Adiabatic limits, Non-multiplicativity of the Signature and the Leray Spectral Sequence

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In this thesis, we first prove an adiabatic limit formula for the \( \eta \)-invariant of a Dirac type operator, which generalizes the recent work of J.-M. Bismut and J. Cheeger. By their work, one is essentially reduced to the study of the large time behavior of the heat kernel. This involves careful analysis of degenerate elliptic operators, which is fashioned after the very recent work of R. Mazzeo and R. Melrose. The analysis enables us to apply perturbation theory to obtain detailed informations about the spectrum of the Dirac operator in the adiabatic limit. A new contribution arises in the adiabatic
limit formula, in the form of a global term coming from the (asymptotically) very small eigenvalues.

We then proceed to show that, for signature operators, these very small eigenvalues have a purely topological significance. In fact, we show that the Leray spectral sequence can be recast in terms of the very small eigenvalues. This is reminiscent of the Hodge theory. Hence the term Hodge-Leray theory. A consequence of the Hodge-Leray theory is a refined adiabatic limit formula for signature operators where the global term is identified with a topological invariant, the signature of a certain bilinear form arising from the Leray spectral sequence.

As an interesting application, we give intrinsic characterizations of the non-multiplicativity of signature. Appendix B contains a proof of a special case of Bismut-Cheeger’s Families Index Theorem for manifolds with boundary, which is used in the applications.
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To my parents and Guofang.
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Introduction

In this thesis, we study the limiting behavior of the \( \eta \)-invariant of Atiyah-Patodi-Singer for Dirac operators on the total space of a fibration, when the metric along base direction is multiplied by a factor \( x^{-2} \) and \( x \to 0 \). This operation of blowing up the metric is called passing to the \textit{adiabatic limit}.

The original motivation comes from a paper of E. Witten, [W], where he considers a family of Dirac operators acting on an even dimensional manifold. The parameter space of his family is a circle and thus, the (odd dimensional) total space of his family is the total space of fibration over the circle. Witten gives an argument relating the holonomy of the determinant line bundle of his family to the adiabatic limit of the \( \eta \)-invariant of the Dirac operator on the total space.

Witten’s result was proved rigorously in [BF] and [C2]. The emphasis in [BF] was on the superconnection formalism of Quillen, [Q], in relation to the proof given in [B] of the local index theorem for families. In [C2], two proofs were given, one based on Duhamel’s principle and second exploited the connection with previous work on conical singularities; see [C1], [C2]. For the case of the signature operator, an expression equivalent to that
considered by Witten had arisen in [C1], when considering the variation of the \(\eta\)-invariant, for a space with isolated conical singularities. Finally, it was emphasized in [C2] that for a fibration of compact manifolds, the \(\eta\)-invariant of the total space can be viewed as a renormalized difference of \(\eta\)-invariants, with coefficients in an infinite dimensional bundle whose fiber is a space of sections along the fiber of the fibration.

Recently J.-M. Bismut and J. Cheeger, [BC1], [BC2], extend the results of these papers to the case in which the base of the fibration is a compact spin manifold of arbitrary dimension. What they found is that the adiabatic limit of the \(\eta\)-invariant of a Dirac operator on the total space is expressible in terms of a canonically constructed differential form, \(\tilde{\eta}\), on the base, which can be viewed as a higher dimensional analogue of the \(\eta\)-invariant. What is more, this \(\tilde{\eta}\) is exactly the boundary correction term in the Families Index Theorem for manifolds with boundary, [BC3].

In their work, they discussed in detail the case when the operators along the fibres are always invertible. Their treatment of the general case (i.e., the kernels of these operators have constant dimension) was less complete and explicit although it did suffice for various applications. For example, one can recover the work of Atiyah-Donnelly-Singer, [ADS], on the Hirzebruch conjecture, and prove results closely related to those of W. Müller, [Mü], and M. Stern, [S].

We study questions related to the noninvertible case, which is essential when we consider signature operators because then the kernel is always nonempty and is given by cohomology. Here some new phenomena occur. Besides the semi-local contribution to the adiabatic limit formula involving
the $\tilde{\eta}$-form, and an expected twisted $\eta$-invariant on the base, there is a
global contribution from the (asymptotically) very small eigenvalues (Cf.
below). It turns out that in the case of signature operator, these eigenvalues
have a purely topological significance. This is our second main result, the
Hodge-Leray theory of the very small eigenvalues, which is a refinement of
the very recent work R. Mazzeo and R. Melrose, [MM]. In particular, it
enables us to identify the global contribution in the adiabatic limit formula
with a topological invariant coming from the Leray spectral sequence of the
fibration.

As an interesting application we give intrinsic characterizations of the
non-multiplicativity of signature (see below), one in terms of the Leray
spectral sequence and the other the adiabatic limit.

Precisely, let us consider a fibration of closed manifolds

$$Y \to M^{2k-1} \xrightarrow{\pi} B.$$  \hspace{1cm} (1)

Equip $M$ with a submersion metric $g_M$,

$$g_M = \pi^* g_B + g_Y,$$

where $g_B$ is the metric on $B$ and $g_Y$ annihilates the orthogonal compliment
of the tangent space to the fibres. Blowing up the metric in the horizontal
direction by a factor $x^{-2}$ gives us a family of metrics $g_x$,

$$g_x = x^{-2} \pi^* g_B + g_Y.$$  

We assume that 1) $M$ is spin so that we can consider the Dirac operator
$D_x$ associated with $g_x$; 2) The bundle of vertical spaces $T^VM$ is also spin so that we have the family of Dirac operators $D_Y$ along the fibers.

It follows that the base space $B$ is also spin and the Dirac operator $D_B$ on the base is well-defined. When $\ker D_Y$ is a vector bundle on $B$, one can consider the twisted Dirac operator $D_B \otimes \ker D_Y$. Here the connection on $\ker D_Y$ is the projection of a unitary connection on the infinite dimensional bundle of smooth spinor sections along the fibres (see Appendix A for details; Cf. also [BC2]). More generally, let $\xi$ be a hermitian vector bundle with a unitary connection on it, we can then consider the above operators twisted by $\xi$. We denote these by the same notations. Let $\eta(D_x)$ be the $\eta$-invariant of $D_x$. The following result is proved in this thesis.

**Theorem 0.1 (Adiabatic limit formula)** Assume that $\ker D_Y$ is a vector bundle on $B$. Assume further that $\dim(\ker D_x)$ stabilizes after some small $x$. Then $\lim_{x \to 0} \eta(D_x)$ exists in $\mathbb{R}$ and

$$\lim_{x \to 0} \eta(D_x) = 2 \int \frac{R_B}{2\pi} \wedge \tilde{\eta} + \eta(D_B \otimes \ker D_Y) + \lim_{x \to 0} \sum_{\lambda_0, \lambda_1 = 0} sgn \lambda_x, \quad (2)$$

where $R_B$ is the curvature tensor of $g_B$.

Here in the first term the $\tilde{\eta}$-from is defined in the same way as in [BC2] (see Appendix A). As we observed in Appendix A, it is well-defined without any assumptions. The last term in the above formula will be explained in a moment. But first we would like to mention that in the invertible case where the spectrum of $D_x$ is uniformly bounded away from 0, the last two terms drop out (see below), and the above formula reduces to Bismut-Cheeger's adiabatic limit formula [BC2]. One of the essential difficulties in the noninvertible case is that there are infinitely many eigenvalues of $D_x$.
approaching 0. However, one still has a nice description of them. Thus in Chapter 2 we will show that (Theorem 1.1) the spectrum of \( D_x \) can be divided into three parts: those uniformly bounded away from 0, those decaying linearly in \( x \) (the small eigenvalues), and those, finitely many in number, decaying at least quadratically (the very small ones). Roughly speaking, the twisted \( \eta \) term in (2) comes from the small eigenvalues, and the last term from the very small ones (the finite sum of the signs of them).

Whereas asking \( \dim(\ker D_x) \) to stabilize is a bit unnatural, it warrants the existence of the adiabatic limit of the \( \eta \)-invariant, which does not happen in general. On the other hand, the limit of the reduced \( \eta \)-invariant, \( \bar{\eta} \), always exists in \( \mathbb{R}/\mathbb{Z} \) ([C2], [BC2]). Hence, we have the following modulo \( \mathbb{Z} \) counterpart of Theorem 0.1.

**Theorem 0.1'** Assume that \( \ker D_Y \) is a vector bundle on \( B \). Then

\[
\lim_{x \to 0} \bar{\eta}(D_x) \equiv \int \hat{A}(\frac{R_B}{2\pi}) \wedge \bar{\eta} + \bar{\eta}(D_B \otimes \ker D_Y) + \frac{1}{2} h \mod \mathbb{Z}, \tag{3}
\]

where \( \frac{1}{2} h \equiv \frac{1}{2} \dim(\ker D_x) \mod \mathbb{Z} \) is a spin cobordism invariant, [AS, V].

An important operator satisfying the hypothesis of Theorem 0.1 is the so-called signature operator \( A \), whose kernel space is identified by the well-known Hodge theory with the cohomology. In this case, as we have indicated, the global contribution turns out to be a topological invariant, a consequence of the following interesting result, to which we refer as the Hodge-Leray theory of the (asymptotically) very small eigenvalues.

**Theorem 0.2** Let \( E_r \) be the limit of space spanned by \( \lambda_x \)-eigenforms associated to eigenvalues \( \lambda_x \) such that \( \lambda_x \) is \( O(x^r) \) \( (r \geq 2) \) in the adiabatic
limit, then \((E_r, x^{-\tau} d)\) forms a spectral sequence which is isomorphic to the Leray spectral sequence of the fibration. Moreover, the \(\ast\) map induced by the metric \(g_x\) gives rise to the duality map.

In [MM], the Leray spectral sequence is related to the asymptotic solutions of the Laplacian in the adiabatic limit. Theorem 0.2 is partly motivated by this result. The analysis involved here is in fact fashioned after [MM].

Theorem 0.2 gives us a refined adiabatic limit formula in the case of the signature operator. Let \((E_r, d_r)\) \((r \geq 2)\) be the \(E_r\)-term of the Leray spectral sequence of the fibration \(Y \to M^{4k-1} \to B\). The orientation gives a natural basis \(\xi_2\) on \(E_2\) (in the sense of Chern-Hirzebruch-Serre [CHS], see Section 4.4) which then induces a basis \(\xi_r\) on \(E_r\) for each \(r > 2\). Consider the pairing

\[
\langle \cdot, \cdot \rangle_r : E_r^p \otimes E_r^q \to \mathbb{R},
\]

\[
\varphi \otimes \psi \mapsto (\varphi \cdot d_r \psi, \xi_r).
\]

It can be verified that \(\langle \cdot, \cdot \rangle_r\) is symmetric when restricted to \(E_r^{2k-1}\). Therefore it gives rise to a symmetric matrix whose signature we will denote by \(\tau_r\). Define \(\tau = \sum_{r \geq 2} \tau_r\).

**Theorem 0.3** Suppose the fibration \(Y \to M^{4k-1} \to B\) is oriented (see Section 4.1), then

\[
\lim_{x \to 0} \eta(A_x) = 2 \int_B L \left( \frac{R_B}{2\pi} \right) \wedge \tilde{\eta} + \eta(A_B \otimes \ker A_Y) + 2\tau,
\]

where \(A_B\) denotes the signature operator on \(B\) and \(A_Y\) the family of signature operators along \(Y\).
Theorem 0.3 combined with the Families Index Theorem for manifolds with boundary [BC3] has the following interesting application. Consider an oriented fibration $Z \to N^{4k} \to B$ such that $B$ is closed but $\partial Z = Y$ and $\partial N = M$. Associated with it is the boundary fibration $Y \to M^{4k-1} \to B$. Let $\text{Sign}^Z$ denote the signature bundle of fibres on $B$. This is a virtual bundle, thus can be considered as a $\mathbb{Z}_2$-graded bundle. It has a natural flat structure. Therefore we can define $\text{sign}(B, \text{Sign}^Z)$, the signature of $B$ with coefficients in the $\mathbb{Z}_2$-graded flat bundle $\text{Sign}^Z$ (see [L]). Let

$$\Delta \overset{\text{def}}{=} \text{sign}(B, \text{Sign}^Z) - \text{sign}(N).$$

This is clearly a topological invariant of the fibration $Z \to N \to B$. Moreover $\Delta$ depends only on the boundary fibration (extended Novikov additivity, see [BC4]). We call $\Delta$ the non-multiplicativity of signature. It measures the deviation of signature for manifolds with boundary from being multiplicative. In the case of an oriented fibration of closed manifolds $Z \to N \to B$, when $\pi_1(B)$ acts trivially on $H^*(Z)$, the Chern-Hirzebruch-Serre Theorem, [CHS], says

$$\text{sign}N = \text{sign}B \text{sign}Z$$

In general, signature for closed oriented manifolds is multiplicative in the sense that

$$\text{sign}N = \text{sign}(B, \text{Sign}^Z).$$

This follows from the Atiyah-Singer index theorems and the signature theorem for twisted coefficients; see [AS], [L]. For manifolds with boundary,
Bismut-Cheeger noted in [BC4] that the index theory is asymptotically multiplicative in the invertible case. Here we have

**Theorem 0.4 (intrinsic characterization of the non-multiplicativity)**

The topological invariant $\tau$, which is defined from the closed fibration $Y \rightarrow M \rightarrow B$, intrinsically characterizes the non-multiplicativity of signature.

That is, whenever there exists another fibration of manifolds with boundary $Z \rightarrow N^n \rightarrow B$ such that $\partial Z = Y$, $\partial N = M$, then $\tau = \Delta$.

Thus we characterize the non-multiplicativity of signature intrinsically in terms of the topological data of the boundary fibration. One can also give an intrinsic characterization in terms of the analytical data.

**Theorem 0.4'** Let

$V_0 =$ limit of space of $A_\pi$-harmonic forms on $M$ in the adiabatic limit,

$V_\pm =$ limit of space of $\lambda_\pi$-eigenforms such that $\lambda_\pi > 0$ (respectively $< 0$) and $\lambda_\pi = O(x^2)$.

Then $V_+ \oplus V_- \oplus V_0 = H^*(B, \mathcal{H}^*(Y))$, and $\tau' \overset{\text{def}}{=} \dim V_+ - \dim V_- = \tau$.

Therefore $\tau' = \Delta$ whenever the fibration $Y \rightarrow M \rightarrow B$ bounds $Z \rightarrow X \rightarrow B$.

The advantage of the adiabatic limit formulation of the intrinsic non-multiplicativity is that it allows an extension to the general Dirac operators, which we omit here.

Before we present the organization of this thesis, let us point out that a natural question arises from Theorem 0.1', that is, whether the assumption that $\ker D_Y$ gives a vector bundle on $B$ is really necessary. In fact one can make sense of all the terms involved in (0.3) without making this assumption. Thus, as we mentioned earlier, the first term is always defined, and
so is the last term. As for the twisted $\eta$ term, if $\dim B$ is even it drops out, while for $\dim B$ odd, one can replace $\ker D_\gamma$ by $\text{Ind} D_\gamma^\dagger$, which is always defined. However, the analysis breaks down here.

The thesis is organized as follows. In Chapter 1, with the work of [BC2] as our starting point, we give a formal argument leading to Theorem 0.1. Then a complete proof is given modulo Theorem 1.1 (the asymptotic behavior of $\text{spec}(D_\gamma)$) and Proposition 1.2 (the uniform asymptotic expansion). These are subsequently established in Chapter 2 and Chapter 3, using respectively Melrose's theory on the degenerate elliptic operators [MM] together with perturbation theory [K], and the finite propagation speed technique of [CGT]. In Chapter 4, we discuss the Hodge-Leray theory and prove the refined adiabatic limit formula for signature operators. We then proceed to give the intrinsic characterizations of non-multiplicativity of the signature. There are two appendices. The first one, included for the convenience of the readers, contains algebraic and geometric preliminaries as well as a review of the work of [BC2]. In the second appendix, we give a proof of a special case of Bismut-Cheeger's Families Index Theorem, which is used in the application.
Chapter 1

Adiabatic limit of $\eta$-invariant

By the work of Bismut-Cheeger the study of the adiabatic limit of the $\eta$-invariant of $D_\omega$ is essentially reduced to the study of the large time behavior of the heat kernel of $D_\omega$. In this chapter, we analyze this large time behavior via Theorem 1.1, which gives detailed information about the spectrum of $D_\omega$ in the adiabatic limit. Theorem 1.1 will be proven in the next chapter. We begin, in Section 1, with the statement of a result from [BC2] and Theorem 1.1. Then we present a formal computation, indicating some ideas behind the proof of Theorem 0.1. The justification for it is given in the following sections.
1.1 Formal computation and outline of its proof

For the clarity of the presentation, we develop all the preliminaries and background in Appendix A. We will use freely the notions and notations there.

By virtue of the local regularity result of [BF] (Cf. Appendix A), the $\eta$-invariant of $D_x$ has a heat kernel representation:

$$\eta(D_x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} t^{-1/2} tr(D_x e^{-tD_x^2}) \, dt.$$  

This involves the contribution from the heat kernel for all the time. For the finite time, by exploitation of the so-called Getzler’s transformation, Bismut-Cheeger showed that $tr(D_x e^{-tD_x^2})$ converges pointwisely uniformly (to certain expression; see Appendix A or [BC2] for detail). It can also be shown that in the invertible case, $spec(D_x)$ being uniformly bounded away from 0, the large time contribution is negligible. In general, it is a direct consequence of the results in [BC2] that

**Proposition 1.1** Without the assumption that $D_Y$ is invertible, one can still find a small positive number $\alpha$ such that

$$\lim_{x \to 0} \eta(D_x) = 2 \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \bar{\eta} + \lim_{x \to 0} \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} tr(D_x e^{-tD_x^2}) \, dt, \quad (1.1)$$

provided either one of the limits exists.

Here the second term is the large time contribution alluded to previously. The above discussion indicates that the information about the spectrum of
$D_x$ in the adiabatic limit is crucial in understanding the large time behavior of the heat kernel. In this respect, we have the following result, the proof of which is deferred to Chapter 2.

**Theorem 1.1** For $x > 0$ the eigenvalues of $D_x$ depend analytically on $x$. Thus there are (countably many) analytic functions $\lambda_x$ which describe the spectrum of $D_x$ (i.e. if we let $\lambda_{x,j}$, $j = 1, 2, \ldots$ be the collection of these analytic functions, for any fixed $x > 0$, $\{\lambda_{x,j}\}_{j=1}^\infty$ is the (unordered) spectrum of $D_x$). Moreover,

**A) (asymptotic behavior)** there exists a positive constant $\lambda_0$ such that either $\lambda_x$ is uniformly bounded away from 0 by $\lambda_0$,

$$|\lambda_x| \geq \lambda_0 > 0,$$

or $\lambda_x$ has a complete asymptotic expansion as $x \to 0$,

$$\lambda_x \sim \lambda_1 x + \lambda_2 x^2 + \cdots,$$

where $\lambda_1 \in \text{spec}(D_B \otimes \ker D_Y)$. In the latter case, the correspondence $\lambda_x \leftrightarrow \lambda_1$ is one-to-one;

**B) (uniform remainder estimate)** if $\lambda_x$ corresponds to $\lambda_1$ and $\lambda_1 \neq 0$, then

$$\lambda_x = \lambda_1 x + x^2 C(x) \lambda_1^2,$$  \hspace{1cm} (1.2)

with $|C(x)| \leq C$ uniformly bounded;

**C) (finiteness)** for any $K > 0$,

$$\#\{\lambda_x \mid \lambda_x \sim \lambda_1 x + \cdots \text{ and } |\lambda_1| \leq K\} < +\infty.$$

In particular, the number of eigenvalues with $\lambda_1 = 0$ (those of at least quadratic decay) is finite.
Granted this, the generalized adiabatic limit formula can be seen formally as follows. From Proposition 1.1

\[
\lim_{x \to 0} \eta(D_x) = 2 \int_B \left( \Lambda \left( \frac{R^B}{2\pi} \right) \right) \Lambda + \lim_{x \to 0} \frac{1}{\sqrt{\pi}} \int_{x^{-\alpha}}^{\infty} t^{-1/2} tr(D_x e^{-tD^2_x}) dt.
\]

And by making a substitution (or rescaling)

\[
\lim_{x \to 0} \int_{C_{x^{-\alpha}}}^{\infty} t^{-1/2} tr(D_x e^{-tD^2_x}) dt = \lim_{x \to 0} \int_{C_{x^{-2\alpha}}}^{\infty} t^{-1/2} tr\left( \frac{1}{x} D_x e^{-t(x^2 D^2_x)^2} \right) dt \quad (1.3)
\]

\[
= \lim_{x \to 0} \left( \int_{C_{x^{2\alpha}}}^{\infty} t^{-1/2} \sum_{|\lambda_x| \geq \lambda_0} \frac{1}{x} \lambda_x e^{-t\left(\frac{1}{x} \lambda_x\right)^2} dt \right.
\]

\[
+ \int_{C_{x^{2\alpha}}}^{\infty} t^{-1/2} \sum_{\lambda_x \sim \lambda_0 \lambda_x \neq \lambda_0} \frac{1}{x} \lambda_x e^{-t\left(\frac{1}{x} \lambda_x\right)^2} dt
\]

\[
+ \int_{C_{x^{2\alpha}}}^{\infty} t^{-1/2} \sum_{\lambda_x \sim \lambda_0 \lambda_x \neq \lambda_0} \frac{1}{x} \lambda_x e^{-t\left(\frac{1}{x} \lambda_x\right)^2} dt
\]

\[
= \int_{C_{x^{2\alpha}}}^{\infty} t^{-1/2} \sum_{|\lambda_x| \geq \lambda_0} \lim_{x \to 0} \frac{1}{x} \lambda_x e^{-t\left(\frac{1}{x} \lambda_x\right)^2} dt
\]

\[
+ \int_{0}^{\infty} t^{-1/2} \sum_{\lambda_x \sim \lambda_0 \lambda_x \neq \lambda_0} \lim_{x \to 0} \frac{1}{x} \lambda_x e^{-t\left(\frac{1}{x} \lambda_x\right)^2} dt
\]

\[
+ \lim_{x \to 0} \left( \int_{C_{x^{2\alpha}}}^{\infty} t^{-1/2} \sum_{\lambda_x \sim \lambda_0 \lambda_x \neq \lambda_0} \frac{1}{x} \lambda_x e^{-t\left(\frac{1}{x} \lambda_x\right)^2} dt \right)
\]

\[= 1 + \Pi + \text{III}.
\]

Here we are ignoring technical problems arising from infinite sums.

In calculating I, note that for \( t \in [C x^{2-\alpha}, +\infty) \) and \( |\lambda_x| \geq \lambda_0, t(x^2 \lambda_x)^2 \geq C \lambda_x^{2-\alpha} \). Therefore, \( \lim_{x \to 0} \frac{1}{x} \lambda_x e^{-t(x^2 \lambda_x)^2} = 0 \).

For II, we have \( \frac{1}{x} \lambda_x \to \lambda_1 \) for some \( \lambda_1 \in \text{spec}(D_B \otimes \ker D_Y) \). Thus \( \lim_{x \to 0} \frac{1}{x} \lambda_x e^{-t(x^2 \lambda_x)^2} = \lambda_1 e^{-t\lambda_1^2} \).
Finally, for III, we use Mellin's formula to get
\[ \int_{C \zeta^2 - \alpha}^{+\infty} t^{-1/2} z \lambda e^{-t(1/2) \lambda^2} dt = \]
\[ \text{sgn}(1/2 \lambda z) \int_{C \zeta^{-\alpha}}^{+\infty} u^{-1/2} e^{-u} du. \]
Hence,
\[ \lim_{z \to 0} \int_{C \zeta^{-\alpha}}^{+\infty} t^{-1/2} tr(D_x e^{-tD_x^2}) dt = \]
\[ 0 + \int_{0}^{+\infty} t^{-1/2} \text{tr}((D_B \otimes \ker D_Y) e^{-t(D_B \otimes \ker D_Y)^2}) dt + \lim_{z \to 0} \sqrt{\pi} \sum_{\lambda \sim \lambda^2 z^2 + \ldots} \text{sgn} \lambda z, \]
and consequently,
\[ \lim_{z \to 0} \eta(D_x) = \int \hat{A} \wedge \hat{\eta} + \eta(D_B \otimes \ker D_Y) + \lim_{z \to 0} \sum_{\lambda \sim \lambda^2 z^2 + \ldots} \text{sgn} \lambda z. \]
The above formal calculation suggests the following.

1) Those eigenvalues which are uniformly bounded away from 0 do not contribute.

2) Those eigenvalues which decay exactly linearly in \( z \) give rise to the twisted \( \eta \)-invariant on the base.

3) The finiteness result in Theorem 1.1 justifies the calculation of the contribution coming from those eigenvalues which decay at least quadratically in \( z \).

The justification of 1) involves estimates from the finite propagation speed technique [CGT], while that of 2) is a consequence of Theorem 1.1. One can in fact combine 1) and 2), reformulating them as a relation between the heat kernel on the total space under the metric shrinking of the fibers and the heat kernel on the base. This is done in the next section.

**Remark** Notice that after rescaling, the time interval for integration in (1.3) is \([Cz^2 - \alpha, \infty)\). So there arises the question of small time convergence. This will be dealt with in Section 3.
1.2 Large time behavior of heat kernels

Motivated by the above formal calculation, we rescale the (large) time interval \([C^{2-\alpha}, \infty)\) back to \([C^{2-\alpha}, \infty)\),

\[
\int_{C^{2-\alpha}}^{+\infty} t^{-1/2} tr(D_{x}e^{-tD_{x}^2}) \, dt = \int_{C^{2-\alpha}}^{+\infty} t^{-1/2} tr\left(\frac{1}{x} D_{x}e^{-t(\frac{1}{x} D_{x})^2}\right) \, dt.
\]

This rescaling corresponds to the metric rescaling

\[x^2 g_x = \pi^* g_B + x^2 g_Y,\]

which shrinks the metric along the fibers. The first step towards the proving of the adiabatic limit formula is the following result, which appears to have an interest of its own.

**Proposition 1.2** Let \(D_0 = D_B \otimes \ker D_Y\). There exists \(N > 0\) sufficiently large such that

\[
|tr'(\frac{1}{x} D_{x}e^{-t(\frac{1}{x} D_{x})^2}) - tr(D_0 e^{-tD_0^2})| \leq \frac{C}{t^N},
\]

where \(tr'\) indicates taking trace over those eigenvalues which decay at most linearly in \(x\).

Taking into account the rest of the eigenvalues (i.e., those decaying quadratically) by using Theorem 1.1, we can rewrite the result in Proposition 1.2 in a nicer, although weaker, form,

\[
|tr(\frac{1}{x} D_{x}e^{-t(\frac{1}{x} D_{x})^2}) - tr(D_0 e^{-tD_0^2})| \leq \left(\frac{C}{t^N} + C'\right)x.
\]

Roughly speaking, this formula says that when the total space collapses to the base (in the sense of Cheeger-Gromov, [CG]) its heat kernels converge to the corresponding heat kernels on the base.
The main technical difficulty in the proof of Proposition 1.2 lies in establishing the dividing line between the large eigenvalues (those do not contribute) and the small ones (those give rise to the twisted $\eta$ term). This is determined by the uniform remainder estimate of Theorem 1.1. It turns out that a workable dividing line is given by $\lambda_0 x^a$ for any $0 < a < 1$. We take $a = 1/2$.

**Proof.** One has

$$tr'\left(\frac{1}{x}D_x e^{-t(\frac{1}{2}D_x)^2}\right) = \sum' \frac{1}{x}\lambda_x e^{-t(\frac{1}{2}\lambda_x)^2},$$

where the summation $\sum'$ runs over all eigenvalues of $D_x$ which decay at most linearly. Also

$$tr(D_0 e^{-tD_0^2}) = \sum \lambda_1 e^{-t\lambda_1^2},$$

where the summation runs over all (nonzero) eigenvalues of $D_0$. Thus

$$tr'\left(\frac{1}{x}D_x e^{-t(\frac{1}{2}D_x)^2}\right) - tr(D_0 e^{-tD_0^2}) = \sum' \frac{1}{x}\lambda_x e^{-t(\frac{1}{2}\lambda_x)^2} - \sum \lambda_1 e^{-t\lambda_1^2}.$$

We now divide our summations into

1) Large eigenvalues.

Define

$$f_x(\lambda) = \begin{cases} \lambda e^{-t\lambda^2} & \text{if } |\lambda| \geq \lambda_0/\sqrt{x} \\ 0 & \text{otherwise} \end{cases}, 
\quad h_x(\lambda) = \begin{cases} \lambda^2 e^{-t\lambda^2} & \text{if } |\lambda| \geq \lambda_0/\sqrt{x} \\ 0 & \text{otherwise} \end{cases}$$

Then the operators $f_x(\frac{1}{x}D_x)$ and $h_x(\frac{1}{x}D_x)$ can be defined by the spectral theorem (see the appendix at the end of this chapter), moreover, by the spectral mapping theorem,

$$|tr(f_x(\frac{1}{x}D_x))| = \left| \sum_{|\lambda_x/\sqrt{x}| \geq \lambda_0/\sqrt{x}} \frac{1}{x}\lambda_x e^{-t(\frac{1}{2}\lambda_x)^2} \right|.$$
\[
\leq \sum_{|\lambda_0/\sqrt{x}| \geq \lambda_0/\sqrt{a}} \frac{1}{x} \lambda \left| e^{-t(\lambda_0/\sqrt{a})^2} \right| \leq \sum_{|\lambda_0/\sqrt{x}| \geq \lambda_0/\sqrt{a}} \left( \frac{1}{x} \lambda_0 \right)^2 e^{-t(\lambda_0/\sqrt{a})^2} = \text{tr}(h_{\lambda}(x)D_x).
\]

Choose a nonnegative function \( \varphi_\lambda(\lambda) \in C^\infty(-\infty, +\infty) \) such that \( \varphi_\lambda(\lambda) \equiv 0 \) for \( |\lambda| \leq \lambda_0/2\sqrt{a} \) and \( \varphi_\lambda(\lambda) \equiv 1 \) for \( |\lambda| \geq \lambda_0/\sqrt{a} \). Further \( \|\varphi_\lambda\|_{C^2} \leq C \) uniformly. Set
\[
H = H_\lambda(\lambda) = \lambda^2 e^{-t\lambda^2} \varphi_\lambda(\lambda).
\]

Clearly we have
\[
\text{tr}(h_{\lambda}(x)D_x)) \leq \text{tr}(H_\lambda(x)D_x).
\]

Let \( k_{H_\lambda} = k_{H_\lambda(x)D_x} \) denote the (Schwartzian) kernel of the operator \( H_\lambda(x)D_x \) (with respect to the metric \( x^2 g_x \)). We want to estimate \( k_{H_\lambda} \) by the finite propagation speed technique of Cheeger-Gromov-Taylor [CGT] (see also the appendix at the end of this chapter).

First of all, note that \( xD_x \) corresponds to the metric \( x^2 g_x \). The sectional curvature of this metric are bounded by \( C\frac{1}{x} \), and the injectivity radius is bounded from below by \( Cx \) (Cf. [C2]). Therefore, by (1.17), one has the pointwise estimates
\[
|k_{H_\lambda}| \leq \frac{C}{x^{n+1}} \sum_{k=0}^{n+1} \int_0^{+\infty} |\hat{H}^{(k)}(s)| ds,
\]

where \( n = \dim M \). Thus it suffices to estimate \( \int_0^{+\infty} |\hat{H}^{(k)}(s)| ds \). To do this, write
\[
\int_0^{+\infty} |\hat{H}^{(k)}(s)| ds = \int_0^{+\infty} \frac{1}{(1+s^2)(1+s^2)} |\hat{H}^{(k)}(s)| ds.
\]

(1.6)
We will show that
\[ |(1 + s^2)\hat{H}^{(k)}(s)| \, ds \leq \frac{C(k)}{t^{(n+5)/2}} e^{-t\lambda_0/4\varepsilon}. \]

By calculus and the basic properties of the Fourier transform, we have
\[
(1 + s^2)\hat{H}^{(k)}(s) = \hat{H}^{(k)}(s) + s^2 \hat{H}^{(k)}(s)
= \hat{H}^{(k)}(s) + (s^2 \hat{H}(s))^{(k)} - 2k(s \hat{H})^{(k-1)} - (k - 1)^2 \hat{H}^{(k-1)}(s)
= \hat{H}^{(k)}(s) + (\hat{H}^n(s))^{(k)} - 2k(\hat{H}^{(k-1)}) - (k - 1)^2 \hat{H}^{(k-1)}(s). \tag{1.7}
\]

Now
\[
|\hat{H}^{(k)}(s)| = |\int e^{-is\lambda}(-i\lambda)^k H(\lambda) d\lambda|
\leq C \int_{|\lambda| \geq \lambda_0/2\sqrt{\varepsilon}} |\lambda^{k+2} e^{-t\lambda^2/2}| d\lambda
\leq 2 \int_{\lambda_0/2\sqrt{\varepsilon}}^{\infty} \frac{C(k)}{t^{(k+1)/2}} \lambda e^{-t\lambda^2/2} d\lambda = \frac{C(k)}{t^{(k+1)/2}} e^{-t\lambda_0^2/4\varepsilon}. \tag{1.8}
\]

The last step comes from the following elementary inequality
\[
\lambda^2 e^{-t\lambda^2} \leq t^{-j} C(j) e^{-t\lambda^2/2}. \tag{1.9}
\]

One can apply the same argument as above to estimate $\hat{H}^{(k-1)}$ and $(\hat{H}^n)^{(k)}$. Plug (1.8) and the corresponding estimates for $\hat{H}^{(k-1)}$ and $(\hat{H}^n)^{(k)}$ in (1.7), and together with (1.5), (1.6), one finds
\[
|k_{H^n\left(1 \frac{D}{x}\right)}| \leq \frac{C}{x^{n+5} t^{(n+5)/2}} e^{-t\lambda_0^2/4\varepsilon}.
\]

Thus one can integrate to obtain
\[
tr\left(H^n\left(1 \frac{D}{x}\right)\right) \leq \frac{C}{x^{n+5} t^{(n+5)/2}} e^{-t\lambda_0^2/4\varepsilon},
\]
where \( p \) denotes the dimension of the base \( B \). Consequently,

\[
| \sum_{|\lambda_z| > \lambda_0/\sqrt{x}} \frac{1}{x} \lambda_z e^{-i\frac{1}{2} \lambda_z^2} | \leq \frac{C}{x \sqrt{(n+\delta)/2}} e^{-i\lambda_0^2/4x}. \tag{1.10}
\]

**Remark** This estimate shows that those eigenvalues \( \lambda_z \) such that \( |\lambda_z| \geq \lambda_0 \sqrt{x} \) do not contribute in the limit. The choice of \( \sqrt{x} \) is not essential here, see the explanation immediately after Proposition 1.2. In fact, if we choose \( x^\beta \) instead and \( \beta \) sufficiently small (e.g. \( 2\beta < \alpha \)), we can use the argument from [C2] (see also [BC2]) to give a simpler proof of this fact.

2) Small eigenvalues.

If \( |\lambda_z| < \lambda_0 \), then by Theorem 1.1, \( \frac{1}{x} \lambda_z \to \lambda_1 \) for some \( \lambda_1 \in \text{spec}(D_0) \). Since we are taking \( tr' \), we can assume \( \lambda_1 \neq 0 \). Now

\[
| \sum_{|\lambda_z| < \lambda_0 \sqrt{x}} \frac{1}{x} \lambda_z e^{-i\frac{1}{2} \lambda_z^2} - \sum_{|\lambda_z| < \lambda_0 \sqrt{x}} e^{-i\lambda_1^2} | \\
\leq | \sum_{|\lambda_z| < \lambda_0 \sqrt{x}} \left( \frac{1}{x} \lambda_z - \lambda_1 \right) e^{-i\frac{1}{2} \lambda_z^2} | + | \sum_{|\lambda_z| < \lambda_0 \sqrt{x}} (e^{-i\frac{1}{2} \lambda_z^2} - e^{-i\lambda_1^2}) | \\
+ | \sum_{|\lambda_z| \geq \frac{\lambda_0}{\sqrt{x}}} \lambda_1 e^{-i\lambda_1^2} | = \text{I} + \text{II} + \text{III}. \tag{1.11}
\]

By the same argument as we show (1.10) (in a simpler situation), it is easy to see that

\[ \text{III} \leq \frac{C}{x^{\beta/2}} e^{-i\lambda_0^2/4x}. \]

To estimate \( \text{I} \) and \( \text{II} \), we have to use the uniform remainder estimate in Theorem 1.1. Note that the remainder there is bounded by a uniform constant times the square of \( \lambda_1 \), which can be arbitrarily large. To cope with this problem, we now use the fact that the summations are over \( |\lambda_z| < \lambda_0 \sqrt{x} \).
and the estimates (2.25), (2.26) in the proof of Theorem 1.1 to show that \( \lambda_1 \) can not grow too fast relative to \( x \).

Since \( \frac{1}{x} \lambda_x \to \lambda_1 \), by taking \( x \to 0 \) in (2.25) where \( \lambda_x \) obeys (2.26), we deduce

\[
|\lambda_1 - \lambda_0| \leq C.
\]

Now \( |\lambda_x| < \lambda_0 \sqrt{x} \) together with (2.26) and (2.25) yields

\[
|\lambda_0| \leq |\lambda_x| + Cx \leq \frac{1}{x} \lambda_x + 2C \leq \frac{\lambda_0}{\sqrt{x}} + 2C \leq \frac{2\lambda_0}{\sqrt{x}}.
\]

Combining the two gives

\[
|\lambda_1| \leq \frac{3\lambda_0}{\sqrt{x}}.
\]

Hence \( \lambda_1 \) grows no faster than \( x^{-1/2} \). It follows from this and the uniform remainder estimate that

\[
\left( \frac{1}{x} \lambda_x \right)^2 = \lambda_1^2 (1 + xC(x)\lambda_1)^2 \geq \frac{\lambda_1^2}{4},
\]

provided \( x \leq \frac{1}{36\lambda_0^2C^2} \). This together with (1.2) implies

\[
I \leq \sum_{\lambda_1 \neq 0} xC \lambda_1^2 e^{-t(\frac{1}{2} \lambda_x)^2} \leq Cx \sum \lambda_1^2 e^{-\lambda_1^2/4}
\]

\[
\leq \frac{C}{tN}.
\]

The last step is by virtue of (1.9) and the standard heat kernel estimates (Cf., say, [CGT]).

For \( \Pi \), notice that by using the inequality \( |e^{-\lambda} - 1| \leq |\lambda| e^{\lambda} \) (which follows by the standard expansion).

\[
|e^{-t(\frac{1}{2} \lambda_x)^2} - e^{-t\lambda_1^2}| = |e^{-t\lambda_1^2} (e^{-t(\frac{1}{2} \lambda_x)^2} - 1)|
\]

\[
\leq tC \lambda_1^2 x e^{-t\lambda_1^2 (1 - x |\lambda_1| |C(x)|)^2}.
\]
Therefore,
\[ \Pi \leq C x t \sum |\lambda|^3 e^{-t\lambda^2/4} \leq \frac{C x}{t^N}. \]

These estimates combined with (1.10) give us the estimate (1.4).

Q.E.D.

**Remark** The above method allows a more general statement. For example, if \( f \in \mathcal{S}([0, +\infty)) \) is in the Schwartz class, then we can show by the same argument (with slight adaptation)

\[ |\text{tr}'(f(\frac{1}{x}D_x)) - \text{tr}(f(D_0))| \leq \frac{C x}{t^N}. \]

An interesting example of such \( f \) would be \( e^{-t\lambda^2} \), which gives rise to the heat kernel. Thus

\[ |\text{tr}'(e^{-t\frac{1}{x}D_x}) - \text{tr}(e^{-tD_0^2})| \leq \frac{C x}{t^N}. \]

An immediate consequence of Proposition 1.2 is

**Corollary 1.1** There exists \( \beta > 0 \) sufficiently small such that

\[
\lim_{x \to 0} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-1/2} \text{tr}(D_x e^{-tD_x^2}) dt
= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-1/2} \text{tr}(D_0 e^{-tD_0^2}) dt + \lim_{x \to 0} \sum_{\lambda \approx \lambda_x \not= \pm} \text{sgn} \lambda_x, \quad (1.12)
\]

if either one of the limits exists in \( \mathbb{R} \).
**Proof.** If one makes a change of variables in the lefthand side of (1.12), one finds

\[
\lim_{x \to 0} \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} t^{-1/2} \text{tr} \left( D_x e^{-tD_x^2} \right) dt = \lim_{x \to 0} \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} t^{-1/2} \text{tr} \left( \frac{1}{x} D_x e^{-t\left(\frac{1}{2} D_x^2\right)} \right) dt = \lim_{x \to 0} \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} t^{-1/2} \text{tr} \left( \frac{1}{x} D_x e^{-t\left(\frac{1}{2} D_x^2\right)} \right) dt + \lim_{x \to 0} \sum_{\lambda \sim \lambda_x} \text{sgn} \lambda_x.
\]

In the last step above, we have used Mellin's formula and the fact that the number of eigenvalues which decay at least quadratically is finite (Cf. Section 1). Now take \(0 < \beta < 1/2N\) and apply Proposition 1.2. We obtain Corollary 1.1.

### 1.3 Small time convergence

In the proceeding section large time behavior of the heat kernel is analyzed by virtue of our detailed knowledge of the (small) eigenvalues. As a result we have obtained (1.12), which furnished a justification for (1) and (2) of Section 1, but only for the time interval \([x^{-2+\beta}, \infty)\), where \(\beta\) is a sufficiently small positive number. Thus we are left with the task of showing that the remaining piece (the small time contribution after the rescaling) is negligible. This is done by extending an argument in [BC2], where they encountered a similar task. Here an important step is to establish a uniform asymptotic expansion.
Proposition 1.3 One has the following uniform pointwise asymptotic expansion,
\[
tr(D_x e^{-tD_x^2}) = \sum_{i=-n}^{N-1} a_i(t)(tx^2)^{i/2} + O((tx^2)^{N/2}),
\]
where \(a_i(t)\)’s are bounded for \(t \geq 1\), and so is \(O(\cdot)\).

The proof of this proposition will be given in Chapter 3. Here we demonstrate that Proposition 1.2 and Proposition 1.3 combined imply the small time convergence.

Proposition 1.4 For any \(\beta > 0\) small, \(2 > \alpha > 0\),
\[
\lim_{x \to 0} \int_{x^{-\alpha}}^{x^{-\alpha+\beta}} t^{-1/2} tr(D_x e^{-tD_x^2}) \, dt = 0.
\]  
(1.14)

Proof. We first rescale the time interval, i.e., make a change of variables in the left-hand side of (1.14). Thus
\[
\lim_{x \to 0} \int_{x^{-\alpha}}^{x^{-\alpha+\beta}} t^{-1/2} tr(D_x e^{-tD_x^2}) \, dt = \lim_{x \to 0} \int_{x^{-\alpha}}^{x^\beta} t^{-1/2} tr\left(\frac{1}{x} D_x e^{-t\left(\frac{1}{x} D_x^2\right)}\right) \, dt.
\]
Now we apply the uniform asymptotic expansion (1.13) to \(tr(D_x e^{-t\left(\frac{1}{x} D_x^2\right)}\)
with \(t/x^2\) as the new time parameter. Thus
\[
tr(D_x e^{-t\left(\frac{1}{x} D_x^2\right)}) = \sum_{i=-n}^{N-1} a_i(t/x^2)(t/x^2)^{i/2} + O(t^{N/2}).
\]

To eliminate the negative powers of \(t\) in the above asymptotic expansion, we put \(u = \frac{t}{x^2} \geq 1\), and rewrite it as
\[
tr(D_x e^{-u D_x^2}) = \sum_{i=-n}^{N-1} a_i(u) u^{i/2} x^i + O(u^{N/2} x^N).
\]
(1.15)
From the proof of Theorem A.3, we know that for fixed $u$, the lefthand side of (1.15) approaches a finite limit as $x \to 0$. Therefore

$$a_i = 0, \quad \text{if } i < 0,$$

and thus

$$tr\left(\frac{1}{x} D_x e^{-t\left(\frac{1}{x} D_x^2\right)^2}\right) = \sum_{i=0}^{N'-1} a_i(t/x^2)^{i/2}x^{-1} + O(t^{N'/2}x^{-1}). \quad (1.16)$$

This is the cancellation in $t$ parameter. To obtain cancellation in $x$ parameter we use the estimate (1.4). Take $M \geq N + 1, N' \geq 2M + 1$ and set $x = t^M \to 0$. By the same reasoning as above (together with Theorem A.1) one finds

$$a_i(t^{1-2M})t^{-M+\frac{i}{2}} = O(1).$$

Or

$$a_i(t^{-1}) = O(t^{M-\frac{i}{2M-1}}).$$

Plugging in (1.16) we finally arrive at

$$tr\left(\frac{1}{x} D_x e^{-t\left(\frac{1}{x} D_x^2\right)^2}\right) = \sum_{i=0}^{N'-1} a'_i t^{(i-1)M-1}x^{-\frac{i+1}{2M-1}} + O(t^{N'/2}/x^{-1}),$$

where $a'_i$ are bounded functions. Consequently

$$\left|\int_{x^{2-\alpha}}^{x^{\beta}} t^{-1/2} tr\left(\frac{1}{x} D_x e^{-t\left(\frac{1}{x} D_x^2\right)^2}\right) dt\right| \leq C x^{\alpha/(4M-2)} \int_{x^{2-\alpha}}^{x^{\beta}} t^{-1/2-M/(2M-1)} dt$$

$$+ \sum_{i=1}^{N'-1} C x^{(i-1)\alpha/(2M-1)} \int_{x^{2-\alpha}}^{x^{\beta}} t^{1/2} dt$$

$$+ C(x^\alpha - x^{2-\alpha})x^{(N'-1)\alpha/2-1}$$

$$\leq C x^{\alpha/(4M-2)} + \sum_{i=1}^{N'-1} C x^{(i-1)\alpha/(2M-1)}(x^\beta/2 - x^{1-\alpha/2})$$

$$+ C(x^\alpha - x^{2-\alpha})x^{(N'-1)\alpha/2-1}.$$
Taking $M \geq \alpha^{-1}$ and $N' \geq 2\alpha^{-1} + 1$ gives us (1.14)

Q.E.D.

As a consequence of Proposition 1.1, Corollary 1.1 and Proposition 1.3, Theorem 0.1 follows.

Appendix to Chapter 1: Finite propagation speed technique

Consider a complete Riemannian manifold $M$. The Laplacian $\Delta$ is a non-negative essentially self-adjoint operator. Thus functions $f(\sqrt{\Delta})$ can be defined by the spectral theorem for unbounded self-adjoint operators, according to the prescription

$$f(\sqrt{\Delta}) = \int_0^\infty f(\lambda) dE_\lambda,$$

where $dE_\lambda$ is the projection valued measure associated with $\sqrt{\Delta}$. The following theorem of Cheeger-Gromov-Taylor, [CGT], gives explicit bounds on the kernel $k_f(\lambda)$.

**Theorem 1.2 (Cheeger-Gromov-Taylor)** Let $M^n$ be complete and $H \leq K_M \leq K$. Fix $x_1, x_2 \in M^n$. Let $d = \rho(x_1, x_2)$ the distance between $x_1$ and $x_2$ and $N$ be an integer such that $4N > n$. Assume that $f$ is a function with the property that up to $N$-th derivatives of its Fourier transform are integrable, $\hat{f}^{(k)} \in L^1(0, \infty), 0 \leq k \leq N$. Let $r_1, r_2 \leq \min(\text{inj}_M, |H|^{-1/2})$, 
\[ |k_f(x_1, x_2)| \leq \frac{c(n)^2}{\pi} \sum_{0 \leq i, j \leq N} r_1^{2i-n/2} r_2^{2j-n/2} \int_{d-r_1-r_2}^{\infty} |\hat{f}(2i+2j)(s)|ds. \quad (1.17) \]

Moreover one also has the estimates for the derivatives,

\[ |k_f(x_1, x_2)|_{\infty, l_1, l_2} \leq \frac{c(n)^2}{\pi} \sum_{0 \leq i \leq N + [l_1/2] + 1} r_1^{2i-n/2} r_2^{2j-n/2} \int_{d-r_1-r_2}^{\infty} |\hat{f}(2i+2j)(s)|ds, \quad (1.18) \]

where \( |\cdot|_{\infty, l_1, l_2} \) denotes the sup norm of up to \( l_1 \)-th derivatives on the first variable and \( l_2 \)-th on the second.

**Proof.** We give the argument in [CGT]. First we construct a nice parametrix for \( \Delta \). To do this we first assume that \( r_j = 1, |K_M| \leq 1 \).

Let \( \phi \) be a smooth function supported on \([0, 1/2]\) with \( \phi|_{[0,1/4]} \equiv 1, |\phi'| < 5 \), and set \( \phi_\epsilon = \phi(r/\epsilon) \). Let

\[ P = \phi_\epsilon(r) \frac{r^{2-n}}{\alpha(n-1)} \quad (P = \phi_\epsilon \log r, n = 2). \]

Finally, let \( \Delta_x(P(x_0, x)) = Q \). Then on \( B_1(x_0) \)

\[ |Q(r)| \leq C_\epsilon r^{2-n}. \]

If \( g \) is a smooth function on \( B_1(x_0) \), then

\[ \int_{B_1(x_0)} P(x_0, x) \Delta g(x)dx = g(x_0) - \int_{B_1(x_0)} Q(x)g(x)dx. \quad (1.19) \]

Fix \( \epsilon \) sufficiently small so that the \( N^2 \) compositions below are defined and apply the standard iteration argument. Thus if we write (1.19) as

\[ P\Delta = I - Q, \]
and set
\[ P = (I + Q + \cdots + Q^{N-1})P, \quad N = \left\lfloor \frac{n}{4} \right\rfloor + 1, \]
\[ Q = Q^N. \]

Then it follows easily that
\[ P^N(\Delta^N g) + P^{N-1}Q(\Delta^{N-1}g) + \cdots + PQ(\Delta g) + Qg = g. \quad (1.20) \]

The kernels on the left hand are continuous and their sup norms are bounded since the metric is boundedly related to the Euclidean metric in normal coordinates, and thus
\[ |g(x_0)| \leq C(n)(\|g\|_{B_1(x_0)} + \|\Delta g\|_{B_1(x_0)} + \cdots + \|\Delta^N g\|_{B_1(x_0)}). \]

Now we only assume that \(|K_M| \leq K\), and
\[ r \leq \min(|K|^{-1/2}, i(x_0)), \quad N = \left\lfloor \frac{n}{4} \right\rfloor + 1. \]

Via scaling, we have
\[ |g(x_0)| \leq C(n)\left(r^{-n/2}\|g\|_{B_1(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_1(x_0)}\right). \]

Next, consider \(\nabla g\). Note that in (1.19), if we take \(4N > n + 1\), then
\[ \nabla g = \nabla P^N(\Delta^N g) + \nabla P^{N-1}Q(\Delta^{N-1}g) + \cdots + \nabla PQ(\Delta g) + \nabla(Qg), \]
and again the kernels on the righthand side are continuous and their sup norms are bounded. Thus
\[ |g(x_0)|_{\infty, 1} \leq C(n)(r^{-n/2}\|g\|_{B_r(x_0)} + r^{-n/2+2}\|\Delta g\|_{B_r(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_r(x_0)}), \]
provided $4N > n + 1$. In general, we have

$$|g(x_0)|_{\infty, l} \leq C(n)(r^{-n/2}\|g\|_{B_r(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_r(x_0)}),$$

provided $4N > n + l$.

To obtain the corresponding $L^2$-estimates, we exploit the unit propagation speed of the wave operator $e^{is\sqrt{\Delta}}$ to get (see [CGT] for details)

$$\|f(\sqrt{\Delta})u\|_{M \setminus B_r(x_2)} \leq \|u\| \frac{1}{\pi} \int_{R-r}^{\infty} |\hat{f}(s)| ds,$$

provided $\text{supp } u \subset B_r(x_2), \ r < R$. Intuitively, this is because it takes a definite amount of time for the effect of a source to be felt in a region at a distance. By the same token

$$\|\Delta^k f(\sqrt{\Delta})\Delta^l u\|_{M \setminus B_r(x_2)} \leq \|u\| \frac{1}{\pi} \int_{R-r}^{\infty} \hat{f}^{(2k+2l)}(s) ds.$$

To obtain the desired estimate for $k_f(x_1, x_2)$, let

$$g = \Delta^2 k_f(x_1, x_2)(u(x_2)) = \int_M \Delta^2 k_f(x_1, x_2)u(x_2)dx_2,$$

for $u$ supported on $B_{r_1}(x_2)$. Apply the elliptic estimate and $L^2$-estimates above,

$$|\Delta^2 k_f(x_1, x_2)(u)| \leq C(n) \sum_{i=0}^{N} r_1^{2i-n/2} \|\Delta^i \Delta^2 k_f(u)\|_{B_r(x_1)}$$

$$= C(n) \sum_{i=0}^{N} r_1^{2i-n/2} \|\Delta^i k_f \Delta^2 u\|_{B_r(x_1)}$$

$$\leq \frac{C(n)}{\pi} \|u\|_{B_{r_1}(x_2)} \sum_{i=0}^{N} r_1^{2i-n/2} \int_{d-r_1-r_2}^{\infty} |\hat{f}^{(2i+2j)}(s)| ds.$$

Since $u$ is arbitrary, we have, for all $x_1$

$$\|\Delta^2 k_f(x_1, x_2)\|_{B_{r_2}(x_2)} \leq \frac{C(n)}{\pi} \sum_{i=0}^{N} r_1^{2i-n/2} \int_{d-r_1-r_2}^{\infty} |\hat{f}^{2i+2j}(s)| ds.$$
Now apply the elliptic estimates again, we obtain (1.17). The estimate for the derivative can be obtained similarly. Q.E.D
Chapter 2

Asymptotic behavior of spectrum

This chapter is devoted to the proof of Theorem 1.1, which gives the asymptotic behavior of the spectrum of $D_x$ as $x \downarrow 0$. Recall that $D_x$ is associated with the metric which is rescaled in the base direction by $x^{-2}$. From the local point of view, the rescaling is making the operator $D_x$ better since the local geometry is simplifying. And this is the reason that for the finite time behavior of the heat kernel one can work effectively with the so called Getzler's transformation. However, from the global point of view, the elliptic operator $D_x$ becomes degenerate as $x \downarrow 0$. This degeneracy of the elliptic operator $D_x$ is the essential difficulty here. The length of this chapter largely comes out of dealing with it.

In Section 2.1, we utilize the theory developed by Melrose et.al. to
treat $D_\varepsilon$. We show that as $L^2$-operators the resolvent of $\frac{1}{\varepsilon}D_\varepsilon$ depends smoothly on the parameter $\varepsilon$ down to $\varepsilon = 0$. This result, combined with the regular perturbation theory, gives us the asymptotic behavior of the small eigenvalues at $\varepsilon = 0$, in Section 2.2. The finiteness result, which is a consequence of ellipticity, is also discussed there. The uniform remainder estimate is established by a deformation argument where the local geometry of the fibration plays a definite role.

### 2.1 Analysis of degenerate elliptic operators

We shall study the degenerate elliptic operator $D_\varepsilon$ by constructing a parametrix which is uniform down to $\varepsilon = 0$. The construction relies on a calculus of pseudodifferential operators well adapted to the type of the degeneracy exhibited by $D_\varepsilon$. A salient feature of this calculus is that the residual operators of the (first) symbol filtration are not compact operators. In fact the novelty here is to find the right tools to probe the residual operators further.

To begin with, recall that any differential operator $L$ on a smooth manifold $X$, $L \in \text{Diff}(X)$, is locally the sum of products of vector fields. Correspondingly, natural subrings of $\text{Diff}(X)$ may be defined as generated in this sense by certain geometrically natural subalgebras of the Lie algebra of all smooth vector fields. From (A.14) of Appendix A, we know that $D_\varepsilon$ is in the subring of $\text{Diff}(\mathbb{R}_+ \times M)$, generated by vector fields of the type (2.1) below, whose elements (when they degenerate) degenerate in a uniform fashion at the boundary of $\mathbb{R}_+ \times M$. In this context, there is still a
good notion of ellipticity, and a class of pseudodifferential operators exists which contains parametrices with compact remainder for elliptic elements in this subring.

The general program of study of this types of pseudodifferential operators originates from R. Melrose, [M3], and further developed in [M1], [M2], and [MM]. In [MM], the Laplacian in the adiabatic limit is analyzed in detail, which provides the model for the first part of our analysis of $D_x$.

Consider the fibration $Y \to M^n \xrightarrow{\pi} B^p$. To incorporate the parameter $x$, let us consider the product manifold $X = [0, \infty) \times M$. On $X$, we have the space $\mathcal{V}$ of $C^\infty$ vector fields which are tangent to the fibers, $M$, of the product structure and which are as well tangent to the fibers of the fibration, $\pi$, above $M_0 = \{ x = 0 \}$. In fact $\mathcal{V}$ is a locally free sheaf of finite rank, therefore $(C^\infty)$ sections of a vector bundle, which we denote by $\mathcal{V}TM$. In local coordinates $y_1, \ldots, y_p, z_1, \ldots, z_{n-p}$ on $M$ where the $y$'s give the coordinates on $B$, a basis of local generating sections of $\mathcal{V}TM$ is given by

$$x \partial_{y_1}, \ldots, x \partial_{y_p}, \partial_{z_1}, \ldots, \partial_{z_{n-p}}. \tag{2.1}$$

The dual, $\mathcal{V}T^*M$, to $\mathcal{V}TM$ plays an important role below. The dual basis to (2.1) is then

$$x^{-1} dy_1, \ldots, x^{-1} dy_p, dz_1, \ldots, dz_{n-p}. \tag{2.2}$$

There is a natural splitting of $\mathcal{V}T^*M$ at $M_0$,

$$\mathcal{V}T^*M = T^*Y \oplus x^{-1} T^*B. \tag{2.3}$$
The family of metrics $g_x$ on $M$ lifts to a non-degenerate inner product on $\nu TM$ and dually on $\nu T^*M$.

Notice that away from $x = 0$ the bundle $\nu TM (\nu T^*M)$ is isomorphic to the lift to $X$ of the tangent bundle $TM$ (the cotangent bundle $T^*M$) but not naturally so. Therefore the significance of this bundle is at $x = 0$.

From (A.14) we see that our operator

$$D_x : C^\infty(X; F(M)) \rightarrow C^\infty(X; F(M))$$

(2.4)

is a $\mathcal{V}$-differential operator (i.e. generated by $\mathcal{V}$, see Section A.4),

$$D_x \in \text{Diff}_\mathcal{V}(X; F(M)).$$

(2.5)

Away from $x = 0$ this corresponds to nothing but a change of basis. Thus again its importance is at $x = 0$.

With the notations out of our way, we can now state the main result of this section (Compare [MM]).

**Theorem 2.1**

1) Let $\lambda_x \in C^\infty([0, \infty))$. Then there is a parametrix $G$ for $D_x - \lambda_x$,

$$(D_x - \lambda_x)G = Id + F,$$

where $F \in x^{-1}C^\infty(Z; \text{End } F(M))$. Moreover, $G$ is uniform in $x$ in the sense that

$$G : C^\infty(X, F(M)) \rightarrow x^{-1}C^\infty(X, F(M)).$$

2) Assume that $\ker D_Y$ gives a vector bundle on $B$. Let $P = P_{\ker D_Y}$ be the orthogonal projection onto $\ker D_Y$, and set $D_0 = P(\sum f_x \nabla f_x)P = D_B \otimes \ker D_Y$ (see (A.14) of section A.3), the Dirac operator on $B$ coupled
to the vector bundle $\ker D_Y$. If $\lambda \notin \text{spec}(D_0)$, then $(\frac{1}{x} D_x - \lambda)^{-1}$ exists for small $x$ and

$$(\frac{1}{x} D_x - \lambda)^{-1} = x(D_x - \lambda x)^{-1} : L^2(M; F(M)) \to L^2(M; F(M))$$

is $C^\infty$ in $x$ down to $x = 0$.

**Remark** Let $H$ be a Hilbert space. A family of bounded linear operators on $H$ can be thought of as a map

$$\mathbb{R} \to L(H),$$

where $L(H)$ denotes the Banach space of bounded linear operators on $H$. The usual definition of derivative extends to this case, thus the smoothness. Consider a map

$$[0,1] \to L(H).$$

When we say that it is $C^\infty$ down to 0 we mean that all its derivatives are uniformly bounded near 0 (or equivalently it extends to $C^\infty$ map in a neighborhood of 0).

We divide the proof of this theorem into several lemmas. The construction of parametrices is approached through their Schwartz kernels. Although these are distributions on $Z = \mathbb{R}_+ \times M \times M$, they live more naturally on a slightly larger manifold, obtained by “blowing up” $Z$ along the fibre diagonal of $M \times M$ in the boundary of $Z$. This process of blowing up, on which the microlocalization takes place, is the essential point here. Our operator, when lifted to the blown up manifold, becomes transversally elliptic. Thus in the first step of the construction, we can use the
standard elliptic theory. The resulting error term, while smooth on the blown up, is not smooth on \( Z \), however. We remove the singular part by killing the Taylor coefficients of the error term at the points being blown up. This is a global construction on a Euclidean bundle over the fibers which distinguishes the fiber-harmonic spinors at the boundary and involves the inversion of certain model operator on the Euclidean space.

The existence of the resolvent follows from this construction and an asymptotic analysis, which is deferred to Section 4.4. The uniformity of the constructed parametrix and the smooth dependence of the resolvent are obtained by showing the tangential and normal regularity separately. While the former is often a consequence of the construction, the latter is considerably harder, involving detailed analysis and various estimates.

We begin with the notion of blow up. Let

\[ Q = \{(0,p,p'); \pi_L(p) = \pi_R(p')\} \subset Z = [0, \infty) \times M \times M \]

where \( \pi_L \) and \( \pi_R \) are the left and right fibrations. The blow up of \( Z \) along the submanifold \( Q \) is (Cf. [M1], [M2])

\[ Z_Q = SNQ \sqcup [Z \setminus Q], \]

where \( SNQ \) is the inward-pointing unit spherical bundle of \( Q \) in \( Z \). As a set it is given by replacing \( Q \) by \( SNQ \). It comes equipped with the ‘blow-down’ map

\[ \pi_Q : Z_Q \to Z, \]

which is the identity away from the ‘front face’ \( SNQ \), which we will denote \( \text{ff}(Z_Q) \). The blown up space \( Z_Q \) has a unique \( C^\infty \) structure such that \( \pi_Q \)
is $C^\infty$, is a diffeomorphism from $Z_Q \setminus \text{ff}(Z_Q)$ to $Z \setminus Q$ and has rank $\dim B + 2 \dim Y + 1$ at $\text{ff}(Z_Q)$.

The front face $\text{ff}(Z_Q)$ is fibred by hemispheres $\pi_Q^{-1}(q)$, $q \in Q$. In fact the projection of $Q$ down to the right factor of $M$ in $Z$ shows that $\text{ff}(Z_Q)$ fibres over $M$ with fibres $Y \times S^p_+$, where $S^p_+$ is the p-dimensional hemisphere,

$$Y \times S^p_+ \to \text{ff}(Z_Q) \to M.$$ 

The hemisphere $S^p_+$ are parameterized non-singularly by the component $\omega$ of the polar coordinates around $Q$,

$$R = (x^2 + |y - y'|^2)^{1/2}, \quad \omega = \left(\frac{x}{R}, \frac{y - y'}{R}\right), \quad y, \ z, \ z'. \quad (2.6)$$

More usefully for computations one can introduce the projective coordinates

$$x, \ u = \frac{y - y'}{x}, \ y, \ z, \ z'. \quad (2.7)$$

valid in the interior of the front face. Since $u$ takes values in $\mathbb{R}^p$, the interior of each fibre $S^p_+$ of the front face has a Euclidean structure.

The significance of $Z_Q$ is mainly related to the following result.

**Lemma 2.1** The closure, $\Delta_Y$, in $Z_Q$ of the submanifold $\{(x, p, p) ; \ x > 0, \ p \in M\}$ is an embedded submanifold which meets the boundary of $Z_Q$ only in the interior of $\text{ff}(Z_Q)$ and does so transversally, and $D_x$ lifts to a differential operator with smooth coefficients on $Z_Q$ which is transversally elliptic with respect to $\Delta_Y$. Moreover, this lift is tangent to the fibres $Y \times \mathbb{R}^p$ of the interior of $\text{ff}(Z_Q)$ with its restriction to a fibre $D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}$.

**Proof.** Check in local coordinates; Cf. [MM].
1) Symbolic Construction.

This is the usual symbolic calculus, carried out uniformly here on the blown-up $Z_Q$ of $Z$.

**Lemma 2.2** There exists $G_1 \in I^{-1}(Z_Q, \Delta \nu; \text{End } F(M))$ which vanishes in a neighborhood of $\partial Z_Q / f(Z_Q)$ such that

$$(D_z - \lambda_z)G - \text{Id} = F_1 \in C^\infty(Z_Q, \text{End } F(M)).$$

**Proof.** In our notations, we are suppressing the density factor, which we choose to be the Riemannian density $dg_z = x^{-p} \, dg$ (or its lift). Lemma 2.1 above gives the lift of our operator $D_z$. As for the Schwartz kernel of the identity operator, on $Z$ it is a Dirac delta section over the diagonal,

$$\text{Id} = x^p \, h \, \delta(y-y') \, \delta(z-z'),$$

where $h$ is an isomorphism on the bundle, and the factors of $x$ comes from the choice of density. Lifted to $Z_Q$, the factors of $x$ just compensate for the homogeneity of the delta function,

$$\text{Id} = h \, \delta(z-z') \, \delta(u).$$

From Lemma 2.1 we know that $D_z$, therefore $D_z - \lambda_z$, is transversally elliptic to $\Delta \nu$, i.e., its symbol restricted to the conormal bundle of $\Delta \nu$, $N(\Delta \nu)$, is invertible. Now repeated application of Theorem A.6 and Theorem A.5 (equivalent to the recursive procedure in the standard symbolic calculus) finishes the proof. The extra vanishing condition is trivial. Q.E.D.

For the simplicity of notation, in what follows, we will use "Hom" to denote "End $F(M)$" and "F" denote "F(M)". (One should not confuse the
spinor bundle $F$ with the error term $F'$s).

2) Model Problem.
It is important to note that the error $F_1$ is not a smoothing operator on $M$, although it is smoothing for $x > 0$ and depends on $x$ smoothly down to $x = 0$. In order to obtain a compact error, one has to kill all the Taylor coefficients of $F_1$ at $\mathcal{H}(Z_Q)$ so that $F_1$ projects downstairs as a smooth function on $Z$. This is essentially reduced to a model problem on Euclidean space as we shall see.

**Lemma 2.3** There exists $G_2 \in \rho^{p-1}C^\infty(Z_Q, \text{Hom}) + \log \rho C^\infty(Z; \text{Hom})$ such that

$$(D_x - \lambda)G_2 - F'_1 = F'_2 + \log \rho \cdot F''_2$$

where $F'_2, F''_2 \in C^\infty(Z; \text{Hom})$ and $F''_2 \equiv 0$ in the sense of Taylor series at $Q$.

**Proof.** In terms of the polar coordinates (2.6), we can take $R$ to be the defining function of the front face, and $\rho = \omega_0$ that of the non-front face of $Z_Q$. Note that

$$x = \rho R.$$ 

Expand $F_1$ in the direction of $R$:

$$F_1 \sim \sum_{i=0}^{\infty} F_{1,i}(\omega, y, z, z') \, R^i,$$

where $F_{1,i} \in C^\infty(S^B \times U \times Y \times Y, F)$ are the Taylor coefficients, and $U$ is a trivializing neighborhood on $B$. It turns out to be advantageous to rewrite
this expansion in terms $x$ (see below),

$$F_1 \sim \sum_{i=0}^{\infty} (\rho^{-i} F_{1,i}) x^i.$$  

The desired kernel will be constructed in the form

$$G_2 \sim \sum_{i=0}^{\infty} x^i e_i(\omega, y, z, z') + \log \rho \ e',$$

with $e_i \in \rho^{i-1} C^\infty(S_+ \times U \times Y \times Y, F)$, and $e' \in C^\infty(Z, F)$, by solving recursively for the $e_i$'s. The logarithmic term will arise naturally in this recursion.

Writing the Taylor series in terms of $x$ introduces extra singularities in the $e_i$'s, but has the advantage that a corresponding series for $(D_x - \lambda_x)G_2$ is easily obtained:

$$(D_x - \lambda_x)G_2 \sim \sum x^i (D_x - \lambda_x)e_i + (D_x - \lambda_x)[\log \rho \ e'].$$

Now in the projective coordinates (2.7),

$$D_x = (D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}) + z B_{(1)}(x, y, z, \partial_z, \partial_u, \partial_y),$$

with

$$B_{(1)} \sim \sum_{j \geq 1} x^{j-1} B_j(y, z, \partial_z, \partial_u, \partial_y).$$

It follows that $D_x - \lambda_x$ has the same form. Now to solve

$$(D_x - \lambda_x)G_2 \sim F_1,$$

we equate their coefficients, obtaining

$$(D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p})e_i = - \sum_{j=0}^{i-1} B_{i-j} e_j + F_{1,i}.$$  \quad (2.8)
To invert the operator on the lefthand side of (2.8), we treat the harmonic and non-harmonic parts on $Y$ separately. Thus, in terms of the splitting

$$C^\infty(\mathfrak{f}(Z_Q), F) = \ker D_Y \oplus \text{Image } D_Y,$$

$D_Y \otimes 1 + 1 \otimes D_{R^p}$ splits accordingly as

$$
\begin{pmatrix}
1 \otimes D_{R^p} & 0 \\
0 & D_Y \otimes 1 + 1 \otimes D_{R^p}
\end{pmatrix}.
$$

The second piece can be more readily inverted. Hence we will first concentrate on the first one. This is the Dirac operator on the flat $R^p$. However, some care must be taken because of the extra singularities in $\epsilon_i$'s mentioned earlier. We observe that the function $\frac{1}{\rho} = \frac{1}{\omega_0}$ is of precisely linear growth in $|u|$, since

$$\omega_0 u = \frac{x y - y'}{R} \frac{x}{x} = \omega'.
$$

Therefore, the setting for our model problem is

$$D_{R^p} v = f \quad \text{on } R^p
$$

for $f \in C^\infty(R^p, F)$ having a growth condition at infinity. We need to solve $v$ with certain control over its growth. This can be done by introducing suitable spaces. Following [MM], we define for $l \in Z$

$$\hat{J}_l = \{ \hat{f} \in C^\infty(R^p - \{0\}, F); \hat{f}(\eta) = |\eta|^{-2q} g(\eta) \text{ for some } q \in N \text{ and } |\hat{f}(\eta)| \leq C|\eta|^{-l} \}.
$$

There is a regularization map (see [MM]),

$$e : \hat{J}_l \rightarrow S'(R^p, F); \quad e(\hat{f})|_{\eta \neq 0} = \hat{f}.$$
Now define \( \mathcal{J}_i \subset S'(\mathbb{R}^p, F) \) by
\[
g \in \mathcal{J}_i \iff \hat{g} = e(\hat{f}) \text{ for some } f \in \hat{\mathcal{J}}_i.
\]

One checks that
\[
D_{\mathbb{R}^p} : \mathcal{J}_i \to \mathcal{J}_{i-1}
\]
is always an isomorphism. Further any \( f \in \mathcal{J}_i \) is smooth on \( \mathbb{R}^p \) with a complete asymptotic expansion at infinity of the form (Cf. [MM])
\[
f(u) \sim \sum_{j \geq 0} |u|^{l-p-j} f_j(\theta) + \log |u| p(u),
\]
where \( p(u) \) is a polynomial of degree \( \leq l-p \). Since \( F_{1,0} \) has compact support on \( u \in \mathbb{R}^p \), \( F_{1,0} \in \mathcal{J}_0 \). From (2.8), one has
\[
(D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}) e_0 = F_{1,0}.
\]

Let \( \mathcal{J}_i^\perp \) be the part of \( \mathcal{J}_i \) that is orthogonal to \( \ker D_Y \). Then it is not hard to see that
\[
D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p} : \mathcal{J}_i^\perp \to \mathcal{J}_i^\perp
\]
is an isomorphism. From these discussions we see that one can solve (2.9) with \( e_0 \in \mathcal{J}_1 + \mathcal{J}_0^\perp \). Solving (2.8) inductively gives
\[
e_i \in \mathcal{J}_{i+1} + \mathcal{P}_{i+1-p},
\]
where \( \mathcal{P}_{i+1-p} \) denotes the space of polynomials of degree \( \leq i + 1 - p \).

In this way we construct a formal power series. Notice that \( u = \frac{v - v'}{x} \) and this means that the series for \( e' \) (corresponding to the \( \mathcal{P} \) terms) is a Taylor series at \( Q \) on \( Z \). Using Borel's lemma to sum this to a smooth
function on \( Z \) and the other series as a smooth function on \( Z_Q \) gives \( G_2 \) as desired. 

Q.E.D.

3) Uniformity.

We shall slightly modify \( G_2 \) before examining its regularity as an operator.

Write \( G_2 = g'_2 + \log \rho \cdot g''_2 \), then

\[
(D_z - \lambda_z)G_2 = (D_z - \lambda_z)g'_2 + (D_z - \lambda_z)(\log \rho \cdot g''_2) = g'_2 + \log \rho \cdot (D_z - \lambda_z)g''_2.
\]

This implies that \( (D_z - \lambda_z)g''_2 \approx 0 \) at \( Q \). Now let

\[
\tilde{G} = G_1 - (G_2 - \log x \cdot g''_2) = G_1 - g'_2 + \log R \cdot g''_2 \\
\in I^{-1}_0(Z_Q, \Delta_z; Hom) + \rho^{p-1}C^\infty(Z_Q, Hom) + \log R C^\infty(Z; Hom).
\]

Then \( \tilde{G} \) is a true parametrix,

\[
(D_z - \lambda_z)\tilde{G} = Id + F_2
\]

where \( F_2 \in x^{p-1}C^\infty(Z, Hom) \). Taking into account the singular density, one has \( F_2 : C^\infty(X, F) \rightarrow x^{-1}C^\infty(X, F) \). We prove the same for \( \tilde{G} \).

**Lemma 2.4** \( \tilde{G} \) is a uniform parametrix for \( D_z - \lambda_z \),

\[
\tilde{G} : C^\infty(X; F) \rightarrow x^{-1}C^\infty(X; F).
\]

**Proof.** For every \( u \in C^\infty(X; F) \), we shall prove that \( x\tilde{G}u \in C^\infty(X, F) \) by showing the tangential and normal regularity at the boundary separately. The tangential regularity is in the form of conormal regularity (see Section A.4),

\[
\tilde{G} : C^\infty(X; F) \rightarrow A(X; F).
\]
That is, for some $s$, $V_1 \cdots V_s(\mathcal{G}u)$ is in the Sobolev space $H^s(X, F)$ for all $V_i$'s in $\mathcal{V}_b(X)$, the space of vector fields on $X$ tangent to $\partial X$ at $\partial X$. The point here is that there is a lifting map (see [MM] for details)

$$ I_L : \mathcal{V}_b(X) \rightarrow \mathcal{V}_b(Z_Q) $$

such that $I_L(V)$ is also tangent to $\Delta$. The integration by parts then gives

$$ V(\mathcal{G}u) = (I_L(V)\mathcal{G})u + \sum \mathcal{G}(W'u), $$

where $W'$ is the transpose of the projection of $I_L(V)$ onto the right factor $M$ in $Z$. Note that because of the special type of the vector fields involved, there is no boundary contribution. Now $\mathcal{G} \in \mathcal{G}$ implies that $I_L(V)\mathcal{G} \in \mathcal{G}$. Hence it suffices to note that any operator with kernel in $\mathcal{G}$ is bounded from $C^\infty(X; F)$ into some fixed Sobolev space.

To show the normal regularity of $\mathcal{G}$, one investigates the holomorphic properties of its Mellin transform

$$ \frac{x^t}{\Gamma(t)} x \mathcal{G} dx dg \in \dot{C}^{-\infty}(Z; Hom). \quad (2.10) $$

We shall show that as a supported distributional density on $Z$, it extends to an entire function of $t$. To see this we lift it to $Z_Q$, noting that $x = \rho R$ and

$$ x dx dg \in x^{1-p} R^p dR d\omega dy dz dz' = R^{1-p} \nu, $$

with $\nu = dR d\omega dy dz dz'$ a smooth density on $Z_Q$. Thus the lift is of the form

$$ \frac{R^t}{\Gamma(t)} \rho^t R^{1-p} \mathcal{G} \nu \in \dot{C}^{-\infty}(Z_Q; Hom). \quad (2.11) $$
Recall that $\tilde{G} = G_1 - g_2' + \log R g_2''$. $G_1$ will contribute an entire term to (2.11), since it is $C^\infty$ up to $R = 0$ and on its support $\rho \neq 0$. For $g_2'$, $\rho^{1-p}g_2'$ is $C^\infty$, and hence we conclude that
\[
\frac{R^t}{\Gamma(t)} \frac{\rho^t}{\Gamma(t)} R \rho^{1-p} g_2^{1+\nu}
\]
is entire. The presence of the factor $\log R$ introduces the possibility of simple poles at the negative integers of
\[
\frac{R^t}{\Gamma(t)} \frac{\rho^t}{\Gamma(t)} \log R R \rho^{1-p} g_2'' \nu.
\]
Combining these two statements we see that the distribution (2.11) is meromorphic in $\mathbb{C}$ with at most double poles at $-N$.

Consider the form of the residues. Since $\frac{\rho^t}{\Gamma(t)}$ is entire and $R^t$ is entire away from the front face, where $\rho^{1-p} \tilde{G}$ is also smooth, the support of the residue must be contained in the front face. Moreover all these residues must be smooth in $z, z', y$ as distributions in $(R, \omega)$. Projecting down to $Z$ gives (2.10), which is therefore at worst meromorphic with residues supported in $Q$. To remove the poles we need only subtract a distribution of the form
\[
L = (\log x)^2 f_2(x, y, z, z', y - y') + (\log x) f_1(x, y, z, z', y - y'),
\]
where $f_1$ and $f_2$ are smooth in $x$ and are chosen to have the correct Taylor series at $x = 0$ to reproduce the residues. Now applying $D_x - \lambda_x$ we find
\[
(D_x - \lambda_x) \tilde{G} = (D_x - \lambda_x)(\tilde{G} - L) + (D_x - \lambda_x)L \in C^\infty(Z, \text{Hom}).
\]
Since $\frac{\rho^t}{\Gamma(t)} x (\tilde{G} - L) dx \ dg_a \ dg$ is entire by construction the same is true of $\frac{\rho^t}{\Gamma(t)} x (D_x - \lambda_x)(\tilde{G} - L) dx \ dg_a \ dg$. This implies that
\[
(D_x - \lambda_x) f_i \doteq 0 \ (i = 1, 2)
\]
in Taylor series at $x = 0$. However, from the support and regularity conditions on the kernels noted above, the operators defined by $f_i$ are of the form

$$\sum_{j=0}^{\infty} x^j P_j(z, z', y, \partial_y).$$

That is, they are differential operators in $y$ with coefficients which are smoothing operators in $z$ and smooth in $y$. Then (2.12) implies that the ranges of these two formal power series operators lie in the null space of $D_x - \lambda_x$ (in the sense of formal power series). By the lemma below, the latter space has finite dimensional coefficients (see 4) below). Hence $f_i$ vanishes as Taylor series in $z$. By definition this means (2.10) is entire. This is the normal regularity of $\mathcal{G}$.

By Theorem A.7 and Proposition A.4, we then have

$$u \in C^\infty(X, F) \implies x\mathcal{G}u \in \mathcal{A}(X, F) \cap \mathcal{A}'(X, F) = C^\infty(X, F).$$

Q.E.D.

4) Asymptotic Calculus.

In the proof of Lemma 2.4, we made use of the fact that the kernel space of $D_x - \lambda_x$ in the sense of power series has finite dimensional coefficients. To make this precise, let us introduce the space of Laurent series [MM],

$$\mathcal{L}(X, F) = \{u \in C^\infty(X, F), x^q u \in C^\infty(X, F), \text{for some } q \in \mathbb{Z}\}/\mathcal{C}^\infty(X, F),$$

where $\mathcal{C}^\infty(X, F)$ is the space of $C^\infty$ sections vanishing to all orders at the boundary. $\mathcal{L}(X, F)$ is a module over the ring of formal Laurent series

$$\mathcal{L} = \mathcal{L}([0, \infty))$$
in the variable \( x \). Our operator \( D_x - \lambda_x \) acts on \( \mathcal{L}(X,F) \) in the obvious way.

**Lemma 2.5** Assume that \( \ker D_Y \) gives a vector bundle on \( B \). Let \( N \subseteq \mathcal{L}(X,F) \) be the null space of \( D_x - \lambda_x \) acting on the Laurent series. Then \( N \) is finite dimensional as an \( \mathcal{L} \)-module. Further,

a) if \( \lambda_x \sim x\lambda_1 + x^2\lambda_2 + \cdots \) and \( \lambda_1 \not\in \text{spec}(D_0) \) then \( N = 0 \) and \( D_x - \lambda_x \) is an isomorphism on \( \mathcal{L}(X,F) \);

b) if \( \lambda_x \) is real-valued, then \( D_x - \lambda_x \) is an isomorphism on the orthogonal complement of \( N \) with respect to the \( \mathcal{L} \)-inner product naturally induced by \( g_x \).

The proof of this lemma is deferred to Section 4.4.

An immediate consequence of this lemma is the existence of the resolvent.

**Proposition 2.1** If \( \lambda \not\in \text{spec}(D_0) \), then the resolvent \( \left( \frac{1}{x} D_x - \lambda \right)^{-1} \) exists for \( x \) sufficiently small. Further, it lies in a certain space \( xG \) defined below.

**Proof.** Take \( \lambda_x = \lambda x \). By previous discussions, there is a

\[
\tilde{G} \in I_0^{-1}(Z_Q, \Delta_Y; H om) + \rho^{p-1} C^\infty(Z_Q; H om) + \log R \cdot x^{p-1} C^\infty(Z, H om)
\]

such that

\[
(D_x - \lambda x) \tilde{G} = Id + F_2
\]

where \( F_2 \in x^{p-1} C^\infty(Z; H om) \). However, we can modify \( \tilde{G} \) so that \( F_2 \in x^p C^\infty(Z; H om) \). This is because the (leading) coefficient of \( x^{p-1} \) in \( F_2 \) must arise directly from the coefficient of \( x^{p-1} \) in \( \tilde{G}_0 - G_1 \), at least away
from $Q$ where this is $C^\infty$. Thus it must be in the range of $D_Y$. In particular, we can remove it by adding to $\tilde{G}$ a term in $x^{p-1}C^\infty(Z; Hom)$. Therefore $F_2 : C^\infty(X, F) \to C^\infty(X, F)$. In fact, $F_2 : C^\infty(\mathbb{R}^*_+, C^\infty(M, F)) \to C^\infty(X, F)$. Let $u \in C^\infty(X, F)$. Then $F_2 u \in C^\infty(X, F) \rightarrow \mathcal{L}(X, F)$. Since $\lambda \not\in \text{spec}(D_0)$, by Lemma 2.5 (Cf. also its proof), there exists $H(u) \in x^{-1}r(C^\infty(X, F)) \subset \mathcal{L}(X, F)$ so that $F_2 u = (D_\alpha - \lambda x)H(u)$ in the sense of formal series. Let $H_i$ be the coefficients of $x^i$ in $H$ ($i \geq -1$). Then $H_i : C^\infty(X, F) \to C^\infty(M, F)$. In fact, $H_i(C^{-\infty}(M, F)) \subset C^\infty(M, F)$. This implies that $H_i$ is an integral operator with kernel $H_i \in C^\infty(M \times M, Hom)$. Using Borel's lemma to sum up the formal series $H$ then provides us a $G_3 \in x^{-1}C^\infty(Z; Hom)$, or $x^{p-1}C^\infty(Z; Hom)$ with respect to the singular density, such that

$$(D_\alpha - \lambda x)(G_3 u) \equiv F_2 u \text{ at } x = 0,$$

for every $u \in C^\infty(X, F)$. Letting $G' = \tilde{G} - G_3$, one has

$$(D_\alpha - \lambda x)G' = Id + F$$

with $F \equiv 0$ at $x = 0$. Clearly $G'$ has the same regularity as $\tilde{G}$,

$$G' : C^\infty(X; F) \to x^{-1}C^\infty(X; F).$$

Note that $(Id + F)^{-1} = Id + \tilde{F}$ for small $x$ and $\tilde{F} \equiv 0$ at $x = 0$. Putting $G = G'(Id + F)^{-1} = G' + G'\tilde{F}$, it satisfies

$$(D_\alpha - \lambda x)G = Id,$$

i.e. $G = (D_\alpha - \lambda x)^{-1} \in G$ where

$$G = \int_0^{-1}(Z_Q, \Delta_V; Hom) + \rho^{p-1}C^\infty(Z_Q; Hom) + x^{p-1}\log R \cdot C^\infty(Z, Hom) + x^{p-1}C^\infty(Z, Hom).$$
And $xG = (\frac{1}{a}D_x - \lambda)^{-1} \in x\mathcal{G}$. Q.E.D.

Thus, the resolvent of our operator will lie in the space $x\mathcal{G}$, for which we have precise description. We proceed next to study the mapping properties of the elements in this space.

5) $L^2$-boundedness of $x\mathcal{G}$.

We shall show that each of

$$xI_0^{-1}(\mathcal{Z}, \Delta_V; Hom), \quad x\rho^{p-1}C^\infty(\mathcal{Z}; Hom),$$
$$x^p \log R \cdot C^\infty(Z, Hom), \quad x^p C^\infty(Z, Hom)$$

defines $L^2$-bounded operators on $L^2(M, F)$. This is clear for $x^p C^\infty(Z, Hom)$. Now we treat

a) $x\rho^{p-1}C^\infty(\mathcal{Z}; Hom)$.

Since we are using the singular measure $dg_x = x^{-p} dy dz$, if $G$ is a Schwartz kernel, then

$$(G\phi)(x, y, z) = \int_M G(x, y, z, y', z')\phi(y', z')dg_x$$
$$= \int_M x^{-p}G\phi dy' dz'.$$

Now if $G \in x\rho^{p-1}C^\infty(\mathcal{Z}; Hom)$, $G' \stackrel{def}{=} x^{-p}G \in x^{-p+1}\rho^{p-1}C^\infty(\mathcal{Z}; Hom) = R^{1-p}C^\infty(\mathcal{Z}; Hom)$. Therefore it suffices to show for those supported near the front face. By using the Schwarz inequality,

$$\int |(G\phi)(x, y, z)|^2 dy' dz' \leq \int (\int |G'(x, y, z, y', z')\phi(y', z')|dy' dz') \times$$
$$\left( \int |G'(x, y, z, y', z')|dy' dz' \right) \times \left( \int |G'(x, y, z, y', z')^{1/2} dy' dz' \right) dy dz.$$
Let
\[ |G'(R, \omega) = \sup_{z, y', z'} |R^{p-1}G'(R, \omega, z, y', z')| \]
in terms of the polar coordinates (2.6) on \( Z_Q \). This is a smooth function of \( R \) and \( \omega \). Therefore
\[
\int |G'(0, y, z, y', z')| dy' dz' = \int |G'(R, \omega, z, y', z')| R^{p-1} dR d\omega
\leq \int |G'(R, \omega)| dR d\omega = C' < \infty.
\]
Plug this into (2.13) to get
\[
\int |(G\phi)(0, y, z)|^2 dy dz \leq C'' \int \int |G'(0, y, z, y', z')| |\phi(y', z')|^2 dy' dz' dy dz
= C' \left( \int |G'(0, y, z, y', z')| dy dz \right) |\phi(y', z')|^2 dy' dz'
\leq C'^2 \int |\phi(y', z')|^2 dy' dz'.
\]
This is the desired \( L^2 \)-boundedness of \( x R^{p-1} C^\infty(Z_Q; Hom) \). From the proof, we see that the singularity here comes from the factor \( R^{1-p} \), which is cancelled in the polar coordinates. The argument clearly carries over for \( x \log R \cdot C^\infty(Z, Hom) \).

b) \( x I_0^{-1}(Z_Q, \Delta_Y; Hom) \).

One reduces it to the \( L^2 \)-boundedness of
\[ x I_0^{-\infty}(Z_Q, \Delta_Y; Hom) \subset x C^\infty(Z_Q, \Delta_Y; Hom). \]

Then, since elements of \( x I_0^{-\infty}(Z_Q, \Delta_Y; Hom) \) vanish on the non-front faces of the boundary, where \( \rho = 0 \), the same argument as above goes through for \( x I_0^{-\infty}(Z_Q, \Delta_Y; Hom) \). For the reduction we need to establish the following lemma.
Lemma 2.6 If \( A \in I^0(Z_Q, \Delta_V; H om) \), and \( \nu \in C^\infty(M, \Omega M) \) is a positive density on \( M \), then there exists a constant \( C > 0 \) and \( B \in I^0(Z_Q, \Delta_V; H om) \), self-adjoint with respect to \( \nu \) such that

\[
A^*A = -B^2 + C + R, \quad R \in I^{-\infty}(Z_Q, \Delta_V; H om),
\]

where \( A^* \) is taken with respect to \( \nu \).

Before we go into the proof of the lemma, we explain a little on the notations. Note that (see [M1])

\[
I^0(Z_Q, \Delta_V; H om) \subset C^{-\infty}(Z_Q; H om) = C^{-\infty}(Z; H om).
\]

Thus every \( G \in I^0(Z_Q, \Delta_V; H om) \) defines

\[
G : \quad C^\infty(X, F) \quad \rightarrow \quad C^\infty(X, F)
\]

\[
\cup \quad \cup
\]

\[
C^\infty(\{x\} \times M, F) \quad \rightarrow \quad C^\infty(\{x\} \times M, F).
\]

The element \( \nu \) gives a positive density on every \( \{x\} \times M \), with respect to which we can define

\[
G^* : C^\infty(\{x\} \times M, F) \rightarrow C^\infty(\{x\} \times M, F).
\]

One checks that \( G^* \) is still in \( I^0(Z_Q, \Delta_V; H om) \) and its symbol behaves like the usual symbols. (This is nothing but simultaneously taking adjoints of the operators for each parameter.)

**Proof of Lemma 2.6.** This follows from the formal properties of the symbolic calculus. Recall the exact sequence of the symbol map

\[
0 \rightarrow I^{-1}(Z_Q, \Delta_V; H om) \rightarrow I^0(Z_Q, \Delta_V; H om) \xrightarrow{\nu} S^*(N^*(\Delta_V), H om) \rightarrow 0
\]

(2.15)
where $N^*(\cdot)$ denotes the conormal bundle, see Theorem A.5. Here we are suppressing the density factor.

Any representative $a \in S^0(N^*(\Delta \nu), \text{Hom})$ of $\sigma_0(A)$ is bounded, so there exists a constant $C$ such that $b_0^2 = C - |a|^2$ for $b_0 \in S^0(N^*(\Delta \nu), \text{Hom})$ real. Using the surjectivity of the symbol map, we can choose $B'_0 \in I^0(Z_Q, \Delta \nu; \text{Hom})$ with $\sigma_0(B'_0) \equiv b_0$, then $B_0 = \frac{1}{2}(B'_0 + (B'_0)^*)$ is self-adjoint and from the symbol calculus

$$R_1 \overset{\text{def}}{=} C - A^*A - B_0^2 \in I^{-1}.$$

Continuing by induction, suppose $B_i \in I^{-i}$ has been chosen for $i = 0, 1, \ldots, N-1$ so that

$$R_N \overset{\text{def}}{=} C - A^*A - (B_0 + \cdots + B_{N-1})^2 \in I^{-N}.$$

The next term $B_N$ should therefore satisfy

$$R_N \equiv 2B_0B_N \mod I^{-N-1}.$$

To do this just choose $\sigma_{-N}(B'_N) = \sigma_{-N}(R_N)/2b_0$ and $B_N = \frac{1}{2}(B'_N + (B'_N)^*)$.

Finally choose $B = \frac{1}{2}(B' + (B')^*)$ where

$$B' \sim \sum_{j=0}^{\infty} B_j \in I^0$$

and we are done. Q.E.D.

With this lemma at our disposal, we can set out to show the $L^2$-boundedness of $xI_0^{-1}(Z_Q, \Delta \nu; \text{Hom})$. Take $\chi \in C^\infty(Z_Q)$ such that $\chi \equiv 1$ on the support of the given element $A$ of $xI_0^{-1}$ and $\chi$ vanishes on the non-front faces of the boundary. Then by (2.14)

$$\|A\phi\|^2 = (\phi, \chi A^* A \chi \phi)$$
\begin{align*}
&= -\|\chi B\phi\|^2 + C\|\chi\phi\|^2 + (\phi, \chi R\phi) \\
&\leq (C + C')\|\phi\|^2, \quad \text{if } \|\chi R\phi\| \leq C'\|\phi\|.
\end{align*}

This reduces the $L^2$-boundedness of $x I_0^{-1}$ to that of $x I_0^{-\infty}$.

6) Smooth Dependence of the Resolvent.

By the principle of uniform boundedness, it suffices to show that for $\phi, \psi \in L^2(M, F)$, $G \in xG$, $F(x) = < G\phi, \psi > \in C^\infty([0, +\infty))$. Here the inner product $<,>$ on $L^2(M, F)$ is induced by $g_M$. First let us note that $F(x) \in L_{loc}([0, +\infty))$. In fact, one has

\begin{equation}
\int |F(x)| |\chi(x)| dx \leq C(\chi) \|\phi\|_2 \|\psi\|_2
\end{equation}

(2.16)

where $C(\chi)$ is a constant depending only on $\chi$, a compactly supported smooth function on $[0, \infty)$. To see this, write out

$$F(x) = \int G(x, y, z, y', z') \phi(y', z') \psi(y, z) dg\, dg'.
$$

Then

$$\int |F(x)| |\chi(x)| dx \leq \int |\chi| |G| |\phi| |\psi| dx \, dg\, dg'.
$$

In polar coordinates we have

$$G \, dx \, dg\, dg' = x^{-p} GR^p dR \, d\omega \, dy \, dz \, dz' = \rho^{-p} G dR \, d\omega \, dy \, dz \, dz'.
$$

Since

$$G \in xG = x I_0^{-1}(Z_Q, \Delta v; Hom) + x \rho^{p-1} C^\infty(Z_Q; Hom)$$

$$+ x^p \log R \cdot C^\infty(Z, Hom) + x^p C^\infty(Z, Hom),$$

\begin{align*}
&= \int G(x, y, z, y', z') \phi(y', z') \psi(y, z) dg\, dg'.
\end{align*}
one finds

\[ \rho^{-p}G \in x\rho^{-p}I_0^{-1}(Z_Q, \Delta_V; \text{Hom}) + R C^\infty(Z_Q; \text{Hom}) \]

\[ + R^p \log R \cdot C^\infty(Z, \text{Hom}) + R^p C^\infty(Z, \text{Hom}). \]

Therefore we obtain (2.16), using, in addition, Lemma 2.6. We now show the tangential regularity.

**Lemma 2.7** \( F(x) \) is conormal to \( \{ x = 0 \} \),

\[ F(x) \in \mathcal{A}([0, +\infty)). \]

**Proof.** To show this, we have to check that \( (x\partial_x)^k F(x) \in L_{loc}([0, +\infty)) \) for any integer \( k \). For \( k = 0 \), this is (2.8). For \( k = 1 \), note that \( x\partial_x \) lifts trivially to \( \mathcal{V}_b(Z_Q) \), tangent to \( \Delta_V \). Since \( G \) (thus \( xG \)) is invariant under the action of this type of vector fields of \( \mathcal{V}_b(Z_Q) \), \( (x\partial_x)G \) is also in \( xG \). Hence

\[ (x\partial_x)F(x) = \langle (x\partial_x)G\varphi, \psi \rangle \in L_{loc}([0, +\infty)). \]

Repeating this argument finishes the proof. Q.E.D.

The next lemma gives the normal regularity.

**Lemma 2.8** The Mellin transform of \( F(x) \),

\[ \frac{x^{t-1}}{\Gamma(t)} F(x) \in \mathcal{C}^{-\infty}([0, +\infty)), \]

extends to an entire function in \( t \in \mathbb{C} \), i.e. \( F(x) \in \mathcal{A}'([0, +\infty)) \).

**Proof.** Note that \( \frac{x^{t-1}}{\Gamma(t)} F(x) = \langle \frac{x^{t-1}}{\Gamma(t)} G\varphi, \psi \rangle \). Hence \( \frac{x^{t-1}}{\Gamma(t)} F(x) \) is holomorphic on the half plane \( \text{Re } t \geq 1 \). Now if \( \varphi \in C^\infty(M, F) \), then Lemma 2.4 implies \( G\varphi \in C^\infty(Z, F) \). Consequently \( \frac{x^{t-1}}{\Gamma(t)} F(x) \) is entire.
For $\phi \in L^2(M, F)$, choose $\phi_\delta \in C^\infty(M, F)$ such that $\|\phi_\delta - \phi\|_2 \to 0$ as $\delta \to 0$. Set

$$F_\delta(x) = \langle G\phi_\delta, \psi \rangle.$$

We claim that $\left\{ \frac{x^{t-1}}{\Gamma(t)} F_\delta(x) \right\}_\delta$ is a normal family of entire functions. Granted this, the limit of the normal family $\left\{ \frac{x^{t-1}}{\Gamma(t)} F_\delta(x) \right\}_\delta$ (which exists by the normality and the principle of uniform boundedness, i.e. the Banach-Steinhaus Theorem) will also be an entire function with values in $C^{-\infty}([0, +\infty))$. On the other hand, one has for $\Re t > 1$, $\frac{x^{t-1}}{\Gamma(t)} F_\delta(x) \to \frac{x^{t-1}}{\Gamma(t)} F(x)$ weakly by (2.8). This proves the lemma.

It remains to establish the claim. We shall show that the family is locally uniformly bounded. For this note that for $\Re t > 1$, and $\chi \in C^\infty([0, \infty))$ compactly supported,

$$\left| \int_0^\infty \frac{x^{t-1}}{\Gamma(t)} F_\delta(x) \chi(x) \, dx \right| \leq \frac{C(\chi) a(\chi) \Re t - 1}{|\Gamma(t)|} \int |F_\delta| \|\chi\| \, dx \leq \frac{C(\chi) a(\chi) \Re t - 1}{|\Gamma(t)|} \|\phi\|_2 \|\psi\|_2,$$

where the last inequality comes from (2.16) (and $\|\phi_\delta\|_2 \leq 2\|\phi\|$). Since $\gamma(t) \overset{def}{=} \frac{t^{t-1}}{\Gamma(t)}$ satisfies the functional equation

$$\frac{d}{dx} \gamma(t) = \gamma(t - 1), \quad (2.17)$$

using integration by parts, we have for $\Re t > -k + 1$,

$$\left| \int_0^\infty \frac{x^{t-1}}{\Gamma(t)} F(x) \chi(x) \, dx \right| \leq \frac{C(\chi) a(\chi) \Re t + k - 1}{|\Gamma(t + k)|} \sum_{i=0}^k \|\partial^i_x F_\delta\|_{L^1_{loc}}. \quad (2.18)$$

We want to estimate $\|\partial^k_x F_\delta\|_{L^1_{loc}}$ in terms of $\|\phi\|_2$ and $\|\psi\|_2$. Without loss of generality let us assume that $F$'s and $F_\delta$'s have compact support. Define

$$F_M(t) \overset{def}{=} \int_0^\infty x^{t-1} F(x) \, dx, \quad F_\gamma(t) = \int_0^\infty \frac{x^{t-1}}{\Gamma(t)} F(x) \, dx.$$
Then
\[ F_\gamma(t) = \frac{F_M(t)}{\Gamma(t)}. \] (2.19)

The following are the basic properties of \( F_M(t) \).

a) \((\partial_x F)_M(t) = -(t - 1)F_M(t - 1)\).

b) \((x \partial_x F)_M(t) = -tF_M(t)\).

c) (Plancherel’s identity) \(< F, \tilde{F} > = \int_{\text{Re} t = 0} F_M(t) \tilde{F}_M(t) dt\).

Verifying a) and b) is just computations, while c) follows from Plancherel’s identity for the Fourier transform, when we make the substitution \( z = e^{-iu} \).

Now \( F(x) \in C^\infty([0, +\infty)) \) implies that \( F_\gamma(t) \) is entire. It follows from (2.19) that \( F_M(t) \) is then meromorphic with at most simple pole at negative integers. If, in addition, \( F(x) \) vanishes near \( x = 0 \), \( F_M(t) \) will then be entire. This can be seen trivially from the definition.

Take \( \varphi(x) \in C^\infty([0, +\infty)) \) identically 1 on \([1, +\infty)\) and 0 on \([0, 1/2]\), and put \( \varphi_\epsilon(x) = \varphi(x/\epsilon) \). Apply Plancherel’s identity to \( \varphi_\epsilon(x) \partial_x^k F_\delta(x) \) to obtain
\[
\int_0^\infty \varphi_\epsilon^2(x) |\partial_x^k F_\delta(x)|^2 dx = \int_{\text{Re} t = 0} (\varphi_\epsilon \partial_x^k F_\delta)_M(t)(\varphi_\epsilon \partial_x^k F_\delta)_M(t) dt.
\]
By the above discussion, \( (\varphi_\epsilon \partial_x^k F_\delta)_M(t)(\varphi_\epsilon \partial_x^k F_\delta)_M(t) \) is entire, which allows us to shift the line of integration \( \text{Re} t = 0 \) to any other such line. Hence
\[
\int_0^\infty \varphi_\epsilon^2(x) |\partial_x^k F_\delta(x)|^2 dx = \int_{\text{Re} t = k+3} (\varphi_\epsilon \partial_x^k F_\delta)_M(t)(\varphi_\epsilon \partial_x^k F_\delta)_M(t) dt.
\]
Letting \( \epsilon \to 0 \) and noting that for \( \text{Re} t > 0 \), \( (\varphi_\epsilon \partial_x^k F_\delta)_M(t) \to (\partial_x^k F_\delta)_M(t) \), we deduce
\[
\int_0^\infty |\partial_x^k F_\delta(x)|^2 dx = \int_{\text{Re} t = k+3} (\partial_x^k F_\delta)_M(t)(\partial_x^k F_\delta)_M(t) dt \tag{2.20}
= \int_{\text{Re} t = k+3} (t - 1)^2 \cdots (t - k - 1)^2(F_\delta)_M(t - k - 1)(F_\delta)_M(t) dt.
\]
To estimate \((F_\delta M(t))\), we use b) above. Thus for any large \(N\),

\[
|((F_\delta)_M(t))| \leq C(1 + |t|)^{-N} \sum_{j=0}^{N} |((x_\partial_x)^j F_\delta)_M(t)|.
\]

By the invariance of \(xG\) under the action of \(x_\partial_x\), \((x_\partial_x)^j F_\delta\) obeys an estimate of the type (2.16), which yields

\[
|((x_\partial_x)^j F_\delta)_M(t)| \leq C a^{\text{Re} t - 1} \|\phi\|_2 \|\psi\|_2,
\]

provided \(\text{Re} t > 1\). Combining these two, we finally arrive at

\[
|((F_\delta)_M(t)| \leq C a^{\text{Re} t - 1} (1 + |t|)^{-N} \|\phi\|_2 \|\psi\|_2.
\]

The claim follows from (2.18), (2.20), and (2.21) by taking \(N > k + 1\).

Q.E.D.

**Proof of Theorem 2.1.** The first statement follows from 1), 2), 3). The second statement is in 4). The smooth dependence of the resolvent on the parameter is a consequence of Lemma 2.7, 2.8 and the following fact (see Theorem A.7),

\[
\mathcal{A}([0, +\infty)) \cap \mathcal{A}'([0, +\infty)) = C^\infty([0, +\infty)).
\]

Q.E.D.

### 2.2 Asymptotic behavior of spectrum

The preceding analysis of the resolvent of \(\frac{i}{\pi} D_x\) bears important implications for the spectrum of \(D_x\) in the adiabatic limit. However, before one can apply regular perturbation theory to obtain the asymptotic behavior
of \( \text{spec}(D_z) \), one needs certain finiteness results. To this end we now take a closer look at our operator by exploring the local geometry of the fibration.

In terms of the decomposition

\[
L^2(M, F) = \text{Image } P^\perp \oplus \text{Image } P,
\]

where \( P = P_{\ker D_Y} \), our operator can be written in the matrix form

\[
\frac{1}{x} D_z = \begin{pmatrix}
A_1 & A_2 \\
A_2 & A_3
\end{pmatrix},
\]

where (see (A.14))

\[
A_1 = x^{-1} D_Y + P^\perp \tilde{D}_B P^\perp + x P^\perp \frac{T}{4} P^\perp,
A_2 = P^\perp \tilde{D}_B P + x P^\perp \frac{T}{4} P,
A_3 = D_0 + x P \frac{T}{4} P.
\]

**Lemma 2.9** a) There exists a constant \( \lambda_0 > 0 \) such that \( \text{spec}(|A_1|) \) is contained in the half line \( \left[ \frac{\lambda_0}{x}, +\infty \right) \).

b) \( A_2 \) (consequently \( A_2^* \)) is bounded on \( L^2(M, F) \).

**Proof.** a) is essentially Proposition 4.41 in [BC2], see also the outline of the proof for Theorem A.3.

b) First note that the commutator \([D_Y, \tilde{D}_B] \overset{def}{=} D_Y \tilde{D}_B + \tilde{D}_B D_Y\) is a first order differential operator which acts fiberwise (Cf. [p.51, BC2]). On the other hand \( P^\perp [D_Y, \tilde{D}_B] P = P^\perp D_Y \tilde{D}_B P \) is fiberwisely of finite rank. Therefore \( P^\perp D_Y \tilde{D}_B P = (P^\perp D_Y P^\perp) P^\perp \tilde{D}_B P \) is bounded on \( L^2(M, F) \). But
\((P^\perp D_YP^\perp)^{-1}\) is also bounded on \(L^2(M,F)\). It follows that \(P^\perp \tilde{D}_BP\), consequently \(A_2 = P^\perp \tilde{D}_BP + \varepsilon P^\perp \frac{\varepsilon}{4} P\), is bounded. \(\quad\text{Q.E.D.}\)

To have a first picture of \(\text{spec}(D_x)\), we deform \(\frac{1}{2} D_x\) to an operator whose spectrum we know much more about. For this purpose, the following lemma is quite useful.

**Lemma 2.10** If \(T(\varepsilon)\) is an analytic family of self-adjoint operators on a Hilbert space. Then its eigenvalue \(\lambda(\varepsilon)\) (which can be arranged to be analytic in \(\varepsilon\)) satisfies

\[
\lambda'(\varepsilon) = (T'(\varepsilon)\phi(\varepsilon), \phi(\varepsilon))
\]

(2.22)

where \(\phi(\varepsilon)\) is a normalized eigenvector associated with the eigenvalue \(\lambda(\varepsilon)\).

In particular,

\[
|\lambda'(\varepsilon)| \leq ||T'(\varepsilon)||.
\]

For the proof, see [K, p.391].

**Lemma 2.11** Consider the operator

\[
T = \begin{pmatrix}
A_1 & 0 \\
0 & A_3
\end{pmatrix}.
\]

Its eigenvalues \(\lambda_x\) are analytic in \(x\) for \(x > 0\). Furthermore for the positive constant \(\lambda_0\) of Lemma 2.9, either

\[
|\lambda_x| \geq \frac{\lambda_0}{x},
\]

(2.23)

or \(\lambda_x\) is analytic at \(x = 0\),

\[
\lambda_x = \lambda_0' + \lambda_1' x + \cdots, \text{ with } \lambda_0' \in \text{spec}(D_0).
\]

(2.24)
Moreover for each $\tilde{\lambda}_x$ there corresponds an eigenvalue $\frac{1}{x} \lambda_x$ of $\frac{1}{x} D_x$ so that

$$\left| \frac{1}{x} \lambda_x - \tilde{\lambda}_x \right| \leq C$$  \hspace{1cm} (2.25)

for a uniform constant $C$.

**Proof.** The eigenvalues of $T$ are either eigenvalues of $A_1$ or that of $A_3$. Accordingly, we call them of type I or II. By the previous lemma, the eigenvalues $\tilde{\lambda}_x$ of type I, i.e. of $A_1$ satisfy

$$|\tilde{\lambda}_x| \geq \frac{\lambda_0}{x}.$$  

On the other hand $A_3 = D_0 + xP_\frac{T}{4}P$ is an entire family of self-adjoint operators (in the sense of Kato). Therefore its eigenvalues $\tilde{\lambda}_x$ (type II) depend analytically on $x$ and we have (2.24). In this case, we also have a uniform remainder estimate

$$\tilde{\lambda}_x = \lambda'_0 + x C(x),$$  \hspace{1cm} (2.26)

with $|C(x)| \leq C$ uniformly. This can be shown by the argument below, using the deformation $A_3(\lambda) = D_0 + \lambda xP_\frac{T}{4}P$ and the fact that $\|P_\frac{T}{4}P\| \leq C$.

Now consider the deformation

$$T(\lambda) = \begin{pmatrix} A_1 & \lambda A_2 \\ \lambda A'_2 & A_3 \end{pmatrix},$$

we have $T(0) = T$ and $T(1) = \frac{1}{x} D_x$. For each fixed $x > 0$, $T(\lambda)$ is an entire family of self-adjoint operators. Therefore its eigenvalues depend analytically on $\lambda$. Thus for each $\tilde{\lambda}_x$ of $T(0)$, there corresponds a $\frac{1}{x} \lambda_x$ of $T(1)$. Moreover (see Lemma 2.10)

$$\left| \frac{1}{x} \lambda_x - \tilde{\lambda}_x \right| \leq \int_0^1 \|T'(\lambda)\| d\lambda \leq C,$$  \hspace{1cm} (2.27)
where the last inequality follows from b) of Lemma 2.9.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1** For \( x > 0 \), \( D_x \) is what is called in [K] the analytic family of type (B) of self-adjoint operators. The first statement follows immediately. Thus the main point here is the study of the behavior of \( \lambda_x \) as \( x \to 0 \). Let us begin with

a) **Regular Perturbation.**

From Theorem 2.1, one knows that for \( \lambda \notin \text{spec}(D_0) \),

\[
\left( \frac{1}{x} D_x - \lambda \right)^{-1} : L^2(M, F) \to L^2(M, F)
\]

is \( C^\infty \). It follows that for any \( \lambda_1 \in \text{spec}(D_0) \), there is \( \epsilon_0 > 0 \) such that

\[
P(x) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_1| = \epsilon_0} \left( \frac{1}{x} D_x - \lambda \right)^{-1} d\lambda
\]

is well-defined and \( C^\infty \) in \( x \) as a family of orthogonal projections. The following lemma says that they are unitarily equivalent to each other.

**Lemma 2.12** Let \( P \) and \( Q \) be (orthogonal) projections on a Hilbert space with \( \|P - Q\| < 1 \), then

\[
W = [1 - (P - Q)^2]^{-1/2} [PQ + (1 - P)(1 - Q)]
\]

is well-defined and invertible (unitary). Furthermore,

\[
W^{-1} = [(1 - Q)(1 - P) + QP][1 - (P - Q)^2]^{-1/2},
\]

\[
W^{-1} PW = Q.
\]
For the proof of this lemma see [p.72, RS].

Lemma 2.12 applied to the family $P(x)$ produces a smooth family of unitary operators $U(x)$ such that

$$U(x)^{-1}P(x)U(x) = P(0) \quad (x \text{ small}).$$

Moreover, $U(x)^{-1}(\frac{1}{x}D_x - \lambda_1 - \epsilon_0)^{-1}U(x)$ is a $C^\infty$ family on $\text{Image } P(0)$:

$$\xrightarrow{\text{Image } P(x)} (\frac{1}{x}D_x - \lambda_1 - \epsilon_0)^{-1} \xrightarrow{\text{Image } P(x)} U(x)$$

$$\xrightarrow{\uparrow U(x)} \xrightarrow{\text{Image } P(0)} \xrightarrow{\text{Image } P(0)}.$$

To apply regular perturbation theory one need to know that $\dim(\text{Image } P(0)) < +\infty$. This is established below.

b) Finiteness.

To finish the proof of our theorem, we note that, being a smooth family of projections,

$$\text{rank } P(x) = \text{rank } P(0).$$

Further, by the discreteness of $\text{spec}(D_x)$ and functional analysis, for $x > 0$,

$$P(x) = \bigoplus_{|\frac{1}{x}\lambda_x - \lambda_1| < \epsilon_0} P_{\lambda_x};$$

(2.28)

where $P_{\lambda_x}$ is the orthogonal projection onto the $\lambda_x$-eigenspace. In other words, $\text{rank } P(x)$ equals to the number of $\lambda_x$ (counted with multiplicity) so that $\frac{1}{x}\lambda_x$ lies in an $\epsilon_0$-ball around $\lambda_1$. For $x$ small such $\lambda_x$ could only correspond to the eigenvalues $\bar{\lambda}_x$ of $A_3$, which obeys

$$\bar{\lambda}_x = \lambda_0' + \lambda_1'x + \cdots$$
with $\lambda'_0 \in \text{spec}(D_0)$. This, together with (2.27) and (2.26), implies that $\text{rank } P(x)$ is bounded from above by the number of eigenvalues of $D_0$ lying in a $(2C + \epsilon_0)$-ball around $\lambda_1$, counted with multiplicity. There are only finitely many of them, yielding

$$\text{rank } P(x) < +\infty,$$

i.e.

$$\dim(\text{Image } P(0)) < +\infty.$$

Thus $U(x)^{-1}(\frac{1}{x} D_x - \lambda_1 - \epsilon_0)^{-1} U(x)$ is a smooth family on a finite dimensional vector space. By standard functional analysis so is its inverse $U(x)^{-1}(\frac{1}{x} D_x - \lambda_1 - \epsilon_0) U(x) = U(x)^{-1}(\frac{1}{x} D_x) U(x) - \lambda_1 - \epsilon_0$. One can then apply the following lemma (Cf. [Sn], Lemma 5.2), showing the existence of complete asymptotic expansions for those $\lambda_x$ with $\frac{1}{x} \lambda_x$ lying near an eigenvalue of $D_0$.

**Lemma 2.13** Let $C(x)$ be a family of symmetric matrices whose element have complete asymptotic expansion in $x$ as $x \to 0$. Then the eigenvalues of $C(x)$ have complete asymptotic expansions.

**Proof.** For any $n$, we can write $C(x) = A_n(x) + B_n(x)$ where $A_n(x)$ is polynomial in $x$ hence analytic and $B_n(x) = O(x^n)$. Since all matrices are symmetric, the difference of eigenvalues of $C(x)$ and those of $A_n(x)$ is $O(x^n)$ and by standard theory (on regular perturbation, [K], [RS]) those of $A_n$ are analytic. Q.E.D.

To show the alternative for the eigenvalues of $D_x$, we use (2.27). Thus
if $\bar{\lambda}_x$ is of type I so that $x|\bar{\lambda}_x| \geq \bar{\lambda}_0$, we have

$$|\lambda_x| \geq x|\bar{\lambda}_x| - Cx \geq \frac{\bar{\lambda}_0}{2} \overset{def}{=} \lambda_0,$$

provided $x \leq \frac{\bar{\lambda}_0}{2C}$. On the other hand, if $\bar{\lambda}_x$ is of type II, we have

$$\left|\frac{1}{x}\lambda_x\right| \leq |\bar{\lambda}_x| + C \leq \lambda_0' + 2C$$

by (2.26). In particular $\{\frac{1}{x}\lambda_x\}$ is uniformly bounded. They must converge as $x \to 0$ to an eigenvalue of $D_0$. Otherwise, a subsequence of $\{\frac{1}{x}\lambda_x\}$ converges to $\lambda_1 \notin spec(D_0)$. This implies that the resolvent $(\frac{1}{x}D_x - \lambda_1)^{-1}$ can not exist no matter how small $x$ is, in contradiction to Theorem 2.1.

It remains to prove (1.2). Choose $\lambda \notin spec(D_0)$ real and consider

$$S(\epsilon) = \begin{pmatrix} A_1 - \lambda & \epsilon A_2 \\ \epsilon A_2^* & A_3 - \lambda \end{pmatrix}.$$ 

Set $T(\epsilon) = S(\epsilon)^{-1}$. One has

$$\frac{d}{d\epsilon} T(\epsilon) = S(\epsilon)^{-1} \frac{dS(\epsilon)}{d\epsilon} S(\epsilon)^{-1} = T(\epsilon) \begin{pmatrix} 0 & A_2 \\ A_2^* & 0 \end{pmatrix} T(\epsilon).$$

By straightforward calculation, one finds

$$T(\epsilon) = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix},$$

where

$$T_1 = (A_1 - \lambda - \epsilon^2 A_2 (A_3 - \lambda)^{-1} A_2^*)^{-1},$$

$$T_2 = -\epsilon (A_1 - \lambda)^{-1} A_2 (A_3 - \lambda - \epsilon^2 A_2^* (A_1 - \lambda)^{-1} A_2)^{-1},$$

$$T_3 = (A_3 - \lambda - \epsilon^2 A_2^* (A_1 - \lambda)^{-1} A_2)^{-1}.$$
From Lemma 2.9, we have

\[ \|T_1\| \leq C x, \quad \|T_2\| \leq C x, \quad \|T_3\| \leq C. \]

It follows then

\[ \left\| \frac{d}{d\epsilon} T(\epsilon) \right\| \leq C x. \]

This, together with (2.22), implies \( |\lambda'(\epsilon)| \leq C x. \) But \( \lambda(1) = (\lambda_x - \lambda)^{-1} \)
and \( \lambda(0) = (\lambda_x - \lambda)^{-1} \) where \( \lambda_x \) is the same as in Lemma 2.11. Thus

\[ \left| (\frac{\lambda_x}{x} - \lambda)^{-1} - (\lambda_x - \lambda)^{-1} \right| \leq C x, \]

or

\[ \left| \frac{\lambda_x}{x} - \lambda_x \right| \leq C x \left| \frac{\lambda_x}{x} - \lambda \right| \left| \lambda_x - \lambda \right|. \quad (2.29) \]

Since \( \lambda_x/x \to \lambda_1 \), we must have \( \lambda_x = \lambda_1 + C(x)x \), where \( |C(x)| \leq C \).

Plugging in (2.29), and using the estimate (2.25) of Lemma 2.11, one obtains

\[ \left| \frac{\lambda_x}{x} - \lambda_1 \right| \leq C x \lambda_1^2. \]

This is (1.2). Q.E.D.
Chapter 3

The uniform asymptotic expansion

In what follows we will also use the notation $D_x e^{-tD_x^2}$ to denote its own kernel, taken with respect to the metric $g$. Therefore $tr(D_x e^{-tD_x^2})$ could mean either the integrated trace or the pointwise trace, depending on the situation. We shall make it clear in the context.

**Proposition 3.1** One has the following pointwise uniform asymptotic expansion (compare [BC2]),

$$tr(D_x e^{-tD_x^2}) = \sum_{i=-n}^{N-1} a_i(t) (tx^2)^{i/2} + O((tx^2)^{N/2}),$$  \hspace{1cm} (3.1)

where $a_i(t)$'s are bounded for $t \geq 1$, and so is $O(\cdot)$.

**Proof.** We first localize the problem, i.e., we transplant the problem to a trivial fibration whose base space is an Euclidean space. Then we construct a parametrix by taking the product of the heat kernels on the
base and along the fibre, and then applying Duhamel's principle. The asymptotic expansion comes from the standard heat expansion.

## 3.1 Localization

Fix a point \( y \in B \), let \( U = B_{\nu_0}(y) \) in the metric \( g_B \) for a fixed small number \( \nu_0 \geq 0 \). Denote by \( K_x(t, y, z, y', z') \) be the kernel of \( D_x e^{-tD_x} \) with respect to the metric \( g = \pi^* g_B + g_Y \).

**Lemma 3.1** \( K_x(t, y, z, y', z') \) decays exponentially as \( x \to 0 \) for those \( y' \not\in U \), i.e., there exist positive constants \( C \) and \( C' \) such that

\[
|K_x(t, y, z, y', z')| \leq C x^{-r} e^{-C'\nu_0 \rho^2/\pi^2}.
\]  

(3.2)

One has the same bounds for the derivatives in \( y, z, y', z' \).

**Proof.** This follows essentially from the theorem of Cheeger-Gromov-Taylor. Let \( \tilde{K}_x(t, y, z, y', z') \) be the kernel of \( D_x e^{-tD_x} \) with respect to the metric \( g_x \), then

\[
K_x(t, y, z, y', z') = x^{-r} \tilde{K}_x(t, y, z, y', z').
\]

By (1.17)

\[
|\tilde{K}_x(t, y, z, y', z')| \leq \frac{C(n)}{\pi^2} \sum_{0 \leq i, j \leq N} \int_0^\infty |f^{(2i+2j)}(s)| ds
\]

where \( r = \min(r_0/3, \text{inj}(M, g)), f(\lambda) = \lambda e^{-t\lambda^2} \overset{\text{def}}{=} \lambda g(\lambda), \) and \( d_x = \text{the distance in } g_x \text{ between } (y, z) \text{ and } (y', z') \geq y', y'/x \geq y, y'/2x + r \) for \( x \) small.

Note that (see [CGT, p.28])

\[
|\tilde{f}^{(k)}(s)| = |\tilde{g}^{(k+1)}(s)| \leq \frac{C(k)}{\pi^{(k+1)/2}} t^{-1/2} e^{-s^2/2t}.
\]
Therefore,

\[ |\tilde{K}_x(t, y, z, y', z')| \leq \frac{C}{\sqrt{t}} \int_{y, y'/z=2}^{+\infty} t^{-1/2} e^{-s^2/8t} dt \]

\[ \leq \frac{C}{\sqrt{t}} e^{-y, y'/2t} e^{-1/32t}. \]

This implies (3.2). The estimates for the derivatives are similar.

Let \( r_0 \) be small so that \( U \) is a trivializing neighborhood of \( \pi \). Consider now the trivial fibration

\[ Y \longrightarrow \tilde{M} = Y \times \mathbb{R}^p \longrightarrow \mathbb{R}^p, \]

\[ \tilde{g} = \tilde{\pi}^* g_{\mathbb{R}^p} + g_Y, \]

where \( g_{\mathbb{R}^p} \) coincides with \( g_B \) in a neighborhood of 0 and is flat outside a compact domain. \( g_Y \) coincides with \( g_{Y, B} \) in a neighborhood of 0 and is constant outside the compact domain. Correspondingly, we have \( \tilde{D}_x \). Let \( \tilde{K}_x(t, y, z, y', z') \) be the kernel of \( \tilde{D}_x e^{-tD_x^2} \) with respect to \( \tilde{g} \). The following lemma says that instead of considering \( D_x \), we can just as well consider \( \tilde{D}_x \).

**Lemma 3.2** \( K_x \) and \( \tilde{K}_x \) differ by an exponentially decaying terms,

\[ |K_x(t, y, z, y', z') - \tilde{K}_x(t, y, z, y', z')| \leq C t e^{-n e^{-C'/t} e^{2t}}. \]

**Proof.** Basically it is because that inside a small neighborhood the two operators coincide while outside the neighborhood the heat kernels decay exponentially. Let \( G_x \) and \( \tilde{G}_x \) be the kernels of \( e^{-tD_x} \) and \( e^{-tD_x} \). Then

\[ K_x = D_x G_x, \quad \tilde{K}_x = \tilde{D}_x \tilde{G}_x. \]
Choose a smooth function $\phi = \phi(y, y')$ such that $\phi(y, y') \equiv 1$ when $y' \in U/3$ and $\phi(y, y') \equiv 0$ when $y' \notin U$. Consider

\[
G_x(t, y, z, y'', z'')\phi(y, y'') - \tilde{G}_x(t, y, z, y'', z'')\phi(y, y'') \\
= \int_0^t \frac{\partial}{\partial s} \int G_x(s, y, z, y', z')\tilde{G}_x(t - s, y', z', y'', z'')\phi(y, y')\phi(y', y'') \, dy' \, dz' \, ds \\
= -\int_0^t \int G_x(s, y, z, y', z')(D_x - \tilde{D}_x)\tilde{G}_x(t - s, y', z', y'', z'') \times \\
\phi(y, y') \phi(y', y'') \, dy' \, dz' \, ds.
\]

Since $D_x = \tilde{D}_x$ in $U$, one has

\[
K_x(t, y, z, y'', z'')\phi(y, y'') - \tilde{K}_x(t, y, z, y'', z'')\phi(y, y'') \\
= -\int_0^t \int D_xG_x(s, y, z, y', z')(D_x - \tilde{D}_x)\tilde{G}_x(t - s, y', z', y'', z'') \times \\
\phi(y, y') \phi(y', y'') \, dy' \, dz' \, ds \\
= -\int_0^t \int_{y', z' \geq r_0} (D_x - \tilde{D}_x)D_xG_x(s, y, z, y', z')\tilde{G}_x(t - s, y', z', y'', z'') \times \\
\phi(y, y') \phi(y', y'') \, dy' \, dz' \, ds.
\]

Apply the previous estimates on the derivatives of $G_x$ and using the fact that as an $L^2$ operator $\tilde{G}_x$ has norm bounded by 1, we find

\[
|K_x(t, y, z, y'', z'')\phi(y, y'') - \tilde{K}_x(t, y, z, y'', z'')\phi(y, y'')| \leq Ctx^{-p}e^{-C'/t\omega^2}.
\]

In particular one has

\[
|K_x(t, y, z, y'', z'') - \tilde{K}_x(t, y, z, y'', z'')| \leq Ctx^{-n}e^{-C'/t\omega^2}
\]

when $y, y'' \leq r_0/3$. For those $y, y''$ such that $y, y'' \geq r_0/3$ each of $K_x$ and $\tilde{K}_x$ are exponentially decaying by (3.2). Combining the two finishes the proof. Q.E.D.
3.2 A “rough” parametrix

By the above result we can replace the original fibration by the trivial one

\[ Y \rightarrow \bar{M} = Y \times \mathbb{R}^p \rightarrow \mathbb{R}^p. \]

We now construct a parametrix for our operator living on the trivial fibration. For the simplicity of notations we will suppress the “bar’s” below.

Recall that the tangent bundle of \( M \) splits as the Whitney sum of its vertical space and its horizontal space by the connection,

\[ TM = T^V \oplus T^H, \]

and the horizontal lift gives an isomorphism

\[ T_y \mathbb{R}^p \xrightarrow{\pi} T^H_{(y,z)} \]

for any \( z \in \pi^{-1}(y) \). Also we have by (A.14),

\[ D_z = D_Y + z \bar{D}_B + z^2 \frac{T}{4}, \quad (3.3) \]

where \( D_Y \) is the family of the signature operators along the fibres and \( \bar{D}_B \) the signature operator on \( \mathbb{R}^p \) lifted via the isomorphism (A.16). Taking the square of (3.4) we have

\[ D_z^2 = D_Y^2 + z[D_Y, \bar{D}_B] + z^2 \bar{D}_B^2 + z^2[D_Y, \frac{T}{4}] + z^3[\bar{D}_B, \frac{T}{4}] + z^4(\frac{T}{4})^2. \]

Recall the basic fact that \([D_Y, \bar{D}_B]\) is a first order differential operator acting fibrewise. Thus we can construct the heat kernel \( F_y(t, x, z, z') \) of \( D_Y^2 + x[D_Y, \bar{D}_B] \). Here the subscript indicates its dependence on the base...
variable. Let $E(t, y, y')$ be the heat kernel of $e^{-tD_B^2}$, originally living on $\mathbb{R}^p$, and lifted to $M$ via the horizontal lift. Consider

$$K_x(t, y, z, y', z') = F_y(t, x, z, z')E(t x^2, y, y'),$$

$$G_x(t, y, z, y', z') = (\partial/\partial t + D^2_x)K_x(t, y, z, y', z').$$

Note that here, and all throughout, we use $y$'s as the coordinates of the base, $\mathbb{R}^p$, and $z$'s as the coordinates of the fibre, $Y$. One has

**Lemma 3.3** $K_x$ is a "rough" parametriz for $\partial/\partial t + D^2_x$, i.e. it satisfies the estimate

$$|\big(\partial/\partial t + D^2_x\big)K_x| \leq C(t x^2)^{-n/2-1/2} x^2$$

for $t \geq 1$.

**Proof.** We calculate,

\[
G_x = (\partial/\partial t F_y)E + F_y(\partial/\partial t E) + (D_B^2 + x[D_B, \bar{D}_B])F_y E + x^2(\bar{D}_B^2 F_y)E \\
+ x^2(\bar{D}_B' F_y)(\bar{D}_B^2 E) + x^2 F_y (\bar{D}_B^2 E) + x^2 F_y (R E) + x^2([D_B, \frac{T}{4}] F_y)E \\
+ x^2 F_y (\bar{D}_B'' E) + x^3([\bar{D}_B, \frac{T}{4}] F_y)E + x^3 F_y (([\bar{D}_B, \frac{T}{4}] E) + x^4(\frac{T}{4})^2 F_y E \\
+ x^2 F_y (\bar{D}_B'' E) + x^3([\bar{D}_B, \frac{T}{4}] F_y)E + x^3 F_y (([\bar{D}_B, \frac{T}{4}] E) + x^4(\frac{T}{4})^2 F_y E(3.5)
\]

Here $\bar{D}_B'$ etc. are first order differential operators acting horizontally, and the curvature factor $R$ comes from the use of the Lichnerowicz formula.

To estimate (3.5), we introduce the following notation. We use $\| \cdot \|_\infty$ to denote the sup norm, and $\| \cdot \|_{\infty, i}$ the sup norm of all the derivatives up to $i$-th
order. Here we make the convention that derivatives are taken with respect to the space variables but not the time variable. Thus \( |F_y'(t, x, z, z')|_{\infty, i_1, i_2, i_3} \) denotes the sup norm of \( F' \) together with its derivatives with respect to \( y' \) up to \( i_1 \)-th order, \( z \) up to \( i_2 \)-th order, \( z' \) up to \( i_3 \)-th order.

First we show that for \( t \geq 1 \),

\[
|F_y'(t, x, z, z')|_{\infty, i_1, i_2, i_3} \leq C(i_1, i_2, i_3).
\]

In fact, for \( i_1 = 0 \), this is the standard heat kernel estimate and follows from, for example, [CGT]. For arbitrary \( i_1 \), note that, since \( \dim \ker D_y \) is assumed to be a (finite) constant and \( D^2_y + x[D_y, \tilde{D}_B] \) is a small perturbation of \( D^2_y \), it follows that all its eigenvalues are bounded away from 0 by a positive constant \( \lambda_0 \) except a finite number of them, which decay in \( x \). Since the number of those is \( \dim \ker D_y \), it follows from Theorem 1.1 and a minimax argument that such eigenvalues must be decaying quadratically. Let \( P \) be the projection on the space spanned by the eigenspaces corresponding to such eigenvalues. \( P \) is a smoothing operator which depends smoothly also on the base variable. Let \( F^0_y(t, x, z, z') \) denote the Schwartz kernel of \( PF_yP \). \( F^0_y(t, x, z, z') \) obeys

\[
|F^0_y(t, x, z, z')|_{\infty, i_1, i_2, i_3} \leq C(i_1, i_2, i_3).
\]

Put \( D_y = D^2_y + x[D_y, \tilde{D}_B] - P \). Then its eigenvalues are bounded away from 0 by \( \lambda_0 \). Therefore if \( F_y(t, x, z, z') \) denotes the heat kernel of \( D_y \), for \( t \geq 1 \),

\[
|F_y(t, x, z, z')|_{\infty, i_2, i_3} \leq C(i_2, i_3)e^{-t\lambda_0}.
\]
Now by Duhamel’s principle,
\[
\frac{\partial}{\partial y} F_y(t, x, z, z') = - \int_0^t \int_Y F_y(s, x, z, z'') \frac{\partial D_Y}{\partial y} F_y(t - s, x, z'', z') \, dz'' \, dt.
\]
To estimate this convolution, we consider the time intervals \( s \leq t/2 \) and \( s \geq t/2 \) separately. If \( s \leq t/2 \), then \( t - s \geq t/2 \). Thinking \( F_y(s, x, z, z'') \) as the kernel of an integral operator on \( L^2(\mathcal{Y}) \) whose norm is bounded by 1, we see that by the elliptic estimate,
\[
\left| \int_Y F_y(s, x, z, z'') \frac{\partial D_Y}{\partial y} F_y(t - s, x, z'', z') \, dz'' \right|_{\infty, \theta_i, \theta_i 3} \\
\leq \|D^2_{Y,1} D^2_{Y,2} \frac{\partial D_Y}{\partial y} F_y(t - s, x, z'', z') \|_{L^2(\mathcal{Y})} \\
\leq C |F_y(t - s, x, z'', z')|_{\infty, 0, 2, 2} \leq C(i_2, i_3) e^{-\lambda_0}.
\]
On the other hand, if \( s \geq t/2 \), note that
\[
\int_Y F_y(s) \frac{\partial D_Y}{\partial y} F_y(t - s) \, dz'' = \int_Y (\frac{\partial D_Y}{\partial y} F_y(s)^* F_y(t - s) \, dz''.
\]
Thus letting \( F_y(t - s, x, z'', z') \) play the role of \( F_y(s, x, z, z'') \), we obtain the same estimate. Hence
\[
\left| \frac{\partial}{\partial y} F_y(t, x, z, z') \right|_{\infty, 0, 2, 2} \leq C(i_2, i_3) t e^{-\lambda_0}.
\]
Or
\[
|F_y(t, x, z, z')|_{\infty, 1, i_2, i_3} \leq C(i_2, i_3).
\]
Proceed inductively, one proves that
\[
|F_y(t, x, z, z')|_{\infty, i_1, i_2, i_3} \leq C(i_1, i_2, i_3).
\]
Combined with (3.6), this gives the estimate we want.
Secondly, for $t x^2$ small, note the standard heat kernel estimate, which follows from the finite propagation speed technique [CGT],

$$|E(t x^2, y, y')|_{\infty, i_1,i_2} \leq C(i_1, i_2)(t x^2)^{-n/2 - i_1/2 - i_2/2}.$$ 

From these discussions we see by (3.5)

$$|G_x| \leq C(t x^2)^{-n/2 - 1/2} x^2,$$

for $t \geq 1$. \hspace{1cm} (3.7) \hspace{1cm} Q.E.D.

Denote

$$G_x \# G_x = - \int_0^t \int G_x(s, y, z, y', z') G_x(t - s, y', z', y'', z'') dy' dz' ds.$$

In general denote

$$G_x^m = \underbrace{G_x \# \cdots \# G_x}_{m+1}.$$ 

A version of Duhamel's principle says

$$e^{-t D_x^2} = \sum_{i=0}^{m-1} (-1)^{i+1} G_x^i \# K_x + G_x^m \# e^{-t D_x^2}. \hspace{1cm} (3.8)$$

To apply this to get the asymptotic expansion, we need estimates on the $m$-fold convolutions $G_x^m$.

**Lemma 3.4** $G_x^m$ improves with $m$,

$$|G_x^m| \leq C(m)(t x^2)^{-n/2 + (m-1)/2} x^2.$$ \hspace{1cm} (3.9)

**Proof.** Let us begin with $G_x \# G_x$. We divide the integral over $[0, t]$ into two, one over $[0, t/2]$ and the other $[t/2, t]$,

$$G_x \# G_x = - \int_0^{t/2} \int G_x(s, y, z, y', z') G_x(t - s, y', z', y'', z'') dy' dz' ds$$

$$- \int_{t/2}^t \int G_x(s, y, z, y', z') G_x(t - s, y', z', y'', z'') dy' dz' ds$$

$$\overset{\text{def}}{=} I + II.$$
By (3.5) and integration by parts we can rewrite I as

\[ I = -\int_0^{t/2} \int K_x(s, y, z, y', z') G'_x(t - s, y', z', y'', z'') \, dy' \, dz' \, ds, \]  

(3.10)

where by the same consideration as in showing (3.7)

\[ |G'_x| \leq C(tx^2)^{-n/2-1}x^4. \]

As an integral operator on the $L^2$ space the norm of $K_x$ is bounded by 1. Therefore,

\[ |I| \leq t/2 \|G'_x(t - s, \cdot, \cdot, y'', z'')\|_2 \]

\[ \leq C(tx^2)^{-n/2}x^2. \]

The estimate for II is similar. In this way one obtains

\[ |G_x\#G_x| \leq C(tx^2)^{-n/2+1}. \]

In estimating the $m$-fold convolution one divides the time interval into $m$ subintervals, on each of them one of the $m$-tuple $t_0 - t_1, \cdots, t_{m-2} - t_m$ is greater than or equal to $t/m$. One lets the kernels corresponding to the other time parameters play the role of $K$ (after integration by parts), and proceeds as for $G_x\#G_x$. Q.E.D.

Now by the standard heat asymptotic expansion and the above estimate we arrive at the uniform asymptotic expansion as claimed.
Chapter 4

Hodge-Leray theory

In Chapter 1, we proved the adiabatic limit formula (1.2) with the global contribution expressed in terms of the very small eigenvalues. Here, confining ourself to signature operators, we identify this global contribution with a topological invariant constructed from the Leray spectral sequence of the fibration. In fact one can recover the Leray spectral sequence from the eigenspaces associated to these eigenvalues. This result, termed the Hodge-Leray theory for the apparent analogy with the Hodge theory, is partly motivated by [MM], where they relate the Leray spectral sequence to the asymptotic solutions.

Included in Section 4.1 is a brief review of signature operators and their relation to Dirac operators. Section 4.2 contains our main results in this chapter, Hodge-Leray theorem, which states that the space of eigenforms corresponding to the eigenvalues which decay polynomially of degree at least \( r \) is isomorphic to the \( E_r \) term of the Leray spectral sequence, and the
adiabatic limit formula of $\eta$-invariant for signature operator. First we recall
the general theory of spectral sequence. Then the space of eigenforms corre-
sponding to asymptotically the same eigenvalues is considered and smooth
parameter-depending basis constructed by the regular perturbation theory
along with Theorem 2.1. From these we construct the spaces used in the
Hodge-Leray theorem. The Hodge-Leray theorem is shown by introducing
an auxiliary complex which incorporates the asymptotic properties. Sec-
tion 4.3 is devoted to the applications of our adiabatic limit formula. First
we introduce the non-multiplicativity of signature and we interpret the the
works of Chern-Hirzebruch-Serre and Atiyah-Singer in this context. Then
we give intrinsic characterizations by virtue of Bismut-Cheeger's Families
Index Theorem for manifolds with boundary and the adiabatic limit for-
mula of $\eta$-invariant. In the last section we will discuss the formal Hodge
decomposition on the space of Laurent series, which is used in Section 2.1
and the proof of the Hodge-Leray Theorem.

4.1 Signature operator

Let $M^n$ be a compact Riemannian manifold. Consider the bundle of differ-
etial forms $\Lambda^* M = \Lambda^* T^* M$. The exterior derivative $d$ acts on it:

$$d : C^\infty(\Lambda^* M) \rightarrow C^\infty(\Lambda^* M).$$

Let $\delta$ be the formal adjoint of $d$, then $d + \delta$ is a first order, self-adjoint,
elliptic differential operator on $\Lambda^* M$ such that

$$(d + \delta)^2 = \Delta,$$
the Hodge Laplacian. The well-known Hodge theorem \([dR]\) is

**Theorem 4.1 (Hodge)** The kernel of \(\Delta\) can be identified with \(H^*(M)\), the cohomology of \(M\). In fact, in every de Rham class, there is a unique harmonic representative.

Now assume that \(M\) is orientable. The orientation together with the metric gives an (oriented) volume form \(dvol\) on \(M\). The Hodge \(*\) operator is defined such that

\[
\alpha \wedge *\beta = (\alpha, \beta) \ dvol
\]

holds for \(\alpha, \beta \in \Lambda^* M\), where \((\alpha, \beta)\) is the induced metric on \(\Lambda^* M\). One checks that \(\delta|_{\Lambda^p} = (-1)^{np+n+1} * d *\). Further, \(*^2|_{\Lambda^p} = (-1)^{np+p}\).

Assume in addition that \(n\) is even. Then \(\tau|_{\Lambda^p} \overset{def}{=} i^p *\) defines an involution on \(\Lambda^* M\), which therefore gives rise to a splitting

\[
\Lambda^* M = \Lambda^*_+ \oplus \Lambda^*_-.
\]

Since \(d + \delta\) actually anti-commutes with \(\tau\), in terms of this splitting,

\[
d + \delta = \begin{pmatrix}
0 & \Lambda^* \\
A & 0
\end{pmatrix},
\]

where

\[
A : C^\infty(\Lambda^*_+) \longrightarrow C^\infty(\Lambda^*_-)
\]

is called the signature operator on the even dimensional manifold \(M\). The terminology comes from the basic fact [AS] that the index of \(A\) gives the signature of \(M\).
If \( \dim M = n \) is odd, such decomposition no longer exists. However, one sets \( \tau|_{\Lambda^p} = i^{p+(n-1)/2} \) and

\[
A = \tau(d + \delta) : C^\infty(\Lambda^* M) \rightarrow C^\infty(\Lambda^* M).
\]

An important fact is that in this case \( A \) is the tangential part of the signature operator on an \((n + 1)\)-dimensional manifold which bounds \( M \). To be more precise, let \( N^{n+1} \) be a compact Riemannian manifold with \( \partial N = M \) and assume that the metric on \( N \) is of product type near the boundary. Then

\[
A_N = \sigma(\partial_u + A_M),
\]

where \( \sigma \) is a bundle isomorphism, \( \partial_u \) is the differentiation along the normal direction of the boundary, and the equality is to be interpreted with the identification (Cf. [APS])

\[
\Lambda_+^*(N) \cong \Lambda^*(M).
\]

The Atiyah-Patodi-Singer theorem in this case yields

\[
\text{sign}(N) = \int_N \mathcal{L} \left( \frac{R_N}{2\pi} \right) - \frac{1}{2} \eta(A_M). \tag{4.1}
\]

Signature operators are Dirac type operators (at least locally so). In fact, one has the bundle isomorphisms (Cf. [Gi]),

\[
\Lambda^* M \cong F(M) \otimes F(M), \quad \Lambda_\pm^* M \cong F_\pm(M) \otimes F(M).
\]
Under these isomorphisms,

\[ A = D_+ \otimes F(M) \quad \text{if } n \text{ is even, or} \]
\[ A = D \otimes F(M \times \mathbb{R}) \quad \text{if } n \text{ is odd.} \]

Suppose now that \( M \) is an oriented odd dimensional manifold which is fibered by (A.9). Here we assume the base \( B \), and the vertical bundle \( T^VM \) are both oriented. (In this case we say the fibration is oriented.) By the Hodge theorem, it is easy to see that all the assumptions in Theorem 0.1 are satisfied by \( A_x \). Therefore the following is an immediate consequence of Theorem 0.1.

**Corollary 4.1** The adiabatic limit of the \( \eta \)-invariant of signature operator always exists. Moreover,

\[
\lim_{x \to 0} \eta(A_x) = 2 \int_B \mathcal{L}(\frac{R}{2\pi}) \wedge \bar{\eta} + \eta(A_B \otimes \ker A_Y) + \lim_{x \to 0} \sum_{\lambda_0, \lambda_1 = 0} \text{sgn} \lambda_z.
\]

For the rest of this chapter, we will develop a tool to identify the last term here with a topological invariant.

### 4.2 Hodge-Leray theory of the very small eigenvalues

In this section we shall prove Theorem 0.2. We will use Serre's construction of the Leray spectral sequence. To relate our space \( E_r \) to the \( E_r \)-term of the Leray spectral sequence, we introduce another complex which incorporates
the asymptotic properties in its filtration. We first recall the general theory of spectral sequence.

Let \((C, d)\) be a filtered differential complex with filtration \(\{F^p(C)\}_{p \in \mathbb{Z}}\), then we can construct differential complexes \((E_i, d_i)\) as follows.

First, set

\[
Z^p_i = F^p(C) \cap d^{-1}(F^{p+i}(C)), \quad -\infty < p < \infty, \ 0 \leq i < \infty,
\]

\[
D^p_i = F^p(C) \cap d(F^{p-i}(C)), \quad -\infty < p < \infty, \ 0 \leq i < \infty,
\]

\[
Z^p_\infty = F^p(C) \cap \ker d, \quad D^p_\infty = F^p(C) \cap d(C).
\]

Clearly

\[
D^p_0 \subset D^p_1 \subset \cdots \subset D^p_\infty \subset Z^p_\infty \subset \cdots \subset Z^p_1 \subset Z^p_0.
\]

Define

\[
E^p_i = Z^p_i / (Z^p_{i-1} + D^p_{i-1}), \quad 1 \leq i \leq \infty,
\]

\[
E^p_0 = Z^p_0 / F^{p+1}(C) + d(F^{p+1}(C)) = F^p(C) / F^{p+1}(C).
\]

The differential \(d\) naturally induces a linear map \(d^p_i : E^p_i \to E^p_{i+1}\) such that \(d^p_{i+1} \circ d^p_i = 0\).

Let \(E_i = \oplus_p E^p_i, \ d_i = \oplus_p d^p_i\). The following proposition is a basic fact of spectral sequence [GHV].

**Proposition 4.1** The projection \(Z^p_i \to E^p_i\) restricts to \(Z^p_{i+1}\) to give a surjective linear map \(Z^p_{i+1} \to H^p(E_i, d_i)\) which in turn induces an isomorphism

\[
E^p_{i+1} \xrightarrow{\cong} H^p(E_i, d_i).
\]
In general, a sequence of differential complexes \((E_i, d_i)\) is called a spectral sequence if \(E_{i+1} \cong H(E_i, d_i)\), and \(E_i\) is usually referred as the \(E_i\)-term of the spectral sequence. Thus a filtered differential complex gives rise to a spectral sequence.

Later we will need to compare several spectral sequences. For that purpose the following basic principle of comparison will be quite useful [GHV].

**Proposition 4.2** Suppose \((C, d)\) and \((C', d')\) are two filtered complexes and there exists a filtration preserving homomorphism \(C \longrightarrow C'\) which also intertwines the differentials. If the homomorphism induces an isomorphism between the \(E_{r_0}\)-terms of the two spectral sequences, then it induces isomorphisms between all \(E_r\)-terms for \(r \geq r_0\).

We now recall Serre's filtration. Consider the fibration \(Y \rightarrow M \xrightarrow{\pi} B\) and basic complex \((C^\infty(\Lambda^*M), d)\). Serre's filtration is defined as follows

\[ \omega^l \in F^i \iff \langle \omega^l(p), \tau_1(p) \wedge \cdots \wedge \tau_l(p) \rangle = 0 \]

whenever \(l - i + 1\) of the tangent vectors \(\tau_j(p)\) are vertical.

In our case, we are given a splitting \(TM = T^H M \oplus T^V M \cong TB \oplus TY\) thus \(T^*M = T^* B \oplus (TY)^*\) and \(\Lambda^*M = \Lambda^* B \otimes \Lambda^* Y\)

Formally, we can use \(a(y, z) \ dy^\alpha \wedge dz^\beta\) (\(y\) local coordinates of \(B\), \(z\) local coordinates of \(Y\)) to indicate such a splitting. Then Serre's filtration is just

\[ F^i = \{a(y, z) \ dy^\alpha \wedge dz^\beta : |\alpha| \geq i\} \]

It is well-known that Serre's filtration gives rise to the Leray spectral sequence of the fibration, \((\bar{E}_i, \bar{d}_i)\).
Consider now complex \((\mathcal{L}(X, Y, \Lambda^*), d)\). It comes equipped with a natural filtration \(F^i = \{\sum_{j \geq i} a_j x^j\}\). By the general theory quoted above, we can construct spectral sequence \((\tilde{E}_i, \tilde{d}_i) = (\oplus_p \tilde{E}_i^p, \oplus_p \tilde{d}_i^p)\). Note that \(x^q : F^i \cong F^{i+q}\) induces \(\tilde{E}_i^p \cong \tilde{E}_i^{p+q}\). Further the diagram

\[
\begin{array}{ccc}
\tilde{E}_i^p & \xrightarrow{\tilde{d}_i^p} & \tilde{E}_i^{p+i} \\
\downarrow x^q & & \downarrow x^q \\
\tilde{E}_i^{p+q} & \xrightarrow{\tilde{d}_i^{p+q}} & \tilde{E}_i^{p+q+i}
\end{array}
\]

commutes, \(x^q \tilde{d}_i^p = \tilde{d}_i^{p+q} x^q\). This implies that \(\tilde{E}_i \cong \tilde{E}_i^0 \otimes \mathcal{L}(x)\) (with the natural gradation induced by that of \(\mathcal{L}(x)\)) and \(\tilde{d}_i^p \cong x^p \tilde{d}_i^0 \otimes 1\).

Explicit calculation showes

\[
\tilde{E}_0 \cong C^\infty(\Lambda^* B \otimes C^\infty(\Lambda^* Z)) \otimes \mathcal{L}(x),
\]

\[
\tilde{E}_1 \cong C^\infty(\Lambda^* B \otimes H^*(Z)) \otimes \mathcal{L}(x),
\]

\[
\tilde{E}_2 \cong H^*(B, H^*(Z)) \otimes \mathcal{L}(x).
\]

By the finite dimensionality of \(\tilde{E}_2\), this spectral sequence must degenerate after finite steps.

**Remark** These quotient modules \((\mathcal{L}(x)\text{-modules})\) can actually be viewed as vector spaces.

To introduce the spaces considered in the Hodge-Leray theorem (Theorem 0.2) we proceed to construct nice bases for certain spaces. These bases will also be used in proving the Hodge-Leray theorem. Let us first introduce some notations.
Let $\Lambda_k$ denote $k$-tuplets of real numbers
\[ \Lambda_k = \{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\} \]
and similarly we use $\Lambda$ to denote infinite sequence of real numbers
\[ \Lambda = \{\lambda_0, \lambda_1, \ldots\}. \]

Also, let $\lambda'_z$ be a function of $z$ which has complete asymptotic expansion at $z = 0$. When we say $\lambda'_z \in \Lambda_k$ we mean that the first $k$ coefficients in the asymptotic expansion of $\lambda'_z$ at $z = 0$ is prescribed by $\Lambda_k$
\[ \lambda'_z \sim \lambda_0 + \lambda_1 z + \cdots + \lambda_{k-1} z^{k-1} + \cdots. \]

Similarly $\lambda'_z \in \Lambda$ means all the coefficients are prescribed by $\Lambda$. Now for the eigenvalues $\lambda_z$ of $A_z$ set $\lambda'_z = \lambda_z / z$ and define
\[ G_{\Lambda_k} = \bigoplus_{\lambda'_z \in \Lambda_k} E(\lambda_z), \quad G_\Lambda = \bigoplus_{\lambda'_z \in \Lambda} E(\lambda_z), \] (4.2)
where $E(\lambda_z)$ is the eigenspace associated to $\lambda_z$. Thus $G_{\Lambda_k}$ is the direct sum of the eigenspaces corresponding to the eigenvalues whose first coefficient in the asymptotic expansion is 0 and whose next $k$ coefficients are prescribed by $\Lambda_k$, and $G_\Lambda$ is the direct sum of the eigenspaces corresponding to the eigenvalues whose first coefficient is 0 and whose other coefficients are all prescribed by $\Lambda$. Obviously
\[ G_\Lambda \subset G_{\Lambda_k} \subset G_{\Lambda_{k-1}}. \]

**Proposition 4.3** Let $\lambda_z$ be an eigenvalue of $A_z$ such that $\lambda_z \sim \lambda_0 z + \lambda_1 z^2 + \cdots$ and let $\Lambda$ be the infinite sequence given by the coefficients of $\lambda_z$. Then
\( G_\Lambda \) is a family of finite dimensional vector space which depends smoothly on \( x \) down to \( x = 0 \). That is, there is an orthonormal basis for \( G_\Lambda \) which is smooth in \( x \) down to \( x = 0 \).

**Proof.** Same analysis as in the proof of Theorem 1.1 reduces it to the finite dimensional case. Therefore it suffices to prove the following lemma

**Lemma 4.1** If \( T(x) \) is a finite dimensional family of self-adjoint operators which depends smoothly on \( x \). Then \( G_{\Lambda_k} = \bigoplus_{\lambda \in \Lambda} E(\lambda_x) \) depends smoothly on \( x \) for all \( k = 1, 2, \ldots, \infty \). (\( \Lambda_\infty = \Lambda \))

**Proof.** Note the slight modification in the definition of \( G_{\Lambda_k} \). The idea is to show by induction that the orthogonal projection \( Q_k(x) \) on \( G_{\Lambda_k} \) depends smoothly on \( x \). Then by Lemma 2.12 we have a smooth family of unitary operators \( V_k(x) \) such that

\[
V_k(x)^{-1}Q_k(x)V_k(x) = Q_k(0).
\]

Therefore the desired basis can be gotten by taking the \( V_k(x) \)-image of an orthonormal basis for Image \( Q_k(0) \).

By the discreteness of \( \text{spec}(T(0)) \) there exists an \( \epsilon > 0 \) such that \( \lambda_0 \) is the only point of \( \text{spec}(T(0)) \) in \( |\lambda - \lambda_0| < \epsilon \). By the first resolvent formula and the smoothness of \( T(x) \)

\[
Q(x) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (T(x) - \lambda)^{-1} d\lambda
\]

is well defined for small \( x \) and is smooth. One easily sees that \( Q(x) = Q_1(x) \).

Now assume that \( Q_{k-1}(x) \) is smooth, then the family

\[
S(x) \overset{\text{def}}{=} V_k(x)^{-1}T(x)V_k(x)
\]
is smooth on Image $Q_{k-1}(0)$ and the eigenvalues of $S(x)$ are the eigenvalues of $T(x)$ having the first $k - 1$ coefficients of their asymptotic expansions prescribed by $\Lambda_{k-1}$. Set

$$R(x) = \frac{S(x) - (\lambda_0 + \lambda_1 x + \cdots + \lambda_{k-1} x^{k-1})}{x^k}.$$ 

This is a smooth family of self-adjoint operators and as above one finds

$$Q_k(x) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_k| = \epsilon} (R(x) - \lambda)^{-1} d\lambda.$$ 

This shows that $Q_k(x)$ is also smooth and the induction is completed.

Since we are in a finite dimensional space the induction process must stop in finite steps, proving it for $k = \infty$.

Now we can proceed to prove the Hodge-Leray theorem of the very small eigenvalues. Consider $\Lambda^0_r = \{0, 0, \ldots, 0\}, \ r \geq 2$. Then $G_{\Lambda^0_r}$ is the direct sum of the eigenspaces associated to eigenvalues which decay at least like $x^r$. From the previous discussion one knows that $G_{\Lambda^0_r}$ is a smooth family of finite dimensional vector spaces, thus it makes sense to consider

$$E_r = \lim_{x \to 0} G_{\Lambda^0_r}$$ 

for $r \geq 2$.

**Theorem 4.2 (Hodge-Leray theorem)** $(E_r, x^{-r} d)$ forms a spectral sequence which is isomorphic to the Leray spectral sequence of the fibration. Moreover, the $*$ map induced by the metric $g_x$ gives rise to the duality map.

**Proof.** We will prove $E_r \cong \tilde{E}_r$ by showing that both of them are isomorphic to $\tilde{E}_r^0$. 


a) $\tilde{E}_r \cong \tilde{E}_r^0$ for all $r \geq 0$.

Consider the natural inclusion:

$$C^\infty(\Lambda^*M) \hookrightarrow \mathcal{L}(\tau^\vee, \Lambda^*)$$

$$a(y, z) dy^\alpha dz^\beta \to x^{\alpha|a(y, z)}(\frac{dy}{x})^\alpha dz^\beta.$$ 

This is clearly filtration preserving. Thus it induces homomorphisms $\iota_r^p : \tilde{E}_r^p \to \tilde{E}_r^p$. Define

$$\iota_r : \tilde{E}_r \to \bigoplus_{p=0}^n \tilde{E}_r^p \to \bigoplus_{p=0}^n \tilde{E}_r^p \to \tilde{E}_r^0,$$

i.e. $\iota_r = \sum_{p=0}^n x^{-p} \iota_r^p$ ($p = \text{dim} B$).

We show that $\iota_r$ is a homomorphism which preserves the differentials.

In fact we know that

$$\tilde{E}_r^p \xrightarrow{\iota_r^p} \tilde{E}_r^{p+r}$$

$$\downarrow \iota_r \quad \downarrow \iota_r^{p+r}$$

$$\tilde{E}_r^p \xrightarrow{d} \tilde{E}_r^{p+r}$$

commutes. Hence $\iota_r^{p+r} d_r^p = \tilde{d}_r^p \iota_r^p$ and $x^q d_r^p = \tilde{d}_r^q x^q$. Therefore

$$x^{-p-r} \iota_r^{p+r} d_r^p = x^{-p-r} \tilde{d}_r^q x^q = d_r^q x^{-p-r}.$$ 

This shows

$$\tilde{E}_r \xrightarrow{d} \tilde{E}_r$$

$$\downarrow \iota_r \quad \downarrow \iota_r$$

$$\tilde{E}_r^0 \xrightarrow{d} \tilde{E}_r^0$$

commutes, i.e. $\iota_r d = \tilde{d}_r$.

Furthermore, the explicit calculation (Cf. p. 81) shows that $\iota_r$ induces isomorphism for $r \leq 2$, therefore it induces isomorphisms on all $r$. 
b) $E_r \cong \tilde{E}_r^0$, for $r \geq 2$.

Recall that the scaled metric $g_z$ induces a bilinear form

$$\mathcal{L}(X, X^*) \times \mathcal{L}(X, X^*) \xrightarrow{\cdot \cdot} \mathcal{L}(x)$$

and $< d, \cdot > = < \cdot, \delta >$. This is the same as saying that $\delta$ is the adjoint of $d$ with respect to $g_z$ for every $z$. (Here of course we are suppressing the $x$-dependence of $\delta$.)

Our operator $A_z$ is, up to sign, $\ast (d + \delta)$. Therefore if $A_z \varphi_z = \lambda_z \varphi_z$, then

$$\|d \varphi_z\|^2 + \|\delta \varphi_z\|^2 = |\lambda_z|^2 \|\varphi_z\|^2.$$  \hspace{1cm} (4.3)

For simplicity we will use the shorthand notation $z \sim x^r$ for $z \in F^r(\mathcal{L}(X, X^*))$. One has (Cf. (4.3))

$$(d + \delta)z \sim x^r \iff dz \sim x^r, \delta z \sim x^r.$$  \hspace{1cm} (4.4)

Basic to our proof is the observation that $(E_r, x^{-r}d)$ does form a spectral sequence, though it is not clear if it comes from a filtered complex. To see this, let us first show that $x^{-r}d$ maps $E_r$ into $E_s$. In fact, by the definition of $E_*$ and (4.3), (4.4), it suffices to show that $d$ leaves each eigenspaces invariant. This can be seen in the following way. Clearly $d$ maps the 0-eigenspace to 0 (by, say, (4.3)). Now let $\lambda_z \neq 0$ be an eigenvalue of $A_z$ and $\varphi_z$ a corresponding eigenform, $A_z \varphi_z = \lambda_z \varphi_z$. But

$$A_z = \pm \ast d \pm d \ast.$$  \hspace{1cm} Thus

$$\varphi_z = \ast d (\pm \varphi_z / \lambda_z) + d \ast (\pm \varphi_z / \lambda_z)$$

$$\xrightarrow{d \varphi_z} \ast d \psi_z + d \ast \psi'_z.$$
Since \( *d\psi_x (d * \psi'_x) \) is coexact (exact) and \( A_x \) takes coexact (exact) forms to coexact (exact) forms, one must have
\[
A_x (*d\psi_x) = \lambda_x (*d\psi_x),
\]
\[
A_x (d * \psi'_x) = \lambda_x (d * \psi'_x).
\]
(4.5)

That is, the eigenspace corresponding to a non-zero eigenvalue splits into the direct sum of coexact and exact eigenspaces. Obviously \( d \) takes exact forms to zero. And (4.6) is the same as
\[
*d(d\psi_x) = \lambda_x(*d\psi_x).
\]

Taking \( d \) of this equation one finds
\[
d * d(*d\psi_x) = \lambda_x d(*d\psi_x), \quad \text{or} \quad d * (d * d\psi_x) = \lambda_x (d * d\psi_x).
\]

This proves the invariance of the \( \lambda_x \)-eigenspace under \( d \).

To see \( E_{r+1} \cong H(E_r, \nu^{-r}d) \), note that by the finite dimensional analog of the Hodge theory
\[
H(E_r, \nu^{-r}d) \cong \ker(\nu^{-r}d) \cap \ker(\nu^{-r}\delta).
\]

Together with (4.3), this implies \( \lambda_x = O(\nu^{r+1}) \) and the converse is easier.

Let \( h_r = \dim E_r \), \( h = \dim E_\infty = \dim H^*(M) \). By Proposition 4.3 we can choose
\[
\varphi_1, \ldots, \varphi_h, \ldots, \varphi_{h_2}
\]
(4.6)
such that \( \varphi_1, \ldots, \varphi_{h_r} \) form a smooth basis for \( E_r \), all \( r \). From (4.4) it is clear that (4.6) induces a linear map
\[
E_r \to \tilde{E}^0_r.
\]
It is a homomorphism between the two differential complex because the two differentials are formally identical. We will show that it is an isomorphism by showing that it is an isomorphism for \( r = 2 \).

1-1: If \( z \in E_r \) s.t. \( z = 0 \) in \( \tilde{E}_r^0 \), then \( z = z_0 + dz_1 \) with \( z_0 \sim x \) and \( dz_0 \sim x^r \), \( z_1 \sim x^{-r+1} \), \( dz_1 \sim x^0 \). But then \( \delta dz_1 = \delta(z - z_0) \sim x \), therefore \( dz_1 \sim x \) and thus \( z \sim x \) i.e. \( z = 0 \) in \( E_r^r \).

onto: Let \( z \in \tilde{E}_r^0 \), then \( z \sim x^0 \), \( dz \sim x^2 \). Consider \( z' = z - \sum_{i=1}^{b_2} z_i \varphi_i, z > \varphi_i, z' \perp E_2 \) and \( dz' \sim x^2 \). By the formal Hodge decomposition (Lemma 2.5) \( z' = (d + \delta)z_1 \), \( z_1 \sim x^{-1} \). Now \( dz' = d\delta z_1 \sim x^2 \Rightarrow \delta z_1 \sim x^2 \), i.e. \( \delta z_1 \in Z_1^1 \). Also \( dz_1 \in D_1^0 \). Hence \( z' = 0 \) in \( \tilde{E}_r^0 \). This show that the map is onto for \( r = 2 \).

Q.E.D.

4.3 Adiabatic limit of \( \eta \)-invariant and non-multiplicativity of signature

In Chapter 1, we proved (1.2), the adiabatic limit formula. There the global contribution is found to be \( \sum_{\lambda_0, \lambda_1 = 0} \lim sgn \lambda_x \). In the light of the Hodge-Leray theorem proved above, it should be possible to characterize this term topologically. This is indeed the case here. To have a topological counterpart of this term, let us first discuss some algebraic aspects of the Leray spectral sequence.

Let \( Y \rightarrow M^n \rightarrow B^p \) be a fibration of closed manifolds. Consider the
Leray spectral sequence \((E_r, d_r)\). One has
\[ E_2 = H^*(B, \mathcal{H}^*(Y)), \quad E_\infty = H^*(M). \]

As a matter of fact \((E_r, d_r)\) has fine algebraic structures as shown in the following theorem [F].

**Theorem 4.3**

1) There is a bigrading on \(E_r\): \(E_r = \bigoplus_{i,j} E_r^{i,j}\), and
\[
    d_r : E_r^{i,j} \rightarrow E_r^{i+r,j-r+1}.
\]
Set \(E_r^k = \bigoplus_{i+j=k} E_r^{i,j}\). Thus \(d_r : E_r^k \rightarrow E_r^{k+1}\).

2) For all \(r \geq 2\), the group \(E_r\) may be endowed with a natural multiplicative structure, i.e., there is a mapping
\[
    E_r^{i,j} \otimes E_r^{k,l} \rightarrow E_r^{i+k,j+l},
\]
\[
    a \otimes b \rightarrow a \cdot b,
\]
which defines the associative and distributive multiplication consistent with the action of differentials,
\[
    d(a \cdot b) = da \cdot b + (-1)^b a \cdot db, \quad \text{if } a \in E_r^k.
\]
The multiplication in \(E_r\) induces that in \(E_{r+1}\). Moreover, the multiplication in \(E_2\) and \(E_\infty\) coincides with their natural ring structures.

Now assume that our fibration is oriented in the sense that both \(B\) and the vertical bundle \(T^V M\) are oriented. The orientation of \(T^V M\) gives a trivialization of the flat line bundle \(\mathcal{H}^{n-p}(Y)\) on \(B\) (note that \(n-p = \dim Y\)). Together with the orientation of \(B\) one finds
\[
    E_2^n = H^p(B, \mathcal{H}^{n-p}(Y)) \cong H^p(B) \cong \mathbb{R}.
\]
Thus the multiplicative structure on $E_2$ gives rise to a Poincaré pairing

$$E_2^k \otimes E_2^{n-k} \longrightarrow \mathbb{R},$$

i.e., $E_2$ is a so called Poincaré ring with differentiation, defined as follows [CHS].

A graded ring $A = \bigoplus_{i=0}^{n} A_i$ is called a Poincaré ring if $\dim A_n = 1$ and the multiplication is graded in the sense that

$$xy = (-1)^{ij}yx \quad \text{if} \quad x \in A_i, y \in A_j.$$

Further the bilinear pairing

$$A_i \otimes A_{n-i} \longrightarrow A_n \cong \mathbb{R}$$

is a Poincaré pairing (i.e. nondegenerate).

A differentiation in a Poincaré ring is a differentiation $d$ of degree +1 such that $dA_{n-1} = 0$ and $d$ is also an anti-derivation,

$$d(xy) = (dx)y + (-1)^i x(dy), \quad \text{if} \quad x \in A_i.$$

It is proved in [CHS] that the derived ring of a Poincaré ring with differentiation is again a Poincaré ring. Thus all $E_r$ for $r \geq 2$ are Poincaré rings, and if we denote $\xi_2$ the natural base for $E_2^n$ induced by the orientations, then it induces a natural base $\xi_r$ for each $E_r^n \ (r \geq 2)$ (Cf. [CHS]). Now consider the bilinear pairing:

$$\tau_r : E_r \otimes E_r \longrightarrow \mathbb{R}$$

$$\tau_r(\varphi \otimes \psi) = < \varphi \cdot d_r \psi, \xi_r > \overset{\text{def}}{=} < \varphi, d_r \psi >$$
It is easy to verify that

\[ \langle \varphi, d, \psi \rangle = (-1)^{(\deg \varphi + 1)(\deg \psi + 1)} \langle \psi, d, \varphi \rangle. \]

In our situation \( n = 4m - 1 \). This implies that \( \tau \), restricted to \( E_r^{2m-1} \), is symmetric. By viewing \( \tau \) as a symmetric matrix one can take its signature, \( \text{sign} \ \tau \). Now define

\[ \tau = \sum_{r \geq 2} \text{sign} \ \tau_r, \]

This is clearly a topological invariant.

**Theorem 4.4 (Adiabatic limit formula)** We have the equality

\[ \tau = \frac{1}{2} \sum_{\lambda_0, \lambda_1 = 0} \lim_{x \to 0} \text{sgn} \lambda_x. \]

Consequently the following adiabatic limit formula holds,

\[ \lim_{x \to 0} \eta(A_x) = 2 \int_B L \left( \frac{R^B}{2\pi} \right) \wedge \tilde{\eta} + \eta(A_B \otimes \ker A_Y) + 2\tau. \]

**Proof.** It suffices to show the first equality (see Corollary 4.1). To do this, let us consider

1) Partial-symmetry of eigenvalues of \( A_x \).

It is enough to consider \( A = *d \pm d^* : \Omega(M) \to \Omega(M) \). Since \( * : \Omega^{\text{even}} \to \Omega^{\text{odd}} \) is an isomorphism, \( A \) is just two copies of its restriction to \( \Omega^{\text{odd}} \). By Hodge decomposition

\[ \Omega^{\text{odd}} = \mathcal{H}^{\text{odd}} \oplus d\Omega^{\text{even}} \oplus \delta\Omega^{\text{even}}. \]

Observe that the operator \( A \) annihilates \( \mathcal{H}^{\text{odd}} \), coincides up to sign with \( d^* \) on \( d\Omega^{\text{even}} \), and with \( *d \) on \( \delta\Omega^{\text{even}} \).
Now $d^* : d\Omega^{2p} \rightarrow d\Omega^{4k-2p-2}, *d : \delta\Omega^{2p} \rightarrow \delta\Omega^{4m-2p}$ and $2p \neq 4m - 2p - 22p = 4m - 2p$ iff $p = m$. Hence we can decompose

$$A = A_0 \oplus A_1 \oplus A_2,$$

where $A_0$ is the zero operator on $\mathcal{H}^{odd}$ and $A_1 = *d$ on $\delta\Omega^{2m}$ and $A_2$ is an operator of the form

$$\begin{pmatrix}
0 & T \\
T^* & 0
\end{pmatrix},$$

therefore having symmetric spectrum.

This shows that

$$\sum_{\lambda_0, \lambda_1 = 0} \frac{1}{2} \lim_{z \to 0} \text{sgn} \lambda_z = \sum_{\lambda_0, \lambda_1 = 0} ' \lim_{z \to 0} \text{sgn} \lambda_z,$$

where $\sum'$ indicates that the summation is over all $\lambda$ which are the eigenvalues of $*d$ on $\Omega^{2m-1}$

2) Since $E_2 = H^*(B, \mathcal{H}(Y))$, it inherits an inner product $(\cdot, \cdot)$, which then induces an inner product on each $E_r$. Define

$$< \varphi, d_r \psi > = (\varphi, *d_r \varphi),$$

then as symmetric mappings the eigenvalues of $\tau_r$ are the same as those of $*d_r$. By Theorem 0.2,

$$d_r \sim x^{-r}d, \quad * \sim *.$$

We see immediately that the first statement of Theorem 4.4 holds. The adiabatic limit formula follows then from ( .2).

Q.E.D.
As an interesting application we now give intrinsic characterizations of the non-multiplicativity of signature for manifolds with boundary.

Chern-Hirzebruch-Serre first studied the multiplicative behavior of signature for closed manifolds [CHS]. They showed that for a fibration \( Z \longrightarrow N^n \longrightarrow B \), we have the multiplicativity

\[
\text{sign} N = \text{sign} B \cdot \text{sign} Z
\]

provided \( Z, B \) are closed manifolds and \( \pi_1(B) \) acts trivially on \( H^*(Z) \). Later Atiyah [A] gave an example showing that this naive sense of multiplicativity does not hold in general. However, he observed that, by the Hirzebruch signature theorem and Atiyah-Singer Family Index Theorem [AS], one does have a generalized multiplicativity,

\[
\text{sign} N = \int_B \mathcal{L}(B) \wedge ch(Sign^Z),
\]

where \( \mathcal{L}(B) \) is a characteristic class which may substitute for the Hirzebruch L-genus, \( Sign^Z \) is the signature bundle of \( Z \) over \( B \) and \( ch(Sign^Z) \) is its Chern character. When \( \pi_1(B) \) acts trivially on \( H^*(Z) \), this reduces to \((4.7)\).

See Introduction for a purely topological formulation of this generalized multiplicativity.

This generalized multiplicativity has a Leray spectral sequence interpretation. Consider the pairing

\[
E^i_r \otimes E^{n-i}_r \longrightarrow \mathbb{R},
\]

\[
\varphi \otimes \psi \longrightarrow \langle \psi, \xi_r \rangle,
\]

where \( E_r \) is the Leray spectral sequence of the fibration and \( \xi_r \) is defined in the same way as before. When \( n = 4k \), the above pairing restricts to
a symmetric pairing on $E_r^{2k}$, whose signature will be denoted by $\sigma_r$. Then (4.8) is equivalent to

$$\sigma_2 = \sigma_{\infty}.$$ 

This follows from [CHS] and the signature theorem for twisted coefficients [L], [AS].

However, for manifolds with boundary, the generalized multiplicativity fails, as can be seen in the following example.

**Example.** (Disk bundles) Let $V^2 \to B^2$ be an oriented 2-plane bundle over an oriented surface. Denote $\Phi$ the Thom class and $\chi(V)$ the Euler number of $V$. Consider its disk bundle $D^2 \to D(V) \to B^2$ with its sphere bundle $S(V)$. By the Thom isomorphism

$$H^{*-2}(B) \xrightarrow{\cup \Phi} H^*(D(V), S(V)),$$

one easily finds

$$\text{sign}(D(V)) = \text{sgn}\chi(V).$$

But $\text{sign}(B) = \text{sign}(D^2) = 0$. In particular, if $V$ is the Hopf bundle, $\text{sign}(D(V)) = 1 \neq 0$.

On the other hand, note that by the extended Novikov additivity [BC4] and (4.8), the difference, called the *non-multiplicativity of signature* (compare also Introduction),

$$\Delta \overset{\text{def}}{=} \int_B \mathcal{L}(B) \wedge \text{ch}(\text{Sign}^2) - \text{sign}N$$

depends only on the associated boundary fibration $Y \to M \to B$, where $Y = \partial Z, M = \partial N$. We assume here $B$ is closed, and we will assume below that $\dim B$ is even.
a symmetric pairing on $E^2_{k}$, whose signature will be denoted by $\sigma_{r}$. Then (4.8) is equivalent to

$$\sigma_{r} = \sigma_{\infty}.$$  

This follows from [CHS] and the signature theorem for twisted coefficients [L], [AS].

However, for manifolds with boundary, the generalized multiplicativity fails, as can be seen in the following example.

**Example.** (Disk bundles) Let $V^{2} \rightarrow B^{2}$ be an oriented 2-plane bundle over an oriented surface. Denote $\Phi$ the Thom class and $\chi(V)$ the Euler number of $V$. Consider its disk bundle $D^{2} \rightarrow D(V) \rightarrow B^{2}$ with its sphere bundle $S(V)$. By the Thom isomorphism

$$H^{*-2}(B) \xrightarrow{u_{\Phi}} H^{*}(D(V), S(V)),$$

one easily finds

$$\text{sign}(D(V)) = \text{sgn}\chi(V).$$

But $\text{sign}(B) = \text{sign}(D^{2}) = 0$. In particular, if $V$ is the Hopf bundle, $\text{sign}(D(V)) = 1 \neq 0$.

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depends only on the associated boundary fibration $Y \rightarrow M \rightarrow B$, where $Y = \partial Z, M = \partial N$. We assume here $B$ is closed, and we will assume below that $\dim B$ is even.
Theorem 4.5 The topological invariant $\tau$, which is defined from the closed fibration $Y \rightarrow M \rightarrow B$, intrinsically characterize the non-multiplicativity of signature. That is, whenever there exists another fibration of manifolds with boundary $Z \rightarrow N^n \rightarrow B$ such that $\partial Z = Y$, $\partial N = M$, then $\tau = \Delta$.

Proof. We start with (4.1), Atiyah-Patodi-Singer’s signature formula for manifold with boundary, and take the adiabatic limit of both sides. The lefthand side remains since signature is a topological invariant. The first term on the righthand side yields, by Lemma A.3,

$$\int_N \mathcal{L}(\frac{R^B}{2\pi i}) \wedge \mathcal{L}(\frac{R^Z}{2\pi i}) = \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge (\int_Z \mathcal{L}(\frac{R^Z}{2\pi i})),$$

while the second term is the adiabatic limit formula (Theorem 4.4). Therefore we obtain

$$\text{sign}(N) = \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge (\int_Z \mathcal{L}(\frac{R^Z}{2\pi i})) - \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge \tilde{\eta} - \frac{1}{2} \eta(A_B \otimes \ker A_Y) - \tau$$

$$= \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge (\int_Z \mathcal{L}(\frac{R^Z}{2\pi i}) - \tilde{\eta}) - \tau,$$

where the $\eta$-term drops out because dim $B$ is even. On the other hand, Bismut-Cheeger’s Families Index Theorem states that $\int_Z \mathcal{L}(\frac{R^Z}{2\pi i}) - \tilde{\eta}$ is a representative for the chern character of the signature bundle $\text{Sign}^Z$. Combining these two finishes the proof.

Q.E.D.

Thus we characterize the non-multiplicativity of signature intrinsically in terms of the topological data of the boundary fibration. One can also give an intrinsic characterization in terms of the analytical data.

Theorem 0.4' Let

$$V_0 = \text{limit of space of } A_x \text{-harmonic forms on } M \text{ in the adiabatic limit},$$
$V_\pm =$ limit of space of $\lambda_\pm$-eigenforms such that $\lambda_\pm > 0$ (respectively $< 0$) and $\lambda_\pm$ is $O(x^2)$.

Then $V_+ \oplus V_- \oplus V_0 = H^*(B, \mathcal{H}(Y))$, and $\tau' \overset{\text{def}}{=} \dim V_+ - \dim V_- = \tau$. Therefore $\tau' = \Delta$ whenever the fibration $Y \to M \to B$ bounds $Z \to X \to B$.

**Remark** One can also study the non-multiplicativity in the "dual" case, i.e. for fibration $Z \to N \to B$ where the fibre $Z$ is closed but the base $B$ has boundary. Theorem 4.5 holds in this case as well. Instead of the Families Index Theorem for manifolds with boundary, a transgression formula for $\tilde{\eta}$ is used.

### 4.4 Formal Hodge decomposition and asymptotic analysis

In this section, we discuss the analysis of formal asymptotic series and prove the formal Hodge decomposition (Lemma 2.5). Consider the manifold $X = [0, \infty) \times M$ with its boundary $M_0 = \{x = 0\}$ and the "compressed form bundle" $\mathcal{V}^\Lambda^*$ (see [MM] and Section 2.1). Formally a smooth section of $\mathcal{V}^\Lambda^*$ is a linear combination of the form

$$a(x, y, z) dz^\alpha \wedge \left(\frac{dy}{x}\right)^\beta$$

with $a(x, y, z)$ a smooth function.

The metric $g_\alpha$ gives rise to a well-defined non-degenerate inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{V}^\Lambda^*$ which in turn gives us a Hodge $\ast$ operator

$$\ast : \mathcal{V}^\Lambda^* \to \mathcal{V} \Lambda^*.$$  \hfill (4.9)
This is nothing but a uniform construction of the \(*\) operators on \(M\) for the parameter \(x\) down to \(x = 0\).

Since \(d \in \text{Diff}_V^*(X, \nu \Lambda^*)\) its formal adjoint \(\delta\) with respect to \(\langle \cdot, \cdot \rangle\) is also a \(V\)-differential operator and \(A_x = * (d + \delta)\) up to a sign. The operator \(A_x\) can be restricted to \(C^\infty(M_0, \nu \Lambda^*)\), the resulting operator \(I(A_x)\), called the indicial operator of \(A_x\) (Cf. [MM]), is just \(A_Y\).

\[
I(A_x)u_0 = (A_x u)|_{x = 0},
\]

where \(u \in C^\infty(X, \nu \Lambda^*)\) such that \(u|_{x=0} = u_0\).

Consider now \(A_x - \lambda_x\) where \(\lambda_x\) is a smooth function of \(x\) and vanishes at \(x = 0\). Thus asymptotically \(\lambda_x = x \lambda_1 + x^2 \lambda_2 + \cdots.\) Then \(I(A_x - \lambda_x) = A_Y\) again and its kernel is the fiber harmonic forms \(\nu E_1\), which can be viewed as a smooth bundle on the base \(B\). Recall that \(A_0 = PA_xP\) is the signature operator on \(B\) coupled to this bundle.

Let

\[
E^{\lambda_1}_2 = \{v \in C^\infty(M_0, \nu \Lambda^*), \; v \in C^\infty(B, \nu E_1) \text{ and } (A_0 - \lambda_1)v = 0\}.
\]

Clearly \(E^{\lambda_1}_2\) is finite dimensional.

**Lemma 4.2** The space \(E^{\lambda_1}_2 \subset C^\infty(M_0, \nu \Lambda^*)\) consists precisely of the boundary values of \(\nu \Lambda^*\)-forms which are "\(\lambda_x\)-harmonic" up to second order error:

\[
E^{\lambda_1}_2 = \{u_0 \in C^\infty(M_0, \nu \Lambda^*); \; \exists u \in C^\infty(X, \nu \Lambda^*)
\text{ with } u|_{M_0} = u_0 \text{ and } (A_x - \lambda_x)u \in x^2C^\infty(X, \nu \Lambda^*)\}.
\]

**Proof.** We mimick the proof of (49) in [MM]. Essentially by definition the space \(\nu E_1\) is characterized as the boundary values of harmonic \(\nu \Lambda^*\)-forms modulo first order error. Thus for any \(u\) with boundary value \(u_0 \in \nu \nu E_1\)
$E_1$, we must have $A_x u|_{M_0} = 0$. Thus

$$A_x u = xv, \ v \in C^\infty(X, V \Lambda^*) \text{ and } A_0 u = P(v).$$

Hence

$$(A_x - \lambda_x)u = x(v - \lambda_1 u) + x^2 v, \ w \in C^\infty(X, V \Lambda^*).$$

By the Hodge Theorem on the fibre, we can choose $u_1 \in C^\infty(X, V \Lambda^*)$ such that restricted to boundary,

$$v - Pv = A_Y u_1 = (A_x - \lambda_x)u_1 + xu_1'.$$

Therefore,

$$(A_x - \lambda_x)(u - xu_1) = x(A_0 - \lambda_1)u_0 + x^2 u_1'.$$

Therefore if $u_0 \in E_2^{\lambda_1}$, $u - xu_1$ is $\lambda_x$-harmonic up to second order error. Conversely, if $u \in C^\infty(X, V \Lambda^*)$, $u|_{M_0} = u_0 \in V E_1$, and $(A_x - \lambda_x)u \in x^2 C^\infty(X, V \Lambda^*)$. Then

$$x(v - \lambda_1 u) \in x^2 C^\infty(X, V \Lambda^*),$$

i.e.

$$(v - \lambda_1 u)|_{M_0} = 0, \text{ or } v|_{M_0} \in V E_1.$$

From above then

$$(A_x - \lambda_x)u = x(A_0 - \lambda_1)u_0 + x^2 u_1',$$

hence $(A_0 - \lambda_1)u_0 = 0.$ \hspace{1cm} Q.E.D.
To show Lemma 2.5 consider the space of Laurent series introduced in Section 2.1, \( \mathcal{L}(X, \nu \Lambda^*) \). This is a module over the formal Laurent series ring \( \mathcal{L} = \mathcal{L}(x) \). By choosing an extension
\[
C^\infty(M_0, \nu \Lambda^*) \rightarrow C^\infty(X, \nu \Lambda^*) \rightarrow \mathcal{L}(X, \nu \Lambda^*),
\]
we can identify \( \mathcal{L}(x) \cdot E^\Lambda_2 \) as a subspace of \( \mathcal{L}(X, \nu \Lambda^*) \). This subspace is finite dimensional over \( \mathcal{L}(x) \) so has a basis \( e_1, \ldots, e_L \). the inner product \( <,>_G \) gives a bilinear form
\[
\mathcal{L}(X, \nu \Lambda^*) \times \mathcal{L}(X, \nu \Lambda^*) \rightarrow \mathcal{L}(x).
\]
Using this to orthonormalize the basis, we obtain a projection operator onto a complement to \( \mathcal{L}(x) \cdot \nu E^\Lambda_2 \):
\[
\pi^\perp_2 : \mathcal{L}(X, \nu \Lambda^*) \rightarrow \mathcal{L}(X, \nu \Lambda^*),
\]
\[
\pi^\perp_2(f) = f - \sum_{j=1}^{L} < f, e_j >_G e_j.
\]
Now our operator \( A_\nu - \lambda_\nu \) also extends. We have

**Lemma 4.3** The operator \( P_2 = \pi^\perp_2(A_\nu - \lambda_\nu) \pi^\perp_2 \) is an isomorphism on \( \pi^\perp_2 \mathcal{L}(X, \nu \Lambda^*) \).

**Proof.** We first show the surjectivity by an inductive argument. Since our operator has the decomposition
\[
A_\nu = A_Y + \nu A_B + \nu^2 V,
\]
the basic idea is to invert each term successively, using respectively the Hodge theory (or ellipticity) along fibers and on the base. Thus if \( f \in \)
\( \pi_2^1 \mathcal{L}(X, \nu \Lambda^*) \) then the leading term \( x^l f_1, l \in I \) must have coefficient \( f_1 \perp \nu E_2^\Lambda \). By the Hodge theorem along fibers we can choose \( u_1 \in C^\infty(M_0, \nu \Lambda^*) \) so that

\[
f_1 = A\nu u_1 + Pf_1.
\]

Therefore \( f' = (A_\nu - \lambda_\nu)(x^l u_1) - f \) has leading term \( x^l Pf_1 \). Now by the ellipticity of \( A_0 \) and the assumption that \( f_1 \perp \nu E_2^\Lambda \) one has \( Pf_1 = A_0 u_{l-1} \) for some \( u_{l-1} \in \nu E_1 \). It follows that

\[
\pi_2^1(A_\nu - \lambda_\nu)(x^l u_1 - x^{l-1} u_{l-1}) - f = O(x^{l+1}).
\]

Let \( u' = \pi_2^1(x^l u_1 - x^{l-1} u_{l-1}) \), then \( (x^l u_1 - x^{l-1} u_{l-1} - u') = \pi_2^1(x^l u_1 - x^{l-1} u_{l-1}) \) and thus by the above lemma (and the definition of \( \pi_2 \)) we have

\[
(A_\nu - \lambda_\nu)(x^l u_1 - x^{l-1} u_{l-1} - u') = O(x^{l+1}).
\]

Hence

\[
\pi_2^1(A_\nu - \lambda_\nu)\pi_2^1 u' = O(x^{l+1}).
\]

Proceeding inductively in the Laurent series we can construct \( u \) such that

\[
\pi_2^1(A_\nu - \lambda_\nu)\pi_2^1 u = f.
\]

To show the injectivity let \( f \in \pi_2^1 \mathcal{L}(X, \nu \Lambda^*) \) such that \( \pi_2^1(A_\nu - \lambda_\nu)\pi_2^1 f = 0 \), i.e., \( (A_\nu - \lambda_\nu)f \in \mathcal{L} \cdot E_2^\Lambda \). By the surjectivity proved above one can find a \( u \in \pi_2^1 \mathcal{L}(X, \nu \Lambda^*) \) so that \( f = (A_\nu - \lambda_\nu) u \). Thus

\[
< (A_\nu - \lambda_\nu)^2 u, u > = 0.
\]

This implies that the leading term of \( f \) must vanish, proving the injectivity.

**Proof of Lemma 2.5.** Certainly \( N \) is finite dimensional \( \mathcal{L} \)-module since \( \pi_2 \) is injective on it. Let \( B \) be the inverse of \( \pi_2^1(A_\nu - \lambda_\nu)\pi_2^1 \) from the
previous lemma. The projection of the matrix equation
\[
\begin{pmatrix}
\pi_2(A_x - \lambda_x)\pi_2 & \pi_2(A_x - \lambda_x)\pi_2^1 \\
\pi_2^1(A_x - \lambda_x)\pi_2 & \pi_2^1(A_x - \lambda_x)\pi_2^1
\end{pmatrix}
\begin{pmatrix}
u' \\
u''
\end{pmatrix}
= 
\begin{pmatrix}
f' \\
f''
\end{pmatrix}
\]
onto the image of \(\pi_2\) is then
\[
Qv = (\pi_2(A_x - \lambda_x)\pi_2 - \pi_2(A_x - \lambda_x)\pi_2^1 B \pi_2^1(A_x - \lambda_x)\pi_2)v
= g = f' - \pi_2(A_x - \lambda_x)\pi_2^1 B f''.
\]
That is, if \((u', u'')\) satisfies the first equation then \(v = u'\) satisfies the second and conversely if \(v\) satisfies the second equation then
\[
\begin{pmatrix}
u' \\
u''
\end{pmatrix}
= 
\begin{pmatrix}
v \\
B f'' - B (\pi_2^1(A_x - \lambda_x)\pi_2)v
\end{pmatrix}
\]
The null space of this self-adjoint operator \(Q\) is \(\pi_2 N\) and by standard argument it is an isomorphism on the orthocomplement in \(\pi_2 L(X, Y^*\Lambda^*)\). This shows that \(A_x - \lambda_x\) is an isomorphism on \(N^\perp\).
Appendix A

Bismut-Cheeger’s adiabatic limit formula

This appendix contains the algebraic and geometric preliminaries, which are mostly taken from [BC2] and [B]. Results that are basic to our discussions, in particular, recent results of Bismut-Cheeger on the adiabatic limit of $\eta$-invariants will also be reviewed. Finally, for use in Chapter 2, we include some facts about the analysis on the manifolds with boundary.

A.1 $\mathbb{Z}_2$-gradings, Clifford algebras and superconnections

A vector space $V$ is called $\mathbb{Z}_2$-graded if it comes equipped with a direct sum decomposition $V = V_0 \oplus V_1$. This decomposition determines and is
determined by an involution $\rho$

$$\rho|_V \overset{\text{def}}{=} (-1)^j.$$

(A.1)

A $\mathbb{Z}_2$-graded vector space $A = A_0 \oplus A_1$, which is also an algebra, is called a $\mathbb{Z}_2$-graded algebra if $A_jA_{j'} \subseteq A_{j+j'}$. We will use subscripts on elements to denote their grading, e.g. $a_j \in A_j$. All such subscripts take values in $\mathbb{Z}_2$.

The supercommutator of $a_j$, $a_{j'}$ is defined as

$$[a_j, a_{j'}] = a_ja_{j'} - (-1)^{jj'}a_{j'}a_j.$$

(A.2)

If $V$ is a $\mathbb{Z}_2$-graded vector space, then $End(V)$ is naturally a $\mathbb{Z}_2$-graded algebra, where $End_0(V)$ consists of endomorphisms commuting with the involution $\rho$ and $End_1(V)$ consists of those anti-commuting with $\rho$. A $\mathbb{Z}_2$-graded representation of a $\mathbb{Z}_2$-graded algebra $A$ on a $\mathbb{Z}_2$-graded vector space $V$ is a grading-preserving homomorphism $\phi: A \rightarrow End(V)$, i.e., $\phi(A_j) \subseteq End_j(V)$.

Given $A, V, \phi$ as above then

$$tr_s(a) = tr(\rho\phi(a))$$

(A.3)

defines a supertrace on $A$. This is a linear map whose kernel contains all supercommutators.

If $A, B$ are $\mathbb{Z}_2$-graded algebras, their $\mathbb{Z}_2$-graded tensor product $A \hat{\otimes} B$ is naturally isomorphic as a vector space to $A \otimes B$. The multiplication on $A \hat{\otimes} B$ satisfies

$$(a \hat{\otimes} b_j)(c_{j'} \hat{\otimes} d) = (-1)^{jj'}ac_{j'} \hat{\otimes} b_jd.$$  

$A \hat{\otimes} B$ inherits a natural $\mathbb{Z}_2$-grading, determined by $\rho_A \otimes \rho_B$. 

The following constructions will be quite useful. Let $\phi$, $\psi$ be representations of $A$, $B$ on $X$, $Y$. Let $C^2 = C \oplus C$ define the grading on $C^2$ and let $J$, $K \in \text{End}_1(C^2)$ denote the involutions,

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Note that $JK = -KJ$. If the grading on $X \otimes Y \otimes C^2$ is determined by the involution $\text{id} \otimes \text{id} \otimes iJK$, then

$$a_j \otimes b_{j'} \mapsto \phi(a_j) \otimes \psi(b_{j'}) \otimes J^j K^{j'}$$

(A.4)

defines a $\mathbb{Z}_2$-graded representation of $A \hat{\otimes} B$.

Let $\phi$, $\psi$ be as above. In addition, let $\psi$ be $\mathbb{Z}_2$-graded and $\rho$ be as in (A.1) for $Y$. Then

$$a_j \hat{\otimes} b \mapsto \phi(a_j) \otimes \rho^j \psi(b)$$

(A.5)

defines a representation of $A \hat{\otimes} B$. If $\phi$ is also $\mathbb{Z}_2$-graded, this representation is $\mathbb{Z}_2$-graded for the natural $\mathbb{Z}_2$-grading of $X \otimes Y$.

Let $V^n$ be an Euclidean space. The complex Clifford algebra $\text{Cl}(V^n)$ is generated by an orthonormal basis $\{e_i\}$ of $V$ satisfying the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$ 

As a vector space $\text{Cl}(V^n)$ can be identified with the complex exterior algebra. The Clifford multiplication is then exterior multiplication minus interior multiplication. The elements $e_\alpha = e_{i_1} \cdots e_{i_j}$, where $\alpha = (i_1, \ldots, i_j)$, $i_1 < \cdots < i_j$, form a basis for $\text{Cl}(V^n)$. The subspaces $\text{Cl}_0(V^n)$, $\text{Cl}_1(V^n)$ spanned
by those \( e_\alpha \) with \(|\alpha|\) even, respectively odd give \( \text{Cl}(V^n) \) the structure of a \( \mathbb{Z}_2 \)-graded algebra. If \( V \) is oriented, the element

\[
\tau = \begin{cases} 
i^k e_1 \cdots e_{2k-1} & \text{if } n = 2k - 1 \\ i^k e_1 \cdots e_{2k} & \text{if } n = 2k \end{cases}
\]

is independent of the choice of \( \{e_i\} \) and satisfies \( \tau^2 = 1 \).

For \( n = 2k \), up to isomorphism, \( \text{Cl}(V^n) \) has a unique irreducible module \( F \), which has dimension \( 2^k \) and is \( \mathbb{Z}_2 \)-graded (by \( \phi(\tau) \)). In fact \( \text{Cl}(V^n) = \text{End}(F) \). We will follow the standard convention and write \( F_+ \oplus F_- \) for \( F_0 \oplus F_1 \).

If \( n = 2k - 1 \), \( \text{Cl}(V^n) \) has two inequivalent irreducible modules, each of dimension \( 2^{k-1} \). For arbitrary \( n \),

\[
e_i \mapsto e_i e_{n+1}
\]

defines an isomorphism, \( \text{Cl}(V^n) \cong \text{Cl}_0(V^n \oplus \mathbb{R}) \). Thus we can regard \( F_+, F_- \) for \( V^n \oplus \mathbb{R} \) as (inequivalent) modules of \( \text{Cl}(V^n) \) (and \( \tau \to \pm \text{Id}_{F_\pm} \)). For \( V^{2k-1} \) oriented, the notation \( tr(a) \) refers to the representation \( F_+ \).

**Lemma A.1** 1) If \( n = 2k \) is even then

\[
tr_\tau(e_\alpha) = \begin{cases} 0 & \text{if } \alpha \neq (1, \cdots, 2k) \\ i^{-k} 2^k & \text{if } \alpha = (1, \cdots, 2k). \end{cases}
\]

2) If \( n = 2k - 1 \) is odd and \(|\alpha| \geq 1\),

\[
tr(e_\alpha) = \begin{cases} 0 & \text{if } \alpha \neq (1, \cdots, 2k - 1) \\ i^{-k} 2^{k-1} & \text{if } \alpha = (1, \cdots, 2k - 1). \end{cases}
\]
Lemma A.2 The irreducible modules of $\text{Cl}(V^n \otimes W^m)$ can be obtained from the irreducible modules $F(V^n)$, $F(W^m)$ for $\text{Cl}(V^n)$, $\text{Cl}(W^m)$ as follows,

1) For $n, m$ both odd, the module $F(V^n) \otimes F(W^m) \otimes \mathbb{C}^2$ defined by (A.4) is irreducible.

2) For $m$ even, the module $F(V^n) \otimes F(W^m)$ defined by (A.5) is irreducible.

The proof of these is fairly easy, and we omit it (see [BC2]).

Finally, we note the effect of scaling the inner product $\langle , \rangle$ on $V$. For any inner product, $\text{Cl}(V^n)$ coincides as a vector space with $\Lambda^*(V) \otimes \mathbb{C}$. Fix an inner product, $\langle , \rangle$ and let $\text{Cl}_x(V^n)$ denote $\Lambda^*(V)$ with Clifford multiplication coming from $x^{-2} \langle , \rangle$. Then the automorphism of $\Lambda^*(V) \otimes \mathbb{C}$ induced by $xv \rightarrow v$, provides a natural isomorphism $\text{Cl}_x(V^n) \cong \text{Cl}(V^n)$. It also provides a natural isomorphism between the orthonormal frames \{xe_i\} for $x^{-2} \langle , \rangle$ and \{e_i\} for $\langle , \rangle$. Thus, although there is no canonical choice for the space $F(V)$ for $\langle , \rangle$, any fixed choice also provides an irreducible module for $\text{Cl}_x(V^n)$ via the above isomorphism.

In the sequel, if $M$ is a Riemannian spin manifold, we will always assume that the space of spinors has been chosen independent of the scaling parameter of the metric. As a consequence, the action of $xe_i \in \text{Cl}_x(T_pM)$ on $F_p$ is independent of $x$ and the Dirac operator $D_x$ corresponding to $x^{-2} \langle , \rangle$ is $xD$, where $D$ is the Dirac operator for $\langle , \rangle$.

A vector bundle $E$ over a manifold $M$ is called $\mathbb{Z}_2$-graded (see [Q]) if it
comes equipped with a Whitney sum decomposition $E = E_0 \oplus E_1$. Consider

$$\Omega(M, E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M, E),$$

the algebra of $E$-valued smooth differential forms with complex coefficients, where $\Omega(M) = \oplus \Omega^p(M)$ is the algebra of smooth differential forms with complex coefficients and $\Omega^0(M, E)$ is the space of smooth sections of $E$. This is a $\Omega(M)$-module. It has a grading with respect to $\mathbb{Z} \times \mathbb{Z}_2$. However, we are primarily concerned with its total $\mathbb{Z}_2$-grading, and so regard it as a $\mathbb{Z}_2$-graded module over $\Omega(M)$.

We will be working in the $\mathbb{Z}_2$-graded algebra of endomorphism valued forms

$$\Omega(M, \text{End}(E)) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M, \text{End}(E)),$$

and we shall write $\omega X$ instead of $\omega \otimes X$. This algebra operates on $\Omega(M, E)$ on the left:

$$(\omega X)(\eta e) = (-1)^{(\deg X)(\deg \eta)} \omega \wedge \eta X(e).$$

The following is easily verified (Cf. [Q]).

**Proposition A.1** In the above way the algebra $\Omega(M, \text{End}(E))$ can be identified with the algebra of operators on $\Omega(M, E)$ which are $\Omega(M)$-linear in the sense that the even and odd components satisfy

$$T'(\omega \alpha) = (-1)^{(\deg T)(\deg \omega)} \omega T'(\alpha).$$

After D. Quillen, [Q], we define a superconnection on $E$ to be an operator $D$ on $\Omega(M, E)$ of odd degree satisfying the derivation property

$$D(\omega \alpha) = (d \omega) \alpha + (-1)^{\deg \omega} \omega D \alpha.$$
for $\omega \in \Omega(M)$ and $\alpha \in \Omega(M, E)$. For example, a connection in $E$ preserving the grading, when extended to an operator on $\Omega(M, E)$ in the usual way, determines a superconnection. Using Proposition A.1 sees that the difference of two superconnections is an odd element of $\Omega(M, \text{End}(E))$.

If $D$ is a superconnection and $\phi \in \Omega(M, \text{End}(E))$, then the supercommutator $[D, \phi]$ as an operator on $\Omega(M, E)$ is linear over $\Omega(M)$, hence an element of $\Omega(M, \text{End}(E))$ by Proposition A.1.

**Proposition A.2** One has

$$d(tr, \phi) = tr[D, \phi].$$

**Proof.** see [Q].

In the sequel, we want to consider infinite dimensional analogues of vector bundles and superconnections. In principle, results discussed above continue to hold, although some care must be taken in their application.

### A.2 Elementary geometry of fibrations

Let

$$Y \to M \to B \quad \text{(A.9)}$$

be a fibration of smooth manifolds. A connection determines a splitting of the tangent bundle of $M$ into its vertical subbundle and horizontal subbundle,

$$TM = T^V M \oplus T^H M.$$ 

Let $P^H$, $P^V$ denote the projections on $T^H$, $T^V$ relative to this splitting.
If $U$ is a locally defined vector field on $B$, we denote by $\tilde{U}$ its horizontal lift. The integrability tensor (or curvature) of $T^H M$ is the 2-form $\mathcal{R}$ on $T^H M$ with values in $T^V M$ defined by

$$\mathcal{R}(\tilde{U}, \tilde{V}) = -[\tilde{U}, \tilde{V}] + [\tilde{U}, \tilde{V}] = -P^V[\tilde{U}, \tilde{V}].$$

Now equip $M$ with a submersion metric $g$ which respects the above splitting,

$$g = \pi^* g_B + g_V.$$

Let $\nabla^B$ be the Levi-Civita connection of $TB$. $\nabla^B$ lifts to a so called Euclidean connection on $T^H M$, which we still denote by $\nabla^B$,

$$\nabla^B_{\tilde{U}} \tilde{V} = \nabla^B_U V,$$

$$\nabla^B_W \tilde{V} = [W, \tilde{V}] \quad \text{if } W \text{ is vertical.}$$

Also let $\nabla^Y$ be the connection on $T^V M$ defined by the projection of the Levi-Civita connection $\nabla^L$ on $TM$. Then $\nabla = \nabla^Y \oplus \nabla^B$ defines a connection on $TM$ which preserves the metric $g$. If we let $T$ be the torsion tensor of this connection and $S$ the difference tensor of $\nabla^L$ and $\nabla$,

$$S = \nabla^L - \nabla.$$ 

One has

1) $T$ takes values in $T^V M$ and it vanishes on $T^V M$.

2) For $W_1, W_2$ vertical

$$< S(W_1)W_2, \tilde{U}_1 > = -< S(W_1)\tilde{U}_1, W_2 > = \frac{1}{2} < A_{\tilde{U}_1}(W_1), W_2 >,$$
where $A$ is the second fundamental form of the fibers.

3) 
\[ \langle S(\tilde{U}_1)W_1, \tilde{U}_2 \rangle = - \langle S(\tilde{U}_1)\tilde{U}_2, W_1 \rangle = \langle S(W_1)\tilde{U}_1, \tilde{U}_2 \rangle = \frac{1}{2} \langle R(\tilde{U}_1, \tilde{U}_2), W_1 \rangle, \]

and all other components of $\langle S(\cdot), \cdot \rangle$ vanish.

It follows easily from these that if $\nabla^{L^x}$ is the Levi-Civita connection of $g_x = x^{-2}\pi^* g_B + g_Y$ and $S^x = \nabla^{L^x} - \nabla$, then

\[ P^H S^x = x^2 P^H S, \quad P^V S^x = P^V S, \]

and hence the limit,

\[ \lim_{x \to 0} \nabla^{L^x} = \nabla^L - P^H S = \nabla + P^V S \]

exists and is in uppertriangular form (with respect to the splitting) with the diagonal entry $\nabla$. Consequently, its curvature is also in uppertriangular form with the diagonal entry $R = \nabla^2$. From this we deduce

**Lemma A.3** Let $P$ be an $O(n)$-invariant polynomial on the Lie algebra $o(n)$, and $R^x, R = R^Y \oplus R^B$ the curvatures of $\nabla^{L^x}, \nabla$ respectively. Then

\[ \lim_{x \to 0} P(R^x / 2\pi i) = P(R / 2\pi i) = P(R^Y / 2\pi i)P(R^B / 2\pi i). \]

Let $\gamma$ be an arbitrary vector bundle over $M$. Associated to $\gamma$ is a vector bundle $\tilde{\gamma}$ over $B$ whose (infinite dimensional) fiber $\tilde{\gamma}_p$ is the space of the smooth sections of $\gamma|_{x^{-1}(p)}$. There is an obvious functorial isomorphism between the space of smooth sections of $\gamma$ and the space of sections of $\tilde{\gamma}$ which are smooth in the appropriate sense.
If $\gamma$ has a connection $\nabla^\gamma$, then we define the associated connection $\nabla^\tilde{\gamma}$ (sometime abbreviated $\tilde{\nabla}$) on $\tilde{\gamma}$ by

$$\nabla^\tilde{U}_U \tilde{s} = \nabla^\gamma_U s,$$

where $\tilde{s}$ denotes the section $s$ of $\gamma$, regarded as a section of $\tilde{\gamma}$. Clearly, the curvature $R^\tilde{\gamma}$ is given by

$$R^\tilde{\gamma}(U_1, U_2) = R^\gamma(\tilde{U}_1, \tilde{U}_2) - \nabla^\gamma_{\tilde{U}_1} \tilde{U}_2 \nabla^\gamma_{\tilde{U}_1} \tilde{U}_2.$$

If $\gamma$ has a hermitian inner product $< , >$, then an inner product can be defined on $\tilde{\gamma}$ via $< , >$ and $dz$, the smoothly varying volume forms on the fibers induced by $g_\gamma$,

$$< \tilde{s}_1, \tilde{s}_2 > = \int_{\pi^{-1}(p)} < s_1, s_2 > dz.$$

Even if $\nabla^\gamma$ is unitary, $\nabla^\tilde{\gamma}$ need not preserve this inner product. For this, one can correct it by subtracting a term involving the mean curvature of the fiber $Y$. In fact, $\nabla^\tilde{\gamma} a = \nabla^\tilde{\gamma} - \frac{1}{4} \sum_i < A_G(e_i), e_i >$ is unitary.

### A.3 Bismut-Cheeger’s adiabatic limit formula

Suppose that $\mathcal{M}$ is a compact connected Riemannian manifold, which is oriented and spin (abbreviated spin manifold). Thus, the $SO(n)$-bundle of oriented orthonormal frames in $TM$ is covered by a $Spin(n)$-bundle $Q$, so that the covering map restricts fiberwisely to the covering projection $Spin(n) \to SO(n)$. Since $Spin(n)$ is canonically embedded in $Cl(\mathbb{R}^n)$, the irreducible module $F$ of $Cl(\mathbb{R}^n)$ gives rise to a hermitian vector bundle
on $M$, also denoted by $F$ (or $F(M)$ when there is a need to indicate the base manifold). The Clifford multiplication on $F$ by a tangent vector is well-defined:

$$TM \to Hom(F, F).$$

Since $SO(n)$ and $Spin(n)$ have the same Lie algebra, the Levi-Civita connection $\nabla$ of $M$ lifts to $Q$, thus inducing a connection on $F$, again denoted by $\nabla$ (or sometimes $\nabla^F$). Let $\{e_i\}$ be a local orthonormal frame on $M$, the relation between the two connections is

$$\nabla_X f = \frac{1}{4} \sum_{i,j} <\nabla_X e_i, e_j > e_i e_j \cdot f,$$

(A.12)

where $f$ is a section of $F$ and "\cdot" indicates Clifford multiplication.

The Dirac operator $D$ acting on $F$ is defined by

$$C^\infty(M, F) \to C^\infty(M, T^*M \otimes F) \to C^\infty(M, TM \otimes F) \to C^\infty(M, F),$$

with the first arrow given by the connection and the second the metric, and the third the Clifford multiplication. Locally,

$$D = e_i \nabla e_i.$$

This is a first order self-adjoint elliptic differential operator whose symbol is given by the Clifford multiplication. Similarly, if $\xi$ is a hermitian vector bundle over $M$ with a unitary connection $\nabla^\xi$, one can define a Dirac type operator on $F \otimes \xi$,

$$D \otimes \xi = e_i (\nabla \otimes \nabla^\xi) e_i = e_i \nabla e_i \otimes 1 + e_i \otimes \nabla^\xi e_i.$$

We call $D \otimes \xi$ the Dirac operator coupled to the connection $\nabla^\xi$ (or twisted Dirac operator). For the simplicity of notation, we will sometimes denote
$D \otimes \xi$ by $D$ alone. This notion generalizes suitably to superconnections [BC2]. Thus suppose $\xi = \xi_+ \oplus \xi_-$ is $\mathbb{Z}_2$-graded with a unitary connection $\nabla^\xi$ preserving the grading. Then $\nabla^\xi$ extends to a superconnection and any other superconnection is of the form $\nabla^\xi + V$ with $V$ an odd element of $\Lambda^*(M) \otimes \text{End}(\xi)$. Write $V = \sum_{j=0}^n V_j$ as a sum of $j$-forms with values in $\text{End}_{j+1}(\xi)$. We define the operator $\mathcal{V}$ on $F(M) \otimes \xi$ by
\[
\mathcal{V} = \sum e_{i_1} \cdots e_{i_j} V_j(e_{i_1}, \ldots, e_{i_j}).
\]

If we assume that $V_j$ takes values in self-adjoint (respectively skew-adjoint) elements of $\text{End}_{j+1}(\xi)$ for $j$ even (respectively $j$ odd), we can again form the self-adjoint operator
\[
D \otimes \xi + \mathcal{V},
\]
which we call the Dirac operator coupled to the superconnection $(\nabla^\xi, V)$.

The $\eta$-function of the Dirac type operator $D$ is defined for $\Re \, s \gg 0$ as
\[
\eta(s) = \frac{1}{\Gamma \left( \frac{s+1}{2} \right)} \int_0^\infty t^{\frac{s-1}{2}} \text{tr}(D e^{-tD^2}) dt.
\]
In terms of the eigenvalues $\lambda$ of $D$, $\eta(s)$ can be written as
\[
\eta(s) = \sum_{\lambda \neq 0} \frac{\text{sgn} \lambda}{|\lambda|^s}.
\]

The parity of $\dim M$ plays a role here. We note that when $\dim M$ is even, $\eta(s)$ vanishes identically. In fact, $\tau$ of (A.6) now gives a well-defined parallel section of $\text{End} F$, anti-commuting with $D$. It follows that pointwisely
\[
\text{tr}(D e^{-tD^2}) = \text{tr}(\tau D e^{-tD^2} \tau) = -\text{tr}(D e^{-tD^2}).
\]
Thus for an even dimensional spin manifold, $tr(De^{-tD^2})$ is identically zero (pointwisely), consequently so is $\eta(s)$. On the other hand, for an odd dimensional spin manifold, we have the following local regularity result due to J.-M. Bismut and D. Freed, [BF].

**Theorem A.1** Pointwisely there is a uniform asymptotic expansion,

$$tr(De^{-tD^2})(p, p) = \sum_{i=0}^{N} b_{\frac{i}{2}+i}(p) t^{\frac{3}{2}+i} + O(t^{\frac{3}{2}+N+1}), \quad (A.13)$$

consequently $\eta(s)$ is holomorphic in $\text{Re } s > -2$.

Of particular interest is the value of $\eta(s)$ at $s = 0$, which is called the $\eta$-invariant of $D$, denoted $\eta(D)$. Its significance can be seen from the following result, [APS].

**Theorem A.2 (Atiyah-Patodi-Singer theorem)** Let $N$ be an even dimensional compact spin manifold with boundary $M$, $\xi$ be a hermitian bundle on $N$ with unitary connection $\nabla^\xi$ and curvature $L^\xi$. Assume that the metric on $N$ and the connection on $\xi$ are of product type near the boundary. Then

$$\text{ind}(D_N) = \int_N \hat{A}\left(\frac{R}{2\pi i}\right) \wedge \text{tr}(e^{-\frac{L^\xi}{2\pi i}}) - \frac{\eta(D_M) + \dim(\ker D_M)}{2},$$

where $D_N$ is the Dirac operator on $N$ together with the Atiyah-Patodi-Singer boundary condition (Cf. Appendix B).

From here we define the reduced $\eta$-invariant

$$\tilde{\eta}(D_M) = \frac{\eta(D_M) + \dim(\ker D_M)}{2}.$$
This is a spectral invariant and in general, it does not depend continuously on the metric. In the process of deformation, it has simple discontinuities caused by the crossing of zero spectrum. When a negative eigenvalue reaches zero, \( \eta \) will jump by 1, and when a positive eigenvalue reaches zero, \( \eta \) will jump by \(-1\). Since \( \eta \) has only integer jumps, its modulo \( \mathbb{Z} \) reduction will depend differentiably on the metric.

Now assume that \( M \) is the total space of a fibration (A.9). The rescaled metric \( g_\varepsilon \) gives rise to a family of Dirac operators \( D_\varepsilon \). The limiting behaviour of \( \eta(D_\varepsilon) \) (or \( \eta(D_\varepsilon) \)) is referred to as the adiabatic limit. As for the existence of such limit, generally we have (Cf. [C2], [BC2])

**Proposition A.3** As a \( \mathbb{R}/\mathbb{Z} \)-valued function, the limit \( \lim_{\varepsilon \to 0} \eta(D_\varepsilon) \) exists.

However, we are more interested in the question of when this limit actually exists in \( \mathbb{R} \), and if it exists, how it relates to the other index-theoretic quantities. This is studied in the recent work of Bismut-Cheeger [BC2] in case \( D_Y \) is invertible, which we now summarize briefly.

Assume from now on that \( \dim M \) is odd. Then from Lemma A.2, we have

\[
F(M) = \pi^* F(B) \otimes F(Y).
\]

Let \( \xi \) be a hermitian bundle on \( M \) with unitary connection \( \nabla^\xi \) and curvature \( L^\xi \) and let \( \nabla^u \) denote the connection on \( F(M) \otimes \xi \) defined by

\[
\nabla^u = \nabla^{F(B)} - \frac{1}{2} \sum <S(e_i) e_i, \cdot>,
\]

where \( \{e_i\} \) is a local orthonormal basis for \( T^V M \). The Dirac operator along the fiber \( D_Y \) is defined as \( D_Y = e_i \nabla e_i \) where \( \nabla \) is the Euclidean connection.
on $T^*M$ defined in Section A.2. We have the following nice formula for $D_z$.

**Lemma A.4** If $\{f_\alpha\}$ is an orthonormal basis for $TB$, then

$$D_z = x \sum_\alpha f_\alpha \nabla^u_{f_\alpha} + D_Y - \frac{x^2}{4} \sum_{\alpha \leq \beta} f_\alpha f_\beta T(f_\alpha, f_\beta).$$ \hspace{1cm} (A.14)

For simplicity, we will denote $\tilde{D}_B = \sum_\alpha f_\alpha \nabla^u_{f_\alpha}$, $T = \sum_{\alpha \leq \beta} f_\alpha f_\beta T(f_\alpha, f_\beta)$.

**Proof.** $D_z$ is defined via the lift of the Levi-Civita connection of $g_z$. (A.14) follows from a straightforward computation by virtue of (A.12) and the results of Section A.2 (see also [BC2]).

We now recall the definition of the Levi-Civita superconnection introduced by Bismut in [B]. The smooth sections of $F(Y) \otimes \zeta$ can be viewed as the smooth sections of a vector bundle over $B$, as observed at the end of Section A.2. Denote this bundle by $H_\infty = H_\infty^+ \oplus H_\infty^-$. The connection $\nabla^u$ induces a connection on $H_\infty$, $\tilde{\nabla}^u$. This is a unitary connection on $H_\infty$ with respect to the natural metric defined by the $L^2$-metric on the fibers of (A.9). If we wish to regard the operator $D_z$ as acting on sections of $F(B) \otimes H_\infty$, we simply write $\tilde{\nabla}^u$ for $\nabla^u$ in (A.14). The Levi-Civita superconnection $B_t$ on $H_\infty$ is the superconnection

$$B_t = \tilde{\nabla}^u + t^{1/2} D_Y - \frac{c(T)}{4t^{1/2}},$$ \hspace{1cm} (A.15)

where $c(T) = \sum_{\alpha \leq \beta} dy^\alpha dy^\beta T(f_\alpha, f_\beta)$, and $dy^\alpha$ denotes the 1-form dual to $f_\alpha$. It follows from (A.14) that $\frac{i}{2}D_z$ is the Dirac operator coupled to the Levi-Civita superconnection $B_{z}$. 
on $T^V M$ defined in Section A.2. We have the following nice formula for $D_\varepsilon$.

**Lemma A.4** If $\{f_\alpha\}$ is an orthonormal basis for $TB$, then

$$D_\varepsilon = \varepsilon \sum_\alpha f_\alpha \nabla^u_{f_\alpha} + D_Y - \frac{\varepsilon^2}{4} \sum_{\alpha \leq \beta} f_\alpha f_\beta T(f_\alpha, f_\beta). \quad (A.14)$$

For simplicity, we will denote $\tilde{D}_B = \sum_\alpha f_\alpha \nabla^u_{f_\alpha}$, $T = \sum_{\alpha \leq \beta} f_\alpha f_\beta T(f_\alpha, f_\beta)$.

**Proof.** $D_\varepsilon$ is defined via the lift of the Levi-Civita connection of $g_\varepsilon$. (A.14) follows from a straightforward computation by virtue of (A.12) and the results of Section A.2 (see also [BC2]).

We now recall the definition of the Levi-Civita superconnection introduced by Bismut in [B]. The smooth sections of $F(Y) \otimes \xi$ can be viewed as the smooth sections of a vector bundle over $B$, as observed at the end of Section A.2. Denote this bundle by $H_\infty = H^+_\infty \oplus H^+_\infty$. The connection $\nabla^u$ induces a connection on $H_\infty$, $\tilde{\nabla}^u$. This is a unitary connection on $H_\infty$ with respect to the natural metric defined by the $L^2$-metric on the fibers of (A.9). If we wish to regard the operator $D_\varepsilon$ as acting on sections of $F(B) \otimes H_\infty$, we simply write $\tilde{\nabla}^u$ for $\nabla^u$ in (A.14). The Levi-Civita superconnection $B_t$ on $H_\infty$ is the superconnection

$$B_t = \tilde{\nabla}^u + t^{1/2} D_Y - \frac{c(T)}{4t^{1/2}}, \quad (A.15)$$

where $c(T) = \sum_{\alpha \leq \beta} dy^\alpha dy^\beta T(f_\alpha, f_\beta)$, and $dy^\alpha$ denotes the 1-form dual to $f_\alpha$. It follows from (A.14) that $\frac{1}{\varepsilon} D_\varepsilon$ is the Dirac operator coupled to the Levi-Civita superconnection $B_{\varepsilon^{-2}}$. 

The asymptotics of kernels associated to the Levi-Civita superconnection exhibit some remarkable cancellations. The first one is expressed in the local index theorem for families, [B], [BF], which says that when \( \dim Y = 2l \) is even
\[
\lim_{t \to 0} \text{tr}_s(e^{-B_t^2}) = \frac{1}{(2\pi i)^l} \int_Y \hat{A}(iR^Y) \text{tr}(e^{-L_t^2}),
\]  
(A.16)
or when \( \dim Y = 2l - 1 \) is odd
\[
\lim_{t \to 0} \text{tr}^{\text{odd}}(e^{-B_t^2}) = \frac{1}{(2\pi i)^l} \int_Y \hat{A}(iR^Y) \text{tr}(e^{-L_t^2}),
\]  
(A.17)
where \( \text{tr}^{\text{odd}} \) indicates taking the odd form part of \( \text{tr} \).

Essential to our discussion are the other two cancellation results, [BC2], [BGS], which state that when \( \dim Y = 2l \) is even
\[
\text{tr}_s[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{1/2}) \text{ as } t \to 0,
\]  
(A.18)
or when \( \dim Y = 2l - 1 \) is odd
\[
\text{tr}^{\text{even}}[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{1/2}) \text{ as } t \to 0,
\]  
(A.19)
where \( \text{tr}^{\text{even}} \) indicates taking the even form part of \( \text{tr} \). We now show that the expressions in (A.18), (A.19) are also well behaved for the large time. We claim that for \( \dim Y = 2l \),
\[
\text{tr}_s[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{-1}) \text{ as } t \to \infty,
\]  
(A.20)
and for \( \dim Y = 2l - 1 \),
\[
\text{tr}^{\text{even}}[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{-1}) \text{ as } t \to \infty.
\]  
(A.21)
In fact, let us define \( \varphi_t : \Lambda^*(B) \to \Lambda^*(B) \) by

\[
\varphi_t|_{\Lambda^i(B)} = \frac{1}{\sqrt{t}}.
\]

Then the key observation is

\[
tr^{\text{even}}\left[(DY + \frac{c(T)}{4t})e^{-t(\tilde{\nabla}^u + \sqrt{t}D^Y - \frac{c(T)}{4\sqrt{t}})^2}\right] = \varphi_t\left(tr^{\text{even}}\left[(DY + \frac{c(T)}{4})e^{-t(\tilde{\nabla}^u + D^Y - \frac{c(T)}{4})^2}\right]\right),
\]

and

\[
tr_s\left[(DY + \frac{c(T)}{4t})e^{-t(\tilde{\nabla}^u + \sqrt{t}D^Y - \frac{c(T)}{4\sqrt{t}})^2}\right] = \varphi_t\left(tr_s\left[(DY + \frac{c(T)}{4})e^{-t(\tilde{\nabla}^u + D^Y - \frac{c(T)}{4})^2}\right]\right).
\]

For \( t \) large, \( (DY + \frac{c(T)}{4})e^{-t(\tilde{\nabla}^u + D^Y - \frac{c(T)}{4})^2} \) is uniformly bounded. Thus, except the 0-form component, \( \varphi_t(tr^{\text{even}}[DY + \frac{c(T)}{4})e^{-t(\tilde{\nabla}^u + D^Y - \frac{c(T)}{4})^2}] \) decays like \( t^{-1} \). But its 0-form component is just \( tr(DY e^{-t(D^Y)^2}) \) which decays exponentially.

As for the other parity, the same reasoning shows that we just have to worry about the 1-form component, which is by Duhamel's principle

\[
\sqrt{t} tr_s\left(DY \tilde{\nabla}^u DY e^{-t(D^Y)^2}\right) = \sqrt{t} tr_s\left[(\tilde{\nabla}^u DY)DY e^{-t(D^Y)^2}\right]
\]

exponentially decaying as well.

By virtue of (A.18 - A.21) we now define a differential form on \( B \), the \( \hat{\eta} \) form

\[
\hat{\eta} = \begin{cases} 
\frac{1}{\sqrt{\pi}} \int_0^\infty tr_s\left[(DY + \frac{c(T)}{4t})e^{-B_t^2}\right] \frac{dt}{2^{1/2}} & \text{if dim } Y = 2l \\
\frac{1}{\sqrt{\pi}} \int_0^\infty tr^{\text{even}}\left[(DY + \frac{c(T)}{4t})e^{-B_t^2}\right] \frac{dt}{2^{1/2}} & \text{if dim } Y = 2l - 1
\end{cases}
\]

For example, the first integral is convergent at 0 because of (A.18), and convergent at \( \infty \) because of (A.20).
The \( \hat{\eta} \) should be viewed as a higher dimensional analogue of the \( \eta \)-invariant. In fact, when \( \dim Y = 2l - 1 \) is odd, its 0-form component is exactly the \( \eta \)-invariant of the Dirac operator along the fibre. We also point out that when \( \dim Y = 2l \) the 1-form component of \( \hat{\eta} \) represents the Quillen connection of the determinant line bundle \( \det D_Y \), which is interpreted by Witten as the covariant anomaly, [W]; see also [BF], [C2], and [F]. We now normalize \( \hat{\eta} \) by defining

\[
\hat{\eta} = \begin{cases} 
\sum \frac{1}{(2\pi i)^j} \hat{[\eta]}_{2j-1} & \text{if } \dim Y = 2l \\
\sum \frac{1}{(2\pi i)^j} \hat{[\eta]}_{2j} & \text{if } \dim Y = 2l - 1 
\end{cases}
\]

Here we decompose the odd (respectively even) form \( \hat{\eta} \) into its homogeneous components \( \hat{[\eta]}_{2j-1} \) (respectively \( \hat{[\eta]}_{2j} \)).

**Theorem A.3 (Bismut-Cheeger)** Assume that \( D_Y \), the Dirac operator along the fiber, is always invertible, then the limit \( \lim_{z \to 0} \hat{\eta}(D_z) = \lim_{z \to 0} \frac{1}{2} \eta(D_z) \) exists in \( \mathbb{R} \) and

\[
\lim_{z \to 0} \hat{\eta}(D_z) = \int_B \hat{A}(\frac{P^B}{2\pi}) \wedge \hat{\eta}.
\]

Moreover,

\[
d\hat{\eta} = \int_Y \hat{A}(\frac{P^Y}{2\pi}) \wedge \text{tr}(e^{-L^t/2\pi i}).
\]

We present an outline of the proof given in [BC2], emphasising on the ideas and techniques we will be using. One is referred to the original paper for further details.

**Outline of Proof.** As we noticed above, the parity of \( \dim B \) (or equivalently, that of \( \dim Y \), since we now always assume that \( \dim M \) is odd)
makes only a slight difference in the discussions, which are otherwise parallel. Thus we assume the dim \( B \) is odd.

1) There is a uniform lower bound for the smallest eigenvalue of \( D_x \).

Set \( E_x = \sum_\alpha f_\alpha (\nabla I^\alpha_{\beta} - \frac{\alpha}{2} f_{\bar{\beta}} T(f_\alpha, f_\beta)) \) then (A.14) implies

\[
D_x^2 = (D_Y)^2 + x[D_Y, E_x] + x^2 E_x^2.
\]

By the results of Section A.2, one easily sees that the operator \([D_Y, E_x]\) is a first order differential operator which acts fibrewise. Thus if \( x \) is small enough

\[
D_x^2 \geq (D_Y)^2 + x[D_Y, E_x] \geq \frac{1}{2} (D_Y)^2,
\]

where the last inequality follows from Garding's inequality for \( D_Y \) and the assumption that \( D_Y \) is always invertible.

2) For \( t \in (0, T] \), we have uniform convergence,

\[
\lim_{z \to 0} tr(D_x e^{-D_x^2 t}) = \frac{\sqrt{\pi}}{(2\pi i)^k} \int_B \hat{A}(i R^B) tr^{\text{even}}((D_Y + \frac{e(T)}{4t}) e^{-B^2} + O(x(1+T)^N))
\]

for some \( N \).

The main technical tool here is the so called Getzler's transformation [G], [BC2]. We first introduce an auxiliary Grassman variable, \( z \) (i.e. \( z^2 = 0 \)), which is odd; see [BC2] for the reason. Once \( z \) has been introduced, instead of considering \( tr(D_x e^{-D_x^2 t}) \), we can consider the part of \( tr(e^{-D_x^2 + z D_x \sqrt{t}}) \) which involves \( z \), denoted by \( tr_z(e^{-D_x^2 + z D_x \sqrt{t}}) \). In fact, an application of Duhamel's principle yields

\[
e^{-D_x^2 + z D_x \sqrt{t}} = e^{-D_x^2 t} + z \sqrt{t} D_x e^{-D_x^2 t}.
\]
Thus,

$$tr_z(e^{-D^2+2Dz}) = z\sqrt{t} tr(Dz e^{-D^2t}).$$

To define Getzler's transformation, one first has to localize the problem. The estimates of [C2], Section 3 and a straightforward generalization of the arguments of [C2], Section 4 show that we can pretend that our base is $\mathbb{R}^{2n-1}$ with a metric flat outside a compact set, the error term being exponentially decaying in $x$ as $x \to 0$. Similarly, we can assume that the bundle is isometrically a product on that region.

By definition, Getzler's transformation, $G_{\delta^{1/2}}$, of a function (usually the square) of a Dirac operator means that we conjugate this operator by the coordinate change in $\mathbb{R}^{2n-1}$,

$$y^\alpha \to \delta^{1/2} y^\alpha,$$

and make the replacement

$$f_\alpha \to \delta^{-1/2} f_\alpha$$

in the Clifford variables and change the Clifford multiplication, $\cdot$, to $\cdot_s$, which satisfies

$$\delta^{-1/2} f_\alpha \cdot_s \delta^{-1/2} f_\beta + \delta^{-1/2} f_\beta \cdot_s \delta^{-1/2} f_\alpha = -2\delta_{\alpha\beta}.$$

The main algebraic ingredient in the application of Getzler's transformation is a Lichnerowicz type formula, which contains no singular terms of $\delta$, Cf. [G], [BC2], and [D]. In [G], the usual Lichnerowicz formula implies that Getzler's transformation of the square of a Dirac operator converges to a Hermite operator (or harmonic oscillator in Physicist's term), whose heat
kernel is explicit. For \( D_x^2 t - z D_x \sqrt{t} \), one does have a Lichnerowicz type formula; see (4.53) of [BC2]. However, it contains a singular term. What is remarkable is that this can be removed by a simple algebraic manipulation. Thus, one first conjugates the operator \( D_x^2 t - z D_x \sqrt{t} \) by
\[
\frac{xy^a f_a}{e^{2(iz^2)^{1/2}}} = 1 + \frac{zy^a f_a}{2(tz^2)^{1/2}},
\]
and then applies Getzler’s transformation \( G_{(iz^2)^{1/2}} \). The formula (4.68) of [BC2] says that
\[
\lim_{z \to 0} G_{(iz^2)^{1/2}} \left[ e^{2(iz^2)^{1/2}} (D_x^2 t - z D_x \sqrt{t}) e^{-2(iz^2)^{1/2}} \right] = \mathcal{H} + B_x^2 - z(\sqrt{t}D_Y + \frac{c(T)}{4\sqrt{t}}),
\]
where \( \mathcal{H} \) is the generalized Hermite operator on \( \mathbb{R}^{2n-1} \) which was considered by Getzler in [G]. Now the same arguments as in [G] show that
\[
tr_z(e^{-D_x^2 t + z D_x \sqrt{t}}) \to tr_z[e^{-\mathcal{H} - B_x^2 + z\sqrt{t}(D_Y + \frac{c(T)}{4t})}]
\]
\[
= \frac{\sqrt{\pi}}{(2\pi i)^k} \hat{A}(iR^p) \wedge z\sqrt{t} tr_z[(D_Y + \frac{c(T)}{4t})e^{-B_x^2}]
\]
uniformly, for \( t \in [\delta, T] \). Or
\[
\frac{1}{2\sqrt{\pi t}} tr(D_x e^{-D_x^2 t}) \to \frac{1}{(2\pi)^k} \int_B \hat{A}(iR^p) tr_z((D_Y + \frac{c(T)}{4t})e^{-B_x^2}) \frac{1}{2t^{1/2}}. \quad (A.23)
\]
To see the convergence is actually uniform for \( t \in [0, T] \), it suffices to show that the coefficient, \( c_x \), in the small time expansion,
\[
\frac{1}{2\sqrt{\pi t}} tr(D_x e^{-D_x^2 t}) = c_x + O(t),
\]
approaches a limit and that the remainder term remains uniformly bounded by \( ct \), independent of \( z \). To this end, note that
\[
z \frac{1}{2\sqrt{\pi t}} tr(D_x e^{-D_x^2 t}) = \frac{1}{2\sqrt{\pi t}^{3/2}} tr_z(e^{-t(D_x^2 - z D_x)})
\]
\[
= \frac{1}{2\sqrt{\pi t}^{3/2}} tr_z e^{-tc_x}, \quad (A.24)
\]
where \( \mathcal{L}_x = G_x e^{\frac{e^{2x}D_x}{2}} (D_x^2 - zD_x \sqrt{i}) e^{-\frac{e^{2x}D_x}{2}} \) is Getzler's transformation of \( D_x^2 - zD_x \), except that we desingularize it by first conjugating the operator. By (4.68) of [BC2] (taking \( u = 1, \sqrt{1/2} = x \)), \( \mathcal{L}_x \) is smooth family of elliptic operators down to \( x = 0 \). (In fact, \( \mathcal{L}_0 = \mathcal{H} + B^2_1 - z(D_Y + \frac{2(T^1)}{4}) \)). Since the asymptotic expansion of \( \text{tr}(e^{-t\mathcal{L}_x}) \) at the origin depends only on the local symbol of \( \mathcal{L}_x \), it follows that the coefficients in this expansion converge as \( x \to 0 \), to those in the expansion for \( \text{tr}(e^{-t\mathcal{L}_0}) \) and that the remainder terms are bounded by \( ct \) independent of \( x \). Putting this together with (A.23) and (A.24) we see that \( c_x \to c_0 \) and that the remainder is bounded by \( ct \) independent of \( x \).

**Remark** The above argument contains a proof of (A.18) and (A.19); see [BC2].

To have a remainder estimate for (A.23), notice that a straightforward generalization of the argument of [C2] shows that, for fixed \( t \), there is an expansion

\[
\text{tr}(D_x e^{-tD_x^2}) \sim \sum_{i=0}^{2k-1} a_i x^{2i - (2k - 1)},
\]

(A.25)

whose coefficients and remainder term are bounded by \( T^N \) for say \( t \in [1, T] \) and some \( N \). Then (A.23) implies that (A.25) does not contain singular terms. The desired estimate follows.

3) Large time contribution is negligible as \( x \to 0 \).

This is shown by using 1) and the finite propagation speed technique [CGT].

From the proof we see that
Corollary A.1 Without the assumption that $D_Y$ is invertible, one can still find a small positive number $\alpha$ such that
\[
\lim_{z \to 0} \frac{1}{2} \eta(D_z) = \int_B \hat{A} \left( \frac{R^B}{2\pi} \right) \land \bar{\eta} + \lim_{z \to 0} \frac{1}{2\sqrt{\pi}} \int_{z^{-\alpha}}^{e^{-D_Z^2}} t^{-1/2} \text{tr}(D_z e^{-D_Z^2}) dt, \quad (A.28)
\]
provided either one of the limits exists.

A.4 Analysis on manifolds with boundary

There are two ways to describe pseudo-differential operators, the symbol description and the Schwartzian kernel description. The two are related by the Fourier transform. The advantage of the Schwartzian kernel description over the symbol description is that it allows a precise description of the singularities of the Schwartzian kernel of pseudo-differential operators. This will be important for the pseudo-differential operators on manifolds with boundary. The Schwartzian kernel of a pseudo-differential operator gives rise to a conormal distribution [Hor] [Me].

Let $X^n$ be a $C^{\infty}$ manifold, $E$ a $C^{\infty}$ vector bundle on $X$, and $Y^k \subset X$ a (closed) submanifold. Let $\mathcal{V} = \mathcal{V}(Y)$ denote the space of vector fields on $X$ which are tangent to $Y$ at $Y$. The space of $\mathcal{V}$-differential operators, $\text{Diff}_\mathcal{V}(X, E)$, is generated by $\mathcal{V}$, i.e. elements of $\text{Diff}_\mathcal{V}(X, E)$ are sums of products of elements in $\mathcal{V}$. The space $I(X, Y; E)$ of distribution sections of $E$, conormal with respect to $Y$, is defined to be the set of distribution sections $u \in \mathcal{D}'(X, E)$ such that $V_1 \cdots V_k u \in H^s(X)$, $V_i \in \mathcal{V}(Y)$ for some real number $s = s(u)$ and all $k$. Here $H^s(X)$ denotes the Sobolev space on $X$. Thus $I(X, Y; E)$ are distributions which have fixed regularities under
the action of $V$-differential operators. The following theorem (see [Hor]) says that such a distribution is locally defined by a symbol via the oscillatory integral.

**Theorem A.4** $u \in I(X, Y; E)$ iff $\phi_j u \in I(X, Y; E)$ for every $\phi_j$ in a partition of unity on $X$. If $X$ is an open set in $\mathbb{R}^n$ and $Y$ is defined by $x' = (x_1, \ldots, x_k) = 0$ while $E = X \times \mathbb{C}^N$, then any $u \in I^m(X, Y; E)$ with compact support is of the form

$$u(x) = \int e^{i<x',\xi'>} a(x'', \xi') d\xi',$$

where $a \in S(\mathbb{R}^{n-k} \times \mathbb{R}^k; \mathbb{C}^N)$, the space of symbols. Conversely, every $u$ of this form is in $I(X, Y; E)$.

The order of symbol gives a filtration on the space of symbols which in term gives a filtration on the space of conormal distribution. The significance of this filtration lies in the following [Hor]

**Theorem A.5** We have the following short exact sequence

$$0 \to I^{m-1}(X, Y; \Omega_X^{1/2} \otimes E) \to I^m(X, Y; \Omega_X^{1/2} \otimes E) \to S^m(N(Y), \Omega_Y^{1/2} \otimes E) \to 0,$$

where $N(Y)$ is the normal bundle of $Y$ in $X$ and $\Omega^{1/2}$ denotes the $1/2$-density bundle. $\sigma$ is called the principal symbol map.

The space of conormal distributions behaves nicely under the action of pseudo-differential operators (Cf. [Hor]).

**Theorem A.6** If $u \in I^m(X, Y; E)$, and $P \in \Psi^m(X, Y; F)$ is properly supported, then $Pu \in I^{m+m'}(X, Y; F)$ and the principal symbol is that of $u$ multiplied by the restriction to $N(Y)$ of the principal symbol of $P$. 
Finally, we note that conormal distributions have singularities only on
the submanifold $Y$ and then only along the normal directions (this can be
made precise by the notion of wave front set of a distribution).

Now suppose $X$ is a $C^\infty$-manifold with boundary. The boundary $\partial X$ is
therefore a closed submanifold of $X$. In this case, the $\mathcal{V}$-differential oper-
ators are called the totally characteristic differential operators. Regarding
the behavior at the boundary, one has three types of distributions. Thus
$C^{-\infty}(\mathring{X})$ are those which have no control at the boundary at all. In gen-
eral, we wish to consider only the distributions having "slow growth" at
the boundary. The space of extendible distributions is defined as the dual
of the space of smooth functions on $X$ vanishing to infinite order at the
boundary,

$$C^{-\infty}(X) = (\mathring{C}^\infty(X))'$$

and the space of supported distributions is defined as the dual of the space
of smooth functions on $X$ (up to boundary),

$$\mathring{C}^{-\infty}(X) = (C^\infty(X))'.$$

The space of almost regular functions

$$\mathcal{A}(X) = I(X \cup X, \partial X)$$

and its supported counterpart

$$\mathring{\mathcal{A}}(X) = I(X \cup X, \partial X) \cap \mathring{C}^{-\infty}(X)$$

play an important role here. Roughly speaking, $\mathcal{A}(X)$ is the space of func-
tions smooth in the interior of $X$ and along $\partial X$, singular only along the
normal direction of $\partial X$. We mention that the relation of $\mathcal{A}(X)$ to totally characteristic operators is similar to the relation of $C^\infty(X)$ to the pseudo-differential operators when $X$ is a compact manifold.

In dealing with regularity on manifold with boundary, of equal importance is the dual of $\mathcal{A}(X)$, denoted $\mathcal{A}'(X)$. Intuitively, $\mathcal{A}'(X)$ consists of distributions having normal derivatives on $\partial X$ of all orders, thus suggesting the following [M2]

**Theorem A.7** On any $C^\infty$-manifold $X$ with boundary,

$$\mathcal{A}'(X) \cap \mathcal{A}(X) = C^\infty(X).$$

A more workable description of the space $\mathcal{A}'(X)$ is furnished by the Mellin transform. For simplicity, we assume that $X = [0, \infty)$. The Mellin transform of a smooth function $u$ on $[0, \infty)$ is defined as

$$u_M(z) = \int_0^\infty x^{z-1} u(x)dx$$

if $\text{Re } z > 0$. This can be extended to distributions ($C^{-\infty}$ or $\mathcal{C}^{-\infty}$) by duality. The normalized Mellin transform is just $\frac{u_M(z)}{\Gamma(z)}$. In general, they are defined only for Re $z$ large.

**Proposition A.4** $\mathcal{A}'(X)$ can be characterized as those distributions $u$ whose normalized Mellin transform $\frac{u_M(z)}{\Gamma(z)} \in C^{-\infty}(X)$ extends to an entire function of $z$.

See [M1] for its proof.
Appendix B

A proof of the Families Index Theorem for manifolds with boundary

The Families Index Theorem for manifolds with boundary is proved in [BC2] by the cone method. There the singularity is essential. Here, with additional assumption, we give a proof along the line of Atiyah-Patodi-Singer [APS]. The Atiyah-Patodi-Singer boundary condition plays an important role.

To be more precise, suppose we are given the following data:

a) A fibration of compact manifolds $Z^{2k} \rightarrow N \rightarrow B$ with base $B$ closed and fibre $Z^{2k}$ having dimension $2k$ and nonempty boundary $\partial Z = Y$. Associated is then the boundary fibration $Y \rightarrow M \rightarrow B$ where $M = \partial N$. 
b) A splitting $TN = T^HN \oplus T^V N$ with $T^V N$ the vertical subspace on which a metric $g^Z$ is given such that $g^Z$ becomes a product metric near $\partial N$. As in [B], we can define connection $\nabla^V$ on $T^V N$ with curvature $R^Z$ (see also Section 1.2).

c) A hermitian bundle $\xi$ with metric and compatible connection whose curvature is $L^\xi$. These are of product type near $\partial N$.

With these data (and additional spin conditions if necessary), we can consider $D^Z$, the family of Dirac operators along $Z$. Let $D_Y$ denote its tangential part near the boundary. We give a proof of the Families Index Theorem for manifolds with boundary in the following case.

**Theorem B.1** Assume that $\ker D_Y$ is a well-defined element in $K^0(B)$. Further, assume the boundary fibration has compact holonomy. Then $\text{Ind}(D^Z_+, P_+)$, the index bundle of $D^Z_+$ with APS boundary condition [APS], is a well-defined element in $K^0(B)$ and its chern character is represented in cohomology by

$$\int_Z \hat{A}(\frac{R^Z}{2\pi}) \wedge \text{tr}(e^{iL^\xi/2\pi}) - \tilde{\eta} - \frac{1}{2} \text{ch}(\ker D_Y).$$

When $B$ is a single point, $\hat{\eta}$ reduces to the usual $\eta$-invariant and the above recovers the Atiyah-Patodi-Singer theorem [APS].

**Proof.** The outline of proof is

1) We show it is possible to define a superconnection for our family of the APS boundary value problems and we establish a heat equation formula for $\text{ch}(\text{Ind}(D^Z_+, P_+))$. 
2) In order to have local convergence in the interior, we have to use
Levi-Civita superconnection of [B]. (See also [Q].) But for the computation
near the boundary, it is much easier to use the unitary superconnection of
[BF], [B]. We patch them on the cylinder.

3) To calculate the contribution from the mixed term, we let the cylinder
become longer and longer and bring in the adiabatic calculation in the spirit
of [C] and [BC2].

4) It is much easier to deal with numbers than with forms. We couple
our forms with $\hat{A}$-genus and employ the meromorphic continuation method
of [APS]. To get back to differential forms, we use a method of Bismut (in
the second proof of [B]).

1) APS boundary problem and heat equation formula for the index

We first consider general first order differential operators of the split
type. Suppose that

$$A : \ C^\infty(\partial Z, E) \rightarrow C^\infty(\partial Z, F)$$

is a first order elliptic differential operator such that in a tubular neighborhood
of the boundary $\partial Z = Y$ it is of the form

$$A = \sigma (\partial_u + B), \quad \text{(B.1)}$$

where $\sigma$ is a bundle isomorphism given by the symbol of $A$ in the direction
$du$, and $B : \ C^\infty(Y, E) \rightarrow C^\infty(Y, E)$ is a self-adjoint first order elliptic
differential operator on $Y$, called the tangential part of $A$. Consider the
spectral projection $P_\pm$ of $B$ onto the eigenspaces with non-negative respec-
tively negative eigenvalues. The Atiyah-Patodi-Singer boundary condition (abbreviated APS boundary condition) for the operator $A$ is $P_+(\phi|_Y) = 0$.

In [APS], they showed that the following are true.

a) Let $C^\infty(Z, E; P_+)$ denote the smooth sections $\phi$ of $E$ such that $P_+(\phi|_Y) = 0$. Then there is a linear operator $Q : C^\infty(Z, F) \longrightarrow C^\infty(Z, E; P_+)$ which is a parametrix for $A : C^\infty(Z, E; P_+) \longrightarrow C^\infty(Z, F)$. i.e., both $AQ - 1, QA - 1$ are smoothing operators.

b) $Q$ extends to a continuous operator $H^{l-1} \rightarrow H^l$ for $l \geq 1$. Therefore

$$A : C^\infty(Z, E; P_+) \longrightarrow C^\infty(Z, F)$$

and

$$A_l : H^l(Z, E; P_+) \longrightarrow H^{l-1}(Z, F) \quad \text{for } l \geq 1$$

are Fredholm operators with the same null-space.

c) Denote $D = (A, P_+)$. When $Z = R_+ \times Y$ is a cylinder, the heat kernels $e^{-tD^*D}$, $e^{-tDD^*}$ are explicit in terms of the spectral resolution $\{\lambda, \varphi_\lambda\}$ of $B$:

$$e^{-tD^*D} = \sum_{\lambda \geq 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{ \exp\left(\frac{-(u - v)^2}{4t}\right) - \exp\left(\frac{-(u + v)^2}{4t}\right) \} \varphi_\lambda(x) \otimes \varphi_\lambda(y)$$

$$+ \sum_{\lambda < 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{ \exp\left(\frac{-(u - v)^2}{4t}\right) + \exp\left(\frac{-(u + v)^2}{4t}\right) \} + \lambda e^{-\lambda(u+v)} \text{erfc}\left(\frac{u + v}{2\sqrt{t}}\right) \} \varphi_\lambda(x) \otimes \varphi_\lambda(y),$$

$$e^{-tDD^*} = \sum_{\lambda < 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{ \exp\left(\frac{-(u - v)^2}{4t}\right) - \exp\left(\frac{-(u + v)^2}{4t}\right) \} \varphi_\lambda(x) \otimes \varphi_\lambda(y)$$

$$+ \sum_{\lambda \geq 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{ \exp\left(\frac{-(u - v)^2}{4t}\right) + \exp\left(\frac{-(u + v)^2}{4t}\right) \}$$

$$- \lambda e^{\lambda(u+v)} \text{erfc}\left(\frac{u + v}{2\sqrt{t}}\right) \} \varphi_\lambda(x) \otimes \varphi_\lambda(y),$$
where \( \text{erfc} \) is the complimentary error function defined by

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} \, du.
\]

Now suppose that \( Z \) is parametrized by a closed manifold \( B \), i.e., \( Z \) is a typical fiber of the fibration (A.9). Correspondingly, we assume that a smooth family \( A_y \) of the form (B.1) has been given on \( Z \), where \( y \in B \) (we sometimes suppress the subscript). We want to study the parameter dependence of the APS boundary problems. We begin with that of the APS boundary conditions.

**Lemma B.1** Assume that \( \dim \ker B_y \) is constant, where \( B_y \) is the tangential part of \( A_y \) (Cf. (B.1)). Then \( P_+ \) is a smooth family of pseudodifferential operators of order 0. Consequently, \( D = (A, P_+) \) is a smooth family of APS boundary value problems. The same for \( D^* = (A^*, P_-) \).

Proof. Since \( B_y \) is a smooth family of self-adjoint elliptic differential operators, by the compactness of \( B \) and the assumption, we can find \( \delta > 0 \), such that for all \( y \in B \),

\[
\text{spec}(B_y) \cap \{ |z| \leq \delta \} = \{ 0 \}.
\]

Consider the Dunford-Schwartz integral

\[
B_+^{-1} = \frac{1}{2\pi i} \int_{\Gamma_+} \lambda^{-1} (B_y - \lambda)^{-1} \, d\lambda,
\]

where \( \Gamma_+ \) is the counterclockwise contour:

\[
\arg \lambda = \pm \pi/4 \text{ if } |\lambda| \geq \delta, \text{ or } |\lambda| = \delta \text{ if } \pi/4 \leq \arg \lambda \leq 7\pi/4.
\]
By the standard theory, $B^{-1}_+$ is a smooth family of pseudodifferential operators of order $-1$. Consequently, $P_+ = BB^{-1}_+$ is a smooth family of pseudodifferential operators of order $0$. Q.E.D.

Next we address the existence of the index bundle.

**Lemma B.2** With the same hypothesis as above, $H^l(Z, E; P_+)$, $H^{l-1}(Z, F)$ are smooth family of Hilbert spaces and

$$D_y : H^l(Z, E; P_+) \rightarrow H^{l-1}(Z, F)$$

is a smooth family of Fredholm operators. Thus $D$ has a well-defined index bundle as an element of $K^0(B)$.

**Proof.** The statement about $H^{l-1}(Z, F)$ is trivial. To prove that of $H^l(Z, E; P_+)$, we fix a $y_0 \in B$ and construct a smooth family of invertible operators

$$\tilde{U}(y) : H^l(Z_y, E; P^y_+) \rightarrow H^l(Z_{y_0}, E; P^{y_0}_+).$$

In fact, since $P^y_+$ is a smooth family of projections, by Lemma 2.12, there exists a smooth family $U(y)$ of invertible operators (on $L^2(y)$) such that

$$U(y)P^y_+U(y)^{-1} = P^{y_0}_+. \quad (B.2)$$

By the explicit formula there, one sees that $U(y)$ is actually pseudodifferential of order $0$. Now we want to extend $U(y)$ to $\tilde{U}(y) : H^l(Z_y) \rightarrow H^l(Z_{y_0})$. Then it is clear that from (B.2) $\tilde{U}(y)$ restricts to

$$H^l(Z_y, E; P^y_+) \rightarrow H^l(Z_{y_0}, E; P^{y_0}_+).$$
To extend $U(y)$, one first deform $U(y)$ to the identity operator and then extend $U(y)$ to be the deformation on the cylinder and identity in the interior. Q.E.D.

By a) and b), $\mathcal{D}$ and $\mathcal{D}^*$ are elliptic boundary problems. Set

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix}.$$ 

We construct the heat kernel of $\mathcal{A}^2$,

$$e^{-t\mathcal{A}^2} = \begin{pmatrix} e^{-t\mathcal{D}^*\mathcal{D}} & 0 \\ 0 & e^{-t\mathcal{D}\mathcal{D}^*} \end{pmatrix}.$$ 

**Proposition B.1** $e^{-t\mathcal{A}^2}$ is a smooth family of heat kernels parametrized by $B$.

**Proof.** We first study the resolvent $(\mathcal{A}^2 - \lambda)^{-1}$, showing that it depends smoothly on $y \in B$ as $L^2$-operators. To do this, we will construct a nice parametrix by patching together the interior parametrix with the boundary parametrix. Let $Q_2$ be the Green's operators for $(\mathcal{A}^2 - \lambda)$ on the double of $Z$. Then it follows from the standard theory that $Q_2$ as a pseudodifferential operator of order $-2$ depends smoothly on $y \in B$. We now study the corresponding operator $Q_1$ for $(\mathcal{A}^2 - \lambda)$ on the half cylinder $Y \times \mathbb{R}_+$.

To solve $(\mathcal{A}^2 - \lambda)f = g$, we expand $f$ and $g$ in terms of the spectral resolution $\{\lambda_1, \phi_{\lambda_1}\}$ of $B$:

$$f(z, u) = \sum f_{\lambda_1}(u)\phi_{\lambda_1}(z), \quad g(z, u) = \sum g_{\lambda_1}(u)\phi_{\lambda_1}(z).$$
We must now solve
\[
\begin{cases}
  -\frac{d^2 f_{\lambda_1}}{du^2} + (\lambda_1^2 - \lambda)f_{\lambda_1} = g_{\lambda_1} \\
  f_{\lambda_1}(0) = 0 \text{ if } \lambda_1 \geq 0 \text{ or } \frac{df_{\lambda_1}}{du}(0) + \lambda_1 f_{\lambda_1}(0) = 0 \text{ if } \lambda_1 < 0
\end{cases}
\]
We take the explicit solution
\[
f_{\lambda_1}(u) = -\frac{1}{2\lambda} \left[ \int_0^u g(v)e^{\lambda(v-u)} \, dv + \int_u^{+\infty} g(v)e^{\lambda(v-u)} \, dv \right] + C,
\]
where \(\lambda = \sqrt{\lambda_1^2 - \lambda}\) with \(\text{Re}\lambda > 0\) and
\[
C = \begin{cases}
  \frac{1}{2\lambda} \int_0^{+\infty} g(v)e^{-\lambda v} \, dv & \text{if } \lambda_1 \geq 0 \\
  \left( \frac{1}{\lambda} + \frac{1}{2\lambda} \right) \int_0^{+\infty} g(v)e^{-\lambda v} \, dv & \text{if } \lambda_1 < 0
\end{cases}
\]
It follows that \(Q_1 = Q_1' + Q_1''\), where \(Q_1'\) is given by convolution in the \(u\)-variable with
\[
P(t) = \frac{\epsilon(t)}{2\sqrt{B^2 - \lambda}} e^{-t\sqrt{B^2 - \lambda}} - \epsilon(-t) \frac{e^{t\sqrt{B^2 - \lambda}}}{2\sqrt{B^2 - \lambda}},
\]
where \(\epsilon(t)\) is the characteristic function of the nonnegative line, and \(Q_1''\) has as kernel that of
\[
\epsilon(t) \frac{e^{-t\sqrt{B^2 - \lambda}}}{\sqrt{B^2 - \lambda}} P_+ + \epsilon(t) \left( \frac{e^{-t\sqrt{B^2 - \lambda}}}{2\sqrt{B^2 - \lambda}} + \frac{e^{-t\sqrt{B^2 - \lambda}}}{2B} \right) P_-.
\]
Hence, \(Q_1\) as \(L^2\)-operators depends smoothly on \(y \in B\). Moreover, the kernel of \(Q_1\) is smooth in all variables away from the diagonal and the boundary.

Now let \(\phi's\) and \(\psi's\) be as [APS] (see also p.141). Put \(Q = \phi_1 Q_1 \psi_1 + \phi_2 Q_2 \psi_2\). Then \(Q\) is a smooth family of \(L^2\)-operators such that \((A^2 - \lambda)Q - \text{Id}\) and \(Q(A^2 - \lambda) - \text{Id}\) are smooth family of smoothing operators. This implies that \((A^2 - \lambda)^{-1}\) is smooth in \(y \in B\) as \(L^2\)-operators. Consequently, so is
\[
e^{-tA^2} = \frac{1}{2\pi i} \int_{\Gamma^+} e^{-\lambda} (A^2 - \lambda)^{-1} d\lambda.
\]
To show that $e^{-tA^2}$ is smooth as smoothing operators, we need the following lemma.

**Lemma B.3** Let $K : B \to \mathcal{L}(L^2(Z), L^2(Z))$ be a smooth family of $L^2$-operators. Suppose that for all $y \in B$, $K(y)$ is actually a smoothing operator with kernel $K(y, z, z')$. If, in addition, we have the uniform estimates

$$|\partial^\alpha \partial^\beta z'K(y, z, z')| \leq C(\alpha, \beta),$$

then $K(y, z, z')$ is also smooth in $y$. i.e., $K$ is actually a smooth family of smoothing operators.

By the construction as in [APS], one can check that $e^{-tA^2}$ satisfies the hypothesis above, yielding the desired result.

We now restrict our attention to the family of Dirac operators. Let $H^\infty$, $H^\infty_\pm$ be $C^\infty$-sections of $F(TZ) \otimes \xi$, $F_{\pm}(TZ) \otimes \xi$. We regard $H^\infty$, $H^\infty_\pm$ as $C^\infty$-sections on $B$ of infinite dimensional vector bundles. Recall that one can define a connection $\nabla$ on $H^\infty_\pm$ via the horizontal lift

$$\nabla_{X} h \overset{\text{def}}{=} \nabla_{X} h.$$

If the holonomy of the fibration is compact, then

$$[\nabla, A] = 0 \quad \text{(B.3)}$$

and $\nabla + \sqrt{t} A$ on $H^\infty \otimes \Lambda^*TB$ is reduced to the unitary superconnection $\nabla^u + \sqrt{t} A$. By (B.3), $(\nabla + \sqrt{t} A)^2 = \nabla^2 + t A^2$ and $\nabla^2 = R$ is a 2-form valued first order differential operator acting fibrewise. By the nilpotency
of the exterior algebra, $e^{-\hat{R}}$ is well-defined and is a form-valued differential operator acting fibrewise. Thus the superconnection heat kernel can be defined as

$$ e^{-(\hat{\nabla}^2 + \sqrt{t}A)^2} = e^{-\hat{R}}(e^{-tA}).$$ \hspace{1cm} (B.4)

However, we only assume that the holonomy of the boundary fibration is compact, or the holonomy of the fibration is compact near the boundary. We have

**Lemma B.4** The superconnection heat kernel $e^{-(\hat{\nabla}^2 + \sqrt{t}A)^2}$ exists for small $t$ and its supertrace $\text{tr} e^{-(\hat{\nabla}^2 + \sqrt{t}A)^2}$ is a smooth differential form of even degree which is closed and represents the renormalized chern character of the index bundle $\overline{ch}(\text{Ind}D)$.

**Notation** $\overline{ch}(\xi) = \text{tr}(e^{-L^2})$ if $L^2$ is the curvature operator of $\xi$. Thus $\overline{ch}$ differs from $ch$ by the factor $\frac{1}{2\pi i}$.

**Proof.** Since the holonomy of the boundary fibration is compact, we can use (B.4) to define the superconnection heat kernel on the half cylinder while in the interior we use the superconnection heat kernel on the double. The two are patched together by the standard procedure, producing a parametrix with an error term exponentially small as $t \to 0$. The first claim follows then from Duhamel's principle. The proof of the second statement is essentially the same as in [B].

For later use, we shall make some computations for our superconnection heat kernels on the half cylinder, $N = M \times \mathbb{R}_+$. Locally we use the coordinates $(y, z, u)$. Denote the pointwise kernel of $e^{-(\hat{\nabla}^2 + \sqrt{t}A)^2}$ by
\[ P_t(y, z, u, y', z', u') \] and set
\[ K(t) = \int_0^\infty \int_Y tr^* (P_t) \, dz \, du. \]

Then by c) on p.132, we find
\[ K(t) = -\sum \frac{sgn \lambda}{2} \text{erfc}(|\lambda| \sqrt{t}) \int_Y e^{-R \varphi_\lambda \cdot \varphi_\lambda} \, dz. \]

Thus
\[ K'(t) = \frac{1}{\sqrt{4\pi t}} \sum \lambda e^{-\lambda^2 t} \int_Y e^{-R \varphi_\lambda \cdot \varphi_\lambda} \, dz \]
\[ = \frac{1}{\sqrt{4\pi t}} tr (D_Y e^{-\left(\tilde{\nabla} + \sqrt{i}D_R\right)^2}). \]

Note that
\[ K(t) \to -\frac{1}{2 \text{ch}(\ker D_Y)} \quad \text{as } t \text{ goes to infinity}. \]

Moreover,
\[ K(t) + \frac{1}{2 \text{ch}(\ker D_Y)} \to 0 \quad \text{exponentially}. \]

One deduce via integration by part,

**Lemma B.5** We have
\[
\frac{1}{2\sqrt{\pi}} \int_0^\infty t^{s-\frac{1}{2}} tr (D_Y e^{-\left(\tilde{\nabla} + \sqrt{i}D_R\right)^2}) \, dt = -s \int_0^\infty t^{s-1} [K(t) + \frac{1}{2 \text{ch}(\ker D_Y)]} \, dt.
\]

(B.5)

2) Levi-Civita superconnection.

From the manifold \( N \), one can construct a closed manifold \( X = N \cup_{\delta N} N \) by the doubling procedure. We shall use \( \mathcal{A} \) again to denote the operators
obtained by the doubling construction. On a closed manifold, of great interest is Bismut's Levi-Civita superconnection
\[ \tilde{\nabla} + \sqrt{t} A - \frac{c(T)}{4\sqrt{t}}. \]

The following cancellation result is proved in [B].

**Theorem B.2** The pointwise supertrace \( \text{tr}_* e^{-\left(\tilde{\nabla} + \sqrt{t} A - \frac{c(T)}{4\sqrt{t}}\right)^2} \) of the heat kernel of the Levi-Civita superconnection has the following asymptotic expansion
\[ \text{tr}_* e^{-\left(\tilde{\nabla} + \sqrt{t} A - \frac{c(T)}{4\sqrt{t}}\right)^2} = \sum_{k=0}^{N} a_k t^{k/2} + O(t^{(N+1)/2}), \]
where \( a_0 = \hat{A}(iR^2) \wedge \text{tr}(e^{-L}). \)

**Remark** Note that \( \left(\tilde{\nabla} + \sqrt{t} A - \frac{c(T)}{4\sqrt{t}}\right)^2 = t \left(\frac{1}{\sqrt{t}} \tilde{\nabla} + A - \frac{c(T)}{4} \right)^2. \) If we let \( \varphi_t \) be the homomorphism from \( \Lambda(B) \) into itself which sends a one-form \( \omega \) to \( \frac{1}{\sqrt{t}} \omega, \) then
\[ \text{tr}_* e^{-\left(\tilde{\nabla} + \sqrt{t} A - \frac{c(T)}{4\sqrt{t}}\right)^2} = \varphi_t(\text{tr}_* e^{-\left(\tilde{\nabla} + A - \frac{c(T)}{4} \right)^2}). \]

From this and the standard elliptic theory, one easily sees the existence of an asymptotic expansion. It also follows that the coefficients in the asymptotic expansion are local.

Thus the Levi-Civita superconnection yields local convergence, which in general does not happen to the unitary superconnection. However, to incorporate the boundary condition, we have to use the unitary superconnection. We now patch them up near the boundary.
Let \( \varphi(u) \) be a smooth function on \([0,1]\) such that it is identically 1 on \([0,a]\) and 0 on \([1-a,1]\), where \(a\) is a small positive number. Consider the superconnection

\[
B = \tilde{\nabla} + \sqrt{i} A - \frac{\varphi(u) T}{4\sqrt{t}}.
\]

The same consideration as in showing Lemma B.4 shows that the superconnection heat kernel for \(B\) also exists for small \(t\).

**Lemma B.6**  \( \operatorname{tr}_t e^{-\left(\tilde{\nabla} + \sqrt{i} A - \frac{\varphi(u) T}{4\sqrt{t}}\right)^2} \) is also a representation of \(\overline{c_h}(\operatorname{Ind} D)\). In fact, we have

\[
\frac{\partial}{\partial \delta}\operatorname{tr}_t e^{-\left(\tilde{\nabla} + \sqrt{i} A - \frac{\varphi(u) T}{4\sqrt{t}}\right)^2} = d(-\operatorname{tr}_t \left[\frac{\varphi(u) T}{4\sqrt{t}} e^{-\left(\tilde{\nabla} + \sqrt{i} A - \frac{\varphi(u) T}{4\sqrt{t}}\right)^2}\right]).
\]

Now let \(e_1\) be the heat kernel of \(\tilde{\nabla} + \sqrt{i} A\) on the cylinder constructed earlier, \(e_2\) the heat kernel of \(B\) on the double of \(Z\). The parametrix \(R\) for \(e^{-B^2}\), the superconnection heat kernel with the APS boundary condition, can be constructed by patching together \(e_1\) with \(e_2\). More precisely, let \(\rho(b,c)\) denote an increasing smooth function of the real variable \(u\), such that \(\rho = 0\) for \(u \leq b\) and \(\rho = 1\) for \(u \geq c\), and define smooth functions \(\phi_1, \phi_2, \psi_1, \psi_2\) by

\[
\phi_2 = \rho(1/4, 1/2), \quad \psi_2 = \rho(1/2, 3/4),
\]

\[
\phi_1 = 1 - \rho(3/4, 1), \quad \psi_1 = 1 - \psi_2.
\]

Note that \(\phi_i = 1\) on the support of \(\psi_i\). We regard these functions of \(u\) as functions on the cylinder \(M \times [0,1]\) and then extend them to \(N\) in the obvious way. Finally we put

\[
R = \phi_1 e_1 \psi_1 + \phi_2 e_2 \psi_2.
\]
One checks that
\[ e^{-B^2} - R = O(e^{-C/t}). \] (B.6)

From this and the above lemma, we have
\[
\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge e^{\hat{A}(I\text{nd}D)} = \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \int_0^1 \int_Y (tr_x P_t)(y, z, u) \phi_1(u) dz du \\
+ \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \int_Z (tr_x e_z) \phi_2 + O(e^{-C/t}).
\]

Because of c) on p.132, it follows that, in the first term, we can replace \( \phi_1 \) by 1 and \( \int_0^1 \) by \( \int_0^\infty \) so that we end up with \( \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge K(t) \). On the other hand
\[ tr_x e_2 = \sum_{k=-n}^N a_k t^{k/2} + O(t^{N+1/2}). \]

Thus,
\[
\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge K(t) \sim \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge e^{\hat{A}(I\text{nd}D)} - \sum_{k=-n}^N a_k t^{k/2} \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \int_Z e^a.
\] (B.7)

3) adiabatic limit.

Let us consider the differential form of even degree (Cf. (B.3))
\[ \hat{\eta}_1(s) = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{e^{-\frac{1}{2}}} tr(DY e^{-(\hat{\gamma} + \sqrt{i}D_Y)^2}) dt. \]

By virtue of (B.3), this integral is convergent for Re \( s \gg 0 \). Standard method shows that \( \hat{\eta}_1(s) \) has a meromorphic extension to the \( s \)-plane. It is possible that \( s = 0 \) might be a simple pole for \( \hat{\eta}_1(s) \). We thus define \( \tilde{\eta}_1(0) \) to be the finite part of \( \hat{\eta}_1(s) \) at \( s = 0 \):
\[ \hat{\eta}_1(0) = (s\hat{\eta}_1(s))'|_{s=0} = (\hat{\eta}_1(s) - \frac{Res_{s=0} \hat{\eta}_1(s)}{s})|_{s=0}. \]
The foregoing discussion together with (B.5) implies that \( \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \hat{\eta}_1(s) \) is actually regular at \( s = 0 \) and its value at \( s = 0 \) is given by

\[
\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \hat{\eta}_1(0) = \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge [\int_Z a_0 - \frac{1}{2} \overline{ch}(\ker D_Y) + \overline{ch}(\text{IndD})],
\]

or

\[
\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \overline{ch}(\text{IndD}) = \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge [\int_Z a_0 - \frac{1}{2} \overline{ch}(\ker D_Y) + \hat{\eta}_1(0)]. \quad (B.8)
\]

Note that if we replace \( D_+^Z \) by \( D_+^Z \otimes \pi^* \zeta \), where \( \zeta \) is a vector bundle on \( B \), we have the same formula (B.8) except that every integrand be wedged by \( ch(\zeta) \). Since \( ch: K^0(B) \rightarrow H^{2*}(B) \) is a rational isomorphism, we have the following consequences:

1) \( \text{Res}_{s=0} \hat{\eta}_1(s) \) is exact. This justifies the way we plug in \( s = 0 \) in the above formulas.

2) \( \overline{ch}(\text{IndD}) \) is represented by

\[
\int_Z a_0 - \frac{1}{2} \overline{ch}(\ker D_Y) - \hat{\eta}_1(0).
\]

Now by the remark following Theorem B.2, we have

\[
a_0|_{Z-Y \times [0,1]} = \hat{A}(iR^Z) \wedge tr(e^{-L^t}).
\]

Thus

\[
\int_Z a_0 = \int_{Z-Y \times [0,1]} \hat{A}(iR^Z) \wedge tr(e^{-L^t}) - \frac{1}{2} \overline{ch}(\ker D_Y) - \hat{\eta}_1(0) + \int_{Y \times [0,1]} a_0.
\]
It remains to identify \( \int_{Y \times [0,1]} a_0 \). We shall calculate it by the adiabatic limit method. The starting point is that instead of using metric \( du^2 + g \) on the cylinder, we use \( \delta^{-2} du^2 + g \). Then the whole discussion goes through and in fact we observe that

1) The index bundle \( \text{Ind}D \) remains unchanged. This follows from the characterization of \( \ker D \) and \( \ker D^* \) in [APS] as the \( L^2 \)-solutions respectively extended \( L^2 \)-solutions on the elongation manifold.

2) In (B.6), \( O(\cdot) \) is uniform in \( \delta \).

Therefore we can take the limit \( \delta \to 0 \), obtaining a representative for \( \overline{ch}(\text{Ind}D) \) where \( a_0 \) is replaced by \( \lim_{\delta \to 0} a_0 \). Recall that \( a_0 \) is the constant term in the asymptotic expansion of \( tr_* e^{-B^2} \) and we can assume that \( B \) lives in \( M \times \mathbb{R} \) since we are only interested in \( a_0 \) restricted to the cylinder. From Section A.1, we know that \( F(M \times \mathbb{R}) = F(M) \otimes C^2 \) with grading given by the involution \( \rho = \text{Id} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Moreover \( X \in TM \) acts like

\[
X \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

whereas \( \frac{\partial}{\partial u} \in T\mathbb{R} \) acts like

\[
\text{Id} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Put \( c_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). In what follows we will abuse notations and suppress the "\( \otimes \)" sign. One has

\[
B = \nabla + i \partial_u c_0 + i D y \sigma - \frac{\varphi(u)c(T)}{4\sqrt{i}} \sigma. \tag{B.9}
\]
At any fixed point \( u_0 \in \mathbb{R} \), consider the coordinate change

\[
v = \frac{u - u_0}{\delta}.
\]

Under this new coordinate, the rescaled metric becomes

\[
dv^2 + g.
\]

Therefore, if we let \( B_5 \) denote the superconnection in the rescaled metric corresponding to (B.9), then

\[
B_5 = \tilde{\nabla} + \sqrt{i} \partial_\sigma e_0 + \sqrt{i} D_\sigma \varphi(u_0 + \delta \nu) c(T) \sigma.
\]

Since this is a smooth family of superconnections, for the pointwise supertrace, one has

\[
\lim_{\delta \to 0} tr_s e^{-B_5^2} \big|_{M \times \{u_0\}} = tr_s e^{-\left(\tilde{\nabla} + \sqrt{i} \partial_\sigma e_0 + \sqrt{i} D_\sigma \varphi(u_0 + \delta \nu) c(T) \sigma \right)^2} = tr_s \left[ e^{-\left(\tilde{\nabla} + \sqrt{i} D_\sigma \varphi(u_0 + \delta \nu) c(T) \sigma \right)^2} e^{-\nu_0^2 \sigma^2} \right] \equiv 0.
\]

This is because the term of which we are taking supertrace does not involve \( e_0 \). By the property of the supertrace (Cf. Lemma A.1), its supertrace vanishes.

Since in the adiabatic limit, the volume grows like \( \delta^{-1} \), we need to compute \( \lim_{\delta \to 0} \frac{d}{d\delta} tr_s e^{-B_5^2} \). This can be done by Duhamel's principle:

\[
\frac{d}{d\delta} tr_s e^{-B_5^2} \big|_{\delta=0,\nu=0} = -tr_s (e^{-B_5^2} \frac{d}{d\delta} (B_5)^2) \big|_{\delta=0,\nu=0} e^{-B_5^2}.
\]

Computation shows

\[
\frac{d}{d\delta} (B_5)^2 \big|_{\delta=0,\nu=0} = \frac{\varphi'(u_0) c(T)}{4} e_0 \sigma.
\]
Thus
\[ e^{-B_0^2 / 4\delta} \frac{d}{ds} \langle B_0 \rangle |_{s=0} e^{-B_0^2} = -\frac{\phi'(u_0)c(T)}{4\delta} e^{-\frac{v(s_0)k(T)}{4\sqrt{t}}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2}. \]

To compute its supertrace, note that the Clifford element must be saturate, i.e., if we let \( e_1, e_2, \ldots, e_n \) be an orthonormal basis of \( TM \), then the supertrace of \( F(M \times \mathbb{R}) \) is nontrivial only on the element \( e_0(e_1 \sigma)(e_2 \sigma) \cdots (e_n \sigma) \),

which involves an odd number (\( n \)) of \( \sigma \)'s. From this fact, and the fact that the trace on \( F(M) \) is nontrivial only on \( \text{Id} \) and \( e_1 \cdots e_n \), we see that
\[ tr[e^{-\frac{\phi'(u_0)}{4\sqrt{4\pi t}} e^{-\frac{v(s_0)k(T)}{4\sqrt{t}}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2}}] = \frac{1}{2\sqrt{\pi}} tr_{\text{even}}[c(T)] e^{-\frac{v(s_0)k(T)}{4\sqrt{t}}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2}]. \]

Hence,
\[ \frac{d}{ds} tr e^{-B_0^2} |_{s=0, u=0} = -\frac{1}{2\sqrt{\pi}} tr_{\text{even}}[c(T)] e^{-\frac{v(s_0)k(T)}{4\sqrt{t}}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2}]. \]

Moreover, the argument in [C2] can easily be adapted in this simple situation to show that
\[ tr e^{-B_0^2} |_{M \times \{u_0\}} = -\frac{1}{2\sqrt{\pi}} tr_{\text{even}}[c(T)] e^{-\frac{v(s_0)k(T)}{4\sqrt{t}}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2} + o(\delta^{3/2}). \]

This implies that
\[ \lim_{\delta \to 0} \int_{Y \times [0,1]} tr e^{-B_0^2} = -\frac{1}{2\sqrt{\pi}} \int_0^1 \int_Y tr_{\text{even}}(\frac{\phi'(u)c(T)}{2\sqrt{t}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2} dz du \]
\[ = -\frac{1}{2\sqrt{\pi}} \int_0^1 \int_Y tr_{\text{even}}(\frac{c(T)}{2\sqrt{t}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2} dz ds. \]

Consequently, if we expand
\[ \frac{1}{2\sqrt{\pi}} \int_0^1 \int_Y tr_{\text{even}}(\frac{c(T)}{2\sqrt{t}} e^{-(\tilde{\gamma} + \sqrt{i}D_Y - \frac{v(s_0)k(T)}{4\sqrt{t}})^2} dz ds = \sum_{k=-n}^N C_k t^{k/2} + O(t(N+1)/2), \]
then
\[ \lim_{\delta \to 0} \int_{Y \times [0,1]} a_0 = -C_0. \]

This shows that \( \partial h(Ind D) \) is represented by
\[ \int_{Z - Y \times [0,1]} \hat{A}(iR^Z) \wedge tr(e^{-Lt}) - \frac{1}{2} \partial h(\ker D_Y) - \hat{\eta}_1(0) - C_0. \]

To finish the proof of Theorem B.1, we just have to show
\[ \hat{\eta} = \hat{\eta}_1(0) + C_0. \quad \text{(B.10)} \]

This can be seen as follows.

For \( t \) small and \( u \) large, consider
\[ I^u(s) = \frac{1}{2\sqrt{\pi}} \int_t^u t^{-1/2} tr^{\text{even}} [(D_Y + \frac{s c(T)}{4\sqrt{t}}) e^{-(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}})^2}] dt. \]

Differentiate with respect to \( s \), one finds
\[ \frac{dI^u}{ds} = \frac{1}{2\sqrt{\pi}} \int_t^u t^{-1/2} tr^{\text{even}} \left[ \frac{c(T)}{4t} e^{-\left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right)^2} \right] dt \]
\[ + \frac{1}{2\sqrt{\pi}} \int_t^u t^{-1/2} tr^{\text{even}} [(D_Y + \frac{s c(T)}{4\sqrt{t}}) 2(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}) c(T) e^{-\left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right)^2}] dt. \]

On the other hand
\[ \frac{d}{dt} e^{-\left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right)^2} = -\frac{1}{\sqrt{t}} \left(D_Y + \frac{s c(T)}{4\sqrt{t}}\right) \left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right) e^{-\left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right)^2}. \]

Therefore,
\[ tr^{\text{even}} [(D_Y + \frac{s c(T)}{4\sqrt{t}}) 2(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}) c(T) e^{-\left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right)^2}] \]
\[ = -tr^{\text{even}} \left[ \frac{c(T)}{2} \frac{d}{dt} e^{-\left(\hat{\nabla} + \sqrt{i} D_Y - \frac{s c(T)}{4\sqrt{t}}\right)^2} \right]. \]
Integration by part gives

\[
\frac{dI^u}{ds} = \frac{1}{2\sqrt{\pi}} \text{tr}_{\text{even}} \left[ \frac{c(T)}{4\sqrt{t}} e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{t}}\right)^2} \right] - \frac{1}{2\sqrt{\pi}} \text{tr}_{\text{even}} \left[ \frac{c(T)}{4\sqrt{u}} e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{u}}\right)^2} \right].
\]

Or

\[
I^u(1) - I^u(0) = \frac{1}{2\sqrt{\pi}} \int_0^1 \text{tr}_{\text{even}} \left[ \frac{c(T)}{4\sqrt{t}} e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{t}}\right)^2} \right] ds
- \frac{1}{2\sqrt{\pi u}} \int_0^1 \text{tr}_{\text{even}} \left[ \frac{c(T)}{4} e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{u}}\right)^2} \right] ds.
\] (B.11)

Note that when \( u \to +\infty \),

\[
I^u(1) \to I(1) = \frac{1}{2\sqrt{\pi}} \int_t^{+\infty} t^{-1/2} \text{tr}_{\text{even}} \left[ (D + \frac{c(T)}{4\sqrt{t}}) e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{t}}\right)^2} \right] dt,
\]

which is convergent by virtue of the cancellation result (A.21). Similarly

\[
I^u(0) \to I(0) = \frac{1}{2\sqrt{\pi}} \int_t^{+\infty} t^{-1/2} \text{tr} [D e^{-\left(\psi + i\sqrt{D}\right)^2}] dt,
\]

which is convergent because of (B.3). On the other hand, for \( u \) large, 

\[ \text{tr} \left[ \frac{c(T)}{4} e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{u}}\right)^2} \right] \]

is uniformly bounded (Cf. Remark after Theorem B.2). Therefore, taking \( u \to +\infty \) in the formula (B.11), one obtains

\[
I(1) - I(0) = \frac{1}{2\sqrt{\pi}} \int_0^1 \text{tr}_{\text{even}} \left[ \frac{c(T)}{2\sqrt{t}} e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{t}}\right)^2} \right] dt. \tag{B.12}
\]

Now when \( t \to 0 \),

\[
I(1) \to \hat{\eta} = \int_0^{+\infty} t^{-1/2} \text{tr}_{\text{even}} \left[ (D + \frac{c(T)}{4\sqrt{t}}) e^{-\left(\psi + i\sqrt{D} - \frac{c(T)}{4\sqrt{t}}\right)^2} \right] dt,
\]

which is convergent because of the cancellation result (A.19). Unfortunately, we do not have a corresponding statement for the unitary superconnection, therefore the same thing does not happen to \( I(0) \). However, we claim that

\[ \hat{\eta}(0) = \text{the finite part of } I(0) \text{ when } t \to 0. \]
To see this, recall that for $\text{Re} s \gg 0$, we have
\[ \hat{\eta}_1(s) = \int_0^{+\infty} t^{s-1/2} \text{tr}[D_\gamma e^{-(\gamma + \sqrt{i} D_\gamma)^2}] dt + \int_0^t t^{s-1/2} \text{tr}[D_\gamma e^{-(\gamma + \sqrt{i} D_\gamma)^2}] dt. \]

For $t$ small, one has the asymptotic expansion
\[ \text{tr}[D_\gamma e^{-(\gamma + \sqrt{i} D_\gamma)^2}] = \sum_{k=-n}^{N} a_k t^{k/2} + O(t^{(N+1)/2}). \]

Plug this in the formula for $\hat{\eta}_1(s)$, we obtain
\[ \hat{\eta}_1(s) = \int_0^{+\infty} t^{s-1/2} \text{tr}[D_\gamma e^{-(\gamma + \sqrt{i} D_\gamma)^2}] dt + \sum_{k=-n}^{N} \frac{a_k}{s+k/2+1/2} t^{s+k/2+1/2} + O(t^{s+1+N/2}). \]

This is how we see the existence of meromorphic extension of $\hat{\eta}_1(s)$. In fact, we can see more. It follows that
\[ \hat{\eta}_1(0) = I(0) + \sum_{k=-n,k\neq-1}^{N} \frac{2a_k}{k+1} t^{k+1} + O(t^{N/2+1}). \]

Our claim follows as well.

Armed with this and formula (B.12), we easily arrive at (B.10). QED.
Bibliography


[D] H. Donnelly, Local index theorem for families.


