 1 \\
P_{n-1}^{n-q,q+1} = \phi, \quad \text{since} \quad n - q + q + 1 = n + 1 > n - 1
\end{aligned}$$

$$\begin{aligned}
P_n^{n-q,q} = \{\epsilon_n \cdot P_{n-1}^{n-q-1,q}\} \\
\oplus \{\epsilon_n \cdot P_{n-1}^{n-1-(q-1),q-1} \cdot \bar{\epsilon}_n\} \quad (1 \le q \le n-1) \qquad (2.3)
\end{aligned}$$

$$P_n^{n,0} = \{\epsilon_n \cdot P_{n-1}^{n-1,0}\}, \quad \text{since} \quad P_{n-1}^{n,-1} = \phi \quad (q = 0) \qquad (2.4)$$

$$P_n^{0,n} = \{\epsilon_n \cdot P_{n-1}^{0,n-1} \cdot \bar{\epsilon}_n\}, \quad \text{since} \quad P_{n-1}^{-1,n} = \phi \quad (q = n) \qquad (2.5)$$

The proof of the proposition (2.1.1) follows easily from the following two lemmas.

Lemma 2.1.2

$$dim_{\mathbf{c}} P_n^{n-q,q} = \begin{pmatrix} n \\ q \end{pmatrix} \qquad q \geq 0$$

Proof: By induction on the dimension n.

$$n = 1 \qquad dim_{\mathbf{c}} P^{1,0} = 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad dim_{\mathbf{c}} P^{0,1} = 1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So lemma (2.1.2) holds.

Suppose claim holds for $n-1$, i.e. we have

$$dim_{\mathbf{c}} P_{n-1}^{(n-1)-q,q} = \begin{pmatrix} n-1 \\ q \end{pmatrix} \qquad 0 \leq q \leq n-1$$

Case 1: $1 \leq q \leq n-1$, from (2.3)

$$\begin{aligned} dim_{\mathbf{c}} P_n^{n-q,q} &= dim_{\mathbf{c}} P_{n-1}^{(n-1)-q,q} + dim_{\mathbf{c}} P_{n-1}^{n-1-(q-1),q-1} \\ &= \begin{pmatrix} n-1 \\ q \end{pmatrix} + \begin{pmatrix} n-1 \\ q-1 \end{pmatrix} \\ &= \begin{pmatrix} n \\ q \end{pmatrix} \end{aligned}$$

Case 2: $q = 0$, from (2.4)

$$dim_{\mathbf{C}} P_n^{n,0} = dim_{\mathbf{C}} P_{n-1}^{(n-1),0} = \begin{pmatrix} n-1 \\ 0 \end{pmatrix} = 1 = \begin{pmatrix} n \\ 0 \end{pmatrix}$$

Case 3: $q = n$, from (2.5)

$$dim_{\mathbf{C}} P_n^{0,n} = dim_{\mathbf{C}} P_{n-1}^{0,(n-1)} = \begin{pmatrix} n-1 \\ n-1 \end{pmatrix} = 1 = \begin{pmatrix} n \\ n \end{pmatrix}$$

Q.E.D of (2.1.2)

Lemma 2.1.3 *Any* $\varphi = \epsilon_1 \cdots \epsilon_n \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q}$ *belongs to* $P_n^{n-q,q}$.

Proof: Recall

$$\mathcal{L}\varphi = -\sum_{k=1}^{n} \epsilon_k \varphi \bar{\epsilon}_k$$

Since $\epsilon_k \cdot \epsilon_k = 0$ so $\mathcal{L}\varphi = 0$ i.e. φ is a primitive element. It is very easy to see $\mathcal{J}\varphi = n - q$, so we only need to show that $\mathcal{H}\varphi = q\varphi$. Using the formula

$$\mathcal{H}(\varphi \cdot \bar{\xi}) = \mathcal{H}(\varphi) \cdot \bar{\xi} + \varphi \cdot \bar{\xi}$$

where $\bar{\xi} \in T^{0,1}$, $T = T^{0,1} \oplus T^{0,1}$.

By induction on q.

If $q = 0$ then $\varphi = \epsilon_1 \cdots \epsilon_n$ and it is obvious that $\mathcal{H}(\varphi) = 0$, and the claim holds.

Assume $\varphi = \epsilon_1 \cdots \epsilon_n \cdot \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_{q-1}}$ and $\mathcal{H}\varphi = (q-1)\varphi$.

Show that $\varphi = \epsilon_1 \cdots \epsilon_n \cdot \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_{q-1}} \bar{\epsilon}_{i_q}$ satisfies $\mathcal{H}(\varphi) = q\varphi$.

$$\begin{aligned}\mathcal{H}(\varphi) &= \mathcal{H}(\epsilon_1 \cdots \epsilon_n \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_{q-1}} \bar{\epsilon}_{i_q}) \\ &= \mathcal{H}(\epsilon_1 \cdots \epsilon_n \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_{q-1}}) \cdot \bar{\epsilon}_{i_q} + \epsilon_1 \cdots \epsilon_n \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_{q-1}} \bar{\epsilon}_{i_q} \\ &= (q-1)\varphi + \varphi = q\varphi\end{aligned}$$

Q.E.D.of (2.1.3)

Since

$$P_n^{n-q,q} = \mathbf{Cl}_n^{n-q,q}$$

Thus

$$dim_{\mathbf{c}} \mathbf{Cl}_n^{n-q,q} = \begin{pmatrix} n \\ q \end{pmatrix}$$

and the set $\{\epsilon_1 \cdots \epsilon_n \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q}\}$ has $\begin{pmatrix} n \\ q \end{pmatrix}$ independent elements of $\mathbf{Cl}_{n,x}^{n-q,q}$, so $\{\epsilon_1 \cdots \epsilon_n \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q}\}$ form a basis for $\mathbf{Cl}_{n,x}^{n-q,q}$.

Using the same arguments we have the description of the subspaces $\mathbf{Cl}_n^{p,q}$ with $|p| + |q| = n$ in the "diamond".

Q.E.D.

2.2 Vanishing theorem for $H_{cl}^{n-q,q}(X, W)$

Note that $H_{cl}^{n-q,q}(X, W) \approx H_{Dol}^0(X, \theta^{n-q}(W^*))$ [8], where W^* denotes the dual vector bundle of W. The results proved in this section recapture a number of known vanishing theorem for holomorphic sections on X, see for example [4], [5], [8]. But the method employed here makes the proof extremely simple.

Definition 2.2.1 *Let W be any holomorphic hermitian vector bundle over X. Define the mean curvature transformation of W by*

$$\begin{aligned} \hat{R}^W : \Gamma(X,W) &\longrightarrow \Gamma(X,W) \\ w &\longmapsto \sum_j R^W_{\epsilon_j,\bar{\epsilon}_j} w \end{aligned}$$

Note that $\hat{R}^W$ is independent of the hermitian basis chosen to define it.

We prove the following formula. If $\varphi \in \Gamma \mathrm{Cl}^{n-q,q}(X,W)$, in the chosen local normal coordinates at $x \in X$, let

$$\varphi = \epsilon_1 \cdots \epsilon_n \cdot \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q} \otimes w$$

Proposition 2.2.2

$$\begin{aligned} (\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi &= \bar{\nabla}^*\bar{\nabla}\varphi \qquad (2.6) \\ &= \nabla^*\nabla\varphi + \frac{1}{2}\Big(\sum_{j=q+1}^{n} Ric_{i_j}\Big)\varphi \\ &\quad + \epsilon_1 \cdots \epsilon_n \cdot \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q} \otimes \sum_j R^W_{\epsilon_j,\bar{\epsilon}_j} w \qquad (2.7) \end{aligned}$$

where Ric_{i_j} denotes the eigenvalue of the Ricci tensor on T(X).

Note that Michelsohn has proved the same formulas by using the bundle of modules over $\mathbf{Cl}(X)$ when W is trivial [8].

Let us prove a lemma first.

Lemma 2.2.3

$$R = \bar{R} + \sum_j R_{\epsilon_j, \bar{\epsilon}_j}$$

Proof: Recall in local normal coordinates

$$\begin{aligned} R &= \sum_{j,k} \epsilon_j \cdot \bar{\epsilon}_k \cdot R_{\bar{\epsilon}_j, \epsilon_k} \\ &= \sum_{k,j} \epsilon_k \cdot \bar{\epsilon}_j \cdot R_{\bar{\epsilon}_k, \epsilon_j} \\ &= \sum_{k,j} (-\bar{\epsilon}_j \epsilon_k - \delta_{jk}) R_{\bar{\epsilon}_k, \epsilon_j} \\ &= \sum_{j,k} \bar{\epsilon}_j \epsilon_k R_{\epsilon_j, \bar{\epsilon}_k} + \sum_j R_{\epsilon_j, \bar{\epsilon}_j} \\ &= \bar{R} + \sum_j R_{\epsilon_j, \bar{\epsilon}_j} \end{aligned}$$

Q.E.D.

Proof of (2.2.2)

Recall $R, \bar{R}$ act as a derivation. Claim $\bar{R}\varphi \equiv 0$. Use the fact $\epsilon_j \cdot \epsilon_j = 0$

$$\begin{aligned} \bar{R}\varphi &= \sum_{j.k} \bar{\epsilon}_j \cdot \epsilon_k R_{\epsilon_j, \bar{\epsilon}_k} (\epsilon_1, \cdots \epsilon_n \cdot \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q} \otimes w) \\ &= \sum_{j.k} \bar{\epsilon}_j \cdot \epsilon_k (R_{\epsilon_j, \bar{\epsilon}_k} \epsilon_1, \cdots \epsilon_n) \cdot \bar{\epsilon}_{i_1} \cdots \bar{\epsilon}_{i_q} \otimes w \end{aligned}$$

$$+\sum_{j.k}\bar{\epsilon}_j\cdot\epsilon_k\epsilon_1,\cdots\epsilon_n R_{\epsilon_j,\bar{\epsilon}_k}(\bar{\epsilon}_{i_1}\cdots\bar{\epsilon}_{i_q}\otimes w)$$
$$=\frac{1}{2}\sum_j R_{ic_j}\bar{\epsilon}_j\cdot\epsilon_j\cdot\epsilon_1\cdots\epsilon_n\bar{\epsilon}_{i_1}\cdots\bar{\epsilon}_{i_q}\otimes w+0$$
$$=0$$

From the lemma, we have

$$R=\bar{R}+\sum_j R_{\epsilon_j,\bar{\epsilon}_j}\quad\text{since}\quad\bar{R}\equiv 0$$
$$R\varphi=\sum_j R_{\epsilon_j,\bar{\epsilon}_j}\varphi$$
$$=\sum_j R_{\epsilon_j,\bar{\epsilon}_j}(\epsilon_1,\cdots\epsilon_n\cdot\bar{\epsilon}_{i_1}\cdots\bar{\epsilon}_{i_q}\otimes w)$$
$$=\frac{1}{2}(\sum_{j=1}^{n}Ric_j-\sum_{j=1}^{q}Ric_{i_j})\varphi+\epsilon_1\cdots\epsilon_n\bar{\epsilon}_{i_1}\cdots\bar{\epsilon}_{i_q}\otimes\sum_j R^W_{\epsilon_j,\bar{\epsilon}_j}w$$
$$=\frac{1}{2}(\sum_{j=q+1}^{n}Ric_{i_j})\varphi+\epsilon_1\cdots\epsilon_n\bar{\epsilon}_{i_1}\cdots\bar{\epsilon}_{i_q}\otimes\sum_j R^W_{\epsilon_j,\bar{\epsilon}_j}w$$

Q.E.D.

We are now in a position to prove the following vanishing theorem of Bochner type. Applying (2.2.2) to $W=X\times\mathbf{C}^r$ we have

Theorem 2.2.4 *If X is a compact Kähler manifold with $dim_{\mathbf{C}}X=n$, and fixed an integer q, $0\le q\le n-1$. If the eigenvalues $Ric_1,\cdots,Ric_n$ of the Ricci tensor satisfy the inequality*

$$Ric_{i_1}+\cdots+Ric_{i_{n-q}}>0\quad\text{for all}\quad i_1<\cdots<i_{n-q}$$

at each point of x, then $H^{n-q,q}_{cl}(X)=0$, $0\le q\le n-1$.

Note that since there is an isomorphism between Clifford and Dolbeault cohomology groups, under the assumption, this theorem proves that X admits no nonzero holomorphic $(n-q)$–form. This theorem is proved in this form in [4].

Theorem 2.2.5 *Let X be a compact Kähler manifold with non-negative Ricci tensor and positive scalar curvature, then*

$$H_{cl}^{n-q,q}(X, K^*) = 0 \qquad \forall q$$

where K^* denotes the anti-canonical line bundle of X i.e. $K^* \approx \bigwedge^n T$.

Proof: Applying (2.2.2) to $W = K^*$, compute R^W in local normal coordinates. If $W \in \Gamma(X, K^*)$ then

$$W = a\epsilon_1 \wedge \cdots \wedge \epsilon_n \qquad a \in \mathbf{C}$$

and R^W acts as a derivation.

$$\begin{aligned} \sum_j R^W_{\epsilon_j,\bar{\epsilon}_j}(a\epsilon_1 \wedge \cdots \wedge \epsilon_n) &= \frac{1}{2}(\sum_{j=1}^n Ric_j) a\epsilon_1 \wedge \cdots \wedge \epsilon_n \\ &= \frac{r}{4} a\epsilon_1 \wedge \cdots \wedge \epsilon_n \end{aligned}$$

where $r = 2\sum_{j=1}^n Ric_j$ is the scalar curvature of X.

Q.E.D.

Theorem 2.2.6 *X as above. Let W be a holomorphic vector bundle on X. If R^W is positive definite, then*

$$H_{cl}^{0,n}(X,W)=0$$

Proof: Let $q=n$ in the (2.2.2) then

$$\varphi=\epsilon_1\cdots\epsilon_n\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes w$$

By (2.6)+(2.7) we have

$$2(\mathcal{D}\bar{\mathcal{D}}+\bar{\mathcal{D}}\mathcal{D})\varphi=(\nabla^*\nabla+\bar{\nabla}^*\bar{\nabla})\varphi+\epsilon_1\cdots\epsilon_n\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes\sum_j R^W_{\epsilon_j,\bar{\epsilon}_j}w$$

Q.E.D.

Now applying (2.2.2) to $E=T^*$ then

$$\sum_j R^{T^*}_{\epsilon_j,\bar{\epsilon}_j}\bar{\epsilon}_k=-Ric_k\bar{\epsilon}_k$$

We obtain following corollary.

Corollary 2.2.7 *Let X be a compact Kähler manifold with negative definite Ricci tensor then*

$$H_{cl}^{0,n}(X,T^*)=0$$

where T^* denotes the cotangent bundle of X.

Note that corollary 2.2.7 proves that X admits no nonzero holomorphic vector field under the curvature assumption. In this form the theorem is proved in [5].

If we apply (2.2.2) to $E = \otimes^p T = \underbrace{T \otimes \cdots \otimes T}_{p}$, it then yields the following

Corollary 2.2.8 *If X is a compact Kähler manifold with positive Ricci tensor, then*

$$H_{cl}^{0,n}(X, \otimes^p T) = 0 \qquad p > 0$$

Corollary [8] Let X be a compact Kähler manifold with $c_1(X) = 0$, then any harmonic form is parallel.

This is because X can carry a Ricci flat metric in this case and let W be any trival bundle, the formula (2.2.2) becomes

$$2\Delta\varphi = (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi$$

2.3 Vanishing theorem for $H_{cl}^{-n+q,q}(X, L)$

In this section we shall give alternate proofs for a series of vanishing theorems for holomorphic line bundles, see for example [4], [5], [8]. More specific references are given in the individual theorems. Clifford formalism makes the proof simple and gives delicate results.

Let L be a holomorphic hermitian line bundle over X. The curvature R^L of L is a closed real (1,1)–form on X. If $\{\epsilon_j, \bar{\epsilon}_j\}$ are local hermitian frames chosen as before, the matrix $(R^L_{\bar{\epsilon}_j,\epsilon_k})$ is hermitian symmetric. If this matrix is positive definite (resp. negative definite) at each point of X, then L is called a positive (resp. negative) hermitian line bundle.

First we show the following formula. If $\varphi \in \Gamma \mathbf{Cl}^{-n+q,q}(X, L)$, in the chosen local normal coordinates at $x \in X$,

$$\varphi = \epsilon_{i_1} \cdots \epsilon_{i_q} \cdot \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \otimes l$$

Proposition 2.3.1

$$(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi = \nabla^*\nabla\varphi + (\sum_{j=1}^{q} \lambda_{i_j})\varphi \tag{2.8}$$

$$= \bar{\nabla}^*\bar{\nabla}\varphi + \sum_{j=q+1}^{n} (\frac{1}{2}Ric_{i_j} - \lambda_{i_j})\varphi \tag{2.9}$$

where $\{\lambda_j\}$ $1 \le j \le n$ *are the eigenvalues for* R^L, *i.e. since* R^L *is hermitian symmetric, we may choose local hermitian frames such that* $R^L_{\epsilon_j,\bar{\epsilon}_k} = \lambda_j \delta_{jk}$.

Note that Michelsohn has proved the similar formulas by using the bundle of modules over $\mathbf{Cl}(X)$ in her paper [8].

Before we prove (2.3.1) we show the following lemma.

Lemma 2.3.2

$$\lambda_j \bar{\epsilon}_j \epsilon_j \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n = \begin{cases} 0 & \textit{if } j = i_k \quad 1 \le k \le q \\ -\lambda_{i_l} \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n & \textit{if } j = i_l \quad q+1 \le l \le n \end{cases}$$

<u>Proof</u>:

Case 1: if $j = i_k \quad 1 \le k \le q$

$$\begin{aligned} &\lambda_j \bar{\epsilon}_j \epsilon_j \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \\ &= \pm \lambda_{i_k} \bar{\epsilon}_{i_k} \epsilon_{i_k} \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \\ &= 0 \qquad \text{since } \epsilon_j \cdot \epsilon_j = 0 \end{aligned}$$

Case 2: if $j = i_l \quad q+1 \le l \le n$

$$\begin{aligned} &\lambda_j \bar{\epsilon}_j \epsilon_j \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \\ &= -\lambda_{i_l} (\epsilon_{i_l} \bar{\epsilon}_{i_l} + 1) \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \\ &= -\lambda_{i_l} \epsilon_{i_l} \bar{\epsilon}_{i_l} \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n - \lambda_{i_l} \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \\ &= -\lambda_{i_l} \epsilon_{i_1} \cdots \epsilon_{i_q} \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \qquad \text{since } \bar{\epsilon}_j \cdot \bar{\epsilon}_j = 0 \text{ and } \epsilon_j \cdot \bar{\epsilon}_k = -\bar{\epsilon}_k \cdot \epsilon_j,\ j \ne k \end{aligned}$$

Q.E.D.

Proof of (2.3.1)

Since

$$R(\epsilon_{i_1} \cdots \epsilon_{i_q} \cdot \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \otimes l) = R^{cl}(\epsilon_{i_1} \cdots \epsilon_{i_q} \cdot \bar{\epsilon}_1 \cdots \bar{\epsilon}_n) \otimes l + \epsilon_{i_1} \cdots \epsilon_{i_q} \cdot \bar{\epsilon}_1 \cdots \bar{\epsilon}_n \otimes R^L l$$

So first we compute that

$$\begin{aligned}
&\bar{R}^{cl}(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n)\\
&=\sum_{j,k}\bar{\epsilon}_j\epsilon_k R^{cl}_{\epsilon_j,\bar{\epsilon}_k}(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n)\\
&=\sum_{j,k}\bar{\epsilon}_j\epsilon_k(R^{cl}_{\epsilon_j,\bar{\epsilon}_k}\epsilon_{i_1}\cdots\epsilon_{i_q})\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\\
&+\sum_{j,k}\bar{\epsilon}_j\epsilon_k\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot(R^{cl}_{\epsilon_j,\bar{\epsilon}_k}\bar{\epsilon}_1\cdots\bar{\epsilon}_n)\\
&=2\sum_{j,k,\beta}\sum_{\alpha=1}^{q}\bar{\epsilon}_j\cdot\epsilon_k\cdot\epsilon_{i_1}\cdots\epsilon_{i_{\alpha-1}}\cdot\epsilon_\beta\cdot\epsilon_{i_{\alpha+1}}\cdots\epsilon_{i_q} < R^T_{\epsilon_j,\bar{\epsilon}_k}\epsilon_{i_\alpha},\bar{\epsilon}_\beta > \bar{\epsilon}_1\cdots\bar{\epsilon}_n\\
&+\sum_{j,k}-\frac{1}{2}Ric_j\bar{\epsilon}_j\cdot\epsilon_j\cdot\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\\
&=\sum_{j,k}-\frac{1}{2}Ric_j\bar{\epsilon}_j\cdot\epsilon_j\cdot\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n \qquad \text{by Lemma 2.3.2}\\
&=\frac{1}{2}\sum_{j=q+1}^{n}Ric_{i_j}\cdot\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{k,\beta}\bar{\epsilon}_j\epsilon_k\epsilon_{i_1}\cdots\epsilon_{i_{\alpha-1}}\cdot\epsilon_\beta\cdot\epsilon_{i_{\alpha+1}}\cdots\epsilon_{i_q} < R^T_{\epsilon_j,\bar{\epsilon}_k}\epsilon_{i_\alpha},\bar{\epsilon}_\beta >\\
&=\sum_{k,\beta}(-1)^{\alpha-1}\bar{\epsilon}_j\epsilon_k\epsilon_\beta\epsilon_{i_1}\cdots\epsilon_{i_{\alpha-1}}\cdot\epsilon_{i_{\alpha+1}}\cdots\epsilon_{i_q} < R^T_{\epsilon_j,\bar{\epsilon}_k}\epsilon_{i_\alpha},\bar{\epsilon}_\beta >\\
&=\sum_{\beta,k}(-1)^{\alpha-1}\bar{\epsilon}_j\epsilon_\beta\epsilon_k\epsilon_{i_1}\cdots\epsilon_{i_{\alpha-1}}\cdot\epsilon_{i_{\alpha+1}}\cdots\epsilon_{i_q} < R^T_{\epsilon_j,\bar{\epsilon}_\beta}\epsilon_{i_\alpha},\bar{\epsilon}_k >\\
&=\sum_{\beta,k}(-1)^{\alpha-1}(-1)\bar{\epsilon}_j\epsilon_k\epsilon_\beta\epsilon_{i_1}\cdots\epsilon_{i_{\alpha-1}}\cdot\epsilon_{i_{\alpha+1}}\cdots\epsilon_{i_q} < R^T_{\epsilon_j,\bar{\epsilon}_k}\epsilon_{i_\alpha},\bar{\epsilon}_\beta >\\
&=-\sum_{k,\beta}\bar{\epsilon}_j\epsilon_k\epsilon_{i_1}\cdots\epsilon_{i_{\alpha-1}}\cdot\epsilon_\beta\epsilon_{i_{\alpha+1}}\cdots\epsilon_{i_q} < R^T_{\epsilon_j,\bar{\epsilon}_k}\epsilon_{i_\alpha},\bar{\epsilon}_\beta >\\
&=0
\end{aligned}$$

$$R^{cl}(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n)$$

$$
\begin{aligned}
&= \bar{R}^{cl}(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n) + \sum_j R^{cl}_{\epsilon_j,\bar{\epsilon}_j}(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n) \\
&= \frac{1}{2}\sum_{j=q+1}^{n} Ric_{i_j}\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n - \frac{1}{2}\sum_{j=q+1}^{n} Ric_{i_j}\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n \\
&= 0
\end{aligned}
$$

Now we can calculate

$$
\begin{aligned}
&R(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l) \\
&= (R^{cl}\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n)\otimes l + \epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes R^L l \\
&= \sum_{j,k}\epsilon_j\cdot\bar{\epsilon}_k\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes R^L_{\bar{\epsilon}_j,\epsilon_k} l \\
&= -\sum_j \lambda_j\epsilon_j\cdot\bar{\epsilon}_j\cdot\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l \\
&= \sum_j \lambda_j(\bar{\epsilon}_j\epsilon_j+1)\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l \\
&= \sum_j \lambda_j\bar{\epsilon}_j\epsilon_j\cdot\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l + \sum_{j=1}^{n}\lambda_j\cdot\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l \\
&= (-\sum_{j=q+1}^{n}\lambda_{i_j})\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l + \sum_{j=1}^{n}\lambda_i\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l \\
&= \sum_{j=1}^{q}\lambda_{i_j}\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l
\end{aligned}
$$

$$
\begin{aligned}
&\bar{R}(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l) \\
&= R(\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l) - \sum_j R_{\epsilon_j,\bar{\epsilon}_j}\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l \\
&= \sum_{j=1}^{q}\lambda_{i_j}\epsilon_{i_1}\cdots\epsilon_{i_q}\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l + (\sum_{j=q+1}^{n}\frac{1}{2}Ric_{i_j} - \sum_{j=1}^{n}\lambda_j)\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l \\
&= \sum_{j=q+1}^{n}(\frac{1}{2}Ric_{i_j} - \lambda_{i_j})\epsilon_{i_1}\cdots\epsilon_{i_q}\cdot\bar{\epsilon}_1\cdots\bar{\epsilon}_n\otimes l
\end{aligned}
$$

Q.E.D.

Recall that we say X is spin, if the first Chern class $c_1(X) \equiv 0 \pmod 2$. Now let X be a compact Kähler manifold, then X is spin if and only if the canonical line bundle K has a square root. (i.e. a complex line bundle L such that $L \otimes L \simeq K$) [4]. We denote such L by $K^{\frac{1}{2}}$.

Applying (2.3.1) to $L = K^{*\frac{1}{2}}$, then $R^{K^{*\frac{1}{2}}}_{\epsilon_j,\bar{\epsilon}_k} = \frac{1}{4}Ric_j\delta_{jk}$ in local normal coordinates, i.e. $\lambda_j = \frac{1}{4}Ric_j, \ \ 1 \leq j \leq n$.

We get

$$
\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla}) + \sum_{j=q+1}^{n}(\frac{1}{2}Ric_{i_j} - \frac{1}{4}Ric_{i_j}) + \sum_{j=1}^{q}\frac{1}{4}Ric_{i_j} \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla}) + \sum_{j=1}^{n}\frac{1}{4}Ric_j \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla}) + \frac{r}{8}
\end{aligned}
$$

where r is the scalar curvature of X.

We then have the following theorem [7] for compact Kähler manifolds.

Theorem 2.3.3 *Let X be a spin manifold, if X carries a metric with positive scalar curvature, then*

$$H^{-n+q,q}_{cl}(X, K^{*\frac{1}{2}}) = 0 \quad \forall q$$

Let X be a Kähler manifold with $c_1(X) = k\alpha, \quad k \in \mathbb{Z}^+$, where $\alpha \in H^2(X;\mathbb{Z})$ is indivisible. We consider the line bundle $L \simeq K^{*\frac{1}{2}} \otimes K^{\frac{p}{2k}} \simeq K^{\frac{p-k}{2k}}$ over X in (2.3.1), $p \in \mathbb{Z}$.

Then

$$R^{K^{\frac{p-k}{2k}}} = -\frac{p-k}{4k} Ric_j \delta_{jk}$$

i.e.

$$\lambda_j = -\frac{p-k}{4k} Ric_j$$

We have

$$\begin{aligned}
&(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&= \nabla^*\nabla\varphi + \sum_{j=1}^{q} \lambda_{i_j}\varphi \\
&= \nabla^*\nabla\varphi + \frac{k-p}{4k}(\sum_{j=1}^{q} Ric_{i_j})\varphi
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&= \bar{\nabla}^*\bar{\nabla}\varphi + \frac{1}{2}\sum_{j=q+1}^{n} Ric_{i_j} - \sum_{j=q+1}^{n} \lambda_{i_j} \\
&= \bar{\nabla}^*\bar{\nabla}\varphi + \frac{1}{2}\sum_{j=q+1}^{n} Ric_{i_j} + \frac{p-k}{4k}\sum_{j=q+1}^{n} Ric_{i_j} \\
&= \bar{\nabla}^*\bar{\nabla}\varphi + \frac{p+k}{4k}(\sum_{j=q+1}^{n} Ric_{i_j})
\end{aligned} \tag{2.11}$$

We are able to prove the following theorem, which will improve the theorem 7.15 of Michelsohn [8] by relaxing the condition of $Ric > 0$ to $Ric \geq 0$ (not all 0).

Theorem 2.3.4 *X as above. If X admits a non-negative Ricci Kähler metric and positive scalar curvature, then*

$$H_{cl}^{-n+q,q}(X, K^{\frac{p-k}{2k}}) = 0 \qquad \forall q$$

wherever $|p| < k$ *and* $p+k$ *is even.*

Proof: (2.10)+(2.11) we get

$$
\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi \\
&+\frac{k-p}{4k}(\sum_{j=1}^{q} Ric_{i_j})\varphi + \frac{k+p}{4k}(\sum_{j=q+1}^{n} Ric_{i_j})\varphi
\end{aligned}
$$

Note that

$$
|p| < k \iff \begin{cases} k > p & \iff k-p > 0 \\ k < -p & \iff k+p > 0 \end{cases}
$$

Case 1: If $p \geq 0$, then $\frac{k+p}{4k} \geq \frac{k-p}{4k}$

We have

$$
\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&\geq (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{k-p}{4k}(\sum_{j=1}^{n} Ric_j)\varphi \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{k-p}{8k} r\varphi
\end{aligned}
$$

Case 2: If $p \leq 0$, then $\frac{k+p}{4k} \leq \frac{k-p}{4k}$

We have

$$
\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&\geq (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{k+p}{4k}(\sum_{j=1}^{n} Ric_j) \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{k+p}{8k} r\varphi
\end{aligned}
$$

Consequently

$$H_{cl}^{-n+q,q}(X, K^{\frac{p-k}{2k}}) = 0 \qquad \forall q$$

Q.E.D.

From (2.10) we have

$$(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi = \nabla^*\nabla\varphi + \frac{k-p}{4k}(\sum_{j=1}^{q} Ric_{i_j})\varphi$$

Let $q = n$, we obtain

Corollary 2.3.5 *X as above.*

If the scalar curvature $r > 0$, then

$$H_{cl}^{0,n}(X, K^{\frac{p-k}{2k}}) = 0 \qquad \textit{for } k > p$$

If $r < 0$, then

$$H_{cl}^{0,n}(X, K^{\frac{p-k}{2k}}) = 0 \qquad \textit{for } k < p$$

From (2.11) we have

$$(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi = \bar{\nabla}^*\bar{\nabla}\varphi + \frac{p+k}{4k}(\sum_{j=q+1}^{n} Ric_{i_j})\varphi$$

Let $q = 0$, we obtain

Corollary 2.3.6 *If $r > 0$, then*

$$H_{cl}^{-n,0}(X, K^{\frac{p-k}{2k}}) = 0 \qquad for\ p > -k$$

If $r < 0$ then

$$H_{cl}^{-n,0}(X, K^{\frac{p-k}{2k}}) = 0 \qquad for\ p < -k$$

These are just the corollaries in [8].

Now applying (2.3.1) to $L \approx K^{*\frac{1}{2}} \otimes \tilde{L}$ and

$$R^{\tilde{L}}_{\epsilon_j, \bar{\epsilon}_k} = \frac{1}{2}\lambda_j \delta_{jk}$$

We get

$$\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{1}{4}\sum_{j=1}^{n} Ric_j + \frac{1}{2}\sum_{j=1}^{q}\lambda_{i_j} - \frac{1}{2}\sum_{j=q+1}^{n}\lambda_{i_j} \\
&\geq (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{r}{8}\varphi - \frac{1}{2}\sum_{j=1}^{n}|\lambda_i| \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{1}{2}(\frac{r}{4} - \sum_{j=1}^{n}|\lambda_i|)
\end{aligned}$$

This gives the theorem of Michelsohn [8].

Theorem 2.3.7 *Let X be a compact Kähler manifold equipped with a spin structure and $\tilde{L}$ be a hermitian line bundle over X. If the scalar curvature* *r satisfies the inequality*

$$\frac{r}{4} > |\lambda_1| + \cdots + |\lambda_n|$$

at each point, then

$$H_{cl}^{-n+q,q}(X, K^{*\frac{1}{2}} \otimes \tilde{L}) = 0 \qquad \forall q$$

We shall prove the following theorem under a weaker assumption. Both the vanishing theorem of Kodaira and the vanishing theorem of Vesentini are special cases of it.

Theorem 2.3.8 *Let L be a holomorphic line bundle and fixed an integer q, $1 \leq q \leq n$. If the eigenvalues $\lambda_1, \cdots, \lambda_n$ of R^L satisfy the inequality*

$$\lambda_{i_1} + \cdots + \lambda_{i_q} > 0 \qquad i_1 < \cdots < i_q$$

at each point of X, where $R^L_{\epsilon_j, \bar{\epsilon}_k} = \lambda_j \delta_{jk}$ then

$$H_{cl}^{-n+q,q}(X, L) = 0 \qquad 1 \leq q \leq n$$

Proof of the theorem is an immediate consequence of (2.3.1). Recall we have a formula

$$(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi = (\nabla^*\nabla)\varphi + (\sum_{j=1}^{q} \lambda_{i_j})\varphi$$

Corollary 2.3.9 *Let L be a holomorphic line bundle, if $\lambda_j > 0$ for all j, then*

$$H_{cl}^{-n+q,q}(X, L) = 0 = H_{Dol}^{0,n-q}(X, L^*) \qquad \forall q \geq 1$$

This is just the vanishing theorem of Kodaira [5] for a negative line bundle.

Corollary 2.3.10 *Let L be a holomorphic line bundle, if $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0$ and $\lambda_1, \cdots, \lambda_k > 0$, then*

$$H_{cl}^{-n+q,q}(X, L) = 0 = H_{Dol}^{0,n-q}(X, L^*) \qquad \forall q \geq n - k + 1$$

This is just the vanishing theorem of Vesentini [5].

Remark: To construct the Clifford cohomology we have been using the left Clifford multiplication. The analagous construction also holds by using right Clifford multiplication.

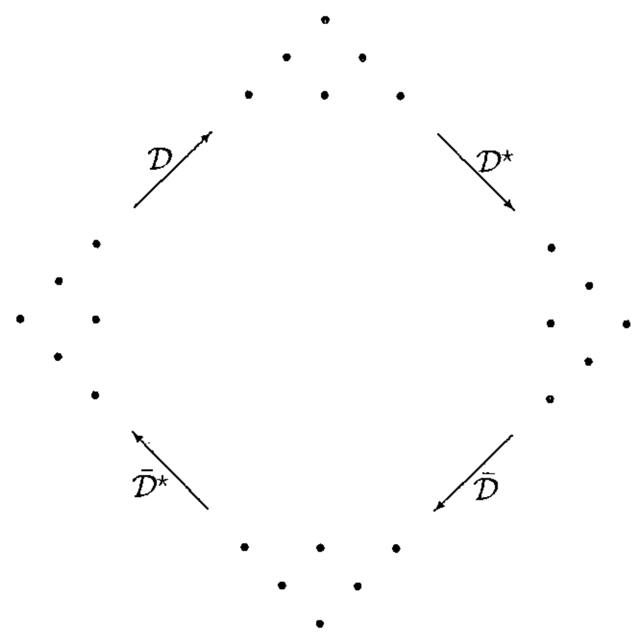
In the latter case, we define

$$\mathcal{D}^{\star}\varphi = \sum_j (\nabla_{\bar{\epsilon}_j}\varphi) \cdot \epsilon_j$$
$$\bar{\mathcal{D}}^{\star}\varphi = \sum_j (\nabla_{\epsilon_j}\varphi) \cdot \bar{\epsilon}_j$$

Since $\mathbf{Cl}^{p,q} \cdot \xi \subseteq \mathbf{Cl}^{p+1,q-1}$ and $\mathbf{Cl}^{p,q} \cdot \bar{\xi} \subseteq \mathbf{Cl}^{p-1,q+1}$, where $\xi \in T^{1,0}$ and $\bar{\xi} \in T^{0,1}$, we have

$$\xrightarrow{\mathcal{D}^\star} \mathbf{Cl}^{p-1,q+1} \xrightarrow{\mathcal{D}^\star} \mathbf{Cl}^{p,q} \xrightarrow{\mathcal{D}^\star} \mathbf{Cl}^{p+1,q-1} \rightarrow$$

Thus within the diamond we get the scheme:



Let

$$\Delta = \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}$$
$$\Delta^\star = \mathcal{D}^\star\bar{\mathcal{D}}^\star + \bar{\mathcal{D}}^\star\mathcal{D}^\star$$

Lemma 2.3.11 *The operator* $\Delta^\star - \Delta$ *acts as a derivation on* $\Gamma\mathrm{Cl}(X, W)$.

Proof: In the chosen local normal coordinates

$$\Delta^\star\varphi = -\sum_{j,k}(R_{\epsilon_k,\bar{\epsilon}_j}\varphi)\cdot\bar{\epsilon}_k\cdot\epsilon_j - \sum_j \nabla_{\epsilon_j}\nabla_{\bar{\epsilon}_j}\varphi$$

$$\Delta\varphi = \sum_{j,k}\bar{\epsilon}_k\cdot\epsilon_j(R_{\epsilon_k,\bar{\epsilon}_j}\varphi) - \sum_j \nabla_{\bar{\epsilon}_j}\nabla_{\epsilon_j}\varphi$$

$$(\Delta^\star - \Delta)\varphi = -\sum_{j,k}(R_{\epsilon_k,\bar{\epsilon}_j}\varphi)\cdot\bar{\epsilon}_k\cdot\epsilon_j - \sum_{j,k}\bar{\epsilon}_k\cdot\epsilon_j(R_{\epsilon_k,\bar{\epsilon}_j}\varphi) - \sum_j R_{\epsilon_j,\bar{\epsilon}_j}\varphi$$

Q.E.D.

If $\varphi = \sigma\otimes w$ where $\sigma\in\Gamma\mathbf{Cl}(X), w\in\Gamma(X,W)$.

$$\begin{aligned}(\Delta^\star - \Delta)\varphi = (\Delta^\star_{cl} - \Delta_{cl})\sigma\otimes w - \sum_{j,k}\sigma\cdot\bar{\epsilon}_k\cdot\epsilon_j\otimes R^W_{\epsilon_k,\bar{\epsilon}_j}w \\ -\sum_{j,k}\bar{\epsilon}_k\cdot\epsilon_j\cdot\sigma\otimes R^W_{\epsilon_k,\bar{\epsilon}_j}w - \sigma\otimes R^W_{\epsilon_j,\bar{\epsilon}_j}w\end{aligned}$$

We know that $kernel\Delta = kernel\Delta^\star$ on $\Gamma(\mathbf{Cl}(X))$ [8], then

$$\begin{aligned}(\Delta^\star - \Delta)\varphi = -\sum_{j,k}\sigma\cdot\bar{\epsilon}_k\cdot\epsilon_j\otimes R^W_{\epsilon_k,\bar{\epsilon}_j}w \\ -\sum_{j,k}\bar{\epsilon}_k\cdot\epsilon_j\cdot\sigma\otimes R^W_{\epsilon_k,\bar{\epsilon}_j}w - \sigma\otimes R^W_{\epsilon_j,\bar{\epsilon}_j}w \qquad (2.12)\end{aligned}$$

We see later that the formula (2.12) which appears so natural in the analysis of Kähler manifolds will play the same role as the Nakano's inequality.

Lemma 2.3.12 *Let L be a negative line bundle over X. There is a Kähler metric on X such that*

$$R^L_{\bar{\epsilon}_j,\bar{\bar{\epsilon}}_k} = -C^2\delta_{jk}$$

where C^2 is a positive function on X.

Proof: Let $g(\cdot\,,\,\cdot)$ be the Kähler metric on X such that $g(\epsilon_j, \bar{\epsilon}_k) = \frac{1}{2}\delta_{jk}$ and $R^L_{\epsilon_j, \bar{\epsilon}_k} = -\lambda_j \cdot \delta_{jk}, \quad \lambda_j > 0, \quad 1 \leq j \leq n$. Define a new Kähler metric on X.

$$\tilde{g}(\cdot\,,\,\cdot) = \frac{1}{2C^2 i} R^L_{\cdot,J\cdot}.$$

Choose a new basis.

$$\begin{cases} \tilde{\epsilon}_j = \frac{C}{\sqrt{\lambda_j}}\epsilon_j \\ \bar{\tilde{\epsilon}}_j = \frac{C}{\sqrt{\lambda_j}}\bar{\epsilon}_j \qquad 1 \leq j \leq n \end{cases}$$

$$\begin{aligned} \tilde{g}(\tilde{\epsilon}_j, \bar{\tilde{\epsilon}}_k) &= \frac{1}{2C^2 i} R^L_{\tilde{\epsilon}_j, J\bar{\tilde{\epsilon}}_k} \qquad \text{since } J\bar{\tilde{\epsilon}}_k = -i\bar{\tilde{\epsilon}}_k \\ &= \frac{1}{2C^2} R^L_{\tilde{\epsilon}_j, \bar{\tilde{\epsilon}}_k} \\ &= \frac{1}{2C^2} R^L_{\frac{C}{\sqrt{\lambda_j}}\epsilon_j, \frac{C}{\sqrt{\lambda_k}}\bar{\epsilon}_k} \\ &= +\frac{1}{2\sqrt{\lambda_j}\sqrt{\lambda_k}}\lambda_j \delta_{jk} \\ &= \frac{1}{2}\delta_{jk} \end{aligned}$$

$$\begin{aligned} R^L_{\tilde{\epsilon}_j, \bar{\tilde{\epsilon}}_k} &= \frac{c^2}{\sqrt{\lambda_j}\sqrt{\lambda_k}} R^L_{\epsilon_j, \bar{\epsilon}_k} \\ &= -\frac{c^2}{\sqrt{\lambda_j}\sqrt{\lambda_k}}\lambda_j \delta_{jk} \\ &= -C^2 \delta_{jk} \end{aligned}$$

Q.E.D.

By considering $\Delta^\star - \Delta$, the proof of the famous Nakano vanishing theorem becomes straightforward.

Theorem 2.3.13 *If L is a negative line bundle over a compact Kähler manifold X, then*

$$H^{p,q}_{cl}(X,L) = 0 \qquad \forall q < 0$$

Proof: We consider $\varphi \in \Gamma \mathrm{Cl}^{p,q}(X,L)$, such that $\mathcal{D}\varphi = \bar{\mathcal{D}}\varphi = 0 \Longleftrightarrow \Delta\varphi = 0$. Without losing generality, we assume $\varphi = \sigma \otimes l$, and apply (2.12) to $W = L$ we have

$$\begin{aligned} \Delta^\star\varphi &= (\Delta^\star - \Delta)\varphi \\ &= -\sum_{j,k} \sigma \cdot \bar{\epsilon}_k \cdot \epsilon_j \otimes R^L_{\epsilon_k,\bar{\epsilon}_j} l \\ &\quad - \sum_{j,k} \bar{\epsilon}_k \cdot \epsilon_j \sigma \otimes R^L_{\epsilon_k,\bar{\epsilon}_j} l - \sigma \otimes \sum_j R^L_{\epsilon_j,\bar{\epsilon}_j} l \end{aligned} \tag{2.13}$$

By (2.3.10) we can assume $R^L_{\epsilon_j,\bar{\epsilon}_k} = -C^2\delta_{jk}$

$$\begin{aligned} \Delta^\star\varphi &= \sum_j C^2\sigma \cdot \bar{\epsilon}_j \cdot \epsilon_j \otimes l + \sum_j C^2 \bar{\epsilon}_j \cdot \epsilon_j \sigma \otimes l\sigma \otimes l + C^2 n\sigma \otimes l \\ &= C^2(\sigma \cdot \sum_j \bar{\epsilon}_j \cdot \epsilon_j + \sum_j \bar{\epsilon}_j \cdot \epsilon_j \cdot \sigma + n^2\sigma) \otimes l \\ &= C^2 \mathcal{H}\sigma \otimes l \\ &= C^2 q\varphi \end{aligned}$$

Since $\Delta^\star$ is non-negative, we see that $\varphi \equiv 0$ for $q < 0$ Q.E.D.

Note that this is basically the proof in Michelsohn's paper [8], but the presentation is more transparent.

We use the following notation. To each (possibly empty) subset $I = \{i_1, \cdots, i_r\} \subseteq \{1, \cdots, n\}$ with complementary subset $\{i_{r+1} \cdots, i_n\}$, we associate $w_I = w_{i_1} \cdots w_{i_r} \bar{w}_{i_{r+1}} \cdots \bar{w}_{i_n}$ where $w_j = -\epsilon_j \bar{\epsilon}_j, \bar{w}_j = -\bar{\epsilon}_j \epsilon_j$ and we denote $|I| = r$. We set $\pi_r = \sum_{|I|=r} w_I$. Suppose σ belongs to $\mathbf{Cl}_n^{r-s,r+s-n}$ then from the proposition 6.4 [8], σ can be written as $\sigma = \pi_r \cdot \eta \cdot \pi_s = \sum_{|I|=r,|J|=s} w_I \cdot \eta \cdot w_J$, where $\eta \in \mathbf{Cl}_n$.

Now we assume L be any holomorphic line bundle and $R^L_{\epsilon_j,\bar{\epsilon}_k} = \lambda_j \delta_{jk}$, $\varphi = \sigma \otimes l \in \mathbf{Cl}_n^{r-s,r+s-n} \otimes L$ then the formula (2.13) becomes

$$\begin{aligned} \Delta^* \varphi &= -\sum_j \sigma \lambda_j \cdot \bar{\epsilon}_j \cdot \epsilon_j \otimes l - \sum_j \lambda_j \bar{\epsilon}_j \cdot \epsilon_j \sigma \otimes l - \sum_j \lambda_j \sigma \otimes l \\ &= -\sum_j \sigma \lambda_j \cdot \bar{\epsilon}_j \cdot \epsilon_j \otimes l + \sum_j \lambda_j \epsilon_j \bar{\epsilon}_j \sigma \otimes l \\ &= \sum_{|I|=r,|J|=s} \Big(\sum_{k=s+1}^{n} \lambda_{j_k} - \sum_{k=1}^{r} \lambda_{i_k} \Big) w_I \cdot \eta \cdot w_J \otimes l \end{aligned} \tag{2.14}$$

If L has the property such that $\lambda_1 \leq \lambda_2 \leq, \cdots, \leq \lambda_n \leq 0$, $\lambda_1, \cdots, \lambda_k < 0$, we know that [5], we can choose a new Kähler metric on X such that $R^L_{\bar{\epsilon}_j,\bar{\bar{\epsilon}}_k} = \lambda'_j \delta_{jk}$. $\{\lambda'_j\}$ satisfies the inequality for any indices $\alpha_1 < \cdots < \alpha_r$ and $\beta_1 < \cdots < \beta_{n-s}$ with $r + s < k$, we have

$$(\lambda'_{\beta_1} + \cdots + \lambda'_{\beta_{n-s}}) - (\lambda'_{\alpha_1} + \cdots + \lambda'_{\alpha_r}) < 0 \tag{2.15}$$

Suppose $\varphi \in \mathbf{Cl}_n^{p,q} \otimes L$, i.e. $p = r - s, q = r + s - n$ or $r = \frac{n+p+q}{2}, s = \frac{n-p+q}{2}$. $r + s = n + q < k$ or $q < -(n - k)$ then (2.15) holds. We already know $\Delta^* \varphi$ is non–negative. Thus combining (2.14) (2.15), we retrieved the following vanishing theorem of Gigante and Girbau.

Theorem 2.3.14 *If X is Kähler and $c_1(L) \leq 0$ with rank $c_1(L) \geq k$, then* $H^{p,q}_{cl}(X, L) = 0$ *for* $q < -(n-k)$.

Chapter 3

In this chapter, as another application of Clifford cohomology, we shall examine the differential geometry, complex structure and topological properties of a spin complex hypersurface M of complex projective space $\mathbf{CP}^{n+1}$. In this case $K^{\frac{1}{2}} \otimes T$ and its dual $K^{*\frac{1}{2}} \otimes T^*$ are the holomorphic vector bundles we will investigate. We discuss the results concerning the relation between the second fundamental form and complex structure of M by proving vanishing theorems for the bundle $K^{*\frac{1}{2}} \otimes T^*$. In the case of an algebraic hypersurface we can also compute the Euler characteristic $\chi(M, K^{\frac{1}{2}} \otimes T)$. Together with $\chi(M, K^{\frac{1}{2}} \otimes T)$ we prove more delicate results.

3.1 Curvature tensor of complex hypersurfaces of $\mathbf{CP}^{n+1}$

If $\mathbf{CP}^{n+1}$ is the $n+1$-dimensional complex projective space with the Fubini–Study metric of constant holomorphic sectional curvature 1, then the curvature tensor R is given by

$$\begin{aligned} < R_{X,Y}W, Z > = \frac{1}{4}\{ & < X, Z >< Y, W > \\ & - < X, W >< Y, Z > \\ & + < X, JZ >< Y, JW > \\ & - < X, JW >< Y, JZ > \\ & + 2 < X, JY >< Z, JW > \} \end{aligned} \tag{3.1}$$

where X, Y, Z and W are any vector fields of $\mathbf{CP}^{n+1}$ and $< \cdot, \cdot >$ denotes the Kähler metric on $\mathbf{CP}^{n+1}$.

Now let M be an n–dimensional complex manifold which is a complex hypersurface of $\mathbf{CP}^{n+1}$. Suppose that M is equipped with the induced metric from $\mathbf{CP}^{n+1}$. By the equation of Gauss [10], the curvature tensor of M is given by

$$\begin{aligned} < R^M_{X,Y}W, Z > = < R_{X,Y}W, Z > + \{ & < AX, Z >< AY, W > \\ & - < AX, W >< AY, Z > + < JAX, Z >< JAY, W > \\ & - < JAX, W >< JAY, Z > \} \end{aligned} \tag{3.2}$$

where R, R^M denote the curvature tensors of $\mathbf{CP}^{n+1}$ and M respectively, and A is a tensor field of type (1,1), which is associated with the second fundamental form Π of M.

Recall: Let M be a submanifold of $\bar{M}$. The second fundamental form is defined by

$$\Pi(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$$

where $\bar{\nabla}, \nabla$ are the covariant differentiations of $\bar{M}$ and M respectively.

Let us express the relation between A and Π in local coordinates at any $x \in M$, when the complex codimension of M is 1.

Let $e_1, \cdots e_n, Je_1, \cdots, Je_n, \xi, J\xi$ be an orthonormal basis of $T_x\bar{M}$ in a neighborhood $U(x)$ of x, where $\xi, J\xi$ are the unit normal tangent vectors of M. Since $< \xi, \xi > \equiv 1$ in $U(x)$, so $< \bar{\nabla}_X \xi, \xi > = 0$. We define A by

$$\bar{\nabla}_X \xi = -AX + (SX)J\xi \tag{3.3}$$

i.e. AX is the orthogonal projection of $\bar{\nabla}_X \xi$ on TM.

$$\Pi(X, Y) = \bar{\nabla}_X Y - \nabla_X Y = h(X, Y)\xi + k(X, Y)J\xi \tag{3.4}$$

where $h(X, Y), k(X, Y)$ are symmetric tensors. From (3.3), we see

$$< \bar{\nabla}_X \xi, Y > = - < AX, Y >$$

But

$$< \bar{\nabla}_X \xi, Y > = - < \bar{\nabla}_X Y, \xi > = -h(X, Y) \qquad \text{by (3.4)}$$

Thus we have

$$h(X, Y) = < AX, Y >$$

In a similar way

$$k(X, Y) = < JAX, Y >$$

It is easy to see A is symmetric with respect to $< \cdot, \cdot >$, and $AJ = -JA$. From a theorem of linear algebra [10] we know that there exists an orthonormal basis $\{e_1, \cdots e_n, Je_1, \cdots, Je_n\}$ such that the matrix of A is diagonal of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix}$$

i.e.

$$\begin{cases} Ae_j = \lambda_j e_j \\ AJe_j = -\lambda_j Je_j \end{cases} \tag{3.5}$$

We extend A to a complex linear transformation of $TM \otimes_R \mathbf{C}$. From (3.5) we have

$$\begin{cases} A\epsilon_j = \lambda_j \bar{\epsilon}_j \\ A\bar{\epsilon}_j = -\lambda_j \epsilon_j \end{cases} \tag{3.6}$$

We extend $< \cdot, \cdot >$, to a complex symmetric bilinear form on $TM \otimes_R \mathbf{C}$. We have

$$\begin{cases} < \epsilon_j, \epsilon_k >= 0 \\ < \epsilon_j, \bar{\epsilon}_k >= \frac{1}{2}\delta_{jk} \end{cases} \tag{3.7}$$

From (3.2), (3.6), (3.7) we obtain the formula for the curvature tensor of M in local coordinates as

$$\begin{aligned} &< R_{\bar{\epsilon}_j, \epsilon_k} \epsilon_l, \bar{\epsilon}_p > \\ &= -\frac{1}{8}(\delta_{pk}\delta_{lj} + \delta_{pl}\delta_{jk}) + \frac{1}{2}\lambda_j \lambda_k \delta_{jp}\delta_{kl} \end{aligned} \tag{3.8}$$

Recall: Let $e_1, Je_1, \cdots e_n, Je_n$ be orthonormal basis at $x \in M$. The Ricci tensor is defined as

$$Ric(X,Y) = \sum_{j=1}^{n} < R_{e_j,X}Y, e_j > + \sum_{j=1}^{n} < R_{Je_j,X}Y, Je_j >$$

If M is a complex hypersurface of $\mathbf{CP}^{n+1}$, from (3.2)

$$Ric(X,Y) = \frac{n+1}{2} < X, Y > -2 < A^2X, Y >$$

Thus we have a formula of Ricci tensor in $\{\epsilon_j, \bar{\epsilon}_k\}$

$$\begin{cases} Ric(\epsilon_j, \epsilon_k) = 0 \\ Ric(\epsilon_j, \bar{\epsilon}_k) = (\frac{n+1}{4} - \lambda_j^2)\delta_{jk} \end{cases} \tag{3.9}$$

and a formula of scalar curvature

$$r = 2\sum_{j=1}^{n} Ric_j = n(n+1) - 4\sum_{j=1}^{n} \lambda_j^2 \tag{3.10}$$

3.2 The vanishing theorems for spin complex hypersurfaces of $\mathbf{CP}^{n+1}$

When we consider the tensor product bundle $K^{*\frac{1}{2}} \otimes T^*$, the formula (2.3.1) becomes useful in the study of the differential geometry of complex hypersurfaces of $\mathbf{CP}^{n+1}$. In this case we have

$$2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D}) = \nabla^*\nabla + \bar{\nabla}^*\bar{\nabla} + \frac{r}{8} + R^{T^*} + \bar{R}^{T^*} \tag{3.11}$$

where r is the scalar curvature of M, T^* is the cotangent bundle of M.

Now we calculate the curvature terms R^{T^*} and $\bar{R}^{T^*}$ in the chosen local normal coordinates. For any $\varphi \in \Gamma\mathrm{Cl}^{-n+q,q}(M, K^{*\frac{1}{2}} \otimes T^*)$,

$$\varphi = \sum_l \sigma_l \otimes \bar{\epsilon}_l$$

where $\sigma_l \in \Gamma\mathbf{Cl}^{-n+q,q}(M, K^{*\frac{1}{2}})$, $\quad \bar{\epsilon}_l \in \Gamma(X, T^*)$, $\quad l \in \{1, \cdots n\}$

$$\begin{aligned}
R^{T^*}\varphi &= R^{T^*}(\sum_l \sigma_l \otimes \bar{\epsilon}_l) \\
&= \sum_l R^{T^*}(\sigma_l \otimes \bar{\epsilon}_l) \\
&= \sum_l \sum_{j,k=1}^{n} \epsilon_j \cdot \bar{\epsilon}_k \cdot \sigma_l \otimes R_{\bar{\epsilon}_j, \epsilon_k} \bar{\epsilon}_l \\
&= 2\sum_l \sum_{j,k,p=1}^{n} \epsilon_j \cdot \bar{\epsilon}_k \cdot \sigma_l \otimes < R_{\bar{\epsilon}_j, \epsilon_k} \bar{\epsilon}_l, \epsilon_p > \bar{\epsilon}_p \\
&= \frac{1}{4}\sum_l \sum_{j,k,p=1}^{n} \epsilon_j \cdot \bar{\epsilon}_k \cdot \sigma_l \otimes (\delta_{lk}\delta_{pj} + \delta_{lp}\delta_{jk})\bar{\epsilon}_p \\
&\quad -\sum_l \sum_{j,k,p=1}^{n} \epsilon_j \cdot \bar{\epsilon}_k \cdot \sigma_l \otimes \lambda_j \lambda_k \delta_{jl} \delta_{kp} \bar{\epsilon}_p \\
&= \frac{1}{4}\sum_l \sum_{p=1}^{n} \epsilon_p \cdot \bar{\epsilon}_l \cdot \sigma_l \otimes \bar{\epsilon}_p + \frac{1}{4}\sum_l \sum_{j=1}^{n} \epsilon_j \cdot \bar{\epsilon}_j \cdot \sigma_l \otimes \bar{\epsilon}_l \\
&\quad -\sum_l \sum_{p=1}^{n} \epsilon_l \cdot \bar{\epsilon}_p \cdot \sigma_l \otimes \lambda_l \lambda_p \bar{\epsilon}_p \\
&= \frac{1}{4}\sum_l \sum_{p=1}^{n} \epsilon_p \cdot \bar{\epsilon}_l \cdot \sigma_l \otimes \bar{\epsilon}_p - \sum_l \sum_{p=1}^{n} \lambda_l \lambda_p \epsilon_l \cdot \bar{\epsilon}_p \cdot \sigma_l \otimes \bar{\epsilon}_p + \frac{1}{4}\sum_{j=1}^{n} \epsilon_j \cdot \bar{\epsilon}_j \varphi
\end{aligned}$$

We have the formula for $R^{T^*}\varphi$.

$$R^{T^*}\varphi = \frac{1}{4}\sum_l \sum_{p=1}^{n} \epsilon_p \cdot \bar{\epsilon}_l \cdot \sigma_l \otimes \bar{\epsilon}_p - \sum_l \sum_{p=1}^{n} \lambda_l \lambda_p \epsilon_l \cdot \bar{\epsilon}_p \cdot \sigma_l \otimes \bar{\epsilon}_p + \frac{1}{4}\sum_{j=1}^{n} \epsilon_j \cdot \bar{\epsilon}_j \varphi \quad (3.12)$$

$$\begin{aligned}
\bar{R}^{T^*}\varphi &= \bar{R}^{T^*}(\sum_l \sigma_l \otimes \bar{\epsilon}_l) \\
&= \sum_l \sum_{j,k=1}^{n} \bar{\epsilon}_j \cdot \epsilon_k \cdot \sigma_l \otimes R_{\epsilon_j, \bar{\epsilon}_k} \bar{\epsilon}_l \\
&= 2\sum_l \sum_{j,k,p=1}^{n} \bar{\epsilon}_j \cdot \epsilon_k \cdot \sigma_l \ \otimes < R_{\epsilon_j, \bar{\epsilon}_k} \bar{\epsilon}_l, \epsilon_p > \bar{\epsilon}_p
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\sum_{l}\sum_{j,k,p=1}^{n}\bar{\epsilon}_j\cdot\epsilon_k\cdot\sigma_l\otimes(\delta_{lj}\delta_{pk}+\delta_{lp}\delta_{jk})\bar{\epsilon}_p \\
&\quad+\sum_{l}\sum_{j,k,p=1}^{n}\bar{\epsilon}_j\cdot\epsilon_k\cdot\sigma_l\otimes\lambda_j\lambda_k\delta_{kl}\delta_{jp}\bar{\epsilon}_p \\
&= -\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\bar{\epsilon}_l\cdot\epsilon_p\cdot\sigma_l\otimes\bar{\epsilon}_p-\frac{1}{4}\sum_{l}\sum_{j=1}^{n}\bar{\epsilon}_j\cdot\epsilon_j\cdot\sigma_l\otimes\bar{\epsilon}_l \\
&\quad+\sum_{l}\sum_{p=1}^{n}\bar{\epsilon}_p\cdot\epsilon_l\cdot\sigma_l\otimes\lambda_p\lambda_l\bar{\epsilon}_p \\
&= -\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\bar{\epsilon}_l\cdot\epsilon_p\cdot\sigma_l\otimes\bar{\epsilon}_p \\
&\quad+\sum_{l}\sum_{p=1}^{n}\lambda_l\lambda_p\bar{\epsilon}_p\cdot\epsilon_l\cdot\sigma_l\otimes\bar{\epsilon}_p+\frac{1}{4}\sum_{j=1}^{n}-\bar{\epsilon}_j\cdot\epsilon_j\varphi
\end{aligned}
$$

In a similar way, we have the formula for $\bar{R}^{T^*}\varphi$.

$$
\bar{R}^{T^*}\varphi=-\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\bar{\epsilon}_l\cdot\epsilon_p\cdot\sigma_l\otimes\bar{\epsilon}_p+\sum_{l}\sum_{p=1}^{n}\lambda_l\lambda_p\bar{\epsilon}_p\cdot\epsilon_l\cdot\sigma_l\otimes\bar{\epsilon}_p+\frac{1}{4}\sum_{j=1}^{n}\bar{\epsilon}_j\cdot\epsilon_j\varphi \quad (3.13)
$$

Note that from lemma 2.5 [8]

$$
\Big(\sum_{j=1}^{n}\epsilon_j\bar{\epsilon}_j+\sum_{j=1}^{n}-\bar{\epsilon}_j\epsilon_j\Big)\varphi=(n-2q)\varphi \quad (3.14)
$$

When $\varphi\in\Gamma\mathrm{Cl}^{-n+q,q}(M)$, by (3.12), (3.13) and (3.14), the formula (3.11) becomes

$$
\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}}+\bar{\mathcal{D}}\mathcal{D})\varphi \\
&=(\nabla^*\nabla+\bar{\nabla}^*\bar{\nabla})\varphi+\frac{r}{8}\varphi+\frac{1}{4}(n-2q)\varphi \\
&+\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\epsilon_p\cdot\bar{\epsilon}_l\cdot\sigma_l\otimes\bar{\epsilon}_p-\sum_{l}\sum_{p=1}^{n}\lambda_l\lambda_p\epsilon_l\cdot\bar{\epsilon}_p\cdot\sigma_l\otimes\bar{\epsilon}_p \\
&-\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\bar{\epsilon}_l\cdot\epsilon_p\cdot\sigma_l\otimes\bar{\epsilon}_p+\sum_{l}\sum_{p=1}^{n}\lambda_l\lambda_p\bar{\epsilon}_p\cdot\epsilon_l\cdot\sigma_l\otimes\bar{\epsilon}_p \qquad (3.15)
\end{aligned}
$$

where $r = n(n+1) - 4\sum_{j=1}^{n} \lambda_j^2$, or

$$\begin{aligned}
&2(\mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D})\varphi \\
&= (\nabla^*\nabla + \bar{\nabla}^*\bar{\nabla})\varphi + \frac{n^2+3n-4q}{8}\varphi - \frac{1}{2}\sum_{j=1}^{n}\lambda_j^2\varphi \\
&+\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\epsilon_p \cdot \bar{\epsilon}_l \cdot \sigma_l \otimes \bar{\epsilon}_p - \sum_{l}\sum_{p=1}^{n}\lambda_l\lambda_p\epsilon_l \cdot \bar{\epsilon}_p \cdot \sigma_l \otimes \bar{\epsilon}_p \\
&-\frac{1}{4}\sum_{l}\sum_{p=1}^{n}\bar{\epsilon}_l \cdot \epsilon_p \cdot \sigma_l \otimes \bar{\epsilon}_p + \sum_{l}\sum_{p=1}^{n}\lambda_l\lambda_p\bar{\epsilon}_p \cdot \epsilon_l \cdot \sigma_l \otimes \bar{\epsilon}_p
\end{aligned} \tag{3.16}$$

Remark

1. The hermitian inner product on $\mathbf{Cl}_n$ is defined by setting

$$(\varphi, \psi) \equiv <\varphi, \bar{\psi}> \qquad \text{for } \varphi, \psi \in \mathbf{Cl}_n$$

where $< \cdot, \cdot >$ denotes the $\mathbf{C}$–bilinear extension of the real inner product on Cl_{2n} to $\mathbf{Cl}_n = Cl_{2n} \otimes_R \mathbf{C}$. The $< \cdot, \cdot >$ on $\mathbf{Cl}_n$ has the property that for any unit vector $e \in \mathbf{R}^{2n}$.

$$< e\varphi, e\psi > = < \varphi, \psi >$$

It follows that for any $v \in \mathbf{C}^{2n} \subseteq \mathbf{Cl}_n$.

$$< v\varphi, \psi > = - < \varphi, v\psi > \qquad \text{on } \mathbf{Cl}_n$$

Consequently

$$(v\varphi, \psi) = -(\varphi, \bar{v}\psi) \tag{3.17}$$

where "–" denotes the complex conjugation.

2. We have the following identity.

$$\| \varphi \|^2 = \| \epsilon_j\varphi \|^2 + \| \bar{\epsilon}_j\varphi \|^2 \tag{3.18}$$

Since $\epsilon_j\bar{\epsilon}_j + \bar{\epsilon}_j\epsilon_j = -1 \quad j \in \{1, \cdots n\}$

$$\begin{aligned}
\| \varphi \|^2 &= (\varphi, \varphi) \\
&= -(\epsilon_j\bar{\epsilon}_j\varphi, \varphi) - (\bar{\epsilon}_j\epsilon_j\varphi, \varphi) \\
&= (\bar{\epsilon}_j\varphi, \bar{\epsilon}_j\varphi) + (\epsilon_j\varphi, \epsilon_j\varphi) \qquad \text{from (3.17)} \\
&= \| \bar{\epsilon}_j\varphi \|^2 + \| \epsilon_j\varphi \|^2
\end{aligned}$$

3. The hermitian inner product on $\mathbf{Cl}_n \otimes W$ is defined as

$$(\sigma_1 \otimes w_1, \sigma_2 \otimes w_2) = (\sigma_1, \sigma_2)^{cl} \cdot (w_1, w_2)^W$$

where $\sigma_i \in \mathbf{Cl}_n, w_i \in W$ and $(\cdot,\cdot)^{cl}, (\cdot,\cdot)^W$ are the hermitian inner products on $\mathbf{Cl}_n, W$ respectively.

In particular if $\varphi \in \mathbf{Cl}_n \otimes T^*$ and $\varphi = \sum_l \sigma_l \otimes \bar{\epsilon}_l$, in chosen local coordinates, we have following formulas which will be needed later.

$$\begin{aligned}
\| \varphi \|^2 &= (\sum_l \sigma_l \otimes \bar{\epsilon}_l, \sum_p \sigma_p \otimes \bar{\epsilon}_p) \\
&= \sum_{l,p} (\sigma_l, \sigma_p)(\bar{\epsilon}_l, \bar{\epsilon}_p) \\
&= \frac{1}{2} \sum_l \| \sigma_l \|^2
\end{aligned}$$

i.e.

$$2\| \varphi \|^2 = \sum_l \| \sigma_l \|^2 \tag{3.19}$$

If $\varphi \in \mathbf{Cl}_n \otimes L$, L is any holomorphic line bundle, then $\varphi = \sigma \otimes l$. Since we can choose local coordinates such that l is a unit vector so

$$\| \varphi \|^2 = \| \sigma \|^2 \tag{3.20}$$

4. Let $\mathbf{V}$ be a vector space and $V_1 \cdots V_l$ be any l vectors of $\mathbf{V}$. We have the inequality.

$$\| \sum_{i=1}^{l} V_i \|^2 \leq l \sum_{i=1}^{l} \| V_i \|^2$$

"=" holds iff $\{V_i\}$ are linearly dependent and $\| V_i \| = \| V_j \| \quad$ for $i \neq j$.

Since

$$\begin{aligned}
\| \sum_{i=1}^{l} V_i \|^2 &= < \sum_{i=1}^{l} V_i, \sum_{j=1}^{l} V_j > \\
&= \sum_{i,j=1}^{l} < V_i, V_j > \\
&\leq \sum_{i,j=1}^{l} \| V_i \| \| V_j \| \\
&\leq \frac{1}{2} \sum_{i,j=1}^{l} (\| V_i \|^2 + \| V_j \|^2) \\
&= \frac{1}{2} \sum_{i=1}^{l} (l \| V_i \|^2 + \sum_{j=1}^{l} \| V_j \|^2) \\
&= \frac{1}{2} (l \sum_{i=1}^{l} \| V_i \|^2 + l \sum_{j=1}^{l} \| V_j \|^2) \\
&= l \sum_{i=1}^{l} \| V_i \|^2
\end{aligned}$$

First we are going to compute the formulas for some terms in (3.16).

$$\begin{aligned}
&(-\frac{1}{4} \sum_{l} \sum_{p=1}^{n} \bar{\epsilon}_l \cdot \epsilon_p \cdot \sigma_l \otimes \bar{\epsilon}_p, \sum \sigma_\alpha \otimes \bar{\epsilon}_\alpha) \\
&= -\frac{1}{8} \sum_{l,\alpha} (\bar{\epsilon}_l \cdot \epsilon_\alpha \cdot \sigma_l, \sigma_\alpha) \\
&= \frac{1}{8} \sum_{l,\alpha} ((\epsilon_\alpha \bar{\epsilon}_l + \delta_{\alpha,\beta}) \sigma_l, \sigma_\alpha)
\end{aligned}$$

$$= \frac{1}{8}\sum_{l,\alpha}(\epsilon_\alpha\bar{\epsilon}_l\sigma_l, \sigma_\alpha) + \frac{1}{8}\sum_l \| \sigma_l \|^2$$
$$= -\frac{1}{8}\sum_{l,\alpha}(\bar{\epsilon}_l\sigma_l, \bar{\epsilon}_\alpha\sigma_\alpha) + \frac{1}{4}\| \varphi \|^2$$
$$= -\frac{1}{8}\| \sum_l \bar{\epsilon}_l\sigma_l \|^2 + \frac{1}{4}\| \varphi \|^2 \qquad (3.21)$$

In a similar way

$$(\frac{1}{4}\sum_l\sum_{p=1}^n \epsilon_p\bar{\epsilon}_l \cdot \sigma_l \otimes \bar{\epsilon}_p, \sum \sigma_\alpha \otimes \bar{\epsilon}_\alpha) = -\frac{1}{8}\| \sum_l \bar{\epsilon}_l\sigma_l \|^2 \qquad (3.22)$$

$$(-\sum_l\sum_{p=1}^n \lambda_l\lambda_p\epsilon_l \cdot \bar{\epsilon}_p \cdot \sigma_l \otimes \bar{\epsilon}_p, \sum_\alpha \sigma_\alpha \otimes \bar{\epsilon}_\alpha)$$
$$= -\frac{1}{2}\| \sum_l \lambda_l\epsilon_l\sigma_l \|^2 + \frac{1}{2}\sum_l \lambda_l^2\| \sigma_l \|^2 \qquad (3.23)$$

$$(\sum_l\sum_{p=1}^n \lambda_l\lambda_p\bar{\epsilon}_p \cdot \epsilon_l \cdot \sigma_l \otimes \bar{\epsilon}_p, \sum_\alpha \sigma_\alpha \otimes \bar{\epsilon}_\alpha) = -\frac{1}{2}\| \sum_l \lambda_l\epsilon_l\sigma_l \|^2 \qquad (3.24)$$

Then from (3.21), (3.22), (3.23), (3.24), the formula (3.16) has the form

$$\begin{aligned} 2(\Delta\varphi, \varphi) &= (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2+3n-4q}{8}\| \varphi \|^2 \\ &- \frac{1}{2}\sum_{j=1}^n \lambda_j^2\| \varphi \|^2 - \frac{1}{4}\| \sum_l \bar{\epsilon}_l\sigma_l \|^2 \\ &- \| \sum_l \lambda_l\epsilon_l\sigma_l \|^2 + \frac{1}{4}\| \varphi \|^2 + \frac{1}{2}\sum_l \lambda_l^2\| \sigma_l \|^2 \end{aligned} \qquad (3.25)$$

Let $M = V^n(d) = \{[Z_0 \cdots Z_{n+1}] \in \mathbf{CP}^{n+1} | P(Z_0 \cdots Z_{n+1}) = 0\}$, where $P\,(Z_0\ \cdots\ Z_{n+1})$ is a homogeneous polynomial of degree d, such that $\nabla P(Z_0 \cdots Z_{n+1}) \neq 0$. We call M a non–singular complex hypersurface of degree d of $\mathbf{CP}^{n+1}$. We know that $c_1(V^n(d)) = (n+2-d)w$, where w is the canonical generator of $H^2(V^n(d), \mathbb{Z})$.

$$V^n(d) \text{ is spin} \Longleftrightarrow n+d \text{ is even.}$$

$\mathbf{CP}^n$ is a submanifold of $\mathbf{CP}^{n+1}$ with degree 1. $\mathbf{CP}^n$ is spin iff n is odd. $\mathbf{CP}^n$ can carry an induced metric with $\lambda_j = 0$ for all j. The formula (3.25) becomes

$$\begin{aligned} 2(\Delta\varphi, \varphi) &= (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2+3n-4q+2}{8}\| \varphi \|^2 - \frac{1}{4}\| \sum_l \bar{\epsilon}_l\sigma_l \|^2 \\ &\geq (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2+3n+2-4q-4n}{8}\| \varphi \|^2 \\ &= (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2-n+2-4q}{8}\| \varphi \|^2 \end{aligned}$$

Let $0 \leq q \leq n-1$

$$2(\Delta\varphi, \varphi) \geq (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{(n-2)(n-3)}{8}\| \varphi \|^2$$

We have the following vanishing theorem for $\mathbf{CP}^n$.

Theorem 3.2.1 *For $n \geq 5$*

$$H_{cl}^{-n+q,q}(\mathbf{CP}^n, K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad \textit{for } 0 \leq q \leq n-1$$

Thus

$$\begin{aligned} \chi(K^{\frac{1}{2}} \otimes T) &= dim_c H_{Dol}^{0,0}(\mathbf{CP}^n, K^{\frac{1}{2}} \otimes T) \\ &= dim_c H_{cl}^{0,n}(\mathbf{CP}^n, K^{*\frac{1}{2}} \otimes T^*) \end{aligned}$$

and

$$H_{cl}^{-n+q,q}(\mathbf{CP}^3, K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad q = 0, 1$$

Recall the definition of Euler characteristic of the holomorphic vector bundle W [10]

$$\chi(W) = \sum_{q=0}^{n}(-1)^q dim_c H_{Dol}^{0,q}(X, W)$$

Now we consider another special case

$$M = Q_n(\mathbf{C}) = \{[Z_0 \cdots Z_{n+1}] \in \mathbf{CP}^{n+1} | \sum_{i=0}^{n+1} Z_i^2 = 0\}$$

call $Q_n(\mathbf{C})$ the complex hypersphere. Since $Q_n(\mathbf{C})$ has degree 2, so $Q_n(\mathbf{C})$ is spin iff n is even. In this case we know that $\lambda_i = \frac{1}{2}$ for all i [10]. The formula (3.25) becomes

$$\begin{aligned} 2(\Delta\varphi, \varphi) &= (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2 + 3n - 4q}{8} \| \varphi \|^2 \\ &- \frac{n}{8} \| \varphi \|^2 - \frac{1}{4} \| \sum_l \bar{\epsilon}_l \sigma_l \|^2 \\ &- \frac{1}{4} \| \sum_l \epsilon_l \sigma_l \|^2 + \frac{1}{2} \| \varphi \|^2 + \frac{1}{2} \| \varphi \|^2 \\ &\geq (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2 - 2n + 4 - 4q}{8} \| \varphi \|^2 \quad (3.26) \end{aligned}$$

Using the fact

$$\| \sum_l \bar{\epsilon}_l \sigma_l \|^2 + \| \sum_l \epsilon_l \sigma_l \|^2 \leq n \sum_l (\| \bar{\epsilon}_l \sigma_l \|^2 + \| \epsilon_l \sigma_l \|^2) = n \sum_l \| \sigma_l \|^2 = 2n \| \varphi \|^2$$

Plugging $q \leq n-1$ into (3.26) we have

$$2(\Delta\varphi,\varphi) \geq (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi,\varphi) + \frac{(n-2)(n-4)}{8}\parallel \varphi \parallel^2$$

Therefore we obtain the following vanishing cohomology theorem for $Q_n(\mathbf{C})$.

Theorem 3.2.2 *For* $n \geq 6$

$$H_{cl}^{-n+q,q}(Q_n(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad for\ 0 \leq q \leq n-1$$

Thus

$$\begin{aligned} \chi(K^{\frac{1}{2}} \otimes T) &= dim_c H^{0,0}_{Dol}(Q_n(\mathbf{C})^n, K^{\frac{1}{2}} \otimes T) \\ &= dim_c H^{0,n}_{cl}(Q_n(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) \end{aligned}$$

and

$$H_{cl}^{-n+q,q}(Q_4(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad 0 \leq q \leq 2$$

$$H^{-n,0}(Q_2(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) = 0$$

Now we estimate (3.25) for the general case, i.e. we assume $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n > 0$, not all λ_j are equal.

$$\begin{aligned} 2(\Delta\varphi,\varphi) &> (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi,\varphi) \\ &\quad + \frac{n^2-n+2-4q}{8}\parallel \varphi \parallel^2 - \frac{n-1}{2}\lambda_1^2\parallel \varphi \parallel^2 \\ &\quad - n(\lambda_1^2 - \frac{1}{4})\sum_l \parallel \epsilon_l\sigma_l \parallel^2 \qquad (3.27) \end{aligned}$$

Case 1: If $\lambda_1^2 \leq \frac{1}{4}$ then (3.27) becomes

$$\begin{aligned} 2(\Delta\varphi, \varphi) &> (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + (\frac{n^2 - n + 2 - 4q}{8} - \frac{n-1}{8}) \| \varphi \|^2 \\ &= (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2 - 2n + 3 - 4q}{8} \| \varphi \|^2 \end{aligned}$$

Suppose $0 \leq q \leq n - 1$

$$2(\Delta\varphi, \varphi) > (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2 - 6n + 7}{8} \| \varphi \|^2 \qquad (3.28)$$

Case 2: If $\lambda_1 > \frac{1}{4}$ then (3.27) becomes

$$\begin{aligned} 2(\Delta\varphi, \varphi) &> (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + (\frac{n^2 - n + 2 - 4q}{8} - \frac{n-1}{8}) \| \varphi \|^2 \\ &\quad -\frac{n-1}{2}\lambda_1^2 \| \varphi \|^2 - 2n(\lambda_1^2 - \frac{1}{4}) \| \varphi \|^2 \\ &= (\nabla^*\nabla\varphi + \bar{\nabla}^*\bar{\nabla}\varphi, \varphi) + \frac{n^2 + 3n + 2 - 4q}{8} \| \varphi \|^2 \\ &\quad -\frac{5n-1}{2}\lambda_1^2 \| \varphi \|^2 \qquad (3.29) \end{aligned}$$

Then (3.28) and (3.29) give the following vanishing theorem.

Theorem 3.2.3 *Let M be a spin complex hypersurface of $\mathbf{CP}^{n+1}$ with induced metric. If the eigenvalues of the second fundamental form of M satisfy*

$$\lambda_j^2 \leq \frac{n^2 - n + 6}{4(5n - 1)} \qquad \textit{for all } j$$

then for $n \geq 5$

$$H_{cl}^{-n+q,q}(M, K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad \textit{for } 0 \leq q \leq n - 1$$

Thus

$$\begin{aligned}\chi(K^{\frac{1}{2}} \otimes T) &= dim_c H^{0,0}_{Dol}(M, K^{\frac{1}{2}} \otimes T) \\ &= dim_c H^{0,n}_{cl}(M, K^{*\frac{1}{2}} \otimes T^*)\end{aligned}$$

3.3 Euler characteristic of the holomorphic vector bundle $K^{\frac{1}{2}} \otimes T$

3.3.1 Chern character, Todd class and Riemann–Roch–Hirzebruch Theorem

The reader who needs more details may consult, for example [3], [11]. Let W be a complex (differential) vector bundle over M, where r = rank W and M is a defferentiable manifold of real dimension m. Let

$$c(W) = 1 + c_1(W) + \cdots + c_r(W)$$

be the total Chern class of W which is an element of the cohomology ring $H^*(M, \mathbf{C})$. The multiplication in this ring is induced by the exterior product of differential forms. One can use the de Rham groups as a representation of cohomology. We introduce a formal factorization

$$c(W) = \prod_{i=1}^{r}(1 + x_i)$$

where $x_i \in H^*(M, \mathbf{C})$. In fact $c_j(W)$ are the elementary symmetric functions of the $(x_1, \cdots x_r)$. Then any formal power series in $x_1 \cdots x_r$ which is symmetric in $(x_1, \cdots x_r)$ is also a power series in $c_1(W), \cdots c_r(W)$. Therefore we define formal power series in the Chern classes of W as follows.

$$\mathcal{T}(W) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

$$ch(W) = \sum_{i=1}^{r} e^{x_i}$$

where $\mathcal{T}(W)$ is called the Todd class of W and $ch(W)$ is called Chern character of W.

Now we recall some functorial properties of the Chern classes, Chern character and Todd class.

Suppose that W and W' are differentiable $\mathbf{C}$–vector bundles over a differentiable manifold X. Then

(1) If $\varphi : Y \longrightarrow X$ is a differentiable mapping where Y is a differentiable manifold, then

$$c(\varphi^*(W)) = \varphi^*(c(W))$$

Consequently

$$\mathcal{T}(\varphi^*(W)) = \varphi^*(\mathcal{T}(W))$$

$$ch(\varphi^*(W)) = \varphi^*(ch(W))$$

where $\varphi^*(W)$ is the pull back vector bundle and $\varphi^*(c(W)), \varphi^*(\mathcal{T}(W)), \varphi^*(ch(W))$ are the pull back of the cohomology classes $c(W), \mathcal{T}(W), ch(W)$ respectively.

(2)

$$c(W \oplus W') = c(W) \cdot c(W') \tag{3.30}$$

Consequently

$$\mathcal{T}(W \oplus W') = \mathcal{T}(W) \cdot \mathcal{T}(W') \tag{3.31}$$

$$ch(W \oplus W') = ch(W) + chW' \tag{3.32}$$

$$ch(W \otimes W') = ch(W) \cdot chW' \tag{3.33}$$

where the product "·" and the addition "+" are in the de Rham cohomology ring $H^*(X, \mathbf{C})$.

(3) If W^* is the dual vector bundle to W then

$$c_j(W^*) = (-1)^j c_j(W)$$

In particular if $W = L$ is a complex line bundle

$$c_1(L^*) = -c_1(L)$$

Thus

$$chL = e^{c_1(L)}$$

$$chL^* = e^{-c_1(L)}$$

Recall the definition of Euler characteristic of the holomorphic vector bundle W

$$\chi(W) = \chi(M, W) = \sum_{q=0}^{n}(-1)^q dim_c H^{0,q}_{Dol}(M, W)$$

Note that $dim_c H^{0,q}_{Dol}(M, W)$ depend only on the complex structures of M and W. However, it is a remarkable fact that $\chi(W)$ can be expressed in terms of topological invariants of the vector bundle W and of the complex manifold M itself. This is the well–known Riemann-Roch theorem of Hirzebruch. We will use it when we compute $\chi(V^n(d), K^{\frac{1}{2}} \otimes T)$.

Theorem (Riemann–Roch–Hirzebruch):

Let M be a compact complex manifold and let W be a holomorphic vector bundle over M. Then

$$\chi(W) = \{ch(W) \cdot \mathcal{T}(T(M))\}[M]$$

where $\mathcal{T}(T(M))$ is the Todd class of the tangent bundle of M; $ch(W)$ is the Chern character of W.

3.3.2 The formulas for $\chi(V^n(d), K^{\frac{1}{2}} \otimes T)$

First let us compute $ch(T\mathbf{CP}^{n+1}), \mathcal{T}(T\mathbf{CP}^{n+1})$. Using the equation [9].

$$T(\mathbf{CP}^{n+1}) \oplus \varepsilon^1 \simeq \underbrace{L \oplus \cdots \oplus L}_{n+2} = \oplus_1^{n+2} L \tag{3.34}$$

where ε^1 means trival line bundle of $\mathbf{CP}^{n+1}$, L means the hyperplane section bundle of $\mathbf{CP}^{n+1}$.

Proposition 3.3.1 *We have the following formulas for* $\mathbf{CP}^{n+1}$.

1. $c(T\mathbf{CP}^{n+1}) = (1+w)^{n+2}$
2. $ch(T\mathbf{CP}^{n+1}) = (n+2)e^w - 1$
3. $\mathcal{T}(T\mathbf{CP}^{n+1}) = (\frac{w}{1-e^{-w}})^{n+2}$

where w is the generator of $c_1(L)$.

Proof:

1. From (3.30), (3.34)

$$\begin{aligned} c(T\mathbf{CP}^{n+1}) \cdot c(\varepsilon^1) &= (c(L))^{n+2} \\ &= (1+w)^{n+2} \quad \text{since } c(\varepsilon') = 1, c(L) = 1 + c_1(L) \end{aligned}$$

2. From (3.32), (3.34)

$$ch(T\mathbf{CP}^{n+1}) + ch(\varepsilon') = (n+2)chL \quad \text{also } ch(\varepsilon') = 1, chL = e^{c_1(L)}$$

3. From (3.31), (3.34)

$$\mathcal{T}(T\mathbf{CP}^{n+1}) \cdot \mathcal{T}(\varepsilon^1) = (\mathcal{T}(L))^{n+2}$$

Note that $\mathcal{T}(L) = \frac{c_1(L)}{1-e^{-c_1(L)}} = \frac{w}{1-e^{-w}}$

Q. E. D.

Let $j : V^n(d) \longrightarrow \mathbf{CP}^{n+1}$ is a holomorphic imbedding, and $\bar{T}$ be the restriction of the tangent bundle of $\mathbf{CP}^{n+1}$ over $V^n(d)$ *i.e.* $\bar{T} = j^*(T\mathbf{CP}^{n+1})$. We get an equation on $V^n(d)$ [6].

$$T(V^n(d)) \oplus L^d = \bar{T} \tag{3.35}$$

where $L^d = \underbrace{L \otimes \cdots \otimes L}_{d}$ restricted to $V^n(d)$.

Proposition 3.3.2 *We have the following formulas for $V^n(d)$.*

1. $ch(T(V^n(d))) = j^*((n+2)e^w - 1 - e^{dw})$

2. $\mathcal{T}(T(V^n(d))) = j^*(\frac{w^{n+2}}{(1-e^{-w})^{n+2}} \cdot \frac{1-e^{-dw}}{dw})$

Proof:

1. By (3.32), (3.35)

$$ch(T(V^n(d))) + j^*(chL)^d = ch(\bar{T})$$

$$(chL)^d = e^{dc_1(L)} = e^{dw}$$

$$\begin{aligned} ch(\bar{T}) &= ch(j^* T\mathbf{CP}^{n+1}) \\ &= j^*(ch(T\mathbf{CP}^{n+1})) \\ &= j^*((n+2)e^w - 1) \end{aligned}$$

follows from proposition (3.3.1)

2. By (3.31), (3.35)

$$\mathcal{T}(TV^n(d)) \cdot j^*(\mathcal{T}(L^d)) = \mathcal{T}(\bar{T})$$

$$\mathcal{T}(L^d) = \frac{c_1(L^d)}{1-e^{-c_1(l^d)}} = \frac{dw}{1-e^{-dw}}$$

Since $c_1(L^d) = dw \quad \mathcal{T}(\bar{T}) = j^*(\mathcal{T}(T\mathbf{CP}^{n+1}))$

Proposition 3.3.3 *Let K be the canonical line bundle over $V^n(d)$. If $V^n(d)$ is spin, i.e. $n+d$ is even, we have*

$$ch(K^{\frac{1}{2}}) = e^{-\frac{n+2-d}{2}j^*w}$$

Proof: Since

$$K = K^{\frac{1}{2}} \otimes K^{\frac{1}{2}}$$

then $chK = chK^{\frac{1}{2}} \cdot chK^{\frac{1}{2}} = (chK^{\frac{1}{2}})^2$. We know that

$$chK = e^{c_1(K)} = e^{-c_1(V^n(d))} = e^{-(n+2-d)j^*w}$$

This gives (3.3.3). Q.E.D.

Theorem 3.3.4

$$2^{2m+1}\chi(V^{2m}(2t), K^{\frac{1}{2}} \otimes T(V^{2m}(2t))$$

$$= \begin{cases} 0 & \textit{when } t < \frac{m+1}{3}. \\ -\sum_{l=0}^{m} \binom{6t}{2l+1} \binom{3t-l-1}{m-l} & \textit{when } \frac{m+1}{3} \leq t < m. \\ \sum_{l=0}^{m} (2m+2) \binom{2m+2}{2l+1} & \\ -\binom{6m}{2l+1} \binom{3m-l-1}{m-l} & \textit{when } t = m. \\ \sum_{l=0}^{m} (2m+2) \binom{2m+4}{2l+1} (m+1-l) & \\ -\binom{2m+2}{2l+1} & \\ -\binom{6m+6}{2l+1} \binom{3m+2-l}{m-l} & \textit{when } t = m+1. \\ (2m+2) \left[\sum_{l=0}^{m} \binom{2t+2}{2l+1} \binom{t-l}{m-l} \right. & \\ \left. + \binom{2t-2}{2l+1} \binom{t-2-l}{m-l} \right] & \\ -\sum_{l=0}^{m} \binom{2t}{2l+1} \binom{t-l-1}{m-l} & \\ -\sum_{l=0}^{m} \binom{6t}{2l+1} \binom{3t-l-1}{m-l} & \textit{when } t \geq m+2. \end{cases}$$

Theorem 3.3.5

$$2^{2m+2}\chi(V^{2m+1}(2t+1), K^{\frac{1}{2}} \otimes T(V^{2m+1}(2t+1)))$$
$$= \begin{cases} 0 & \textit{when } t < \frac{m}{3}. \\ -\sum_{l=0}^{m+1} \binom{6t+3}{2l} \binom{3t+1-l}{m+1-l} & \textit{when } \frac{m}{3} \leq t < m. \\ \sum_{l=0}^{m+1} (2m+3) \binom{2m+3}{2l} & \\ -\binom{6m+3}{2l} \binom{3m+1-l}{m+1-l} & \textit{when } t = m. \\ \sum_{l=0}^{m+1} (2m+3) \binom{2m+5}{2l} (m+2-l) & \\ +\binom{2m+3}{2l} & \\ -\binom{6m+9}{2l} \binom{3m+4-l}{m+1-l} & \textit{when } t = m+1. \\ \sum_{l=0}^{m+1} (2m+3) \left[\binom{2t+3}{2l} \binom{t+1-l}{m+1-l} \right. & \\ \left. -\binom{2t-1}{2l} \binom{t-1-l}{m+1-l} \right] & \\ -\sum_{l=0}^{m+1} \binom{6t+3}{2l} \binom{3t+1-l}{m+1-l} & \\ +\sum_{l=0}^{m+1} \binom{2t+1}{2l} \binom{t-l}{m+1-l} & \textit{when } t \geq m+2. \end{cases}$$

where $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$

Idea of the proof:

We know that

$$\begin{aligned}
&\chi(V^n(d), K^{\frac{1}{2}} \otimes T(V^n(d))) \\
&= \{j^*((n+2)e^w - 1 - e^{dw})\frac{(\frac{w}{2})^{n+2}}{(sinh\ \frac{w}{2})^{n+2}}\frac{2sinh\ \frac{d}{2}\ w}{dw}\}[V^n(d)] \\
&= \{((n+2)e^w - 1 - e^{dw})\frac{(\frac{w}{2})^{n+2}}{(sinh\ \frac{w}{2})^{n+2}}2sinh\ \frac{d}{2}\ w\}[\mathbf{CP}^{n+1}]
\end{aligned}$$

Let

$$f(w) = ((n+2)e^w - 1 - e^{dw})\frac{(\frac{w}{2})^{n+2}}{(sinh\frac{w}{2})^{n+2}}2sinh\ \frac{d}{2}\ w \tag{3.36}$$

In fact $f(w)$ is a power series in w i.e. $f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}w^k$ and $w^{n+1}[\mathbf{CP}^{n+1}] = 1, \varphi[\mathbf{CP}^{n+1}] = 0$, if $\varphi \in H^p(\mathbf{CP}^{n+1}, \mathbf{C})$, $p < n+1$. Thus by applying the Cauchy integral formula for the analytic functions,

$$\begin{aligned}
\chi(V^n(d), K^{\frac{1}{2}} \otimes T(V^n(d))) &= \frac{f^{n+1}(0)}{(n+1)!} \\
&= \frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{f(Z)}{Z^{n+2}}dZ
\end{aligned}$$

where $f^{(n)}(Z)$ means n-th derivative of $f(Z)$. We denote w by Z. let $f(Z) = f_1(Z) + f_2(Z)$ in (3.37) where

$$f_1(Z) = ((n+2)cosh\ Z - 1 - cosh\ dZ)(\frac{\frac{Z}{2}}{sinh\ \frac{Z}{2}})^{n+2}(2sinh\ \frac{d}{2}Z)$$

$$f_2(Z) = ((n+2)sinh\ Z - sinh\ dZ)(\frac{\frac{Z}{2}}{sinh\ \frac{Z}{2}})^{n+2}(2sinh\ \frac{d}{2}Z)$$

For dimensional reasons, in fact we have

for $n = 2m$

$$\chi(V^{2m}(2t), K^{\frac{1}{2}} \otimes T(V^{2m}(2t)) = \frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{f_1(Z)}{Z^{2m+2}}dZ$$

for n=2m+1

$$\chi(V^{2m+1}(2t+1), K^{\frac{1}{2}} \otimes T(V^{2m+1}(2t+1)) = \frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{f_2(Z)}{Z^{2m+3}}dZ$$

The calculation of $\frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{f_1(Z)}{Z^{2m+2}}dZ$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{f_1(Z)}{Z^{2m+2}}dZ \\
&= \frac{1}{2^{2m+1}}\{\frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{(2m+2)cosh\ Z\ sinh\ tZ}{(sinh\ \frac{Z}{2})^{2m+2}}dZ \\
&-\frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{sinh\ tZ}{(sinh\ \frac{Z}{2})^{2m+2}}dZ \\
&-\frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{cosh\ 2tZ\ sinh\ tZ}{(sinh\ \frac{Z}{2})^{2m+2}}dZ\}
\end{aligned}$$

Let $u = \frac{Z}{2}$ $\quad du = \frac{1}{2}dZ$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|Z|=\varepsilon} \frac{f_1(Z)}{Z^{2m+2}}dZ \\
&= \frac{1}{2^{2m+1}}\{\frac{1}{2\pi i}\int_{|u|=\eta} \frac{(2m+2)2cosh\ 2u\ sinh\ 2tu}{(sinh\ u)^{2m+2}}du \\
&-\frac{1}{2\pi i}\int_{|u|=\eta} \frac{2sinh\ 2tu}{(sinh\ u)^{2m+2}}du \\
&-\frac{1}{2\pi i}\int_{|u|=\eta} \frac{2cosh\ 4tu\ sinh\ 2tu}{(sinh\ u)^{2m+2}}du\}
\end{aligned}$$

Using the identities

$$2sinh\ x\ cosh\ y = sinh\ (x+y) + sinh\ (x-y)$$

$$sinh\,(-x) = -sinh\,x$$

$$\frac{1}{2\pi i}\int_{|Z|=\varepsilon}\frac{f_1(Z)}{Z^{2m+2}}dZ$$
$$= \frac{1}{2^{2m+1}}\{\frac{1}{2\pi i}\int_{|u|=\eta}\frac{(2m+2)[sinh\,(2t+2)u + sinh\,(2t-2)u]}{(sinh\,u)^{2m+2}}du$$
$$-\frac{1}{2\pi i}\int_{|u|=\eta}\frac{sinh\,6tu}{(sinh\,u)^{2m+2}}du$$
$$-\frac{1}{2\pi i}\int_{|u|=\eta}\frac{sinh\,2tu}{(sinh\,u)^{2m+2}}du\}$$

The calculation reduces to computing the integral

$$\frac{1}{2\pi i}\int_{|u|=\varepsilon}\frac{sinh\,2pu}{(sinh\,u)^{2m+2}}du$$
$$= \frac{1}{2\pi i}\int_{|u|=\varepsilon}\sum_{l=0}^{p-1}\begin{pmatrix}2p\\2l+1\end{pmatrix}\frac{(sinh\,u)^{2l+1}(cosh\,u)^{2p-2l-1}}{(sinh\,u)^{2m+2}}du$$
$$= \frac{1}{2\pi i}\int_{|w|=\eta}\sum_{l=0}^{p-1}\begin{pmatrix}2p\\2l+1\end{pmatrix}W^{-m-1+l}(1+W)^{p-l-1}dW$$
$$= \frac{1}{2\pi i}\int_{|w|=\eta}\sum_{l=0}^{p-1}\begin{pmatrix}2p\\2l+1\end{pmatrix}W^{-m-1+l}\sum_{k=0}^{p-l-1}\begin{pmatrix}p-l-1\\k\end{pmatrix}W^k dW$$
$$= \frac{1}{2\pi i}\int_{|w|=\eta}\sum_{l=0}^{p-1}\sum_{k=0}^{p-l-1}\begin{pmatrix}2p\\2l+1\end{pmatrix}\begin{pmatrix}p-l-1\\k\end{pmatrix}W^{-m-1+l+k}dW$$
$$= \begin{cases}\sum_{l=0}^{p-1}\begin{pmatrix}2p\\2l+1\end{pmatrix}\begin{pmatrix}p-l-1\\m-l\end{pmatrix} & p \geq m+1\\ 0 & \text{otherwise}\end{cases}$$

Note that $W = sinh^2\,u$.

In a similar way one can compute $\frac{1}{2\pi i}\int_{|Z|=\varepsilon}\frac{f_2(Z)}{Z^{2m+3}}dZ$.

3.4 Some applications

We already knew that Clifford cohomology groups $H_{cl}^{p,q}(M,W)$, which are defined by the Kähler metric, reflect the complex structures of M and W, while $\chi(W)$ is a topological invariant. The Clifford formalism gives a natural way to study relations among the complex structure, the Riemannian geometry and the topological properties of the manifold. The followings are some results in this vein.

Theorem 3.4.1 *Let M be the complex projective space $\mathbf{CP}^n$, n odd. Then for $n \geq 5$*

$$H_{cl}^{-n+q,q}(\mathbf{CP}^n, K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad \forall q$$

and

$$H_{cl}^{-n+q,q}(\mathbf{CP}^n, K^{*\frac{1}{2}} \otimes T^*) = \begin{cases} 0 & q = 0,1 \\ dim_c H_{cl}^{-1,2} = dim_c H_{cl}^{0,3} \end{cases}$$

This is an easy consequence of the theorems (3.2.1) and (3.3.5). Since for $n \geq 5$

$$\chi(\mathbf{CP}^n, K^{\frac{1}{2}} \otimes T) = dim_c H_{cl}^{0,n}(\mathbf{CP}^n, K^{*\frac{1}{2}} \otimes T^*)$$

Using theorems (3.2.2) and (3.3.4) we get the following result.

Theorem 3.4.2 *For $n \geq 6$, n even,*

$$H_{cl}^{-n+q,q}(Q_n(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) = 0 \qquad \forall q$$

and

$$H_{cl}^{-n+q,q}(Q_4(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) = \begin{cases} 0 & q = 0,1,2 \\ dim_c H_{cl}^{0,4} + 1 = dim_c H_{cl}^{-1,3} \end{cases}$$

$$H_{cl}^{-n+q,q}(Q_2(\mathbf{C}), K^{*\frac{1}{2}} \otimes T^*) = \begin{cases} 0 & q = 0 \\ dim_c H_{cl}^{0,2} = dim_c H_{cl}^{-1,1} \end{cases}$$

Finally the theorems (3.2.3), (3.3.4) and (3.3.5) give the following theorem.

Theorem 3.4.3 *Let M be a complex hypersurface imbedded in $\mathbf{CP}^{n+1}$ with the induced metric and assume $c_1(M)$ is even. If the eigenvalues of the second fundamental form of M satisfy*

$$\lambda_j^2 \leq \frac{n^2 - n + 6}{4(5n-1)}$$

for all j, then for $n \geq 5$, degree $(M) < [\frac{n+2}{3}]$.

This follows from the following lemma which can be proved by elementary estimates in the theorems (3.3.4) and (3.3.5).

Lemma 3.4.4 *If $V^n(d)$ be a spin manifold, then for $n \geq 5$*

$$\chi(V^n(d), K^{\frac{1}{2}} \otimes T) = \begin{cases} 0 & 0 < d < \frac{n+2}{3} \\ < 0 & \frac{n+2}{3} \leq d \leq n \end{cases}$$

In particular, when M has complex dimension $n = 5$ or $n = 7$, we prove the following rigidity result for complex projective space $\mathbf{CP}^n$.

Corollary 3.4.5 *Suppose that M is as in theorem 3.4.3. If the eigenvalues of the second fundamental form of M satisfy*

$$\begin{aligned} & \lambda_j^2 \leq \frac{13}{48} \qquad \textit{when } n = 5 \\ \textit{or} \quad & \lambda_j^2 \leq \frac{6}{17} \qquad \textit{when } n = 7 \end{aligned}$$

for all j, then M has degree 1, i.e. M is the complex projective space $\mathbf{CP}^n$.

In the situation when M has complex dimension $n = 6, n = 8$, or $n = 10$, we get the estimate on the eigenvalues of the second fundamental form of M which guarantees M has degree 2.

Corollary 3.4.6 *Suppose that M is as in theorem 3.4.3. If the eigenvalues of the second fundamental form of M satisfy*

$$\begin{aligned} & \lambda_j^2 \leq \frac{9}{29} \qquad \textit{when } n = 6 \\ \textit{or} \quad & \lambda_j^2 \leq \frac{31}{78} \qquad \textit{when } n = 8 \\ \textit{or} \quad & \lambda_j^2 \leq \frac{24}{49} \qquad \textit{when } n = 10 \end{aligned}$$

for all j then M is an algebraic manifold with degree 2, i.e. M is the complex hypersphere $Q_n(\mathbf{C})$.

Bibliography

[1] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology 3, (Suppl. 1) (1964), pp. 3-38.

[2] M. F. Atiyah and I. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. , 69 (1963), pp. 422-433.

[3] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Grundlehren der math. Wiss. , Springer, Berlin-Herdelberg-New York, 1966.

[4] N. Hitchin, *Harmonic spinors*, Advances in Math. 14 (1974), pp. 1-55.

[5] S. Kobayashi, *Differential Geometry of Complex Vector Bundle*, Princeton University Press, Princeton, New Jersey, 1987.

[6] H. B. Lawson and M. L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, New Jersey, 1989.

[7] A. Lichnerowicz, *Spineurs harmoniques*, C.R. Acad. Sci. Paris, Ser. A-B 257 (1963), pp. 7-9.

[8] M. L. Michelsohn, *Clifford and spinor cohomology of Kähler manifolds*, Amer. J. of Mathematics, Vol. 102 (1980), No. 6, pp. 1083-1146.

[9] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1974.

[10] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. 85 (1967), pp. 246-266.

[11] R. O. Wells, *Differential Analysis on Complex Manifolds*, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.