On Generalizations of Jørgensen's Inequality for Kleinian Groups and Some Topics on Quasiconformal Extension

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in Mathematics

Stae University of New York
at
Stony Brook

August 1990

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at Stony Brook
The Graduate School
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Abstract of Dissertation

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This dissertation consists of two parts. The main part is the first one which deals with the generalizations of Jørgensen's inequality. The second part deals with some problems related to qusiconformal extensions.

We denote the field of complex numbers by $\mathbb{C}$. A Möbius transformation g is a one-to-one meromorphic function of the extended complex plane $\overline{\mathbb{C}}$ onto itself. It is necessarily of form $g(z) = \frac{az+b}{cz+d}$ $(ad - bc = 1)$. Denote the group of all Möbius transformations by **Möb**. It is naturally isomorphic to $PSL(2, \mathbb{C})$. We define the trace of a transformation $g(z) = \frac{az+b}{cz+d}$ to be $\mathbf{tr}(g) = a + d$. It is clear that the trace of g is well defined up to sign.

In 1976, Jørgensen obtained the following very important results:

Let two Möbius transformations f and g generate a non-elementary discrete group. Then

$$|\,\mathbf{tr}^2(f)-4|+|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|\geq 1.$$

In this dissertation, we prove a general version of Jørgensen's inequality as follows:

For arbitrary rational numbers R and r, $0<R,r\leq\frac{1}{2}$, there exist positive numbers $\alpha(R,r)$ and $\beta(R,r)$ having the following properties: Let Möbius transformations f and g generate a non-elementary discrete group. If $\mathbf{tr}^2(f)\neq 4\cos^2 r\pi$ and $\mathbf{tr}(fgf^{-1}g^{-1})\neq -2\cos 2R\pi$, then

$$|\,\mathbf{tr}^2(f)-4\cos^2 r\pi|+|\,\mathbf{tr}(fgf^{-1}g^{-1})+2\cos 2R\pi|\geq\alpha(R,r),$$

in addition if $\mathbf{tr}(f)\neq 0$ then

$$|\,\mathbf{tr}^2(f)-4\cos^2 r\pi|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})+2\cos 2R\pi|\geq\beta(R,r).$$

For some specific R and r, we get the estimates of $\alpha(R,r)$ and $\beta(R,r)$ including some sharp results. For example: *Let Möbius transformations f and g generate a non-elementary discrete group. If $\mathbf{tr}(f)\neq 0$, then*

$$|\,\mathbf{tr}^2(f)|+|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|>2(\sqrt{2}-1)=0.8284\cdots$$

and

$$|\,\mathbf{tr}^2(f)|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|>0.1354.$$

If in addition $\mathbf{tr}(g)\neq 0$, then

$$|\,\mathbf{tr}^2(f)|+|\,\mathbf{tr}^2(g)|+|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|>1.5407$$

and

$$\tfrac{1}{2}\{|\,\mathbf{tr}^2(f)|+|\,\mathbf{tr}^2(g)|\}+|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|>0.9706.$$

Also we get the following result which generalizes the Shimizu-Leutbecher's lemma:

For an arbitrary rational number $R \in (0, \frac{1}{2}]$, *there exists a positive constant* $\sigma(R)$ *having the following property: If the group generated by* $f(z) = z + 1$ *and* $g(z) = \frac{az+b}{cz+d}$ $(ad - bc = 1, c \neq 0)$ *is non-elementary discrete and* $c^2 \neq \mp 4\cos^2 R\pi$, *then*

$$|c^2 \pm 4\cos^2 R\pi| \geq \sigma(R).$$

We know $\sigma(\frac{1}{2}) = 1$. By elementary calculation $\sigma(\frac{1}{3}) > 0.7548$, $\sigma(\frac{1}{4}) > 0.2654$ and $\sigma(\frac{1}{6}) > 0.1181$. That means

$$|c^2 \pm 1| > 0.7548, \quad |c^2 \pm 2| > 0.2654, \quad |c^2 \pm 3| > 0.1181$$

provided that the left sides are non-zero.

In the end of Part One we give some conjectures.

In Part Two we derive some lower bounds for the inner radius of universal Teichmüller space from an Ahlfors inequality. We give improved estimates for the dilatation of the Beuring-Ahlfors extension of quasisymmetric automorphisms of the real line. At last, with the help of the sigular integral, we obtain some general quasiconformal extension and univalency criteria for analytic functions $f(z)$ defined in the upper half plane or in the unit disk. Unlike all the previous criteria, our results contain an arbitrary analytic function $a(z)$. Specific choices of $a(z)$ will yield many new interesting univalency criteria. Some of them generalize the results of Ahlfors, Harmelin and Krzyz.

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To my beloved parents
Tan Shen and Zhao Aifeng

Acknowledgments

Thanks are due to many people.

First I would like to thank my research advisor, Professor Irwin Kra, for his inspiring guidance, valuable advice and kind help beyond the study of mathematics.

Next, I owe a deep dept of gratitude to Bernie Maskit for his generous helpful and illuminating conversations.

My sincere thanks also go to the faculty, secretaries and all other members of the Mathematics Department at Stony Brook, without whose moral support this dissertation would not have been possible.

Finally, I deeply thank the Alfred P. Sloan Foundation for awarding me a Doctoral Dissertation Fellowship during the 1988-89 academic year.

Part One

Generalizations of Jørgensen's Inequality

Chapter 1

Introduction and Preliminary Remarks

§1.1. We denote the field of complex numbers by $\mathbb{C}$. A Möbius transformation is a one-to-one meromorphic function of the extended complex plane $\overline{\mathbb{C}}$ onto itself. We can write a Möbius transformation in the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Denote the group of all the Möbius transformations by **Möb**. The group of two by two complex matrices with determinant 1 is denoted by $SL(2, \mathbb{C})$. There is a natural topology on $SL(2, \mathbb{C})$ which identifies it with a closed subspace of $\mathbb{C}^4$. Let $PSL(2, \mathbb{C})$ be the quotient group obtained from $SL(2, \mathbb{C})$ by identifying a matrix with its negative. So $PSL(2, \mathbb{C})$ is endowed with the quotient topology. There is a natural isomorphism between **Möb** and $PSL(2, \mathbb{C})$. So we will consider a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(ad - bc = 1)$ as an element of $PSL(2, \mathbb{C})$ and also as the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$.

Let $g(z) = \frac{az+b}{cz+d}$ define such a Möbius transformation. We define the trace of g to be $\mathbf{tr}(g) = a + d$. It is clear that the trace of g is well defined up to sign on **Möb**.

If $g \neq I$, we call g

$$\begin{array}{ll} \text{elliptic} & \text{if } 0 \leq \mathbf{tr}^2(g) < 4, \\ \text{parabolic} & \text{if } \mathbf{tr}^2(g) = 4, \\ \text{loxodromic} & \text{if } \mathbf{tr}^2(g) \notin [0, 4]. \end{array}$$

The loxodromic elements with $\mathbf{tr}^2(g) > 4$ are called hyperbolic.

Definition 1: Let G be a subgroup of **Möb**. G is said to be a discrete group if the corresponding subgroup of $PSL(2, \mathbb{C})$ is a discrete set in the topology mentioned above.

G is not discrete if and only if there is a sequence $g_n(z) = \frac{a_n z + b_n}{c_n z + d_n} \in$ G such that $\{g_n\}$ is distinct and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are all bounded.

Let

$$\mathrm{H}^3 = \{w = z + tj : \ z \in \mathbb{C} \text{ and t} > 0\}.$$

Then H^3 is a hyperbolic space under the Poncaré metric $\frac{1}{t}|dw| = \frac{1}{t}(|dz|^2 + |dt|^2)^{\frac{1}{2}}$. For $g(z) = \frac{az+b}{cz+d}$, we can define the Poncaré extension of g by

$$\begin{aligned} g(w) &= (aw + b)(cw + d)^{-1} \\ &= \frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2 + |ad - bc|tj}{|cz + d|^2 + |c|^2 t^2}. \end{aligned}$$

Then g acts as an isometry on H^3.

When g is not identity and not parabolic, g has two fixed points in $\overline{\mathbb{C}}$. Then the hyperbolic line in H^3 connecting these two points is called the axis of g.

Definition 2: Let G be a discrete subgroup of **Möb**. We say that G acts *discontinuously* at $z_o \in \overline{\mathbb{C}}$ provided there is a neighborhood K of z_o such that

$$g(K) \cap K \neq \emptyset$$

for only finitely many $g \in G$.

The *region of discontinuity* $\Omega = \Omega(G)$ is the largest (open) set on which G acts discontinuously. Its complement $\Lambda = \Lambda(G) = \overline{\mathbb{C}} - \Omega$ is called the *limit set* of G. We will call G *Kleinian* if $\Omega \neq \emptyset$ and $|\Lambda| > 2$; *elementary* if $|\Lambda| \leq 2$.

§1.2. It is very interesting to ask whether a group generated by two Möbius transformations is discrete or not. A special case is when one elemennt is parabolic. Because the discreteness is invariant under conjugation , we can always assume the parabolic element is of form $z \mapsto z+1$. The following classical result is called Shimizu-Leutbecher's lemma:

Proposition 1: *If the group generated by* $f(z) = z+1$ *and* $g(z) = \frac{az+b}{cz+d}$ *(where* $ad - bc = 1$*) is discrete and* $c \neq 0$*, then*

$$|c| \geq 1. \tag{1.1}$$

From this lemma we see: if $0 < |c| < 1$, then the group $< f, g >$ is not discrete.

Jørgensen generalized [J-1] the above lemma as follows:

Proposition 2: *Let f and g be two Möbius transformations that generate a non-elementary discrete group. Then*

$$|\,\mathbf{tr}^2(f)-4|+|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|\geq 1. \tag{1.2}$$

It is clear that $\mathbf{tr}(fgf^{-1}g^{-1})$ is well defined. If $f(z)=z+1$ and $g(z)=\frac{az+b}{cz+d}$ $(ad-bc=1)$, then $\mathbf{tr}^2(f)=4$ and $\mathbf{tr}(fgf^{-1}g^{-1})=2-c^2$. That means that when f is parabolic Jørgensen's inequality reduces to the Shimizu-Leutbecher inequality.

Jørgensen's inequality is important because it has many beautiful applications. For example, Jørgensen [J-2] proved that a non-elementary group is discrete if and only if any two of its elements generate a discrete group. Many other applications of Jørgensen's inequality appear in Beardon's book [B].

In chapter 3, we obtain another generalization of the Shimizu-Leutbecher lemma.

§1.3. There are many way to generalize Jørgensen's inequality. Let

$$f(z)=u^2z,\ (u\in\mathbb{C},\, u\neq 0,\pm 1)\ \ g(z)=\frac{az+b}{cz+d}.$$

Define the Shimizu-Leutbecher's sequence

$$g_1=g\circ f\circ g^{-1},\quad g_n=g_{n-1}\circ f\circ g_{n-1}^{-1}=\frac{a_nz+b_n}{c_nz+d_n}\,(n>1). \tag{1.3}$$

Set $z_n = b_n c_n$ and $\beta = (u - \frac{1}{u})^2$, then

$$z_{n+1} = -\beta z_n(z_n + 1).$$

Define

$$F_1(t) = -\beta t(t+1)\ , \quad F_n(t) = F \circ F_{n-1}(t)\ . \tag{1.4}$$

Set $z = bc$, then $F_n(z) = z_n$. Brooks and Metalski [B-M] proved that to every fixed point of the polynomial $F_n(t)$ there corresponds an inequality involving $\mathbf{tr}^2(f)$ and $\mathbf{tr}(fgf^{-1}g^{-1})$. We see that $F_1(t)$ has fixed points $t_1 = 0$ and $t_2 = -(1 + \frac{1}{\beta})$. If $r = |\beta|(1 + |z|) < 1$, then $|F_1(z) - t_1| = |z - t_1| \cdot r$. By induction, we have

$$|F_n(z) - t_1| \leq |z - t_1| \cdot r^n$$

Therefore

$$\lim_{n \to \infty} F_n(z) - t_1 = 0. \tag{1.5}$$

If $< f, g >$ is non-elementary, then $F_n(z) - t_1 \neq 0$ for any n. That means $\{z_n\}$ is distinct and convergent. Then we can find a subset of $< f, g >$ which is not discrete. So the hypothesis of discreteness of $< f, g >$ implies

$$|\beta|(1 + |z|) \geq 1. \tag{1.6}$$

Because $\beta = \mathbf{tr}^2(f) - 4$ and $\beta z = 2 - \mathbf{tr}(fgf^{-1}g^{-1})$, this is just Jørgensen's inequality. But what will be happen if we apply the same trick to the other fixed point of $F_1(t)$? By calculation

$$F_n(z) - t_2 = (F_{n-1}(z) - t_2)[\beta + 2 - \beta(F_{n-1}(z) - t_2)].$$

So if $r = |\beta+2| + |\beta(z-t_2)| < 1$, then $|F_n(z)-t_2| = |z-t_2| \cdot r$. By induction

$$|F_n(z) - t_2| \leq |z - t_2| \cdot r^n.$$

Therefore

$$\lim_{n\to\infty} F_n(z) - t_2 = 0. \tag{1.7}$$

If $F_n(z) - t_2 \neq 0$ for any n, then same argument shows that the discreteness of $< f, g >$ implies

$$|\beta + 2| + |\beta(z - t_2)| \geq 1. \tag{1.8}$$

Because $\beta + 2 = \mathbf{tr}^2(f) - 2$ and $\beta(z - t_2) = \mathbf{tr}(fgfg^{-1}) - 1$, this is the new inequality

$$|\,\mathbf{tr}^2(f) - 2| + |\,\mathbf{tr}(fgfg^{-1}) - 1| \geq 1. \tag{1.9}$$

But the trouble we meet here is the possibility that $F_n(z) - t_2 = 0$ for some n even though $z \neq t_2$. In this case z_n is a constant for large n. So we can not trivially get this new result. In chapter 4 we will get rid this possibility and prove (1.9).

The approach of Gehring and Martin is slightly different. They analyze a dynamically defined region in the complex plane given by iteration theory of quadratic polynomials and produce new inequalities from its geometry as follows [**G-M**]. For $\beta \in \mathbb{C}$ define the quadratic mapping $R_\beta(z) = -\beta z + z^2$, and let $R_{\beta^n}(z)$ denote the n^{th} iterate of R_β. The filled in Julia set for R_β is the bounded perfect set

$$D(\beta) = \{z \in \mathbb{C} : \ \{R_{\beta^n}(z)\}_{n=1}^{\infty} \text{ is a bounded set } \},$$

and the eventually periodic points which are not eventually the fixed point zero is the countable subset of $D(\beta)$ defined by

$$P^*(\beta) = \{z \in \mathbb{C} : \ \{R_{\beta^n}(z)\}_{n=1}^{\infty} \text{ is a finite set not containing zero } \}.$$

Given Möbius transformations f and g, set

$$\beta = \beta(f) = \mathbf{tr}^2(f) - 4 \ \text{ and } \ \gamma = \gamma(f,g) = \mathbf{tr}(fgf^{-1}g^{-1}) - 2,$$
$$\Theta f(g) = gfg^{-1} \ \text{ and } \ \Theta f^{n+1}(g) = \Theta f(\Theta f^n(g)).$$

They have established the following general result.

Proposition 3: *If $< f, g >$ is a kleinian group, then*

$$\gamma \notin D(\beta) - P^*(\beta). \tag{1.10}$$

Moreover if $\gamma \in P^*(\beta)$, then there is a nontrivial relation in $< f, g >$ of one of the following types

$$\Theta f^n(g) = \Theta f^m(g), \ \ n \geq m + 2 \quad \text{or} \quad f^k \Theta f^n(g) f^{-k} = \Theta f^m(g), \ \ k \neq 0. \tag{1.11}$$

This result contains many inequalities including Jørgensen's inequality and some of our results in chapter 4.

For some other generalizations of Jørgensen's inequality, the reader can refer to [Gi], [J-K] and [R].

§1.4. Let Möbius transformations f and g generate a non-elementary discrete group. In this work we obtain some inequalities similar to Jørgensen's.

It is natural to ask the following very general question: *Do there exist some constants a, b and c other than 4, 2 and 1 such that*

$$|\,\mathbf{tr}^2(f) - a| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - b| \geq c? \tag{1.12}$$

Also we can ask another even stronger question: *Do there exist some constants α, β and γ such that*

$$|\,\mathbf{tr}^2(f) - \alpha| \cdot |\,\mathbf{tr}(fgf^{-1}g^{-1}) - \beta| \geq \gamma? \tag{1.13}$$

In chapter 2, we will give a positive answer to these questions and obtain some estimations for c and γ.

The basic idea is follows: First we choose two elements from $< f, g >$ and apply the Jørgensen's inequality on them. Then we use the Jørgensen's Lie product transformation (defined below) to get some new Möbius transformations which still generate a non-elementary discrete group. Applying the Jørgensen's inequality once again and comparing two inequalities, we will get some new results. But it is very hard to find the sharp lower bounds of (1.12) and (1.13) for specific a, b and α and β.

§1.5. A very important tool used in this thesis is the Jørgensen's Lie product [J-3]. Let the matrices A and B represent f and g in $SL(2,\mathbb{C})$ respectively. Suppose $\mathbf{tr}(ABA^{-1}B^{-1}) \neq 2$. (This hypothesis always holds when $< f, g >$ is non-elementary discrete.) Because $\mathbf{det}(AB - BA) = 2 - \mathbf{tr}(ABA^{-1}B^{-1}) \neq 0$, the matrix $AB - BA$ defines a Möbius transformation. Denote it by $\phi = AB - BA$, which is called the Jørgensen's Lie product of

A and B. We know that ϕ is elliptic element of order 2 with the following properties :

$$\phi^{-1}A\phi = A^{-1}, \ \ \phi^{-1}B\phi = B^{-1}. \tag{1.14}$$

The group $< A, B >$ has index at most two in the group $< A, B, \phi >$ and thus both groups are simultaneously discrete or nondiscrete.

Suppose $< A, B >$ is nonelementary discrete, Jørgensen chose A and $B\phi$ as new generators to obtain the following result [J-3]:

Proposition 4. *Let f and g be two Möbius transformations that generate a non-elementary discrete group. Then*

$$|\,\mathbf{tr}^2(f) - 4| + |\,\mathbf{tr}(fgfg^{-1}) - 2| \geq 1. \tag{1.15}$$

Here we want to ask: *Can we choose some other generators?*

Because the trace function is not well defined in **Möb**, we have to be careful when we use this function. Notice that when we regard ϕ as a matrix of $SL(2, \mathbb{C})$, $\phi^2 = -I$. If we take $F = AB\phi$ and $G = A^{-1}$, then

$$\begin{aligned}
&\mathbf{tr}^2(F) = \mathbf{tr}(AB\phi)^2 + 2 = -\,\mathbf{tr}(ABA^{-1}B^{-1}) + 2,\\
&\mathbf{tr}(FGF^{-1}G^{-1}) = \mathbf{tr}(ABAB^{-1}),\\
&\mathbf{tr}(FGFG^{-1}) = -\,\mathbf{tr}(A^2) = -\,\mathbf{tr}^2(A) + 2.
\end{aligned} \tag{1.16}$$

Notice when $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 2$ and $\mathbf{tr}(fgfg^{-1}) \neq 2$, the inequality (1.15) still holds. Applying this result to $< AB\phi, A^{-1} >$ we get:

Proposition 5. *Suppose that the Möbius transformations f and g generate a discrete group. If* $\mathbf{tr}(f) \neq 0$ *and* $\mathbf{tr}(fgfg^{-1}) \neq 2$*, then*

$$|\,\mathbf{tr}^2(f)| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2| \geq 1. \tag{1.17}$$

This result is very important for the proof in chapter 4.

In fact Proposition 5 is equivalent to Jørgensen's inequality under the Jørgensen Lie product transformation. Let $< f, g >$ be discrete. If $\mathbf{tr}(f) \neq 0$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 2$, then we can aplly Proposition 5 to $F = A$ and $G = B\phi$ to obtain

$$|\,\mathbf{tr}^2(f)| + |\,\mathbf{tr}(fgfg^{-1}) + 2| \geq 1. \tag{1.18}$$

Now suppose $< f, g >$ is discrete and non-elementary. Because the hypothesis of non-elementary implies $\mathbf{tr}(fgf^{-1}g^{-1}) - 2$, $\mathbf{tr}(fgfg^{-1}) - 2$ are both nonzero, we can apply inequality (1.18) to $F = AB\phi$ and $G = A^{-1}$ to obtain

$$|\,\mathbf{tr}^2(f) - 4| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \geq 1. \tag{1.19}$$

This is exactly the Jørgensen's inequality.

Chapter 2

General Theorem and Some Estimates

§2.1. To answer the questions in §1.4, we have the following general theorem.

Theorem 1. *For arbitrary rational numbers R and r, $0 < R, r \leq \frac{1}{2}$, there exist positive number $\alpha(R,r)$ and $\beta(R,r)$ having the following properties: Let Möbius transformations f and g generate a non-elementary discrete group. If $\mathbf{tr}^2(f) \neq 4\cos^2 r\pi$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq -2\cos 2R\pi$, then*

$$|\,\mathbf{tr}^2(f) - 4\cos^2 r\pi| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2\cos 2R\pi| \geq \alpha(R,r), \qquad (2.1)$$

if in addition $\mathbf{tr}^2(f) \neq 0$, then

$$|\,\mathbf{tr}^2(f) - 4\cos^2 r\pi| \cdot |\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2\cos 2R\pi| \geq \beta(R,r). \qquad (2.2)$$

Remark: Let us consider the group generated by

$$f(z) = \frac{(\lambda^2 + \lambda^{-2})z - 2}{2z - (\lambda^2 + \lambda^{-2})}, \quad g(z) = -\lambda^2 z \quad (\lambda \neq 0, 1) \tag{2.3}$$

For real λ, Jørgensen showed that $< f, g >$ is discrete and non-elementary [**J-3**]. It is clear that $\mathbf{tr}(f) = 0$, and $\mathbf{tr}(fgf^{-1}g^{-1}) - 2 = 4/(\lambda - \frac{1}{\lambda})^2 \to 0$ (as $\lambda \to \infty$). So the hypothesis of $\mathbf{tr}(f) \neq 0$ is necessary for Theorem 1 when $R = \frac{1}{2}$. But we do not know whether the hypothesis of $\mathbf{tr}(f) \neq 0$ can be omitted when $R \neq \frac{1}{2}$.

§2.2. Proof of theorem 1

Suppose that $< f, g >$ is discrete and non-elementary. Let matrices A and B be representatives of f and g in $SL(2, \mathbb{C})$ respectively. We can define the Lie product $\phi = AB - BA \in SL(2, \mathbb{C})$. Groups $< AB\phi, A^{-1} >$ and $< A, B\phi >$ are always discrete but not necessarily non-elementary. However we have the following observation: *Let Möbius transformations F and G generate a discrete group, if* $\mathbf{tr}(FGF^{-1}G^{-1}) \neq 2$ *and* $\mathbf{tr}(FGFG^{-1}) \neq 2$, *then $< F, G >$ satisfies the Jørgensen inequality.* Let $F = AB\phi$ and $G = A^{-1}$, then

$$\mathbf{tr}^2(F) = -\mathbf{tr}(ABA^{-1}B^{-1}) + 2, \quad \mathbf{tr}(FGF^{-1}G^{-1}) = \mathbf{tr}(ABAB^{-1}),$$

$$\mathbf{tr}(FGFG^{-1}) = -\mathbf{tr}^2(A) + 2. \tag{2.4}$$

Let $F = A$ and $G = B\phi$, then [**J-3**]

$$\mathbf{tr}(FGF^{-1}G^{-1}) = \mathbf{tr}(ABAB^{-1}), \quad \mathbf{tr}(FGFG^{-1}) = \mathbf{tr}(ABA^{-1}B^{-1}). \tag{2.5}$$

So we can always apply the Jørgensen inequality to $< AB\phi, A^{-1} >$ under the condition of $\mathbf{tr}(A) \neq 0$ and to $< A, B\phi >$ without assumption. In the following, if there is no danger of confusion, we still write f and g in stead of A and B. For any rational number $r \in (0, \frac{1}{2}]$, let $r = \frac{q}{n}$ where n and q are positive integers and coprime. First we suppose $< F, G >$ is non-elementary discrete and $F^n \neq I$. By an elementary calculation, we have

$$\mathbf{tr}(F^n G F^{-n} G^{-1}) - 2 = \prod_{j=1}^{n-1} \left(\mathbf{tr}^2(F) - 4\cos^2 \frac{j\pi}{n}\right) \cdot \Big(\mathbf{tr}(FGF^{-1}G^{-1}) - 2 \Big),$$

$$\mathbf{tr}(F^n G F^n G^{-1}) - 2 = \prod_{j=1}^{n-1} \left(\mathbf{tr}^2(F) - 4\cos^2 \frac{j\pi}{n}\right) \cdot \Big(\mathbf{tr}(FGFG^{-1}) - 2 \Big),$$

$$\mathbf{tr}^2(F^n) - 4 = \prod_{j=1}^{n-1} \left(\mathbf{tr}^2(F) - 4\cos^2 \frac{j\pi}{n}\right) \cdot \Big(\mathbf{tr}^2(F) - 4 \Big).$$

So $< F^n, G >$ also satisfies Jørgensen's inequality. Then

$$|\,\mathbf{tr}^2(F^n) - 4| + |\,\mathbf{tr}(F^n G F^{-n} G^{-1}) - 2| \geq 1. \tag{2.6}$$

It follows

$$\left|\prod_{j=1}^{n-1} \left(\mathbf{tr}^2(F) - 4\cos^2 \frac{j\pi}{n}\right)\right| \cdot \{|\,\mathbf{tr}^2(F) - 4| + |\,\mathbf{tr}(FGF^{-1}G^{-1}) - 2|\} \geq 1. \tag{2.7}$$

When $\frac{q}{n} = \frac{1}{2}$, we get

$$|\,\mathbf{tr}^2(F)| \cdot \{|\,\mathbf{tr}^2(F) - 4| + |\,\mathbf{tr}(FGF^{-1}G^{-1}) - 2|\} \geq 1. \tag{2.8}$$

When $\frac{q}{n} \neq \frac{1}{2}$, we get

$$\begin{aligned} &\left|\mathbf{tr}^2(F) - 4\cos^2 \frac{q\pi}{n}\right|^2 \cdot \left|P\left(\mathbf{tr}^2(F)\right)\right| \\ &\qquad \cdot \Big\{|\,\mathbf{tr}^2(F) - 4| + |\,\mathbf{tr}(FGF^{-1}G^{-1}) - 2|\Big\} \geq 1, \end{aligned} \tag{2.9}$$

where

$$P\left(\mathbf{tr}^2(F)\right) = \prod_{1\le j\le n-1\ j\ne q,n-q}\left(\mathbf{tr}^2(F) - 4\cos^2\frac{j\pi}{n}\right).$$

It is clear that

$$\left|P\left(\mathbf{tr}^2(F)\right)\right| \le \prod_{1\le j\le n-1\ j\ne q,n-q}\left(\left|\mathbf{tr}^2(F) - 4\cos^2\frac{q\pi}{n}\right| + 4\left|\cos^2\frac{q\pi}{n} - \cos^2\frac{j\pi}{n}\right|\right)$$

$$\le \left(\left|\mathbf{tr}^2(F) - 4\cos^2\frac{q\pi}{n}\right| + 4\right)^{n-3}.$$

Define the polynomial

$$P_r(t) = (t+4)^{n-3}.$$

Thus

$$\left|\mathbf{tr}^2(F) - 4\cos^2 r\pi\right|^2 \cdot P_r\left(\left|\mathbf{tr}^2(F) - 4\cos^2 r\pi\right|\right)$$
$$\cdot\left\{|\,\mathbf{tr}^2(F) - 4| + |\,\mathbf{tr}(FGF^{-1}G^{-1}) - 2|\right\} \ge 1.\ \ (r \ne \frac{1}{2}) \qquad (2.10)$$

If positive integer $p \ne q$ and $p < n$, then

$$\left|4\cos^2\frac{p}{n}\pi - 4\cos^2 r\pi\right|^2 \cdot P_r\left(\left|4\cos^2\frac{p}{n}\pi - 4\cos^2 r\pi\right|\right) \ge 4^{n+1}n^{-4} > 1\ \ (n \ge 3). \qquad (2.11)$$

So the hypothesis of $F^n \ne I$ for (2.10) can be replaced by $\mathbf{tr}^2(F) \ne 4\cos^2 r\pi$, $\mathbf{tr}(fgf^{-1}g^{-1}) \ne 2$ and $\mathbf{tr}(fgfg^{-1}) \ne 2$.

Similarly, if $\mathbf{tr}^2(F) \ne 4\cos^2 R\pi$ then

$$|\,\mathbf{tr}^2(F)| \cdot \{|\,\mathbf{tr}^2(F) - 4| + |\,\mathbf{tr}(FGF^{-1}G^{-1}) - 2|\} \ge 1,\ \ (R = \frac{1}{2}) \qquad (2.12)$$

or

$$\left|\mathbf{tr}^2(F) - 4\cos^2 R\pi\right|^2 \cdot P_R\left(\left|\mathbf{tr}^2(F) - 4\cos^2 R\pi\right|\right)$$
$$\cdot\left\{|\,\mathbf{tr}^2(F) - 4| + |\,\mathbf{tr}(FGF^{-1}G^{-1}) - 2|\right\} \ge 1.\ \ (R \ne \frac{1}{2}) \ (2.13)$$

Applying (2.12) and (2.13) to $F = f$ and $G = g\phi$ we get : if $\mathbf{tr}^2(f) \neq 4\cos^2 R\pi$ then

$$|\,\mathbf{tr}^2(f)| \cdot \{|\,\mathbf{tr}^2(f) - 4| + |\,\mathbf{tr}(fgfg^{-1}) - 2|\} \geq 1, \;\; (R = \frac{1}{2}) \qquad (2.14)$$

or

$$\left|\mathbf{tr}^2(f) - 4\cos^2 R\pi\right|^2 \cdot P_R\left(\left|\mathbf{tr}^2(f) - 4\cos^2 R\pi\right|\right) \cdot \left\{|\,\mathbf{tr}^2(f) - 4| + |\,\mathbf{tr}(fgfg^{-1}) - 2|\right\} \geq 1. \;\; (R \neq \frac{1}{2}) \qquad (2.15)$$

Now assume that $< f, g >$ is non-elementary discrete, $\mathbf{tr}(fgf^{-1}g^{-1}) \neq -2\cos 2R\pi$ and $\mathbf{tr}(f) \neq 0$. We can apply (2.14) and (2.15) to $< fg\phi, g^{-1} >$ to get

$$|\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \cdot \{|\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2| + |\,\mathbf{tr}^2(f)|\} \geq 1, \;\; (R = \frac{1}{2}) \qquad (2.16)$$

or

$$\left|\mathbf{tr}(fgf^{-1}g^{-1}) + 2\cos 2R\pi\right|^2 \cdot P_R\left(\left|\mathbf{tr}(fgf^{-1}g^{-1}) + 2\cos 2R\pi\right|\right) \cdot \left\{|\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2| + |\,\mathbf{tr}^2(f)|\right\} \geq 1. \;\; (R \neq \frac{1}{2}) \qquad (2.17)$$

Let

$$x = |\,\mathbf{tr}^2(f) - 4\cos^2 r\pi|, \;\; \text{and} \;\; y = |\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2\cos 2R\pi|.$$

If $< f, g >$ is non-elementary discrete, $\mathbf{tr}^2(f) \neq 4\cos^2 r\pi, 0$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq -2\cos 2R\pi$ then we have

$$x(x + y + 8) \geq 1 \quad \text{or} \quad x^2 P_r(x)(x + y + 8) \geq 1 \qquad (2.18)$$

and

$$y(x + y + 8) \geq 1 \quad \text{or} \quad y^2 P_R(y)(x + y + 8) \geq 1. \qquad (2.19)$$

Because the polynomials $P_r(x)$ and $P_R(y)$ have positive coefficients, it is easy to see that $x + y$ and xy have positive lower bounds depending only on

R and r. When $\mathbf{tr}(f) = 0$ and $r \neq \frac{1}{2}$, (2.1) is trivial. So $\alpha(R, r)$ and $\beta(R, r)$ exist. This finishes the proof of Theorem 1.

§2.3. For some specific R and r we have:

Corollary 1. *Let Möbius transformations f and g generate a non-elementary discrete group. If $\mathbf{tr}(f) \neq 0$, then*

$$|\mathbf{tr}^2(f)| + |\mathbf{tr}(fgf^{-1}g^{-1}) - 2| > 2(\sqrt{2} - 1) = 0.8284\cdots \tag{2.20}$$

and

$$|\mathbf{tr}^2(f)| \cdot |\mathbf{tr}(fgf^{-1}g^{-1}) - 2| > 0.1354. \tag{2.21}$$

Moreover if in addition $\mathbf{tr}(g) \neq 0$, then

$$|\mathbf{tr}^2(f)| + |\mathbf{tr}^2(g)| + |\mathbf{tr}(fgf^{-1}g^{-1}) - 2| > 1.5407 \tag{2.22}$$

and

$$\frac{1}{2}\{|\mathbf{tr}^2(f)| + |\mathbf{tr}^2(g)|\} + |\mathbf{tr}(fgf^{-1}g^{-1}) - 2| > 0.9706 \tag{2.23}$$

Corollary 2. *Let Möbius transformations f and g generate a non-elementary discrete group. If $\mathbf{tr}^2(f) \neq 1$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 1$, then*

$$|\mathbf{tr}^2(f) - 1| + |\mathbf{tr}(fgf^{-1}g^{-1}) - 1| > 0.9032 \tag{2.24}$$

if in addition $\mathbf{tr}(f) \neq 0$, then

$$|\mathbf{tr}^2(f) - 1| \cdot |\mathbf{tr}(fgf^{-1}g^{-1}) - 1| > 0.2039. \tag{2.25}$$

Corollary 3. *Let Möbius transformations f and g generate a non-elementary discrete group. If $f^4 \neq I$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 0$, then*

$$|\mathbf{tr}^2(f) - 2| + |\mathbf{tr}(fgf^{-1}g^{-1})| > 0.6131, \tag{2.26}$$

$$|\mathbf{tr}^2(f)-2|\cdot|\mathbf{tr}(fgf^{-1}g^{-1})|>0.09398, \tag{2.27}$$

$$|\mathbf{tr}^2(f)-2|+|\mathbf{tr}(fgf^{-1}g^{-1})-2|>0.6757, \tag{2.28}$$

$$|\mathbf{tr}^2(f)-2|\cdot|\mathbf{tr}(fgf^{-1}g^{-1})-2|>0.03819. \tag{2.29}$$

§2.4. Proof of Corollaries

First we suppose $<F,G>$ is non-elementary discrete and $\mathbf{tr}(G)\neq 0$, then $<F,G^2>$ is also non-elementary. So

$$|\mathbf{tr}^2(F)-4|+|\mathbf{tr}(FG^2F^{-1}G^{-2})-2|\geq 1$$

or

$$|\mathbf{tr}^2(F)-4|+|\mathbf{tr}^2(G)|\cdot|\mathbf{tr}(FGF^{-1}G^{-1})-2|\geq 1. \tag{2.30}$$

Now if $\mathbf{tr}(f)\neq 0$, let $F=fg\phi$ and $G=f^{-1}$. From (2.34) we get

$$|\mathbf{tr}(fgf^{-1}g^{-1})+2|+|\mathbf{tr}^2(f)|\cdot|\mathbf{tr}(fgfg^{-1})-2|\geq 1. \tag{2.31}$$

Then applying (2.31) to $<f,g\phi>$ yields

$$|\mathbf{tr}(fgfg^{-1})+2|+|\mathbf{tr}^2(f)|\cdot|\mathbf{tr}(fgf^{-1}g^{-1})-2|\geq 1. \tag{2.32}$$

Let

$$x=|\mathbf{tr}^2(f)| \text{ and } y=|\mathbf{tr}(fgf^{-1}g^{-1})-2|.$$

then

$$|\mathbf{tr}(fgfg^{-1})+2|=|\mathbf{tr}^2(f)-\mathbf{tr}(fgf^{-1}g^{-1})+2|\leq x+y.$$

Thus

$$xy+x+y\geq 1$$

So

$$x \geq \frac{1-y}{1+y} \quad \text{or} \quad y \geq \frac{1-x}{1+x}. \tag{2.33}$$

Thus

$$x+y \geq x+\frac{1-x}{1+x} = \frac{1+x^2}{1+x} \geq 2(\sqrt{2}-1). \tag{2.34}$$

If the equality holds, then $x = y = \sqrt{2}-1$. Jørgensen and Kiikka proved if Jørgensen's inequality holds with the equal sign, then f must be elliptic or parabolic. [**J-K**] So we get

$$\mathbf{tr}^2(f) = \sqrt{2}-1. \tag{2.35}$$

However, (2.35) means f is an elliptic element of infinite order, this contradicts discreteness. That proves (2.23).

Now From (2.8) and (2.16), we get

$$x(x+y+4) \geq 1 \quad \text{and} \quad y(x+y+4) \geq 1. \tag{2.36}$$

So we have

$$xy \geq 1-4x-x^2 \quad \text{and} \quad xy \geq 1-4y-y^2. \tag{2.37}$$

If x and y are both bigger than $\sqrt{2}-1$. we have

$$xy \geq (\sqrt{2}-1)^2 > 0.1715\cdots. \tag{2.38}$$

If $x \leq \sqrt{2}-1$, then

$$xy \geq \max_{0\leq x\leq\sqrt{2}-1} \{1-4x-x^2, \frac{x(1-x)}{1+x}\} > 0.1354634. \tag{2.39}$$

If $y \leq \sqrt{2}-1$, then

$$xy \geq \max_{0\leq y\leq\sqrt{2}-1} \{1-4y-y^2, \frac{y(1-y)}{1+y}\} > 0.1354634. \tag{2.40}$$

Moreover if in addition $\mathbf{tr}^2(g) \neq 0$, applying the above result to group $< f, g^2 >$ we have

$$|\,\mathbf{tr}^2(f)| \cdot |\,\mathbf{tr}^2(g)| \cdot |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| > 0.1354634. \tag{2.41}$$

So

$$\begin{aligned} &|\,\mathbf{tr}^2(f)| + |\,\mathbf{tr}^2(g)| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \\ &\qquad > 3\{|\,\mathbf{tr}^2(f)| \cdot |\,\mathbf{tr}^2(g)| \cdot |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2|\}^{\frac{1}{3}} \\ &\qquad > 1.5407, \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} &\frac{1}{2}\{|\,\mathbf{tr}^2(f)| + |\,\mathbf{tr}^2(g)|\} + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \\ &\qquad > \sqrt{\frac{0.1354634}{|\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2|}} + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \\ &\qquad > 0.9706. \end{aligned} \tag{2.43}$$

That finishes the proof of the Corollary 1.

Now we suppose $\mathbf{tr}^2(f) \neq 1$, $\mathbf{tr}^2(f) \neq 0$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 1$, we have

$$|\,\mathbf{tr}^2(f) - 1|^2 \cdot \{|\,\mathbf{tr}^2(f) - 4| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2|\} \geq 1. \tag{2.44}$$

and

$$|\,\mathbf{tr}(fgf^{-1}g^{-1}) - 1|^2 \cdot \{|\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2| + |\,\mathbf{tr}^2(f)|\} \geq 1. \tag{2.45}$$

Let

$$x = |\,\mathbf{tr}^2(f) - 1| \quad \text{and} \quad y = |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 1|.$$

Then

$$x^2(x + y + 4) \geq 1 \quad \text{and} \quad y^2(x + y + 4) \geq 1. \tag{2.46}$$

Let

$$D = \{(x,y) : x, y > 0,\ x^2(x+y+4) \geq 1 \text{ and } \ y^2(x+y+4) \geq 1\},$$

$$L_1 = \{(x,y) : x, y > 0,\ x^2(x+y+4) = 1\},$$

$$L_2 = \{(x,y) : x, y > 0,\ y^2(x+y+4) = 1\}.$$

It is clear that $\min_D \{x+y\}$ and $\min_D \{xy\}$ only occur on the boundary of D. Observing that $x+y$ and xy are decreasing functions of x (or of y) on the curve L_1 (or on L_2). So only at the common point of L_1 and L_2 can they attain their minimal values in D. Solving for the coordinates of this point, we get $x = y = 0.4516059\cdots$. Thus

$$x + y > 0.9032118\cdots, \quad \text{and} \quad xy > 0.2039478\cdots . \tag{2.47}$$

If $\mathbf{tr}(f) = 0$, then (2.24) is trivial. That proves Corollary 2.

Now we suppose $f^4 \neq I$ and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 0$, we have

$$|\,\mathbf{tr}^2(f)| \cdot |\,\mathbf{tr}^2(f) - 2|^2 \cdot \{|\,\mathbf{tr}^2(f) - 4| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2|\} \geq 1 \tag{2.48}$$

and

$$|\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \cdot |\,\mathbf{tr}(fgf^{-1}g^{-1})|^2 \cdot \{|\,\mathbf{tr}(fgf^{-1}g^{-1}) + 2| + |\,\mathbf{tr}^2(f)|\} \geq 1. \tag{2.49}$$

Let

$$x = |\,\mathbf{tr}^2(f) - 2| \quad \text{and} \quad y = |\,\mathbf{tr}(fgf^{-1}g^{-1})|.$$

We have

$$x^2(x+2)(x+y+4) \geq 1 \quad \text{and} \quad y^2(y+2)(x+y+4) \geq 1. \tag{2.50}$$

Solving the equations

$$x^2(x+2)(x+y+4)=1 \text{ and } y^2(y+2)(x+y+4)=1, \tag{2.51}$$

we get roots $x=y=0.3065629\cdots$. Thus

$$x+y>0.6131258\cdots, \quad xy>0.0939808\cdots, \tag{2.52}$$

and (2.26) and (2.27) follow.

If let

$$x=|\,\mathbf{tr}^2(f)-2| \text{ and } y=|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|,$$

then from (2.48) and (2.49), we have

$$x^2(x+2)(x+y+2)\geq 1 \text{ and } y(y+2)^2(x+y+6)\geq 1. \tag{2.53}$$

Also applying (2.20) and (2.21) to $<f^2,g>$ we have

$$|\,\mathbf{tr}^2(f)-2|^2+|\,\mathbf{tr}^2(f)|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|\}>2\sqrt{2}-2. \tag{2.54}$$

$$|\,\mathbf{tr}^2(f)-2|^2\cdot|\,\mathbf{tr}^2(f)|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|\}>0.1354634. \tag{2.55}$$

So

$$x^2+xy+2y>2\sqrt{2}-2 \text{ and } x^2(x+2)y>0.1354634. \tag{2.56}$$

Solving (2.53) and (2.56). we get

$$x+y>0.6757646, \quad xy>0.0381988. \tag{2.57}$$

That proves Corollary 3.

Chapter 3

Some Applications

§3.1. We can use some inequalities as non-discrete criteria for Möbius groups. For example, consider the group generated by f and g in (2.3) of chapter 2. Let $t = \lambda^2 + \lambda^{-2}$. If $|t| < 0.3108$, then

$$|\mathbf{tr}^2(fg)| \cdot |\mathbf{tr}(fgf^{-1}g^{-1}) - 2| = \left|\frac{2t}{t-2}\right|^2 < 0.13546. \tag{3.1}$$

Notice $< f, g > = < fg, g >$ and $fgf^{-1}g^{-1} = (fg)g(fg)^{-1}g^{-1}$. By (2.39) and (2.40), we conclude that the group $< f, g >$ is not discrete when $0 < |\lambda^2 + \lambda^{-2}| < 0.3108$.

§3.2. The first important application is to generalize Shimizu-Leutbecher's lemma. We consider the group generated by

$$f(z) = z + 1, \quad g(z) = \frac{az + b}{cz + d} \quad (ad - bc = 1, c \neq 0). \tag{3.2}$$

It is well-known if $< f, g >$ is discrete then $|c| \geq 1$. George proved the following result: Let c be real or pure imaginary and $1 < |c| < 2$, then only when $|c| = |2 \cos R\pi|$ (R rational), can $< f, g >$ be discrete [**Ge**]. It is clear that $\mathbf{tr}^2(f) = 4$ and $\mathbf{tr}(fgf^{-1}g^{-1}) = 2 + c^2$. Let rational numbers $R, r \in (0, \frac{1}{2}]$. Suppose that $c^2 + 4\cos^2 R\pi \neq 0$, then from (2.2)

$$\left|c^2 + 4\cos^2 R\pi\right| \geq \frac{\beta(R,r)}{4\sin^2 r\pi}. \tag{3.3}$$

If $c^2 - 4\cos^2 R\pi \neq 0$, applying (3.3) to $< f, g\phi >$ we have

$$\left|c^2 - 4\cos^2 R\pi\right| \geq \frac{\beta(R,r)}{4\sin^2 r\pi}. \tag{3.4}$$

So we obtain the following theorem which generalizes Shimizu-Leutbecher's lemma.

Theorem 2 *For an arbitrary rational number $R \in (0, \frac{1}{2}]$, there exists a positive constant $\sigma(R)$ having the following property: If the group generated by $f(z) = z + 1$ and $g(z) = \frac{az+b}{cz+d}$ $(ad - bc = 1, c \neq 0)$ is non-elementary discrete and $c^2 \neq \mp 4\cos^2 R\pi$, then*

$$\left|c^2 \pm 4\cos^2 R\pi\right| \geq \sigma(R). \tag{3.5}$$

We know $\sigma(\frac{1}{2}) = 1$. By elementary calculation $\sigma(\frac{1}{3}) > 0.7548$, $\sigma(\frac{1}{4}) > 0.2654$ and $\sigma(\frac{1}{6}) > 0.1181$. That means

$$|c^2 \pm 1| > 0.7548, \quad |c^2 \pm 2| > 0.2654, \quad |c^2 \pm 3| > 0.1181 \tag{3.6}$$

provided that the left sides are non-zero. When $R = \frac{1}{2}$, Theorem 2 reduces to Shimizu-Leutbecher's lemma.

§3.3. Now we try to get the geometric meaning of theorem 1. Let $f = \frac{az+b}{cz+d}$ $(ad - bc = 1)$. Then the matrx $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$.

We define the norm of f to be $\|A\| = \sqrt{|a|^2+|b|^2+|c|^2+|d|^2}$. After the choosing of A representing f, we define the norm of $f-I$ to be $\|A-I\| = \sqrt{|a-1|^2+|b|^2+|c-1|^2+|d|^2}$. Because the choices of A is not unique, $\|f-I\|$ is not well-defined. However we have the following

Theorem 3 *Let Möbius transformations f and g generate a non-elementary discrete group. If $f^2 \neq I$, then*

$$|\,\mathbf{tr}(f)| \cdot \|f\| \cdot \|g-I\| > 0.36805. \tag{3.7}$$

where $\|g-I\|$ is to be interpreted as $\|B-I\|$ for either choice of matrix B in $SL(2,\mathbb{C})$ representing g.

Proof. Let matrices A and B be defined as above. First we suppose that A and B are neither parabolic nor $\pm I$. Following Waterman's argument [**W**], we may conjugate A and B by $T \in SL(2,\mathbb{C})$ such that the common perpendicular of the axes of TAT^{-1} and TBT^{-1} is the j-axis and $\|TAT^{-1}\| \leq \|A\|$, $\|TBT^{-1}\| \leq \|B\|$. It is clear that $\|TBT^{-1}-I\| \leq \|B-I\|$. So we can assume

$$A = \begin{pmatrix} c & \alpha s \\ -s/\alpha & c \end{pmatrix}, \; c = \cos(u+iv), \; s = \sin(u+iv), \; \alpha \in \mathbf{C}-\{0\},$$

$$B = \begin{pmatrix} \tilde{c} & \beta\tilde{s} \\ -\tilde{s}/\beta & \tilde{c} \end{pmatrix}, \; \tilde{c} = \cos(x+iy), \; \tilde{s} = \sin(x+iy), \; \beta \in \mathbf{C}-\{0\}. \tag{3.8}$$

Then

$$\begin{aligned} |\,\mathbf{tr}^2(A)| \cdot |\,\mathbf{tr}(ABA^{-1}B^{-1})-2| &= 4|c|^2 \cdot \left| s\tilde{s}\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\right|^2 \\ &\leq 4|cs\tilde{s}|^2\left(|\alpha^2|+\frac{1}{|\alpha|^2}\right)\cdot\left(|\beta^2|+\frac{1}{|\beta|^2}\right), \end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
&|\,\mathbf{tr}^2(A)| \cdot \|A\|^2 \cdot \|B-I\|^2 \\
&\quad = 4|c|^2 \left[2|c|^2 + |s|^2 \left(|\alpha^2| + \frac{1}{|\alpha|^2}\right)\right] \cdot \left[2|\bar{c}-1|^2 + |\bar{s}|^2 \left(|\beta^2| + \frac{1}{|\beta|^2}\right)\right] \\
&\quad \geq |\,\mathbf{tr}^2(A)| \cdot |\,\mathbf{tr}(ABA^{-1}B^{-1}) - 2| > 0.1354634. \qquad (3.10)
\end{aligned}
$$

So

$$|\,\mathbf{tr}(A)| \cdot \|A\| \cdot \|B-I\| > \sqrt{0.1354634} > 0.36805. \qquad (3.11)$$

When one of A and B is parabolic, we can assume by conjugation with an element of $SU(2,\mathbb{C})$ [**B**, p108], that

$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$|\,\mathbf{tr}(A)| \cdot \|A\| \cdot \|B-I\| \geq 2\|A-I\| \cdot \|B-I\| \geq 2. \qquad (3.12)$$

$\square$

The geometric meaning of Theorem 3 is very clear. If we use the quantity $|\,\mathbf{tr}(f)| \cdot \|f\|$ to measure the distance from f to the set of elements of order 2. Then (3.7) means that if f is too closed to the set of elements of order 2 then g can not be too closed to the identity. Compare with the famous inequality (due to Gehring)

$$\|f-I\| \cdot \|g-I\| \geq 2-\sqrt{3}$$

it is possible that $\|f-I\|$ is very largr while $|\,\mathbf{tr}(f)| \cdot \|f\|$ is arbitrarily small.

Also, from (2.2) we can get:

Theorem 4 *For any non-elementary discrete group, let*

$$\eta_n = \max_{g \neq I \; q \leq n} \frac{\beta(\frac{1}{2}, \frac{q}{n})}{\sqrt{2}(\|g - I\|^2 + \|g - I\|)}. \tag{3.13}$$

If $f^n \neq I$, then

$$\left|\mathbf{tr}^2(f) - 4\cos^2 \frac{q\pi}{n}\right| \cdot \|f\|^2 > \eta_n. \tag{3.14}$$

Inequality (3.14) clearly shows: In any non-elementary discrete group, the elliptic elements are isolated not only in the sense of norm, also in the sense in trace.

Chapter 4

Some Sharp Results

§4.1. It is difficult to get the accurate value of $\alpha(R,r)$ and $\beta(R,r)$ for specific R and r. In this chapter, we obtain some sharp estimates as follows:

Theorem 5. *Suppose that the Mõbius transformations f and g generate a discrete group. If $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 1$, then*

$$|\,\mathbf{tr}^2(f) - 2| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 1| \geq 1; \tag{4.1}$$

if $\mathbf{tr}(fgf^{-1}g^{-1}) = 1$ and $\mathbf{tr}^2(f) \neq 2$,then

$$|\,\mathbf{tr}^2(f) - 2| > \frac{1}{2}. \tag{4.2}$$

Theorem 6. *Suppose that the Mõbius transformations f and g generate a discrete group. If $\mathbf{tr}^2(f) \neq 1$, then*

$$|\,\mathbf{tr}^2(f) - 1| + |\,\mathbf{tr}(fgf^{-1}g^{-1})| \geq 1; \tag{4.3}$$

if $\mathbf{tr}^2(f) = 1$, then

$$|\,\mathbf{tr}(fgf^{-1}g^{-1})| > \frac{1}{2} \quad \text{or} \quad \mathbf{tr}(fgf^{-1}g^{-1}) = 0; \tag{4.4}$$

and

$$|\,\mathbf{tr}(fgf^{-1}g^{-1})-1| > \frac{1}{2} \quad \textit{or} \quad \mathbf{tr}(fgf^{-1}g^{-1}) = 1. \tag{4.5}$$

Theorem 7. *Suppose that the Mõbius transformations f and g generate a discrete group. If $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 1$, then*

$$|\,\mathbf{tr}^2(f)-\mathbf{tr}(fgf^{-1}g^{-1})| + |\,\mathbf{tr}(fgf^{-1}g^{-1})-1| \geq 1; \tag{4.6}$$

if $\mathbf{tr}(fgf^{-1}g^{-1}) = 1$ and $\mathbf{tr}^2(f) \neq 1$, then

$$|\,\mathbf{tr}^2(f)-1| > \frac{1}{2}. \tag{4.7}$$

Remark 1. If we take

$$f(z) = iz, \qquad\qquad g(z) = 2z,$$

then the lower bounds of (4.1) and (4.6) are attained. We see also (4.3) is sharp by taking

$$f(z) = -z, \qquad\qquad g(z) = \frac{z-1}{z+1}. \tag{4.8}$$

Recent work of Maskit in response to a question of Gehring and Martin (see [**G-M**]) shows that inequalities (4.1), (4.3) and (4.6) are sharp for non-elementary groups [**M1**]. However, the lower bounds $\frac{1}{2}$ of (4.2), (4.4), (4.5) and (4.7) are not the best possible. In fact, the above theorems are equivalent under the Lie product transformation.

Remark 2. Take the discrete group G_1 generated by

$$f(z) = iz, \qquad g(z) = \frac{z-1}{z+1}, \tag{4.9}$$

the discrete group G_2 generated by

$$f(z)=\frac{z-i}{z+i}, \qquad g(z)=-\frac{z-i}{z+i} \tag{4.10}$$

and the discrete group G_3 generated by

$$f(z)=\frac{z-i}{z+i}, \qquad g(z)=\frac{iz+1}{z+i}. \tag{4.11}$$

The group G_i shows that hypothesis for Theorem i+4 is necessary.

Remark 3. Gehring and Martin have worked on the same problem and have established a general result which contains many inequalities including (4.1), (4.6) as well as Jørgensen's inequality [**G-M**].

§4.2. Proof of the theorems.

The proof is based on the following lemma:

Lemma *Suppose that the Möbius transformations f and g generate a discrete group. If* $\mathbf{tr}(fgfg^{-1})\neq 1$, *then*

$$|\,\mathbf{tr}^2(f)-2|+|\,\mathbf{tr}(fgfg^{-1})-1|\geq 1; \tag{4.12}$$

if $\mathbf{tr}(fgfg^{-1})=1$ and $\mathbf{tr}^2(f)\neq 2$,then

$$|\,\mathbf{tr}^2(f)-2|>\frac{1}{2}. \tag{4.13}$$

We will prove the lemma in §4.3. In this section we use it to prove Theorem 5 to 7.

If $\mathbf{tr}(fgfg^{-1}) = 2$, then there is nothing to prove. Next we suppose that $\mathbf{tr}(fgfg^{-1}) \neq 2$. So we can define a new Möbius transformation ϕ as before, which is the Lie product of f and g.

Now applying the Lemma to f and $g\phi$ yields Theorem 5.

Applying the Lemma to $fg\phi$ and f^{-1} yields (4.3) and (4.4), while applying (4.4) to f and $g\phi$ yields (4.5). Then applying Theorem 1 to $fg\phi$ and f^{-1}, we have : if $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 1$, then

$$|\mathbf{tr}(fgf^{-1}g^{-1})| + |\mathbf{tr}(fgfg^{-1}) - 1| \geq 1; \tag{4.14}$$

if $\mathbf{tr}(fgfg^{-1})$=1 and $\mathbf{tr}(fgf^{-1}g^{-1}) \neq 0$, then

$$|\mathbf{tr}(fgf^{-1}g^{-1})| > \frac{1}{2}. \tag{4.15}$$

Replacing f and g by f and $B\phi$ in (4.14) and (4.15), we get Theorem 7.

If we continue the same trick, we come back to (4.1), (4.3) or (4.12).

§4.3. Proof of the Lemma

For the proof of Lemma, we need only to consider the case where f is elliptic or loxodromic. Let f and g be represented respectively by the matrices A and B in $SL(2, C)$. The inequalities are invariant under conjugation and so we can assume A is of the form

$$A = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \quad (u \neq -1, 0, 1) \text{ and } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1) .$$

Then

$$\mathbf{tr}^2(A) = \left(u + \frac{1}{u}\right)^2, \quad \mathbf{tr}(ABAB^{-1}) = ad\left(u - \frac{1}{u}\right)^2 + 2$$

and

$$\mathbf{tr}(ABA^{-1}B^{-1}) = u^2 + \frac{1}{u^2} - ad\left(u - \frac{1}{u}\right)^2. \tag{4.16}$$

Now we suppose that $\mathbf{tr}(ABAB^{-1}) \neq 1$ and (4.12) is false. Then

$$\begin{aligned} r &= |\,\mathbf{tr}^2(A) - 2| + |\,\mathbf{tr}(ABAB^{-1}) - 1| \\ &= \left|u^2 + \frac{1}{u^2}\right| + \left|1 + ad(u - \frac{1}{u})^2\right| < 1. \end{aligned} \tag{4.17}$$

Let

$$B_1 = B,$$

$$B_{n+1} = A_n B_n A_n^{-1} B_n^{-1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} \quad (n = 1, 2, 3, \ldots).$$

Then we have

$$\begin{aligned} a_{n+1} &= a_n d_n (1 - u^2) + u^2, \\ b_{n+1} &= a_n b_n (u^2 - 1), \\ c_{n+1} &= c_n d_n \left(\frac{1}{u^2} - 1\right), \\ d_{n+1} &= a_n d_n \left(1 - \frac{1}{u^2}\right) + \frac{1}{u^2}. \end{aligned} \tag{4.18}$$

Hence

$$1 + a_{n+1} d_{n+1} \left(u - \frac{1}{u}\right)^2 = \left[1 + a_n d_n \left(u - \frac{1}{u}\right)^2\right] \left[u^2 + \frac{1}{u^2} - 1 - a_n d_n \left(u - \frac{1}{u}\right)^2\right]. \tag{4.19}$$

Let $t = u^2 + \frac{1}{u^2}$ and $z_n = 1 + a_n d_n (u - \frac{1}{u})^2$, then (4.19) becomes

$$z_{n+1} = z_n(t - z_n). \tag{4.20}$$

So

$$|z_2| \le |z_1|(|t| + |z_1|) = |z_1| r. \tag{4.21}$$

Then by induction, we have

$$|z_{n+1}| \le |z_n| r \le |z_1| r^n. \tag{4.22}$$

Observe that z_n tends to zero as n tends to ∞. So from (4.18)

$$a_{n+1} = \frac{u^2(u^2 - z_n)}{u^2 - 1} \to \frac{u^4}{u^2 - 1} \ (n \to \infty), \tag{4.23}$$

$$d_{n+1} = \frac{(u^2 z_n - 1)}{u^2(u^2 - 1)} \to \frac{-1}{u^2(u^2 - 1)} \ (n \to \infty). \tag{4.24}$$

Now we consider two cases.

Case 1. Suppose that $z_n \neq 0$ for all n. Then (4.22) implies that the z_n are distinct and hence that the same is true of a_n and d_n.

If f is elliptic, then $|u| = 1$. Therefore for some constant K

$$|b_{n+2}| = |u^2 - z_n||b_{n+1}| \le (1 + Kr^n)|b_{n+1}|$$

and

$$|c_{n+2}| = |1 - u^2 z_n||c_{n+1}| \le (1 + Kr^n)|c_{n+1}|. \tag{4.25}$$

Hence the sequences $\{b_n\}, \{c_n\}$ are bounded.

If f is loxodromic, then $|u| \neq 1$. Notice that the distinctness of $\{z_n\}$ implies for any n, $b_n \neq 0$. So we can choose an integer $k = k(n)$ such that

$$1 \le |u^{2k} b_n| \le |u^2| + \left|\frac{1}{u^2}\right|. \tag{4.26}$$

Thus

$$A^k B_n A^{-k} = \begin{pmatrix} a_n & u^{2k} b_n \\ u^{-2k} c_n & d_n \end{pmatrix}$$

has bounded entries .

Hence we can find a subsequence of $< A, B >$ which is distinct and bounded. This contradicts discreteness.

The above argument explicitly follows Jørgensen. (Also see [**B**].)

Case 2. Suppose that z_n=0 for some n.

The hypothesis $\mathbf{tr}(fgfg^{-1}) \neq 1$ means $z_1 \neq 0$. From (4.20) we can find an integer N such that

$$z_n = 0 \quad \text{for } n > N \quad \text{and} \quad z_n \neq 0 \;\text{ for }\; n \leq N. \tag{4.27}$$

So

$$t = z_N \neq 0 \;\text{ and }\; |t| = \frac{1}{2}(|t| + |z_N|) \leq \frac{1}{2}(|t| + |z_1|) < \frac{1}{2}. \tag{4.28}$$

Also

$$a_n = \frac{u^4}{u^2 - 1} \quad \text{and} \quad d_n = \frac{-1}{u^2(u^2-1)} \quad \text{for } \; n > N + 1.$$

Thus we get a matrix M in $< A, B >$ of the form

$$M = \begin{pmatrix} \frac{u^4}{u^2 - 1} & b^* \\ c^* & \frac{-1}{u^2(u^2-1)} \end{pmatrix}, \tag{4.29}$$

where $b^* c^* = -\frac{u^4 - u^2 + 1}{(u^2-1)^2}$.

By computation

$$M^2 = \begin{pmatrix} \dfrac{u^8}{(u^2-1)^2} + b^*c^* & * \\ * & \dfrac{1}{u^4(u^2-1)^2} + b^*c^* \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{u^6+u^4+1}{u^2-1} & * \\ * & -\dfrac{u^6+u^2+1}{u^4(u^2-1)} \end{pmatrix}. \tag{4.30}$$

Now let B_1 be replaced by M^2 and return to the iteration of B_n. Let z_n be defined as above corresponding to the new sequence. Then

$$z_1 = 1 + a_1 d_1 \left(u - \frac{1}{u}\right)^2$$

$$= 1 - \frac{(u^6+u^4+1)(u^6+u^2+1)}{u^6}$$

$$= -\frac{u^{12}+u^{10}+u^8+2u^6+u^4+u^2+1}{u^6}$$

$$= -\left[\left(u^6+\frac{1}{u^6}\right)+\left(u^4+\frac{1}{u^4}\right)+\left(u^2+\frac{1}{u^2}\right)+2\right]$$

$$= -\left[\left(u^2+\frac{1}{u^2}\right)^3+\left(u^2+\frac{1}{u^2}\right)^2-2\left(u^2+\frac{1}{u^2}\right)\right]$$

$$= -t(t^2+t-2) \tag{4.31}$$

and

$$z_2 = z_1(t - z_1) = -t^2(t^2 + t - 2)(t^2 + t - 1)$$
$$= -t^2(t-1)(t+2)(t^2 + t - 1). \tag{4.32}$$

Let

$$D_1 = \left\{ t: \ 0 < |t| \le \frac{1}{2} \quad \text{and} \quad |t(t-1)(t+2)(t^2+t-1)| < 1 \right\}. \tag{4.33}$$

If $t \in D_1$, we have $0 < |z_2| < |t|$. From (4.20)

$$|z_n|(|t| - |z_n|) \le |z_{n+1}| \le |z_n|(|t| + |z_n|).$$

By induction we get

$$0 < |z_n| \le |z_2| < |t| \quad \text{and} \quad |t| + |z_n| \le |t| + |z_2| < 1 \quad \text{for} \quad n \ge 2. \tag{4.34}$$

So we can go back to the case 1. That means if $t \in D_1$, then $< A, B >$ is not discrete.

By simple computation, we see that

$$D_1 \supset \left\{ t: \ 0 < |t| \le \frac{1}{2} \quad \text{and} \quad \frac{-\pi}{4} \le \mathbf{arg}(t) \le \frac{\pi}{4} \right\}. \tag{4.35}$$

Now from (4.29)

$$(A^{-1}M)^2 = \begin{pmatrix} \dfrac{u^6}{(u^2-1)^2} + b^*c^* & * \\ * & \dfrac{1}{u^4(u^2-1)^2} + b^*c^* \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{u^4+1}{u^2-1} & * \\ * & -\dfrac{u^4+1}{u^2(u^2-1)} \end{pmatrix}. \tag{4.36}$$

We redefine the matrix B to be

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A^3(A^{-1}M)^2 = \begin{pmatrix} \dfrac{u^3(u^4+1)}{u^2-1} & * \\ * & -\dfrac{u^4+1}{u^5(u^2-1)} \end{pmatrix}. \tag{4.37}$$

Then

$$\begin{aligned} \mathbf{tr}^2(B) = (a+d)^2 &= \frac{(u^4+1)^2(u^8-1)^2}{(u^2-1)^2u^{10}} \\ &= \frac{(u^4+1)^4(u^2+1)^2}{u^{10}} \\ &= \left(u^2+\frac{1}{u^2}\right)^2 \left(u^2+\frac{1}{u^2}+2\right) \\ &= t^4(t+2) \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} ad &= -\frac{(u^4+1)^2}{u^2(u^2-1)^2} \\ &= \frac{-t}{t-2}. \end{aligned} \tag{4.39}$$

If we set $F = A^2$, then from (4.16)

$$\begin{aligned} \mathbf{tr}(BFB^{-1}F^{-1}) = \mathbf{tr}(FBF^{-1}B^{-1}) &= u^4 + \frac{1}{u^4} - ad\left(u^4+\frac{1}{u^4}-2\right) \\ &= t^2 - 2 + \frac{t^2}{t-2}(t^2-4) \\ &= t^3 + 3t^2 - 2. \end{aligned} \tag{4.40}$$

When $0 < |t| \leq \frac{1}{2}$, we have $\mathbf{tr}^2(B) \neq 0$ and

$$\begin{aligned} \mathbf{tr}(BFBF^{-1}) - 2 &= \mathbf{tr}^2(B) - \mathbf{tr}(BFB^{-1}F^{-1}) - 2 \\ &= t^5 + 2t^4 - t^3 - 3t^2 \neq 0. \end{aligned} \tag{4.41}$$

So applying Proposition 5 of chapter 1 to B and F yields

$$|\,\mathbf{tr}^2(B)| + |\,\mathbf{tr}(BFB^{-1}F^{-1}) + 2| \geq 1. \tag{4.42}$$

Thus

$$|t^4(t+2)| + |t^3 + 3t^2| \geq 1. \tag{4.43}$$

Let

$$D_2 = \{t : \ 0 < |t| \leq \frac{1}{2} \quad \text{and} \quad |t^4(t+2)| + |t^3 + 3t^2| < 1\}. \tag{4.44}$$

If $t \in D_2$, then $< A, B >$ is not discrete.

By simple computation, we see that

$$D_2 \supset \left\{t : \ 0 < |t| \leq \frac{1}{2} \quad \text{and} \quad \frac{\pi}{4} \leq \mathbf{arg}(t) \leq \frac{7\pi}{4}\right\}. \tag{4.45}$$

Combining (4.35) with (4.45), we conclude that $< A, B >$ is not discrete. So we get (4.12).

If $\mathbf{tr}(ABAB^{-1})$=1, then $z_1 = 0$. The same argument yields $t \notin D_1 \cup D_2$. So $t = 0$ or $|t| > \frac{1}{2}$ and (4.13) follows. This finishes the proof of the Lemma.

Chapter 5

Some Conjectures

Because

$$|4\cos^2 r_1\pi - 4\cos^2 r_2\pi| + |\cos 2R_1\pi - \cos 2R_2\pi| \leq 4\pi(|r_1 - r_2| + |R_1 - R_2|),$$

thus

$$|\alpha(R_1, r_1) - \alpha(R_2, r_2)| \leq 4\pi(|r_1 - r_2| + |R_1 - R_2|).$$

So the function $\alpha(R, r)$ in Theorem 1 is a continuous function of the rational numbers R and r. Can we extend it to the whole region $(0 < R \leq \frac{1}{2}, 0 < r \leq \frac{1}{2})$? Are (2.1) and (2.2) still valid for irrational R or r? Setting $R = \frac{1}{2}$ in (2.2) , we have

$$|\,\mathbf{tr}^2(f) - 4\cos^2 r\pi| \cdot |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \geq \beta(\frac{1}{2}, r).$$

As we said in chapter 3, this inequality implies the isolated behavior of non-elementary discrete groups near elliptic Möbius transformations. But we know that every element is isolated in a discrete group. So we have the first conjecture as follows:

For every complex number a, there exists a positive number $\gamma(a)$ such that if $< f, g >$ is non-elementary discrete and $\mathbf{tr}^2(f) \neq a$, then

$$|\,\mathbf{tr}^2(f)-a|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})-2|\geq\gamma(a). \tag{5.1}$$

provided some auxiliary conditions are satisfied.

Observing that (2.1) is valid for $r=0$ and $R=\frac{1}{2}$, so we also hope to extend Theorem 1 to $R=0$ or $r=0$ and suggest the following conjecture:

There exist positive constants α_1 and β_1 such that if $< f,g >$ is non-elementary discrete and f, $fgf^{-1}g^{-1}$ are not parabolic then

$$|\,\mathbf{tr}^2(f)-4|+|\,\mathbf{tr}(fgf^{-1}g^{-1})+2|\geq\alpha_1, \tag{5.2}$$

$$|\,\mathbf{tr}^2(f)-4|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})+2|\geq\beta_1, \tag{5.3}$$

provided some auxiliary conditions are satisfied.

On the other hand, many estimations are not best possible. For example, when $\mathbf{tr}^2(f)\neq 1$ and $\mathbf{tr}(fgf^{-1}g^{-1})\neq 1$, by carefully calculating we can get a better result:

$$|\,\mathbf{tr}^2(f)-1|+|\,\mathbf{tr}(fgf^{-1}g^{-1})-1|>0.96\cdots. \tag{5.4}$$

We hope to find the sharp lower bounds of the following inequalities:

$$|\,\mathbf{tr}^2(f)-a|+|\,\mathbf{tr}(fgf^{-1}g^{-1})-b|\geq ? \tag{5.5}$$

$$|\,\mathbf{tr}^2(f)-a|\cdot|\,\mathbf{tr}(fgf^{-1}g^{-1})-b|\geq ? \tag{5.6}$$

where $a=0,1,2,3$; $b=-2,-1,0,1,2$. All these inequalities correspond to elliptic elements of low order.

The results established also suggest the following very interesting question: For what kinds of discrete groups $< f,g >$ will the left hand sides of inequalities (1.2), (4.1), (4.3) or (1.17) be zero?

The left hand side of (1.2) is zero if $f(z)=z+1$, $g(z)=z+\tau(\tau$ nonreal). Here f and g are both parabolic and $< f,g >$ has signature (1,0).

The left hand side of (4.1) is zero if $f(z) = iz$, $g(z) = \frac{z-1}{z+1}$. Here f and g are both of order 4 and $< f, g >$ has signature (0,3;2,3,4).

The left hand side of (4.3) is zero if $f(z) = \frac{z-i}{z+i}$, $g(z) = -\frac{z-i}{z+i}$. Here f and g are both of order 3 and $< f, g >$ has signature (0,3;2,3,3).

Finally the left hand side of (1.17) is zero if $f(z) = -z$, $g(z) = \frac{1}{z}$. Here f and g are both of order 2 and $< f, g >$ has signature (0,3;2,2,2).

In each of the four cases discussed above, the group $< f, g >$ is elementary. From these observations, we are led to make the following conjecture:

Let $< F, G >$ *be an elementary discrete group with* $\mathbf{tr}(F) = \mathbf{tr}(G)$*. If* $< f, g >$ *is discrete, then*

$$|\,\mathbf{tr}^2(f) - \mathbf{tr}^2(F)| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - \mathbf{tr}(FGF^{-1}G^{-1})| \geq 1, \qquad (5.7)$$

provided some appropriate auxiliary conditions are satisfied.

For example, if we take $F(z) = -z$, $G(z) = -z + 1$, the signature of $< F, G >$is (0,3;2,2,∞). Then (5.7) will become

$$|\,\mathbf{tr}^2(f)| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| \geq 1. \qquad (5.8)$$

In the chapter 2 we have: *If* $< f, g >$ *is non-elementary discrete and* $\mathbf{tr}(f) \neq 0$*, then*

$$|\,\mathbf{tr}^2(f)| + |\,\mathbf{tr}(fgf^{-1}g^{-1}) - 2| > 2(\sqrt{2} - 1) = 0.828\cdots. \qquad (5.9)$$

The sharp lower bound of (5.9) is still unknown.

At last we hope to find $|a| > 4$, $|b| > 2$ and $c > 0$ or $|\alpha| > 4$ and $|\beta| > 2$ and $\gamma > 0$ satisfying (1.12) or (1.13).

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Part Two

Some Topics on Quasiconformal Extensions

Chapter 1

The Inner Radius of Universal Teichmüller Space

§1.1. Introduction and main results

Let f be a holomorphic function defined on unit disc $U = \{z : |z| < 1\}$ and $S_f = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$ be its Schwarzian derivative. Let $k \in (0,1)$. In 1973, Ahlfors proved if we can choose an auxiliary function v satisfying $v_{\bar{z}/v^2} \neq 0$ for $|z| < 1$ and $v \to \infty$ as $|z| \to 1$, then the inequality

$$\left|\frac{1}{2}S_f + v^2 - v_z\right| \leq k|v_{\bar{z}}| \quad (|z| < 1) \tag{1.1}$$

implies the existence of a quasiconformal extension of f to the whole complex plane. [**A**]

If f is defined on the upper half plane $H = \{z : Im(z) > 0\}$. Following the Ahlfors' idea, define

$$g(z) = f(z) + \frac{f'(z)}{v(z) - \frac{1}{2}\frac{f''(z)}{f'(z)}} \quad (Im(z) > 0), \tag{1.2}$$

where $v_{\bar{z}}/v^2 \neq 0$ for $Im(z) > 0$ and $v(z) \to \infty$ as $Im(z) \to 0$.

Then by computation

$$\left|\frac{g_z}{g_{\bar{z}}}\right| = \frac{\left|\frac{1}{2}S_f + v^2 - v_z\right|}{|v_{\bar{z}}|}. \tag{1.3}$$

So if

$$\left|\frac{1}{2}S_f + v^2 - v_z\right| \leq k|v_{\bar{z}}| \;\; (Im(z) > 0), \tag{1.4}$$

then $g(\bar{z})$ $(Im(z) < 0)$ is a quasiconformal extension of $f(z)$.

Writing $\frac{1}{2}v$ instead of v, (1.1) and (1.4) becomes

$$\left|S_f - \left(v_z - \frac{1}{2}v^2\right)\right| \leq k|v_{\bar{z}}| \quad (0 < k < 1). \tag{1.5}$$

Let A be any simply connected domain in $\overline{\mathbb{C}}$ of hyperbolic type. We define the *Poincaré density* ρ_A of A by

$$\rho_A(z) = \frac{|h'(z)|}{1 - |h(z)|^2},$$

where $h(z)$ is any conformal mapping of A onto the unit disc U. For complex- valued functions ϕ on A we set the norm

$$||\phi||_A = \sup_{z \in A} \frac{|\phi(z)|}{\rho_A(z)^2}.$$

Let $F(z)$ be any meromorphic function on A. Following the idea of Nehari **[N]**, Lehto defined the *inner radius of univalence* $\sigma_I(A)$ as the supremum of the numbers $a \geq 0$ with the property that $F(z)$ is injective whenever $\|S_F\|_A \leq a$ [**O. L1**].

Let $g(z)$ be defined in U (or in H) and $A = g(U)$ (or $g(H)$) be a quasidisc. Let $\sigma_I(A)$ be the inner radius of univalence for A. Assume that $f(z)$ is any meromorphic function on U (or in H). It is clear that the inner radius of univalence $\sigma_I(A)$ is also the supremum of the numbers $a \geq 0$ with the property that $f(z)$ is injective whenever $\|S_f - S_g\|_U \leq a$ (or $\|S_f - S_g\|_H \leq a$). In the other words, $\sigma_I(A)$ is the *inner radius of universal Teichmüller space* with respect to S_g.

There are many papers concerning $\sigma_I(A)$, for example, those published by Calvis [**C**] and Lehtinen [**M. L1**] [**M. L2**] [**M. L3**]. In this chapter we want to show that the Ahlfors inequality is a very powerful tool for investigating $\sigma_I(A)$. Some special choices of v can yield valuable lower bounds for $\sigma_I(A)$ including some well-known results. In fact we obtain the following results:

Theorem 1. Let $g(z)$ be holomorphic in U and $A = g(U)$. Then

$$\sigma_I(A) \geq 2 - 2 \sup_{|z|<1} \left| z(1 - |z|^2) \left(\frac{g''}{g'} - \frac{2g'}{g+c} \right) \right| \tag{1.6}$$

and

$$\sigma_I(A) \geq 2 \inf_{|z|<1} \left| \frac{zg'}{g} \right| \left(1 - \left| \frac{zg'}{g} - 1 \right| \right) \tag{1.7}$$

where c is any complex number.

Theorem 2. Let $g(z)$ be holomorphic in H and $A = g(H)$. Then

$$\sigma_I(A) \geq 2 - 4 \sup_{Im(z)>0} \left| y \left(\frac{g''}{g'} - \frac{2g'}{g+c} \right) \right| \tag{1.8}$$

and

$$\sigma_I(A) \geq 2 \inf_{Im(z)>0} \left|\frac{zg'}{g}\right| \left(1 - \left|\frac{zg'}{g} - 1\right|\right). \tag{1.9}$$

where c is any complex number.

Remark: Let $g(z) = z + \frac{1}{4}z^2$ $z \in U$ and $A = g(U)$. Then from (1.7)

$$\sigma_I(A) \geq 2 \inf_{|z|<1} \left|\frac{1+\frac{1}{2}z}{1+\frac{1}{4}z}\right| \left(1 - \left|\frac{\frac{1}{4}z}{1+\frac{1}{4}z}\right|\right) > \frac{8}{9}.$$

But for $c = 0$ from (1.6)

$$\sigma_I(A) \geq 2 - 2 \sup_{|z|<1} \left|z(1-|z|^2) \cdot \frac{1}{2+z}\right| > 1.13029.$$

By carefully choosing the constant c, we can get the better estimation for $\sigma_I(A)$

§1.2. Proofs of Theorems

Because the proofs are routine, we omit the details.

(i) In the case of $A = g(U)$, for any complex number c, choose

$$v = \frac{g''}{g'} - \frac{2g'}{g+c} + \frac{2\bar{z}}{1-|z|^2}.$$

Then (1.5) becomes

$$\left|S_f - S_g + \frac{2\bar{z}}{1-|z|^2}\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right)\right| \leq \frac{2k}{(1-|z|^2)^2}.$$

So

$$\sigma_I(A) \geq 2 - 2 \sup_{|z|<1} \left|z(1-|z|^2)\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right)\right|. \tag{1.10}$$

Let $c = \infty$, we have

$$\left| S_f - S_g + \frac{2\bar{z}}{1-|z|^2}\left(\frac{g''}{g'}\right) \right| \leq \frac{2k}{(1-|z|^2)^2}, \tag{1.11}$$

and

$$\sigma_I(A) \geq 2 - 2\sup_{|z|<1}\left| z(1-|z|^2)\frac{g''}{g'} \right|. \tag{1.12}$$

The inequality (1.11) was first obtained by Epstein under some additional assumptions and was proved by Pommerenke later.[**P**]

Now choose

$$v = \frac{g''}{g'} - \frac{2g'}{g(1-|z|^{-2})}\ .$$

Then (1.5) becomes

$$\left| S_f - S_g - \frac{2\bar{z}g'(zg'-g)}{g^2(1-|z|^2)^2} \right| \leq \left| \frac{2kzg'}{g(1-|z|^2)^2} \right|.$$

Thus

$$\sigma_I(A) \geq 2\inf_{|z|<1}\left|\frac{zg'}{g}\right|\left(1 - \left|\frac{zg'}{g} - 1\right|\right). \tag{1.13}$$

(ii) In the case of $A = g(H)$, choose

$$v = \frac{g''}{g'} - \frac{2g'}{g+c} - \frac{2}{z-\bar{z}}.$$

Then (1.5) becomes

$$\left| S_f - S_g + \frac{2}{z-\bar{z}}\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right) \right| \leq \frac{2k}{|z-\bar{z}|^2}.$$

We get

$$\sigma_I(A) \geq 2 - 4\sup_{Im(z)>0}\left| y\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right) \right| \tag{1.14}$$

and in the special case for $c = \infty$

$$\sigma_I(A) \geq 2 - 4 \sup_{Im(z)>0} \left| y \left(\frac{g''}{g'} \right) \right|. \tag{1.15}$$

If we choose

$$v = \frac{g''}{g'} - \frac{2g'}{g(1 - \frac{\bar{z}}{z})} .$$

Then (1.5) becomes

$$\left| S_f - S_g - \frac{2\bar{z}g'(zg' - g)}{g^2(z - \bar{z})^2} \right| \leq \left| \frac{2kzg'}{g(z - \bar{z})^2} \right| .$$

Thus

$$\sigma_I(A) \geq 2 \inf_{Im(z)>0} \left| \frac{zg'}{g} \right| \left(1 - \left| \frac{zg'}{g} - 1 \right| \right) . \tag{1.16}$$

This inequality makes sense only for $|\frac{zg'}{g} - 1| < 1$. If we take

$$g(z) = z^k = exp(k\mathbf{log}z) \ \ (z \in H, \ |k - 1| < 1, \ \mathbf{log}\, i = \frac{1}{2}\pi i) ,$$

$A = g(H)$ is a spiral-like domain for non-real k. Because $\frac{zg'}{g} = k$, we have

$$\sigma_I(A) \geq 2|k|(1 - |k - 1|). \tag{1.17}$$

When k is real, Lehtinen and Lehto obtained [**O. L2**]

$$\sigma_I(A) = 2k(1 - |k - 1|). \tag{1.18}$$

We do not know whether (1.17) is sharp for non-real k.

§1.3. A general formula

Generally, let $h(z)$ be any quasiconformal self-mapping of the whole plane. Let $g(z)$ be holomorphic in H or U. Define $\tau(z) = h(\bar{z})/h(z)$ for $A = g(H)$ and $\tau(z) = h(\frac{1}{\bar{z}})/h(z)$ for $A = g(U)$. We choose

$$v = \frac{g''}{g'} - \frac{2g'}{g(1-\tau)}.$$

Then

$$\sigma_I(A) \geq 2 \inf \frac{|g'|(|\bar{\partial}(g\tau)| - |\partial(g\tau)|)}{|g - g\tau|^2\eta^2}, \tag{1.19}$$

where $\eta = \frac{1}{2y}$ or $\frac{1}{1-|z|^2}$.

Let $h(z)$ be any quasiconformal extension of $g(z)$, denote $g^* = g(\bar{z})$ or $g(\frac{1}{\bar{z}})$, then

$$\sigma_I(A) \geq 2 \inf \frac{|g'|(|\bar{\partial}g^*| - |\partial g^*|)}{|g - g^*|^2\eta^2}. \tag{1.20}$$

This is just another form of Lehto's result. [**O. L2**, page 121]

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Chapter 2

The Dilatation of Beuring–Ahlfors Quasiconformal Extension

§2.1. Introduction and theorems

Let $\mu(x)$ be ρ-quasisymmetric, $1 \leq \rho < \infty$, that is, $\mu(x)$ is a continuous increasing function mapping the real line onto itself and satisfying

$$\frac{1}{\rho} \leq \frac{\mu(x+t)-\mu(x)}{\mu(x)-\mu(x-t)} \leq \rho \tag{2.1}$$

for all x and $t \neq 0$.

In 1956, Beuring and Ahlfors [**B–A**] using the formulas

$$\begin{aligned} u(x,y) &= \frac{1}{2}\int_0^1 [\mu(x+ty)+\mu(x-ty)]dt, \\ v(x,y) &= \frac{r}{2}\int_0^1 [\mu(x+ty)-\mu(x-ty)]dt \ \ (r>0) \end{aligned} \tag{2.2}$$

constructed the function $w(z) = u(x,y) + iv(x,y)(z = x + iy)$, which is a quusiconformal mapping from the upper half-plane onto itself having $\mu(x)$ as its boundary value. It is of interest to estimate the dilatation $K(z)$of $w(z)$.

Beuring and Ahlfors [**B-A**] first proved for some r

$$K \leq \rho^2. \tag{2.3}$$

For large ρ, (2.3) is not the best possible. In 1966, Reed [**R**] proved for $r = 1$

$$K \leq 8\rho. \tag{2.4}$$

(2.4) means for any r, the order of K with respect to ρ is not bigger than one.

In 1983, Li Zhong [**Li**] improved (2.4) as follow:

$$K \leq 4.2\rho \quad (r = 1). \tag{2.5}$$

Also in 1983, Lehtinen [**Le1**] obtained

$$K \leq 2\rho \quad (r = 1). \tag{2.6}$$

In 1966, Ahlfors [**A2**] also proved that the Beuring-Ahlfors extension function with r=1 is quasi-symmetric, i.e. there exists a constant A such that

$$\frac{1}{A} d(z_1, z_2) \leq d(w(z_1), w(z_2)) \leq A d(z_1, z_2) \tag{2.7}$$

for any z_1, z_2 in the upper half-plane, where d(.,.) denotes the non-Euclidean distance.

Ahlfors obtained

$$A \leq 4\rho^2(\rho + 1). \tag{2.8}$$

In this chapter we refine the Beuring-Ahlfors technique and obtained the following theorems:

Theorem 1. *Let $\mu(x)$ be a ρ-quasisymmetric function. Then the dilatation $K(z)$ of the Beuring-Ahlfors extension with r=1 satisfies the inequalities*

$$K \leq 2\rho - \frac{7(\rho - 1)}{6(\rho + 1)} \tag{2.9}$$

and

$$K < 2\rho - 2 + O\left(\frac{1}{\rho}\right) \tag{2.10}$$

for sufficiently large ρ.

Theorem 2. *Let $\mu(x)$ be a ρ-quasisymmetric function and let $w(z)$ be the Beuring-Ahlfors extension of $\mu(x)$ for r=1. Then*

$$\frac{1}{2\rho} d(z_1, z_2) \leq d(w(z_1), w(z_2)) \leq 2\rho\, d(z_1, z_2) \tag{2.11}$$

for any z_1, z_2 in the upper-half plane.

Remark. When $\rho = 1$ and $\mu(0) = 0$, then for any $t \neq 0$ and x we have

$$\mu(x + t) + \mu(x - t) - 2\mu(x) \equiv 0.$$

So the second derivative $\frac{d^2}{dx^2}\mu(x) \equiv 0$ *a.e.*, which means $\mu(x) = cx$ $(c > 0)$. we easily see that $w(z)$=$c(x + \frac{1}{2}yi)$ $(c > 0)$. It is evident that $K(z) = 2$ and

$$\lim_{z_1 \to z_2 \Im z_1 = \Im z_1 > 0} \frac{d(w(z_1), w(z_2))}{d(z_1, z_2)} = 2.$$

Therefore the coefficient 2 of ρ either in theorem 1 or 2 cannot be replaced by any smaller number.

§2.2. Lemma. *Let $\mu(x)$ be a ρ-quasisymmetric function normalized by $\mu(0) = 0$ and $\mu(1) = 1$. Then*

$$(1+2\rho)\xi + \beta\eta \geq 1 + \beta, \tag{2.12}$$

$$(1+2\rho)\beta\eta + \xi \geq 1 + \beta, \tag{2.13}$$

where $\beta = -\mu(-1)$, $\xi = 1 - \int_0^1 \mu(t)dt$ and $\eta = 1 + \beta^{-1}\int_{-1}^0 \mu(t)dt$.

Proof. Taking $x = t > 0$ in (2.1) we have $(1+\rho)\mu(t) \geq \mu(2t)$. Thus

$$(1+\rho)\int_0^1 \mu(t)dt \geq \int_0^1 \mu(2t)dt = \frac{1}{2}\int_0^1 \mu(t)dt + \frac{1}{2}\int_1^2 \mu(t)dt.$$

Therefore

$$(1+2\rho)\int_0^1 \mu(t)dt \geq \int_1^2 \mu(t)dt \tag{2.14}$$

Substituting $1 - \mu(1-t)$ for $\mu(t)$, we get

$$(1+2\rho)\left[1 - \int_0^1 \mu(t)dt\right] \geq \ 1 - \int_{-1}^0 \mu(t)dt.$$

This yields (2.12). Similarly, substituting $1 + \beta^{-1}\mu(t-1)$ for $\mu(t)$ yields (2.13).

§2.3. Proof of theorem 1.

Because of linear invariance we only need to estimate $K(z)$ for $x = 0$ and $y = 1$ and $\mu(x)$ normalized by $\mu(0) = 0$ and $\mu(1) = 1$. Thus the dilatation

$K = K(i)$ with $r = 1$ satisfying the equation [**A-B**]

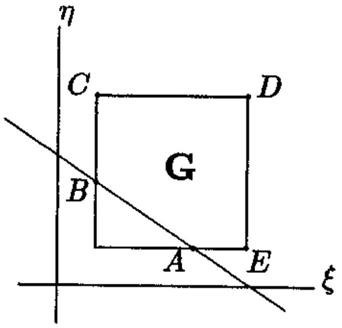
$$K + \frac{1}{K} = \frac{1}{\xi + \eta}\left[\frac{1}{\beta}(1+\xi^2) + \beta(1+\eta^2)\right] \equiv F(\xi,\eta,\beta), \tag{2.15}$$

where

$$\beta \le \rho, \quad \frac{1}{1+\rho} \le \xi, \eta \le \frac{\rho}{1+\rho}.$$

Furthermore, we can suppose $\beta \ge 1$, otherwise consider $-\frac{1}{\beta}w(-\bar{z})$.

Let $\overline{G}$ be a closed domain bounded by a polygon $ABCDE$. The side AB lies on the line of $(1+2\rho)\xi + \beta\eta = 1 + \beta$; other sides BC, CD, DE and AE lie on the lines of $\xi = \frac{1}{1+\rho}$, $\eta = \frac{\rho}{1+\rho}$, $\xi = \frac{\rho}{1+\rho}$ and $\eta = \frac{1}{1+\rho}$ respectively. It is sufficient to look at the maximum of $F(\xi,\eta,\beta)$ in $\overline{G}$.



By calculating we have

$$\frac{\partial F}{\partial \eta} = \frac{\beta(\xi+\eta)^2 - (\beta + \frac{1}{\beta})(1+\xi^2)}{(\xi+\eta)^2}, \quad \frac{\partial^2 F}{\partial^2 \eta} = \frac{2(\beta + \frac{1}{\beta})(1+\xi^2)}{(\xi+\eta)^3}.$$

Since $\frac{\partial^2 F}{\partial^2 \eta} > 0$, the max of F in $\bar{G}$ is in $CD \cup AE \cup AB$. Since $\frac{\partial^2 F}{\partial^2 \xi} > 0$, the max of F in $\bar{G}$ is in $DE \cup BC \cup AB$. So the max is in $AB \cup \{C, D, E\}$. Since $\frac{\partial F}{\partial \eta} < 0$ in BC and $\frac{\partial F}{\partial \xi} < 0$ in AE, the max is in $AB \cup \{D\}$.

When $(\xi,\eta) \in AB$, then $\beta\eta = 1+\beta-(1+2\rho)\xi$ and

$$F(\xi,\eta,\beta) = 2\cdot\frac{1+\beta+\beta^2-(1+\beta)(1+2\rho)\xi+(1+2\rho+2\rho^2)\xi^2}{1+\beta-(1+2\rho-\beta)\xi}$$
$$\equiv 2W(\xi,\beta). \tag{2.16}$$

But $W(\xi,\beta)$ is a convex function of either β or ξ, therefore the max of $W(\xi,\beta)$ must occur when $\xi = \frac{1}{1+\rho}$, $\beta = \rho$, or

$$\xi = \frac{1+\rho+\rho\beta}{(1+\rho)(1+2\rho)}, \quad 1 \le \beta \le \rho.$$

Hence we only need to consider the following cases:

Case 1: When (ξ,η) lies at the point D. Then $\xi = \eta = \frac{\rho}{1+\rho}$,

$$F\left(\frac{\rho}{1+\rho},\frac{\rho}{1+\rho},\beta\right) = \left(\beta+\frac{1}{\beta}\right)\cdot\frac{\rho^2+(1+\rho)^2}{2\rho(1+\rho)}$$
$$\le 2\rho+\frac{1}{2\rho}-\frac{(\rho-1)(2\rho^3+4\rho^2+2\rho+1)}{2\rho^2(\rho+1)}\,. \tag{2.17}$$

Case 2: When (ξ,η) lies at the point B and $\beta=\rho$. Then $\xi = \frac{1}{1+\rho}$, $\eta = \frac{\rho}{1+\rho}$,

$$F\left(\frac{1}{1+\rho},\frac{\rho}{1+\rho},\rho\right) = 2\rho-2+\frac{2}{\rho}+\frac{2}{1+\rho}-\frac{2}{(1+\rho)^2}$$
$$= 2\rho+\frac{1}{2\rho}-\frac{(\rho-1)(4\rho^2+5\rho+3)}{2\rho(\rho+1)^2}\,. \tag{2.18}$$

Case 3: When (ξ,η) lies at the point A. Then $\xi = \frac{1+\rho+\beta\rho}{(1+\rho)(1+2\rho)}$, $1 \le \beta \le \rho$, $\eta = \frac{1}{1+\rho}$,

$$F\left(\frac{1+\rho+\beta\rho}{(1+\rho)(1+2\rho)},\frac{1}{1+\rho},\beta\right)$$
$$= \frac{\beta+\frac{1}{\beta}+\frac{\beta}{(1+\rho)^2}+[\frac{1+\rho+\beta\rho}{(1+\rho)(1+2\rho)}]^2}{\frac{2+3\rho+\beta\rho}{(1+\rho)(1+2\rho)}}$$

$$\leq (1+\rho)(1+2\rho)\cdot\frac{\beta+\frac{1}{\beta}+\frac{\beta}{(1+\rho)^2}+\frac{1}{\beta}[\frac{1+\rho+\rho^2}{(1+\rho)(1+2\rho)}]^2}{2+3\rho+\beta\rho}$$
$$\equiv (1+\rho)(1+2\rho)Y(\beta,\rho)\ . \qquad (2.19)$$

Denote $\lambda(\rho)=[\frac{1+\rho+\rho^2}{(1+\rho)(1+2\rho)}]^2$. Then

$$(2+3\rho+\beta\rho)^2\frac{\partial Y}{\partial\beta}=(2+3\rho)\left[1+\frac{1}{(1+\rho)^2}\right]-[1+\lambda(\rho)]\left[\frac{2\rho}{\beta}+\frac{1}{\beta^2}(2+3\rho)\right]\ .$$

When β increases from 1 to ρ, the sign of $\frac{\partial Y}{\partial\beta}$ changes only once. Hence

$$\max_{1\leq\beta\leq\rho}Y(\beta,\rho)=\max\{Y(1,\rho),Y(\rho,\rho)\},$$

and

$$(1+\rho)(1+2\rho)Y(1,\rho)=\frac{9}{8}\rho+1+\frac{1}{1+\rho}-\frac{9}{16(1+2\rho)}-\frac{9}{16(1+2\rho)^2}$$
$$=2\rho+\frac{1}{2\rho}-\frac{(\rho-1)(7\rho^4+13\rho^3+4\rho^2-2\rho-1)}{2\rho(1+\rho)(1+2\rho)^2}\ , \qquad (2.20)$$

$$(1+\rho)(1+2\rho)Y(\rho,\rho)=2\rho-3+\frac{1}{\rho}+\frac{15}{2+\rho}-\frac{3}{1+2\rho}-\frac{4}{1+\rho}+\frac{2}{(1+\rho)^2}$$
$$=2\rho+\frac{1}{2\rho}-\frac{(\rho-1)(12\rho^4+26\rho^3+23\rho^2+9\rho+2)}{2\rho(\rho+1)^2(\rho+2)(2\rho+1)}\ . \qquad (2.21)$$

From (2.17), (2.18), (2.20) and (2.21) we have

$$K+\frac{1}{K}\leq 2\rho+\frac{1}{2\rho}-\frac{7(\rho-1)}{6(\rho+1)} \qquad (2.22)$$

and

$$K+\frac{1}{K}\leq 2\rho+\frac{1}{2\rho}-2+O\left(\frac{1}{\rho}\right) \qquad (2.23)$$

for sufficiently large ρ. Inequalities (2.9) and (2.10) follow.

§2.4. Proof of theorem 2.

Because the non-Eucleadian metric is also a linear invariant we only need to prove

$$\frac{1}{2\rho} \leq \left| \frac{1}{v(i)} \cdot \frac{dw}{dz}(i) \right| \leq 2\rho \tag{2.24}$$

for $\mu(0) = 0$ and $\mu(1) = 1$. Similarly, we suppose $\beta \geq 1$. From (2.2)

$$v(i) = \frac{1}{2} \int_0^1 [\mu(t) - \mu(-t)]dt = \frac{1}{2}(1+\beta) - \frac{1}{2}(\xi + \beta\eta).$$

Then

$$\frac{1+\beta}{2(1+\rho)} \leq v(i) \leq \frac{\rho(1+\beta)}{2(1+\rho)}.$$

From [**A**]

$$|w_z(i)|^2 = \frac{1}{8}[(1+\xi^2) + \beta^2(1+\eta^2) + 2\beta(\xi+\eta)].$$

Then

$$\begin{aligned} |w_z(i)|^2 &\leq \frac{1}{8}\left[1 + \frac{\rho^2}{(1+\rho)^2} + \beta^2\left(1 + \frac{\rho^2}{(1+\rho)^2}\right) + \frac{4\beta\rho}{1+\rho}\right] \\ &= \frac{(1+\beta)^2(2\rho^2+2\rho+1) - 2\beta}{8(1+\rho)^2}, \\ |w_z(i)|^2 &\geq \frac{1}{8}(1+\beta^2). \end{aligned}$$

Hence

$$\begin{aligned} \left| \frac{1}{v(i)} \cdot \frac{dw}{dz}(i) \right|^2 &\leq \frac{(1+\beta)^2(2\rho^2+2\rho+1) - 2\beta}{8(1+\rho)^2} \cdot \frac{4(1+\rho)^2}{(1+\beta)^2} \\ &= \frac{1}{2}\left[2\rho^2 + 2\rho + 1 - \frac{2\beta}{(1+\beta)^2}\right] \\ &\leq \frac{1}{2}\left[2\rho^2 + 2\rho + 1 - \frac{2\rho}{(1+\rho)^2}\right], \end{aligned} \tag{2.25}$$

$$\left| \frac{1}{v(i)} \cdot \frac{dw}{dz}(i) \right|^2 \geq \frac{1}{8}(1+\beta^2) \cdot \frac{4(1+\rho)^2}{\rho^2(1+\beta)^2} \geq \frac{(1+\rho)^2}{4\rho^2}. \tag{2.26}$$

From theorem 1

$$K \leq 2\rho - \frac{\rho - 1}{\rho + 1} = \frac{2\rho^2 + \rho + 1}{\rho + 1} \, .$$

Then

$$\frac{K}{K+1} \leq \frac{2\rho^2 + \rho + 1}{2\rho^2 + 2\rho + 2} \, . \tag{2.27}$$

For $dw = w_z dz + w_{\bar z} d\bar z$, we have

$$\begin{aligned} |dw| &\leq \left(1 + \left|\frac{w_{\bar z}}{w_z}\right|\right) |w_z dz| = \frac{2K}{K+1} |w_z dz| \, , \\ |dw| &\geq \left(1 - \left|\frac{w_{\bar z}}{w_z}\right|\right) |w_z dz| = \frac{2}{K+1} |w_z dz| \, . \end{aligned}$$

Then

$$\begin{aligned} \left|\frac{1}{v(i)} \cdot \frac{dw}{dz}(i)\right|^2 &\leq \frac{4K^2}{(K+1)^2} \cdot \left|\frac{1}{v(i)} \cdot \frac{dw}{dz}(i)\right|^2 \\ &\leq 2\left(\frac{2\rho^2 + \rho + 1}{2\rho^2 + 2\rho + 2}\right)^2 \left[2\rho^2 + 2\rho + 1 - \frac{2\rho}{(1+\rho)^2}\right] \\ &= 4\rho^4 - \frac{2(\rho - 1)(2\rho^5 + 12\rho^4 + 15\rho^3 + 13\rho^2 + 5\rho + 1)}{2(1+\rho)^2(1+\rho+\rho^2)^2} \\ &\leq 4\rho^2, \end{aligned} \tag{2.28}$$

$$\left|\frac{dw(i)}{v(i)dz}\right|^2 \geq \frac{4}{(K+1)^2} \cdot \left|\frac{w_z(i)}{v(i)}\right|^2 \geq \frac{4}{(2\rho+1)^2} \cdot \frac{(1+\rho)^2}{4\rho^2} > \frac{1}{4\rho^2}, \tag{2.29}$$

and that completes the proof.

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Chapter 3

Quasiconformal Extension and Univalency Criteria

§3.1. Introduction and main results

Following Ahlfors [A2] and Anderson–Hinkkanen [A–H], Harmelin [H] recently obtained a univalency criterion for analytic functions $f(z)$ in the upper half plane U. It says:

Suppose that $f(z)$ is analytic and $f'(z) \neq 0$ in U and satisfies the inequality

$$\left|2y\frac{f''(z)}{f'(z)} - c\right| \leq k, \quad \textit{for } y = I_m z > 0, \tag{3.1}$$

where c is some given complex number with $|c| \leq k$. If $k < 1$, then $f(z)$ is univalent in U and has a k–quasiconformal extension to the whole plane. If $k = 1$ and $|c| < 1$, then (3.1) *implies that $f(z)$ is univalent in U.*

Our initial observation is that the condition $|c| \leq k$ can be dropped from Harmelin's criterion. Later we find that the above constant c can be replaced by some analytic functions. So it is natural to ask the following question:

Do there exist some analytic functions $a(z)$ and $c(z)$ related to $f(z)$ such that the inequality

$$|(\bar{z}-z)a(z)+c(z)| \leq k < 1, \quad \textit{for } I_m z > 0 \tag{3.2}$$

implies that $f(z)$ is univalent in U and has a $k-$quasiconformal extension to the whole plane?

In this chapter, we will give a positive answer. In fact, with the help of singular integrals, we have obtained the following results.

Theorem 1. *Let $f(z)$ be analytic and $f'(z) \neq 0$ in U. Let $w_0 \overline{\in} f(U)$. Assume there exists an analytic function $a(z)$ and some constant $c \neq 0$ such that*

$$\left|(\bar{z}-z)a(z)+\frac{cf'(z)e^{-\int a(z)dz}}{(f(z)-w_0)^2}-1\right| \leq k, \tag{3.3}$$

or

$$\left|(\bar{z}-z)a(z)+cf'(z)e^{-\int a(z)dz}-1\right| \leq k, \tag{3.4}$$

If $k<1$, then $f(z)$ is univalent in U and has a $k-$quasiconformal extension to the whole plane. If $k=1$, then $f(z)$ is also univalent in U.

When $f(z)$ is analytic in the unit disk $B=\{z:|z|<1\}$, we have also obtained the following similar result.

Theorem 2. *Let $f(z)$ be analytic and $f'(z) \neq 0$ in B. Let $w_0 \overline{\in} f(B)$. Assume there exists an analytic function $a(z)$ and some constant $c \neq 0$ such* *that*

$$\left|z(1-|z|^2)a(z)+\left(\frac{cf'(z)e^{-\int a(z)dz}}{(f(z)-w_0)^2}-1\right)|z|^2\right|\le k, \tag{3.5}$$

or

$$\left|z(1-|z|^2)a(z)+\left(cf'(z)e^{-\int a(z)dz}-1\right)|z|^2\right|\le k, \tag{3.6}$$

If $k<1$, then $f(z)$ is univalent in B and has a $k-$quasiconformal extension to the whole plane. If $k=1$, then $f(z)$ is also univalent in B.

By choosing specific $a(z)$, we get many interesting univalency criteria. Here are some examples from Theorem 1: $(I_m z>0)$

$$\left.\begin{array}{l}\mathbf{1.}\ \ |cf'(z)-1|\le k, \\[2ex] \mathbf{2.}\ \ \left|\dfrac{cf'(z)}{(f(z))^2}-1\right|\le k,\end{array}\right\}\qquad \Leftarrow\ a(z)=0.$$

$$\mathbf{3.}\ \ \left|\frac{\bar{z}}{z}+c\frac{f'(z)}{z}-2\right|\le k, \qquad \Leftarrow\ a(z)=\frac{1}{z}.$$

$$\mathbf{4.}\ \ \left|\frac{\bar{z}}{z}+czf'(z)\right|\le k, \qquad \Leftarrow\ a(z)=-\frac{1}{z}.$$

$$\mathbf{5.}\ \ \left|(2y+c)\frac{f'(z)}{f(z)}\mp i\right|\le k, \qquad \Leftarrow\ a(z)=\pm\frac{f'(z)}{f(z)}.$$

$$\mathbf{6.}\ \ \left|2y\frac{f''(z)}{f'(z)}+c-i\right|\le k, \qquad \Leftarrow\ a(z)=\frac{f''(z)}{f'(z)}.$$

$$\mathbf{7.}\ \ \left|2y\frac{f''(z)}{f'(z)}+c(f'(z))^2+i\right|\le k, \qquad \Leftarrow\ a(z)=-\frac{f''(z)}{f'(z)}.$$

$$\textbf{8.}\quad \left|2y\frac{f''(z)}{f'(z)} - \frac{\bar{z}}{z}i + cz\right| \le k, \qquad \Leftarrow \; a(z) = \frac{f''(z)}{f'(z)} - \frac{1}{z}.$$

$$\textbf{9.}\quad \left|2y\frac{f'''(z)}{f''(z)} + cf'(z)f''(z) - 1\right| \le k, \quad \Leftarrow \; a(z) = -\frac{f'''(z)}{f''(z)}.$$

From Theorem 2, we have ($|z| < 1$)

$$\left.\begin{array}{ll} \textbf{10.} & |cf'(z) - 1| \le \dfrac{k}{|z|^2}, \\[3ex] \textbf{11.} & \left|\dfrac{cf'(z)}{(f(z))^2} - 1\right| \le \dfrac{k}{|z|^2}, \end{array}\right\} \qquad \Leftarrow \; a(z) = 0.$$

$$\textbf{12.}\quad \left|z(1-|z|^2)\frac{f'(z)}{f(z)} + c|z|^2\frac{f'(z)}{f(z)} \mp |z|^2\right| \le k, \qquad \Leftarrow \; a(z) = \pm\frac{f'(z)}{f(z)}.$$

$$\textbf{13.}\quad \left|z(1-|z|^2)\frac{f''(z)}{f'(z)} + (c-1)|z|^2\right| \le k, \qquad \Leftarrow \; a(z) = \frac{f''(z)}{f'(z)}.$$

$$\textbf{14.}\quad \left|z(1-|z|^2)|z|^2\frac{f''(z)}{f'(z)} + (c(f'(z))^2 + 1)|z|^2\right| \le k, \quad \Leftarrow \; a(z) = -\frac{f''(z)}{f'(z)}.$$

$$\textbf{15.}\quad \left|z(1-|z|^2)\frac{f'''(z)}{f''(z)} + (cf'(z)f''(z) - 1)|z|^2\right| \le k, \quad \Leftarrow \; a(z) = -\frac{f'''(z)}{f''(z)}.$$

Remark 1. Criterion 10 is equivalent to $|f'(z) - 1| \le \frac{k}{|z|^2}$, which was obtained by Krzyz under an additional assumption $f'(0) = 1$ in 1976. [K]

Remark 2. We know that the function $f(z) = \frac{1}{c}(z + \gamma e^{iz})$ ($|\gamma| > 1$) is not univalent in U. However $|cf'(z) - 1| = |\gamma|e^{-y}$. So the constant k in criterion 1 can not be replaced by any number bigger than 1. The function

$f(z) = \frac{z}{c(z^2-\gamma)}$ $(|\gamma| > 1)$, which is not univalent in B, shows that criterion 11 is also sharp in the sense of k.

Remark 3. From the proof, the restriction $c \neq 0$ can be removed from criteria 6 and 13, which generalize the results of Harmelin and Ahlfors [A2] respectively.

Remark 4. In order to generalize the criterion of Anderson and Hinkkanen [A–H], we need further to consider the following question:

Find the relation between analytic functions $a(z)$ and $c(z)$ such that the inequality

$$|(\bar{z} - z)^2 a(z) + c(z)| \leq k < 1, \quad \textit{for } I_m z > 0 \tag{3.7}$$

implies that $f(z)$ is univalent in U and has a $k-$quasiconformal extension to the whole plane.

Remark 5. We will only prove Theorem 1, because the proof of Theorem 2 is similar (but relatively easier).

§3.2. Proof of Theorem 1

The proof is based on the following proposition.

Proposition. *Suppose that $a(z)$ and $c(z)$ are analytic in U and satisfy*

$$|(\bar{z} - z)a(z) + c(z)| \leq k < 1, \quad z \in U. \tag{3.8}$$

Set

$$\mu(z) = \begin{cases} 0\,, & \textit{for } z \in U; \\ (z - \bar{z})a(\bar{z}) + c(\bar{z})\,, & \textit{for } z \in L = \{z : I_m z < 0\}. \end{cases} \tag{3.9}$$

Let $F(z)$ *be any* $k-$*quasiconformal homeomorphism of* $\mathbb{C}$ *with dilatation* $\mu(z)$. *Then*

$$\frac{F''(z)}{F'(z)} = a(z) + \frac{c'(z)}{c(z)+1}, \quad \textit{for} \quad z \in U. \tag{3.10}$$

We postpone the proof of the proposition to the later.

Now suppose that $f(z)$ satisfies (3.3) or (3.4). Set

$$c(z) = \frac{cf'(z)e^{-\int a(z)dz}}{(f(z)-w_0)^2} - 1 \quad \text{or} \quad c(z) = cf'(z)e^{-\int a(z)dz} - 1. \tag{3.11}$$

First we assume $k < 1$. Then $a(z)$ and $c(z)$ satisfy (8). Let $F(z)$ be a $k-$quasiconformal homeomorphism of $\mathbb{C}$ with dilatation of the form (3.9). Then from the proposition

$$\frac{F''(z)}{F'(z)} = a(z) + \frac{c'(z)}{c(z)+1} = \begin{cases} \dfrac{f''(z)}{f'(z)} - \dfrac{2f'(z)}{f(z)-w_0}, \\ \\ \textit{or} \ \dfrac{f''(z)}{f'(z)}. \end{cases} \quad \textit{for} \quad z \in U. \tag{3.12}$$

In both cases, we have

$$S_F(z) = S_f(z), \quad \text{for} \quad z \in U. \tag{3.13}$$

Where $S_f(z) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ is the Schwarzian derivative of $f(z)$. So there exist some complex constants α, β, γ, and δ such that

$$f(z) = \frac{\alpha F(z) + \beta}{\gamma F(z) + \delta}, \quad \text{for} \ z \in U. \tag{3.14}$$

It is clear that (3.14) is a quasiconformal extension formula for $f(z)$.

Next we consider the case $k = 1$. For $n = 1, 2, 3, \cdots$, define

$$\mu_n(z) = \begin{cases} 0 & \text{for} \ z \in U; \\ \frac{n}{n+1}\{(z-\bar{z})a(\bar{z}) + c(\bar{z})\}, & \text{for} \ z \in L, \end{cases} \tag{3.15}$$

then $|\mu_n(z)| < \frac{n}{n+1}$. Let $F_n(z)$ be a k-quasiconformal homeomorphism of C with dilatation $\mu_n(z)$ and agree with $f(z)$ at three points of U, where $f(z)$ attains distinct values. Then $\{F_n(z)\}$ is a normal family in U. We may choose a subsequence $\{F_{n_k}(z)\}$ locally uniformly converging to an analytic univalent function $F(z)$ in U. So

$$\frac{F''(z)}{F'(z)} = \lim_{n_k\to\infty} \frac{F''_{n_k}(z)}{F'_{n_k}(z)} \quad \text{for} \quad z \in U. \tag{3.16}$$

But

$$\frac{F''_n(z)}{F'_n(z)} = \frac{n}{n+1}a(z) + \frac{\frac{n}{n+1}c'(z)}{\frac{n}{n+1}c(z)+1} \to a(z) + \frac{c'(z)}{c(z)+1} \quad (\text{as } n\to\infty), \quad \text{for } z \in U. \tag{3.17}$$

Again, we obtain (3.14) which implies that $f(z)$ is univalent in U.

§3.3. Lemmas

To prove the proposition, we need some lemmas.

For positive numbers r and ε, set

$$z_r = -i\sqrt{r^2+1}, \quad D(r) = \{z : |z - z_r| < r\}, \quad B(t,\varepsilon) = \{z : |z-t| < \varepsilon\}.$$

Note when $z \in \partial D(r)$

$$\bar{z} = \bar{z}_r + \frac{r^2}{z - z_r}, \quad d\bar{z} = -\frac{r^2}{(z-z_r)^2}dz.$$

Let $h(z)$ be a continuous function in L, we define

$$\int\int_L \frac{h(z)}{(z-t)^2} dz\wedge d\bar{z} = \lim_{r\to\infty,\ \varepsilon\to 0} \int\int_{D(r)\setminus B(t,\varepsilon)} \frac{h(z)}{(z-t)^2} dz \wedge d\bar{z}$$
$$\int\int_L \frac{h(z)}{z-t} dz\wedge d\bar{z} = \lim_{r\to\infty,\ \varepsilon\to 0} \int\int_{D(r)\setminus B(t,\varepsilon)} \frac{h(z)}{z-t} dz \wedge d\bar{z}$$

Lemma 1. *Let $a(z)$ be analytic in U. Then*

$$\frac{1}{2\pi i}\int\int_L \frac{a(\bar{z})}{(z-t)^2}dz\wedge d\bar{z} = \begin{cases} a(t), & \text{if } t\in U,\\ 0, & \text{if } t\in L.\end{cases} \tag{3.18}$$

Proof: When $t \in U$

$$\begin{aligned}\frac{1}{2\pi i}\int\int_{D(r)} \frac{a(\bar{z})}{(z-t)^2}dz\wedge d\bar{z} &= -\frac{1}{2\pi i}\int_{\partial D(r)} \frac{a(\bar{z})}{z-t}d\bar{z}\\ &= \frac{1}{2\pi i}\int_{\partial D(r)} \frac{a\left(\bar{z}_r+\frac{r^2}{z-z_r}\right)}{z-t}\cdot\frac{r^2}{(z-z_r)^2}dz\\ &= -a\left(\bar{z}_r+\frac{r^2}{t-z_r}\right)\cdot\frac{r^2}{(t-z_r)^2}\\ &= -a\left(\frac{it\sqrt{r^2+1}-1}{t+i\sqrt{r^2+1}}\right)\cdot\frac{r^2}{(t+i\sqrt{r^2+1})^2}\to a(t)\ \ (\text{as } r\to\infty).\end{aligned} \tag{3.19}$$

When $t \in L$

$$\begin{aligned}\frac{1}{2\pi i}\int\int_{D(r)\setminus B(t,\epsilon)} \frac{a(\bar{z})}{(z-t)^2}dz\wedge d\bar{z} &= -\frac{1}{2\pi i}\int_{\partial D(r)} \frac{a(z)}{z-t}d\bar{z}+\frac{1}{2\pi i}\int_{|z-t|=\epsilon}\frac{a(z)}{z-t}d\bar{z}\\ &= I_1+I_2.\end{aligned} \tag{3.20}$$

Then

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{\partial D(r)} \frac{a\left(\bar{z}_r+\frac{r^2}{z-z_r}\right)}{z-t}\cdot\frac{r^2}{(z-z_r)^2}dz = 0\\ I_2 &= \frac{a(\bar{t})}{2\pi i}\int_{|z-t|=\epsilon}\frac{d\bar{z}}{z-t}+\frac{a'(\bar{t})}{2\pi i}\int_{|z-t|=\epsilon}\frac{\overline{z-t}}{z-t}d\bar{z}+o(|z-t|^2)\to 0,\ (\text{as } \varepsilon\to 0).\end{aligned} \tag{3.21}$$

□

In the proof of the next two lemmas, we will use the following fact:

Let D be a Jordan domain containing ∞. Suppose that $a(z)$ is analytic in $\overline{D}$ and has a zero of at least second order at ∞. Then for $t \in D$,

$$\int_{\partial D} a(z)\log(z-t)dz = 2\pi i \int_{\infty}^{t} a(z)dz, \tag{3.22}$$

where $\int_{\infty}^{t}$ means any simple path from ∞ to t.

Lemma 2. *Let $a(z)$ be analytic in U. Then for $t \in U$*

$$\frac{1}{2\pi i}\int\int_{L} \frac{a(\bar{z})}{z-t} dz \wedge d\bar{z} = \int_{+\infty i}^{t} a(z)dz, \tag{3.23}$$

provided that the integral in the left hand exists.

Proof:

$$\begin{aligned}\frac{1}{2\pi i}\int\int_{D(r)} \frac{a(\bar{z})}{z-t} dz \wedge d\bar{z} &= \frac{1}{2\pi i}\int_{\partial D(r)} a(\bar{z})\log(z-t)d\bar{z} \\ &= -\frac{1}{2\pi i}\int_{\partial D(r)} a\left(\bar{z}_r + \frac{r^2}{z-z_r}\right)\log(z-t)\cdot\frac{r^2}{(z-z_r)^2}dz \\ &= -\int_{\infty}^{t} a\left(\bar{z}_r + \frac{r^2}{z-z_r}\right)\cdot\frac{r^2}{(z-z_r)^2}dz \\ &= \int_{\bar{z}_r}^{\bar{z}_r+\frac{r^2}{t-z_r}} a(\zeta)d\zeta \quad \left(\zeta = \bar{z}_r + \frac{r^2}{z-z_r}\right) \\ &\to \int_{+\infty i}^{t} a(\zeta)d\zeta, \ (\text{as } r \to \infty). \end{aligned} \tag{3.24}$$

□

Lemma 3. *Let $a(z)$ be analytic in U. Then for $t \in L$*

$$\frac{1}{2\pi i}\int\int_{L} \frac{a(\bar{z})}{z-t} dz \wedge d\bar{z} = \int_{+\infty i}^{\bar{t}} a(z)dz, \tag{3.25}$$

provided that the integral in the left hand exists.

Proof: Assume that $B(t,\varepsilon) \subset D(r)$.

$$\frac{1}{2\pi i}\int\int_{D(r)\setminus B(t,\varepsilon)}\frac{a(\bar{z})}{z-t}dz\wedge d\bar{z}=\frac{1}{2\pi i}\int_{\partial D(r)}a(\bar{z})\log|z-t|^2d\bar{z}$$
$$-\frac{1}{2\pi i}\int_{|z-t|=\varepsilon}a(\bar{z})\log|z-t|^2d\bar{z}=I_1-I_2. \qquad (3.26)$$

Denote $\tilde{t}=z_r+\frac{r^2}{\bar{t}-\bar{z}_r}$. Then for $z\in\partial D(r)$,

$$\log|z-t|^2=\log\frac{z-t}{z-z_r}+\log(\bar{z}_r-\bar{t})+\log(z-\tilde{t}).$$

So

$$\begin{aligned}I_1&=-\frac{1}{2\pi i}\int_{\partial D(r)}a\left(\bar{z}_r+\frac{r^2}{z-z_r}\right)\log(z-\tilde{t})\cdot\frac{r^2}{(z-z_r)^2}dz\\&=-\int_{\infty}^{\tilde{t}}a\left(\bar{z}_r+\frac{r^2}{z-z_r}\right)\cdot\frac{r^2}{(z-z_r)^2}dz\\&=\int_{\bar{z}_r}^{z_r+\frac{r^2}{\bar{t}-z_r}}a(\zeta)d\zeta\quad\left(\zeta=\bar{z}_r+\frac{r^2}{z-z_r}\right)\\&\rightarrow\int_{+\infty i}^{\bar{t}}a(\zeta)d\zeta\ (\text{as } r\rightarrow\infty).\end{aligned} \qquad (3.27)$$

Obviously

$$I_2\rightarrow 0,\ (\text{as } \varepsilon\rightarrow 0). \qquad (3.28)$$

□

§3.4. Proof of the proposition

First we assume that for some $p>2$,

$$\mu(z)\in L^p(\mathbb{C}). \qquad (3.29)$$

For $g(z)\in L^p(\mathbb{C})$, operators $\mathbf{P}$, $\mathbf{T}$ and $\mathbf{T}\mu$ are defined by [A1]

$$\begin{aligned}\mathbf{P}g(t)&=\frac{1}{2\pi i}\int\int_{\mathbb{C}}g(z)\left(\frac{1}{z-t}-\frac{1}{z}\right)dz\wedge d\bar{z},\\\mathbf{T}g(t)&=\frac{1}{2\pi i}\int\int_{\mathbb{C}}\frac{g(z)}{(z-t)^2}dz\wedge d\bar{z},\\\mathbf{T}\mu(g)&=\mathbf{T}(\mu g).\end{aligned} \qquad (3.30)$$

Let $\mathbf{I}$ be the identity operator. As we know, $\|T\|_p \to 1$ as $p \to 2$. Let p satisfy $k\|T\|_p < 1$. Then $(\mathbf{I} - \mathbf{T}\mu)^{-1}$ exists. Denote

$$h(z) = (\mathbf{I} - \mathbf{T}\mu)^{-1} T\mu. \tag{3.31}$$

Since $\mu(h+1) \in L^p$,

$$F(z) = \mathbf{P}[\mu(h+1)] + z \tag{3.32}$$

is well defined.

We claim that $F(z)$ is a $k-$quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu(z)$. To see that, for $n = 1, 2, 3, \cdots$, define

$$\mu_n(z) = \begin{cases} \mu(z) & \text{for } |z| \leq n; \\ 0, & \text{for } |z| > n. \end{cases}$$

$$h_n(z) = (\mathbf{I} - \mathbf{T}\mu_n)^{-1} T\mu_n,$$

$$F_n(z) = \mathbf{P}[\mu_n(h_n + 1)] + z. \tag{3.33}$$

From [A1], $F_n(z)$ is a $k-$quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu_n(z)$ and satisfies $F_n(0) = 0$, $F_n(z) = z + O(1)$ $(z \to \infty)$. Note as $n \to \infty$, $\|h_n - h\|_p \to 0$. Then $F_n(z) \to F(z)$ in L^P sense. However $\{F_n(z)\}$ is a normal family in $\mathbb{C}$. We may choose a subsequence locally uniformly converging to a $k-$quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu(z)$. Obviously the limiting function must be $F(z)$.

Next we want to prove that $F(z)$ satisfies (3.10). From [A1] the distributional derivative

$$\begin{aligned} \{F(z)\}_z = h(z) + 1 &= 1 + \mathbf{T}\mu(z) + \mathbf{T}\mu\mathbf{T}\mu(z) + \mathbf{T}\mu\mathbf{T}\mu\mathbf{T}\mu(z) + \cdots \\ &= \sum_{n=0}^{\infty} (\mathbf{T}\mu)^n (z). \end{aligned}$$

For $z \in U$ and $n = 1, 2, 3, \cdots$, we formally define

$$g_n(z) = \left(\int_{+\infty i}^{z} a(z_n) \int_{+\infty i}^{z_n} a(z_{n-1}) \int_{+\infty i}^{z_{n-1}} \cdots \int_{+\infty i}^{z_2} a(z_1)\right) dz_1 \cdots dz_{n-1} dz_n, \tag{3.35}$$

Then

$$g_n(t) = \int_{+\infty i}^{t} a(z) g_{n-1}(z) dz \tag{3.36}$$

When $t \overline{\in} \mathbb{R}$

$$\mathbf{T}\mu(t) = \frac{1}{2\pi i} \int\int_L \frac{(z - \bar{z})a(\bar{z}) + c(\bar{z})}{(z - t)^2} dz \wedge d\bar{z}$$

$$= \frac{1}{2\pi i} \int\int_L \frac{a(\bar{z})}{z - t} dz \wedge d\bar{z} + \frac{1}{2\pi i} \int\int_L \frac{(t - \bar{z})a(\bar{z}) + c(\bar{z})}{(z - t)^2} dz \wedge d\bar{z}$$

$$= \begin{cases} g_1(t) + c(t), & \text{if } t \in U, \\ g_1(\bar{t}). & \text{if } t \in L. \end{cases} \tag{3.37}$$

$$\mathbf{T}\mu\mathbf{T}\mu(t) = \frac{1}{2\pi i} \int\int_L \frac{a(\bar{z}) g_1(\bar{z})}{z - t} dz \wedge d\bar{z}$$

$$+ \frac{1}{2\pi i} \int\int_L \frac{(t - \bar{z})a(\bar{z})g_1(\bar{z}) + c(\bar{z})g_1(\bar{z})}{(z - t)^2} dz \wedge d\bar{z}$$

$$= \begin{cases} g_2(t) + c(t)g_1(t), & \text{if } t \in U, \\ g_2(\bar{t}). & \text{if } t \in L. \end{cases} \tag{3.38}$$

By induction

$$(\mathbf{T}\mu)^n(t) = \begin{cases} g_n(t) + c(t)g_{n-1}(t), & \text{if } t \in U, \\ g_n(\bar{t}). & \text{if } t \in L. \end{cases} \tag{3.39}$$

Clearly the existence of $g_n(z)$ is guaranteed by the condition $\mu \in L^p$. Define $g(z) = \sum_{n=1}^{\infty} g_n(z)$ $(z \in U)$. Since $g(\bar{z}) = h(z) \in L^p(L)$, it follows that $g(z) \in L^p(U)$. Then the distributional derivative of g in U is

$$g_z = \sum_{n=1}^{\infty} g_n'(z) = a(z) \sum_{n=2}^{\infty} g_{n-1}'(z) + a(z) = a(g+1). \tag{3.40}$$

Now for $z \in U$

$$h = g + gc + c. \tag{3.41}$$

So the distributional derivative of h in U is

$$h_z = g_z + g_z c + g c_z + g_c = (1+c)(1+g)\left(a + \frac{c_z}{1+c}\right) = (h+1)\left(a + \frac{c_z}{1+c}\right). \tag{3.42}$$

Note that the last expression is an analytic function. Therefore

$$\frac{F''(z)}{F'(z)} = \frac{h'(z)}{h(z)+1} = a(z) + \frac{c'(z)}{c(z)+1}, \quad \text{for } z \in U. \tag{3.43}$$

Because any two $k-$quasiconformal homeomorphisms of $\mathbb{C}$ with same dilatation only differ by an integral linear transformation, we have finished the proof in the case $\mu(z) \in L^p(\mathbb{C})$.

In general, for $n = 1, 2, 3, \cdots$, set

$$\mu_n(z) = \begin{cases} 0\,, & \text{for } z \in U; \\ \dfrac{n}{n + (-i\bar{z})^{1+\delta}}\{(z-\bar{z})a(\bar{z}) + c(\bar{z})\} & \text{for } z \in L. \end{cases} \tag{3.44}$$

where δ is a small positive number satisfying $k < \cos\frac{\pi\delta}{2}$.

Then for some $p > 2$

$$\mu_n(z) \in L^p(\mathbb{C}) \quad \text{and} \quad |\mu_n(z)| \leq \frac{k}{\cos\frac{\pi\delta}{2}} < 1. \tag{3.45}$$

Let $F_n(z)$ be a quasiconformal homeomorphism of $\mathbb{C}$ with dilatation $\mu_n(z)$ and fixes 0, 1, and ∞. Again $\{F_n(z)\}$ is a normal family. Thus there exists a subsequence $\{F_{n_k}(z)\}$ locally uniformly converging to a $k-$quasiconformal homeomorphism $F(z)$ with dilatation $\mu(z)$. Denote $\rho_n(z) = \frac{n}{n+(-iz)^{1+\delta}}$ $(z \in U)$. Now for $z \in U$

$$\begin{aligned} \frac{F''(z)}{F'(z)} &= \lim_{n_k \to \infty} \frac{F''_{n_k}(z)}{F'_{n_k}(z)} \\ &= \lim_{n_k \to \infty} \left\{ \rho_{n_k}(z)a(z) + \frac{\rho'_{n_k}(z)c(z) + \rho_{n_k}(z)c'(z)}{\rho_{n_k}(z)c(z)} \right\} = a(z) + \frac{c'(z)}{c(z)+1}. \end{aligned} \tag{3.46}$$

This completes the proof of the Proposition.

§3.5. Some other results

It is not hard to derive the following results from the proposition.

Theorem 3. *Let $f(z)$ be analytic, $f'(z) \neq 0$ and $I_m f(z) > 0$ in U. If there exists an analytic function $a(z)$ such that*

$$\left|(\bar{z} - z)a(z) + c(f(z))^{\delta} f'(z) e^{-\int a(z)dz} - 1\right| \leq k < 1, \tag{3.47}$$

where c and δ are constants with $c \neq 0$ and $|\delta| < 1$. Then $f(z)$ is univalent in U and has a $k|\delta|-$quasiconformal extension to the whole plane.

Theorem 4. *Let $f(z)$ be analytic, $f'(z) \neq 0$ and $I_m f(z) > 0$ in B. If there exists an analytic function $a(z)$ such that*

$$\left|z(1 - |z|^2)a(z) + \left(c(f(z))^{\delta} f'(z) e^{-\int a(z)dz} - 1\right)|z|^2\right| \leq k < 1, \tag{3.48}$$

where c and δ are constants with $c \neq 0$ and $|\delta| < 1$. Then $f(z)$ is univalent in B and has a $k|\delta|-$quasiconformal extension to the whole plane.

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