Rationality of Limiting $\eta$-Invariants of Collapsed 3-Manifolds

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Rationality of Limiting $\eta$-Invariants of Collapsed 3-Manifolds

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We study the topological aspects of a sufficiently collapsed 3-manifold with bounded covering geometry. Based on the recent work of Cheeger and Gromov on collapsing Riemannian manifolds and F-structure theory, we find the existence of injective F-structures on such manifolds.

As an application, we prove a Cheeger and Gromov conjecture on rationality of limiting $\eta$-invariants associated to a volume collapse with bounded covering geometry in the 3-dimensional case. Another application is that we are able to give an explicite residue formula for computing the limiting $\eta$-invariants in terms of the information derived from an injective F-structure. We also discuss the uniqueness of injective F-structures and obtain a finite-
ness result on limiting \( \eta \)-invariants.
To my parents and Bin
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Introduction

In this thesis, we study the rationality of limiting $\eta$- invariants for collapsed Riemannian manifolds. These were defined and previously studied in [CG2]. We prove a conjecture of Cheeger and Gromov (see below) in the 3-dimensional case.

Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with bounded sectional curvature, $|K| \leq 1$. Let $\alpha(g)$ denote one of the following geometric quantities associated to $g$: the upper bound on the injectivity radii; the volume of $M$; or the diameter of $M$. If $\alpha$ is sufficiently small relative to the sectional curvature, we will say that $\alpha(g)$ is sufficiently collapsed. If $M$ admits a family of metrics $\{g_\delta\}$ so that the family $\{\alpha(g_\delta)\}$ converges to zero as $\delta \to 0$, one says that $M$ admits an $\alpha$-collapse.

Understanding the interplay between the collapsing geometry and the topology of $M$ has been very fruitful in recent years. This can be viewed as studying the questions complimentary to those of controlling the topology by putting bounds on certain geometric quantities (for instance, Cheeger’s finiteness theorem [Ch]). The basic questions are:

1) What kind of structures and invariants can be attached to a sufficiently $\alpha$-collapsed manifold?

2) Does a sufficiently $\alpha$-collapsed manifold imply an $\alpha$-collapse?

Starting with [Gr1], there has been considerable progress in this field; for instance, Gromov’s almost flat manifolds ([Gr1]) for sufficiently small di-
ameters, the F-structure theory for uniformly small injectivity radii ([CG3], [CG4]) and the bundle structure theorems for manifolds which collapse to a manifold of lower dimension ([Fu1], [Fu2]) and more recently, the Nil-structure theorem ([CFG]).

Important progress has been made, concerning the implication of a volume collapse, in the work of Cheeger and Gromov in [CG1], [CG2] and [CG5]. There, they gave generalizations of Gauss-Bonnet theory to open manifolds of bounded sectional curvature and finite volume (note that such manifolds have their volume collapsed at infinity).

A volume collapse \( \{g_\delta\} \) with bounded curvature is said to have bounded covering geometry (BCG) if the pull-back metrics of the family \( \{g_\delta\} \) on the universal covering of \( M \) have a uniform lower bound on their injectivity radii.

Let \( N \) be a compact oriented \((4k-1)\)-manifold. Consider the \( \eta \)-invariant of \( N \) in the sense of Atiyah, Patodi and Singer ([APS]). Cheeger and Gromov [CG2] proved that the limit of \( \eta \)-invariants with respect to a volume collapse with BCG exists and is a topological invariant (i.e., it does not depend on the specific collapsing sequence of metrics). We call this limit the limiting \( \eta \)-invariant and denote it by \( \eta_{(2)}(N) \). Cheeger and Gromov made the following conjecture.

**Conjecture I:** The limiting \( \eta \)-invariant \( \eta_{(2)}(N) \) is a rational number.

The difficulty in proving Conjecture I stems from the fact that the metrics in a given volume collapse with BCG may not be obviously related
to one another. In addition, it is very difficult to compute the $\eta$-invariant explicitly except in circumstances which are special in one way or another. In this thesis, we prove Conjecture I for 3-manifolds.

**Theorem A.** Let $N$ be a compact oriented 3-manifold. Suppose $N$ admits a volume collapse with BCG, then the limiting $\eta$-invariant $\eta_3(N)$ is a rational number.

Our approach to Theorem A is to show that sufficiently collapsed 3-manifolds with BCG admit a certain topological structure, the injective $F$-structure. Roughly an $F$-structure can be thought of as a family of local torus actions satisfying a certain consistency condition. It was introduced in [CG3]. An $F$-structure partitions the underlying manifold into orbits. An injective $F$-structure is a $F$-structure satisfying the condition that the fundamental group of each orbit injects into the fundamental group of the total space. We will prove

**Theorem B.** There exists a constant $\epsilon > 0$ such that if a compact 3-manifold $N$ admits a Riemannian metric $g$ satisfying:

- (i) $|K_g(N)| \leq 1$
- (ii) $\text{Injrad}(N, g) < \epsilon$
- (iii) $\overline{\text{co}}(N, g) \geq 1$ (BCG)

then either $N$ admits an injective $F$-structure or $N$ (possibly some doubling cover of it) is diffeomorphic to a lens space. In particular, a compact 3-manifold $N$ admits a volume collapse with BCG if and only if $N$ admits an
injective $F$-structure.

An $F$-structure is said to be polarized, if the local orbits have the same dimension as the local group which acts. We observe that when specialized to the case of an injective $F$-structure the volume collapse constructed using a polarized $F$-structure as in [CG3] has BCG. This together with Theorem B implies that an injective $F$-structure on a 3-manifold fully reflects the volume collapse with BCG. A consequence of this is that one can compute $\eta_2(N)$ with respect to a volume collapse compatible with an injective $F$-structure (in general, the limiting $\eta$-invariant associated to a volume collapse compatible with a polarized $F$-structure $\mathcal{F}$ is called the limiting $\eta$-invariant associated to $\mathcal{F}$ and is denoted by $\eta(N, \mathcal{F})$. $\eta(N, \mathcal{F})$ is independent of invariant volume collapse with or without BCG). By filling in a compact 4-manifold $M$ and extending the injective $F$-structure to $M$ by using the so-called equivariant plumbing technique, the residue theory for the secondary characteristic classes as developed in [Ya] implies the rationality of $\eta(N, \mathcal{F})$ and thus the rationality of $\eta_2(N)$.

In proving Theorem B our starting point is the result obtained in [CG4]. There Choeger and Gromov associate an $F$-structure $\mathcal{F}_g$ (topological structure) to a sufficiently injectivity radius collapsed metric $g$. $\mathcal{F}_g$ is constructed on the scale of small injectivity radius of the manifold, priori, it reflects neither small volume nor BCG. Using some theorems from 3-dimensional topology and some property of $\mathcal{F}_g$ which depends on BCG, we find the existence of an injective $F$-structure satisfying the BCG condition. Note that the existence of an injective $F$-structure may not be deduced from the
geometrical construction in [CG4] (also see example 7).

We also discuss the uniqueness of injective F-structures on a 3-manifold. Two injective F-structures on a 3-manifold are said to be weakly isomorphic, if they are different only on a saturated area which is homeomorphic to $T^2 \times I$ (see Definition 9). Based on the classification theorem for graph manifolds obtained in [Wa1] and [Wa2], we obtain the following.

**Theorem C.** Let $N$ be a closed orientable 3-manifold which is not homeomorphic to either $S^2 \times S^1$ or a nilmanifold. Suppose $N$ admits an injective T-structure $\mathcal{F}$. Then $\mathcal{F}$ is unique up to weak isomorphism.

From our approach to Theorem A, not only does one get the rationality of the limiting $\eta$-invariant, but one can compute explicitly the value $\eta^{(2)}(N)$ and see its topological meaning in the 3-dimensional case.

In the second part of this thesis, we give the explicit residue formula for $\eta^{(2)}(N)$ in terms of topological data derived from the injective F-structure (see Theorem 11). Theorem C implies that except for a few cases such a residue formula is intrinsic in a certain sense. For instance, we express $\eta^{(2)}(N)$ for $N$ an injective Seifert manifold in terms of Seifert invariants (we refer to [Or] for Seifert manifolds).

**Theorem D.** Let $N = \{b; (a, g); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$ be the Seifert invariant of an injective Seifert manifold $N$. Then

$$
\eta^{(2)}(N) = -\sum_{i=1}^{r} s_i + \epsilon(b, r) + \frac{1}{3} (b + r) + \frac{1}{3} \sum_{i=1}^{r} \sum_{j=1}^{s_i} b_{ij} + \frac{1}{3} \sum_{i=1}^{r} \frac{\alpha_i s_i - 1}{\alpha_i} \quad (1)
$$
where \( \frac{\alpha_i}{\alpha_i - \beta_i} = [b_{i,1}, \ldots, b_{i,s_i}] \) is the continued fraction \( (b_{i,j} \geq 2) \), \( \alpha_{i,j} = b_{i,j}\alpha_{i,j-1} - \alpha_{i,j-2}; \alpha_{i,0} = 1, \alpha_{i,1} = b_{i,1} \) and
\[
\epsilon(b, r) = \begin{cases} 
-1 & \text{if } b + r > 0 \\
+1 & \text{if } b + r \leq 0.
\end{cases}
\]

A consequence of Theorem D is that the invariants \( \eta_{(2)}(N) \) can take any rational number in 3-dimension. However, using Gromov’s pre-compactness theorem, the result in [Fu2] and Theorem D, we prove a finiteness result for limiting \( \eta \)-invariants as follows:

**Theorem E.** For a real number \( D > 0 \), let \( M^3(D) \) be the collection of closed orientable 3-manifolds which admit volume collapses with BCG for which the diameters stay bounded by \( D \). Then \( \{ \eta_{(2)}(N) \mod Z | N \in M^3(D) \} \) is a finite set.

Note that \( M^3(D) \) contains infinitely many diffeomorphic types for any \( D > 0 \). For instance, 3-dimensional nilmanifolds.

The result in the last part of this thesis is about a bounded version of the main result in [CG5]. We will use this result to study rationality of geometric signatures for open manifolds (see below). First, let us recall the following.

**Theorem 12([CG5])** Let \( M^n \) be a complete manifold with bounded sectional curvature, \( |K| \leq 1 \). Given \( X \subset M^n, 0 < r \leq 1 \), there is a sub-
manifold $U^n$ with smooth boundary $\partial U^n$ such that for some constant $c(n)$ depending on $n$,

$$X \subset U^n \subset T_r(X)$$

$$\text{Vol}(\partial U) \leq c(n)\text{Vol}(T_r(X))r^{-1}$$

$$\|I_{\partial U}\| \leq c(n)r^{-1}$$

Moreover $U^n$ can be chosen to be invariant under $I(r, X)$, where $I(r, X)$ denotes the group of isometries of $T_r(X)$ which fix $X$.

We show the following fact:

**Theorem F.** Under the same assumptions as in Theorem 12. Let $D = \text{diam}(X) = \sup\{\text{dist}(x, y) | x, y \in X\}$. Then there exists a constant, $c(n, r, D)$, depending on $n$, $r$ and $D$, such that

$$\text{diam}(\partial U) \leq c(n, r, D).$$

An immediate consequence of Theorem F is what can be viewed a bounded version of Cheeger and Gromov’s good chopping theorem (see Theorem 14).

In fact, our results in this thesis can be further exploited to prove a Cheeger and Gromov conjecture on rationality of geometric signatures. We now explain this.

By the work of Cheeger and Gromov, the Gauss-Bonnet-Chern theorem has been extended to complete manifolds of bounded sectional curvature and finite volume ([CG1] and [CG2]). They considered the geometric Euler

numbers and the geometric signatures of such manifolds. These are defined as the integrals of the Euler form and the signature form on these manifolds. (Note that examples show that the geometric signatures of a 4k-manifold can be an arbitrary real number in general.) They proved that, under the assumption that the metrics have bounded covering geometry over a neighborhood of infinity (denoted by $\widetilde{\text{geo}}_\infty(M) \geq 1$), the geometric Euler number is a homotopy invariant and the geometric signature is a proper homotopy invariant. More precisely, they are equal to the $L_2$-Euler number and $L_2$-signature, denoted by $\chi_{(2)}(M)$ and $\sigma_{(2)}(M)$ respectively. Using the so-called good chopping technique ([CG5]), Cheeger and Gromov proved that the geometric Euler numbers are always integers (without assuming BCG). Furthermore they made the following conjecture.

**Conjecture II**: $\sigma_{(2)}(M)$ is a rational number.

It turns out that using Theorem A, Theorem E, Theorem 14 and the results in [CFG], we can prove Conjecture II for 4-manifolds under an additional assumption that the manifolds have finite diameters at infinity. A complete manifold $M$ is said to has finite diameter at infinity, if there is point $p \in M$ so that $\sup_{r>0}\{\sum_{\alpha} \text{diam}((T_1(\partial B_r(p)))_\alpha)\} < \infty$, where the sume runs over all components of the 1-tubular neighborhood of the metric sphere $\partial B_r(p)$. We will only give the outline of the proof in §3.2. The detailed argument will appear elsewhere.

The thesis is organized as follows.

In §1.1, we briefly review the results in [CG3], [CG4], [Fu1] and [Fu2]
which we shall use in the sequel. We also introduce injective F-structures and show that a volume collapse compatible with a polarized F-structure $\mathcal{F}$ has BCG if and only if $\mathcal{F}$ is injective (Theorem 5).

In §1.2, we give a quick proof of Theorem A by assuming Theorem B (proved in §1.3) and a result from §1.6. Some readers may wish to read §1.3 and §1.6 before reading §1.4 and §1.5.

In §1.3 we first identify the collapsed 3-manifolds as graph manifolds in general, and the collapsed 3-manifolds with bounded diameters as Seifert manifolds and solvable manifolds up to a double covering (Propositions 4 and 5). Then we give the characterization of injective F-structures on 3-manifolds (Theorem 8). Furthermore, we give a sufficient and necessary condition which guarantees that a polarized F-structure which can be modified to be injective (Theorem 9). Based on this we prove Theorem B.

In §1.4, we prove Theorem C.

In §1.5, we describe the idea of an alternative proof of that a limiting $\eta$-invariant is a topological invariant in dimension 3. This uses mainly Theorem A, B and C (note that this is totally different from the one given in [CG2]).

Given a pair $(N, \mathcal{F})$, we fill in $N$ with a compact 4-manifold $M$ and extend the polarized F-structure of $N$ to $M$ by using the invariant plumbing technique (Theorem 10). This is accomplished in §1.6.

Section §2.1 is devoted to explicit residue formulas for $\eta_\phi(N)$. We carry out the general residue theory of [Ya] in our special circumstance. Theorem D is proved and the general residue formula is given (Theorem 11).

Theorem E is proved in §2.2.
In §3.1 we prove Theorem F.

Finally in §3.2 we describe the outline of our approach to Conjecture II in 4-dimensional case.
Chapter 1

Rationality of Limiting $\eta$-invariants of 3-manifolds

1.1 Collapsing and F-structures

In this section we will review the fundamental results on collapsed Riemannian manifolds and F-structures obtained in [CG3], [CG4], [Fu1] and [Fu2]. Then we will discuss some basic properties of injective F-structures.

Let $M$ be a manifold. A sequence of complete Riemannian metrics $\{g_t\}$ on $M$ is said to determine an injectivity radius (or volume) collapse, if $\{g_t\}$
satisfies the conditions

(i) \(|K_g(M)| \leq 1\),

(ii) \(\lim_{i \to \infty} \sup \{\text{InjRad}(x, g_i) \mid x \in M\} = 0\) (or \(\lim_{i \to \infty} \text{Vol}(M, g_i) = 0\)).

A volume collapse \(\{g_i\}\) is said to have bounded diameter, if \(\text{diam}(M, g_i) \leq D\) for some constant \(D > 0\).

A volume collapse \(\{g_i\}\) is said to have BCG, if the pull back metrics of \(\{g_i\}\) on the universal covering of \(M\) have a lower bound (say at least 1) on the injectivity radius.

The topological aspect of collapsing phenomena on a manifold \(M\) is to a large extent captured by the so-called F-structures which were first introduced in [CG3].

**Definition 1** Let \(M\) be \(C^\infty\) manifold. An F-structure \(\mathcal{F}\) on \(M\) stands for \(\{(\tilde{U}_i, U_i, T^k_i, \phi_i, \psi_i)\}\) (called an atlas for \(\mathcal{F}\)) such that,

(i) \(\{U_i\}\) is a locally finite open cover of \(M\),

(ii) \(\pi_i : \tilde{U}_i \to U_i\) is a finite Galois covering with Galois (deck transformation) group \(G_i\).

(iii) \(T^k_i\) is a \(k_i\)-dimensional torus and \(\phi_i : T^k_i \to \text{Diff}(\tilde{U}_i)\) is an effective and smooth action,

(iv) \(\psi_i : G_i \to \text{Aut}(T^k_i)\) is a homomorphism satisfying

\[ g_i(\phi_i(\gamma_i)(x)) = \phi_i((\psi_i(g_i)(\gamma_i))(g_i x)) \]

for each \(g_i \in G_i, \gamma_i \in T^k_i\) and \(x \in \tilde{U}_i\),

(v) if \(U_i \cap U_j \neq \emptyset\), then there is a common covering \(V_{ij}\) of \(\pi_i^{-1}(U_i \cap U_j)\) and \(\pi_j^{-1}(U_j \cap U_j)\) so that the lifting actions of \(T^k_i\) and \(T^k_j\) on \(V_{ij}\) commute.
Remark 1 In view of (iv) in Definition 1, the orbits of the local action $\phi_i$ on $U_i$ are well defined. By (v) in Definition 1, the orbits of $\mathcal{F}$ at the point $x \in M$ are defined as the union of all the orbits of $\phi_i$ through $x$. The rank of $\mathcal{F}$ at $x \in M$ is defined as the dimension of the orbit through $x$. $\mathcal{F}$ has positive rank, if all orbits of $\mathcal{F}$ have positive dimensions.

Definition 2 A T-structure on $M$ is an F-structure such that we can choose $\tilde{U}_i = U_i$ for all $i$ as in Definition 1.

Definition 3 An F-structure $\mathcal{F}$ is called pure, if for all $i, j, k_i = k_j$. $\mathcal{F}$ is called polarized, if the local action $\phi_i$ has finite isotropy group at all points of $\tilde{U}_i$.

Definition 4 A polarization of $\mathcal{F}$ is a collection of connected subgroups $H_i \subset T^{k_i}$ such that the dimension of each $H_i$-orbit is equal to $\text{dim} H_i$. If $H_i$ is a compact subgroup of $T^{k_i}$ for all $i$, we call the polarization a polarized substructure of $\mathcal{F}$.

Definition 5 A T-structure $\mathcal{F} = \{(U_i, T^{k_i}, \phi_i)\}$ is isomorphic to the T-structure $\mathcal{F}' = \{(U'_i, T^{k_i}, \phi'_i)\}$, if there is a diffeomorphism $\phi$ of $N$ such that $(U_i, T^{k_i}, \phi_i)$ is isomorphic to $(U'_i, T^{k_i}, \phi'_i)$ under the restriction of $\phi$ for all $i$.

The relations between positive rank F-structures and the injectivity radius collapse on a manifold $M$ is given in the following theorems.

Theorem 1 ([CG3]) Let $(M, g)$ be a complete $n$-manifold with $|K| \leq 1$. There exists a constant $c(n)$ depending only on $n$, so that if the injectivity
radii are smaller than $c(n)$ everywhere on $M$, then $M$ admits a positive rank $F$-structure $F_\theta$ which is almost compatible with the metric $g$. Conversely, suppose $M$ admits a positive rank $F$-structure. Then one is able to construct an invariant injective radius collapse.

If a manifold $M$ admits a polarized $F$-structure, then one can obtain a volume collapse.

**Theorem 2 ([CG4])** (1) Let $M$ be a manifold and $F$ be a polarized $F$-structure of $M$. Then $M$ admits an invariant volume collapse.

(2) Suppose $M$ admits a polarized $F$-structure $F$ outside a compact subset $C$ of $M$, then $M$ admits a complete metric $g$ with bounded sectional curvature, finite volume and compatible with $F$.

**Remark 2** The converse of Theorem 2 may not be true in general. In fact, an affirmative answer would solve an open question on vanishing minimal volume (see [Gr2]).

The next two theorems are not needed in proving Theorem A but only in proving Proposition 5 in §1.3 and Theorem E in §2.2. Some readers may omit them on the first reading.

**Theorem 3 ([Fu2])** Let $M_i$ be a sequence of closed $n$-manifolds satisfying the following conditions:

(i) $|K(M_i)| \leq 1$

(ii) $\text{diam}(M_i) \leq D$.

Assume there exists a metric space $Y$ such that $\lim_{i \to +\infty} d_H(M_i, Y) = 0$. Then we have the following:
There exists a smooth manifold $N$ with a $C^{1,\alpha}$-metric $g_N$, on which there is a smooth and isometric action of the orthogonal group $O(n)$ such that

(1) $Y$ is isometric to $N/O(n)$,

(2) For each $p \in N$, the isotropy group is an extension of a torus $T^k$ by a finite group.

**Theorem 4 (F2)** Under the same assumption as in Proposition 3, for sufficiently large $i$, there exists maps $f_i : M_i \rightarrow Y$ and $\tilde{f}_i : FM_i \rightarrow N$ satisfying:

(1) $\tilde{f}_i$ is a fibration with fiber infranilmanifold $\approx N_i/\Lambda$.

(2) The structure group is contained in

$$\frac{CentN_i}{CentN_i \cap \Lambda} \cong Aut\Lambda$$

(3) $\tilde{f}_i$ is an almost Riemannian submersion. In other words, for each $V \in TM_i$ perpendicular to the fiber of $\tilde{f}_i$, we have

$$e^{-o(i)} < \frac{|\tilde{f}_i(V)|}{|V|} < e^{o(i)}$$

where $o(i)$ satisfies $\lim_{i \rightarrow \infty} o(i) = 0$.

(4) The following diagram commutes:

$$\begin{array}{ccc}
\tilde{f}_i & \rightarrow & N \\
FM_i & \downarrow & \downarrow \\
\pi_i & & \pi_i \\
\downarrow & & \downarrow \\
f_i & \rightarrow & Y = N/O(n)
\end{array}$$
Note that (2) of Theorem 4 implies that, for sufficiently large $i$, $N_i$ admits a pure positive rank $F$-structure which is given by the projection of structure group in (2). Note that in general such pure $F$-structure is not polarized.

For examples of $F$-structures we refer to [CG3], [CG4] and [Fu3].

To study a volume collapse with BCG, we introduce the injectivity of an $F$-structure as follows.

**Definition 6** Let $\mathcal{F} = \{ (\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i) \}$ be a positive rank $F$-structure of $N$. For any $x \in N$ and $(\tilde{U}_{io}, U_{io}, T^{k_{io}}, \phi_{io}, \psi_{io}) \supset O_x$ ($\dim(O_x) = k_{io}$), consider the following diagram:

\[
\begin{array}{ccc}
\phi_{io} & \rightarrow & (\tilde{U}_{io}, \tilde{x}) \\
(T^{k_{io}}, e) & \nearrow \phi_{io} & \downarrow \pi \\
\tilde{\phi}_{io} = \pi \circ \phi_{io} & \subset (U_{io}, x) & \subset (N, x).
\end{array}
\]

$F$ is called injective if the induced map $(\tilde{\phi}_{io})_* : \pi_1(T^{k_{io}}, e) \rightarrow \pi_1(N, x)$ is injective.

**Remark 3** The injective $F$-structure is to some degree a generalization of an injective torus action studied in [CR1], [CR2], and [LR].

Injective $F$-structures are polarized. The basic topological feature of an injective $F$-structure is the local splitting result for its universal covering space.
**Lemma 1** Let $\mathcal{F}$ be an injective $F$-structure of $M$, let $\pi : \tilde{M} \to M$ be the universal covering. For any $x \in M$ there exists an invariant tubular neighborhood $U$ of $O_x$ such that $\pi^{-1}(U)$ is homeomorphic to $D^{n-k} \times R^k (k = \dim(O_x))$.

**Remark 4.** To be consistent with usage, Lemma 1 can be viewed as the local version of the so-called splitting result for the injective torus actions in [CR1] and [CR2].

**Proof of Lemma 1.** First we assume that $\mathcal{F}$ is a $T$-structure. For any $x \in M$, let $(U, T^k, \phi)$ be a stratum of $\mathcal{F}$ with $U \supset O_x$. Let $\Gamma$ be the finite isotropy subgroup of $O_x$ and let $S \approx D^{n-k}$ be the slice of $O_x$ at $x$. Then $O_x$ determines the invariant tubular neighborhood $U$ which is diffeomorphic to $U \approx D^{n-k} \times_\Gamma T^k$.

The universal covering of $U$ is $D^{n-k} \times R^k$ and the covering group $R^k$ of $T^k$ acts on $\pi^{-1}(U)$ by the addition of the $R^k$-factor with the following commutative diagram

\[
\begin{array}{ccc}
R^k \times (D^{n-k} \times R^k) & \rightarrow & D^{n-k} \times R^k \\
\pi \times \pi \downarrow & & \pi \downarrow \\
T^k \times (D^{n-k} \times_\Gamma T^k) & \rightarrow & D^{n-k} \times_\Gamma T^k.
\end{array}
\]

We claim that $\pi^{-1}(U)$ is simply connected (thus it is diffeomorphic to $D^{n-k} \times R^k$). This follows from $\pi_1(U) \approx \pi_1(O_x)$ and the injectivity of the local $T^k$-action.
Now consider $\mathcal{F}$ to be a general $F$-structure. Take a stratum $(\tilde{U}, U, T^k, \phi, \psi)$ which contains $O_x$ ($dim O_x = k$). Pick up $\tilde{x} \in \tilde{U}$ with $\pi(\tilde{x}) = x$. Clearly, we can first find a tubular neighborhood of $O_x$ as before and project it down by $\pi$ to obtain the desired result. Q.E.D.

The basic geometrical consequence of the existence of an injective $F$-structure is the existence of a compatible volume collapse with BCG.

**Theorem 5** (1) Suppose $M$ admits an injective $F$-structure. Then $M$ admits an invariant volume collapse with BCG.

(2) Suppose $M$ (open) admits an injective $F$-structure outside some compact subset, then $M$ admits an invariant metric $g$ satisfying

1. $|K_g(M)| \leq 1$,
2. $Vol(M, g) < +\infty$,
3. $g \circ o_\infty(M, g) \geq 1$.

**Proof.** Given an injective $F$-structure $\mathcal{F}$, the construction for a volume collapse $\{g_\delta\}(0 \leq \delta \leq 1)$ compatible with $\mathcal{F}$ is given in [CG3]. What we shall do is observe the BCG, that is actually a consequence of the local splitting property of $\mathcal{F}$. Let $\pi : \tilde{M} \rightarrow M$ be the universal covering, let $\tilde{g}_\delta = (\pi)^*(g_\delta)$ be the pull-back metric. Fix a point $\tilde{x} \in \tilde{M}$ and $x = \pi(\tilde{x})$. Let $O_x$ be the orbit through $x$ and $T^+_x$ be the orthogonal complement of the subspace $T_x O_x$ in $T_x M$ with respect to the initial metric $g_1$. We can find $\rho_x > 0$ such that $S_x = exp_x B^1_{\rho_x}(\approx D^{n-k})(k = dim(O_x))$ is a slice of $O_x$ where $B^1_{\rho_x}$ is the ball of radius $\rho_x$ in $T^1_x$. Let $U$ be the invariant neighborhood determined by $S_x$. Applying Lemma 1, $\pi^{-1}(U) \approx S \times R^k$. 


Let $\tilde{g}^1_{\delta}$ denote the restriction of $\tilde{g}_{\delta}$ to $S_x$. Following the proof of Theorem 2 in [CG4], one sees that the metrics $\{\tilde{g}_{\delta}\}$ converge to a $C^\infty$ product metric $\tilde{g}_0 = \tilde{g}_0' + \tilde{g}_0''$, where $\tilde{g}_0'' = fg_e$ ($f$ is a $C^\infty$ function on the $S_x$, $g_e$ is the Euclidean metric and $\tilde{g}_0'$ is the limit of $\{\tilde{g}_{\delta}^1\}$). If $F$ is pure, then $\tilde{g}_0' = g^1_{\delta} = g^1_1$ $(0 \leq \delta \leq .1)$. It follows that $\text{InjRad}(\tilde{x}, \tilde{g}_{\delta}) \geq \rho_x$ as $\delta \to 0$. In the case of $F$ not being pure, one still has $\text{InjRad}(\tilde{x}, \tilde{g}_{\delta}) \geq \rho_x$ since here the $\tilde{g}_{\delta}^1$ is obtained basically by spanning $g^1_1$.

Now it is clear that if the initial invariant metric $g_1$ satisfies the condition that the $\rho_x$ has a uniform lower bound for all $x \in M$, then $\{g_{\delta}\}$ has BCG. The result in [CG3] implies one can always construct such an invariant metric $g_1$. We refer to [CG3] for further details. Q.E.D.

Examples of injective $F$-structures.

Example 1 The category of the manifolds which admits a pure injective $F$-structure coincides with the category of the so-called injective Seifert fiber space with fiber either a flat manifold or a almost flat manifold. Seifert fiber space were first introduced in [CR1] and [CR2]. Seifert fiber space include the following:

1) classical Seifert space with infinite fundamental groups,

2) flat manifolds and almost flat manifolds(=infranilmanifolds). An $n$-dimensional flat manifold $M$ admits a natural injective $T^n$-structure. By Bieberbach’s theorem, there exists a finite covering of $M$ which is a flat torus $T^n$. It is clear that the multiplication of $T^n$ determines the pure $T^n$-structure which is also injective.
Similarly, an almost flat manifolds $M$ admits an injective F-structure given by the center of a nilpotent group acting on a finite covering of it (see [Gr1]).

The following example is very important.

**Example 2** All the 3-manifolds (or a double covering) which admits an injective F-structure can be obtained as follows:

Take finite Seifert fiber spaces with torii boudary components, $N_1, ..., N_r$. Suppose none of $N_i$ is homeomorphic to a solid torus and the Seifert fiber structure of each $N_i$ is trivial near $\partial N_i$. Form a closed by identifying the boundary components of $N_i$'s in pairs (we assume the total number of boundary components are even). Then these $S^1$- Seifert fibering on $N_i$ generate a polarized T-structure which is actually injective (see Theorem 8).

**Example 3** Let $M^n$ be a closed oriented manifold and let $f : K(\pi, 1)$ be the classifying map, where $\pi \approx \pi_1(M^n)$. We call $M^n$ essential, if the fundamental class, $[M^n] \in H_n(M, R)$ satisfies $f_*([M^n]) \neq 0$. Suppose $\mathcal{F}$ is a pure positive rank F-structure on an essential manifold $M^n$. Then $\mathcal{F}$ is injective. (The proof of this is given in the Appendix of [CG3]).

**Example 4** Let $M$ be a complete Riemannian manifold satisfying:

(i) $-b^2 \leq K \leq -a^2$, for some constant $a, b > 0$,

(ii) $Vol(M) < \infty$.

Then $M$ has finite topological type, i.e., there exists a compact manifold
with boundary $M_0$, such that $M$ is diffeomorphic to the interior of $M_0$. Moreover each component of $\partial M_0$ is an infranilmanifold. Thus $M$ admits an injective $\mathcal{F}$-structure outside a compact subset of $M$. We refer to [BGS] for more details.

1.2 Proof of Rationality of Limiting $\eta$-invariants
Modulo §1.3 and §1.6

In this section, by employing the results in §1.3 and §1.6, we will give a simple proof of Theorem A. We also explain our idea to approach to Theorem A. First, let us recall the following result in [CG2].

Theorem 6 ([CG2]) Let $N$ be a compact orientable $(4k - 1)$ manifold. Suppose $N$ admits a volume collapse $\{g_\epsilon\}$ with BCG. Then the limiting $\eta$-invariant $\eta_{(2)}(N)$, defined by

$$\eta_{(2)}(N) = \lim_{\epsilon \to 0} \eta(N, g_\epsilon)$$

exists (and thus $\eta_{(2)}(N)$ is a topological invariant).

Our starting point in proving Theorem A is Theorem 1, Theorem 5, Theorem 6 and the result in [Ya].

In [Ya], D. Yang took a different approach from [CG2] to realize the global and topological aspects of the secondary geometric invariants (in particular, $\eta$-invariant). Yang’s approach depends on the result on collapsing geometry in [CG3]. Essentially, he generalized the residue formula
for characteristic numbers of closed manifolds in [Bo] and [BC], which expresses the characteristic numbers in terms of the data from the zero point set of a global Killing vector field, to compact manifolds whose boundary admits a polarized F-structure. Specializing to the \(\eta\)-invariant, with a little work, his result can be stated as follows.

**Theorem 7 (Ya)** Let \(N\) be an orientable compact \((4k - 1)\)-manifold and let \(F\) be a polarized F-structure of \(N\). If \(\{g_\delta\}\) is a volume collapse compatible with \(F\), then the limiting \(\eta\)-invariant,

\[
\eta(N, F) = \lim_{\delta \to 0} \eta(N, g_\delta)
\]

(1.1)

exists and is independent of this specific invariant volume collapse chosen. Thus \(\eta(N, F)\) is a topological invariant depending only on \(N\) and \(F\), which is called the limiting \(\eta\)-invariant associated to \(F\). Further, suppose \(N\) is the boundary of some compact manifold \(M\) and \(F\) extends over \(M\), \(\tilde{F}\). Let \(Z = \cup Z_i\) be the singularity of polarization. Then

\[
\eta(N, F) = \lim_{\delta \to 0} \eta(N, g_\delta) = \sigma(M) + \sum_i \text{Res}(\alpha, Z_i)
\]

(1.2)

where \(\sigma(M)\) is the signature of \(M\) and \(\text{Res}(\alpha, Z_i)\) is the residue of the cochain \(\alpha\), which is determined by the data from \(F\), at the component \(Z_i\) of \(Z\). In particular, if all orbits of \(\tilde{F}\) are closed submanifolds of \(M\), then \(\eta(N, F)\) is a rational number.

Note that \(\eta(N, F)\) is computable for an invariant volume collapse as long as \(N\) is a boundary and \(F\) is extendable. In addition, it follows from
the concrete construction for $\alpha$ (see [Ya]) that $\eta(N, F)$ is a rational number if all orbits of $\tilde{F}$ are closed submanifolds.

Now consider $N$ as in Theorem 6. To get the rationality of $\eta(2)(N)$ one needs to somehow find a method for computing $\eta(2)(N)$ explicitly. We need to find a "good" volume collapse with BCG by means of $\eta(2)(N)$ computable. Motivated by Theorem 6, Theorem 5 and Theorem 7, we would like to show the following:

From a sufficiently volum-collapsed metric $g$ with BCG, one finds an injective $T$-structure $F$. Using $F$, one constructs an invariant volume collapse with BCG. Consequently, $\eta(2)(N) = \eta(N, F)$. By filling $N$ with a compact $4k$-manifold $M$ and extending $F$ to $M$ one gets the rationality of $\eta(2)(N)$.

Two questions arise here:

1) does a volume collapse with BCG imply the existence of injective $F$-structures $F$?

2) can one always fill in a $(4k - 1)$-manifold $N$ with a compact $4k$-manifold $M$ and extend $F$ over $M$?

In our special circumstances (3-dimensional case) the affirmative answer to 2) is not hard to achieve. However, 1) turns out to be considerably more difficult to establish.

The difficulties in 1) have two different sources. One is whether or not a sufficiently volume collapse metric leads to the existence of a polarized $F$-structure (see Introduction). In fact, according to Theorem 2, the affirmative answer would solve an open question on existence of the critical vanishing minimal volume (see [Gr2]). However, it is not hard to see that
a sufficiently collapsed 3-manifold always admits a polarized F-structure (Corollary 1). Another difficulty is illustrated by the following observation:

3) The construction of $\mathcal{F}_g$ from a sufficiently injective radii collapsed metric $g$ depends only on the local geometry of $g$ ([CG4]). Although $\mathcal{F}_g$ fully reflects the injectivity radii collapse, $\mathcal{F}_g$ does not take into account either the small volume or BCG. Consequently, one can not expect $\mathcal{F}_g$ to be injective (see Example 7 and the comment 3) in §1.5).

4) Except for few cases, $\mathcal{N}$ admits a unique injective F-structure (if it admits any at all). Note that $\mathcal{N}$ may admit infinitely many different polarized F-structures. Thus the existence of an injective F-structure is a global and topological constraint on $\mathcal{N}$ (see §1.4).

Intuitively, what we need to do is first find some local property for $\mathcal{F}_g$ which reflects the BCG. Then use this to get the existence of an injective F-structure (the global topological information).

Following the construction for $\mathcal{F}_g$ in [CG4], one sees that locally, $\mathcal{F}_g$ is injective. By taking into account BCG, we find that the $S^1$-orbits of $\mathcal{F}_g$ are not homotopically contractible globally. It turns out, in the 3-dimensional situation, that such a local property is enough to guarantee the existence of an injective F-structure (here we assume $\pi_1(\mathcal{N})$ is infinite). Our basic method is to modify $\mathcal{F}_g$ to be injective and our basic tools are some theorems from 3-dimensional topology.

Now by assuming Theorem B and Theorem 10, we give a simple proof of Theorem A as follows.
**Proof of Theorem A.** Let $N$ be a closed oriented 3-manifold which admits a volume collapse $\{g_\delta\}$ with BCG ($\delta \to 0$).

First, according to Theorem B, we can find an injective T-structure $\mathcal{F}$ on $N$. Using $\mathcal{F}$ we construct an invariant volume collapse $\{\tilde{g}_\delta\}$ with BCG on $N$ (Theorem 5). From Theorem 6 and Theorem 7, we obtain

$$\eta(\mathcal{O})(N) = \lim_{\delta \to 0} \eta(N, g_\delta) = \lim_{\delta \to 0} \eta(N, \tilde{g}_\delta) = \eta(N, \mathcal{F}).$$

From Theorem 10, we fill in $N$ with a compact 4-manifold $M_N$ and extend $\mathcal{F}$ to $M_N$ so that all orbits of the extension are closed submanifolds. Finally we get the rationality of $\eta(\mathcal{O})(N)$ by applying Theorem 7 to $\eta(N, \mathcal{F})$. Q.E.D.

### 1.3 Collapsed 3-Manifolds and Injective T-structures

Based on the general results in §1.1, we study systematically the collapsing phenomena and F-structures on 3-manifolds. Our main goal is to establish Theorem B.

**a. Polarized T-structures on 3-manifolds**

The basic fact of an F-structure on a 3-manifold is the following:

**Proposition 1** Any positive rank F-structure $\mathcal{F}$ on a 3-manifold contains a polarized substructure.

A consequence of Theorem 1 and Proposition 1 is the following.
Corollary 1 Let $(N, g)$ be a complete 3-manifold with $|K| \leq 1$. There exists a constant $\epsilon > 0$, so that if the injectivity radii is smaller than $\epsilon$ everywhere on $N$, then $N$ admits a polarized $F$-structure $\mathcal{F}_g$ which is almost compatible with the metric $g$.

Remark 5 Corollary 1 fails in higher dimensions. See Example 1.9 in [CG3].

Proof of Proposition 1. Let $\mathcal{F} = \{(\tilde{U}_\alpha, U_\alpha, T^{k_\alpha}, \phi_\alpha)\}$. If there is a stratum $(\tilde{U}_\alpha, U_\alpha, T^{k_\alpha}, \phi_\alpha)$ with $k_\alpha = 3$, then $U_\alpha = N$ and $N$ is a $T^3$-manifold up to finite covering. Since the $T^3$-action $\phi_\alpha$ has no fixed points, it follows that there is an $S^1$-subgroup of $T^3$ acting on $N$ (or a finite cover of $N$) with only finite isotropy groups; that is, $\mathcal{F}$ contains a polarized $S^1$-substructure.

Now we assume all $k_\alpha \leq 2$. Let $Z$ be a component of the singular set of $\mathcal{F}$ and let $(\tilde{U}_1, U_1, T^2, \phi_1, \psi_1), ..., (\tilde{U}_r, U_r, T^2, \phi_r, \psi_r)$ be all the strata of $\mathcal{F}$ which contain $Z$. By taking a common cover $U$ of $\tilde{U}_1, ..., \tilde{U}_r$ and lifting the $T^2$-action on $\tilde{U}_i$ over $U$, we may assume $\mathcal{F}$ is locally a $T$-structure (i.e., we assume $Z \subset \tilde{U}_i \subseteq U$). Since $\dim N = 3$ and $\dim Z \geq 1$, it is easy to see that $Z$ consists of a single $S^1$-orbit; i.e., $Z \approx S^1$ is isolated. Endowed with an invariant metric, one can find an invariant tubular neighborhood $T_\rho(Z)$ with radius $\rho$ satisfying the conditions: (i) $T_\rho(Z)$ contains no other components of the singular set of $\mathcal{F}$; (ii) $T_\rho(Z) \approx D^2 \times S^1$ (slice theorem), and (iii) $T_\rho(Z) \subset \bigcap_{i=1}^r \tilde{U}_i$. Let $S^1$ be a subgroup of $T^2$ without fixed points under $\phi_1$. By replacing the strata $(\tilde{U}_1, U_1, T^2, \phi_1), ..., (\tilde{U}_r, U_r, T^2, \phi_r)$ in $\mathcal{F}$ by $(T_\rho(Z), S^1, \phi_1|_{S^1}, \psi_1), (\tilde{U}_1', U_1, T^2, \phi_1, \psi_1), ..., (\tilde{U}_r', U_r, T^2, \phi_r, \psi_r)$ with $U_i' = U_i - T_{\rho/2}(Z)$, one gets a substructure of $\mathcal{F}$ which is polarized near $Z$. 
A T-structure is easier to handle than an F-structure. We will see that an F-structure on an orientable 3-manifold has the same orbit structure as some T-structure (Proposition 3). In this sense one can reduce an F-structure to a T-structure in dimension 3. In fact, let us observe the following.

**Proposition 2** Let $\pi : M' \to M$ be a finite covering and let $\mathcal{F}$ be an $F$-structures of $M$. Then $\mathcal{F}$ lifts to an $F$-structure $\mathcal{F}'$. Moreover, $\mathcal{F}$ has the following properties if and only if $\mathcal{F}'$ does:

(i) $\mathcal{F}$ is polarized;

(ii) $\mathcal{F}$ is pure;

(iii) $\mathcal{F}$ is injective.

**Proof.** Take any atlas $\{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ for $\mathcal{F}$. For each $i$, consider the local pull-back covering bundle:

$$
\begin{array}{cccc}
\pi^* & & & \\
\pi^*(\tilde{U}) & \rightarrow & \tilde{U}_i \\
\pi_i \downarrow & & \pi_i \downarrow \\
\pi & & \\
\pi^{-1}(U_i) & \rightarrow & U_i \\
\end{array}
$$

Since $\pi$ is a finite covering, the pull-back $\pi^*(\tilde{U})$ is a finite covering over $\tilde{U}$. Thus the $T^{k_i}$-action $\phi_i$ on $\tilde{U}$ lifts to a (unique) $T^{k_i}$-action $\phi'_i$ (as the
covering group of $T^k_i$) on $\pi^*(\tilde{U})$ with the commutative diagram:

$$
\begin{array}{ccc}
\phi_i' \\
T^k_i \times \pi^*(\tilde{U}_i) & \rightarrow & \pi^*(\tilde{U}_i) \\
1 \times \pi^* \downarrow & & \pi^* \downarrow \\
\phi_i & & \\
T^k_i \times (\tilde{U}_i) & \rightarrow & \tilde{U}_i
\end{array}
$$

Now we need to check that the collection $\{\pi^*(\tilde{U}_i), \pi^{-1}(U_i), T^k_i, \phi_i'\}$ satisfies Definition 1. This is straightforward from the above commutative diagrams and the definition of $\{(\tilde{U}_i, U_i, T^k_i, \phi_i, \psi_i)\}$. Q.E.D.

A consequence of Proposition 2 is that, as far as the properies (i), (ii) and (iii) are concerned in practice, one can always assume the manifolds are orientable (see Proposition 3).

Given a polarized F-structure $F$ on an orientable 3-manifold, let $N'$ and $N''$ be the union of 1-dimensional orbits and 2-dimensional orbits respectively. Let $N' = \bigcup N_i$ and $N'' = \bigcup X_j$ where $N_i$ and $X_j$ are connected components. Since $N$ is orientable a 2-dimensional orbit of $F$ is a torus and each $X_j$ is homeomorphic to $T^2 \times I$ with $T^2 \times t$ corresponding to the 2-orbit ($I$ is a closed interval). Thus we may view $N$ as decomposed into pieces $N_1, ..., N_r$, which satisfy the conditions:

(i) Each $N_i$ is a Seifert fiber space with tori boundary components such that the Seifert fiber structure is trivial near the boundary,

(ii) By identifying the corresponding boundary components of $N_i$’s in pairs, these $S^1$-Seifert fibrations produce $T^2$-orbits near the identified bound-
aries.

We call the above natural decomposition of the pair \((N, \mathcal{F})\) and denote it by \(D(N, \mathcal{F}) = \{N_1, \ldots, N_r\}\).

To find a T-structure whose natural decomposition coincides with the natural decomposition of \(\mathcal{F}\), it is enough to realize that, locally, a Seifert fiber structure coincides with the orbits of a local \(S^1\)-action. This is true because an \(S^1\)-fibered solid torus (and also a solid Klein bottle) admits an \(S^1\)-action leaving the fibration invariant (see [Sc]). As a summary of the above discussion, we give the following proposition.

**Proposition 3** The orbit structure of a polarized F-structure on an orientable 3-manifold \(N\) coincides with the orbit structure of some T-structure on \(N\).

Note that \(K^2 \times S^1\) (\(K^2\) a Klein bottle) supports an F-structure of rank three but does not admit a T-structure of rank three. Thus the orientability is a necessary condition in Proposition 3.

To characterize the 3-manifolds which admit polarized T-structures, we recall graph manifolds as follows.

**Definition 7** Let \(N\) be a compact 3-manifold. Suppose \(N\) decomposes into pieces \(N_1, \ldots, N_r\) satisfying the following conditions

(i) each \(N_i\) is a manifold with torus boundary components,

(ii) each \(N_i\) supports an \(S^1\)-bundle which is trivial near \(\partial N_i\).

Then \(N_G = \{N_1, \ldots, N_r\}\) is said to be a graph structure of \(N\) and \(N\) is called a graph manifold with the graph structure \(N_G\).
The topological classification for graph manifolds was obtained in [Wa1] and [Wa2]. We shall use it to discuss the uniqueness of injective $F$-structures on 3-manifolds in §1.4.

Note that the natural decomposition $D(N, \mathcal{F}) = \{N_1, ..., N_r\}$ of a pair $(N, \mathcal{F})$ may not be a graph structure since the Seifert fibration on $N_i$ may not be an $S^1$-bundle structure (e.g. suppose $N_i$ has exceptional orbits). But if one cuts out a tubular neighborhood around each exceptional orbit of $N_i$ (i.e., further decomposes $N_i$), then one can actually obtain a graph structure from $D(N, \mathcal{F})$.

On the other hand, for a given graph structure $N_G = \{N_1, ..., N_r\}$, by assigning an $S^1$-structure on each $N_i \in N_G$ (which is not necessarily the original $S^1$-bundle structure) one then constructs a polarized $T$-structure which is called a polarized $T$-structure associated to the graph structure $N_G$ (note that this associated polarized $T$-structure may not have rank two near $\partial N_i$'s).

As a summary of the above discussion one can characterize the 3-manifolds which admit polarized $F$-structures as follows.

**Proposition 4** Let $N$ be an orientable 3-manifold and let $\mathcal{F}$ be a polarized $T$-structure. Then $N$ is a graph manifold and $\mathcal{F}$ is associated to the graph structure of $N$.

The next proposition is about the sufficiently collapsed 3-manifolds with bounded diameter; it is needed only to prove Theorem E.

**Proposition 5** For each real number $D > 0$, there exists a constant, $c_3(D) >$
0, depending only on $D$, such that if a 3-manifold $N$ satisfies the conditions:

$$|\text{sectional curvature}| \leq 1, \text{diameter} \leq D, \text{and Volume} < c_3(D),$$

then $N$ admits a pure polarized $F$-structure. Moreover, $N$ is diffeomorphic to either a Seifert manifold or a solve manifold up to a double covering.

To prove Proposition 5, we need a lemma as follows.

**Lemma 2** Let $G$ be a compact Lie group acting on a manifold. If the principal orbits have codimension 2, then the orbit space is a manifold (with boundary).

**Proof.** See 4.1 Lemma in [Br]. Q.E.D.

**Proof of Proposition 5.** We first assume that $N$ admits a pure polarized $T^k$-structure $\mathcal{F}$. If $k = 1$, this amounts to saying that $N$ admits an $S^1$-foliation; i.e., $N$ is a Seifert fibre space ([Sc]). If $k = 2$, then the orbit space $N/\mathcal{F} \approx S^1$ since $\mathcal{F}$ is pure polarized. Thus the projection $\pi : N \to N/\mathcal{F}$ is actually a bundle map with fiber either $T^2$ or $K^2$ (Klein bottle), equivalently, $N$ (or its double covering if $N$ is not orientable) is solvable.

Now we show that $N$ does admit a pure polarization. We prove by contradiction. Supposing the opposite, we then obtain a sequence of closed 3-manifolds $\{N_i\}$ satisfying the conditions: 1) $|K(N_i)| \leq 1$, 2) $diam(N_i) \leq D$, 3) $Vol(N_i) \leq \frac{1}{i}$, and 4) $N_i$ does not admit any pure polarized $F$-structure. By Gromov's pre-compactness theorem one can chose a subsequence of $\{N_i\}$
which converges to a metric space \( Y \) under the Hausdorff distance. Note that 3) implies \( \dim(Y) < 3 \). We may assume that \( \lim_{i \to +\infty} d_H(N_i, Y) = 0 \). Applying Theorem 3, \( Y \approx N/O(3) \) with \( \dim(Y) \leq 2 \). From Lemma 2, \( Y \) is a manifold (with boundary). If \( \dim(Y) = 2 \), Theorem 4 shows that \( N_i \) admits an \( S^1 \)-fibration and hence a Seifert manifold ([Sc]) (otherwise \( Y \) is not a manifold) for \( i \) sufficiently large. This contradicts our assumption 4) above. If \( \dim(Y) = 1 \), then \( Y \approx S^1 \) or \([0,1]\). In the first case \( N_i \) is a fiber bundle with fiber a 2-infranilmanifold (Theorem 4); that is, either \( T^2 \) or \( K^2 \). So \( N_i \) admits a pure polarized \( T^2 \)-structure and this contradicts 4) again. If \( Y \approx [0,1] \), the \( N_i \) are actually \( T^2 \)-manifolds without fixed points. It is easy to see that one can pick up an \( S^1 \)-subgroup of \( T^2 \) which also has no fixed points; i.e., \( N_i \) admits an \( S^1 \) polarization. Again this contradicts 4). By now our proof is complete.

Q.E.D.

b. Characterization of injective \( F \)-structures on 3-manifolds

We first observe the simple case where a 3-manifold \( N \) admits a pure injective \( F \)-structure. In such a case, \( N \), or possibly its double covering, is either a Seifert manifold (for rank one) or a solve manifold (for rank two). Note that a solve manifold is the total space of a \( T^2 \)-bundle over \( S^1 \) and the \( T^2 \)-bundle structure is a pure injective \( T^2 \)-structure. Moreover we have

**Proposition 6** Let \( N \) be a Seifert manifold and \( F \) the pure \( S^1 \)-structure which coincides with the Seifert fiber structure. Then \( F \) is injective if and only if \( \pi_1(N) \) is infinite.

**Proof.** See [Or].

Q.E.D.
The next lemma is crucial for establishing the characterization of 3-manifolds which admit injective \( F \)-structures (Theorem 9).

Let \( \Sigma \) be a boundary component of a 3-manifold \( N \). \( \Sigma \) is called incompressible if \( \Sigma \) is not \( S^2 \) or \( P^2 \) and the natural map \( \pi_1(\Sigma) \to \pi_1(N) \) is injective.

**Lemma 3** The boundary of a Seifert fibre space \( N_s \) is incompressible unless \( N_s \) is homeomorphic to a solid torus or a solid Klein bottle.

**Proof.** See Corollary 3.3 in [Sc]. Q.E.D.

The criterion for a polarized \( T \)-structure to be injective is given below.

**Theorem 8** Let \( F \) be a polarized \( T \)-structure on a 3-manifold \( N \). Then \( F \) is injective if and only if the natural decomposition \( D(N,F) \) of \( F \) contains no piece \( N_i \) which is homeomorphic to a solid torus.

**Proof.** One direction in the proof is obvious. Suppose \( F \) is injective. Let \( D(N,F) = \{N_1,\ldots,N_r\} \) be the natural decomposition of \( F \). From Lemma 3, the injectivity of \( F \) implies that no \( N_i \) is a solid torus.

To prove the opposite direction we need some results from the combinatorial group theory.

Let \( G \) be a group, and let \( A \) and \( B \) be subgroups of \( G \) with \( \phi : A \to B \) an isomorphism. The HNN extension of \( G \) relative to \( A, B \) and \( \phi \) is the
group
\[ G^* = \langle G, t; t^{-1}at = \phi(a), a \in A \rangle. \] (1.3)

The group \( G \) is called the base of \( G^* \), \( t \) is called the stable letter, and \( A \) and \( B \) are called the associated subgroups. The relation between \( G \) and \( G^* \) is the following:

**Lemma 4** The group \( G \) is embedded in \( G^* \) by the natural inclusion map.

**Proof.** See Theorem 2.1 in [LV]. Q.E.D.

HNN extensions arise in the following topological context. Let \( X \) be a space, and \( U \) and \( V \) be subspaces. Suppose \( \pi_1(U) \to \pi_1(X) \) and \( \pi_1(V) \to \pi_1(X) \) are both injective. Let \( h : U \to V \) be a homeomorphism. Construct a new space \( Z \) by identifying \( U \) and \( V \) via \( h \). Then \( \pi_1(Z) \) is an HNN-extension of \( \pi_1(X) \) associated to \( \pi_1(U) \) and \( \pi_1(V) \). Note that here \( t \) represents a path in \( X \) from \( u \) to \( h(u) \) (\( u \in U \)).

**Corollary 2** Let \( i : X \to Z \) be the natural inclusion. Then the induced map \( i_* : \pi_1(X) \to \pi_1(Z) \) is injective.

A different situation from above is when \( V \) is a subspace of another space \( Y \). We still suppose \( \pi_1(V) \to \pi_1(Y) \) is injective. Let \( Z \) be the space formed by identifying \( U \) and \( V \) via \( h \). According to Van Kampen’s theorem, the fundamental group \( \pi_1(Z) \) is:

\[ \pi_1(Z) = \langle \pi_1(X) * \pi_1(Y) : \pi_1(U) = \pi_1(V), h_* > \] (1.4)
Correspondingly, let $B$ be a subgroup of another group $H$. The group $G^*$ is called the free product of $G$ and $H$ with amalgamation $\phi$, if

$$G^* = \langle G \ast H, \ A = B, \phi \rangle.$$  

(1.5)

**Lemma 5** $G$ and $H$ are embedded in $G^*$ by the natural inclusions.

**Proof.** See Theorem 2.6 in [LV].

Q.E.D.

**Corollary 3** Let $i_1 : X \to Z$ and $i_2 : Y \to Z$ be the natural inclusions. Then the induced maps $i_1_\ast : \pi_1(X) \to \pi_1(Z)$ and $i_2_\ast : \pi_1(Y) \to \pi_1(Z)$ are injective.

Now we continue our proof of Theorem 8.

Assume each $N_i$ in the natural decomposition $\{N_1, ..., N_r\}$ of $(N, F)$ is not homeomorphic to a solid torus. We shall show that $F$ is injective.

First we claim that the $S^1$-fiberation of $N_i$ is injective. This can be seen from Lemma 3. Note that assuming the $S^1$-fiber is not injective implies that the $\partial N_i$ must be compressible. Consequently, $N_i$ is a solid torus (Lemma 3).

When glueing $\{N_i\}$ together, these $S^1$-fibers generate $T^2$-orbits around $\partial N_i$'s. We need to show that these $T^2$-orbits are also injective. We start at $N_1$ and glue one component of $\partial N_1$ with its partner. There are two possibilities. One is that the partner is another component of $\partial N_1$ itself. In this case, since $\partial N_1$ is incompressible, we use the HNN extension (Corollary 2) to conclude that the produced $T^2$-orbits are injective. For the other case we use (Corollary 3) to get the same conclusion. Note that the boundary of
the glueing result still remains incompressible in both cases. By repeating
the process finitely many times we then complete the proof. Q.E.D.

c. Existence of injective $F$-structures on 3-manifolds

We begin to prove our main result, Theorem B.

First, we take the constant $\epsilon = c(3)$, where $c(3)$ is given in Theorem
1. By Theorem 1 one finds a positive rank $F$-structure $F_g$ which is almost
compatible with $g$. From Propositions 1, 2 and 3, we can assume $N$ is
orientable and $F_g$ is a polarized $T$-structure. As we have already explained
in §1.2, $F_g$ may not be injective in general (also see Example 9 and 3) in
§1.5). What we shall do is to modify $F_g$ so as to obtain an $F$-structure
which is injective.

Let $D(N,F_g) = \{N_1, ..., N_r\}$ be the natural decomposition of $(N,F_g)$.
If each $N_i$ is not homeomorphic to a solid torus then $F_g$ is already injective
(Theorem 8). Thus the appearance of solid torii in $D(N,F)$ violates the
injectivity of $F_g$. We shall get rid of these defect by changing the $S^1$-
structure on $N_i$ which is homeomorphic to a solid torus.

Lemma 6 Let $N_1$ be a Seifert fiber space with $r$ disjoint torus boundary
components such that the fiber structure is trivial near the boundary. Form
a manifold $N$ with boundary by attaching solid tori $D_1 \times S^1, ..., D_r \times S^1$ to $N_1$
along their boundaries ($r' < r$). If the $S^1$-fiber of $N_1$ is not homotopically
contractible in each $D_i \times S^1$ and $N$ is not a solid torus, then $N$ admits an
injective $S^1$-structure.

Proof. Since the $S^1$-fiber of $N_1$ is not homotopically contractible in
each $D_i \times S^1$, it follows that the $S^1$-fibration of $N_1$ extends over $N$. So $N$ is a Seifert fibre space with boundary ($r' < r$). We need to show that the $S^1$-fiber has infinite order in $\pi_1(N)$. Since $N$ is not a solid torus, $\partial N$ is incompressible (Lemma 3). Thus $\pi_1(\partial N) \to \pi_1(N)$ is injective. This implies that the $S^1$-fiber of $N$ is injective. Q.E.D.

Before proceeding further, we give an interesting example.

**Example 5 ([So], [CG1]).** This example shows that when considered as an graph manifold, a solid torus may have a rather complicated standard decomposition. First, let $\Sigma$ denote the surface formed by removing three disjoint disks from $S^2$. Take two copies of $D^2 \times S^1$ and attach them to $\Sigma \times S^1$ along their boundaries. We require that the attachment satisfies the conditions that the $S^1$-factor of $\Sigma \times S^1$ is not identified with $\partial D^2$, and $\partial N$ is compressible. Note that $N$ is a Seifert fiber space with boundary. Applying Lemma 3 one recognizes $N$ as a solid torus.

Now taking $r$ copies, $\Sigma_1 \times S^1, ..., \Sigma_r \times S^1$ of $\Sigma \times S^1$, and $r + 1$ solid tori, $D_0 \times S^1, ..., D_r \times S^1$, we play the following game. The first step is to attach $D_0 \times S^1$ and $D_1 \times S^1$ to $\Sigma_1 \times S^1$ as above and to denote the result (a solid torus) by $N_1$. The second step is to attach $N_1$ and $D_2 \times S^1$ to $\Sigma_2 \times S^1$. Continuing in this manner, it is obvious that one actually obtains the natural decomposition of $(D \times S^1, \mathcal{F})$ for some polarized $T$-structure $\mathcal{F}$.

Motivated by Example 5, we introduce the following.
Definition 8 Let $\mathcal{D}(N, F)$ be the natural decomposition. A maximal solid torus chain of $\mathcal{D}(N, F)$ is a subset $T_{\text{max}}$ in $\mathcal{D}(N, F)$ such that the total space of $T_{\text{max}}$ is homeomorphic to a solid tori and $T_{\text{max}}$ is a maximal subset with this property.

Note that $\mathcal{D}(N, F)$ may contain no solid torus chain at all.

From the proof of Theorem 8 one observes that each maximal solid torus contains at least one $D_{i_0} \times S^1$ for some $i_0$. To form a maximal solid torus chain, one starts with $D_{i_0} \times S^1$ and looks at its gluing partner. If the partner is also a solid torus, then the maximal solid torus chain consists of a single $D_{i_0} \times S^1$ (and $N$ is homeomorphic to a lens space). Otherwise one attaches $D_{i_0} \times S^1$ to the partner and denotes the result by $N_1$. Next one attaches all the solid torus, which are the partners of $N_1$, to $N_1$ and denotes the result by $N_2$. If $N_2$ is not a solid tori then we claim that the maximal solid torus chain is $D_{i_0} \times S^1$. This can be seen easily from Theorem 8. Otherwise we then repeat the above program by starting at the solid torus $N_2$. After a finite number of steps one obtains the maximal solid torus chain containing $D_{i_0} \times S^1$. Clearly, if two distinct maximal solid torus chains in $\mathcal{D}(N, F)$ have nonempty intersection, then $N$ is formed by gluing two solid tori along their boundaries. It is well-known that such a space is homeomorphic to a lens space (which includes $S^3$ and $S^2 \times S^1$).

We have actually proved the following lemma.

Lemma 7 Let $T_{\text{max}}$ and $T'_{\text{max}}$ be two distinct maximal solid torus chains of $\mathcal{D}(N, F)$. If $T_{\text{max}} \cap T'_{\text{max}} \neq \emptyset$, then $N$ is homeomorphic to a lens space.
Theorem 9 Let \( D(N, \mathcal{F}) \) be the natural decomposition in which all distinct maximal solid tori chains are disjoint. Suppose all \( S^1 \)-orbits of \( \mathcal{F} \) are not homotopically contractible in \( N \). Then \( N \) admits an injective \( T \)-structure.

**Proof.** Let \( \{T_{\text{max}, k}\} \) be the collection of all maximal solid torus chains of \( D(N, \mathcal{F}) \). We obtain a new decomposition \( D_1(N) \) for \( N \) by simply replacing \( \{T_{\text{max}, k}\} \) in \( D(N, \mathcal{F}) \) by their total spaces \( \{D_k \times S^1\} \). Attaching all \( D_k \times S^1 \) in \( D_1(N) \) to their partners, we get two possible results. One is that the result is \( N \) itself. In this case all \( \{D_k \times S^1\} \) attach to a single \( N_0 \).

It follows from the condition that the \( S^1 \)-fibration extends to an injective \( S^1 \)-structure of \( N \) (Lemma 6). Another possible result is that we obtain the decomposition \( D_2(N) \) of \( N \) satisfying the condition of Theorem 8. This can be seen as follows. First, each piece in \( D_2(N) \) is not a solid torus (otherwise it would contradict the maximal solid tori chains we defined). Secondly, our assumption that all \( S^1 \)-orbits of \( \mathcal{F} \) are not homotopically contractible implies that each piece in \( \mathcal{F}_2(N) \) admits an injective Seifert fiber structure (Lemma 6). We then complete the proof by applying Theorem 8. Q.E.D.

Now we can finish the proof of Theorem B.

From Theorem 3 we only need to check that the \( S^1 \)-orbit of \( \mathcal{F}_g \) is not homotopically contractible. Note that so far we have not used anything related to BCG. But now this property enters. Suppose there is an \( S^1 \)-orbit \( O_x \) of \( \mathcal{F}_g \) which is homotopically trivial. From the concrete construction for \( \mathcal{F}_g \) given in [CG4], one sees that \( O_x \) is not homotopically trivial in a metric ball of certain size at \( x \). Say, \( B_r(x) \supset O_x \) for some constant \( 0 < r < 1 \). Let
\[ \pi : \tilde{N} \to N \] be the universal covering \( \tilde{N} \) of \( N \) and \( \tilde{g} = \pi^*g \) the pull-back metric. Pick up a point \( \tilde{x} \) in \( \pi^{-1}(x) \) and denote by \( \tilde{O}_x \) the lifting of \( O_x \) at \( \tilde{x} \). Then \( \tilde{O}_{\tilde{x}} \subset B_r(\tilde{x}) \) since \( O_x \) is trivial in \( N \). Note that the injectivity radius of \( \tilde{M} \) at \( \tilde{x} \) is at least 1 (BCG). It follows that \( \tilde{O}_{\tilde{x}} \) is homotopically trivial in \( B_r(\tilde{x}) \). So is the projection \( O_x \) of \( \tilde{O}_{\tilde{x}} \) in \( B_r(x) \). We then get an obvious contradiction. Q.E.D.

We conclude this section by giving several examples. The first one will show that Theorem B is sharp.

**Example 6.** Let \( S^3 \) be the standard 3-sphere. Take a sequence of prime integers \( p_i \) with \( p_i \to +\infty \). For each \( p_i \), define a free \( \mathbb{Z}_{p_i} \)-action on \( S^3 \) as follows: first, parametrize \( S^3 \subset C^1 \times C^1 \) and let \( \varphi_{p_i} \) be the generator of \( \mathbb{Z}_{p_i} \), then the action is given by \( \varphi_{p_i}(\rho_1 e^{2\pi i \theta_1}, \rho_2 e^{2\pi i \theta_2}) = (\rho_1 e^{2\pi i \theta_1/p_i}, \rho_2 e^{2\pi i \theta_2/p_i}) \). It is well-known that the quotient \( S^3/Z_{p_i} \) is the lens space \( L(p_i, 1) \). Since \( Z_{p_i} \) acts as isometries, the standard metric of \( S^3 \) projects to a metric on \( L(p_i, 1) \) of sectional curvature = 1. Note that \( \text{Vol}(L(p_i, 1)) = \text{Vol}(S^3)/p_i \to 0 \) as \( p_i \to +\infty \).

The next example shows that in general, one cannot expect to obtain injective F-structures by means of geometrical constructions as given in both [CG4] and [Fut3].

**Example 7.** Take the standard \( S^2 \) and \( S^1 \) and make \( S^2 \times S^1 \). Consider
the $S^1$ subgroup $H$ of $SO(3)$:

$$H = \{ \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid 0 \leq t < 2\pi \}. $$

$T^2 = H \times S^1$ acts on $S^2 \times S^1$ by $H$ on the first factor and multiplication on the second factor. Let $R^1_\theta$ be a $R^1$-subgroup of $T^2$, determined by an irrational angle $\theta$. We obtain a volume collapse $\{g_\delta\}$ (with bounded diameter) by simply shrinking the standard product metric on $S^2 \times S^1$ along the orbits of $R^1_\theta$. Note that the universal covering space of $S^2 \times S^1$ is $S^2 \times R^1$.

It follows that the pull-back metrics have a uniform lower bound on injectivity radii. Since $R^1_\theta$ is dense in $T^2$, the limiting space of this collapse is a closed interval. This implies that for any sufficiently collapsed metric $g_\delta$, the F-structure $\mathcal{F}_{g_\delta}$, constructed from $g_\delta$ by using either the local short geodesics technique ([CG4]) or the frame-bundle technique ([F2]), has to contain 2-dimensional orbits. In particular, $\mathcal{F}_{g_\delta}$ is not injective.

### 1.4 Uniqueness of Injective F-structures on 3-manifolds

Using the classification results for Seifert manifolds and for graph manifolds obtained in [Se] and in [Wa1] and [Wa2] respectively, we discuss the uniqueness of injective T-structures and prove Theorem C.
The question on the uniqueness of injective $F$-structures on a closed orientable 3-manifold $N$ is suggested by the following observation:

Suppose $N$ admits two injective $F$-structures $\mathcal{F}_1$ and $\mathcal{F}_2$. From our previous work one gets

$$\eta(N, \mathcal{F}_1) = \eta_{(2)}(N) = \eta(N, \mathcal{F}_2).$$  \hspace{1cm} (1.6)

The explicit residue formula for limiting $\eta$-invariant associated to a polarized $T$-structure shows that $\eta(N, \mathcal{F})$ is computed using the topological data derived from $(N, \mathcal{F})$.

To be precise, let $\mathcal{D}(N, \mathcal{F}) = \{N_1, \ldots, N_r\}$ be the natural decomposition. An $S^1$-orbit of $\mathcal{F}$ is said to be an exceptional orbit if it is an exceptional orbit for some $N_i$ (as a Seifert fiber space). The rational portion of $3\eta(N, \mathcal{F})$ is determined by the exceptional orbits of $\mathcal{F}$ and the integer portion of $3\eta(N, \mathcal{F})$ is contributed by the twisting around the boundaries of the $N_i$’s (see §1.7 for further details).

Thus (1.6) suggests that in some sense $\mathcal{F}_1$ may not be very “different” from $\mathcal{F}_2$. It turns out that except for few cases, if a closed orientable 3-manifold admits an injective $F$-structure $\mathcal{F}$ then it is unique up to weak isomorphism (see Definition 9). Before giving the precise definition for weak isomorphism, let us first describe all the exceptional cases.

**Example 8** Let $N$ be a 3-nilmanifold. The action of the center of $N$ on $N$ gives a (unique) pure $S^1$-structure $\mathcal{F}_1$ which is injective. Note that $N$ may also be viewed as a torus bundle over $S^1$ and thus $N$ supports a pure $T^2$-structure $\mathcal{F}_2$ which is also injective. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ have different
ranks \( \mathcal{F}_1 \) is not isomorphic to \( \mathcal{F}_2 \).

**Example 9** Let \( N \) be a solve manifold. \( N \) is the total space of a \( T^2 \)-bundle over \( S^1 \), \( \pi : T^2 \to N \to S^1 \). Take a finite open cover, \( U_1, \ldots, U_r (r \geq 2) \) of the base \( S^1 \) and denote \( \tilde{U}_i = \pi^{-1}(U_i) \). By choosing pairwise different \( S^1 \)-actions \( \phi_i \) on \( \tilde{U}_i \), we then obtain a mixed injective T-structure \( \mathcal{F} = \{(\tilde{U}_i, S^1, \phi_i)\}_{i=1}^r \). It is obvious that in such a way one can construct infinitely many non-isomorphic classes of injective T-structures on \( N \).

Note that all the injective T-structures in above examples do not have exceptional orbits. The next example is not of the case.

**Example 10** Any \( S^1 \)-action without a fixed point on \( S^2 \times S^1 \) is injective. Thus \( S^2 \times S^1 \) supports infinitely many non-isomorphic pure injective \( S^1 \)-structures. Note that the Seifert invariants corresponding to the \( S^1 \)-action is \( \{b, (0, 0), (\alpha, \beta), (\alpha, \alpha - \beta)\} \).

Two exceptional orbits of a T-structure \( \mathcal{F} \) on an oriented manifold \( N \) are said to be conjugate if their corresponding Seifert invariants \( (\alpha_1, \beta_1) \) and \( (\alpha_2, \beta_2) \) satisfying \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \alpha_2 - \beta_2 \). Our explicit residue formula in §1.6 will show that a pair of conjugate exceptional orbits in \( \mathcal{F} \) contributes only integers to \( 3\eta(N, \mathcal{F}) \).

**Definition 9** Let \( N \) be a 3-manifold and let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two injective T-structures. Let \( \mathcal{F}(N, \mathcal{F}_i) = \{N_{i,1}, \ldots, N_{i,r_i}\} \) be the natural decompositions of \( \mathcal{F}_i \) \( (i = 1, 2) \). \( N_i \) is called a trivial piece if \( N_i \) is homeomorphic to \( T^2 \times I \). \( \mathcal{F}_1 \) is said to be weakly isomorphic to \( \mathcal{F}_2 \) if there is an automorphism \( \phi \) of \( N \).
such that for each non-trivial piece $N_{1,j}$, there is an $N_{2,k(j)}$ such that $\phi|_{N_{1,j}} : N_{1,j} \to N_{2,k(j)}$ is an embedding which prevents the Seifert fiber structure.

**Remark 6** Note that 2) actually says that if $\mathcal{F}_1$ is weakly isomorphic to $\mathcal{F}_2$, then they are different only on the trivial pieces. Thus weakly isomorphic injective T-structures have the same exceptional orbits. The explicit residue formula for $\eta(N, \mathcal{F})$ in §1.7 will show that $\eta(N, \mathcal{F})$ depends only on the weak isomorphism class of $\mathcal{F}$.

Note that the injective T-structures in Example 9 are in the same weak isomorphism class since their natural decompositions contain only trivial pieces. We shall see later that Examples 8 and 10 are the only cases where a 3-manifold $N$ supports more than one non weak isomorphism classes of injective T-structures. First, let us check this for the simplest case where the injective T-structure is pure.

**Proposition 7** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two pure injective $T^k$-structure on a manifold $N$ ($k = 1, 2$). Suppose $N$ is not diffeomorphic to $S^2 \times S^1$. Then $\mathcal{F}_1$ is isomorphic to $\mathcal{F}_2$ up to isomorphism.

**Proof.** First we check the rank one case; that is, the case when $N$ is an injective Seifert manifold. It is well-known that a large Seifert manifold supports a unique Seifert fibration up to isomorphism (see Theorem 6 of §5, [Or]). Since $N$ is not homeomorphic to $S^2 \times S^1$ and $\pi_1(N)$ is not finite, one easily checks that $N$ is actually a large Seifert manifold. Consequently, $N$ admits a unique injective $S^1$-structure.
In the case of rank two, $N$ is a solve manifold. By the classification result for solve manifolds ([RV]) one concludes with the uniqueness of injective $T^2$-structures. Q.E.D.

Now we are ready to prove Theorem C.

**Proof of Theorem C.** Suppose $N$ is homeomorphic neither to $S^2 \times S^1$ nor a nilmanifold. Let $\mathcal{F}_i$ be two injective $T$-structures on $N$ and let $D(N, \mathcal{F}_i) = \{N_{i,1}, ..., N_{i,r_i}\}$ be the natural decomposition of $\mathcal{F}_i$, ($i = 1, 2$). If $r_1 = 1$, then either $N = N_1$, or $N$ is homeomorphic to $T^2 \times I$, or $N$ is not homeomorphic to $T^2 \times I$. The first two cases are actually reduced to Proposition 7. The last possibility will be treated in the case of $r_i \geq 2$.

Now assume either $r_1 = 1$ and $N_1$ is not homeomorphic to $T^2 \times I$, or $r_i \geq 2$ (*).

First we put an additional assumption that both $D(N, \mathcal{F}_1)$ and $D(N, \mathcal{F}_2)$ contain no trivial piece. Under (*) and the above additional assumption, it is easy to check that by cutting out a tubular neighborhood around each exceptional orbit of $N_i$ one can actually obtain a so-called simple graph structure in the sense of [Wa1] and [Wa2] (also see §8 of [Or]). Consequently, there is an automorphism $\phi$ of $N$ which is isotopic to the identity map such that $\phi(N_{1,j}) = N_{2,j}$ and $r_1 = r_2$ (see Theorem 5 of §8, [Or]).

Note that from Proposition 7 each $N_{i,j}$ admits a unique Seifert fiber structure up to isomorphism. Thus $\mathcal{F}_1$ is actually isomorphic to $\mathcal{F}_2$.

Now we allow $D(N, \mathcal{F}_i)$ to contain trivial pieces. What we shall do is glue the trivial pieces in $D(N, \mathcal{F}_i)$ to produce a simple decomposition as in
the previous case.

We first glue the trivial pieces in each $D(N, F_i)$ which are supposed to glue to each other. Clearly the results are disjoint trivial pieces. Then we attach each of these disjoint trivial pieces to a non-trivial piece in $D(N, F_i)$ (there are two choices here). We denote the result by $\{\tilde{N}_{i,1}, \ldots, \tilde{N}_{i,\tilde{r}_i}\} (i = 1, 2)$. Since none of $\tilde{N}_{i,j}$ are trivial, from our previous discussion we conclude that there is an automorphism $\phi$ of $N$ such that $\phi(\tilde{N}_{1,i}) = \tilde{N}_{2,i}$ and $\tilde{r}_1 = \tilde{r}_2$. This implies that $F_1$ is weakly isomorphic to $F_2$. Q.E.D.

1.5 Further Discussion

Besides the uniqueness of injective T-structures, there is another question arising from our proof of Theorem A. To state it clearly, let us recall the following:

Let $\{g_\delta\}$ be a volume collapse with BCG on a compact orientable 3-manifold. Then $\eta(2)(N) = \lim_{\delta \to 0} \eta(N, g_{\delta})$. For a fixed sufficiently collapsed metric $g_\delta$, one constructs a polarized F-structure $F_\delta$ which is almost compatible with $g_\delta$ (Corollary 1). Then,

$$\eta(2)(N) \approx \eta(N, g_\delta) \approx \eta(N, F_\delta). \quad (1.7)$$

Note that the first approximation in (1.7) follows from the estimate below which was established in [CG2]:

$$|\eta(2)(N) - \eta(N, g_\delta)| \leq C \cdot Vol(N, g_\delta)$$

The second approximation follows from the fact that $g_\delta$ is sufficiently collapsed and almost compatible with $F_\delta$. Here “almost compatible” means
that there exists an invariant metric closed to \( g_6 \) (see [CG4]).

On the other hand, by performing the modification process on \( F_6 \) as in the proof of Theorem 9, one obtains an injective \( F \)-structure \( F \). From \( \eta(N, F) = \eta(g_6)(N) \) and (1.7) we deduce that

\[
\eta(N, F_6) \approx \eta(N, F).
\]

(1.8)

Note that (1.8) may suggest that \( F_6 \) is weakly isomorphic to \( F \) when \( g_6 \) is sufficiently collapsed. (Here, we assume \( N \) is neither homeomorphic to \( S^2 \times S^1 \) nor a nilmanifold and thus the weak isomorphism class of \( F \) is independent of \( \delta \) (Theorem C).) The following example supports this.

**Example 11** Let \( N \) be an injective Seifert manifold. To simplify our discussion below we assume \( N \) has a single exceptional orbit \( O \). As a model, we may form \( N \) as follows:

Let \( \Sigma \) be a torus with a disc removed and let \( N_1 = \Sigma \times S^1 \). Glueing in a twisted manner a solid torus \( D \times S^1 \) to \( N_1 \) along their boundaries, one obtains a closed injective Seifert manifold \( N \) with a single exceptional orbit which is the central orbit of \( D \times S^1 \).

Now we start with an invariant metric \( g_1 \) and simply shrink \( g_1 \) along the Seifert fiber while keeping \( g_1 \) fixed on the orthogonal direction to the orbits. Thus we obtain an invariant volume collapse \( \{g_6\} \) with BCG (Theorem 5).

Then we fix a sufficiently collapsed metric \( g_6 \) and construct \( F_6 \) as in [CG4]. Note that by the invariance of \( g_6 \) the exceptional orbit \( O \) is the shortest closed geodesic in a neighborhood \( U \) of \( O \). From the concrete construction of \( F_6 \) as in [CG4] one sees that \( F_6 \) is actually isomorphic to \( F \).
on $U$ and on $N_1$. On $N - (U \cup N_1)$ which is homeomorphic to $T^2 \times I$, $\mathcal{F}_\delta$ is actually a mixed $T$-structure just as in Example 9. Thus we see that in this situation $\mathcal{F}_\delta$ is actually weakly isomorphic to $\mathcal{F}$.

We would like to make the following comments.

1) Note that the above discussion for Example 11 is valid for any 3-manifold $N$ as long as the volume collapse with BCG is compatible with an injective $T$-structure on $N$. Although a general volume collapse $\{g_\delta\}$ with BCG may not be constructed as in Example 11, it can be thought of as a perturbation of an invariant volume collapse with respect to an injective $T$-structure. Thus the associated polarized $T$-structures $\mathcal{F}_\delta$ is "almost" weakly isomorphic to $\mathcal{F}$ in the sense that $\mathcal{F}_\delta$ is weakly isomorphic to $\mathcal{F}$ as $\delta \to 0$. Consequently,

$$\lim_{\delta \to 0} \eta(N, \mathcal{F}_\delta) = \eta(N, \mathcal{F}) \quad (1.9)$$

2) Based on the observation in 1) and our previous work in §1.3, §1.4 and §1.6, we can give a topological proof of Theorem 6 for 3-manifolds that is totally different from [CG2].

We first assume $N$ is not homeomorphic to either $S^2 \times S^1$ or a nilmanifold. We start with a volume collapse $\{g_\delta\}$ with BCG on a closed oriented 3-manifold $N$. For each sufficiently small $\delta$, we obtain a polarized $T$-structure $\mathcal{F}_\delta$ (Corollary 1). Since each metric $g_\delta$ has BCG, we can modify each $\mathcal{F}_\delta$ to an injective $T$-structure $\mathcal{F}$ as in the proof of Theorem B (note that by our assumption on $N$, $\mathcal{F}$ is unique up to weak isomorphism
(Theorem C)). From (1.7) and (1.9) we obtain
\[ \lim_{\delta \to 0} \eta(N, g_\delta) = \lim_{\delta \to 0} \eta(N, \mathcal{F}_\delta) = \eta(N, \mathcal{F}) \]  
(1.10)
This implies that the limiting \( \eta \)-invariant with respect to a volume collapse with BCG is a topological invariant.

As for when \( N \) is either \( S^2 \times S^1 \) or a nilmanifold, although \( N \) admits non-weak isomorphic classes of injective \( T \)-structures, a simple computations show that \( \eta(N, \mathcal{F}) \) is independent of \( \mathcal{F} \) as along as \( \mathcal{F} \) is injective. Now the proof is complete.

3) Note that in Example 11, one actually gets a maximal solid torus chain of \((N, \mathcal{F}_\delta)\) on \( N - (U \cup N_i) \). In fact, this maximal solid torus chain is a trivial one, i.e., except for the initial \( U \), all other pieces in this chain are homeomorphic to \( T^2 \times I \). It is conceivable that more complicated maximal solid torus chains may be formed in the polarized \( T \)-structure constructed from geometry as in \([CG4]\). Also, since \( \mathcal{F}_\delta \) contains a maximal solid torus chain for all sufficiently small \( \delta \), one sees again that an injective \( T \)-structure cannot be produced from the geometrical construction given in \([CG4]\).

1.6 Filling 3-manifolds via Equivariant Plumbings

In part c of this section we will prove the following fact
Theorem 10 Let $N$ be a closed orientable 3-manifold and let $\mathcal{F}$ be a polarized $T$-structure of $N$. Then there is a 4-manifold $M_N$ and a $T$-structure $\tilde{\mathcal{F}}$ satisfying:

1) $\partial M_N \approx N$,
2) $\tilde{\mathcal{F}}|_N \approx \mathcal{F}$,
3) all orbits of $\tilde{\mathcal{F}}$ are closed submanifolds of $M_N$.

Note that any closed orientable 3-manifold can be viewed as a boundary since the cobordism ring of such manifolds is trivial. To insure that we can extend $\mathcal{F}$ to $M_N$, we will construct $M_N$ by the so-called equivariant plumbing technique.

a. Equivariant Plumbings

Let us recall the equivariant plumbing technique from [Or]. The principal $S^1$-bundles over a closed orientable 2-manifold $Y$ are classified by $H^2(Y, \mathbb{Z}) = \mathbb{Z}$. Denote the associated $D^2$-bundles indexed by $b \in \mathbb{Z}$ as $\xi = (M_b, \pi, Y), \pi : M_b \to Y$. The compact 4-manifold $M_b$ has the homotopy type of $Y$. Let the zero section $\nu : M_b \to Y$ represent the positive generator $\alpha \in H_2(M_b, \mathbb{Z})$. Then the self intersection number $\alpha \cdot \alpha = b$ is the Euler number.

Given two such bundles $\xi_i = (M_{b_i}, \pi_i, Y_i)(i = 1, 2)$, we plumb them together as follows. Choose 2-disks $B_i \subset Y_i$ and the bundles over them $\pi^{-1}(B_i)$. Since they are trivial bundles there are natural identifications $\mu_i : D^2 \times D^2 \to \xi_i$. Let $s : D^2 \times D^2 \to D^2 \times D^2, s(x, y) = (y, x)$ be the reflection and define $f : \pi^{-1}(B_1) \to \pi^{-1}(B_2)$ by $f = \mu_2 \circ s \circ \mu_1^{-1}$. Pasting $\xi_1$
and $\xi_2$ together along $\pi^{-1}(B_1)$ and $\pi^{-1}(B_1)$ by the map $f$ is called plumbing. It yields a topological 4-manifold with corners that may be smoothed. The resulting smooth structure is independent of the choices involved.

A graph $\Gamma$ is a finite, 1-dimensional, connected simplicial complex. Let $A_0, \ldots, A_n$ denote its vertices. A star is a contractible graph where at most one vertex, say $A_0$, is connected with more than two other vertices. If there is such a vertex, call it the center. A weighted graph is a graph where a non-negative integer $g_i$ (the genus) and an integer $b_i$ (the weight) is associated with each vertex $A_i$.

Given a weighted graph $\Gamma$ we define a compact 4-manifold $M_\Gamma$ as follows: For each vertex $(A_i, g_i, b_i)$ take a $D^2$-bundle $\xi_i = (Y_{b_i}, \pi_i, M_i)$ where $M_i$ is a closed, orientable 2-manifold of genus $g_i$. If an edge connects $A_i$ and $A_j$ in $\Gamma$ then perform plumbing on $\xi_i$ and $\xi_j$. If $A_i$ is connected with more than one other vertex, choose pairwise disjoint disks on $M_i$ to perform the plumbing. Finally smooth the resulting manifold to obtain $M_\Gamma$.

We shall now define an $S^1$-action on the building blocks. For $g > 0$ let $S^1$ act trivially on the base and as a rotation on each fiber. For $g = 0$ we define $S^1$-actions on $\xi = (M_b, \pi, S^2)$ in general. A plumbing is equivariant if the trivializing maps $\mu_i$ and identifying map $s$ are equivariant.

b. Seifert manifolds as the results of plumbings

In his classical paper, Seifert [Se] classified the class of closed 3-manifolds (Seifert manifolds) satisfying the conditions:

(S1) the manifold decomposes into a collection of simple closed curves called fibers so that each point lies on a unique fiber,
(S2) each fiber has a tubular neighborhood $U$ consisting of fibers so that $U$ is a fibred solid torus. A fibred solid torus is finitely covered by a trivial fibred solid torus.

His main result states that a Seifert manifold $N$ is determined up to a fiber-preserving homeomorphism by the following Seifert invariants:

$$N = \{ b; (\epsilon, g); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r) \}$$  \hspace{1cm} (1.11)

where $g$ is the genus of the base space $Y$ of $N$, the symbols $\epsilon = o, o_1, n_1$ and $n_2$ represent the situation when both $N$ and $Y$ are orientable, $N$ is orientable and $Y$ is not, $N$ is not orientable and $Y$ is, neither $N$ nor $Y$ are orientable respectively. $(\alpha_i, \beta_i)$ is the Seifert invariant of the exceptional orbit which is determined as follows. An $S^1$-orbit $O$ is called exceptional, if its isotropy group is nontrivial, say $Z_p$. In a tubular neighborhood $D \times S^1$ of $O$ as in (S2), the $S^1$-fibration is equivalent to the orbits of the local $S^1$-action (after normalization):

$$\theta(r, \theta_1, \theta_2) = (r, \theta_1 + q\theta, \theta_2 + p\theta)$$

Then the oriented Seifert invariant of $O$ is determined by

$$\alpha = p, \quad \beta q \equiv 1 \mod \alpha \quad 0 < \beta < \alpha$$

If $N$ is orientable then the fiber preserving homeomorphisms preserves orientation. Note that a change of orientation of $N$ gives the Seifert invariants

$$-M = \{ -b - r; (\epsilon, g); (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_r, \alpha_r - \beta_r) \}$$  \hspace{1cm} (1.12)
Lemma 8 Consider the star $S$ below with each $b_{i,j} \geq 2$ and $g_{i,j} = 0$ except for the center:

$$
(-b_{1,1}) \cdots (-b_{1,2}) \cdots \cdots (-b_{1,s_1})
\downarrow
(-b_{2,1}) \cdots (-b_{2,2}) \cdots \cdots (-b_{2,s_2})
\vdots
(-b_{r,1}) \cdots (-b_{r,2}) \cdots \cdots (-b_{r,s_r})
$$

(1.13)

The result, $N_S = \partial M_S$, of the equivariant boundary plumbing has Seifert invariants

$$
N_S = \{ b_i(o,g); (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s) \}
$$

(1.14)

where

$$
\frac{\alpha_j}{\alpha_j - \beta_j} = [b_{j,1}, \ldots, b_{j,s_j}], \quad j = 1, \ldots, r.
$$

are the continued fractions. Conversely, every Seifert manifold as in (1.14) is the result of an equivariant plumbing according to a star $S$ as in (1.13).

Proof. See Lemma 3 and Corollary 5, §2 of [Or].

Q.E.D.

Remark 7 Lemma 8 is valid for the Seifert manifolds whose base spaces are non-orientable. Let $\xi = (M_{-b-r}, \pi, Y)$ be the $D^2$-bundle corresponding to the center $[-b-r,g]$ in (1.13) and let $\bar{\xi} = (M_{-b-r}, \bar{\pi}, \bar{Y})$ have the same meaning as $\xi = (M_{-b-r}, \pi, Y)$ except that $\bar{Y}$ is non-orientable. It is not hard to see that if in the plumbing process of Lemma 1.13, we replace $M_{-b-r}$ by $\bar{M}_{-b-r}$ and leave all others unchanged, then the plumbing result is the Seifert manifold whose Seifert invariant is $N = \{ b_i(n_2,g); (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r) \}$ (i.e., the base space of $N$ is non-orientable).
Remark 8 Let $N = \{b; (o, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$. Then $-N = \{-b - r; (o, g); (\alpha_1, \alpha_1 - \beta_1), ..., (\alpha_r, \alpha_r - \beta_r)\}$. If we denote by $M_N$ and $M_{-N}$ the fillings for $N$ and $-N$ respectively as in Lemma 8, then $M_N \neq M_{-N}$ since $|\sigma(M_N)| \neq |\sigma(M_{-N})|$ (see Theorem D). This implies that $M_N$ depends not only on the topology of $N$ but also on the orientation of $N$.

**c. Proof of Theorem 10**

Let $(N, \mathcal{F})$ be as in Theorem 10. From now on, we will assume the natural decomposition of $(N, \mathcal{F})$ has the following form

$$\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_k; \phi_1, ..., \phi_l\}$$

where

1) Each $N_e$ is an orientable Seifert fiber space (with boundary) such that the Seifert fiber structure is trivial near $\partial N_e$ and $\mathcal{F}$ is generated by these Seifert fibration of $N_e$'s ($1 \leq e \leq k$).

and

2) the total number of boundary components of $\{N_e\}_{e=1}^k$ is $2l$. For each normalized pair $((\partial N_e)_f, (\partial N_e')_f)$ ($(\partial N_e)_f = S^1_e \times S^1_f$) which are glued via $\phi_f$, we may assume the glueing map $\phi_f$ ($1 \leq f \leq l$) is given by the matrix

$$\phi_f = \begin{pmatrix} u_f & v_f \\ p_f & q_f \end{pmatrix} : S^1_e \times S^1_f \to S^1_{e'} \times S^1_f; \quad \text{det}(\phi_f) = -1 \quad (1.15)$$

where the second factors are fibers. We will form $M_N$ in four steps.

Step 1. By 1) one can close up each component of $\partial N_e$ by attaching a solid torus $D_e \times S^1_f$ to $(\partial N_e)_f$ in the obvious way. We denote by $\tilde{N}_e$ the
closed Seifert manifold produced from $N_e$. According to the classification result in [Se] we may assume $\tilde{N}_e$ is given by

$$
\tilde{N}_e = \{b_e; (e, g); (\alpha_{e,1}, \beta_{e,1}), ..., (\alpha_{e,se}, \beta_{e,se})\}
$$

where $\epsilon_e = 0$ if the base of $\tilde{N}_e$ is orientable. Otherwise $\epsilon_e = n_2$.

Step 2. Fill in each $\tilde{N}_e$ with a 4-manifold $M_{\tilde{N}_e}$ as in Lemma 8 or Remark 7 depending on $\epsilon_e = 0$, or $n_2$.

Step 3. Form $M_N$ by making plumbings among the $M_{\tilde{N}_e}$'s according to the $\phi_f$ ($1 \leq f \leq l$). For each $\phi_f$ we glue $M_{\tilde{N}_e}$ with $M_{\tilde{N}_e'}$ by making equivariant plumbings from $\partial D_e \times \partial D_f = D_{e} \times S^1_f$ to $\partial D_{e'} \times \partial D_f = D_{e'} \times S^1_f$ successively according to the linear graph

$$
(-c_{f,1}) \quad (-c_{f,2}) \quad \cdots \quad (-c_{f,s_f})
$$

which is determined by the continuous fraction $p_f/q_f = [c_{f,1}, ..., c_{f,s_f}]$.

Note that the boundary of the total space of the sequence of plumbings from $D_e \times D_f$ to $D_{e'} \times D_f$ is the lens space $L(p_f, q_f)$.

Clearly, $\partial M_N$ has a decomposition $(N_1, ..., N_k, \phi'_1, ..., \phi'_l)$ where

$$
\phi'_f = \begin{pmatrix}
u_f' \\ u_f' \\ p_f \\ q_f \\
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\ 0 & 1 \\ c_{f,s_f} & 1 \\ 1 & 0 
\end{pmatrix} \cdots \begin{pmatrix}
0 & 1 \\ 1 & 0 \\ c_{f,1} & 1 
\end{pmatrix} \begin{pmatrix}
-1 & 0 \\ 0 & 1 \\ c_{f,1} & 1 
\end{pmatrix}
$$

To see $\partial M_N \approx N$, it is enough to show the following:

**Lemma 9** $\phi_f$ is isotopic to $\phi'_f$ ($1 \leq f \leq l$).

**Proof.** Since

$$
det \begin{pmatrix}
u_f & u_f' \\ p_f & q_f \\
\end{pmatrix} = det \begin{pmatrix}
u_f & u_f' \\ p_f & q_f \\
\end{pmatrix} = -1
$$
i.e.,
\[ u_f q_f - p_f v_f = -1 \tag{1.16} \]
\[ u'_f q_f - p_f v'_f = -1, \tag{1.17} \]

by subtracting (1.17) from (1.16), one gets
\[ (u_f - u'_f)q_f = (v_f - v'_f)p_f. \tag{1.18} \]

Since \( p_f \) and \( q_f \) are coprime integers and \( p_f((u_f - u'_f)q_f) \), then \( p_f((u_f - u'_f) \).

Put \( u_f - u'_f = p_f m_f \). It follows that \( v_f - v'_f = m_f q_f \). Therefore we define an isotopy \( \phi_{f,t} \) by
\[
\phi_{f,t} = \begin{pmatrix}
  u'_f + tm_f p_f & v'_f + tm_f q_f \\
  p_f & q_f
\end{pmatrix} \quad 0 \leq t \leq 1.
\]

Q.E.D.

Step 4. Extending \( \mathcal{F} \) over \( M_N \). First, we observe the following:

(i) \( M_N = (\bigcup_{e=1}^k M_{\tilde{N}_e}) \cup (\bigcup_{f=1}^j M_{L(p_f, q_f)}) \),

(ii) \( M_{\tilde{N}_e} \cap M_{\tilde{N}_{e'}} = \emptyset, (e \neq e') \)

(iii) \( M_{\tilde{N}_e} \cap M_{L(p_f, q_f)} = \emptyset \) or \( D_e \times D_f \).

Let \( \mathcal{F}_e \) be the restriction of \( \mathcal{F} \) to \( N_e \) and let \( \mathcal{F}'_f \) be the restriction of \( \mathcal{F} \) to \( L(p_f, q_f) \) (note that \( \mathcal{F}'_f \) is actually the T-structure given in c of §2.1 with specifying \( a_1 = a_2 = 0 \)).

We then make the extension of \( \mathcal{F} \) over \( M_N \) by simply extending \( \mathcal{F}_e \) over \( M_{\tilde{N}_e} \) and \( \mathcal{F}'_f \) over \( M_{L(p_f, q_f)} \). Note that the above extension are not unique in any sense. Clearly, all orbits of the extended T-structure \( \tilde{\mathcal{F}} \) are closed submanifolds of \( M_N \) (note that \( \tilde{\mathcal{F}} \) may not have positive rank).

Now our proof of Theorem 10 is complete. Q.E.D.
Chapter 2

Computation of Limiting \( \eta \)-invariants in Dimension 3

2.1 Residue Formula of Limiting \( \eta \)-invariants in 3-dimension

This section is devoted to explicit topological residue formulas for \( \eta^{(2)}(N) \) where \( N \) is an oriented closed 3-manifold which admits a volume collapse with BCG. We shall express \( \eta^{(2)}(N) \) in terms of the data from an injective T-structure \( \mathcal{F} \) of \( N \); more precisely, the data associated to the natural decomposition of the pair \( (N, \mathcal{F}) \). Since in most cases such decompositions
are unique up to weak isomorphism (Theorem C), the corresponding residue formulas are intrinsic (e.g. Theorem D).

We should point out that the methods used in this section are valid for computing $\eta(N, \mathcal{F})$ where $\mathcal{F}$ is a polarized T-structure (possibly without BCG).

As we have already explained in §1.2, we shall carry out the formula (1.2) of Theorem 7. We organize our computation in the following order:

a. residue formula for $N = \{b; (o, g); (\alpha, \beta)\}$,

b. residue formula for Seifert manifolds,

c. residue formula for $\eta(L(p, q), \mathcal{F})$,

d. residue formula for graph manifolds.

a. Residue formula for $N = \{b; (o, g); (\alpha, \beta)\}$

Before giving the residue formula for general injective Seifert manifolds, let us first consider the simpler case of Seifert manifolds which have only a single exceptional orbit.

**Proposition 8** Given an oriented injective Seifert manifold $N = \{b; (o, g), (\alpha, \beta)\}$, then

$$\eta_{(2)}(N) = -s + \epsilon(b) + \frac{1}{3} \sum_{i=1}^{s} b_i + \frac{\alpha_{s-1}}{3\alpha}$$  \hspace{1cm} (2.1)

where $\frac{\alpha}{\alpha-\beta} = [b_1, ..., b_s]$, $\alpha_i = b_i\alpha_{i-1} - \alpha_{i-2}$ with $\alpha_0 = 1$ and $\alpha_1 = b_1$, and $\epsilon(b) = -1$ or $1$ if $b > -1$ or $b \leq -1$ respectively.

**Remark 9** Note that $M_N$ is not homeomorphic to $M_{-N}$ (see Remark 8). Thus $\eta_{(2)}(-N) = -\eta_{(2)}(N)$ cannot be seen from (2.1).
Proof. By applying Lemma 8, we obtain $M_N$. The injective $S^1$-action on $N$ extends to $M_N$. Note that $M_N$ is formed by plumbing the bundles $\xi = (M_{-b-1}, \pi, Y)$, $\xi_i = (M_{-b_i}, \pi_i, S^2)$ successively. Decomposing each base

$$S_i^2 = B_{i,1} \times D_{i,1} \cup_{\phi_i} B_{i,2} \times D_{i,2},$$

where

$$\phi_i = \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}$$

we can express $M_N$ as

$$M_N = M_{-b-1} \perp B_{1,1} \times D_{1,1} \bigcup_{\phi_1} B_{1,2} \times D_{1,2}

s \downarrow

B_{2,1} \times D_{2,1} \bigcup_{\phi_2} B_{2,2} \times D_{2,2}

s \downarrow

...$$

$$B_{s,1} \times D_{s,1} \bigcup_{\phi_s} B_{s,2} \times D_{s,2}$$

and

$$\begin{pmatrix} \alpha_{s-1} & \beta_{s-1} \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ b_s & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}.$$

Let $X$ be the velocity vector field of the $S^1$-action of $M_N$ and let $X_i$ be the restriction of $X$ on $B_{i,1} \times D_{i,1}$ ($1 \leq i \leq s$). Choosing multipolar coordinates on $B_{i,1} \times D_{i,1}$ we write

$$X_i = \alpha_i \frac{\partial}{\partial \theta_{i,1}} - \alpha_{i+1} \frac{\partial}{\partial \theta_{i,2}}, \quad \alpha_0 = 1, \alpha_1 = b_1.$$ 

We claim that $\alpha_i > \alpha_{i-1} > 0$. We prove this by induction. Assume $\alpha_k > \alpha_{k-1}$ for $k = 1, \ldots, i-1$. Since $b_i \geq 2$ and $\frac{\alpha_i-1}{\alpha_i} < 1$, then $\alpha_i = b_i \alpha_{i-1} - \alpha_{i-2} =$
\(a_i(b_i - \frac{a_i-1}{a_i}) > a_i\). From \(a_i > 0\) it follows that the fixed point set of \(X\) consists of \(Z_0 \cup \{0_{i,1} \times 0_{i,1}, 0_{2,i} \times 0_{2,1}, ..., 0_{s,i} \times 0_{s,1}, 0_{s,2} \times 0_{s,2}\}\), where \(Z_0 \approx Y\) and each \(0_{i,j} \times 0_{i,j}\) is the center of \(B_{i,j} \times D_{i,j}\). According to formula (1.2) of Theorem 7,

\[
\eta(N) = \sigma(M_N) + Res(X, Z_0) + \sum_{i=1}^s Res(X, 0_{i,1} \times 0_{i,2}) + Res(X, 0_{s,2} \times 0_{s,2}) \tag{2.2}
\]

We shall compute each term in (2.2) in the following lemmas.

**Lemma 10** \(\sigma(M_N) = -s + \epsilon(b)\).

**Proof.** From §2 of [Or] one sees that \(\sigma(M_N)\) is equal to the signature of the following matrix:

\[
A = \begin{pmatrix}
-b - 1 & 1 \\
1 & -b_1 & 1 \\
1 & & \ddots & 1 \\
1 & & & -b_s
\end{pmatrix},
\]

and \(A\) is negative definite if and only if \(-b - 1 < 0\). So we only need to show that if \(-b - 1 \geq 0\) then \(\text{sig}(A) = -s + 1\). First assume \(-b - 1 > 0\); that is, \(-b - 1 \geq 1\). Given two \(n \times n\) matrices \(A_1\) and \(A_2\). If there is a invertible matrix \(C\) such that \(A_1 = CA_2CT\), then we say that \(A_1\) is congruent to \(A_2\).
and denote this by "$A_1 \sim A_2$". It is easy to see that

$$A \sim \begin{pmatrix} -b - 1 & 0 \\ 0 & -b_1 + \frac{1}{b_1 - 1} & 1 \\ & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}.$$  

Since $b_1 \geq 2$ and $-b - 1 \geq 1$, then $-b_1 + \frac{1}{b_1 - 1} < 0$. Consequently, the matrix

$$A_1 = \begin{pmatrix} -b_1 + \frac{1}{b_1 - 1} & 1 \\ 1 & -b_2 & 1 \\ & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}$$

is negative definite; i.e., $\text{sig}(A) = -s + 1$.

Now we consider $b = -1$, i.e., $-b - 1 = 0$. In this case

$$A \sim \begin{pmatrix} -b_1 & 0 \\ 0 & \frac{1}{b_1} & 0 \\ & 0 & -b_2 + \frac{1}{b_1} & 1 \\ & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}.$$  

Note that $-b_2 + \frac{1}{b_1} < 0$ since $b_1 \geq 2$. By the same reason as above we conclude $\text{sig}(A) = -s + 1$. Q.E.D.
The next lemma is concerned with the local residue in the following situations:

1) a pure rank one structure at an isolated singular point,

2) a pair of pure structures at the isolated singular $S^1$-orbit.

Take $D^2_x \times D^2_y \subset R^2 \times R^2$ ($D^2$ is a 2-disc with radius 2) and let $(\gamma_1, \theta_1, \gamma_2, \theta_2)$ be the multipolar coordinate of $R^4$. Put

$$X = a \frac{\partial}{\partial \gamma_1} + b \frac{\partial}{\partial \gamma_2} \quad (ab \neq 0)$$

$$X_i = a_i \frac{\partial}{\partial \gamma_1} + b_i \frac{\partial}{\partial \gamma_2} \quad (i = 1, 2).$$

Suppose $a_1 b_1 \neq 0$ and $b_2 \neq 0$. Put $U_1 = D^2_x \times D^2_1$ and $U_2 = D^2 \times D^2 - U_1$. The local models of 1) and 2) are $(U_1, X)$ and $\{(U_1, X_1), (U_2, X_2)\}$ respectively. Their singularities are $0 \times 0$ and $(0 \times \partial D^2_1) \cup (\partial D^2_1 \times 0)$ respectively.

Lemma 11 Suppose $(U, X)$ and $\{(U_1, X_1), (U_2, X_2)\}$ are given as above. Under the standard orientation of $R^4$,

$$Res(X, 0 \times 0) = \frac{a}{b} + \frac{b}{a},$$

$$Res(\{X_1, X_2\}, 0 \times \partial D^2_1) = -\frac{a_1}{b_1} + \frac{a_2}{b_2}, \quad (2.3)$$

$$Res(\{X_1, X_2\}, \partial D^2_1 \times 0) = \frac{a_1}{b_1} - \frac{a_2}{b_2}.$$ 

**Proof.** See [Ya]. Q.E.D

Now we can finish the proof of Proposition 8. From Lemma 11 we get

$$Res(X, 0_{i,1} \times 0_{i,2}) = \frac{a_{i,1}}{a_i} + \frac{a_{i,2}}{a_{i+1}} = b_i + \frac{a_{i+1}}{a_i} - \frac{a_{i-1}}{a_i}. \quad (2.4)$$

Substituting Lemma 10, (2.4) and $Res(X, Z_0) = \frac{b_{i+1}}{3}$ into (2.2), a simplification of (2.2) gives (2.1). Q.E.D
b. Residue formula for Seifert manifolds

First, let us restate Theorem D as follows.

**Theorem D.** Let \( N = \{b; (o, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\} \) be an injective Seifert manifold, then

\[
\eta_{(2)}(N) = -\sum_{i=1}^{r} s_i + \epsilon(b, r) + \frac{1}{3} (b + r) + \frac{1}{3} \sum_{i=1}^{r} \sum_{j=1}^{s_i} b_{ij} + \frac{1}{3} \sum_{i=1}^{r} \frac{\alpha_i s_{i-1}}{\alpha_i} \tag{2.5}
\]

where \( \frac{\alpha_i}{\alpha_i - \beta_i} = [b_{i,1}, ..., b_{i,s_i}] \), \( \alpha_{i,j} = b_{i,j} \alpha_{i,j-1} - \alpha_{i,j-2}; \alpha_{i,0} = 1, \alpha_{i,1} = b_{i,1} \) and

\[
\epsilon(b, r) = \begin{cases} 
-1 & \text{if } b + r > 0 \\
+1 & \text{if } b + r \leq 0.
\end{cases}
\]

Note that an injective Seifert manifold admits a volume collapse with BCG (Theorem 5). Thus (2.5) makes sense. Also by Theorem 6, for any volume collapse \( \{g_s\} \) with BCG, the limit \( \lim_{s \to 0} \eta(N, g_s) \) is given by (2.5).

**Corollary 4** Given \( N = \{b; (o, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\} \) and \( N' = \{b'; (o, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\} \). Then

\[
3\eta_{(2)}(N) = 3\eta_{(2)}(N') \mod Z. \tag{2.6}
\]

**Remark 10** Corollary 4 is valid for \( N = \{b; (n_2, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\} \).

By the same argument as in the proof of Theorem D one can get

\[
\eta_{(2)}(N) = \sigma(M_N) + \frac{1}{3} (b + r) + \frac{1}{3} \sum_{i=1}^{r} \sum_{j=1}^{s_i} b_{ij} + \frac{1}{3} \sum_{i=1}^{r} \frac{\alpha_i s_{i-1}}{\alpha_i} \tag{2.7}
\]

where \( \frac{\alpha_i}{\alpha_i - \beta_i} = [b_{i,1}, ..., b_{i,s_i}] \), \( \alpha_{i,j} = b_{i,j} \alpha_{i,j-1} - \alpha_{i,j-2}; \alpha_{i,0} = 1, \alpha_{i,1} = b_{i,1} \).
Remark 11 Due to the same reason as was given in Remark 8, the anti-invariance of $\eta(2)(N)$ under the change of orientation is not obvious from (2.5). However, we can show $\eta(2)(N) = -\eta(2)(-N)$ as follows by using the anti-invariance in Proposition 8.

Given $N = \{b; (o, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$, then $-N = \{b; (o, g); (\alpha_1, \alpha_1 - \beta_1), ..., (\alpha_r, \alpha_r - \beta_r)\}$. Assume $b + r > 0$. Putting $N_1 = \{b; (o, g); (\alpha_1, \beta_1)\}$ and $N_i = \{0; (o, g); (\alpha_i, \beta_i)\}$ we can write (compare Proposition 8 with Theorem D)

$$\eta(2)(N) = \sum_{i=0}^{r} \eta(2)(N_i) + (r - 1)$$

and

$$\eta(2)(-N) = \sum_{i=0}^{r} \eta(2)(-N_i) - (r - 1).$$

Therefore $\eta(2)(-N) = -\eta(2)(N)$ since $\eta(2)(-N_i) = -\eta(2)(N_i)$ for all $i$.

Proof of Theorem D. From Lemma 8 we have a 4-manifold $M_N$ such that $\partial M_N \cong N$. Note that the $S^1$-action on $N$ which gives the Seifert fiber structure extends uniquely over $M_N$. Applying formula (1.2) of Theorem 7, and using essentially same argument as in the proof of Proposition 8, we deduce the following:

$$\eta(2)(N) = \sigma(M_N) + \frac{1}{3} \sum_{i=1}^{r} \sum_{j=1}^{s_i} b_{i,j} + \frac{1}{3} (b + r) + \sum_{i=1}^{r} \frac{\alpha_i \alpha_i - 1}{3 \alpha_i}.$$
lowing symmetric matrix $\phi$

\[
\begin{pmatrix}
-b - r & 1 & & & & & \\
- b_{1,1} & 1 & & & & & \\
1 & - b_{1,2} & & & & & \\
 & & \ddots & & & & \\
 & & & 1 & & & \\
 & & & & \ddots & & \\
 & & & & & 1 & - b_{1,s_1} \\
\end{pmatrix}
\begin{pmatrix}
- b_{2,1} & 1 & & & & & \\
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & - b_{1,s_2} & & & \\
 & & & \ddots & & & \\
 & & & & 1 & - b_{1,s_r} \\
\end{pmatrix}
\]

where each unfilled entry equals zero. Since $b_{i,j} \geq 2$ for all $i, j$ this matrix is easily seen to be negative definite if and only if $-b - r < 0$. What we shall do is to show that if $-b - r \geq 0$ then $\sigma(M_N) = - \sum_{i=1}^{r} s_i + 1$. We first prove an algebraic lemma.

**Lemma 12** Suppose we are given

\[
A = \begin{pmatrix}
A_1 - \alpha_0 E_{11}^1 & -\alpha_0 E_{12}^1 & \cdots & -\alpha_0 E_{1r}^1 \\
-\alpha_0 E_{21}^1 & A_2 - \alpha_0 E_{22}^1 & \cdots & -\alpha_0 E_{2r}^1 \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_0 E_{r1}^1 & -\alpha_0 E_{r2}^1 & \cdots & A_r - \alpha_0 E_{rr}^1
\end{pmatrix}
\]
where

\[
A_i = \begin{pmatrix}
-b_{i,1} & 1 & & & \\
1 & -b_{i,2} & 1 & & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -b_{i,n_i}
\end{pmatrix}_{m_i \times m_i},
\]

and \(E^k_{ij} = (a_{ij})\) is the \(m_i \times m_j\) matrix with \(a_{k1} = 1\) and all other entries equal to zero. Suppose \(x_0 > 0\) and \(b_{ij} \geq 2\). Then \(A\) is negative definite.

**Proof.** We proceed by induction on \(r\). The case of \(r = 1\) is just Lemma 10. Assume Lemma 12 holds for \(r - 1\). In what follows we shall perform a congruence transformation on \(A\) to make \(A^1\) diagonal and \(E^1_{1i}\) and \(E^1_{i1}\) vanish simultaneously \((1 \leq i \leq r)\). In the first step we see the following

\[
A \sim A^1 = \begin{pmatrix}
A^1_1 & -a_1 E^2_{12} & \cdots & -a_1 E^2_{1r} \\
-a_1 E^2_{21} & A_2 - (a_0 - a_1) E^3_{22} & \cdots & -(a_0 - a_1) E^3_{2r} \\
& \ddots & \ddots & \ddots \\
& -(a_1 E^2_{r1} & -(a_0 - a_1) E^3_{r2} & A_r - (a_0 - a_1) E^r_{rr}
\end{pmatrix}
\]

where

\[
A^1_1 = \begin{pmatrix}
-(b_{1,1} + a_0) & & & \\
& -(b_{1,2} - \tilde{a}_1) & 1 & & \\
& & 1 & -b_{1,3} & \\
& & & \ddots & 1 \\
& & & & 1 - b_{1,n_1}
\end{pmatrix}_{m_1 \times m_1}.
\]
\[ \tilde{a}_1 = \frac{1}{b_{1,1} + \alpha_0}, \; \tilde{a}_1 = \frac{\alpha_0}{b_{1,1} + \alpha_0} \text{ and } \alpha_1 = \frac{\alpha_0^2}{b_{1,1} + \alpha_0}. \text{ In the second step we get} \]
\[ A \sim A^1 \sim A^2 = \begin{pmatrix}
A_1^2 & -a_2 E_{12}^3 & \cdots & -a_2 E_{1r}^3 \\
-a_2 E_{21}^3 & A_2 - (\alpha_0 - \alpha_1 - \alpha_2) E_{22}^1 & \cdots & (\alpha_0 - \alpha_1 - \alpha_2) E_{2r}^1 \\
\vdots & \vdots & \ddots & \vdots \\
-a_2 E_{r1}^3 & (\alpha_0 - \alpha_1 - \alpha_2) E_{r2}^1 & \cdots & A_r - (\alpha_0 - \alpha_1 - \alpha_2) E_{rr}^1
\end{pmatrix}
\]
where
\[ A_1^2 = \begin{pmatrix}
-(b_{1,1} + \alpha_0) & \cdots \\
(b_{1,2} - \tilde{a}_1) & \ddots \\
-(b_{1,3} - \tilde{a}_2) & \cdots & 1 \\
1 & \cdots & 1 \\
1 & -b_{i,s_1} & \cdots \end{pmatrix}_{m_1 \times m_1}
\]
\[ \tilde{a}_2 = \frac{1}{b_{1,2} - \tilde{a}_1}, \; a_2 = \frac{\alpha_0^2}{b_{1,2} - \tilde{a}_1} \text{ and } \alpha_2 = \frac{\alpha_0^2}{b_{1,2} - \alpha_2}. \text{ After the } s_1 \text{-steps we find} \]
\[ A \sim A^{s_1} = \begin{pmatrix}
A_1^{s_1} & 0 & \cdots & 0 \\
0 & A_2 - (\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{22}^1 & \cdots & (\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{2r}^1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & (\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{r2}^1 & \cdots & A_r - (\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{rr}^1
\end{pmatrix}
\]
with
\[ A_1^{s_1} = \begin{pmatrix}
-(b_{1,1} + \alpha_0) & \cdots \\
(b_{1,2} - \tilde{a}_1) & \ddots \\
-(b_{1,3} - \tilde{a}_2) & \cdots & -(b_{i,s_1} - \tilde{a}_{s_1-1}) \\
\end{pmatrix}_{m_1 \times m_1} \]
where
\[
\alpha_1 = \frac{\alpha^2}{b_{1,1} + \alpha_0}, \quad \alpha_{i+1} = \frac{a_i}{b_{1,i+1} - \tilde{a}_i},
\]
\[
a_1 = \frac{\alpha_0}{b_{1,1} + \alpha_0}, \quad a_{i+1} = \frac{a_i}{b_{1,i+1} - \tilde{a}_i} \quad (i = 1, \ldots, s_1 - 1).
\]
\[
\tilde{a}_1 = \frac{1}{b_{1,1} + \alpha_0}, \quad \tilde{a}_{i+1} = \frac{1}{b_{1,i+1} - \tilde{a}_i}.
\]
To apply the inductive assumption on \(A_s^1\) we only need to verify
\[
\alpha_0 - \sum_{i=1}^{s_1} \alpha_i > 0. \tag{2.8}
\]
We first estimate \(\alpha_i\) by starting at \(\alpha_2\);
\[
\alpha_2 = \frac{a_1^2}{b_{1,2} - \tilde{a}_1} = \frac{(\frac{\alpha_0}{b_{1,1} + \alpha_0})^2}{b_{1,2} - \frac{1}{b_{1,1} + \alpha_0}} = \frac{\alpha_1}{b_{1,2}(b_{1,1} + \alpha_0) - 1}.
\]
Put
\[
c(\alpha_0) = \frac{1}{b_{1,2}(b_{1,1} + \alpha_0) - 1}
\]
and we have \(\alpha_2 = \alpha_1 c(\alpha_0)\). We claim that \(\alpha_k < \frac{\alpha_1 c(\alpha_0)}{2^{k-2}}\) for \(k = 2, \ldots, s_1\). We prove this by induction. Assume \(\alpha_k < \frac{\alpha_1 c(\alpha_0)}{2^{k-2}}\). Then
\[
\alpha_{k+1} = \frac{a_k^2}{b_{1,k+1} - \tilde{a}_k} = \frac{(\frac{\alpha_{k-1}^2}{b_{1,k} - \tilde{a}_{k-1}})^2}{b_{1,k+1} - \frac{1}{b_{1,k} - \tilde{a}_{k-1}}} = \frac{\alpha_{k-1}}{b_{1,k+1}(b_{1,k} - \tilde{a}_{k-1}) - 1} \leq \frac{\alpha_k}{b_{1,k+1}(b_{1,k} - \tilde{a}_{k-1}) - 1} \leq \frac{\alpha_k}{b_{1,k+1}(b_{1,k} - \tilde{a}_{k-1}) - 1} \leq \frac{\alpha_1 c(\alpha_0)}{2^{k-1}},
\]
since \(b_{i,j} \geq 2\) and \(\tilde{a}_i < 1/2\) and hence \(b_{1,k+1}(b_{1,k} - \tilde{a}_{k-1}) - 1 \geq 2\). Then we estimate (2.7) as follows:
\[
\alpha_0 - \sum_{i=1}^{s_1} \alpha_i > \alpha_0 - \alpha_1 (1 + c(\alpha_0) \sum_{i=2}^{s_1} \frac{1}{2^{i-2}})
\]
\[
> \alpha_0 - \alpha_1 (1 + 2c(\alpha_0))
\]
\[
= \alpha_0 - \frac{\alpha_0^2}{b_{1,1} + \alpha_0} \cdot (1 + \frac{\alpha_0}{b_{1,1}(b_{1,2} + \alpha_0) - 1})
\]
\[
= \alpha_0 (1 - \frac{\alpha_0}{b_{1,1} + \alpha_0} \cdot \frac{b_{1,1}(b_{1,2} + \alpha_0) + 1}{b_{1,1}(b_{1,2} + \alpha_0) - 1}) \geq 0
\]
if
\[
\frac{c_0}{b_{1,1} + c_0} \cdot \frac{b_{1,1}(b_{1,2} + c_0) + 1}{b_{1,1}(b_{1,2} + c_0) - 1} \leq 1
\]
\[\iff \alpha_0(b_{1,1}b_{1,2} + b_{1,1}\alpha_0 + 1) \leq (b_{1,1} + \alpha_0)(b_{1,1}b_{1,2} + b_{1,1}\alpha_0 + 1) \quad (2.9)\]
\[\iff 2\alpha_0 \leq b_{1,1}^2 \alpha_0 + b_{1,1}^2 b_{1,2} - b_{1,1}.
\]
The last inequality in (2.8) is obvious since \(b_{i,j} \geq 2\). Since \(\alpha_0 - \sum_{i=1}^{n_1}\alpha_i > 0\), our proof is complete by our inductive assumption. Q.E.D.

Now we return to the proof of Theorem D.

First assume \(b + r < 0\), i.e., \(b + r \leq -1\). It is easy to see that
\[
\phi \sim \begin{pmatrix}
-b - r & 0 & \cdots & 0 \\
0 & A_1 + \frac{1}{b + r} E_{11}^1 & \cdots & \frac{1}{b + r} E_{1r}^1 \\
0 & \frac{1}{b + r} E_{21}^1 & \cdots & \frac{1}{b + r} E_{2r}^1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{1}{b + r} E_{r1}^1 & \cdots & A_r + \frac{1}{b + r} E_{rr}^1
\end{pmatrix}
\]
Since \(b + r \leq -1\), by applying Lemma 12 we obtain \(\text{sig}(\phi) = -\sum_{i=1}^{r} s_i - 1\).

Now consider \(b + r = 0\). By performing the same diagonalization procedure to \(\phi\) as done in the proof of Lemma 12 we find
\[
\phi \sim \begin{pmatrix}
A_1^{s_1} & 0 & \cdots & 0 \\
0 & A_2 - \alpha_0 E_{22}^1 & \cdots & -\alpha_0 E_{2r}^1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & -\alpha_0 E_{r2}^1 & \cdots & A_r - \alpha_0 E_{rr}^1
\end{pmatrix}
\]
where

$$A_{1}^{{\alpha _{1} \alpha _{2} \ldots \alpha _{s_{1}}}} = \begin{pmatrix}
-b_{1,1} & \frac{1}{b_{1,1}} & \cdots & \frac{1}{b_{1,s_{1}}} \\
\alpha_{2} & -(b_{1,2} - \tilde{a}_{2}) & \cdots & \frac{1}{b_{1,s_{1}} - \tilde{a}_{s_{1} - 1}} \\
\vdots & \ddots & \ddots & \vdots \\
\alpha_{s_{1}} & \frac{1}{b_{1,s_{1}} - \tilde{a}_{s_{1} - 1}} & \cdots & -(b_{1,1} - \tilde{a}_{s_{1}} - 1)
\end{pmatrix}_{m_{1} \times m_{1}}$$

$$\alpha_{0} = b_{1,1} - \sum_{i=2}^{s_{1}} \alpha_{i}; \ \tilde{a}_{2} = a_{2} = \alpha_{2} = \frac{1}{b_{1,1}}; \ \tilde{a}_{k} = \frac{1}{b_{1,k} - \tilde{a}_{k-1}}; \ \alpha_{k} = \frac{a_{k-1}}{b_{1,k} - \tilde{a}_{k-1}} \quad (k = 3, \ldots, s_{1}).$$

By using essentially the same method as used in the proof of Lemma 12, one gets \( \alpha_{k} < \frac{1}{2^{k-2}}. \) Thus

$$\alpha_{0} = b_{1,1} - \sum_{i=2}^{s_{1}} \alpha_{i} = b_{1,1} - \alpha_{2}(1 + \sum_{i=3}^{s_{1}} \frac{1}{2^{k-2}}) > b_{1,1} - \frac{2}{b_{1,1}} \geq 1. \quad (b_{1,1} \geq 2)$$

Consequently \( \text{sig}(\phi) = -\sum_{i=1}^{r} s_{i} + 1 \) (Lemma 12). The proof of Theorem 2 is now complete. \( \text{Q.E.D.} \)

c. Residue formula for \( \eta(L(p,q),\mathcal{F}) \)

To prepare for deriving the topological residue formula for limiting \( \eta \)-invariants in general, we need to compute the limiting \( \eta \)-invariant associated with a non-pure polarized \( T \)-structure on a lens space.

Taking two copies of a solid torus \( D \times S^{1} \) and parameterizing them in multipolar coordinates \( (r_{i}, \theta_{i,1}, \theta_{i,2}) \) \((i = 1, 2)\), one forms a lens space \( L(p,q) \) by gluing \( D_{1} \times S_{1}^{1} \) with \( D_{2} \times S_{2}^{1} \) along their boundaries by the matrix

$$\begin{pmatrix}
u
u
p
q
\end{pmatrix} = \begin{pmatrix}
-1 & 0
0 & 1
b_{r} & 1
1 & 0
\end{pmatrix} \cdots \begin{pmatrix}
0 & 1
0 & 1
1 & 0
b_{1} & 1
\end{pmatrix}.$$
where \( p = [b_1, ..., b_s] \). Let

\[
X_i = a_i \frac{\partial}{\partial \theta_{i,1}} + \tilde{a}_i \frac{\partial}{\partial \theta_{i,1}} \quad (i = 1, 2)
\]

be defined in a neighborhood \( U_1 \) of \( D_i \times S^1 \). Assume \( \tilde{a}_1 \tilde{a}_2 \neq 0 \). Then \( \mathcal{F}(X_1, X_2) \) determines a non-pure polarized \( T \)-structure of \( L(p, q) \).

We shall compute the precise value for the limiting \( \eta \)-invariant associated to \( \mathcal{F}(X_1, X_2) \) in the fashion of [Ya] (we refer to [Ya] for more details).

First we fill in \( L(p, q) \) with a 4-manifold \( M_N \) by using equivariant plumbing. Suppose we are given a linear graph \( \Gamma[b_1, ..., b_s] \):

\[
(-b_1) \quad (-b_2) \quad \cdots \quad (-b_s)
\]

The following lemma is elementary.

**Lemma 13** The boundary of the equivariant linear plumbing \( M_\Gamma \) according to the graph \( \Gamma[b_1, ..., b_s] \) above, is the lens space \( L(p, q) \) where

\[
\frac{p}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots - \frac{1}{b_s}}} = [b_1, ..., b_s], \quad b_i \geq 2. \quad (2.10)
\]

**Proof.** See Lemma 1 of §2, [Or].

Q.E.D.
To see $M_{L(p,q)}$ clearly, we express it as follows.

$$M_{L(p,q)} = B_{1,1} \times D_{1,1} \bigcup_{\phi_1} B_{1,2} \times D_{1,2}$$

$$s \downarrow$$

$$B_{2,1} \times D_{2,1} \bigcup_{\phi_2} B_{2,2} \times D_{2,2}$$

$$s \downarrow$$

$$\cdots$$

$$B_{r,1} \times D_{r,1} \bigcup_{\phi_r} B_{r,2} \times D_{r,2}$$

and

$$\phi_i = \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}.$$  

We make the following observation:

1) $\sigma(M_{L(p,q)}) = -s - 1$. (Lemma 10).

2) An $S^1$-action on $B_{1,1} \times \partial D_{1,1}(B_{r,2} \times \partial D_{r,2}) \subset L(p,q)$ extends to a unique $S^1$-action over $M_{L(p,q)}$.

3) Consider the natural inclusion map $(i,j) : B_{i,j} \times D_{i,j} \hookrightarrow R^2 \times R^2 = R^4$ ($1 \leq i, j \leq r$). Clearly, either $(i,j)$ does preserve orientation for all $(i,j)$ or does not for all $(i,j)$. The orientation convention of $M_{L(p,q)}$ is taken so that $(i,1)$ is a orientation preserving under the standard orientation on $R^4$.

From 2) one may view $X_1$ ($X_2$) as defined on a neighborhood of $B_{1,1} \times \partial D_{1,1}$ ($B_{r,2} \times \partial D_{r,2}$). We may choose $\tilde{F} = \{(U_0, X_0), (U_1, X_1), (U_2, X_2)\}$
where $U_0$ is the interior of $M_{t(r,q)}$ and $X_0$ is generated by

$$\frac{\partial}{\partial \theta_{1,1}} + \frac{\partial}{\partial \theta_{1,1}}$$

on $B_{1,1} \times D_{1,1}$.

Let $X_0^{(i)}$ denote the restriction of $X_0$ to $B_{i,1} \times D_{i,1}$ and put

$$X_0^{(i)} = \alpha_i \frac{\partial}{\partial \theta_{i,1}} - \alpha_{i+1} \frac{\partial}{\partial \theta_{i,2}}.$$

We then have $0 < \alpha_i = b_i \alpha_{i-1} - \alpha_{i-2}; \alpha_0 = 1$ and $\alpha_1 = 1$ (i = 2, ..., s) (see page 54).

The stratified set of $M_{t(r,q)}$ subordinate to $\{U_1, U_2, U_3\}$ and compatible with $\tilde{\mathcal{F}}$, is $\{M_0, M_1, M_2, M_{(0,1)}, M_{(0,2)}\}$ where

$$M_1 = B_{1,1}(1/2) \times A_{1,1}; \quad A_{1,1} = D_{1,1} - D_{1,1}(1/2)$$

$$M_2 = B_{r,2}(1/2) \times A_{r,2}; \quad A_{r,2} = D_{r,2} - D_{r,2}(1/2)$$

$$M_0 = M(p,q) - M_1 \cup M_2$$

$$M_{(0,1)} = M_0 \cap \bar{M}_1 = B_{1,1}(1/2) \times \partial D(1/2)$$

$$M_{(0,2)} = M_0 \cap \bar{M}_2 = B_{r,2} \times \partial D_{r,2}(1/2)$$

and the singularity $Z$ of $\tilde{\mathcal{F}}$ is $Z_0 \cup Z_{(0,1)} \cup Z_{(0,2)}$, where

$$Z_0 = \{0_{1,1} \times 0_{1,1}, 0_{2,1} \times 0_{2,1}, ..., 0_{r,1} \times 0_{r,1}, 0_{r,2} \times 0_{r,2}\} \subseteq M_0$$

$$Z_{(0,1)} = 0_{1,2} \times \partial D_{1,2}(1/2) \subseteq M_{(0,1)}$$

$$Z_{(0,2)} = 0_{r,2} \times \partial D_{r,2}(1/2) \subseteq M_{(0,2)}.$$

$B_{1,1}(1/2)$ is the ball of radius 1/2, and $0_{i,1} \times 0_{i,1}$ is the center of $B_{i,1} \times D_{i,1}$.

Under the same orientation convention as in 3), from Lemma 11 one
gets
\[ Res(X_0^{(i)}, 0_{i,1} \times 0_{i,1}) = \frac{\alpha_i}{\alpha_{i+1}} + \frac{\alpha_{i+1}}{\alpha_i} = b_i + \frac{\alpha_i}{\alpha_{i+1}} - \frac{\alpha_{i+1}}{\alpha_i} \]
\[ Res(\{X_0^{(1)}, X_1\}, 0_{1,2} \times \partial D_{1,2}(1/2)) = -\frac{\alpha_1}{\alpha_1} + \frac{\alpha_2}{\alpha_1} \]  \( (2.11) \)
\[ Res(\{X_0^{(s)}, X_2\}, 0_{r,2} \times \partial D_{s,2}(1/2)) = -\frac{\alpha_s}{\alpha_s} + \frac{\alpha_{s-1}}{\alpha_s}. \]

Substituting 1) and (2.10) into formula (1.2) of Theorem 7, one deduces:
\[ \eta(L(p,q), \mathcal{F}) = \sigma(M_{L(p,q)}) + \frac{1}{3} \sum_{i=1}^{s} Res(X_0^{(i)}, 0_{i,1} \times 0_{i,1}) \]
\[ + Res(\{X_0^{(1)}, X_1\}, 0_{1,2} \times \partial D_{1,2}(1/2)) \]
\[ + Res(\{X_0^{(s)}, X_2\}, 0_{r,2} \times \partial D_{s,2}(1/2)) \]
\[ = -s - 1 + \frac{1}{3} \sum_{i=1}^{s} (b_i + \frac{\alpha_i}{\alpha_{i+1}} - \frac{\alpha_{i+1}}{\alpha_i}) - \frac{\alpha_1}{3\alpha_1} + \frac{\alpha_2}{3\alpha_1} - \frac{\alpha_s}{3\alpha_s} + \frac{\alpha_{s-1}}{3\alpha_s} \]
\[ = -s - 1 + \frac{1}{3} \sum_{i=1}^{s} b_i - \frac{1}{3} (\frac{a_1}{b_1} + \frac{a_2}{b_2}). \]  \( (2.12) \)

We summarize the above discussion in the following lemma.

**Lemma 14**  Given \( L(p,q) \) and \( \mathcal{F}(X_1,X_2) \) as above, then under the orientation convention,
\[ \eta(L(p,q), \mathcal{F}(X_1,X_2)) = -s - 1 + \frac{1}{3} \sum_{i=1}^{s} b_i - \frac{1}{3} (\frac{a_1}{b_1} + \frac{a_2}{b_2}). \]  \( (2.13) \)

**d. Residue formula for graph manifolds**

Let \( N \) be an oriented closed 3-manifold and let \( \mathcal{F} \) be an injective T-structure of \( N \). Taking the natural decomposition \( \mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_k, \phi_1, ..., \phi_l\} \) of \( (N, \mathcal{F}) \) as in part c of §1.6, we fill in \( N \) with the 4-manifold \( M_N \) and extend \( \mathcal{F} \) to \( \tilde{\mathcal{F}} \) over \( M_N \). We recall the following:
1) There are submanifolds $M_{\tilde{N}_e}$ and $M_{L(p_f,q_f)}$ of $M_N$ where $M_{\tilde{N}_e}$ is the filling of the Seifert manifold $\tilde{N}_e = \{b_e; (e, g_e); (\alpha_{e,1}, \beta_{e,1}), \ldots, (\alpha_{e,r_e}, \beta_{e,r_e})\}$ as in Lemma 8 and $M_{L(p_f,q_f)}$ is the filling of the lens space $L(p_f, q_f)$ as in Lemma 13. Note that $M_{\tilde{N}_e} \cap M_{L(p_f,q_f)} \approx D_e \times D_f$.

2) Denote by $F_e$ and $F_f$ the restriction of $F$ to $N_e$ and $L(p_f,q_f)$ respectively. Then the restriction $\tilde{F}_e$ of $\tilde{F}$ to $M_{\tilde{N}_e}$, is as in Theorem D and the restriction $\tilde{F}_f$ of $\tilde{F}$ on $M_{L(p_f,q_f)}$, is as in Lemma 13 with specifying $a_{e,1} = a_{e,2} = 0$. From formula (1.2) of Theorem 7 we can write

$$\eta_2(M_N) = \sigma(M_N) + \frac{1}{3} \left( \sum_{e=1}^{k} \text{Res}(M_{\tilde{N}_e}, \tilde{F}_e) + \sum_{j=1}^{l} \text{Res}(M_{L(p_f,q_f)}, \tilde{F}_f) \right).$$  \hspace{1cm} (2.14)

From the proof of Theorem D and Lemma 13, we have

$$\text{Res}(M_{\tilde{N}_e}, \tilde{F}_e) = \frac{1}{3} \left( \sum_{i=1}^{r_e} \sum_{j=1}^{n_e} b_{e,i,j} + b_e + r_e + \sum_{i=1}^{r_e} \frac{\alpha_{e,i} \beta_{e,i}}{\alpha_{e,i}} \right)$$

$$\text{Res}(M_{L(p_f,q_f)}, \tilde{F}_f) = \frac{1}{3} \sum_{j=1}^{l} \varepsilon_f c_{f,j}.$$  \hspace{1cm} (2.15)

where $\varepsilon_f = 1$, or $-1$ depending on whether or not the induced orientation on $L(p_f,q_f)$ agrees with the orientation convention of $L(p_f,q_f)$ (see 3), part c of §2.1. Substituting (2.15) into (2.14) and simplifying (2.14) we then obtain the following theorem.

**Theorem 11** Let $N$ be an oriented compact 3-manifold and let $F$ be an injective $T$-structure. For the natural decomposition $D(N,F) = \{N_1, \ldots, N_l, \phi_1, \ldots, \phi_l\}$ of $(N,F)$ as above, we have

$$\eta_2(N) = \eta(N,F)$$

$$= \sigma(M_N) + \frac{1}{3} \left[ \sum_{e=1}^{k} \left( \sum_{i=1}^{r_e} \sum_{j=1}^{n_e} b_{e,i,j} + (b_e + r_e) \right) \right.$$  \hspace{1cm} (2.16)

$$\left. + \sum_{i=1}^{r_e} \frac{\alpha_{e,i} \beta_{e,i}}{\alpha_{e,i}} \right] + \frac{1}{3} \sum_{j=1}^{l} \varepsilon_f c_{f,j}.$$
Remark 12 If $F$ has no exceptional orbits that is, each $N_i$ has no exceptional orbit as a Seifert fiber space for all $i$ then $3\eta_2(N)$ is an integer. Equivalently, the rational portion of $3\eta_2(N)$ is determined by the exceptional orbits of $F$.

Corollary 5 Under the same assumption as in Theorem 11, if $N$ is a solv manifold then $3\eta_2(N) = 0 \mod Z$.

Proof. To apply Theorem 4, we only need to find an injective $T$-structure of $N$ without exceptional orbits. Note that an oriented 3-dimensional solv manifold $N_\#$ is a $T^2$-bundle over $S^1$ with monodromy matrix:

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(\phi) = -1.$$

$N$ is formed by identifying the boundary of $[0, 1] \times T^2$ via $\varphi$. Writing $T^2$ as $S^1_1 \times S^1_2$, we choose an injective $T$-structure $F_\#_0$ of $N_\#$ as follows. Let $\partial/\partial \theta_i (i = 1, 2)$ be the velocity vector of rotation on $S^1_i$ and let $U$ be an invariant neighborhood of $\{0\} \times T^2$ in $N_\#$. Since $F_\#_0$ is determined by $\{((\epsilon, 1 - \epsilon) \times T^2, \partial/\partial \theta_2), (U, \partial/\partial \theta_1, \partial/\partial \theta_2)\}$, then $F_\#_0$ has no exceptional orbits. Note that the natural decomposition of $(N_\#, F_\#_0)$ is given by:

$$D(N, F_\#_0) = \{[0, 1] \times S^1_1 \times S^1_2, \phi\}. \quad (2.17)$$

Q.E.D.

2.2 A finiteness Result for Limiting $\eta$-Invariants

For a real number $D > 0$, let $M^3(D)$ be the collection of closed orientable 3-manifolds which admit a volume collapse with BCG. According to Theorem
$E, \{\eta_{(2)}(N) \mod mZ \mid N \in \mathcal{M}^3(D)\}$ is a finite set. Before giving the proof of Theorem E, we make some remarks.

**Remark 13** 1) The set $\mathcal{M}^3(D)$ contains infinitely many topological types for any $D > 0$. For instance, a 3-nilmanifold admits a diameter collapse (hence a volume collapse) with BCG and there are infinitely many non-differmorphis class of 3-nilmanifolds.

2) The size of $\{\eta_{(2)}(N) \mod mZ \mid N \in \mathcal{M}^3(D)\}$ depends on the number $D$. For example, if $D < \exp(-\exp(\exp 9))$, then $\{\eta_{(2)}(N) \mod mZ \mid N \in \mathcal{M}^3(D)\} = \{0, \frac{1}{3}, \frac{2}{3}\}$ (Corollary 5) since the elements in $\mathcal{M}^3(\exp(-\exp(\exp 9)))$ are nilmanifolds ([Gr1]).

**Proof of Theorem E.** First, from Proposition 5 and Theorem B we identify the elements in $\mathcal{M}^3(N)$ as either injective Seifert manifolds or solve manifolds. Since $3\eta_{(2)}(N) \mod Z = 0$ (Corollary 5), we only need to consider the elements in $\mathcal{M}^3(D)$ which are injective Seifert manifolds.

We proceed by contradiction. Assuming the opposite, we then have a sequence of injective Seifert manifolds $\{N_i\}$ in $\mathcal{M}^3(D)$ such that $\{\eta_{(2)}(N_i) \mod Z\}$ is an infinite set. By Gromov’s pre-compactness theorem ([GLP]), we may assume $\{N_i\}$ converges to a lower-dimensional metric space $Y$, since each $N_i$ admits a sufficiently volume collapsed metric with diameter bounded by $D$.

We split the following proof according to $dim(Y)$.

Case I. $dim(Y) = 0$. This amounts to saying that $diam(N_i) \to 0$. It follows from a well-known result in [Gr1] that all but finitely many of $\{N_i\}$ are identified as infranilmanifolds. Thus $\{\eta_{(2)}(N_i) \mod Z\} = \{0, 1/3, 2/3\}$
for $i$ sufficiently large (Corollary 4). This contradicts our assumption on
\{N_i\}.

Case II. $\dim(Y) = 1$. $Y$ is homeomorphic to either an interval $[0,1]$ or $S^1$. Note that if $Y \approx S^1$, then $N_i$ are solve manifolds and Thoerem E have been proved in this case (Corollary 5). If $Y \approx [0,1]$, then $N_i$ are trivial $T^2$-bundles with isolated $S^1$-orbits over 0 and 1 (Theorem 6). So the $N_i$ are actually $T^2$-manifolds for large $i$. Since $\pi_1(N_i)$ are not finite we then identify $N_i$ as either $S^2 \times S^1$ or $T^3 ([Ne])$. Thus we get the same contradiction as in case I.

Case III. $\dim(Y) = 2$. In this case, the limit space $Y$ can be viewed as the base space of the the injective Seifert manifold $N_i$ for all large $i$ (Proposition 5). Note that $Y$ is a orbifold since $Y \approx N/O(3)$ as in Theorem 3. Denote by $y_1, ..., y_k$ all the singular points of $Y$. Thus each $N_i$ has exactly $k$ exceptional orbits and the same base space. Write

$$N_i = \{b_i; (e, g); (\alpha_{i,1}, \beta_{i,1}), ..., (\alpha_{i,k}, \beta_{i,k})\}$$  \tag{2.18}$$

where $g$ is the genus of $Y$. From Corollary 4 one sees that the invariant $b_i$ contributes only integers to $3\eta_2(N_i)$. Therefore, to show $\{\eta_2(N_i) \mod Z\}$ is finite it is enough to show that the Seifert invariants (2.18) is finite up to the invariant $b_i$ for all $i$.

For convenience, we define the $\epsilon$-norm, "$\| \|_{\epsilon}$", for a Seifert manifold

$$N = \{b; (e, g); (\alpha_1, \beta_1), ..., (\alpha_k, \beta_k)\}.$$  

We define

$$\|N\|_{\epsilon} = \max\{\alpha_1, ..., \alpha_k\}.$$  \tag{2.19}$$
We claim that $\{\|N_i\|_s\}$ is a finite set (thus, (2.18) is finite up to the invariants $h_i$). We prove this by contradiction.

Assuming the opposite, we then have a sequence $\{(\alpha_{i,j}, \beta_{i,j})\}((\alpha_{i,j}, \beta_{i,j}) \in N_i)$ such that $\alpha_{i,j} \to \infty (j \to \infty)$. We can also assume that the exceptional orbits in $N_i$ corresponding to the $(\alpha_{i,j}, \beta_{i,j})$ are the preimages of some fixed $y_j$, say, $y_1$.

Choose a small metric balls $B_\delta(y_1)$ (with $\delta$ fixed) at $y_1$ such that $\pi_i^{-1}(B_\delta(y_1))$ in $N_i$ is a solid torus for all $i$ sufficiently large. Note that for sufficiently large $i$, $B_\delta(y_1)$ can be viewed as the base space of the $S^1$-action on $\pi_i^{-1}(B_\delta(y_1))$ determined by the exceptional orbit invariants $(\alpha_{i,1}, \beta_{i,1})$ with the isotropy group $Z_{\alpha_{i,1}}$. Since $\text{diam}(\pi_i^{-1}(B_\delta(y_1))) \leq D$ and $\alpha_{i,1} \to \infty$, the limiting of the orbit spaces $\pi^{-1}(B_\delta(y_1))/Z_{\alpha_{i,1}}$ must be of dimension 1. This contradicts $\text{dim}(B_\delta(y_1)) = \text{dim}(Y) = 2$.

By now the proof of Theorem E has been completed. Q.E.D.
Chapter 3

Bounded Good Choppings

3.1 Bounded Good Choppings

In this section, we will prove Theorem F. As an application of Theorem F, we obtain a bounded version of Cheeger-Gromov's existence of good choppings theorem ([CG5]). First, let us recall the main result in [CG5] as follows.

Theorem 12 ([CG5]). Let $M^n$ be a complete manifold with bounded sectional curvature, $|K| \leq 1$. Given $X \subset M^n, 0 < r \leq 1$, there is a submanifold $U^n$ with smooth boundary $\partial U^n$ such that for some constant $c(n)$
depending on $n$,

\[ X \subset U^n \subset T_r(X) \quad (3.1) \]

\[ \text{Vol}(\partial U) \leq c(n)\text{Vol}(T_r(X)) r^{-1} \quad (3.2) \]

\[ ||II\partial U|| \leq c(n)r^{-1} \quad (3.3) \]

Moreover $U^n$ can be chosen to be invariant under $I(r, X)$, where $I(r, X)$ denote the group of isometries of $T_r(X)$ which fix $X$.

**Remark 14** If $M^n$ admit a positive rank $F$-structure $F$ and $X$ is invariant, then $U^n$ can be chosen invariant under $F$.

**Theorem F.** Under the same assumptions as in Theorem 12. Let \( D = \text{diam}(X) = \sup\{\text{dist}(x, y) | x, y \in X\} \). Then there exists a constant, \( c(n, r, D) \), depending on $n$, $r$ and $D$, such that

\[ \text{diam}(\partial U) \leq c(n, r, D). \]

An important application of Theorem F is a bounded version of the following Cheeger-Gromov's good chopping theorem.

**Theorem 13** ([CG5]) Let $M^n$ be complete, $|K| \leq 1$, $\text{Vol}(M^n) < \infty$. Then $M^n$ admits an exhaustion $M^n = \bigcup_{i=1}^{\infty} M^n_i$, by manifolds with smooth boundary, such that

\[ \lim_{j \to \infty} \text{Vol}(\partial M^n_i) = 0, \]

\[ ||II_{M^n_i}|| \leq c(n). \]

In order to state a bounded version of Theorem 12, we define the following.
Definition 10 Let $M$ be a complete manifold. $M$ is said have finite diameter at infinity, if there is a point $p \in M$ so that $\sup_{r>0}\{\sum_{\alpha} \text{diam}(T_1(\partial B_r(p)))_{\alpha}\} < \infty$, where $B_r(p)$ is the metric ball of radius $r$ with center at $p$ and $(T_1(\partial B_r(p)))_{\alpha}$ is a component of the 1-tubular neighborhood of $\partial B_r(p)$.

Let $S$ be a connected submanifold of $M$. For $x, y \in S$, let $\gamma_{x,y}$ be a path in $S$ from $x$ to $y$. We define

$$d_{st}(x, y) = \inf_{\gamma_{x,y}} \{L(\gamma_{x,y})\}.$$ 

Theorem 14 Under the same assumption as in Theorem 13. Suppose $M$ has finite diameter at infinity. Then the good chopping choppings as in Theorem 12 can be choosen so that the set $\{\text{diam}_s(\partial M_i)\}$ is bounded. Where $\text{diam}_s(\partial M_i) = \sum_{\alpha} \text{diam}_s((M_i)_{\alpha})$ and the sum runs over all components of $\partial M_i$.

Proof. That $M$ has finite diameter at infinity means that there is a constant $D > 0$ and a point $p \in M$ so that $\sup_{r>0}\{\sum_{\alpha} \text{diam}(T_1(\partial B_r(p)))_{\alpha}\} \leq D$. Since $T_1(\partial B_r(p))$ has at most $[D]$ (the integer part of $D$) components, applying Theorem F to $\partial B_r(p)$, we complete the proof. Q.E.D.

We need several lemmas to prove Theorem F. The first lemma is about the local isolate property of $\partial U^n$.

Lemma 15 Under the same assumption as Theorem 12. Then there is a constant $\rho(n, r) > 0$ depending only on $n$ and $r$ so that for $x \in \partial U^n$, $B_{\rho(n, r)}(x) \cap \partial U^n$ is connected.
Proof. The proof is based on the concrete construction for \( U^n \) given in [CG5]. Let us first recall the following from [CG5]:

Let \( g \) be the Riemannian metric of \( M \). Then there exists a Riemannian metric \( \tilde{g} \) such that

\[
\left( \frac{5}{6} \right) g \leq \tilde{g} \leq \left( \frac{6}{5} \right) g
\]

Denote by "\( || \cdot || \)" the norm determined by \( \tilde{g} \). Clearly, it is enough to prove the lemma only for the metric \( \tilde{g} \).

Roughly, \( \partial U^n \) is a level set of the smooth function \( F : M \to R^1 \) which satisfies the conditions:

(i) there are constants \( 0 < \delta(n) < 1 \) and \( \epsilon(n) > 0 \) depending only on \( n \) such that

\[
|| \text{grad} F(x) || \leq 2 \quad x \in M, \\
|| \text{grad} F(x) || \geq \epsilon(n) \quad x \in F^{-1}([0, \delta(n)r]),
\]

(ii) \( \partial U^n = F^{-1}(y), \ y \in [\frac{1}{3} \cdot \delta(n)r, \frac{2}{3} \cdot \delta(n)r] \).

(iii) \( \partial U^n \) has a tubular neighborhood \( F^{-1}([y - \frac{1}{4} \cdot \delta(n)r, y + \frac{1}{4} \cdot \delta(n)r]) \).

where the

Then we observe the following:

For \( x \in \partial U^n \), let \( x' \) be in the \( \text{grad}F \)-flow line through \( x \) such that \( x' \in F^{-1}([0, y - \frac{1}{4} \cdot \delta(n)r] \cup [y + \frac{1}{4} \cdot \delta(n)r, \delta(n)r]) \). Let \( \gamma : [0, l] \to M \) be the integral curve of \( \frac{\text{grad}F}{||\text{grad}F||} \), \( \gamma(0) = x \) and \( \gamma(l) = x' \). We claim

\[
\frac{\delta(n)r}{8} \leq l
\]
Our claim is proved in the following:

\[
\frac{\delta(n)r}{4} \leq |F(x) - F(x')| = |F(\gamma(0)) - F(\gamma(l))| \\
= \int_{0}^{l} |\dot{\gamma}(t)||dt = \int_{0}^{l} |\nabla F(\gamma(t)), \dot{\gamma}(t)| |dt \\
\leq \int_{0}^{l} ||\nabla F(\gamma(t))|| |dt \leq 2 \cdot l \quad \text{(by (3.4))}.
\]

This implies that starting at \( \partial U^n \), one can flow along the integral curves of \( \nabla F \) at least \( \frac{\delta(n)r}{8} \) units without hitting \( \partial U^n \) again.

Now we begin the proof. We shall first put an additional assumption that the injectivity radii is at least 1 everywhere on \( M \). Then we will explain how we can reduce this case by lifting to the tangent space.

First we observe that Lemma 15 is trivial where \( M^n \) is Euclidean space and \( F^{-1}(x) \) are hyperplanes for \( x \in F^{-1}([0, \delta(n)r]) \). Since in that situation the \( (\nabla F) \)-flow are straight lines perpendicular to \( F^{-1}(x) \) and \( l = \text{dist}(x, x') \). By (3.4) we can choose \( \rho(n, r) = \frac{\delta(n)r}{8} \).

Roughly, the conditions that \( \text{geo}(M) \geq 1 \) (our additional assumption) and \( ||II_{\partial U^n}|| \leq c(n)r^{-1} \) ((3.3)) imply that by multiplying the metric by a large number one is able to reduce the general case to the above trivial situation.

Now we proceed by contradiction. Assuming the opposite, we then find a sequence of the quadrupla \( (M_i, U_i, g_i, B_{1/i}(x_i)) \) which satisfies the conditions:

1) \( (M_i, U_i, g_i) \) satisfies the conditions of Lemma 15,

2) \( B_{1/i}(x_i) \cap \partial U^n \) has at least two components \( (x \in \partial U_i) \)

Rescaling the metric \( g_i \) by \( i \), we actually have the sequence \( (M_i, U_i, i \cdot g_i, x_i) \) satisfying:
1') \(|K(M_i)| \to 0, \text{InjRad}(M_i) \to \infty\) and ||II_{\partial U_i}|| \to 0.

2') \(B_1(x_i) \cap \partial U_i\) has at least two components \((x \in \partial U_i)\)

Note that for sufficiently large \(i\), (3.5) becomes

\[2 \ll \ell_i,\]  \tag{3.6}

Consider the pointed Hausdorff limit \((M_0, x_0)\) of \((M_i, x_i)\) and \((\partial U_0, x_0)\) of \((\partial U_i, x_i)\) respectively. From 1') we see that \(M_0\) is Euclidean space and \(\partial U_0\) is a hyperplane in \(M_0\). Then (3.6) implies that \(B_1(x_0) \cap \partial U_0 = \emptyset\). This contradicts 2').

Now we consider \(M\) in general, i.e., we do not have \(\text{geo}(M) \geq 1\). For any \(x \in \partial U^n\), consider the tangent space at \(x\), \(T_x(M)\) which equipt with the pull-back metric \(g'\) under the exponential map, \(\exp_x : T_x(M) \to M\). Let \(B_{\frac{r}{2}}(0)\) be the metric ball at \(0 \in T_x(M)\) of radius \(\frac{r}{2}\). Then \((B_{\frac{r}{2}}(0), g')\) has bounded geometry, i.e., \(\forall z \in B_{\frac{r}{2}}(0), \text{InjRad}(T_z(M), z) \geq 1\). By lifting \(B_{\frac{r}{2}}(x) \cap \partial U^n\) to \(B_{\frac{r}{2}}(0)\), applying previous result we then obtain the desired constant \(\rho(n, r)\).

Q.E.D

Lemma 16 Under the same condition as in Theorem 12. Then there is a constant \(\sigma(n, r) > 0\) depending only on \(n\) and \(r\), such that each component of \(B_{\sigma}(x) \cap \partial U^n\) has the diameter, \(\text{diam}_s\), less than 1.

Proof. The proof of Lemma 16 is essentially the same as the proof of Lemma 15. So we omit the details. We first observe that Lemma 16 is true for \(M\) a Euclidean space and \(\partial U^n\) a hyperplane. By rescaling the metric of \(M^n\) by a sufficiently large number we may assume \(M^n\) and \(\partial U^n\) are almost as in above situation. Therefore Lemma 16 is true in general (Otherwise
Lemma 17 Let $M^n$ be a complete manifolds with $\text{Ric}(M^n) \geq 1$. For $D > 0$ and $r > 0$, there is natural number $N(n, r, D)$ depending on $n, r$ and $D$, so that if $U$ is a submanifold with boundary such that if $\text{diam}(U) \leq D$, then $\partial U$ can be covered by $N(n, r, D)$ balls of radius $r$ with centers in $\partial U^n$.

**Proof.** See the Covering Lemma in [CG1].

**Proof of Theorem F.** First we take $\epsilon(n, r) = \min\{\rho(n, r), \sigma(n, r)\}$ where $\rho(n, r)$ and $\sigma(n, r)$ are as in Lemma 15 and 16 respectively. Thus we have

1) $B(x_i) \cap \partial U^n$ is connected (Lemma 15),

2) $\text{diam}_s(B(x_i) \cap \partial U^n) \leq 1$ (Lemma 16).

Since $\text{diam}(T_1(X)) \leq D$, we can find at most $N(n, r, D)$ metric balls of radius $\epsilon(n, r)$, $B(x_1), \ldots, B(x_{N(n, r, D)})$ ($x_i \in \partial U^n$) which covers $\partial U^n$ (Lemma 17). Bombining with 1) and 2), we deduce

$$\text{diam}_s(\partial U^n) \leq \sum_{i=1}^{N(n, r, D)} \text{diam}_s(B(x_i) \cap \partial U^n) = N(n, r, D).$$

Now the proof is complete.

Q.E.D.

3.2 Some Progress on Conjecture II

Based on the results in this thesis we have made some progress in proving Conjecture II for 4-manifolds (see Introduction). In fact, using Theorem
A, Theorem E and Theorem 14 we can prove the following result:

Let \((M, g)\) be an oriented complete 4-manifold, \(|K_g| \leq 1\), \(Vol(M, g) < \infty\) and \(\overline{\text{geo}}(M) \geq 1\). Suppose \(M\) has finite diameter at infinity. Then the geometric signature, defined by

\[
\sigma_2(M) = \int_M P_L(\Omega)
\]

is a rational number.

Note that there are 4-manifolds of infinite topological type which satisfy the above conditions. Also if one can eliminate the assumption that \(M\) has finite diameter at infinity, then one would prove Conjecture II for 4-manifolds. Here we only give the outline of the proof. More detailed argument will appear elsewhere.

**Sketch of Proof.** Let us start with the general situation. Let \(M\) be a 4k-complete manifold, \(|K| \leq 1\), \(Vol(M^n) < \infty\). Consider the geometric signature \(\sigma(M, g)\) of \((M, g)\), defined by

\[
\sigma(M, g) = \int_M P_L(\Omega)
\]

Suppose \(M\) has BCG in a neighborhood of infinity, \(\overline{\text{geo}}(M, g) \geq 1\). As we have already explained in Introduction, the above integral, also denoted by \(\sigma_2(M)\), is a proper homotopy invariant.

Taking a good choppings \(\{M_j\}\) of \(M\) and then applying the Atiyah-Patodi-Singer index formula ([APS]) to \(\{M_j\}\), one relates the \(\sigma_2(M)\) to
the asymptotic behavior of the sequence $\eta(\partial M_j, g_j)$ ($g_j$ the induced metric on $\partial M_j$).

\[
\sigma(\partial M) = \int_M P_L(\Omega) = \lim_{i \to \infty} \int_M P_L(\Omega)
= \lim_{i \to \infty} (\sigma(M_i) - \eta(\partial M_i, g_i) - II(\partial M_i, g_i))
= \lim_{i \to \infty} (\sigma(M_i) - \eta(\partial M_i, g_i))
\]

(3.7)

where $II(\partial M_j, g_i)$ are certain locally computable expression involving the second fundamental form of $\partial M_j$. Note that it follows from Theorem 12 that $\lim_{i \to \infty} II(\partial M_i, g_i) = 0$. Since we are interested in the rationality of $\sigma(\partial M)$ (Conjecture II), we take for convenience $\text{mod } Z$ on both sides of (3.7),

\[
\sigma(\partial M) \text{ mod } Z = \lim_{i \to \infty} (\eta(\partial M_i, g_i) \text{ mod } Z).
\]

(3.8)

To prove Conjecture II one needs to study the asymptotic behavior of the sequence $\{\eta(\partial M_j, g_j) \text{ mod } Z\}$. Here the difficulties are: (a) there are not obvious relations among $\partial M_i$ in general; (b) computation for $\eta(\partial M_j, g_i)$.

Now consider the case when $k = 1$. In order to reduce the difficulty (a) and (b), we suppose the 4-manifold $M$ has finite diameter at infinity, i.e., there is a constant $D > 0$ and $p \in M$ such that $\sup_{r>0} \{\sum_{\alpha} \text{diam}(\{T_r(\partial B_r(p))\}_{\alpha})\} \leq D$. We claim that there exists a good choppings $\{M_i\}$. satisfying the following conditions:

1) there exists a positive integer $K(D)$ depending only on $D$, such that each $\partial M_i$ has at most components. We number its components as $(\partial M_i)_1, ..., (\partial M_i)_K$ where we allow that some components are empty set.

2) the sequence of the $j$th-component $\{(\partial M_i)_j\}$ converges to a metric
space $X_j$ under the Hausdorff distance, $\lim_{i \to \infty} \text{dist}_H((\partial M_i)_j, X_j) = 0$. Note that $X_j$ is a topological manifold of dimension less or equals to 2. 1) can be seen from the proof of Theorem F, and 2) can be proved by using Gromov's precompactness theorem and a simple induction argument. Since

$$\eta(\partial M_i, g_i) \mod Z = \sum_{j=1}^{K(D)} \eta((\partial M_i)_j, g_{i,j}) \mod Z,$$

it is enough to prove rationality of (3.8) for the case when $K(D) = 1$.

We split the rest proof according to $dim(X)$ (note that $X = X_1$).

Case I. $dim(X) = 0$ or 2 or $X \approx S^1$.

We can show that, in this case, $\pi_1(\partial M_i)$ is infinite. We also observe that $(\partial M_i, g_i)$ has BCG. Here we use the condition that $||H\beta M_i||$ is bounded and the Gauss Lemma. The consequence of these two facts is that each $\partial M_i$ admits an injective F-structure for $i$ sufficiently large (Theorem B), i.e., $\eta(\partial M_i)$ makes sense. Thus, as we has already seen in §1.5,

$$\lim_{i \to \infty} \eta(\partial M_i, g_i) = \lim_{i \to \infty} \eta(\partial M_i). \quad (3.9)$$

From (3.8) and (3.9), we have

$$\sigma(2)(M) = \lim_{i \to \infty} \eta(\partial M_i) \mod Z \quad (3.10)$$

From $\text{diam}(\partial M_i, g_i) \leq D$ we conclude that $\{\eta(\partial M_i) \mod Z\}$ is a finite set (Theorem E). From (3.10) and Theorem A, we obtain the rationality of $\sigma(2)(M)$.

Case II. $X \approx I$ where $I$ is an interval.

We then identify that each $\partial M_i$ is a lens space for $i$ sufficiently large (see the proof of Proposition 5).
First, by applying Theorem 1 one construct, depending on $g$, a positive rank $T$-structure $\mathcal{F}_g$ outside some compact subset of $M$. We can assume that our previous good chopping is compactible with $\mathcal{F}_g$ and the metric $g$ is invariant ([CFG]). Since $\partial M_i$ converges to an interval, we observe the following:

3) $\mathcal{F}_g$ is pure $T^2$-structure in a neighborhood of $\partial M_i$. Let $O_i$ be a singular $S^1$-orbit on $\partial M_i$. Then the singular set $Z$ of $\mathcal{F}_g$ has non-empty intersection with $\partial M_i$ as $i$ sufficiently large.

4) the diameters of the above $T^2$-orbits around $O_i$ converges to zero as $i \to \infty$.

By studying the structure of the singularity $Z$ and using the result in [CFG], we can show that in this case the $T^2$ orbits of $\mathcal{F}_g$ which are near to $Z$ have definite size diameters (note that here we need the condition BCG). This contracts 4).
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