Some Rigidity and Pointwise Pinching Theorems in Riemannian Geometry

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Abstract of the Dissertation

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We study manifolds with positive sectional curvature which are pointwise δ-pinched, or which have 2-nonnegative curvature operator, and obtain the following results: (1). The complex projective space $\mathbf{CP}^m$ $(m > 1)$ has only one metric(up to rescaling), such that it is pointwise $\frac{1}{4}$-pinched. (2). If M is a pointwise δ-pinched (2m+1)-dimensional manifold where $\delta \geq 2m(m-1)/[m(8m-5)+3]$, then the second Betti number $b_2 = 0$. (3). If there is $\phi(\neq 0) \in H^2(M;\mathbf{R})$ on a δ-pinched 2m-dimensional manifold M, such that, $\phi^{\wedge^m} = 0$ in $H^{2m}(M;\mathbf{R})$, then $\delta < 4m(m-1)^2/[16m(m-1)^2+3(2m-1)]$. (4). We classify all compact 4-manifolds with 2-nonnegative curvature operator.

To my father, mother, brother and sisters

in Beijing

To my lovely wife Yanning

and delightful daughter Nina

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§0. Introduction

This thesis is concerned with the geometry or topology of riemannian manifolds with positive sectional curvature or positive curvature operator. It is divided into two parts. In the first part, we study pointwise δ-pinched manifolds and derive a rigidity theorem for metrics on complex projective space. In the second part, we extend results of Hamilton to manifolds with 2-nonnegative curvature operator.

PART I. A compact (simply-connected) riemannian manifold M with positive sectional curvature is said to be pointwise δ-pinched ($\delta > 0$), if we can find a positive function k defined on M, such that, at each point $x \in M$, the following inequality

$$\delta k(x) \leq K_x \leq k(x)$$

is satisfied for all sectional curvatures K_x of M at x. If $k(x)$ happens to be constant, then we say M is globally δ-pinched or simply δ-pinched.

Much work has been done on the problem of determining whether or not for a given manifold M, one can find a metric on M such that M has positive sectional curvature and is (pointwise) δ-pinched for certain δ. We would like to list some results which are relevant to this thesis: There is a classifying theorem proved by Berger [$\mathbf{B}_1$] which states that any $\frac{1}{4}$-pinched manifold is either homeomorphic to a sphere or isometric to a symmetric space of rank 1. Following the classification of symmetric spaces of rank 1, we know that only $\mathbf{CP}^m$ has nonzero second Betti number; see Helgason [**H**]. There are examples of manifolds with positive sectional curvature and nonzero second Betti number in dimension 7 and 13 [$\mathbf{B_2}$][**W**]. There is also the work due to Berger [$\mathbf{B_3}$], improved by A. Tsagas [$\mathbf{T_1}$][$\mathbf{T_2}$], which shows that for some specific values of δ $(0 < \delta < 1)$, certain manifolds can not carry a δ-pinched metrics because of their second real cohomology group.

Recently, M. Micallef and J. Moore gave a beautiful result concerning manifolds which are pointwise δ-pinched [**M-M**]: If M is a pointwise δ-pinched manifold for $\delta > \frac{1}{4}$, then M is homeomorphic to a sphere.

Here we have proved the following result which is related to their work:

Theorem A. *If M is pointwise $\frac{1}{4}$-pinched, and the dimension of M is bigger then 2, then either $H^2(M;\mathbf{R}) = 0$, or M is isometric to $\mathbf{CP}^m$ with the standard metric (up to rescaling).*

Corollary. $\mathbf{CP}^m$ $(m > 1)$ *has only one metric(up to rescaling), such that, it is pointwise $\frac{1}{4}$-pinched.*

Remark. Theorem A is also obtained independently by M. Micallef [**M**] and W. Seaman [**S**].

The following theorems will improve the work of Berger and Tsagas:

Theorem B. *If M is a pointwise δ-pinched (2m+1)-dimensional manifold* *where*

$$\delta \geq \frac{2m(m-1)}{m(8m-5)+3},$$

then $b_2 = 0$.

Theorem C. *If there is* $\phi(\neq 0) \in H^2(M;\mathbf{R})$ *on a* δ*-pinched 2m-dimensional manifold* M*, such that,*

$$\underbrace{\phi \wedge \phi \wedge \cdots \wedge \phi}_{m-times} = 0$$

in $H^{2m}(M;\mathbf{R})$*, then*

$$\delta < \frac{4m(m-1)^2}{16m(m-1)^2 + 3(2m-1)}.$$

Note that Berger's result is $\delta > \frac{2(m-1)}{8m-5}$ when the dimension is 2m+1 [$\mathbf{B_3}$]. Tsagas improved this result to $\delta > \frac{2(2m-1)(m-1)^2 m}{(8m-5)(2m-1)(m-1)m+3}$ [$\mathbf{T_1}$]. Tsagas' result for the 2m-dimensional case is that $\delta \leq \frac{2(2m-1)(m-1)^2 m}{8(2m-1)(m-1)^2 m+3}$ [$\mathbf{T_2}$]. However, there appears to be a mistake in Tsagas' proofs for both cases. It occurs when estimating a real function of the form $F = a + b - c$, where a, b, and c are nonnegative, he enlarged b while enlarging c. The author has fixed this error in his proof.

PART II. A curvature operator R_M can be defined via the curvature tensor R as follows:

Fix $x \in M$ and choose any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_xM. Then we define

$$R_M(e_i \cdot e_j) = \sum_{1 \le k < l \le n} R_{ijkl} e_k \cdot e_l \ ,$$

where $e_i \cdot e_j$ is denoted as the Clifford multiplication of e_i and e_j $(i \neq j)$.

It is clear that R_M is a symmetric bilinear operator on $\wedge^2 TM$ (under the natural identification of the Clifford algebra $Cl(M)$ and $\wedge^* TM$).

R_M is said to be **k-nonnegative** (resp. **k-positive**) if the sum of the first k eigenvalues of R_M is ≥ 0 (resp. > 0). It is easy to see that when the dimension of M is 3, R_M is 2-nonnegative $\Longleftrightarrow$ the Ricci curvature is ≥ 0.

There is quite a long history to the study of metrics on homotopy spheres under the conditions of either δ-pinching or nonnegative (positive) curvature operator. The main results are the following: For δ-pinched manifolds, it was first proved by Gromoll that M^n is diffeomorphic to S^n for some $\delta(n)$. Later, it was shown that $\delta(n)$ does not depend on the dimension and that $\delta > 0.8$ is sufficient, cf [C-E]. However, if the given manifold is pointwise δ-pinched, then such δ depends on n, and $\delta \to 1$ as $n \to \infty$ [**R**][**Hu**][**Ma**]. On the other hand, R. Hamilton studied manifolds with nonnegative curvature operator and classified all manifolds (up to diffeomorphism) with nonnegative Ricci curvature when $n = 3$ [$\mathbf{H_1}$] or nonnegative curvature operator when $n = 4$ [$\mathbf{H_2}$].

Our results are:

Theorem D. *If M^4 has a metric with 2-nonnegative curvature operator, then a finite covering space of M^4 is diffeomorphic to one of the spaces* $\mathbf{S^4}, \mathbf{CP^2}$, $\mathbf{S^3} \times \mathbf{S^1}, \mathbf{S^2} \times \mathbf{S^2}, \mathbf{S^2} \times \mathbf{S^1} \times \mathbf{S^1}$, *or* $\mathbf{T^4}$.

Theorem E. *If M^4 is a pointwise δ-pinched manifold with $\delta \geq \frac{\sqrt{13}}{6+\sqrt{13}} \approx 0.3754$, then M^4 is diffeomorphic to $\mathbf{S}^4$ or $\mathbf{RP}^4$.*

REMARK 1. Hamilton's result for 3-manifolds together with Theorem D give a classification of (compact) manifolds M^n with 2-nonnegative curvature operator, when $n = 3, 4$.

REMARK 2. Theorem E can not be directly derived from Theorem D, because we still do not know what $\delta(< 0.4)$ implies the 2-nonnegativity of the curvature operator.

§1. Preliminaries

Suppose M^n is an n-dimensional manifold with a riemannian metric $\langle\ ,\ \rangle$ and Levi-Civita connection ∇. Then the curvature tensor R is defined by

$$R_{v,w} = -\nabla_v \nabla_w + \nabla_w \nabla_v + \nabla_{[v,w]}\,. \tag{1}$$

We can define the Clifford bundle $Cl(M)$ as the tensor algebra $\sum_{i=0}^{\infty} \otimes^i(TM)$ modulo the relation $v \otimes v = -\langle v, v\rangle$ for any $v \in TM$. ∇ and R act on $Cl(M)$ like derivatives, i.e., for any $\phi, \psi \in Cl(M)$, we have

$$\nabla(\phi \cdot \psi) = (\nabla\phi) \cdot \psi + \phi \cdot (\nabla\psi), \tag{2}$$

and

$$R(\phi \cdot \psi) = (R\phi) \cdot \psi + \phi \cdot (R\psi). \tag{3}$$

The main tool we will use in Part 1 is The General Bochner Identity

$$D^2 = \nabla^* \nabla + \mathcal{R}. \tag{4}$$

Here D is the Dirac operator, $\nabla^*\nabla$ is the connection Laplacian for the real Clifford bundle $Cl(M)$, and $\mathcal{R}$ is a certain curvature operator. Using a local orthonormal tangent frame field $\{e_1, e_2, \ldots, e_n\}$ with $(\cdot)$ the Clifford multiplication, for any $\phi \in Cl(M)$, we define

$$\mathcal{R}(\phi) = -\sum_{1 \le i < j \le n} e_i \cdot e_j \cdot R_{e_i, e_j}(\phi). \tag{5}$$

(Usually we omit $(\cdot)$ if it causes no confusion). For more details, refer to *Spin Geometry* written by Professors B. Lawson and M-L. Michelsohn [**L-M**].

§2. Some notation and lemmas for Part I

To simplify the notation in the following computations, we denote $\langle R_{e_i,e_j}e_k, e_l\rangle$ by $(ijkl)$, and in the 4-tuple $(ijkl)$, when i is replaced by $\alpha+\beta$, this means that e_i is replaced by $e_\alpha + e_\beta$. The sectional curvature of a tangent 2-plane spanned by e_i and e_j is $(ijij)$ or denoted as $K_{i,j}$.

Lemma 2.1 *For any 2-form ϕ which can be written as*

$$\phi = \sum_{i=1}^{m} a_i e_{2i-1}e_{2i} \tag{6}$$

in a local orthonormal tangent frame field $\{e_1, \ldots, e_n\}$, where $m = [\frac{n}{2}]$, then

$$\begin{aligned} \langle \mathcal{R}(\phi), \phi\rangle &= \sum_{i=1}^{m} a_i^2 \sum_{l\neq 2i-1,2i} (K_{2i-1,l} + K_{2i,l}) \\ &- \sum_{1\leq i<j\leq m} 4a_ia_j((2i-1), 2i, (2j-1), 2j). \end{aligned} \tag{7}$$

($\mathcal{R}$ is invariant under the choice of any local frame field.)

This lemma has different versions $[\mathbf{B_3}], [\mathbf{T_1}]$. The advantage of this proof is that the terms in the calculation can be more easily handled and better estimates can be obtained.

Proof. Using the property that $\mathcal{R}$ is $C^\infty(M)$-linear and symmetric on $Cl(M)$, we have

$$\begin{aligned} \langle \mathcal{R}(\phi), \phi\rangle &= \left\langle \mathcal{R}(\sum_{i=1}^{m} a_i e_{2i-1}e_{2i}), \sum_{i=1}^{m} a_i e_{2i-1}e_{2i} \right\rangle \\ &= \sum_{i=1}^{m} a_i^2 \langle \mathcal{R}(e_{2i-1}e_{2i}), e_{2i-1}e_{2i}\rangle + \sum_{1\leq i<j\leq m} 2a_ia_j\langle \mathcal{R}(e_{2i-1}e_{2i}), e_{2j-1}e_{2j}\rangle. \end{aligned} \tag{8}$$

By definition,

$$\langle \mathcal{R}(e_{2i-1}e_{2i}), e_{2i-1}e_{2i}\rangle = -\sum_{1\leq k<l\leq m} \langle e_k e_l R_{e_k,e_l}(e_{2i-1}e_{2i}), e_{2i-1}e_{2i}\rangle, \tag{9}$$

and

$$-\langle e_k e_l R_{e_k,e_l}(e_{2i-1}e_{2i}), e_{2i-1}e_{2i}\rangle = \langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_k e_l e_{2i-1}e_{2i}\rangle. \tag{10}$$

(Here we used the properties that $\langle \phi, \psi\rangle = \langle e_i\phi, e_i\psi\rangle$ and $e_ie_i = -1$.)

If $k = 2i-1$ and $l = 2i$ or $\{e_k, e_l\}$ has no common element with $\{e_{2i-1}, e_{2i}\}$, then $e_ke_le_{2i-1}e_{2i}$ belongs to either $\wedge^0TM$ or $\wedge^4TM$ by the identification of $Cl(M)$ with $\wedge^*TM$. Using the property that

$$R_{e_i,e_j}: \qquad \wedge^pTM \longmapsto \wedge^pTM \qquad 0\leq p\leq n, \tag{11}$$

we know that in this case

$$\langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_ke_le_{2i-1}e_{2i}\rangle = 0. \tag{12}$$

If $\{e_k, e_l\}$ has only one common element with $\{e_{2i-1}, e_{2i}\}$, say $k = 2i-1$, then

$$\begin{aligned}
&\langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_ke_le_{2i-1}e_{2i}\rangle \\
=\ &\langle R_{e_{2i-1},e_l}(e_{2i-1}e_{2i}), e_le_{2i}\rangle \\
=\ &\langle (R_{e_{2i-1},e_l}e_{2i-1})e_{2i} + e_{2i-1}R_{e_{2i-1},e_l}(e_{2i}), e_le_{2i}\rangle \\
=\ &\langle ((2i-1), l, (2i-1), s)e_se_{2i}, e_le_{2i}\rangle - \langle R_{e_{2i-1},e_l}e_{2i}, e_{2i-1}e_le_{2i}\rangle \\
=\ &((2i-1), l, (2i-1), l) \\
=\ &K_{2i-1,l}\,.
\end{aligned} \tag{13}$$

Continuing in this way, we finally get

$$\langle \mathcal{R}(e_{2i-1}e_{2i}), e_{2i-1}e_{2i}\rangle = \sum_{l\neq 2i-1,2i} (K_{2i-1,l} + K_{2i,l})\,. \tag{14}$$

For $i \neq j$, by definition, we have

$$\begin{aligned}\langle \mathcal{R}(e_{2i-1}e_{2i}), e_{2j-1}e_{2j}\rangle &= -\sum_{l\leq k<l\leq m} \langle e_k e_l R_{e_k,e_l}(e_{2i-1}e_{2i}), e_{2j-1}e_{2j}\rangle \\ &= \sum_{l\leq k<l\leq m} \langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_k e_l e_{2j-1}e_{2j}\rangle. \end{aligned} \tag{15}$$

For the same reason as before, if $k = 2j-1$ and $l = 2j$ or if $\{e_k, e_l\}$ has no common element with $\{e_{2j-1}, e_{2j}\}$, then

$$\langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_k e_l e_{2j-1}e_{2j}\rangle = 0. \tag{16}$$

If $\{e_k, e_l\}\cap\{e_{2j-1}, e_{2j}\} \neq \emptyset$, say $l = 2j$, but $k \neq 2j-1$, and if $k \notin \{2i-1, 2i\}$, then it is easy to see that (16) is still true.

If $k = 2i-1$, then

$$\begin{aligned} &\langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_k e_l e_{2j-1}e_{2j}\rangle \\ =\ &\langle R_{e_{2i-1},e_{2j}}(e_{2i-1}e_{2i}), e_{2i-1}e_{2j}e_{2j-1}e_{2j}\rangle \\ =\ &\langle (R_{e_{2i-1},e_{2j}}e_{2i-1})e_{2i} + e_{2i-1}R_{e_{2i-1},e_{2j}}e_{2i}, e_{2i-1}e_{2j-1}\rangle \\ =\ &((2i-1), 2j, 2i, (2j-1)) \\ =\ &-(2j, (2i-1), 2i, (2j-1)). \end{aligned} \tag{17}$$

If $k = 2i$, similarly we have

$$\langle R_{e_k,e_l}(e_{2i-1}e_{2i}), e_k e_l e_{2j-1}e_{2j}\rangle = -(2i, 2j, (2i-1), (2j-1)). \tag{18}$$

By Bianchi's Identity, we see that

$$\begin{aligned} & -(2j, (2i-1), 2i, (2j-1)) - (2i, 2j, (2i-1), (2j-1)) \\ = \ & ((2i-1), 2i, 2j, (2j-1)) \\ = \ & -((2i-1), 2i, (2j-1), 2j). \end{aligned} \tag{19}$$

We get $-((2i-1), 2i, (2j-1), 2j)$ again when $l = 2j-1$ and $k = 2i-1, 2i$.

This shows that

$$\langle \mathcal{R}(e_{2i-1}e_{2i}), e_{2j-1}e_{2j}\rangle = -2((2i-1), 2i, (2j-1), 2j). \tag{20}$$

Combining $(8), (14)$ and (20) gives (7). qed.

Let K_i $\quad i = 1, 2, 3, \ldots 8.$ denote the sectional curvature of planes in

T_xM defined as follows:

$$K_1 = \tfrac{1}{4}((2i-1)+2i,(2j-1)+2j,(2i-1)+2i,(2j-1)+2j),$$
$$K_2 = \tfrac{1}{4}(-(2i-1)+2i,-(2j-1)+2j,-(2i-1)+2i,-(2j-1)+2j),$$
$$K_3 = \tfrac{1}{4}((2j-1)+2i,(2i-1)+2j,(2j-1)+2i,(2i-1)+2j),$$
$$K_4 = \tfrac{1}{4}-((2j-1)+2i,-(2i-1)+2j,-(2j-1)+2i,-(2i-1)+2j),$$
$$K_5 = ((2i-1),2j,(2i-1),2j),$$
$$K_6 = ((2j-1),2i,(2j-1),2i),$$
$$K_7 = ((2i-1),2i,(2i-1),2i),$$
$$K_8 = ((2j-1),2j,(2j-1),2j),$$

and for $i \neq j$, set

$$K^{i,j} = \sum_{\substack{k=2i-1,2i \\ l=2j-1,2j}} K_{k,l}.$$

Then we have the following.

Lemma 2.2. *When $i \neq j$, the following equality holds:*

$$\begin{aligned} & K^{i,j} - 2((2i-1),2i,(2j-1),2j) \\ = \ & \tfrac{1}{3}[2(K_1+K_2)+4(K_3+K_4)+2(K_5+K_6)-2(K_7+K_8)]. \end{aligned} \tag{21}$$

<u>Proof</u>. Remembering that, for example,

$$\begin{aligned} & (-(2i-1)+2i,-(2j-1)+2j,-(2i-1)+2i,-(2j-1)+2j) \\ & \quad = \langle R_{-e_{2i-1}+e_{2i},-e_{2j-1}+e_{2j}}(-e_{2i-1}+e_{2i}),-e_{2j-1}+e_{2j}\rangle, \end{aligned}$$

and that the curvature tensor R is a multi-linear 4-tuple, we have

$$4K_1 \ = \ ((2i-1)+2i,(2j-1)+2j,(2i-1)+2i,(2j-1)+2j)$$

$$
\begin{aligned}
= \;& ((2i-1),(2j-1),(2i-1),(2j-1)) + (2i,(2j-1),2i,(2j-1)) \\
& +((2i-1),2j,(2i-1),2j) + (2i,2j,2i,2j) \\
& +((2i-1),(2j-1),2i,2j) + ((2i-1),2j,2i,(2j-1)) \\
& +(2i,(2j-1),(2i-1),2j) + (2i,2j,(2i-1),(2j-1)) \\
& +((2i-1),(2j-1),2i,(2j-1)) + ((2i-1),2j,2i,2j) \\
& +(2i,(2j-1),(2i-1),(2j-1)) + (2i,2j,(2i-1),2j) \\
& +((2i-1),(2j-1),(2i-1),2j) + (2i,(2j-1),2i,2j) \\
& +((2i-1),2j,(2i-1),(2j-1)) + (2i,2j,2i,(2j-1)). \qquad (22)
\end{aligned}
$$

Replacing $(2i-1)$ $(i.e., e_{2i-1})$ by $-(2i-1)$ $(i.e., -e_{2i-1})$ and $(2j-1)$ $(i.e., e_{2j-1})$ by $-(2j-1)$ $(i.e., -e_{2j-1})$ in (22) shows

$$
\begin{aligned}
4K_2 &= (-(2i-1)+2i, -(2j-1)+2j, -(2i-1)+2i, -(2j-1)+2j) \\
&= \text{the first four rows minus the last four rows} \\
&\quad\; \text{in the right hand side of (22).} \qquad (23)
\end{aligned}
$$

Consequently,

$$
2K_1+2K_2 = K^{i,j}+2[((2i-1),(2j-1),2i,2j)+((2i-1),2j,2i,(2j-1))]. \quad (24)
$$

Exchanging $(2i-1)$ with $(2j-1)$ in (24), we get

$$
\begin{aligned}
2K_3+2K_4 &= ((2i-1),(2j-1),(2i-1),(2j-1)) + (2i,2j,2i,2j) \qquad (25) \\
&\quad +((2i-1),2i,(2i-1),2i) + ((2j-1),2j,(2j-1),2j) \\
&\quad +2[((2j-1),(2i-1),2i,2j) + ((2j-1),2j,2i,(2i-1))].
\end{aligned}
$$

Taking $(24) + 2(25)$ yields

$$
\begin{aligned}
& 2(K_1+K_2)+4(K_3+K_4) \\
= \; & 3[((2i-1),(2j-1),(2i-1),(2j-1))+(2i,2j,2i,2j)] \\
& +((2i-1),2j,(2i-1),2j)+(2i,(2j-1),2i,(2j-1)) \\
& +2[((2i-1),2i,(2i-1),2i)+((2j-1),2j,(2j-1),2j)] \\
& +2[-((2i-1),(2j-1),2i,2j)+((2i-1),2j,2i,(2j-1))] \\
& -4((2i-1),2i,(2j-1),2j) \\
= \; & 3K^{i,j}-2(K_5+K_6)+2(K_7+K_8)-6((2i-1),2i,(2j-1),2j),
\end{aligned}
$$

here we have used the basic properties of R that the first two entries or the last two entries are anti-commutative with each other and that the first two entries are commutative with the last two entries. We also have used the first Bianchi's identity to simplify the result.

Lemma 2.3. (Berger [$\mathbf{B_3}$]) *If M is pointwise δ-pinched, then*

$$
|\,(ijkl)\,| \leq \frac{2}{3}(1-\delta)k(x), \tag{26}
$$

for all i, j, k, and l mutually distinct.

Lemma 2.4. *If A is a $2n \times 2n$ real symmetric matrix whose diagonal entries are all equal and invariant under Ad_g ($g \in U(n) \subset SO(2n)$ under the natural imbedding), then $A = aI_{2n}$ for some constant a.*

<u>Proof</u>. Let

$$A = \begin{pmatrix} a & & * \\ & \ddots & \\ * & & a \end{pmatrix},$$

we will show that $a_{1j} = 0$ for $j \neq 1$.

If $\qquad j = 2i-1 \qquad i \neq 1$,

Let

$$g = \begin{pmatrix}
\cos\theta & & & & & \overbrace{\sin\theta}^{2i-1} & & & & \\
& \cos\theta & & & & & \sin\theta & & & \\
& & 1 & & & & & & & \\
& & & \ddots & & & & & & \\
& & & & 1 & & & & & \\
-\sin\theta & & & & & \cos\theta & & & & \\
& -\sin\theta & & & & & \cos\theta & & & \\
& & & & & & & 1 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & 1
\end{pmatrix} \in U(n),$$

then

$$g^{-1}Ag = \begin{pmatrix} \cos\theta & 0\cdots0 & -\sin\theta & 0\cdots0 \\ & & & \\ & * & & \\ & & & \end{pmatrix} \begin{pmatrix} a & \cdots & a_{1,2i-1} & \cdots \\ \vdots & \ddots & & \\ a_{1,2i-1} & & a & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} \cos\theta & \\ 0 & \\ \vdots & * \\ -\sin\theta & \\ 0 & \\ \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} a\cos\theta - a_{1,2i-1}\sin\theta & \cdots & a_{1,2i-1}\cos\theta - a\sin\theta & \cdots \\ & & & \\ & * & & \\ & & & \end{pmatrix} \begin{pmatrix} \cos\theta & \\ 0 & \\ \vdots & * \\ -\sin\theta & \\ 0 & \\ \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} a\cos^2\theta - 2a_{1,2i-1}\sin\theta\cos\theta + a\sin^2\theta & \cdots \\ & \\ * & \\ & \end{pmatrix}$$

$$= \begin{pmatrix} a - a_{1,2i-1}\sin 2\theta & \cdots \\ & \\ * & \\ & \end{pmatrix}.$$

Since θ is arbitrary and $(g^{-1}Ag)_{11} = (A)_{11}$, we should have $a_{1,2i-1} = 0$.

Choosing

$$g = \begin{pmatrix} \cos\theta & & & & & & \overbrace{\sin\theta}^{2i} & & & \\ & \cos\theta & & & & -\sin\theta & & & & \\ & & 1 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & 1 & & & & & \\ & \sin\theta & & & & \cos\theta & & & & \\ -\sin\theta & & & & & & \cos\theta & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{pmatrix},$$

we can show that $a_{1,2i} = 0$.

The proof for the rest cases follows in this way. qed.

§3. Rigidity Theorem for CP^m

Proof of Theorem A. We first show that if $\delta = \frac{1}{4}$, then $\mathcal{R}$ is nonnegative, i.e.,

$$\langle \mathcal{R}(\phi), \phi \rangle \geq 0 \tag{27}$$

for any 2-form ϕ. Because $D = d + d^*$, where d is the exterior derivative and d^* is the formal adjoint of d, the kernel of D consists of all harmonic forms. If ϕ is harmonic, by (4), we have

$$\begin{aligned} 0 &= \int_M \{ \langle \nabla^* \nabla \phi, \phi \rangle + \langle \mathcal{R}\phi, \phi \rangle \} \\ &= \int_M \{ \langle \nabla\phi, \nabla\phi \rangle + \langle \mathcal{R}\phi, \phi \rangle \}. \end{aligned} \tag{28}$$

Then either $\phi = 0$ or ϕ is parallel, because $\langle \nabla\phi, \nabla\phi \rangle$ is nonnegative.

The inequality (27) will be proved in two cases:

Case 1. $\qquad n = 2m + 1$.

In this case, the proof will be deferred to §4. Actually, we will prove that for even smaller δ, (27) is still true.

Case 2. $\qquad n = 2m$.

From Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned} \langle \mathcal{R}\phi, \phi \rangle &= \sum_{1 \leq i < j \leq m} [(a_i^2 + a_j^2) K^{i,j} - 4 a_i a_j ((2i-1), 2i, (2j-1), 2j)] \\ &\geq \sum_{1 \leq i < j \leq m} \left[(a_i^2 + a_j^2) 4 \times \frac{1}{4} - \frac{8}{3} \mid a_i a_j \mid (1 - \frac{1}{4}) \right] k(x) \\ &= \sum_{1 \leq i < j \leq m} (\mid a_i \mid - \mid a_j \mid)^2 k(x) \geq 0. \end{aligned} \tag{29}$$

If ϕ is harmonic, then $\langle \mathcal{R}\phi, \phi \rangle$ and $\langle \nabla\phi, \nabla\phi \rangle$ have to be zero for all $x \in M$. This means $\mid a_i \mid = \mid a_j \mid =$a, for some constant a. If we assume ϕ is non-zero, then by choosing an orientation of TM and rescaling ϕ, we can assume that

$$\phi = e_1 e_2 + \cdots + e_{2m-1} e_{2m}.$$

Therefore, from (29), we can see that

$$K^{i,j} - 2((2i-1), 2i, (2j-1), 2j) = 0 \qquad\qquad i \neq j.$$

Hence by Lemma 2.3 and the pinching assumption,

$$K_{k,l} = \frac{1}{4} k(x)$$

for any $k \in \{2i-1, 2i\}$, $l \in \{2j-1, 2j\}$ and $i \neq j$.

On another hand, by Lemma 2.2 and lemma 2.3,

$$0 = 2(K_1+K_2)+4(K_3+K_4)+2(K_5+K_6)-2(K_7+K_8) \geq 16\times\frac{1}{4}k(x)-2(K_7+K_8) \geq 0,$$

so we must have

$$K_{2i-1,2i} = k(x) \qquad\qquad \text{for all } i.$$

Therefore, with respect to any orthonormal basis $\{e_1, \ldots e_{2m}\}$ which "diagonalizes" ϕ, we have

$$Ric_x = \begin{pmatrix} \frac{m+1}{2}k(x) & & * \\ & \ddots & \\ * & & \frac{m+1}{2}k(x) \end{pmatrix}. \tag{30}$$

We know that $e_1e_2 + \cdots + e_{2m-1}e_{2m}$ is invariant under the action of $U(m)$, i.e., for any $g \in U(m)$, if

$$(e_1^{'}, \dots e_{2m}^{'}) = g(e_1, \dots e_{2m}),$$

then

$$e_1^{'}e_2^{'} + \cdots + e_{2m-1}^{'}e_{2m}^{'} = e_1e_2 + \cdots + e_{2m-1}e_{2m}.$$

This means that under another basis $\{e_1^{'}, \dots, e_{2m}^{'}\}$ which is conjugate to $\{e_1, \dots, e_{2m}\}$ by $U(m)$, the Ricci tensor still has the same form as in (30). In another words, Ric_x satisfies the conditions of Lemma 2.4. Hence,

$$Ric_x = \frac{m+1}{2}k(x)I_{2m} ,$$

which shows that M is Einstein, and its scalar curvature $\frac{m(m+1)}{2}k(x)$ is constant. By the classification theorem for (globally) $\frac{1}{4}$-pinched manifolds, M is isometric to CP^m (or $\overline{CP^m}$).

qed.

§4. Smaller δ for the second Betti number b_2

Proof of Theorem B. Let ϕ be a harmonic 2-form on M^{2m+1}, and suppose that locally we can write ϕ as

$$\phi = \sum_{i=1}^{m} a_i e_{2i-1} e_{2i}.$$

where $\{e_1, \ldots e_{2m+1}\}$ is an orthonormal tangent frame field. (This can be done on an open-dense subset of M).

Using Bochner formula (4) on ϕ, we have

$$0 = \langle D^2(\phi), \phi \rangle = -\frac{1}{2} \triangle |\phi|^2 + \langle \nabla \phi, \nabla \phi \rangle + \langle \mathcal{R}\phi, \phi \rangle, \tag{31}$$

where $\triangle$ is the usual Laplacian operator, and

$$\nabla \phi = \sum_{i=1}^{2m+1} (\nabla_{e_i} \phi) \otimes e_i.$$

Multiplying by $|\phi|^{2m-2} = (\sum_{i=1}^{m} a_i^2)^{m-1}$ on the both sides of formula (31), and integrating them give

$$0 \geq \int_M |\phi|^{2m-2} \{ | \nabla \phi|^2 + \langle \mathcal{R}\phi, \phi \rangle \}, \tag{32}$$

since

$$\int_M -\frac{1}{2} |\phi|^{2m-2} \triangle (|\phi|^2) = \frac{m-1}{2} \int_M |\phi|^{2m-4} \left| \nabla (|\phi|^2) \right|^2 \geq 0,$$

and the equality holds iff $\sum_{i=1}^{m} a_i^2$ is a constant.

Locally,

$$\begin{aligned} \nabla \phi &= \sum_{i=1}^{m} \nabla (a_i e_{2i-1} e_{2i}) \\ &= \sum_{i=1}^{m} [(\nabla a_i) e_{2i-1} e_{2i} + a_i \nabla (e_{2i-1} e_{2i})]. \end{aligned}$$

Since

$$\langle(\bigtriangledown a_i)e_{2i-1}e_{2i}, (\bigtriangledown a_j)e_{2j-1}e_{2j}\rangle = 0 \qquad \text{if } i \neq j \ ,$$

and

$$\langle(\bigtriangledown a_i)e_{2i-1}e_{2i}, a_j \bigtriangledown (e_{2j-1}e_{2j})\rangle = 0 \qquad \text{for any } i, j \ ,$$

$$\begin{aligned}\langle\bigtriangledown\phi, \bigtriangledown\phi\rangle &= \sum_{i=1}^{m}\langle\bigtriangledown a_i, \bigtriangledown a_i\rangle \\ &+ \left\langle\sum_{i=1}^{m} a_i \bigtriangledown (e_{2i-1}e_{2i}), \sum_{i=1}^{m} a_i \bigtriangledown (e_{2i-1}e_{2i})\right\rangle.\end{aligned}$$

Then we have the following.

Lemma 4.1

$$(\sum_{i=1}^{m} a_i^2)^{m-1}(\sum_{i=1}^{m}\langle\bigtriangledown a_i, \bigtriangledown a_i\rangle) \geq m^{m-2}\langle\bigtriangledown(\prod_{i=1}^{m} a_i), \bigtriangledown(\prod_{i=1}^{m} a_i)\rangle. \tag{33}$$

The equality holds iff $a_1^2 = a_2^2 = \cdots = a_m^2$.

<u>Proof</u>.

$$\begin{aligned}&\langle\bigtriangledown(\prod_{i=1}^{m} a_i), \bigtriangledown(\prod_{i=1}^{m} a_i)\rangle \\ =\ & \langle\sum_{i=1}^{m}\prod_{j\neq i} a_j \bigtriangledown a_i, \sum_{i=1}^{m}\prod_{j\neq i} a_j \bigtriangledown a_i\rangle \\ =\ & \frac{1}{2}\sum_{1\leq i<j\leq m} a_1^2\cdots\hat{a_i^2}\cdots\hat{a_j^2}\cdots a_m^2\langle\bigtriangledown a_i^2, \bigtriangledown a_j^2\rangle + \sum_{i=1}^{m}\prod_{j\neq i} a_j^2\langle\bigtriangledown a_i, \bigtriangledown a_i\rangle \\ \leq\ & \sum_{1\leq i<j\leq m} a_1^2\cdots\hat{a_i^2}\cdots\hat{a_j^2}\cdots a_m^2(a_j^2\langle\bigtriangledown a_i, \bigtriangledown a_i\rangle + a_i^2\langle\bigtriangledown a_j, \bigtriangledown a_j\rangle) + \sum_{i=1}^{m}\prod_{j\neq i} a_j^2\langle\bigtriangledown a_i, \bigtriangledown a_i\rangle \\ =\ & \sum_{i=1}^{m}\sum_{j=1}^{m} a_1^2\cdots\hat{a_i^2}\cdots a_m^2\langle\bigtriangledown a_j, \bigtriangledown a_j\rangle \\ \leq\ & \frac{1}{m^{m-2}}(\sum_{i=1}^{m} a_i^2)^{m-1}\sum_{j=1}^{m}\langle\bigtriangledown a_j, \bigtriangledown a_j\rangle.\end{aligned}$$

It needs an explanation for one line in the above estimate, i.e.,

$$\langle \nabla a_i^2, \nabla a_j^2 \rangle \leq 2(a_j^2 \langle \nabla a_i, \nabla a_i \rangle + a_i^2 \langle \nabla a_j, \nabla a_j \rangle).$$

But this is just an application of Hölder's inequality, i.e.,

$$\langle \nabla a_i^2, \nabla a_j^2 \rangle = 4 \langle a_j \nabla a_i, a_i \nabla a_j \rangle \leq 2(a_j^2 | \nabla a_i|^2 + a_i^2 | \nabla a_j|^2).$$

It is easy to see that if $a_i^2 = a_j^2$ for all i, j, the equality holds. qed.

Lemma 4.2

$$(\sum_{i=1}^m a_i^2)^{m-1} \left\langle \sum_{i=1}^m a_i \nabla (e_{2i-1} e_{2i}), \sum_{i=1}^m a_i \nabla (e_{2i-1} e_{2i}) \right\rangle \geq m^{m-2} \prod_{i=1}^m a_i^2 \langle \nabla e_{2m+1}, \nabla e_{2m+1} \rangle, \tag{34}$$

and the equality can not hold at all $x \in M$.

Proof.

$$\begin{aligned}
& (\sum_{i=1}^m a_i^2)^{m-1} \left\langle \sum_{i=1}^m a_i \nabla (e_{2i-1} e_{2i}), \sum_{i=1}^m a_i \nabla (e_{2i-1} e_{2i}) \right\rangle \\
\geq \;& (\sum_{i=1}^m a_i^2)^{m-1} \sum_{j=1}^{2m+1} \langle \sum_{i=1}^m a_i \nabla (e_{2i-1} e_{2i}), e_j e_{2m+1} \rangle^2 \\
= \;& (\sum_{i=1}^m a_i^2)^{m-1} \sum_{i=1}^m a_i^2 [\langle \nabla e_{2i-1}, e_{2m+1} \rangle^2 + \langle \nabla e_{2i}, e_{2m+1} \rangle^2] \\
= \;& (\sum_{i=1}^m a_i^2)^{m-1} \sum_{i=1}^m a_i^2 [\langle e_{2i-1}, \nabla e_{2m+1} \rangle^2 + \langle e_{2i}, \nabla e_{2m+1} \rangle^2] \\
\geq \;& (m-1)^{m-1} \sum_{i=1}^m \prod_{j=1}^m a_j^2 [\langle e_{2i-1}, \nabla e_{2m+1} \rangle^2 + \langle e_{2i}, \nabla e_{2m+1} \rangle^2] \\
\geq \;& m^{m-2} \sum_{i=1}^m \prod_{j=1}^m a_j^2 [\langle e_{2i-1}, \nabla e_{2m+1} \rangle^2 + \langle e_{2i}, \nabla e_{2m+1} \rangle^2] \\
= \;& m^{m-2} \prod_{j=1}^m a_j^2 \langle \nabla e_{2m+1}, \nabla e_{2m+1} \rangle \, ,
\end{aligned}$$

here we have used the inequality $(\sum_{i=1}^m a_i^2)^{m-1} \geq (m-1)^{m-1}\prod_{j\neq i} a_j^2$ for the last second inequality and $(m-1)^{m-1} \geq m^{m-2}$ for the last one. The equality holds iff either $a_i = 0$ or $\bigtriangledown(e_{2i-1}e_{2i}) = 0$ for all i . But neither can happen at all $x \in M$.

qed.

Let $\underbrace{\phi\wedge\phi\wedge\cdots\wedge\phi}_{m-times}$ denote the 2m-form obtained by taking the m-fold wedge product. Consider this in the Clifford bundle. Then we define

$$\psi = \frac{\omega}{m!}(\underbrace{\phi\wedge\phi\wedge\cdots\wedge\phi}_{m-times}),$$

where ω is the volume form. Locally, $\psi = \prod_{i=1}^m a_i e_{2m+1}$. Since $\underbrace{\phi\wedge\phi\wedge\cdots\wedge\phi}_{m-times}$ is d-closed, $D(\underbrace{\phi\wedge\phi\wedge\cdots\wedge\phi}_{m-times}) \in \wedge^{2m-1}TM$, and we have in the Clifford bundle that

$$D\left[\frac{\omega}{m!}(\underbrace{\phi\wedge\phi\wedge\cdots\wedge\phi}_{m-times})\right] = \frac{\omega}{m!}D(\underbrace{\phi\wedge\phi\wedge\cdots\wedge\phi}_{m-times}) \quad \in \wedge^2 TM,$$

which implies that locally

$$\begin{aligned}
D\psi = D(a_1\cdots a_m e_{2m+1}) &= \sum_{i=1}^{2m+1} e_i \bigtriangledown_{e_i}(a_1\cdots a_m e_{2m+1}) \\
= \sum_{i=1}^{2m} \bigtriangledown_{e_i}(a_1\cdots a_m)e_i e_{2m+1} &+ a_1\cdots a_m \\
\cdot\left[\sum_{1\leq i<j\leq 2m}(\langle\bigtriangledown_{e_i}e_{2m+1},e_j\rangle - \langle\bigtriangledown_{e_j}e_{2m+1},e_i\rangle)e_ie_j + \sum_{i=1}^{2m}\langle\bigtriangledown_{e_{2m+1}}e_{2m+1},e_i\rangle e_{2m+1}e_i\right]&.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle D(a_1\cdots a_m e_{2m+1}), D(a_1\cdots a_m e_{2m+1})\rangle \\
\leq &\sum_{i=1}^{2m+1}(\bigtriangledown_{e_i}(a_1\cdots a_m) - a_1\cdots a_m\langle\bigtriangledown_{e_{2m+1}}e_{2m+1},e_i\rangle)^2
\end{aligned}$$

$$
\begin{aligned}
&+a_1^2\cdots a_m^2\sum_{1\le i<j\le 2m}(\langle\nabla_{e_i}e_{2m+1},e_j\rangle-\langle\nabla_{e_j}e_{2m+1},e_i\rangle)^2\\
\le\ & 2\langle\nabla(a_1\cdots a_m),\nabla(a_1\cdots a_m)\rangle+2a_1^2\cdots a_m^2\langle\nabla e_{2m+1},\nabla e_{2m+1}\rangle\\
=\ & 2\langle\nabla(a_1\cdots a_m e_{2m+1}),\nabla(a_1\cdots a_m e_{2m+1})\rangle
\end{aligned}
$$

i.e., $|D\psi|^2\le 2|\nabla\psi|^2$, which together with (4) :

$$\int_M\langle D\psi,D\psi\rangle=\int_M\{\langle\nabla\psi,\nabla\psi\rangle+\langle\mathcal{R}(\psi),\psi\rangle\}$$

yields

$$
\begin{aligned}
\int_M\langle\nabla\psi,\nabla\psi\rangle &\ge \int_M\langle\mathcal{R}(\psi),\psi\rangle\\
=\int_M\langle Ric(\psi),\psi\rangle &\ge \int_M 2m\delta|\psi|^2. \qquad (35)
\end{aligned}
$$

Putting formulas (32)−(35) together with Lemma 2.1 and 2.3, we establish

$$0>\int_M\left\{2m^{m-1}a_1^2\cdots a_m^2\delta+(\sum_{i=1}^m a_i^2)^{m-1}\left[2(2m-1)\delta\sum_{i=1}^m a_i^2-\frac{8}{3}\sum_{1\le i<j\le m}a_ia_j(1-\delta)\right]\right\}.$$

Then we define

$$F(\delta)=2m^{m-1}a_1^2\cdots a_m^2\delta+(\sum_{i=1}^m a_i^2)^{m-1}\left[2(2m-1)\delta\sum_{i=1}^m a_i^2-\frac{8}{3}\sum_{1\le i<j\le m}a_ia_j(1-\delta)\right]$$

and want to show that $F(\delta)\ge 0$ if δ is big enough. Hence δ should be smaller than some certain number. Note that

$$F(\delta)\ge 0\Longleftrightarrow\begin{cases}\delta\ge\dfrac{\frac{8}{3}\sum_{1\le i<j\le m}a_ia_j(\sum_{i=1}^m a_i^2)^{m-1}}{2(2m-1)(\sum_{i=1}^m a_i^2)^m+\frac{8}{3}\sum_{1\le i<j\le m}a_ia_j(\sum_{i=1}^m a_i^2)^{m-1}+2m^{m-1}a_1^2\cdots a_m^2}\\ \text{or all } a_i=0.\end{cases}$$

We define

$$G_1(a_1,\dots,a_m)=\frac{\frac{8}{3}\sum_{1\le i<j\le m}a_ia_j(\sum_{i=1}^m a_i^2)^{m-1}}{2(2m-1)(\sum_{i=1}^m a_i^2)^m+\frac{8}{3}\sum_{1\le i<j\le m}a_ia_j(\sum_{i=1}^m a_i^2)^{m-1}+2m^{m-1}a_1^2\cdots a_m^2}$$

for $a_i \geq 0$.

By a simple trick that both $f/(f+g)$ and f/g get the maximum value at the same point as the functions, we only need to locate the point where G gets the maximum value, here

$$G(a_1,\dots,a_m) \stackrel{def.}{=} \frac{\frac{8}{3}\sum_{1\leq i<j\leq m} a_i a_j (\sum_{i=1}^m a_i^2)^{m-1}}{2(2m-1)(\sum_{i=1}^m a_i^2)^m + 2m^{m-1}a_1^2\cdots a_m^2}$$

for $a_i \geq 0$.

Proposition. The function G_1 assumes maximum value at $a_1 = a_2 = \cdots = a_m$.

Proof. As we mentioned before, we only need to locate the point for G. Using the homogeneity of G on all a_i, we assume that $\sum_{i=1}^m a_i^2 = m$, then,

$$G(a_1,\dots,a_m) = \frac{\frac{4}{3}\sum_{1\leq i<j\leq m} a_i a_j}{(2m-1)m + a_1^2\cdots a_m^2}.$$

Applying the Lagrange's method of multipliers on it, we have

$$\frac{\frac{4}{3}(\sum_{k\neq i} a_k)[(2m-1)m + a_1^2\cdots a_m^2] - \frac{8}{3}\sum_{1\leq k<l\leq m} a_k a_l a_1^2\cdots a_i\cdots a_m^2}{[(2m-1)m + a_1^2\cdots a_m^2]^2} = \lambda a_i,$$

or

$$\begin{aligned} &\{(\sum_{k\neq i} a_k)[(2m-1)m + a_1^2\cdots a_m^2] - 2\sum_{1\leq k<l\leq m} a_k a_l a_1^2\cdots a_i\cdots a_m^2\}a_j \\ = \; &\{(\sum_{k\neq j} a_k)[(2m-1)m + a_1^2\cdots a_m^2] - 2\sum_{1\leq k<l\leq m} a_k a_l a_1^2\cdots a_j\cdots a_m^2\}a_i. \quad (36) \end{aligned}$$

Adding $a_i a_j[(2m-1)m + a_1^2\cdots a_m^2]$ on the both sides of (36), and simplifying it, we obtain

$$\begin{aligned} &(\sum_{k\neq i} a_k)[(2m-1)m + a_1^2\cdots a_m^2](a_i - a_j) = 0 \\ \Longrightarrow \quad & a_i = a_j = 1. \end{aligned}$$

Hence,

$$G_1(1,\ldots,1) = \frac{\frac{8}{3}m^{m-1}\frac{m(m-1)}{2}}{2(2m-1)m^m + \frac{8}{3}m^{m-1}\frac{m(m-1)}{2} + 2m^{m-1}} = \frac{2m(m-1)}{(8m-5)m+3}$$

is the possible maximum value in the interior of our region. We want to show that it is the maximum, and that therefore δ has to be smaller than this number.

Suppose that G_1 gets maximum at some $a_i = 0$, say $a_m = 0$. Then, assume that $\sum_{i=1}^{m-1} a_i^2 = m-1$,

$$G_1(a_1,\ldots,a_{m-1},0) = \frac{\frac{4}{3}\sum_{1\le i<j\le m-1} a_i a_j}{(2m-1)(m-1) + \frac{4}{3}\sum_{1\le i<j\le m-1} a_i a_j},$$

which achieves maximum iff at $a_1 = \cdots = a_{m-1} = 1$. However,

$$G_1(1,\ldots,1,0) = \frac{\frac{4}{3}\frac{(m-1)(m-2)}{2}}{(2m-1)(m-1) + \frac{4}{3}\frac{(m-1)(m-2)}{2}} = \frac{2(m-2)}{8m-7},$$

which is smaller.

qed.

<u>Proof of Theorem C.</u> Given ϕ as a harmonic 2-form, we choose as before a local frame field $\{e_1,\ldots,e_{2m}\}$, such that

$$\phi = \sum_{i=1}^{m} a_i e_{2i-1} e_{2i}.$$

We have already shown that

$$\langle \nabla\phi, \nabla\phi\rangle = \sum_{i=1}^{m} \langle \nabla a_i, \nabla a_i\rangle + \left\langle \sum_{i=1}^{m} a_i \nabla (e_{2i-1}e_{2i}), \sum_{i=1}^{m} a_i \nabla (e_{2i-1}e_{2i}) \right\rangle.$$

Then we want to estimate the second term at right hand side in terms of the first one. Choose another local frame field $\{e'_1, \ldots, e'_{2m}\}$ centered at x, such that,

$$e'_i(x) = e_i(x) \qquad and \qquad \nabla e'_i|_x = 0.$$

Under this coordinate,

$$\phi = \sum_{i=1}^{m} a'_i e'_{2i-1} e'_{2i} + \sum_{(k,l) \neq (2i-1,2i)} a'_{kl} e'_k e'_l,$$

where $a'_{kl}(x) = 0$. Then

$$\langle \nabla\phi, \nabla\phi \rangle = \sum_{i=1}^{m} \langle \nabla a'_i, \nabla a'_i \rangle + \sum_{(k,l) \neq (2i-1,2i)} \langle \nabla a'_{kl}, \nabla a'_{kl} \rangle \qquad \text{at } x.$$

It is very easy to verify that $\nabla a_i = \nabla a'_i$ at x by evaluating $\nabla\phi$ at $e_{2i-1}e_{2i}(= e'_{2i-1}e'_{2i})$ on both local frame fields. Hence,

$$\left\langle \sum_{i=1}^{m} a_i \nabla (e_{2i-1}e_{2i}), \sum_{i=1}^{m} a_i \nabla (e_{2i-1}e_{2i}) \right\rangle = \sum_{(k,l) \neq (2i-1,2i)} \langle \nabla a'_{kl}, \nabla a'_{kl} \rangle \quad \text{at } x.$$

Since ϕ is harmonic, $D\phi = 0$. The coefficient of e'_{2i-1} for $D\phi$ is

$$\nabla_{e'_{2i}} a'_i + \sum_{k \neq 2i-1,2i} (-1)^{\epsilon_k} \nabla_{e'_k} a'_{k,2i-1} \qquad \epsilon_k = 0, 1.$$

The coefficient of e'_{2i} is similar. The coefficient of $e_{2i-1}e_{2i}e_k$ $(k \neq 2i-1, 2i)$ is

$$(-1)^{\epsilon_1} \nabla_{e'_k} a'_i + (-1)^{\epsilon_2} \nabla_{e'_{2i-1}} a'_{2i,k} + (-1)^{\epsilon_3} \nabla_{e'_{2i}} a'_{2i-1,k} \qquad \epsilon_1, \epsilon_2, \epsilon_3 = 0, 1.$$

By the inequality

$$a^2 \leq p(b_1^2 + \cdots + b_p^2) \qquad \text{if} \quad a = b_1 + \cdots + b_p,$$

we have

$$\sum_{i=1}^{m} \langle \nabla a'_i, \nabla a'_i \rangle \leq 2(m-1) \sum_{(k,l) \neq (2i-1,2i)} \langle \nabla a'_{kl}, \nabla a'_{kl} \rangle.$$

Hence,

$$\langle \nabla\phi, \nabla\phi\rangle \geq \frac{2m-1}{2(m-1)} \sum_{i=1}^{m} \langle \nabla a_i, \nabla a_i\rangle.$$

Following the argument shown before, we establish an inequality

$$0 > \int_{\mathrm{M}} \left\{ \frac{2m-1}{2(m-1)} m^{m-2} \langle \nabla(a_1 \dots a_m), \nabla(a_1 \dots a_m)\rangle + (\sum_{i=1}^{m} a_i^2)^{m-1} \langle \mathcal{R}\phi, \phi\rangle \right\}. \tag{37}$$

The equality can not hold because it requires that $a_1 = \dots = a_m$=constant, which contradicts that ψ is a zero class in $H^{2m}(M;\mathbf{R})$.

In [$\mathbf{T_2}$], Tsagas shows that

$$\int_M \langle \nabla(a_1 \dots a_m), \nabla(a_1 \dots a_m)\rangle \geq \int_M 2m\delta a_1^2 \dots a_m^2. \tag{38}$$

Substituting (7) and (38) in (37) gives

$$\begin{aligned} 0 > \int_M \quad & \left\{ \frac{2m-1}{m-1} m^{m-1} \delta a_1^2 \dots a_m^2 \right. \\ & \left. + (\sum_{i=1}^{m} a_i^2)^{m-1} \left[4(m-1)\delta \sum_{i=1}^{m} a_i^2 - \frac{8}{3} \sum_{1 \leq i < j \leq m} |a_i a_j| (1-\delta) \right] \right\}. \end{aligned}$$

This leads to considering the maximum value of

$$\frac{\frac{8}{3}\sum_{1\leq i<j\leq m} a_i a_j (\sum_{i=1}^m a_i^2)^{m-1}}{4(m-1)(\sum_{i=1}^m a_i^2)^m + \frac{8}{3}\sum_{1\leq i<j\leq m} a_i a_j (\sum_{i=1}^m a_i^2)^{m-1} + \frac{2m-1}{m-1} m^{m-1} a_1^2 \cdots a_m^2},$$

for $a_i \geq 0$.

Similarly we can show that the maximum can be obtained at $a_1 = \dots = a_m = 1$. Hence,

$$\begin{aligned} \delta \quad &< \quad \frac{\frac{8}{3}\frac{m(m-1)}{2} m^{m-1}}{4(m-1)m^m + \frac{8}{3}\frac{m(m-1)}{2} m^{m-1} + \frac{2m-1}{m-1} m^{m-1}} \\ &= \quad \frac{4m(m-1)^2}{16m(m-1)^2 + 3(2m-1)}. \end{aligned}$$

qed.

Remark 1. Theorem C can not be extended to the pointwise case, because we have to estimate the first non-zero eigenvalue of the Laplacian operator $\triangle$ by using the δ-pinching condition.

Remark 2. For a given 2-form ϕ, at an arbitrary $x \in M$, generally we can not find a local orthonormal frame field $\{e_1, \ldots, e_n\}$ on its neighborhood, such that, ϕ can be diagonized. However, the measure of such x's is zero. By the continuity of its norm and gradient, we are sure that the global estimates are valid.

§5. 4-manifolds with 2-nonnegative curvature operator

In this section, we will prove Theorem D. The outline of the proof is similar to Hamilton's work [**H₂**]. Here we will adopt his notation, and use his results with some modification.

Let V be a vector bundle on an n-dimensional manifold M, which is isomorphic to the tangent bundle TM, with an isometry $\{u_a^i\}$ between V and TM. Let $\{u_a^i\}$ satisfy the evolution equation

$$\frac{\partial}{\partial t} u_a^i = g^{ij} R_{jk} u_a^k \ ,$$

where g^{ij} is the inverse of a metric g_{ij} on M and R_{jk} is the Ricci curvature of M. We pull back the Levi-Civita connection ∇ and the curvature tensor R on V via $\{u_a^i\}$. Then, under the unnormalized evolution equation

$$\frac{\partial}{\partial t} g = -2Rc \ ,$$

the pull-back curvature tensor R_M satisfies the heat equation

$$\frac{\partial}{\partial t} R_M = \triangle R_M + R_M^2 + R_M^{\#} \ , \tag{39}$$

where R_M is treated as a linear operator from $\wedge^2 TM$ to $\wedge^2 TM$, $R_M^{\#}$ is an abbreviation of $R_M \# R_M$ and generally $A \# B$ is defined by

$$(A\#B)_{\alpha\beta} = C_{\alpha\gamma\eta} C_{\beta\delta\theta} (A)_{\gamma\delta} (B)_{\eta\theta} \ ,$$

where $C_{\alpha\beta\gamma}$'s are the Lie structure constants related to a standard orthonormal basis of $\wedge^2(V) \cong so(n)$, which are fully anti-symmetric, i.e., $C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma} = C_{\beta\gamma\alpha}$, etc..

To simplify the problem, Hamilton shows that, instead of studying the partial differential equation (39), one can study the related ordinary differential equation

$$\frac{d}{dt}R_M = R_M^2 + R_M^{\#} \tag{40}$$

and he shows that if the solutions of the ODE stay in certain closed convex subset $X \subset Sym(\wedge^2 V)$, so does the solutions of the PDE.

Because we have to take derivatives of some functions which are not quite differentiable, we should give the definition of derivatives:

Definition 5.1. *If $f(t)$ is a Lipshitz function of t, we say*

$$\frac{df}{dt} \le c \qquad if \qquad \limsup_{h\downarrow 0} \frac{f(t+h)-f(t)}{h} \le c \ ; \tag{41}$$

$$\frac{df}{dt} \ge c \qquad if \qquad \liminf_{h\downarrow 0} \frac{f(t+h)-f(t)}{h} \ge c \ . \tag{42}$$

Lemma 5.2 $[\mathbf{H_2}]$ *If $f(a) \le g(a)$ and $df/dt \le dg/dt$ on $a \le t \le b$, then $f(b) \le g(b)$.*

Let G be a smooth function of $t \in R$ and $y \in R^k$ and $f(t) = sup\{G(t,y) : y \in Y\}$, where Y is a compact set of R^k. Then $f(t)$ is Lipshitz, and we have

Lemma 5.3. $[\mathbf{H_2}]$

$$\frac{d}{dt}f(t) \le sup\{\frac{\partial}{\partial t}G(t,y) : y \in Y(t)\} \ , \tag{43}$$

where $Y(t) = \{y : G(t,y) = f(t)\}$.

When we say that R_M is k-**positive** (resp. k-**nonnegative**), we mean that the sum of its first k eigenvalues is > 0 (resp. ≥ 0). Hamilton observed that the positivity of R_M is preserved by equation (39). Actually, we will show that more can be preserved by (39).

Lemma 5.4. *The* 2*-positivity (resp.* 2*-nonnegativity) of* R_M *is preserved by equation* (39).

<u>Proof</u>. By Hamilton's work, we only need to show that the 2-positivity (resp. 2-nonnegativity) of R_M is preserved by equation (40).

Let R_M be diagonized as

$$R_M = \begin{pmatrix} r_{11} & & \\ & \ddots & \\ & & r_{ll} \end{pmatrix}, \qquad where \quad l = \frac{n(n-1)}{2} \;,$$

then by lemma 5.3, $r_{11} + r_{22}$ satisfies the following differential inequality

$$\frac{d(r_{11}+r_{22})}{dt} \geq r_{11}^2 + r_{22}^2 + \sum_{\eta>2} C_{12\eta}^2 (r_{11}+r_{22}) r_{\eta\eta} + \sum_{\gamma>\eta>2} (C_{1\gamma\eta}^2 + C_{2\gamma\eta}^2) r_{\gamma\gamma} r_{\eta\eta} \;. \tag{44}$$

If $r_{11}+r_{22} > 0$ (resp. ≥ 0) at $t = 0$, then $r_{11}+r_{22} > 0$ (resp. ≥ 0) for all $t > 0$, since the right hand side is nonnegative. qed.

Let r be the scalar curvature and $\tilde{R}_M$ be the traceless part of R_M, the work done by Hamilton [$\mathbf{H_2}$] and Huisken [**H**] shows that if $|\, \tilde{R}_M \,| \leq Cr^{1-\delta}$

preserved by equation (40) for some constant C and $\delta > 0$, then M is diffeomorphic to a space form with constant positive sectional curvature.

However the problem is very difficult for general dimensions. But it is rather easy for $n = 4$, because there is a canonical decomposition of

$$\wedge^2 = \wedge_+^2 + \wedge_-^2$$

determined by the volume form ω. This gives a decomposition of R_M as

$$R_M = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix} .$$

By computation, we have

$$R_M^{\#} = 2 \begin{pmatrix} A^{\#} & B^{\#} \\ {}^tB^{\#} & C^{\#} \end{pmatrix} .$$

each block above is a 3×3 submatrix.

Therefore, equation(40) can be decomposed as

$$\frac{d}{dt}A = A^2 + 2A^{\#} + B\,{}^tB \,,$$

$$\frac{d}{dt}B = AB + BC + 2B^{\#} \,,$$

$$\frac{d}{dt}C = C^2 + 2C^{\#} + {}^tBB \,.$$

Remark. $B^{\#}$ is not always same as in the definition, but has some signs altered.

Let B be diagonized with eigenvalues $0 \leq b_1 \leq b_2 \leq b_3$, and let $a_1 \leq a_2 \leq a_3$ and $c_1 \leq c_2 \leq c_3$ be the eigenvalues of A and C respectively.

Set $a = a_1 + a_2 + a_3$, $b = b_1 + b_2 + b_3$, and $c = c_1 + c_2 + c_3$.

Now we assume that M is a 4-dimensional manifold with 2-nonnegative curvature operator.

Lemma 5.5.

$$
\begin{aligned}
&\frac{d}{dt} a_1 \geq a_1^2 + 2b_1^2 + 2a_2 a_3 \ , \\
&\frac{d}{dt} c_1 \geq c_1^2 + 2b_1^2 + 2c_2 c_3 \ , \\
&\frac{d}{dt} a_3 \leq a_3^2 + b_3^2 + 2a_1 a_2 \ , \\
&\frac{d}{dt} c_3 \leq c_3^2 + b_3^2 + 2c_1 c_2 \ , \\
&\frac{d}{dt}(b_2 + b_3) \leq (a_2 + c_2) b_2 + (a_3 + c_3) b_3 + 2(b_2 + b_3) b_1 \ , \\
&\frac{d}{dt}(a_1 + a_2) \geq a_1^2 + a_2^2 + 2b_1^2 + 2(a_1 + a_2) a_3 \ , \\
&\frac{d}{dt}(c_1 + c_2) \geq c_1^2 + c_2^2 + 2b_1^2 + 2(c_1 + c_2) c_3 \ .
\end{aligned}
\tag{45}
$$

<u>Proof</u>. The first five inequalities were proven in $[\mathbf{H_2}]$ by using lemma 5.3. The last two inequalities can be proven in the same pattern, because

$$a_1 + a_2 = inf\{\langle A(u_1), u_1\rangle + \langle A(u_2), u_2\rangle \mid u_1, u_2 \in \wedge_+^2, u_1 \perp u_2, |u_1| = |u_2| = 1\} \ ,$$

and

$$c_1 + c_2 = inf\{\langle C(v_1), v_1\rangle + \langle C(v_2), v_2\rangle \mid v_1, v_2 \in \wedge_-^2, v_1 \perp v_2, |v_1| = |v_2| = 1\} \ .$$

qed.

Lemma 5.6

$$\frac{d}{dt}(a - 2b + c) \geq (a_1 + 2b_1 + c_1)(a - 2b + c) \ . \tag{46}$$

Remark. This lemma was proven in [**H₂**] under the stronger condition on R_M.

Proof. Hamilton has shown that

$$\frac{d}{dt}(a-2b+c) \geq tr(A+2B+C)\#(A-2B+C) \ .$$

First, we observe that $A-2B+C$ is always 2-nonnegative by applying R_M to any unit vectors $(x_1,-x_1)$ and $(x_2,-x_2) \in (\wedge_+^2, \wedge_-^2)$, where $x_1 \perp x_2$, under the bases of $\wedge_+^2$ and $\wedge_-^2$ which make B be diagonal and nonnegative.

Let P and Q be two symmetric 3×3 metrices. Q is 2-nonnegative. Let p_1 be the smallest eigenvalue of P and q_1, q_2, q_3 be the eigenvalues of Q. Then,

$$trP\#Q = \frac{1}{2}[p_{11}(q_2+q_3)+p_{22}(q_1+q_3)+p_{33}(q_1+q_2)] \geq p_1 q \ .$$

Applying this to $P=A+2B+C$ and $Q=A-2B+C$ with the facts that $p_1 \geq a_1+2b_1+c_1$ and $q=tr(A-2B+C)=a-2b+c$ completes the proof.

qed.

Now Hamilton's argument can be brought to our case without too much change. The only difference is that we have to use $(a_1+a_2)/2$ (resp. $(c_1+c_2)/2$) for the estimates instead of a_1 (resp. c_1).

More precisely, we will list all theorems parallel to Hamilton's along with proofs if they are necessary.

Let E be the set of all symmetric bilinear forms R^* on the Lie algebra $so(n)$. By the canonical identification of $so(n)$ with $\wedge^2 TM$, we see that $R_M \in E$.

Definition 5.7. We say that a subset $Z \subseteq E$ is a pinching set if

(1) Z is closed and convex.

(2) Z is invariant under the action of the Lie group $O(n)$.

(3) Z is invariant under the flow of the ODE

$$\frac{d}{dt}R^* = (R^*)^2 + (R^*)^\# .$$

(4) $|\ \widetilde{R^*}\ | \leq Cr^{1-\delta}$ for some C and all $R^* \in Z$.

Theorem 1. If we choose successively constants G, H, J, K and L large enough, δ, ϵ and θ small enough, then the set of $Z \subseteq \{R^* \text{ is 2-positive}\}$ defined by the inequalities

(1) $(b_2 + b_3)^2 \leq G(a_1 + a_2)(c_1 + c_2)$,

(2) $a_3 \leq H(a_1 + a_2)$, *and* $c_3 \leq H(c_1 + c_2)$,

(3) $(b_2 + b_3)^{2+\delta} \leq J(a_1 + a_2)(c_1 + c_2)(a - 2b + c)^\delta$,

(4) $(b_2 + b_3)^{2+\epsilon} \leq K(a_1 + a_2)(c_1 + c_2)$,

(5) $a_3 \leq (a_1 + a_2)/2 + L(a_1 + a_2)^{1-\theta}$, *and* $c_3 \leq (c_1 + c_2)/2 + L(c_1 + c_2)^{1-\theta}$.

is a pinching set for the flow (40) in the sense of definition 5.7. Moreover, every R_M which is 2-positive lies in some such set Z.

Remark. At this moment, we only discuss M^4 with 2-positive curvature operator. The 2-nonnegative case will be treated later.

Proof. All other claims can be proven exactly same as in Theorem 7.1 of [$\mathbf{H_2}$]. The only thing we have to show is that the inequalities are preserved by (40) for some chosen constants. The corresponding lemma we need is

Lemma 5.8.

$$\frac{d}{dt}\ln a_1 \geq 2b_1 + 2a_3 + \frac{(b_1-a_1)^2}{a_1} + 2\frac{a_3}{a_1}(a_2-a_1) \quad \text{if } a_1 > 0. \tag{47}$$

$$\frac{d}{dt}\ln c_1 \geq 2b_1 + 2c_3 + \frac{(b_1-c_1)^2}{c_1} + 2\frac{c_3}{c_1}(c_2-c_1) \quad \text{if } c_1 > 0. \tag{48}$$

$$\frac{d}{dt}\ln(a_1+a_2) \geq \frac{(a_1-a_2)^2}{2(a_1+a_2)} + 2b_1 + 2a_3 + \frac{1}{2(a_1+a_2)}[2b_1-(a_1+a_2)]^2. \tag{49}$$

$$\frac{d}{dt}\ln a_3 \leq a_3 + a_1 + a_2 + \frac{b_3^2}{a_3} + \frac{2a_1a_2 - a_3(a_1+a_2)}{a_3}. \tag{50}$$

$$\frac{d}{dt}\ln(b_2+b_3) \leq 2b_1 + a_3 + c_3 - \frac{b_2}{b_2+b_3}[(a_3-a_2)+(c_3-c_2)]. \tag{51}$$

$$\frac{d}{dt}\ln(c_1+c_2) \geq \frac{(c_1-c_2)^2}{2(c_1+c_2)} + 2b_1 + 2c_3 + \frac{1}{2(c_1+c_2)}[2b_1-(c_1+c_2)]^2. \tag{52}$$

$$\frac{d}{dt}\ln c_3 \leq c_3 + c_1 + c_2 + \frac{b_3^2}{c_3} + \frac{2c_1c_2 - c_3(c_1+c_2)}{c_3}. \tag{53}$$

$$\frac{d}{dt}\ln(a-2b+c) \geq a_1 + 2b_1 + c_1. \tag{54}$$

Proof. This is just the rewriting of Lemma 5.5 and Lemma 5.6. qed.

Now we will prove the first three inequalities listed in the Theorem 1 in 4 cases:

Case 1: $a_1 \geq a_2/3$ and $c_1 \geq c_2/3$.

In this case, a_1 and c_1 are positive and bounded from below, Hamilton [**H₂**] shows that for suitable constants, the inequalities where $a_1 + a_2$ and $c_1 + c_2$ are replaced by a_1 and c_1 are preserved under the flow of the ODE . Then reverse replacements give our inequalities.

Case 2: $a_1 < a_2/3$ and $c_1 \geq c_2/3$.

Case 3: $a_1 \geq a_2/3$ and $c_1 < c_2/3$.

Case 4: $a_1 < a_2/3$ and $c_1 < c_2/3$.

We will only prove the second case. Other cases follow in the same way.

Actually, we will prove that the inequalities are preserved by the Ricci flow when $c_1 + c_2$ is replaced by c_1.

(1). By formulas (48), (49) and (51), we have

$$
\begin{aligned}
\frac{d}{dt}\ln\frac{(a_1+a_2)c_1}{(b_2+b_3)^2} \geq\ & \frac{(a_1-a_2)^2}{2(a_1+a_2)} + \frac{1}{2(a_1+a_2)}[2b_1-(a_1+a_2)]^2 \\
& + \frac{(c_1-b_1)^2}{c_1} + 2\frac{c_3}{c_1}(c_2-c_1) \\
& + 2\frac{b_2}{b_2+b_3}[(a_3-a_2)+(c_3-c_2)] \\
\geq\ & 0\ .
\end{aligned}
$$

the inequality $(b_2+b_3)^2 \leq G(a_1+a_2)c_1$ is preserved for any constant G.

(2). Since $\quad a = trA = trC = c,$

$$
c_1 \leq \frac{1}{3}(c_1+c_2+c_3) = \frac{1}{3}(a_1+a_2+a_3) \leq a_3\ ,
$$

and

$$
(b_2+b_3)^2 \leq G(a_1+a_2)c_1\ ,
$$

we see that

$$\frac{b_3^2}{a_3} \leq G(a_1 + a_2) \ .$$

Then by formulas (49) and (50), we have

$$\frac{d}{dt} \ln \frac{a_3}{a_1 + a_2} \leq (G+1)(a_1 + a_2) - a_3 \ , \tag{55}$$

because all other omitted terms are nonpositive, so is $[2a_1a_2 - a_3(a_1+a_2)]/a_3$: Recall that $a_1 + a_2 \geq 0$ and if $a_1 \leq 0$, then $a_2 \geq 0$ and

$$\frac{2a_1a_2 - a_3(a_1 + a_2)}{a_3} \leq 0 \ .$$

If $a_1 > 0$, then

$$\frac{2a_1a_2 - a_3(a_1 + a_2)}{a_3} = \frac{a_1(a_2 - a_3) + a_2(a_1 - a_3)}{a_3} \leq 0 \ .$$

Then we consider function $f = a_3/(a_1 + a_2) - H$, where H $(\geq G+1)$ is a fixed constant. We want to show that if $f \geq 0$, then $df/dt \leq 0$. This means that when $f < 0$ initially, then it remains so.

$$\begin{aligned} \frac{df}{dt} &= \frac{a_3}{a_1 + a_2} \frac{d}{dt} \ln \frac{a_3}{a_1 + a_2} \\ &\leq \frac{a_3}{a_1 + a_2} [H(a_1 + a_2) - a_3] = -a_3 f \ . \end{aligned}$$

The proof for $c_3 \leq Hc_1$ is similar. Because $a_1 + a_2 \leq 2c_3$, we only need that $H \geq 2G + 1$.

(3). By the assumption that $a_1 < \frac{1}{3}a_2$, we have

$$\frac{(a_1 - a_2)^2}{2(a_1 + a_2)} > \frac{1}{8}(a_1 + a_2) \geq \frac{a_3}{8H} \geq \frac{a_3 - a_1}{16H} \ .$$

We also have

Lemma 5.9. *If* $(b_2+b_3)^2 \le G(a_1+a_2)c_1$ *and* $c_3 \le Hc_1$*, then for* $\delta \le \min(1/16H, 1/\sqrt{3GH})$*, we have*

$$\frac{(c_1-b_1)^2}{c_1} + \frac{2b_2}{b_2+b_3}(c_3-c_2) \ge \delta(c_3-c_2) \ .$$

Proof. We consider two cases:

Case 1: $b_1 \le c_1/2$. In this case we have

$$\frac{(c_1-b_1)^2}{c_1} \ge \frac{c_1}{4} \ge \frac{c_3}{4H} > \frac{c_3-c_2}{4H} \ .$$

Case 2: $b_1 \ge c_1/2$. In this case, since

$$a_1+a_2 \le a_1+a_2+a_3 \le 3c_3 \le 3Hc_1 \ ,$$

and $b_2+b_3 \le \sqrt{G(a_1+a_2)c_1} \le \sqrt{3GH}c_1$, we get

$$\frac{2b_2}{b_2+b_3} \ge \frac{2b_2}{\sqrt{3GH}c_1} \ge \delta \ .$$

qed.

As a consequence we see that

$$\frac{d}{dt}\ln\frac{(a_1+a_2)c_1}{(b_1+b_2)^2} \ge \delta[(a_3-a_1)+(c_3-c_1)] \ .$$

Since we also have

$$\frac{d}{dt}\ln\frac{b_1+b_2}{a-2b+c} \le (a_3-a_1)+(c_3-c_1) \ ,$$

we conclude that the inequality

$$(b_2+b_3)^{2+\delta} \le J(a_1+a_2)c_1(a-2b+c)^\delta$$

will be preserved for any constant J.

As one may expected, eventually, a_1 will be $\geq a_2/3$. Then we come back to case 1, and choose new G, H, J bigger and η smaller which have bounded ratios with the old ones. We can see that the new constants are also good in case 2. The remaining cases are handled in the same way.

Lemma 5.10. *There exists $\eta > 0$, such that, on the set defined by inequality* (3) *in Theorem 1, we have $b \leq (1-\eta)a$.*

Proof. If $b \leq a/2 = c/2$, it is trivial. If $b > a/2$, then $b_2 + b_3 \geq \frac{2}{3}b > a/3$ and for some constant k we have

$$a^{2+\delta} \leq ka^2(a-b)^\delta \ ,$$

which makes $a \leq k'(a-b)$ for some k', or $b \leq (1-\eta)a$ for some η.

Corollary 5.11. *There exists $\lambda > 0$, such that, on the set defined by inequality* (3) *in Theorem 1, we have*

$$\frac{(a_1-a_2)^2}{2(a_1+a_2)} + \frac{1}{2(a_1+a_2)}[2b_1 - (a_1+a_2)]^2 + \frac{2b_2}{b_2+b_3}(a_3-a_2) \geq \lambda a \ . \quad (56)$$

Proof. We will prove inequality (56) in several cases:

Case 1: $2b_1 \leq (1-\frac{\eta}{6})(a_1+a_2)$. In this case, we have

$$\frac{1}{2(a_1+a_2)}[2b_1-(a_1+a_2)]^2 \geq \frac{\eta^2}{72}(a_1+a_2) \geq \frac{\eta^2}{72(H+1)}a \ .$$

Case 2: $2b_1 > (1-\frac{\eta}{6})(a_1+a_2)$. Then we have to consider two subcases:

Subcase 1: If $a_1 \leq (1-\frac{\eta}{6})a_2$, then

$$\frac{(a_1-a_2)^2}{2(a_1+a_2)} \geq \frac{\eta^2}{72}\frac{a_2^2}{(a_1+a_2)} \geq \frac{\eta^2}{288}(a_1+a_2) \geq \frac{\eta^2}{288(H+1)}a \ .$$

Subcase 2: If $a_1 > (1-\frac{\eta}{6})a_2$, then we claim that $a_2 < (1-\frac{\eta}{6})a_3$. If it is not true, by the assumptions, we have

$$\begin{aligned} b &\geq 3b_1 > \frac{3}{2}(1-\frac{\eta}{6})(a_1+a_2) > \frac{3}{2}(1-\frac{\eta}{6})(2-\frac{\eta}{6})a_2 \\ &\geq \frac{3}{2}(1-\frac{\eta}{6})^2(2-\frac{\eta}{6})a_3 \geq (1-\frac{\eta}{6})^2(1-\frac{\eta}{12})a > (1-\eta)a \ , \end{aligned}$$

when η is sufficiently small. This contradicts to Lemma 5.10. Hence,

$$\frac{2b_2}{b_2+b_3}(a_3-a_2) \geq \frac{(1-\frac{\eta}{6})(a_1+a_2)}{a}\frac{\eta}{3}a_3 \geq \frac{(1-\frac{\eta}{6})\frac{\eta}{6}}{6(H+1)}a \ .$$

Choosing λ the smallest among these numbers completes the proof.

qed.

(4). Now we prove that inequality (4) in Theorem 1 is preserved: By Lemma 5.7 and Corollary 5.11, we have

$$\frac{d}{dt}\ln\frac{(a_1+a_2)(c_1+c_2)}{(b_2+b_3)^2} \geq \lambda a \ .$$

We also have

$$\frac{d}{dt}\ln(b_2+b_3) \leq 2b_1+a_3+c_3 < 4a \ .$$

Choose $\epsilon = \lambda/4$, we see that

$$\frac{d}{dt}\ln\frac{(a_1+a_2)(c_1+c_2)}{(b_2+b_3)^{2+\epsilon}} \geq 0 \ ,$$

and it follows that the inequality $(b_2+b_3)^{2+\epsilon} \leq K(a_1+a_2)(c_1+c_2)$ is preserved for any K.

Corollary 5.12. *For some constants k and $\theta > 0$, we have $b_3^2 \leq k(a_1+a_2)^{1-\theta}a_3$ on the previous set.*

(5). The proof for inequality (5) is almost identical with Hamilton's. However, it seems that a print error occurs in his paper [**H**$_2$].

Lemma 5.13. *Let* $f = [a_1 + a_2 + L(a_1 + a_2)^{1-\theta}]/2a_3$, *we can show that if* $\theta > 0$ *is made small enough and if* L *then made large enough, we will have* $df/dt \geq 0$ *for* $f \leq 1$, *which shows that the set* $f \geq 1$ *is preserved.*

Proof. Rewriting formula (45) in Lemma 5.5, we have

$$\frac{d}{dt}\ln(a_1 + a_2) \geq \frac{a_1 + a_2}{2} + 2a_3 \ ,$$

then

$$\frac{d}{dt}\ln(a_1+a_2+L(a_1+a_2)^{1-\theta}) \geq \frac{a_1 + a_2 + (1-\theta)L(a_1 + a_2)^{1-\theta}}{a_1 + a_2 + L(a_1 + a_2)^{1-\theta}}(\frac{a_1 + a_2}{2}+2a_3) \ .$$

With $b_3^2 \leq k(a_1 + a_2)^{1-\theta}a_3$ and formula (50), we get

$$\frac{d}{dt}\ln a_3 \leq a_3 + a_1 + a_2 + k(a_1 + a_2)^{1-\theta}.$$

It is easy to see that

$$\frac{a_1 + a_2 + (1-\theta)L(a_1 + a_2)^{1-\theta}}{a_1 + a_2 + L(a_1 + a_2)^{1-\theta}} \geq 1 - \theta L(a_1 + a_2)^{-\theta},$$

and

$$\frac{a_1 + a_2}{2} + 2a_3 \leq 3a_3 \leq 3H(a_1 + a_2).$$

Then we get

$$\begin{aligned} \frac{d}{dt}\ln f &\geq (1 - \theta L(a_1 + a_2)^{-\theta})(\frac{a_1 + a_2}{2} + 2a_3) - (a_3 + a_1 + a_2 + k(a_1 + a_2)^{1-\theta}) \\ &\geq a_3 - \frac{a_1 + a_2}{2} - (3H\theta L + k)(a_1 + a_2)^{1-\theta}. \end{aligned}$$

Now if $f \leq 1$, then $a_3 - (a_1 + a_2)/2 \geq \frac{L}{2}(a_1 + a_2)^{1-\theta}$. So $df/dt \geq 0$ provided $\frac{L}{2} \geq 3H\theta L + k$. This will hold if we first make θ so small that $3H\theta \leq 1/4$ and then make L so large that $L \geq 4k$.

qed.

Now we consider a manifold M^4 with 2-nonnegative curvature operator. But we have more general results. Let E be a vector bundle over M^n, f a section of E.

Lemma 5.14 [$\mathbf{H_2}$]. *Suppose $\partial f/\partial t = \triangle f + \phi(f)$. Let $s(f)$ be a convex function on the bundle invariant under parallel translation whose level curves $s(f) \leq c$ are preserved by the ODE $df/dt = \phi(f)$. Then the inequality $s(f) \leq c$ is preserved by the PDE for any constant c. Furthermore if $s(f) < c$ at one point at time $t = 0$, then $s(f) < c$ everywhere on M for all $t > 0$.*

Corollary 5.15. *Let R^* be a symmetric bilinear form on $so(n)$ satisfying* (40)*, which is 2-nonnegative, then either R^* is 2-positive when $t > 0$, or R^* is nonnegative.*

Proof. Let $r_1 + r_2$ be the sum of the first two eigenvalues of R^*, then $r_1 + r_2$ is a convex function on M. By Lemma 5.14, if $r_1 + r_2 \geq 0$ at any point of M when $t = 0$, then it remains so when $t > 0$, and if $r_1 + r_2 > 0$ at some point when $t = t_0$, then $r_1 + r_2 > 0$ at everywhere when $t > t_0$. Suppose $r_1 + r_2 \equiv 0$

for $0 \leq t \leq t_1$, then $r_1 + r_2$ satisfies the following partial differential equation

$$\frac{\partial(r_1 + r_2)}{\partial t} = \triangle(r_1+r_2)+r_1^2+r_2^2+\sum_{\eta>2} C_{12\eta}^2(r_1+r_2)r_{\eta\eta}+\sum_{\gamma>\eta>2}(C_{1\gamma\eta}^2+C_{2\gamma\eta}^2)r_\gamma r_\eta \ .$$

The right hand side is nonnegative, hence it should be identically zero on M, in particular, we have $r_1^2 = r_2^2 = 0$ at $\forall x \in M$, which implies that R^* is nonnegative.

qed.

Now we can quote Hamilton's result [$\mathbf{H_2}$].

§6. Pointwise δ-pinched 4-manifolds are space forms

When we considered M^4 which was pointwise δ-pinched, our first thought was that although we need $\delta \geq 0.4$ to make the curvature operator nonnegative, we might get a 2-nonnegative curvature operator by using a smaller δ. Unfortunately, we failed. Nevertheless, we can find a smaller δ to make the whole process work. The number is given in Theorem E. Actually, this number can be improved slightly with some extra work. But we do not think it is so significant.

We explain how it works. From the proof in §5, we see that, under the condition of 2-positive R_M, all we need are the conditions:

(1) $a_1 + a_2 > 0$,

(2) $c_1 + c_2 > 0$,

(3) $A - 2B + C$ is 2-nonnegative, and $a - 2b + c > 0$

for all $t \geq 0$. The first two inequalities are automatically preserved by the Ricci flow by formulas (49) and (52). The third one is not. One way to cure this problem is to show that if

$$(b_2 + b_3)^2 \leq (a_1 + a_2)(c_1 + c_2)$$

at $t = 0$, then it remains so for all $t > 0$ by the proof for formula (1) in Theorem 1. Hence, we have

$$2(b_2 + b_3) \leq a_1 + a_2 + c_1 + c_2 \qquad \text{for all } t \geq 0,$$

or

$$a_1 + a_2 + c_1 + c_2 - 2(b_2 + b_3) \geq 0 \qquad \text{for all } t \geq 0,$$

which means that $A - 2B + C$ is 2-nonnegative for all $t \geq 0$.

Lemma 6.1. *Given an orthogonal basis* $\{\phi_1, \phi_2, \phi_3\}$ *for* $\wedge^2_+ R^4$ *and* $\{\psi_1, \psi_2, \psi_3\}$ *for* $\wedge^2_- R^4$, *where all vectors have norm* $\sqrt{2}$, *we can find an orthonormal basis* $\{e_1, e_2, e_3, e_4\}$ *for* R^4, *such that,*

$$\phi_1 = e_1e_2 - e_3e_4, \qquad \phi_2 = e_1e_3 + e_2e_4, \qquad \phi_3 = e_1e_4 - e_2e_3;$$

$$\psi_1 = e_1e_2 + e_3e_4, \qquad \psi_2 = e_1e_3 - e_2e_4, \qquad \psi_3 = e_1e_4 + e_2e_3$$

with their signs altered.

Proof. First, recall that every 2-form in $\wedge^2 R^n$ can be identified with an $n \times n$ anti-symmetric matrix. The change of the basis for R^n implies the change of the matrix representation under the associated adjoint operation.

Let $\Phi_1 = \phi_1 + \psi_1$. Φ_1 is a 4×4 anti-symmetric matrix which can be diagonized, i.e., we can find an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for R^4, such that, $\Phi_1 = ae_1e_2 + be_3e_4$. Let $\omega = e_1e_2e_3e_4$ be the volume form. Then

$$\phi_1 = \frac{\Phi_1 + \omega\Phi_1}{2} = \frac{a-b}{2}(e_1e_2 - e_3e_4), \qquad \psi_1 = \frac{\Phi_1 - \omega\Phi_1}{2} = \frac{a+b}{2}(e_1e_2 + e_3e_4).$$

Since $\|\phi_1\| = \|\psi_1\| = \sqrt{2}$, by changing the signs of ϕ_1 and ψ_1, we have

$$\phi_1 = (e_1e_2 - e_3e_4) \quad \textit{and} \quad \psi_1 = (e_1e_2 + e_3e_4).$$

Let

$$g = \begin{pmatrix} \cos\theta & \sin\theta & & \\ -\sin\theta & \cos\theta & & \\ & & \cos\eta & \sin\eta \\ & & -\sin\eta & \cos\eta \end{pmatrix} \in SO(4).$$

It is easy to check that ϕ_1 and ψ_1 are invariant under Ad_g. That is equivalent to say that if $(e'_1, e'_2, e'_3, e'_4) = g(e_1, e_2, e_3, e_4)$, then

$$\phi_1 = (e'_1 e'_2 - e'_3 e'_4) \quad and \quad \psi_1 = (e'_1 e'_2 + e'_3 e'_4).$$

Let $\phi_2 = a(e_1e_3 + e_2e_4) + b(e_1e_4 - e_2e_3)$ and $\psi_2 = c(e_1e_3 - e_2e_4) + d(e_1e_4 + e_2e_3)$. Let $\Phi_2 = \phi_2 + \psi_2$. We have that as an element of $so(4)$,

$$\Phi_2 = \begin{pmatrix} & & -a-c & -b-d \\ & & b-d & -a+c \\ a+c & -b+d & & \\ b+d & a-c & & \end{pmatrix}.$$

We want to show that by choosing θ and η, we can get

$$g^{-1}\Phi_2 g = \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{57}$$

To simplify the computation, we introduce the submatrices of g and Φ_2:

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad B = \begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix} \text{ and } C = \begin{pmatrix} -a-c & -b-d \\ b-d & -a+c \end{pmatrix}.$$

Then by the multiplication rule on submatrices, we see

$$
\begin{aligned}
g^{-1}\Phi_2 g &= \begin{pmatrix} {}^tA & \\ & {}^tB \end{pmatrix}\begin{pmatrix} & C \\ -{}^tC & \end{pmatrix}\begin{pmatrix} A & \\ & B \end{pmatrix} \\
&= \begin{pmatrix} & {}^tAC \\ -{}^tB\,{}^tC & \end{pmatrix}\begin{pmatrix} A & \\ & B \end{pmatrix} \\
&= \begin{pmatrix} & {}^tACB \\ -{}^tB\,{}^tCA & \end{pmatrix}.
\end{aligned}
$$

Therefore, equation(57) is reduced to find θ and η, such that,

$$
{}^tACB = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \tag{58}
$$

Since $\det C = (a+c)(a-c) + (b+d)(b-d) = \frac{1}{2}(\|\phi_2\|^2 - \|\psi_2\|^2) = 0$, carefully choosing θ and η confirms our claim on equation(58).

Then, under the new orthonormal basis for R^4, ϕ_3 should be $\pm\,(e_1e_4 - e_2e_3)$ and $\psi_3 = \pm\,(e_1e_4 + e_2e_3)$.

qed.

Lemma 6.2. *If $\delta \geq 1/4$, then we have $a_1 + a_2 \geq 0$ and $c_1 + c_2 \geq 0$. The equalities hold only if $\delta = 1/4$.*

Proof. Choose two eigenvectors ϕ_1 and ϕ_2 with respect to a_1 and a_2 respectly. We assume that $\|\phi_1\| = \|\phi_2\| = \sqrt{2}$. By Lemma 6.1, we can find a locally orthonormal frame field $\{e_1, e_2, e_3, e_4\}$, such that,

$$\phi_1 = e_1e_2 - e_3e_4 \quad \textit{and} \quad \phi_2 = e_1e_4 - e_2e_3.$$

Then

$$\begin{aligned}
2(a_1 + a_2) &= \langle R_M(\phi_1), \phi_1\rangle + \langle R_M(\phi_2), \phi_2\rangle \\
&= R_{1212} + R_{3434} - 2R_{1234} + R_{1414} + R_{2323} - 2R_{2314} \\
&= R_{1212} + R_{3434} + R_{1414} + R_{2323} + 2R_{3124}
\end{aligned}$$

Applying Lemma 2.3, we get

$$2(a_1 + a_2) \geq 4\delta - \frac{4}{3}(1 - \delta) \geq 0 \quad \text{if} \quad \delta \geq \frac{1}{4}.$$

the equality holds only if $\delta = \frac{1}{4}$.

Similarly, we have same statement for $c_1 + c_2$.

qed.

Lemma 6.3. *If* $\delta \geq \frac{\sqrt{13}}{6+\sqrt{13}}$, *then*

$$(b_2 + b_3)^2 \leq (a_1 + a_2)(c_1 + c_2)\,. \tag{59}$$

Proof. Let ϕ_1, ϕ_2, ψ_1 and ψ_2 be the eigenvectors of norm $\sqrt{2}$ with respect to a_1, a_2, c_1 and c_2 respectly. By Lemma 6.2, we can find a locally orthonormal frame field $\{e_1, e_2, e_3, e_4\}$, such that,

$$\phi_1 = e_1e_2 - e_3e_4,\ \phi_2 = e_1e_4 - e_2e_3,\ \psi_1 = e_1e_2 + e_3e_4,\ \psi_2 = e_1e_4 + e_2e_3.$$

Then using Lemma 2.3, we obtain

$$
\begin{aligned}
4(a_1+a_2)(c_1+c_2) &= \langle R_M(\phi_1),\phi_1\rangle + \langle R_M(\phi_2),\phi_2\rangle + \langle R_M(\psi_1),\psi_1\rangle + \langle R_M(\psi_2),\psi_2\rangle \\
&= (R_{1212}+R_{3434}+R_{1414}+R_{2323})^2 - 4R_{3124}^2 \\
&\geq 16\delta^2 - \frac{16}{9}(1-\delta)^2 \qquad (60)
\end{aligned}
$$

To estimate b_2, let $\phi \in \wedge^2 TM$ and $\psi \in \wedge^2 TM$ be the 2-forms of norm $\sqrt{2}$ with respect to b_2. By Lemma 6.1, we can write

$$\phi = e_1e_2 - e_3e_4,\ \psi = e_1e_2 + e_3e_4,$$

then

$$2b_2 = \langle R_M(\phi),\psi\rangle = R_{1212} - R_{3434} \leq 1-\delta. \qquad (61)$$

Similarly, we have $2b_3 \leq 1-\delta$.

Hence, by inequalities (60) and (61), the proof is reduced to determine δ by the following inequality:

$$
\begin{aligned}
& 4\delta^2 - \frac{4}{9}(1-\delta)^2 \geq (1-\delta)^2 \\
\iff\ & 36\delta^2 \geq 13(1-\delta)^2 \\
\iff\ & 6\delta \geq \sqrt{13}(1-\delta) \\
\iff\ & \delta \geq \frac{\sqrt{13}}{6+\sqrt{13}}
\end{aligned}
$$

qed.

Lemma 6.4. *If M^4 has positive sectional curvature, then $a-2b+c>0$.*

Proof. Let ϕ_i and ψ_i be the eigenvectors of norm 1 with respect to b_i, $i = 1, 2, 3$. Since

$$a = \sum_{i=1}^{3} \langle R_M(\phi_i), \phi_i \rangle \quad \textit{and} \quad c = \sum_{i=1}^{3} \langle R_M(\psi_i), \psi_i \rangle \, .$$

we have

$$a - 2b + c = \sum_{i=1}^{3} \langle R_M(\phi_i - \psi_i), \phi_i - \psi_i \rangle \, . \tag{62}$$

By Lemma 6.1, we see that $\phi_i - \psi_i$ can be written as $\sqrt{2} e_j e_k$ for $j, k \in \{1, 2, 3, 4\}$, hence, the right hand side of equation (62) is just the sum of some sectional curvatures. By the assumption, $a - 2b + c > 0$.

qed.

If $\delta \geq \frac{\sqrt{13}}{6+\sqrt{13}}$, by Lemma 6.2 - 6.4, we know that $a_1 + a_2 > 0$, $c_1 + c_2 > 0$, $A - 2B + C$ is 2-nonnegative, and $a - 2b + c > 0$. Then the proof follows exactly same as in §5.

References

[B_1] M. Berger, *Sur les variétés à courbure positive de diamètre minimum.* Comment Math. Helv. **35**(1), 1961, 28-34.

[B_2] _________, *Les variétés Riemanninnes homogènes normales simplement connexes a courbure strictement positive.* Ann. Scuola Norm. Sup. Pisa, XV 1961, 179-246.

[B_3] _________, *Sur quelques variétés Riemanninnes sufisamment pincèes.* Bul. Soc. Math. France, **88**, 1960, 57-71.

[C-E] J. Cheeger and D. Ebin, *Comparison theorem in Riemannian Geometry*, North-Holland, Amsterdam, 1975, p.118.

[H_1] R. Hamilton, *Three-manifolds with Positive Ricci Curvature* J. Differential Geometry **17**, 1982, 255-306.

[H_2] ___________, *Four-manifolds with Positive Curvature Operator*, J. Differential Geometry **24**, 1986, 153-179.

[H] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[Hu] G. Huisken, *Ricci Deformation of the Metric on a Riemannian Manifold*, J. Differential Geometry **21**, 1985, 47-62.

[L-M] B. Lawson and M-L. Michelsohn, *Spin Geometry*, Princeton Press, 1989.

[Ma] C. Margerin, *Pointwise Pinched Manifolds are Space Forms*, Proc.

Symp. Pure Math., Vol. **44** 1986, 307-328.

[M] M. Micallef, *Private conversation.*

[M-M] M. Micallef and J. Moore, *Minimal Two-spheres and the Topology of Manifolds with Positive Curvature on Totally Isotropic Two-planes*, Ann. of Math. (2) **127**, 1988, no.1, 199-227.

[R] E. Ruh, *Riemannian Manifolds with Bounded Curvature Ratios*, J. Differential Geometry **17**, 1982, 643-653.

[S] W. Seaman, *Existence and Uniqueness of Algebraic Curvature Tensors with Prescribed Properties and an Application to the Sphere Theorem*, preprint.

[T_1] G. Tsagas, *An Improvement of the Method of Lichnerowicz-Bochner on the Betti Numbers and Curvature of a Compact Manifold*, Ann. Mat. Pura Appl. (4) **83**, 1969, 227-234.

[T_2] ________, *On the Second Cohomology Group of a Pinched Riemannian Manifold*, Ann. Mat. Pura Appl. (4) **86**, 1970, 299-311.

[W] N. R. Wallach, *Compact Homogeneous Riemannian Manifolds with Strictly Positive Curvature*, Ann. Math., **96**, 1972, 277-295.