The symplectic structure of submanifolds of Kähler manifolds of non-positive curvature

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Abstract of the Dissertation

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Building on work of D. McDuff we investigate properties of special manifold pairs. Let P be a simply connected complete Kähler 2n-dimensional manifold of non-positive curvature . Denote by ω the Kähler form on P and by ω_0 the standard symplectic form on $\mathbb{R}^{2n}$. A submanifold Q of (P, ω) is said to be symplectic if ω restricts to a symplectic form on Q and is said to be isotropic if the restriction of ω to Q is identically zero. Let Q be a totally geodesic connected properly embedded submanifold. Some of our main results are :

- if Q is complex (therefore symplectic) of dimension $2k$ then (P, Q, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \mathbb{R}^{2k}, \omega_0)$, where $\mathbb{R}^{2k}$ is a symplectic linear subspace of $\mathbb{R}^{2n}$.
- if Q is isotropic of dimension k, then (P, Q, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \mathbb{R}^{k}, \omega_0)$, where $\mathbb{R}^{k}$ is an isotropic linear subspace of $\mathbb{R}^{2n}$.

The proof involves studying the local structure of a Liouville vector field ξ which vanishes on an isotropic submanifold Q. Some of the eigenvalues of its linear part at the singular points are zero and the remaining ones are in resonance . This is a very interesting question which appears not to have been looked at before. We show that

- there is a C^1-smooth linearizing conjugation between the Liouville vector field ξ and its linear part.

To do this we construct Darboux coordinates adapted to the unstable foliation which is provided by the Center Manifold Theorem. We then apply recent linearization results due to G. Sell.

To Tristán Schmidt

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Chapter 1

1.1 Introduction

Gromov's discovery of the existence of exotic symplectic structures on $\mathbb{R}^{2n}$ has raised many questions. For example one can look for more explicit examples of exotic structures , or one can try to find invariants which would distinguish exotic structures from the standard one. It is also important to investigate properties of the standard structure itself in order to get a deeper understanding of its nature. This is what we do in the present thesis. Building on work of D. McDuff we investigate properties of special manifold pairs.

McDuff proved a global version of the Darboux Theorem which states that the Kähler form ω on a simply connected complete Kähler 2n-dimensional manifold P of non-positive curvature is diffeomorphic to the standard symplectic form ω_0 on $\mathbb{R}^{2n}$. This means in particular that the symplectic structure on a Hermitian symmetric space of non-compact type is standard. She also showed that if L is a totally geodesic connected properly embedded Lagrangian submanifold of such a P then P is symplectomorphic to the cotangent bundle T^*L with its usual symplectic structure.

In this work we extend McDuff's results to other special submanifolds of the Kähler manifold (P,ω). Recall that a submanifold Q of P is said to be symplectic if ω restricts to a symplectic form on Q and is said to be isotropic if the restriction of ω to Q is identically zero.

Let Q be a totally geodesic connected properly embedded submanifold of (P,ω). Then we prove:

Theorem 2.1.1 If Q is complex (therefore symplectic) of dimension $2k$ then (P,Q,ω) is symplectomorphic to $(\mathbb{R}^{2n},\mathbb{R}^{2k},\omega_0)$, where $\mathbb{R}^{2k}$ is a symplectic linear subspace of $\mathbb{R}^{2n}$.

and

Theorem 4.1.1 If Q is isotropic of dimension k, then (P,Q,ω) is symplectomorphic to $(\mathbb{R}^{2n},\mathbb{R}^{k},\omega_0)$, where $\mathbb{R}^k$ is an isotropic linear subspace of $\mathbb{R}^{2n}$.

The crucial point in the proof of Theorem 2.1.1 is to show that the symplectomorphism constructed by McDuff takes a totally geodesic symplectic submanifold Q into a symplectic linear subspace of $\mathbb{R}^{2n}$. But in Example 2.2 we show that this is no longer the case when Q is isotropic. Thus we have to change strategy to deal with this case.

Theorem 4.1.1 appears as a natural extension of the theorems proved by McDuff, and some of the proof goes through without much change. However there is one crucial place where the argument breakes down and it is necessary to study the local structure of a Liouville vector field ξ which vanishes on an isotropic submanifold Q. This is a very interesting question which appears not to have been looked at before.

A homeomorphism which linearizes a vector field in a neighborhood of a singular point can not always be chosen to be smooth, because of resonant eigenvalues of the linear part. (In our particular case the eigenvalues 1 and 1/2 are in resonance). See Definition 3.5.1. The situation becomes even more complicated when the linear part of the equation at the singular point has eigenvalues on the imaginary axis (for example 0 in our case). We overcome this obstacle by constructing Darboux coordinates adapted to the foliation provided by the Center Manifold Theorem. This allows us to treat the isotropic submanifold Q as a parameter set. Further, although the vector field ξ has resonant eigenvalues , using the fact that ξ is Liouville we show that indeed it has no resonant terms in its Taylor expansion and therefore, by recent results of G. Sell, we deduce

Theorem 3.1.1 There is a C^1-smooth linearizing conjugation between the Liouville vector field ξ and its linear part.

This Thesis is organized as follows : in Chapter 2 we discuss the symplectic case and describe an example. In Chapter 3 we study the properties of the Liouville vector field ξ, and construct the Darboux coordinates mentioned above, finally we apply Sell's theorem to prove Theorem 3.1.1. We complete the discussion of the isotropic case in Chapter 4.

1.2 Basic definitions and notation

Let P be a $2n$-dimensional differentiable manifold . A *symplectic structure* on P is a closed nondegenerate differentiable 2-form on P, i.e.

$$d\omega = 0$$

and

$$\forall X \neq 0 \; \exists Y : \omega(X,Y) \neq 0 \; (X, Y \in T_pP).$$

The pair (P, ω) is called a *symplectic manifold.*

A diffeomorphism $\Phi : (P_1, \omega_1) \to (P_2, \omega_2)$ between symplectic manifolds is said to be a *symplectomorphism* if $\Phi^*\omega_2 = \omega_1$.

If Q is a submanifold of (P, ω), the orthogonal space to the tangent space of Q with respect to the form ω, at each point q in Q is defined to be

$$(T_qQ)^{\perp_\omega} = \{v \in T_qP \; : \; \omega(v,w) = 0 \; \forall w \in T_qQ\}$$

Q is called *isotropic* if for every $q \in Q \quad T_qQ$ is contained in $(T_qQ)^{\perp_\omega}$. Note that this means that the restriction of the symplectic form ω from P to Q is identically zero. An isotropic submanifold Q is called *Lagrangian* if $dimQ = \frac{1}{2}\, dimP$. Q is called *coisotropic* if for every $q \in Q \quad T_qQ$ contains $(T_qQ)^{\perp_\omega}$. Q is called *symplectic* if ω restricts to a symplectic form on Q.

(Equivalently , if for every $q \in Q \quad T_qQ \cap (T_qQ)^{\perp_\omega} = 0$).

A smooth vector field of (P, ω) is called *Liouville* if its flow φ_t satisfies $\varphi_t^* \omega = e^t \omega$ (or equivalently when $\mathcal{L}_\xi \omega = \omega$). In particular a Liouville field is conformal vector field.

A *Poisson structure* on a differentiable manifold P is defined by choosing a bilinear map from $C^\infty(P, \mathbb{R}) \times C^\infty(P, \mathbb{R})$ into $C^\infty(P, \mathbb{R})$ called the *Poisson bracket* and denoted by $(f, g) \rightarrow \{f, g\}$ which satisfies the following properties

1. it is skewsymmetric: $\{g, f\} = -\{f, g\}$

2. it satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

3. it satisfies the Leibniz identity:

$$\{f_1 f_2, g\} = \{f_1, g\} f_2 + f_1 \{f_2, g\}$$

A manifold equipped with this structure is called a *Poisson manifold.* Properties 1. and 3. imply that the Poisson bracket is a derivation in each of its arguments. Thus for each function h there is a vector field ξ_h such that $\xi_h f = \{f, h\}$ for all f. ξ_h is called the *hamiltonian vector field generated by* h.

A map $\Psi : P_1 \rightarrow P_2$ between Poisson manifolds is defined to be a *Poisson mapping* if $\{f \circ \Psi, g \circ \Psi\}_1 = \{f, g\}_2 \circ \Psi$ for all $f, g \in C^\infty(P_2, \mathbb{R})$.

A *Poisson submanifold* is a submanifold Q in a Poisson manifold P with a Poisson structure for which the inclusion is a Poisson mapping . Such structure , if it exists , is unique (see [16]).

Poisson manifolds appear as a natural extension of symplectic manifolds. A symplectic manifold may be described as a manifold carring a Poisson structure which is locally isomorphic to the standard one on $\mathbb{R}^{2n}$.

Chapter 2

2.1 The symplectic case

Let P be a simply connected, complete Kähler manifold of nonpositive curvature of dimension $2n$, and let Q be a totally geodesic, complex (therefore symplectic), connected, properly embedded, $2k$-dimensional submanifold of P. Denote by ω the Kähler form on P.

Our first result is

Theorem 2.1.1 *(P, Q, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \mathbb{R}^{2k}, \omega_0)$, where $\mathbb{R}^{2k}$ is a symplectic subspace of $\mathbb{R}^{2n}$ and ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$*

McDuff proved that the Kähler form on a simply connected complete Kähler manifold of nonpositive curvature is diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$ [9] [10]. The proof of our theorem consist in verify that the symplectomorphism that she constructed takes the submanifold Q into a symplectic subspace of $\mathbb{R}^{2n}$.

Here is a sketch of the proof :

Pick a point x_o in $Q \subset P$ and let $\rho(x)$ be the distance from x to x_0. By using a Hessian comparison theorem for manifolds of nonpositive curvature McDuff shows that the 2-form $\omega_\rho = -dJd\rho^2$ is symplectic and that $G_\rho \geq 4G$, where G_ρ is the Levi form $G_\rho(X,Y) = \omega_\rho(X,JY)$ and G is the original Kähler metric.

Applying Moser's method (see [11]) to the family of forms
$\tau_t = t\omega + (1-t)\omega_\rho$ $0 \leq t \leq 1$, she shows that (P,ω) is symplectomorphic to (P,ω_ρ). In Proposition 2.1.4 we will prove that this diffeomorphism, which we call Φ_1, preserves Q.

Then McDuff constructs a symplectomorphism Φ_2 from (P,ω_ρ) to
$(\mathbb{R}^{2n},\omega_0)$. To do this , she shows that the Liouville vector field ξ_ρ defined by $\xi_\rho \lrcorner \omega_\rho = -Jd\rho^2$ is diffeomorphic to the radial vector field ξ_0 on $\mathbb{R}^{2n}$ given in polar coordinates by $\frac{r}{2}\frac{\partial}{\partial r}$. Further this diffeomorphism takes ω_ρ to a symplectic form which is linearly diffeomorphic to ω_0. In Proposition 2.1.5 we show that ξ_ρ is tangent to Q. Consequently Φ_2 takes Q into a symplectic linear subspace, which we may clearly suppose to be $\mathbb{R}^{2k}$.

¶The crucial point in the proof that the diffeomorphisms Φ_1 and Φ_2 preserve the submanifold Q is the following :

Proposition 2.1.2 *The ω- and ω_ρ-orthogonal spaces to the tangent space T_qQ are equal at each point q of Q , i.e.* $(T_qQ)^{\perp_\omega} = (T_qQ)^{\perp_{\omega_\rho}}$.

For the proof we need the

Lemma 2.1.3 *For every vector v which is orthogonal to T_qQ with respect to the metric G we have $v(\rho^2) = d\rho^2(v) = 0$.*

Proof: Let γ be a smooth curve through q tangent to v , i.e. $\gamma : (-\varepsilon, \varepsilon) \to P$ is a smooth map such that $\gamma(0) = q$ and $\dot{\gamma}(0) = v$, where $v \perp T_qQ$. By the proof of Toponogov's theorem [3], $\rho^2(\gamma(0))$ has a critical point at $t = 0$, namely a minimum. Therefore $d\rho^2(v) = \frac{d}{dt}\rho^2(\gamma(t))|_{t=0} = 0$. □

Denote by $\bar{\nabla}$ the Levi-Civita connection on P , by ∇ the induced connection on the submanifold Q and by J the almost complex structure on P , which comes from its structure as a complex manifold. Denote by $< X, Y >$ the metric $G(X,Y)$ and by $< X, Y >_\rho$ the metric $G_\rho = \omega_\rho(X, JY)$.

Now we prove the proposition 2.1.2.

Proof: Let Y be a vector in $T_qQ^{\perp_\omega}$ and let X be a vector in T_qQ , i.e. Y and X are such that $\omega(X,Y) = 0$. Extend them in a neighborhood of q in P . Then

$$dJd\rho^2(X,Y) = X(Jd\rho^2(Y)) - Y(Jd\rho^2(X)) - Jd\rho^2[X,Y]$$

Now $Jd\rho^2(Y) = d\rho^2(JY) = 0$ by Lemma 2.1.3 since $Y \in (T_qQ)^{\perp_\omega}$ if and only if JY is G-orthogonal to T_qQ. Therefore

$$\begin{aligned} dJd\rho^2(X,Y) &= -Y(d\rho^2(JX)) - d\rho^2(J[X,Y]) \\ &= -Y(JX(\rho^2)) - (J[X,Y])\rho^2 \end{aligned}$$

by definition of J and of $d\rho^2$.

Furthermore, since $\tilde{\nabla}$ is torsion free $-J[X,Y] = -J\tilde{\nabla}_X Y + J\tilde{\nabla}_Y X$ and $Y(JX(\rho^2)) = -JX(Y\rho^2) - [Y,JX]\rho^2$. Therefore

$$\begin{aligned} dJd\rho^2(X,Y) &= -JX(Y\rho^2) - [Y,JX]\rho^2 - (J[X,Y])\rho^2 \\ &= -(\tilde{\nabla}_Y JX)\rho^2 + (\tilde{\nabla}_{JX} Y)\rho^2 - (J\tilde{\nabla}_X Y)\rho^2 + (J\tilde{\nabla}_Y X)\rho^2 \end{aligned}$$

The first and the last term cancel each other because the almost complex structure J is parallel with respect to the connection $\tilde{\nabla}$ since P is Kähler , (i.e. $\tilde{\nabla} J = 0 \Leftrightarrow J\tilde{\nabla}_Y X = \tilde{\nabla}_Y JX$).

Thus $dJd\rho^2(X,Y) = (\tilde{\nabla}_{JX} Y)\rho^2 - J(\tilde{\nabla}_X Y)\rho^2$.

Now let Z be a vector field tangent to Q , then $< Y, Z >= 0$, so that $X < Y, Z >= 0$. Therefore

$$- < \tilde{\nabla}_X Y, Z > = < Y, \tilde{\nabla}_X Z > = < Y, \nabla_X Z > = 0.$$

The second equality holds because the second fundamental form $s(X,Z) = \tilde{\nabla}_X Z - \nabla_X Z$ of Q is identically zero since Q is totally geodesic submanifold and the third because $\nabla_X Z$ is tangent to Q.

Since Q is a complex submanifold, JX is also tangent to Q, therefore analogously we obtain $< \tilde{\nabla}_{JX} Y, Z >= 0$. Since Z was arbitrary we have that the vectors $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_{JX} Y$ are orthogonal to TQ with respect to the metric.

Therefore , Lemma 2.1.3 implies that $(\tilde{\nabla}_{JX} Y\rho^2) = d\rho^2(\tilde{\nabla}_{JX} Y) = 0$ and $(\tilde{\nabla}_X Y\rho^2) = d\rho^2(\tilde{\nabla}_{JX} Y) = 0$

Hence $\omega_\rho(X,Y) = -dJd\rho^2(X,Y) = 0.$ □

McDuff applies Moser's method to the family of symplectic forms $\tau_t = t\omega + (1-t)\omega_\rho$, $0 \leq t \leq 1$, to construct the diffeomorphism Φ_1 such that $\Phi_1^*\omega = \omega_\rho$. In her proof she shows that the family of vector fields u_t that the method provides is complete with respect to the metric G. Here she used the fact that the curvature is nonpositive.

Proposition 2.1.4 *Φ_1 preserves the totally geodesic symplectic submanifold Q.*

Proof: It suffices to check that the family of vector fields u_t is tangent to Q. The u_t are defined by the equation $u_t \lrcorner \tau_t = -\beta$ where β is some 1-form which satisfies $\dot{\tau}_t = d\beta$, where $\dot{\tau}_t$ denotes $\frac{d\tau}{dt} = \omega - \omega_\rho$. McDuff takes $\beta = \frac{1}{2}Jd\rho^2 + \lambda$ where λ is obtained by integrating ω along the geodesic emanating from x_0. Precisely , let $\phi_r : S^{2n-1} \to P$ be the map $y \to (r,y)$, where $(r,y) \in (0,\infty) \times S^{2n-1}$ are geodesic polar coordinates on $P - \{x_0\}$ and let ∂ denote the radial vector field $\frac{\partial}{\partial r}$. Define λ by

$$\begin{aligned} \phi_r^*\lambda &= \int_0^r \phi_s^*(\partial \lrcorner \omega)ds \\ \lambda(\partial) &= 0 \end{aligned}$$

Then $d\lambda = \omega$.

Let X_q be a vector G-orthogonal to T_qQ . Then $X_q = (\phi_r)_*v$ for some

$v \in S^{2n-1}$ and some $r \in (0, \infty)$, and

$$\begin{aligned} \lambda(X_q) &= \lambda((\phi_r)_* v) \\ &= \int_0^r (\partial \lrcorner \omega)((\phi_r)_* v) ds \\ &= \int_0^r \omega(\partial, (\phi_r)_* v) ds \end{aligned}$$

The vector field $W_s \stackrel{\text{def}}{=} (\phi_s)_* v$ is a Jacobi field . Since $W_0 = 0$ and $W_r = X_q$ is G-orthogonal to Q then $W_s(q) \perp_G Q$ for all $0 \leq s \leq r$ and $q \in Q$. Therefore $JW_s(q) \perp_G Q$ since J preserves TQ because Q is a complex submanifold.

∂ is tangent to the geodesics on P emanating from x_0. Thus because Q is totally geodesic $\partial(q) \in T_q Q$ for all $q \in Q$. Then we have that $\omega(\partial, W_s) = 0$ for all $s \in [0, r]$. Therefore $\lambda(X_q) = 0$ for all $X_q \perp_G T_q Q$. It follows from this and from Lemma 2.1.2 that $\beta(X) = 0$ for all $X \in (TQ)^{\perp_G}$.

Now if we denote by $< \cdot, \cdot >_t$ the norm associated to τ_t, (i.e. $< X, Y >_t = \tau_t(X, JY)$) we have that $< u_t, X >_t = 0$ for all $X \in (TQ)^{\perp_G}$. It follows by Proposition 2.1.2 that $TQ^{\perp_G} = TQ^{\perp_t}$ for all $t \in [0, 1]$. Therefore u_t is tangent to Q for all $t \in [0, 1]$. □

Proposition 2.1.5 *The Liouville vector field ξ_ρ is tangent to Q.*

<u>Proof:</u> Observe that $d\rho^2(X) = J(\xi_\rho \lrcorner \omega_\rho)(X) = \omega_\rho(\xi_\rho, JX) = < \xi_\rho, X >_\rho$ This together with Lemma 2.1.3 implies $< \xi_\rho, X >_\rho = 0$ for all X G-orthogonal to Q and for all $q \in Q$. Hence $< \xi_\rho, X > = 0$ for all X G-

orthogonal to Q by Proposition 2.1.2 . □

Consequently the diffeomorphism Φ_2 takes the submanifold Q into a linear subspace of $\mathbb{R}^{2n}$. This completes the proof of Theorem 2.1.1 .

2.2 An example

We now construct an example that illustrates why is not possible to use McDuff's symplectomorphism to obtain a similar result for the case of a totally geodesic isotropic submanifold.

Consider $\mathbb{R}^4$ with the metric G given by the cartesian product of the Poincaré metric on $\mathbb{R}^2$ with the standard metric on $\mathbb{R}^2$. In polar coordinates (r, θ, s, ϕ) :

$$G = dr^2 + (sinh\, r)^2\, d\theta^2 + ds^2 + s^2\, d\phi^2$$

and the associated Kähler form

$$\omega = (sinh\, r)\, dr \wedge d\theta + s\, ds \wedge d\phi.$$

A source of examples of totally geodesic isotropic submanifolds is provided by the rays through the origin. McDuff constructed a symplectomorphism $\Phi : (\mathbb{R}^4, \omega) \rightarrow (\mathbb{R}^4, \omega_0)$ which fixes the origin. The diffeomorphism Φ^{-1} takes the radial Liouville vector field ξ_0 on $(\mathbb{R}^4, \omega_0)$, which is tangent to the geodesic rays through the origin, to the vector field $\xi = \Phi_*^{-1}\xi_0$ which is a

Liouville field of ω because

$$\begin{aligned} d(\xi \lrcorner \omega) &= d(\Phi_*^{-1}(\xi_0) \lrcorner \omega) = d(\Phi^{-1})^*(\xi_0 \lrcorner \Phi^* \omega) \\ &= (\Phi^{-1})^* d(\xi_0 \lrcorner \omega_0) = (\Phi^{-1})^* \omega_0 \\ &= \omega. \end{aligned}$$

But we claim that in $(\mathbb{R}^2 \times \mathbb{R}^2, G)$ no Liouville vector field points in the direction of the rays , which are the only geodesics through the origin. Hence Φ^{-1} fails to preserve the property of being totally geodesic.

Proof: If $R = \sqrt{r^2 + s^2}$, a general radial vector field through the origin can be written as

$$f \frac{\partial}{\partial R}$$

for some real function $f = f(r, \theta, s, \phi)$.

Suppose that the radial vector field were a Liouville field for ω. Because

$$f \frac{\partial}{\partial R} \lrcorner \omega = f \frac{r}{R} sinh\, r d\theta + f \frac{s^2}{R} d\phi$$

and

$$d(f \frac{\partial}{\partial R} \lrcorner \omega) = \omega$$

we find that the coefficient corresponding to $ds \wedge d\theta$ on the left hand side of the equation must equal zero. Thus:

$$\frac{\partial f}{\partial s} \frac{r}{R} sinh\, r \; + f r\, sinh\, r\, (-\frac{s}{R^3}) = 0$$

which implies

$$\frac{\partial f}{\partial s} = f \frac{s}{R^2} \qquad (2.1)$$

Further the coefficient of $ds \wedge d\phi$ must equal s, then

$$\frac{\partial f}{\partial s}\frac{s^2}{R} + f\frac{2s}{R} - f\frac{s^3}{R^3} \quad = \quad s \tag{2.2}$$

From 2.1 and 2.2 we get

$$\frac{\partial f}{\partial s}\frac{s^2}{R} + f\frac{2s}{R} - f\frac{r}{R^3} = s$$

therefore

$$f \quad = \quad \frac{R}{2} \tag{2.3}$$

Further, we find that the coefficient corresponding to $dr \wedge d\theta$ must equal $sinh\ r$, then

$$\frac{\partial f}{\partial r}\frac{r\, sinh\, r}{R} + f(\frac{sinh\, r}{R} + \frac{r\, cosh\, r}{R} - \frac{r^2\, sinh\, r}{R^3}) \quad = \quad sinh\, r \tag{2.4}$$

and the coefficient corresponding to $dr \wedge d\phi$ must be zero ,then we get

$$\frac{\partial f}{\partial r}\frac{s^2}{R} - f\frac{r s^2}{R^3} = 0$$

which implies

$$\frac{\partial f}{\partial r} \quad = \quad f\frac{r}{R^2} \tag{2.5}$$

From 2.4 and 2.5 it follows that

$$f \quad = \quad \frac{R\, sinh\, r}{sinh\, r + r\, cosh\, r} \tag{2.6}$$

But 2.3 and 2.6 are incompatible , therefore the claim follows. □

¶Consequently , in order to obtain a result similar to Theorem 2.1.1 for the case of an isotropic totally geodesic submanifold we have to change strategy.

Chapter 3

3.1 The local structure of a Liouville vector field

INTRODUCTION

In this chapter we study the local structure of a Liouville vector field of a 2n-dimensional Kähler manifold (P, Ω) , which vanishes on a k-dimensional isotropic submanifold Q of P . We shall not assume that the manifold P has nonpositive curvature, or that Q is totally geodesic.

Recall that a Liouville vector field of (P, Ω) is a smooth vector field ξ such that $\mathcal{L}_\xi \Omega = \Omega$. Its flow φ_t is a conformal transformation of Ω such that $\varphi_t^* \Omega = e^t \Omega$.

We shall assume that at every point q in Q , $\xi(q) = 0$ and the eigenvalues of its 1-jet are 1, 1/2 and 0. This means that in some coordinates

$$J_q^1(\xi) = \sum_{i=1}^{k}(0x_i\frac{\partial}{\partial x_i} + 1y_i\frac{\partial}{\partial y_i}) + \sum_{r=k+1}^{n}(\frac{1}{2}x_r\frac{\partial}{\partial x_r} + \frac{1}{2}y_r\frac{\partial}{\partial y_r})$$

where Q is given by $\{y_i = 0\}_{i=1}^{k} \cap \{x_r = 0\}_{r=k+1}^{n} \cap \{y_r = 0\}_{r=k+1}^{n}$.

Our goal is to prove

Theorem 3.1.1 *There is a C^1-smooth linearizing conjugation between the Liouville vector field ξ and its linear part.*

This means that there is a local diffeomorphism on a neighborhood of q in P which carries the trajectories of the flow generated by the vector field ξ to the trajectories of the flow of its linear part, preserving the direction of motion.

We proceed as follows:

In §3.2 we verify that the flow generated by the Liouville vector field ξ is normally hyperbolic at Q , and that Q is an invariant manifold consisting of fixed points. This means that we can apply the Center Manifold theorem which states that under these conditions P is smoothly foliated by strong unstable submanifolds W_q which are transverse to Q , i.e. $P = \bigcup_{q \in Q} W_q$. Since ξ is tangent to the W_q , we may treat Q as a parameter set and analyze ξ by looking at the family ξ_q of its restrictions to the W_q.

In §3.3 we show that the fibers W_q of the strong unstable foliation are coisotropic and, using this fact, we construct at any point q of Q Darboux coordinates adapted to the foliation.

We study the relation of the Liouville vector field ξ to the Darboux coordinates in §3.4 .

Using this coordinates together with the recipe given by the Poincaré -Dulac theorem [2] we show, in §3.5 , that for each q in Q the vector field ξ_q on W_q contains no quadratic resonant monomials in its Taylor expansion.

Hence we can apply Sell's linearization theorem [13] , which guarantees that for each q , the vector field ξ_q is at least C^1-conjugate to its linear part in a vicinity of the singular point q in Q , and that the linearizing conjugation Φ_q depends smoothly on the parameter q. Putting the Φ_q together, we obtain a C^1-diffeomorphism which conjugates ξ to its linearization.

3.2 The Center Manifold Theorem

¶An *invariant manifold* of a vector field ξ and of the corresponding differential equation $\dot{x} = \xi(x)$ is a submanifold of the phase space P which is tangent to the vector field at each of its points.

Let x_0 be a singular point of a differential vector field ξ (i.e. the vector field ξ vanishes at the point x_0) . By linearizing P in some neighborhood $N(x_0)$ of x_0 , we can consider ξ to be a map $\xi : N(x_0) \to T_{x_0}P \equiv \mathbb{R}^N$. If A denotes the derivative of the mapping $\xi : x \mapsto \xi(x)$,then the equation $\dot{x} = Ax$ is called *the linearization of the equation* $\dot{x} = \xi(x)$*at the singular point* x_0. The vector field Ax is the linear part of the field ξ at x_0. The eigenvalues of A are called *the eigenvalues of* ξ *at the singular point* x_0.

¶Let us denote by φ_t the flow generated by the vector field ξ.

An invariant submanifold S is *normally hyperbolic* if the restriction of the tangent bundle of P to S splits into the continuous subbundles $TP|_S = N^u \oplus TS \oplus N^s$, each of which is invariant by the derivative of φ_t , $D\varphi_t$, and if for each q there is a number $\tau = \tau(q) > 0$ such that

a) $D\varphi_t$ expands N^u more sharply than it expands anything in TS.

b) $D\varphi_t$ contacts N^s more sharply than it contracts anything in TS.

More formally, the submanifold S is *r-normally hyperbolic* if φ_t is C^r , and for all q in S and $0 \leq k \leq r$

$$a)\ m(D\varphi_\tau|_{N_q^u}) > \|D\varphi_\tau|_{T_qS}\|^k$$

$$b)\ \|D\varphi_\tau|_{N_q^s}\|^k < m(D\varphi_\tau|_{T_qS})$$

where $m(L) = inf\{\|Lv\| : \|v\| = 1\}$ is the minimum norm of the linear map L and $\|L\| = sup\{\|Lv\| : \|v\| = 1\}$ is the norm of L.

Theorem 3.2.1 *Let φ_t be a C^r flow on P , for $r \geq 1$, with an r-normally hyperbolic submanifold S consisting of fixed points . Assume further that the splitting $TP|_S = N^u \oplus TS \oplus N^s$ is C^r-smooth. Then*

i) there exist C^r locally φ_t-invariant submanifolds W_S^u and W_S^s tangent at S to $N^u \oplus TS$ and $TS \oplus N^s$ respectively.

ii) W_S^u has a φ_t-invariant fibration $\{W_q^{uu}, q \in S\}$ over S whose fibers are tangent to N^u at S ,i.e. $W_S^u = \bigcup_{q\in S} W_q^{uu}$.

iii) each W_q^{uu} is a C^r-manifold and the map $\pi : W_S^u \to S$ given by $\pi(W_q^{uu}) = q$ is C^r. Points of W_q^{uu} are characterized by the fact that the distance from $\varphi_t(p)$ to $\varphi_t(q)$ goes to zero exponentially fast.

In fact , $d(\varphi_t(p), \varphi_t(q)) \leq Ce^{-kt}$, where C is a constant and $k = \|D\varphi_t(q)|_{N_q^u}\|$.

iv) similarly for the fibration $\{W_q^{ss}, q \in S\}$ of W_S^s.

This theorem is a less general version of the theorem stated in [12] . For a proof see [6].

W_S^u and W_S^s are called the *unstable* and *stable manifolds of* S ; W_q^{uu} and W_q^{ss} are called the *strong unstable* and *strong stable leaves* of the φ_t-invariant fibration through q.

¶In our context, because the Liouville vector field ξ vanishes on the isotropic submanifold Q of (P,Ω) , Q is an invariant manifold of the field ξ which consist of fixed points . The eigenvalues of ξ at a singular point $q \in Q$ are $\{0,1,1/2\}$, so that $m(D\varphi_\tau|_{N_q^u}) = e^{1/2}$ and $\|D\varphi_\tau|_{T_qQ}\| = e^0 = 1$. Therefore Q is r-normally hyperbolic for all $r \in \mathbb{N}$, and the theorem we just stated applies.

Note that $W_Q^u = Q$ so we get a smooth fibration $\pi : P \to Q$ with fibers W_q^{uu} . From now on, we write $W_q = W_q^{uu}$.

3.3 Construction of the Darboux coordinates

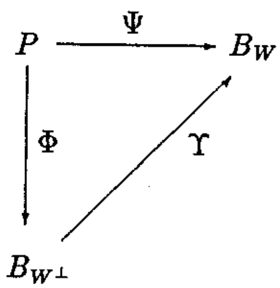
In this section we construct Darboux coordinates adapted to the W_q . More precisely, our aim is to construct coordinates $x_1, ..., x_n, y_1, ..., y_n$ near a point q in Q such that

- $$\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{r=k+1}^{n} 2dx_r \wedge dy_r$$

- $$J_q^1(\xi) = \sum_{i=1}^{k}(0x_i\frac{\partial}{\partial x_i} + 1y_i\frac{\partial}{\partial y_i}) + \sum_{r=k+1}^{n}(\frac{1}{2}x_r\frac{\partial}{\partial x_r} + \frac{1}{2}y_r\frac{\partial}{\partial y_r})$$

- $$W_q = \{p \in P : x_i(p) = x_i(q)\}$$

We show in Proposition 3.3.2 that the submanifolds W_q are coisotropic. This implies that each W_q is itself foliated by isotropic manifolds $W_q^\perp$ where $T_x(W_q^\perp) = (T_x W_q)^\perp$. Then we can form, at least locally, the quotient manifolds $B_W = P/W$ and $B_{W^\perp} = P/W^\perp$ and consider the following diagram



We will show that there are unique Poisson structures on B_W and $B_{W^\perp}$ such that the maps Ψ and Φ are Poisson morphisms and we will then construct the desired coordinates by lifting functions from B_W and $B_{W^\perp}$ to P.

¶Observe that at any point q in Q , the tangent space to P at q can be written as $T_qP = T_qQ \oplus N_q$, where $N_q = (T_qQ)^{\perp\sigma}$.

Lemma 3.3.1 N_q *is a coisotropic vector subspace of* T_qP .

Proof: Consider a non-zero vector v in $N_q^{\perp\Omega}$. Then for all vectors u in N_q we have $\Omega(v,u) = 0$ which implies that $G(Jv,u) = 0$. Thus Jv belongs to $N_q^{\perp G} = T_qQ$. But $J(T_qQ) \subset TQ^{\perp G} = N_Q$ since T_qQ is an isotropic vector space . Hence $v = -J(Jv)$ belongs to N_q . Therefore $N_q^{\perp\Omega} \subset N_q$, which means that N_q is coisotropic. □

Let us denote by φ_s the flow of the Liouville vector field ξ and by π the projection that maps W_q into q . The definition of the strong unstable foliation W_q implies that:

1. $\pi \circ \varphi_s = \pi$, (since if p belongs to W_q for some q so does $\varphi_s(p)$, therefore $\pi(\varphi_s(p)) = \pi(p) = q$) ;
2. if Y is a vector tangent to W_q at a point p , then $(\varphi_s)_*Y$ is tangent to W_q at $\varphi_s(p)$;
3. if $X \in (T_pW_q)^{\perp\Omega}$ then $(\varphi_s)_*X \in (T_{\varphi_s(p)}W_q)^{\perp\Omega}$,(since $\Phi_s^*\Omega = e^s\Omega$).

We can now prove the

Proposition 3.3.2 *The W_q are coisotropic submanifolds of (P,Ω).*

Proof: Assume by contradiction , that for some p in W_q ,there is a non-zero vector v in $(T_pW_q)^{\perp\Omega}$ that is not tangent to T_pW_q . Let α be a smooth curve tangent to v at p , i.e. $\alpha : (-\varepsilon, \varepsilon) \to P$ is a smooth map such that $\alpha(0) = p$ and $\dot\alpha(0) = v$. Since $d(\phi_s(x), \pi(x)) \to 0$ as $s \to -\infty$,

$$d(\varphi_s(\alpha(t)), \pi(\varphi_s(\alpha(t))) = d(\varphi_s(\alpha(t)), \pi(\alpha(t)) \to 0$$

Therefore

$$\lim_{s\to-\infty} \varphi_s(\alpha(t)) = \lim_{s\to-\infty} \pi(\varphi_s(\alpha(t)) = \pi(\alpha(t))$$

pointwise .

Define $\beta(t) = \pi(\varphi_s(\alpha(t))$. This is a curve in Q whose tangent vector at $t = 0$ is $\pi_*(v)$ and hence is non-zero. Further

$$\lim_{s\to-\infty} (\varphi_s)_*\dot{\alpha}(0) = \lim_{s\to-\infty} (\pi \circ \varphi_s)_*\dot{\alpha}(0) = \lim_{s\to-\infty} \pi_*(\dot{\alpha}(0)) = \dot{\beta}(0)$$

By 2. and 3. above, $(\varphi_s)_*(v)$ does not belong to $T_{\varphi_s(p)}W_q$ but it belongs to $(T_{\varphi_s(p)}W_q)^{\perp_\Omega}$. By continuity $\dot{\beta}(0)$ $(= \lim_{s\to-\infty}(\varphi_s)_*(v))$ does not belong to N_q but it belongs to $(N_q)^{\perp_\Omega}$, however this is impossible since N_q is a coisotropic subspace. □

Let us denote by W the foliation of P by the strong unstable manifolds W_q, where W_q is the leaf through a point q in Q. In view of the fact that W is a coisotropic foliation we have

Lemma 3.3.3 $(TW)^\perp$ *considered as a subbundle of* TW *is integrable.*

Proof: Let X_1 , X_2 be vectors fields on W which are sections of $TW^\perp$. We will show that $[X_1, X_2]$ is a section of $TW^\perp$.

Denote by $\Omega|_W$ the pullback of Ω to W. $\Omega|_W$ is a closed form and since W is coisotropic $\Omega|_W(X_i, Y) = 0$ for $i = 1, 2$ and for all Y on W. Now, we have for any vector field Z on W

$$0 = d\Omega|_W(X_1, X_2)$$

$$\begin{aligned} &= X_1\Omega|_W(X_2,Z) - X_2\Omega|_W(X_1,Z) + Z\Omega|_W(X_1,X_2) \\ &\quad -\Omega|_W([X_1,X_2],Z) + \Omega|_W([X_1,Z],X_2) - \Omega|_W([X_2,Z],X_1) \end{aligned}$$

Because

$$\begin{aligned} 0 &= \Omega|_W(X_2,Z) = \Omega|_W(X_1,Z) = \Omega|_W(X_1,X_2) \\ &= \Omega|_W([X_1,Z],X_2) = \Omega|_W([X_2,Z],X_1) \end{aligned}$$

then $\Omega|_W([X_1,X_2],Z)$ must be zero as well, since Z was arbitrary it follows that $[X_1,X_2]$ is a section of $TW^\perp$. □

Consequently, $TW^\perp$ is the tangent bundle to a foliation of W, which we denote by $W^\perp$. Now let q be a point in the isotropic submanifold Q, and let U be a neighborhood of q in P sufficiently small so that the foliations of U defined by $W^\perp$ and W are simple, i.e. the set of the leaves of the foliation are smooth manifolds and the correspondings projections

$$\Phi : U \longrightarrow \frac{P\cap U}{W^\perp \cap U} = B_{W^\perp}$$

and

$$\Psi : U \longrightarrow \frac{P\cap U}{W \cap U} = B_W$$

are submersions.

Then we have

Proposition 3.3.4 *1. There is a unique Poisson structure on $B_{W^\perp}$ such that Φ is a Poisson morphism.*

2. There is a unique Poisson structure on B_W such that Ψ is a Poisson morphism.

In order to prove this proposition we need the following

Definition 3.3.5 *A differentiable function f defined on P is said to be a first integral of a distribution F (or of the foliation defined by F on P) if and only if, for every differentiable section X of F, $Xf = 0$.*

Lemma 3.3.6 *Let F be a smooth and completely integrable distribution. Then a smooth function f is a first integral of F if and only if its restriction to each leaf of the foliation defined by F is constant.*

Proof: Let X be a smooth section of F, if the restriction of f to each leaf of this foliation is constant, then $Xf = 0$ since X is tangent to F.

Conversely, assume that f is a first integral of F. Let S be a k-dimensional leaf of F. For each point x of S there exist k-differentiable sections $X_1, ..., X_k$ of F, defined on a neighborhood of x, whose values at x form a basis of T_xS . Then $X_if = 0$ for all $i = 1, ..., k$. Thus $df|_S$ is constant. □

Lemma 3.3.7 *A function f is a first integral of a distribution F if and only if the Hamiltonian vector field X_f , with hamiltonian function f, is a section of $F^{\perp}$.*

Proof: f is a first integral of $F \Leftrightarrow$ for all sections X of F , $Xf = 0 \Leftrightarrow 0 = df(X) = \{X_f, X\} = \Omega(X_f, X)$ for all X tangent to $F \Leftrightarrow X_f$ is a section of $F^\perp$

□

We can now prove Proposition 3.3.4

Proof: Let F denote the distribution tangent to W or $W^\perp$. Then the distribution $F^\perp$ equals either $W^\perp$ or W and so is integrable. Let Θ denote the projection $P \to P/F = B_F$, so that Θ is either Φ or Ψ .

If $\hat{f}$ and $\hat{g}$ are smooth functions on B_F , then $f = \hat{f} \circ \Theta$ and $g = \hat{g} \circ \Theta$ are smooth functions on P and their restrictions to each fiber $\Theta^{-1}(b)$, $b \in B_F$ are constant functions. Since $\Theta^{-1}(b)$ are the leaves of the foliation defined by F, Lemma 3.3.6 implies that f and g are first integrals of the distribution tangent to F . Therefore the Hamiltonian vector fields X_f and X_g generated by f and g are sections of $F^\perp$ by Lemma 3.3.7 . Since $F^\perp$ is integrable, $X_{\{f,g\}_P} = [X_f, X_g]$ is a smooth section of $F^\perp$, and so by Lemma 3.3.7 , $\{f, g\}_P$ is a first integral of TF.

Lemma 3.3.6 now implies that $\{\hat{f} \circ \Theta, \hat{g} \circ \Theta\}_P$ is constant on each $\Theta^{-1}(b)$, $b \in B^F$, so that it is the pullback of some function on B_F , which we call $\{\hat{f}, \hat{g}\}_{B_F}$. Therefore there exist a Poisson bracket $\{\hat{f}, \hat{g}\}_{B_F}$ on B_F such that

$$\Theta^*\{\hat{f}, \hat{g}\}_{B_F} = \{\hat{f} \circ \Theta, \hat{g} \circ \Theta\}_P$$

Clearly this bracket satisfies the Jacobi identity. □

At each point $b \in B_F$ the value of $\{f, g\}$ and hence of X_f depends only on the differential of f, so there is a bundle map $\Lambda^F : T^*B_F \to TB_F$ such that $X_f = \Lambda^F \circ df$ for all f.

The rank of the Poisson structure at a point $b \in B_F$ is defined to be the rank of $\Lambda^F : T^*B_F \to TB_F$.

We have the isomorphisms:

$$T_b^*B_F \cong \frac{T_x^*P}{T_x^*F} \cong (F_x)^\circ \cong F_x^\perp$$

$b = \Theta(x)$ where $(F_x)^\circ$ is the annihilator of F_x (i.e. the set of 1-forms that vanish on F_x).

This isomorphism may be extended into an isomorphism of the exterior algebras which maps the Poisson tensor Λ^F of B_F at Θ_x into the bilinear 2-form $\Omega_{F^\perp}$ induced by Ω on the leaf of the foliation defined by $F^\perp$.

Consequently the rank of the Poisson structure of $B_{W^\perp}$ at $\Phi(x)$ is equal to the rank of the 2-form $\Omega|_W$ induced by Ω on the leaf W_x through x of the foliation defined by W , which equals $2n - 2k$.

Similarly the rank of the Poisson structure of B_W at $\Psi(x)$ equals the rank of $\Omega|_{W_x^\perp}$, which is zero since $W_x^\perp$ is isotropic.

Hence there are k coordinates functions $\check{x}_i$ $i = 1, ..., k$ on B_W such that their Poisson brackets vanish , i.e. $\{\check{x}_i, \check{x}_j\} = 0$; $i, j = 1, ..., k$.

¶Definition of the functions x_i :

Define on P, the functions $x_i = \check{x}_i \circ \Psi$; $i, j = 1, ..., k$. Observe that each of the functions x_i is constant when restricted to each leaf of the foliation defined by W . Thus if $x = \Psi(q) \in B$ we have

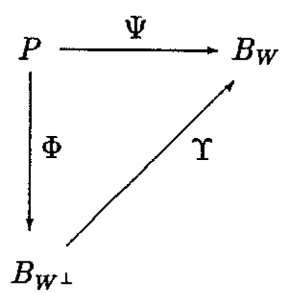
$$\Psi^{-1}(x) = W_q \cap U = \{p \in U \ : \ x_i(p) = x_i(q)\}.$$

Remark 3.3.8 *The functions x_i ; $i = 1, ..., k$ are first integrals of W by Lemma 3.3.6 and for each $i = 1, ..., k$ the Hamiltonian vector field ξ_{x_i} generated by x_i, is a section of $W^\perp$. Hence the $\{\xi_{x_i}\}_{i=1}^{k}$ span $TW^\perp$.*

Because the foliations of U defined by W and $W^\perp$ are simple and for each $p \in U$ we have

$$\frac{T_pP}{T_pW} \cong \frac{T_pP/T_pW^\perp}{T_pW/T_pW^\perp}$$

we can define a function $\Upsilon : B_{W_\perp} = P/W^\perp \to B_W = P/W$ such that the diagram



commutes. Therefore we have k functions on $B_W^\perp$ defined by $\hat{x}_i = \check{x}_i \circ \Upsilon$ $i = 1, ..., k$.

¶Now, for each q in $Q \cap U$ we may form the quotient manifold

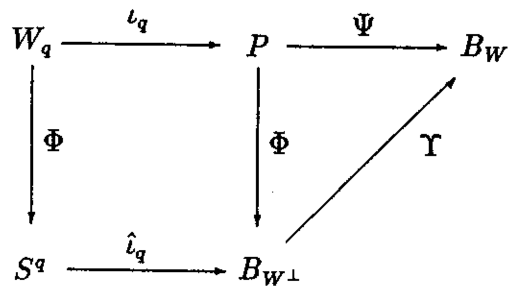
$$S^q = \frac{W_q \cap U}{W_q^\perp \cap U}$$

contained in $B_{W^\perp}$. Observe that since $\Phi(W_q \cap U) = S^q$ we have

$$S^q = \{b \in B_{W^\perp} \ : \ \hat{x}_i(b) = \hat{x}_i(\Phi(q))\}$$

The tangent space to S^q at a point $b \in B_{W^\perp}$ may be identified with $T_pW_q/T_pW_q^\perp$ for some $p \in P$ such that $\Phi(p) = b$.

Thus we have the commutative diagram



Let σ^q be the 2-form defined on $T_pW_q/T_pW_q^\perp$ by

$$\sigma^q(X + W_q^\perp, Y + W_q^\perp) = \Omega|_{W_q}(X, Y)$$

where X , Y lie in TW_q .

Note that

1. σ^q is well defined : (i.e. it does not depend on the point chosen on the leaf $TW_q^\perp$)

   Proof: For any section X of $TW_q^\perp$, we have

   $$\mathcal{L}_X(\Omega|_{W_q}) = d(X \lrcorner \Omega|_{W_q}) + X \lrcorner d(\Omega|_{W_q}) = 0$$

   since X lies in $TW_q^\perp$ and $d\Omega|_{W_q} = 0$. $\square$

2. σ^q is non-degenerate :

   Proof: Given $Y \in TW_q - TW_q^\perp$ there exist $X \in TW_q - TW_q^\perp$ such that $\Omega(X, Y) \neq 0$ therefore $\sigma^q(X + W_q^\perp, Y + W_q^\perp) \neq 0$. $\square$

3. σ^q is closed :

Proof:

$$d\sigma^q(X + W_q^\perp, Y + W_q^\perp, Z + W_q^\perp) = \Phi^*(d\sigma^q)(X,Y,Z)$$

$$= d(\Phi^*\sigma^q)(X,Y,Z) = d\Omega|_{W_q}(X,Y,Z) = 0$$

□

It follows that there is a naturally determined symplectic structure on S^q for each $q \in Q \cap U$, whose pullback to W_q is $\Omega|_{W_q}$.

Proposition 3.3.9 *For each $q \in Q \cap U$, S^q is a Poisson submanifold of the Poisson manifold $B_{W^\perp}$.*

Lemma 3.3.10 *S is a Poisson submanifold of a Poisson manifold B if and only if each tangent space T_bS contains the image of $\Lambda_b : T_b^*B \to T_bB$, i.e. if and only if all hamiltonian vector fields are tangent to S.*

Proof: Given functions $\hat{f}$ and $\hat{g}$, extend them to functions f and g on B . The tangency condition implies that the restriction of $\{f,g\} = X_f(g)$ to S depends only on $\hat{f}$ and $\hat{g}$ so there is an induced bracket operation on S for which the inclusion $\iota : S \hookrightarrow B$ is a Poisson mapping.

Conversely , if the tangency condition fails , the bracket of extended functions depends upon the extensions , so the restriction map $C^\infty(B) \to C^\infty(S)$ cannot define a homomorphism for any Lie algebra structure on

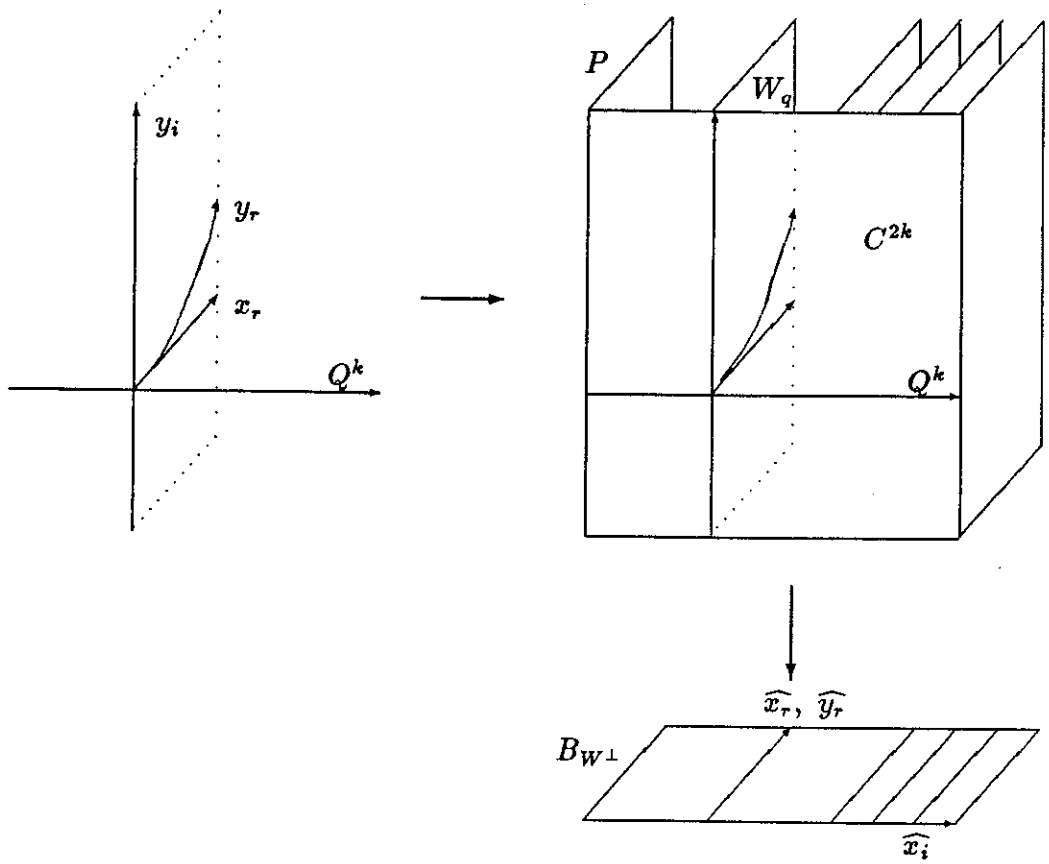
$C^\infty(S)$. □

We now prove the proposition 3.3.9

Proof: Given any function $\widehat{f}$ on $B_{W^\perp}$, the function $f = \widehat{f} \circ \Phi$ defined on P is constant on $W_q^\perp$ for each q. Since the hamiltonian vector fields $\{\xi_{x_i}\}_{i=1}^k$ span $TW^\perp$ by Remark 3.3.8 we have that $\xi_{x_i} f = 0 \; \forall i = 1, ..., k$, therefore $\{x_i, f\}_P = 0 \; \forall i$. This implies that the derivative of the x_i in the direction of the hamiltonian vector field generated by f is zero, i.e. $\xi_f(x_i) = 0$, which implies that $dx_i(\xi_f) = 0$. Therefore the hamiltonian vector field ξ_f is tangent to W_q for each q. Hence the hamiltonian vector field generated by $\widehat{f}$, $\xi_{\widehat{f}} = \Phi_*(\xi_f)$ is tangent to $S^q = \Phi(W_q)$. Since $\widehat{f}$ was an arbitrary function the proposition holds. □

We can consider q as a parameter and use Arnold's method to find Darboux coordinates on each S^q separately in such a way that they vary smoothly with respect to the parameter q. Denote them by x_r^q, y_r^q, $r = k+1, ..., n$. We can choose them such that $\{x_r^q, y_r^q\} = 2\delta_{rs}$.

Now define on $B_{W^\perp}$ functions $\widehat{x_r}, \widehat{y_r}$, $r = k+1, ..., n$ so that the restriction $\widehat{x_r}|_{S^q}$ of $\widehat{x_r}$ to S^q equals x_r^q and the restriction $\widehat{y_r}|_{S^q}$ of $\widehat{y_r}$ to S^q equals y_r^q.



Proposition 3.3.11 *The functions $\widehat{x_r}$, $\widehat{y_r}$, $r = k+1, ..., n$ together with the functions $\widehat{x_i}$, $i = 1, ..., k$ form a coordinate system which satisfy the following bracket relations :* $\{\widehat{x_s}, \widehat{y_r}\} = 2\delta_{sr}$

$$\{\widehat{y_s}, \widehat{y_r}\} = \{\widehat{x_s}, \widehat{x_r}\} = \{\widehat{y_s}, \widehat{x_i}\} = \{\widehat{x_s}, \widehat{x_i}\} = \{\widehat{x_j}, \widehat{x_i}\} = 0$$

for all $s, r = k+1, ..., n$ and $i, j = 1, ..., k$.

Proof: By the Proposition 3.3.9 , the hamiltonian vector fields $\xi_{\widehat{y_r}}$ and $\xi_{\widehat{x_r}}$ generated repectively by the functions $\widehat{y_r}, \widehat{x_r}$, $r = k+1, ..., n$ are tangent to S^q. Since each $\widehat{x_i}$ is constant on S^q we have that

$$0 = \xi_{\widehat{y_r}}\widehat{x_i} = \{\widehat{x_i}, \widehat{y_r}\}_{B_W^\perp}$$

and

$$0 = \xi_{\widehat{x_r}}\widehat{x_i} = \{\widehat{x_i}, \widehat{x_r}\}_{B_W^\perp}$$

for all $i = 1, ..., k$ and $r = k+1, ..., n$. Also

$$\{\widehat{x_j}, \widehat{x_i}\}_{B_W^\perp} = \{\check{x}_j, \check{x}_i\}_{B_W} \circ \Upsilon = 0 \ .$$

And finally

$$\{\widehat{x_r}, \widehat{y_s}\}_{B_W^\perp} \circ \iota_q = \{x_r^q, y_s^q\} = 2\delta_{sr}.$$

Therefore for all $b \in S^q$ and all $q \in Q \cap U$ we have $\{\widehat{x_r}, \widehat{y_s}\}_{B_W^\perp}(b) = 2\delta_{sr}$. The other brackets relations can be obtained in a similar way. □

¶<u>Definition of the functions x_r, y_r :</u>

We define the functions

$x_i = \widehat{x_i} \circ \Phi$, $i = 1, ..., k$

$x_r = \widehat{x_r} \circ \Phi$, $y_r = \widehat{y_r} \circ \Phi$; $r = k+1, ..., n.$

Since Φ is a Poisson morphism, it follows that for each p in U :

$\{y_s, x_r\}_P(p) = \{\widehat{y_s} \circ \Phi, \widehat{x_r} \circ \Phi\}_P(p) = \{\widehat{y_s}, \widehat{x_r}\}_{B_{W^\perp}} \Phi(p) = 2\delta_{sr}\Phi(p)$

$\{y_s, y_r\}_P(p) = \{\widehat{y_s} \circ \Phi, \widehat{y_r} \circ \Phi\}_P(p) = \{\widehat{y_s}, \widehat{y_r}\}_{B_{W^\perp}} \Phi(p) = 0$

In a similar manner we get the remaining bracket relations.

Summarizing we have, on the neighborhood U functions x_i , x_r , y_r $i = 1, ..., k$; $r = k+1, ..., n$ satisfying the relations :

$$\clubsuit \begin{cases} \{x_r, y_s\} = 2\delta_{rs} \\ \{y_s, y_r\} = \{x_s, x_r\} = \{y_s, x_i\} = \{x_s, x_i\} = \{x_j, x_i\} = 0 \\ \text{for all } s, r = k+1, .., n \text{ and } i, j = 1, ..., k \end{cases}$$

Further the coisotropic leaf through the point $q \in Q \cap U$, is given by $W_q = \{p \in P \ : \ x_i(p) = x_i(q)\}$.

Let us denote by C the set of points in U whose x_r and y_r-coordinates $r = k+1, ..., n$ vanish . Notice that this is the union of the isotropic leaves $W_q^\perp$ intersecting the isotropic submanifold $Q \cap U$. We want to show that there is a neighborhood V of q_0 contained in C, such that the restriction of Ω to V is symplectic . Since being symplectic is an open condition it suffices to show the following

Proposition 3.3.12 *The restriction of Ω to the tangent space to C at q_0 is nondegenerate.*

Proof: Let ξ_{x_r} and ξ_{y_r} ; $r = k+1, ..., n$ be the hamiltonian vector fields corresponding to the hamiltonian functions x_r , y_r respectively, i.e. $\xi_{x_r} \lrcorner \Omega = dx_r$, $\xi_{y_r} \lrcorner \Omega = dy_r$ for all r.

Let X be a vector tangent to C at q_0 ,and take any extension of X to a neighborhood of q_0. Since Ω depends only on the values of the vector field at the point , we have that

$\Omega(X,\xi_{x_r}) = Xx_r = 0$ and $\Omega(X,\xi_{y_r}) = Xy_r = 0$ for all $r = k+1,...,n$ because $x_r = y_r = 0$ on C and $X \in T_{q_0}C$. Therefore X is Ω-orthogonal to the vectors $\xi_{x_r}(q_0)$, $\xi_{y_r}(q_0)$ for all $r = k+1,...,n$. Note that these vectors span a subspace complementary to $T_{q_0}C$ since their projections onto $B_{W^\perp}$ span $T(S_q)$. Therefore, if X were Ω-orthogonal to the whole $T_{q_0}C$, X would be orthogonal to $T_{q_0}P$, which is not possible because Ω is nondegenerate. □

By Darboux Theorem there are symplectic coordinates in a neighborhood V of the point q_0 in C.

Let us denote them by $(\bar{x}_1,...,\bar{x}_k,\bar{y}_1,...,\bar{y}_k)$. It is possible to choose them such that $\bar{x}_i(p) = x_i(p)$, $p \in V$ and such that $\frac{\partial}{\partial \bar{y}_i}$ is tangent to $\frac{\partial}{\partial \hat{y}_i}$.

¶Definition of the functions y_i :

Extend the coordinates ($\bar{x}_i$ and) $\bar{y}_i$, $i = 1,...,k$ from V to U using the hamiltonian flows of x_r , y_r ; $r = k+1,...,n$ in the following way :

Denote by $g_{x_r}^{t_r}$, $g_{y_r}^{s_r}$; $r = k+1,...,n$ the hamiltonian flows with hamiltonian functions x_r , y_r respectively.

Because the Poisson brackets of the hamiltonian functions are constant (♣-relations) , their flows commute. Therefore every point p in some neighborhood $U_1 \subset U$ of the point $q_0 \in Q$, can be uniquely represented in the form

$$p = g_{x_{k+1}}^{t_{k+1}} \circ ... \circ g_{x_n}^{t_n} \circ g_{y_{k+1}}^{s_{k+1}} \circ ... \circ g_{y_n}^{s_n}(p') \stackrel{\text{def}}{=} \theta(p')$$

where p' belongs to the neighborhood $V \subset C$ and s_r , $t_r \in \mathbb{R}$; $r = k+1,...,n$.

Define : $y_i(p) = \bar{y}_i(p')$ and $c_i(p) = \bar{x}_i(p')$, $i = 1,...,k$.

We claim that the values of the extensions c_i of $\bar{x}_i$ agree with those of the old functions x_i

Proof: For the ♣-relations $0 = \{c_i, x_r\} = \xi_{x_r} c_i$ and $0 = \{c_i, y_r\} = \xi_{y_r} c_i$ $i = 1, ..., k$; $r = k+1, ..., n$ imply that each c_i is a first integral of the hamiltonian functions x_r , y_r ; $r = k+1, ..., n$ and so is constant along the corresponding hamiltonian flows . Similar remarks apply to the x_i. Thus $x_i(p') = c_i(p') = c_i(\theta(p')) = c_i(p)$ and $x_i(p') = x_i(\theta(p')) = x_i(p)$ so that $x_i(p) = c_i(p)$. □

¶Proof that the coordinates are symplectic

Because of the way the functions $\bar{y}_i$ $i = 1, ..., k$ were extended , each of this extensions y_i is invariant with respect to the flows $g^{t_r}_{x_r}$, $g^{s_r}_{y_r}$; $r = k+1, ..., n$.

Thus the Poisson brackets of the y_i with the x_r , y_r are equal to zero : $\{y_i, x_r\} = \{y_i, y_r\} = 0$ $i = 1, ..., k$; $r = k+1, ..., n$.

Therefore the hamiltonian flows $g^{s}_{y_i}$ with hamiltonian functions y_i $i = 1, ..., k$ commute with the $g^{t_r}_{x_r}$, $g^{s_r}_{y_r}$.

Since the hamiltonian flows preserve the symplectic form Ω, we have that the values of Ω on the hamiltonian vector fields at p are the same of those at p' , and these equal the values of the Poisson brackets, i.e.

$$\{y_i, y_r\}(p) = \Omega(\xi_{y_i}(p), \xi_{y_r}(p)) = \Omega(\xi_{y_i}(p'), \xi_{y_r}(p')) = \{y_i, y_r\}(p').$$

Further because $0 = \{x_r, y_i\} = \Omega(\xi_{x_r}, \xi_{y_i})$, and similarly $0 = \{y_r, y_i\} = \Omega(\xi_{y_r}, \xi_{y_i})$ $i = 1, ..., k$; $r = k+1, ..., n$, the functions x_r and y_r are first integrals of the hamiltonian vector fields ξ_{y_i}. Therefore the ξ_{y_i} restrict to hamiltonian vector fields on the symplectic manifold $(V, \Omega|_V)$ and the corresponding hamiltonian functions are $y_i|_V = \bar{y}_i$.

Thus in the whole neighborhood $U_1 \subset U$, the Poisson brackets of the y_i with themselves is the same as the Poisson brackets of this coordinates in (V, Ω), i.e. $\{y_i, y_r\}(p') = \{\bar{y}_i, \bar{y}_r\}(p')$.

Hence on the neighborhood $U_1 \subset P$ of q_0 $\{y_i, y_r\}(p) = \{\bar{y}_i, \bar{y}_r\}(p') = 0$. Similarly we get $\{y_i, x_r\}(p) = \delta_{ir}$. We already had $\{x_i, x_r\}(p) = 0$.

The Poisson bracket of the coordinate functions determine the shape of the symplectic form uniquely , thus

$$\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{r=k+1}^{n} 2dx_r \wedge dy_r$$

in the neighborhood U_1 of q_0 .

3.4 The relation with the Liouville vector field

In this section we study the relation of the Liouville vector field ξ to the Darboux coordinates adapted to the foliation W_q. Our aim is to show that the linear part of ξ at the point q_0 is diagonal with respect to these coordinates.

Let us write $\xi = L(\xi) + \eta$ where $L(\xi)$ denotes the linear part of ξ at q_0 and η contains terms of order ≥ 2 only. Then, for $q_0 \in Q$ we have that

$d(\eta \lrcorner \Omega)(q_0) = 0$ and $d(L(\xi) \lrcorner \Omega)(q_0) = \Omega(q_0)$. Therefore since

$d(\xi \lrcorner \Omega) = d(L(\xi) \lrcorner \Omega) + d(\eta \lrcorner \Omega) = \Omega$ and Ω is linear it follows that

$d(\eta \lrcorner \Omega) = 0$ and $d(L(\xi) \lrcorner \Omega) = \Omega$.

Now choose new coordinates $(z_1, ..., z_{2n})$ in a neighborhood of q_0 in P which depend linearly on the coordinates (x_i, y_i, x_r, y_r) such that $L(\xi)$ is diagonal with respect to the $(z_1, ..., z_{2n})$. We will take $z_i = x_i$ for $i = 1, ..., k$. Then

$$L(\xi) = \sum_{\ell=1}^{2n} \lambda_\ell z_\ell \frac{\partial}{\partial z_\ell}.$$

Further, because ξ preserves the submanifolds W_q and $W_q^\perp$ each of the subspaces $T_q(W_q)$ and $T_q(W_q^\perp)$ have a basis consisting of some of the $\frac{\partial}{\partial z_\ell}$ which diagonalize $L(\xi)$. We will assume that $\{\frac{\partial}{\partial z_j}\}_{j=k+1}^{2k}$ span $T_q(W_q^\perp)$. Therefore

$$\frac{\partial}{\partial y_i} = \sum_{j=k+1}^{2k} A_{ij} \frac{\partial}{\partial z_j}$$

and the transformation matrix has the form

$$\begin{pmatrix} \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial w_s} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_i} \\ \frac{\partial}{\partial z_j} \\ \frac{\partial}{\partial z_\alpha} \end{pmatrix}$$

$$\text{where} \begin{cases} i = 1, ..., k \\ j = k+1, ..., 2k \\ s, \alpha > 2k \end{cases} \text{and } \frac{\partial}{\partial w_s} = \begin{cases} \frac{\partial}{\partial x_r} & for\ s = k + r, \\ \frac{\partial}{\partial y_r} & for\ s = n + r, \\ where & r = k + 1, .., n \end{cases}$$

Thus

$$\begin{aligned} x_i &= z_i \\ y_i &= \sum_{k<i\leq 2k} a_i^j z_i + \sum_{\alpha>2k} a_\alpha^j z_\alpha \\ x_r &= \sum_{\alpha>2k} b_\alpha^r z_\alpha \\ y_r &= \sum_{\alpha>2k} c_\alpha^r z_\alpha \end{aligned}$$

for some constants $a_i^j, a_\alpha^j, b_\alpha^r, c_\alpha^r$.

Thus with respect to the new coordinates

$$\Omega = \sum_{i=1,j=k+1}^{k,2k} a_j^i dz_i \wedge dz_j + \sum_{i=1,\alpha>k}^{k} a_\alpha^i dz_i \wedge dz_\alpha + 2\sum_{\alpha,\beta} b_\alpha^r c_\beta^r dz_\alpha \wedge dz_\beta$$

Hence we obtain

$$\begin{aligned} d(\mathrm{L}(\xi) \lrcorner \Omega) &= \sum (\lambda_i + \lambda_j) a_j^i dz_i \wedge dz_j + \sum (\lambda_i + \lambda_\alpha) a_\alpha^i dz_i \wedge dz_\alpha \\ &\quad + 2\sum (\lambda_\alpha + \lambda_\beta) b_\alpha^r c_\beta^r dz_\alpha \wedge dz_\beta \end{aligned}$$

Because $\lambda_i = 0$ for $i = 1, \ldots, k$, $d(\mathrm{L}(\xi) \lrcorner \Omega) = \Omega$ implies that $\lambda_j = 1$ for all $a_j^i \neq 0$, $\lambda_\alpha = 1$ for all $a_\alpha^i \neq 0$ and $\lambda_\alpha + \lambda_\beta = 1$ for all $b_\alpha^r c_\beta^r \neq 0$ By hypothesis there are k eigenvalues equal to 1 and $2n - 2k$ equal to $1/2$. Since the matrix corresponds to a change of coordinates, the submatrix (a_i^j) is non-singular. Therefore all the $\lambda_j = 1$, and all a_α^i are zero.

Hence the eigenspace for $L(\xi)$ corresponding to the eigenvalue $\lambda = 1/2$ is exactly the space spanned by $\{\frac{\partial}{\partial x_r}, \frac{\partial}{\partial y_r}\}_{r=k+1}^{n}$ and the eigenspace corresponding to the eigevalue $\lambda = 1$ is exactly the space spanned by $\{\frac{\partial}{\partial y_i}\}$.

3.5 The linearizing conjugation

In order to prove that the Liouville vector field ξ is smoothly conjugate to its linear part, we shall use a linearization theorem due to G.Sell [13], which extends the linearization theorem of Sternberg to the case of a vector field with resonant eigenvalues.

We shall exploit the fact that the vector ξ is Liouville , together with the explicit algorithm that Sell's Theorem provides to compute a lower bound for the order of smoothness of the conjugacy.

In the Darboux coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ about a singular point q_0 , adapted to the strong unstable coisotropic foliation constructed in §2 , the isotropic submanifold Q is given by

$$Q \cap U = \{y_i = 0\}_{i=1}^{k} \cap \{x_r = 0\}_{r=k+1}^{n} \cap \{y_r = 0\}_{r=k+1}^{n} ,$$

the leaf through a point q in $Q \cap U$ is given by

$$W_q \cap U = \{p \in U \ : \ x_i(p) = x_i(q) \ i = 1, ..., k\}$$

and all the coordinate functions vanish at the singular point q_0.

For each q the integral curves of the vector field $\xi|_{W_q}$ satisfy the following equation

$$(I) \qquad\qquad \dot{w} = A(q)w + F(w, q)$$

where $A(q)$ and $F(w, q)$ depend smoothly on $q \in Q \cap U$ and $w \in U$, where $w = (x_{k+1}, ..., x_n, y_1, ..., y_n)$. Furthermore the matrix $A(q)$ does not have zero eigenvalues , that is, $A(q)$ is hyperbolic. Because the eigenvalues are positive $A(q)$ is said to be unstable.

In order to go further we need the following

Definition 3.5.1 *A collection of non-zero eigenvalues is resonant if one of them is an integral linear combination (with nonnegative coefficients whose sum is at least two) of the others.*

i.e. Let $\lambda_1, ..., \lambda_N$ *be a set of non-zero eigenvalues repeated with multiplicities and let* $m = (m_1, ..., m_N)$ *be nonnegative integers.*

Define $|m| = \sum m_i$ *and* $\gamma(\lambda_i, m) = \lambda_i - \sum m_r \lambda_r$.

Then if a relation $\gamma(\lambda_i, m) = 0$ *holds for* $|m| \geq 2$ *the eigenvalues are said to be in resonance , and* $|m|$ *is called the order of resonance.*

If $(z^1, ..., z^N)$ are coordinates with respect to the basis $(e_1, ..., e_N)$, let z^m stand for $z_1^{m_1} ... z_N^{m_N}$.

Definition 3.5.2 *The vector valued monomial* $z^m e_i$ *is resonant if* $\gamma(\lambda_i, m) = 0$ *and* $|m| \geq 2$.

When A is hyperbolic, let $\sum^+(A)$ denote those eigenvalues λ of A with $Re\lambda > 0$ and $\sum^-(A)$ those with $Re\lambda < 0$. If $\sum^i(A) \neq \emptyset$ where $i = +$ or $-$, *the spectral spread* is defined to be

$$\rho^i = \frac{max\{|Re\lambda| \ : \ \lambda \in \sum^i(A)\}}{min\{|Re\lambda| \ : \ \lambda \in \sum^i(A)\}}.$$

Definition 3.5.3 *The r-smoothness of A is the largest integer* $K \geq 0$ *such that*

1. $r - K\rho^- \geq 0$, *if* $\sum^+(A) = \emptyset$

2. $r - K\rho^+ \geq 0$, *if* $\sum^-(A) = \emptyset$

3. There exist positive numbers M, N with $r = M + N$, $M - K\rho^+ \geq 0$, $N - K\rho^- \geq 0$ if $\sum^+(A) \neq \emptyset$ and $\sum^-(A) \neq \emptyset$.

Now suppose that the following condition holds for some integer $r \geq 2$. (This is condition "B" in [13]).

$$(II)\begin{cases} D^j F(q_0, q) = 0 & \text{for } O \leq j \leq r-1 \\ \text{and } Re\gamma(\lambda, m) \neq 0 & \text{for all } \lambda \in \sum A(q) = \sum^+ A(q) \cup \sum^- A(q), \\ \text{for all } m \text{ with } |m| = r & \text{and for all } q \in \hat{V} \text{ neighborhood of } q_0 \end{cases}$$

Then Sell's Theorem asserts that there is a C^K-smooth linearizing conjugation $x = z + \Phi(z, q)$ between (I) and $\dot{z} = A(q)z$, where Φ varies smoothly in terms of the parameter q , and is of class C^K in z , where K is the r-smoothness of $A(q_0)$.

Samovol (see [14]) proves a more general theorem which implies Sell's theorem, but we use Sell's version because it describes the dependence of the linearizing conjugation on the parameter set.

¶In the situation we are considering , the eigenvalues of the Liouville vector field ξ (i.e. the eigenvalues of $A(q)$) at the singular point q satisfy the integral relations

$$\lambda_i = 1\lambda_r + 1\lambda_s$$

where $\lambda_i = 1$, $i = 1, ..., k$ $\lambda_r = \lambda_s = \frac{1}{2}$, $r, s = k+1, ..., n$.

According to definition 3.5.2 , the possible resonant monomials in each fiber are

$$(III) \qquad x_r x_s \frac{\partial}{\partial y_i} \ , \ y_r y_s \frac{\partial}{\partial y_i} \ , \ x_r y_s \frac{\partial}{\partial y_i}$$

for $i = 1, ..., k$; $r, s = k+1, ..., n$.

Recall that the linear part $L(\xi)$ of the Liouville vector field ξ is

$$L(\xi) = \sum_{i=1}^{k} y_i \frac{\partial}{\partial y_i} + \sum_{r=k+1}^{n} (\frac{1}{2} x_r \frac{\partial}{\partial x_r} + \frac{1}{2} y_r \frac{\partial}{\partial y_r})$$

Thus

$$\xi = L(\xi) + \sum_{r=k+1}^{n} E_r \frac{\partial}{\partial x_r} + \sum_{\ell=1}^{n} F_\ell \frac{\partial}{\partial y_\ell}$$

where for each $q = (x_1, ..., x_k)$ the functions $E_r(q, \cdot)$, $F_\ell(q, \cdot)$ vanish to higher order than $(\sum_{r=k+1}^{n} |x_r|^2 + \sum_{l=1}^{n} |y_\ell|^2)^{1/2}$.

Denote by ξ^q the restriction of ξ to W_q

Proposition 3.5.4 *For each q the Taylor expansion of ξ^q contains no resonant quadratic terms.*

Proof: The Taylor expansions of ξ^q fit together to give a Taylor expansion of the vector field ξ in terms of the coordinates y_i , x_s , y_s , $i = 1, ..., k$; $s = k+1, ..., n$, with coefficients which are functions of the x_i , $i = 1, ..., k$.

It suffices to show that this expansion has no terms of the form (III).

Notice that the vector field $\eta = \xi - L(\xi)$ is hamiltonian since

$$\mathcal{L}_\eta \Omega = d([\xi - L(\xi)] \lrcorner \Omega) = \Omega - \Omega = 0$$

Let H be a hamiltonian function for η (i.e. $\eta \lrcorner \Omega = dH$)

If η (therefore ξ) had any resonant monomial , then the usual Taylor expansion of the hamiltonian function H in terms of all the x_l, y_l would contain nonzero terms of the type :

$$x_i x_r x_s \ , \ x_i x_r y_s \ , \ x_i y_r y_s$$

for $1 \leq i \leq k$; $k+1 \leq s, r \leq n$. Consequently η would also contain terms of the type

$$x_i x_s \frac{\partial}{\partial x_r} \ , \ x_i x_s \frac{\partial}{\partial y_r} \ , \ x_i y_s \frac{\partial}{\partial x_r} \ \text{or} \ x_i y_s \frac{\partial}{\partial y_r}$$

which is impossible since the functions E_r and F_l do not contain terms which depend linearly on x_s or y_s , $k+1 \leq s \leq n$.

Note that for each leaf W_q , the x_i are constants and so are coefficients. □

¶<u>Construction of the conjugacy</u>

Proposition 3.5.4 implies that

$$D^0 F(q_0, q) = D^1 F(q_0, q) = 0$$

Hence condition (II) holds for $r = 2$ at least.

Since the eigenvalues of $A(q_0)$ are 1 or 1/2 , the spectral spread ρ^+ equals 2. Thus , because $\sum^- A(q_0) = \emptyset$ the r-smoothness of $A(q_0)$ is the largest integer K , $K \geq 0$, such that $2 - K2 \geq 0$. Thus $K = 1$. Hence Sell's theorem guarantees that the linearizing conjugation Φ is at least of class C^1. This completes the proof of Theorem 3.1.1.

Chapter 4

4.1 The isotropic case

In this chapter we extend McDuff's results to other special submanifolds of a Kähler manifold, the isotropic ones.

As in Chapter 1 we assume that P is a complete, simply connected, Kähler manifold of dimension $2n$ with non-positive sectional curvature and we denote the Kähler form by ω. Now let Q be a totally geodesic isotropic properly embedded k-dimensional submanifold . Our aim is to prove the

Theorem 4.1.1 *(P, Q, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \mathbb{R}^k, \omega_0)$, where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$*

This Theorem is a natural extension for the extreme cases proved by McDuff when the dimension of Q is zero (i.e. Q is a point) and when the dimension of Q is n (i.e. Q is lagrangian).

We proceed as follows :

In §4.2 we construct a diffeomorphism Φ_1 which replaces the given Kähler form by an equivalent form which arises from the Levi form of the distance function ρ to the submanifold Q.

In §4.3 we compute the 1-jet of the Liouville vector field ξ_ρ defined by $\xi_\rho \lrcorner \omega_\rho = -\frac{1}{2} J d\rho^2$ and apply the results obtained in Chapter 3.

We complete the proof, by showing that the symplectic form ω_ρ is diffeomorphic to the standard one in $\mathbb{R}^{2n}$. We call this diffeomorphism Φ_2.

4.2 Construction of Φ_1

We need a comparison theorem which estimates the Levi form of the distance function from $x \in P$ to Q , in terms of the original Kähler metric. The arguments here are very similar to those in [9] .

Define $\rho(x) = dist(x, Q)$ for x in P , and let $G_\rho(X,Y) = -dJd\rho^2(X, JY)=$ the Levi form of the function ρ^2, where J is the canonical almost complex structure on TP . Denote by G the Kähler metric on P . Then we have :

Lemma 4.2.1 *There is a constant* $\epsilon > 0$ *such that* $G_\rho \geq \epsilon\, G$.

Proof: We will compare G_ρ with the Levi form of r^2 on $\mathbb{C}^n$ where $r(z) = dist(z, \mathbb{R}^k)$; $z \in \mathbb{C}^n$, $\mathbb{R}^k \subset \mathbb{C}^n$. Because G is a Kähler metric

$$G_\rho(X,X) = D^2\rho^2(X,X) + D^2\rho^2(JX,JX)$$

where $D^2\rho(X,Y) = X(Yf) - (\nabla_X Y)f =$ Hessian of ρ^2.

We also have $D^2\rho^2 = 2[d\rho \otimes d\rho + \rho\, D^2\rho]$ (see [4]).

Let $x \in P - Q$ and denote by ∂ the gradient vector field of ρ with respect to the metric G. Let X be a non-zero vector at x orthogonal to ∂ . Then since $X\rho = 0$

$$\begin{aligned} D^2\rho(X,\partial(x)) &= -(D_X\partial)\rho = -< D_X\partial, grad\,\rho > \\ &= -< D_X\partial, \partial > = -\frac{1}{2}X < \partial, \partial > = 0. \end{aligned}$$

Thus the $G-$orthogonal direct sum $P_x = span\partial(x) \oplus \partial^{\perp}$ is also orthogonal with respect to the form $D^2\rho$. Therefore because

$$D^2\rho(\partial,\partial) = \partial(\partial\rho) - (\nabla_{\partial}\partial)\rho = \partial(1) = 0 \ ,$$

it is enough to prove that there is a constant $\epsilon > 0$ so that $D^2\rho(X,X) \geq \epsilon$ for all X such that $\|X\|_G = 1$, and $X \perp_G \partial$.

Let $b = \rho(x)$. Then X is tangent to the level surface $S(b) = \{y \ : \ \rho(y) = b\}$, and there is a curve $\zeta : (-a,a) \to S(b)$ such that $\dot{\zeta}(0) = X$. Define a 1-parameter family of curves $\{\gamma_s\}$ such that $\gamma_s : [0,b] \to P$ and γ_s is the unique normal geodesic from Q to $S(b)$ such that $\gamma_s(b) = \zeta(s)$ for all s. These geodesics are all perpendicular to Q since $\text{dist}(\zeta(s),Q) = b$. If we denote the Jacobi vector field along γ_0 by $W_X(t)$

$$W_X(t) = \frac{d}{ds}|_{s=0}\gamma_s(t)$$

then $W_X(b) = X$ and $W_X(t) \perp \dot{\gamma}_0(t)$ for all t. (In particular $Y = W_X(0)$ is tangent to Q).

Since the length function $L(s)$ of $\{\gamma(s)\}$ is constant , the second variation formula gives

$$\begin{aligned} 0 = L"(0) &= < \nabla_X X, \dot\gamma_0 > - < \nabla_Y Y, \dot\gamma_0 > \\ &+ \int_0^b < \dot W_X, \dot W_X > - < R(W_X, \dot\gamma_0)W_X, \dot\gamma_0 > dt \end{aligned}$$

Now $\nabla_Y Y$ is tangent to Q since Q is totally geodesic, and so $< \nabla_Y Y, \dot\gamma_0 >= 0$. Therefore, because of our assumption that

$$- < R(W_X, \dot\gamma_0)W_X, \dot\gamma_0 > \quad geq\ 0,$$

we have

$$- < \nabla_X X, \dot\gamma_0 > \geq \int_0^b \|\dot W_X\|^2 dt.$$

But

$$D^2\rho(X,X) = X(X\rho) - (\nabla_X X)\rho = - < \nabla_X X, \partial >= - < \nabla_X X, \dot\gamma_0 >,$$

where the second equality holds because $X\rho = 0$ and the third because $\dot\gamma_0 = \partial$ along $\dot\gamma_0$. Thus

$$D^2\rho(X,X) \geq \int_0^b \|\dot W_X\|^2 dt \geq 0$$

Sub-Lemma : If X is a unit vector at x , which is perpendicular to ∂ and such that $D^2\rho(X,X) \leq \epsilon/b$ where $b = \rho(x)$, then $\|W_X(0)\| \geq 1 - \sqrt{\epsilon}$ and $\|W_X(0) - U_X(0)\| < \sqrt{\epsilon}$, where $U_X(t)$ is parallel translate of X along γ.

Proof: Let $Y_X(t) = W_X(t) - U_X(t)$. Then $\dot Y(t) = \dot W_X(t)$, $Y_X(b) = 0$, and

$$|\frac{d}{dt}\|Y_X(t)\|| = \frac{| < \dot Y, Y > |}{\|Y\|} \leq \|\dot Y_X\| = \|\dot W_X\|.$$

Therefore

$$\|Y_X(0)\| = |\int_0^b \frac{d}{dt}\|Y_X(t)\|dt| \le \int_0^b \|\dot{Y}\|dt \le (b\int_0^b \|\dot{W}_X\|^2 dt)^{\frac{1}{2}} \le \sqrt{\epsilon}.$$

Thus $\|W_X(0)\| \ge 1 - \sqrt{\epsilon}$ since $\|U_X(t)\| = 1$, $\forall t$. □

As in [9] , in order to complete the proof, it suffices to show that if X is such that $\|X\| = 1$, $X \perp \partial$ and $X \perp J\partial$, then at least one of $D^2\rho(X,X)$ or $D^2\rho(JX,JX)$ is $> \epsilon/b$.

For suppose not. Then both $D^2\rho(X,X)$ and $D^2\rho(JX,JX)$ are $< \epsilon/b$. The sub-lemma implies

(A) $\|W_X(0)\|$ and $\|W_{JX}(0)\|$ are both $\ge 1 - \sqrt{\epsilon}$, and

$$\begin{aligned} \|JW_X(0) - W_{JX}(0)\| &\le \|JW_X(0) - JU_X(0) + U_{JX}(0) - W_{JX}(0)\| \\ &\le \|JW_X(0) - JU_X(0)\| + \|U_{JX}(0) - W_{JX}(0)\| \\ (B) \qquad\qquad &\le 2\sqrt{\epsilon} \end{aligned}$$

Now $W_X(0)$ and $W_{JX}(0)$ are both tangent to Q by construction. On the other hand $JW_X(0)$ is not tangent to Q ; suppose it were , then we would have $\|W_X(0)\|^2 = \omega(JW_X(0), W_X(0)) = 0$ since Q is isotropic , therefore the inequalities (A) and (B) are incompatible when ϵ is sufficiently small. □

Thus G_ρ is a Kähler metric on P , tamed by J ; let us denote by $\omega_\rho = -\frac{1}{2}dJd\rho^2$ the corresponding Kähler form.

We now prove the

Proposition 4.2.2 (P, Q, ω) *is symplectomorphic to* (P, Q, ω_ρ).

Proof: We apply Moser's Method to the one parameter family of 2-forms $\tau_t = t\omega_\rho + (1-t)\omega$, $0 \leq t \leq 1$, which are symplectic because they all tame J. This method will provide a family of vector fields u_t which integrates locally to a family of maps $\Phi_t : P \to P$ such that $\Phi_t^* \tau_t = \omega$. Since P is non-compact, the crucial step is to check that the u_t are complete, i.e. that the Φ_t are diffeomorphisms. Then Φ_1 will be the required symplectomorphism.

To begin with , we construct an explicit form β which vanishes on Q and is such that $\omega_\rho - \omega = d\beta$.

Let $S(\nu Q)$ denote the unit sphere bundle of Q and let $\phi : \mathbb{R}^+ \times S(\nu Q) \to P \setminus Q$ be the map $(t, v_q) \to exp_q(tv_q)$, $\phi_0(v_q) = q$ which is a diffeomorphism.

Define β by

$$\begin{aligned} \phi_r^* \beta &= \int_0^r \phi_s^*(\partial \lrcorner (\omega_\rho - \omega)) ds \\ \beta(\partial) &= 0. \end{aligned}$$

By the Cartan Homotopy Formula

$$\begin{aligned} \phi_r^*(d\beta) &= d(\phi_r^* \beta) = d(\int_0^r \phi_s^*(\partial \lrcorner (\omega_\rho - \omega)) ds) \\ &= \phi_r^*(\omega_\rho - \omega) - \phi_0^*(\omega_\rho - \omega) - \int_0^r \phi_s^*(\partial \lrcorner d(\omega_\rho - \omega)) ds \end{aligned}$$

But $\phi_0^*(\omega_\rho - \omega) = 0$ since Q is isotropic ($\phi_0^*\omega = \omega|_Q = 0$) and $\phi_0^*\omega_\rho = 0$ by definition . Further $d(\omega_\rho - \omega) = 0$. Thus , since ϕ is a diffeomorphism , we have that $\omega_\rho - \omega = d\beta$ (where $\beta = 0$ on Q).

Solving the equation $u_t \lrcorner \tau_t = -\beta$ we obtain a family of vector fields u_t , $t \in [0,1]$. In order to prove that u_t is complete we need to estimate the size of β. For this purpose write in polar coordinates on $P \setminus Q$:

$$\Omega = \omega_\rho - \omega = dr \wedge \alpha + \gamma$$

where α is a 1-form and γ is a 2-form such that $\partial \lrcorner \alpha = 0$ and $\partial \lrcorner \gamma = 0$.

By Lemma 4.2.1 since $\|X\|_\rho \geq \epsilon \|X\|$ we have that

$$1 = \|\omega_\rho\|_\rho = \sup_{\|X\|=\|Y\|=1} |\omega_\rho(X,Y)| \geq \sup_{\|X\|=\|Y\|=\frac{1}{\epsilon}} |\omega_\rho(X,Y)| = \epsilon^2 \|\omega_\rho\|$$

Therefore $\|\omega_\rho\| \leq \frac{1}{\epsilon^2}$, then $\|\Omega\| = \|\omega_\rho - \omega\| \leq \frac{1}{\epsilon^2} + 1$

$$\|\Omega - \gamma\| = \sup_{\|X\|=\|Y\|=1} |(\Omega - \gamma)(X,Y)| = \sup_{\|X\|=\|Y\|=1} |dr \wedge \alpha(X,Y)|$$

taking $Y = \partial$ we get

$$\sup_{\|X\|=1} |\Omega(X,\partial)| = \sup_{\|X\|=1} |\alpha(X)| = \|\alpha\|.$$

It follows that $\|\alpha\| \leq \|\omega_\rho - \omega\| \leq \frac{1}{\epsilon^2} + 1$. Hence $\|\alpha\|$ is bounded.

Now for each $v \in S(\nu Q)$ define the function

$$f_v(s) = \frac{\|(\phi_s)_*(v)\|_G}{s}.$$

As in the proof of Rauch Comparison theorem the function

$$s \longrightarrow \frac{\|(\phi_s)_*(v)\|_G^2}{s^2}$$

is increasing (since $f_v(s) > 0$ and this implies $[f_v(s)]' > 0$).

Now we are in a position to estimate the size of β

$$\begin{aligned}
\|\beta((\phi_r)_*(v)\| &= |\int_0^r \phi_s^*(\partial \lrcorner (\omega_\rho - \omega))(v) ds| \\
&\leq \int_0^r \|(\omega_\rho - \omega)(\partial, (\phi_s)_*(v)\|_G ds \\
&= \int_0^r \|\alpha((\phi_s)_*(v))\|_G ds \\
&\leq \int_0^r C s f_v(s) ds \leq C\frac{r^2}{2} f_v(r) = \frac{C}{2} r \|(\phi_r)_*(v)\|_G.
\end{aligned}$$

Thus $\|\beta\| = O(r)$.

To finish the proof of the proposition , denote by $\|\cdot\|_t$ the norm given by $\|X\|_t^2 = \tau_t(X, JX)$. Again by the Lemma 4.2.1 we have

$$\|u_t\|_G \leq \|u_t\|_t = \|\beta\|_t \leq \|\beta\|_G = O(r).$$

Because P is complete with respect to the metric G, the vector field u_t can be integrated to a family of diffeomorphisms Φ_t such that

$$\frac{d\Phi_t}{dt} = u_t \circ \Phi_t \ , \ \Phi_0 = id$$

Then as we expected $\Phi_1^*\omega_\rho = \omega$ because $\frac{d}{dt}\tau_t = \omega_\rho - \omega = d\beta$ and

$$\mathcal{L}_{u_t}\tau_t = \imath_{u_t} d\tau_t + d(\imath_{u_t}\tau_t) = 0 + d(u_t \lrcorner \tau_t) = -d\beta$$

imply

$$\Phi_1^*\omega_\rho - \omega = \int_0^1 \frac{d}{dt}(\Phi_t^*\tau_t) dt = \int_0^1 \Phi_t^*(\frac{d}{dt}\tau_t + \mathcal{L}_{u_t}\tau_t) = 0.$$

□

4.3 Construction of Φ_2

¶In this section we first compute the 1-jet of the Liouville vector field ξ_ρ at a point $q \in Q$.

Let $z_j = x_j + i\,y_j$, $j = 1, ..., n$ be complex normal coordinates about the point $q \in Q$, chosen so that

$$T_q Q = \{\cap_{j=k+1}^{n} Ker\, dx_j\} \cap \{\cap_{j=1}^{n} Ker\, dy_j\},$$

To calculate the 1-jet of ξ_ρ at q define the function ρ_0 near q by

$\rho_0(p) = G_0$-distance of q from $Q_0 = (\sum_{j=1}^{n} y_j^2 + \sum_{r=k+1}^{n} x_r^2)^{1/2}$ near q.

Let ζ be the vector field given by the formula $\zeta \lrcorner (dJd\rho_0^2) = Jd\rho_0^2$. Then

$$\zeta = \sum_{i=1}^{k} y_i \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{r=k+1}^{n} (x_r \frac{\partial}{\partial x_r} + y_r \frac{\partial}{\partial y_r}),$$

and we want to show that $J_q^1(\xi_\rho = J_q^1(\zeta)$. Since $J_q^1((\xi_\rho)_q)$ depends on $J_q^3(\rho^2)$ we just have to show that ρ^2 and ρ_0^2 have the same 3-jet at q. This follows if $\rho = \rho_0 + O(\rho_0^3)$ near q, i.e. if ρ and ρ_0 have the same 2-jet. Since, if G_0 is the flat metric , we know that $J_q^1(G) = J_q^1(G_0)$, where J_q^k denotes the kth-jet at q. Because the Christoffel symbols Γ_{jk}^i depend on the 1-jet of the metric, the geodesic equation $\ddot{x}_i + \Gamma_{jk}^i \dot{x}_j \dot{x}_k = 0$ implies that the exponential maps $exp_q : T_q P \rightarrow P$ corresponding to G and G_0 have the same 2-jet at q. Because Q is totally geodesic , it is 2-tangent at q to the submanifold $Q_0 \stackrel{\text{def}}{=} \{y_j = 0\}_{j=1}^{n} \cap \{x_r = 0\}_{r=k+1}^{n}$.

Thus there is a local diffeomorphism h of P that takes Q into Q_0 and is such that $J_q^2(h) = J_q^2(Id)$. It follows that $\psi = exp_G \circ h_*$ and $\psi_0 = exp_{G_0} \circ h_*$ have the same 2-jet at q. Thus $J_q^1(\psi_* \zeta) = J_q^1(\psi_{0*} \zeta)$

But $\rho(p) = G$-dist of p from Q $= h_* G$-dist of $h(p)$ from $h(Q) = Q_0$, and if we define $\hat{\rho}$ by $\hat{\rho}(p) = G_0$-dist of $h(p)$ from Q_0 it is not hard to check using the above remarks that both $\rho - \hat{\rho}$ and $\hat{\rho} - \rho_0$ are $O(\rho_0^3)$.

Thus we just proved the

Proposition 4.3.1 *The 1-jet of ξ_ρ is*

$$J_q^1(\xi) = \sum_{i=1}^{k}(0x_i\frac{\partial}{\partial x_i} + 1y_i\frac{\partial}{\partial y_i}) + \sum_{r=k+1}^{n}(\frac{1}{2}x_r\frac{\partial}{\partial x_r} + \frac{1}{2}y_r\frac{\partial}{\partial y_r})$$

where $x_j + i\,y_j$, $j = 1, ..., n$ are complex normal coordinates about the point $q \in Q$, chosen so that

$$T_qQ = \{\cap_{j=k+1}^{n} Ker\, dx_j\} \cap \{\cap_{j=1}^{n} Ker\, dy_j\},$$

Remark 4.3.2 *1. It follows from Theorem 3.1.1 Chapter 3 that ξ_ρ is C^1-conjugate to its linear part.*

2. Notice that in the normal complex coordinates on which we just computed the 1-jet of ξ

$$\diamondsuit \qquad \omega_\rho = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{r=k+1}^{n} 2dx_r \wedge dy_r$$

only at the point q. Instead in the Darboux coordinates which we constructed on Chapter 3 the equality $\diamondsuit$ holds on the whole neighborhood U_1 of q.

¶Now consider in $(\mathsf{R}^{2n}, \mathsf{R}^k)$ the Liouville vector field

$$\zeta = \sum_{j=1}^{k} y^j \frac{\partial}{\partial y^j} + \sum_{r=k+1}^{n} \frac{1}{2}x^r\frac{\partial}{\partial x^r} + \sum_{s=k+1}^{n} \frac{1}{2}y^s\frac{\partial}{\partial y^s}$$

for the standard symplectic form ω_0 where

$(...x^i..,..y^j..,..x^r..,..y^s...); i,j = 1,...,k\ ,\ r,s = k+1,...,n$ are global Darboux coordinates adapted to the foliation tangent to the Liouville vector field ζ.

Proposition 4.3.3 *Any symplectic form ϖ on R^{2n} for which the vector field ζ is Liouville , is diffeomorphic to the standard symplectic form ω_0 by a diffeomorphism which is the identity on $Q_0 = \{y_j = x_r = y_s = 0\} = \mathrm{R}^k$).*

Proof: Let us denote by ϕ_t the flow generated by ζ. Let p be a point with coordinates $p = (..x^i..,..y^j..,..x^r..,..y^s..)$. Then

$$\phi_t(p) = (..x^i..,..e^t y^j..,..e^{\frac{t}{2}} x^r..,..e^{\frac{t}{2}} y^s..) \qquad (*)$$

Thus ϕ_t preserves each term $dx^i \wedge dx^j$, $dx^r \wedge dy^i$, etc. in the expresion of ϖ with respect to this coordinates , so that we can consider each of these terms separately. Note also that because ζ is Liouville for ϖ its flow ϕ_t satisfies

$$\phi_t^* \varpi = e^t \varpi \qquad (**)$$

We now claim:

1. The coefficients of $dx^i \wedge dy^j$; $dx^r \wedge dy^s$; $dx^r \wedge dx^s$; $dy^r \wedge dy^s$ in ϖ are constant along the orbits of ϕ_t. Therefore since $\varpi|_Q = \omega_0|_Q$ they equal those of ω_0.

   Proof: By $(*)$ we have that

$$\begin{aligned} \phi_t^*(a dx^i \wedge dy^j)(p) &= a(\phi_t(p)) dx^i \wedge d(e^t y^j) \\ &= e^t a(\phi_t(p)) dx^i \wedge dy^j \end{aligned}$$

and $(**)$ implies $\phi_t^*(adx^i \wedge dy^j)(p) = e^t a(p) dx^i \wedge dy^j$. Thus the coefficients in ϖ of the term $dx^i \wedge dy^j$ are constant along the flow ϕ_t as claimed.

Now consider a term of the form $dx^r \wedge dy^s$. By $(*)$ we have that

$$\begin{aligned}\phi_t^*(adx^r \wedge dy^s)(p) &= a(\phi_t^*(p))d(e^{t/2}x^r) \wedge d(e^{t/2}y^s)\\ &= e^t a(\phi_t^*(p))dx^r \wedge dy^s\end{aligned}$$

and $(**)$ implies $\phi_t^*(adx^r \wedge dy^s)(p) = e^t a(p) dx^r \wedge dy^s$

Hence the coefficients corresponding to $dx^r \wedge dy^s$ are constant along the flow ϕ_t . Similarly for the others. □

2. The coefficients of $dy^i \wedge dy^j$; $dy^j \wedge dx^r$ and $dy^j \wedge dy^s$ equal zero.

Proof: Assume by contradiction that a term of the form $dy^i \wedge dy^j$ has non-zero coefficients. Then by $(*)$ we have

$$\begin{aligned}\phi_t^*(ady^j \wedge dy^i)(p) &= a(\phi_t)d(e^t y^j) \wedge d(e^t y^j)\\ &= e^{2t} a(\phi_t(p))dy^j \wedge dx^r\end{aligned}$$

but $(**)$ implies $\phi_t^*(ady^j \wedge dy^i)(p) = e^t a(p) dy^j \wedge dy^i$

It follows that $a(\phi_t(p)) = e^{-t}a(p)$, so that it blows up as t goes to $-\infty$. Since $\phi_t(p)$ tends to a point of Q as t goes to $-\infty$, this is not allowed. Therefore these coefficients must equal zero.

Suppose now that the coefficients of $dy^j \wedge dx^r$ are different from zero. Then by $(*)$ we have

$$\begin{aligned} \phi_t^*(ady^j \wedge dx^r)(p) &= a(\phi_t^*)d(e^t y^j) \wedge d(e^{t/2}x^r) \\ &= e^{\frac{3t}{2}} a(\phi_t^*(p))dy^j \wedge dx^r \end{aligned}$$

but $(**)$ implies $\phi_t^*(ady^j \wedge dx^r)(p) = e^t a(p)dy^j \wedge dx^r$.

It follows that $a(\phi_t(p)) = e^{-t/2}a(p)$, so that it blows up as t goes to $-\infty$. Since $\phi_t(p)$ tends to a point of Q as t goes to $-\infty$, this is not allowed. Therefore these coefficients must equal zero.

Similarly for the coefficients of $dy^j \wedge dy^s$ □

3. The coefficients of $dx^i \wedge dx^r$ and of $dx^i \wedge dy^s$ satisfy $a(\phi_t(p)) = e^{t/2}a(p)$ and the ones corresponding to $dx^i \wedge dx^j$ satisfy $a(\phi_t(p)) = e^t a(p)$.

Proof: by $(*)$ we have

$$\begin{aligned} \phi_t^*(adx^i \wedge dx^r)(p) &= a(\phi_t^*(p))dx^i \wedge d(e^{t/2}x^r) \\ &= e^{\frac{t}{2}} a(\phi_t^*(p))dx^i \wedge dx^r \end{aligned}$$

form $(**)$ we get $\phi_t^*(adx^i \wedge dx^r)(p) = e^t a(p)dx^i \wedge dx^r$. Thus $a(\phi_t(p)) = e^{t/2}a(p)$. The calculations are analogous for the remaining terms. □

Consequently we can write

$$\varpi = \omega_0 + \sum A_{ir} dx^i \wedge dx^r + \sum B_{ir} dx^i \wedge dy^r + \sum C_{ij} dx^i \wedge dx^j$$

where :

$$\spadesuit \begin{cases} A_{ir}(\phi_t(p)) & = e^{t/2} A_{ir}(p) \qquad\qquad i,j = 1,...,k \\ B_{ir}(\phi_t(p)) & = e^{t/2} B_{ir}(p) \qquad\qquad r = k+1,...,n \\ C_{ij}(\phi_t(p)) & = e^{t} C_{ij}(p) \end{cases}$$

Recall that ϖ is assumed to be closed. An easy calculation shows that $\varpi^n = \omega_0^n$ so that ϖ is always nondegenerate. In fact each $\tau_t = \omega_0 + t(\varpi - \omega_0)$, $t \in [0,1]$ also satisfies the equation $\tau_t^n = \omega_0^n$. Thus we can apply Moser's method to the family of symplectic forms τ_t, $t \in [0,1]$.

If we take

$$\beta = \zeta \lrcorner (\varpi - \omega) = \sum_i (\sum_r \frac{1}{2}(x^r A_{ir} + y^r B_{ir})) dx^i = \sum_i \beta_i dx^i$$

then $\dot{\tau}_t = \varpi - \omega = d\beta$.

Let

$$u_t = \sum_i (a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i}) + \sum_r (a^r \frac{\partial}{\partial x^r} + b^r \frac{\partial}{\partial y^r})$$

be the vector field that the method provides (i.e. $u_t \lrcorner \tau_t = -\beta$)

From

$$\begin{aligned} -\sum_i \beta_i dx^i &= a^i dy^i + [t(-A_{ir} a^r - B_{ir} b^r - C_{ij} a^j + C_{ji} a^j) - b^i] dx^i \\ &\quad -(b^r - t A_{ij} a^i) dx^r + (a^r + t B_{ir} a^i) dy^r \end{aligned}$$

we get $a^i = 0$ which implies $a^r = b^r = 0$. Hence $b^i = \beta_i$ and $\beta = 0$ on Q. It remains to show that the u_t is in effect complete. For this purpose , if $z = (x^i, y^j, x^r, y^s)$ we consider $\|z\| = \max(|x^i|, |y^j|, |x^r|, |y^s|)$. By $(*)$ we have $\phi_{-t}(z) = (x^i, e^{-t}y^j, e^{-t/2}x^r, e^{-t/2}y^r)$ and by ♠

$$A_{ir}(z) = e^{t/2}A_{ir}(\phi_{-t}(z))$$

$$B_{ir}(z) = e^{t/2}B_{ir}(\phi_{-t}(z)).$$

Then

$$\begin{aligned}
\beta_\ell(z) &= \sum_r \frac{1}{2}(x^r A_{ir} + y^r B_{ir})(z) \\
&= \sum_r \frac{1}{2}(x^r(z)A_{ir}(z) + y^r(z)B_{ir}(z)) \\
&= \sum_r \frac{1}{2}(e^{t/2}x^r(\phi_{-t}(z))e^{t/2}A_{ir}((\phi_{-t}(z)) + e^{t/2}y^r((\phi_{-t}(z))e^{t/2}B_{ir}(\phi_{-t}(z))) \\
&= e^t \sum_r \frac{1}{2}(x^r A_{ir} + y^r B_{ir})(\phi_{-t}(z)) \\
&= e^t \beta_\ell(\phi_{-t}(z)
\end{aligned}$$

Consider the orbit $z(t) = (x_0^i, y_t^j, x_0^r, y_0^s)$ of u_t through $z(0) = (x_o^i, y_o^j, x_o^r, y_o^s)$. We claim that there are constants K_1 and K_2, depending on the orbit, such that $|\beta_\ell(z(t))| \leq K_1 \|z\| + K_2$ for all $z(t) = (x_0^i, y_t^j, x_0^r, y_0^s)$, where $K_2 = \max|\beta_\ell|$ on points where $|y_t^j| \leq \max(1, |x_0^i|)$ and $K_1 = \max|\beta_\ell|$ on the set $S = \{(x_o^i, y^j, x^r, y^s)$ with $|y^j|,\ |x^r|,\ |y^r|\ \leq 1\}$.

Since u_t involves only the $\frac{\partial}{\partial y_j}$, if $\max|y_t^j| \leq \max(1, |x_0^i|)$ there is nothing to prove. For any other point $z(t) = (x_0^i, y_t^j, x_0^r, y_0^s)$ on the orbit, $\|z(t)\| = \max_j |y_t^j|$ and for some τ $\phi_{-\tau}(z(t)) = (x_o^i, e^{-\tau}y_t^j, e^{-\tau/2}x_o^r, e^{-\tau/2}y_o^s)$ belongs to the compact set S. Therefore $|\beta_\ell(\phi_{-\tau}(z(t))| \leq K_2 e^\tau$, $\forall \ell = 1, \ldots, k$.

By hypothesis $\exists j \in \{1, ..., K\}$ such that $\|z(t)\| = |y_t^j| \geq |y_t^i|$ for $1 \leq i \leq k$. We may take τ to be such that $e^{-\tau}|y_t^1| = 1$, i.e. $\phi_\tau(z(t)) \in S$. Hence $\|z(t)\| = |y_t^1| = e^\tau$.

Therefore $|\beta_t(z)| \leq K_2\|z(t)\|$ as required.

It follows that u_t is complete. Therefore it can b e integrated to a one-parameter family of diffeomorphisms ψ_t such that $\psi_t^*\tau_t = \omega_0$.

Hence $\psi_1^*\varpi = \omega$. □

¶Since ξ_ρ is the gradient of ρ^2 with respect to the metric G_ρ, then ξ_ρ is G_ρ-perpendicular to the level surfaces ρ =constant. By Lemma 4.2.1

$$\|\xi_\rho\|_G \leq \|\xi_\rho\|_{G_\rho} = \| -\frac{1}{2}Jd\rho^2\|_{G_\rho} \leq \| -\frac{1}{2}Jd\rho^2\|_G = \rho\|d\rho\|_G = \rho.$$

Therefore , because P is complete with respect to the metric G, ξ_ρ integrates up to a complete flow .

By Remark 4.3.2 ξ_ρ is C^1-conjugate to its linear part ζ.

Because ξ_ρ is complete this local conjugacy may be extended to a C^1-diffeomorphism from P to $\mathbb{R}^{2n}$, which pushes ξ_ρ forward to ζ. Note that this diffeomorphism is C^∞ on Q.

Then ω_ρ is pushed forward to a form ϖ, for which ζ is a Liouville field. By Proposition 4.3.3 ϖ is diffeomorphic to ω_0 by a diffeomorphism preserving the linear subspace Q_0. Hence ω_ρ is symplectomorphic to ω_0 by a C^1-diffeomorphism φ that takes the submanifold Q into the linear subspace Q_0.

We now show that there exist a C^∞-diffeomorphism

$$\Phi_2 : (P, Q, \omega_\rho) \to (\mathbb{R}^{2n}, \mathbb{R}^k, \omega_0).$$

We can choose a diffeomorphism $\tilde{\varphi}$ arbitrary close to φ in the fine C^1-topology, i.e. given any function $\epsilon = \epsilon(x) > 0$ we can make $|d\varphi(x) - d\tilde{\varphi}(x)| < \epsilon(x)$ (see [5]). $\tilde{\varphi}$ pushes forward ω_ρ to a symplectic form $\hat{\omega}$ on $\mathbb{R}^{2n}$. Since φ is C^∞ on Q $\tilde{\varphi}$ can be chosen to be equal to φ on Q. There is an $\epsilon = \epsilon(n)$ such that $\|\hat{\omega} - \omega_0\| \leq \epsilon$ for all x. Let $\tau_t = \omega_0 + t(\hat{\omega} - \omega_0)$, $t \in [0,1]$. It follows from a straight forward calculation that $[\tau_t]^n \neq 0$ for all $t \in [0,1]$. Therefore all the τ_t are not degenerate. Hence we can apply Moser's method to the family of symplectic forms τ_t.

If we take

$$\beta(x) = \int_0^{r(x)} (\frac{\partial}{\partial r} \lrcorner (\hat{\omega} - \omega_0))dt,$$

where $\frac{\partial}{\partial r}$ is the radial vector field on $\mathbb{R}^{2n}$. Then $\dot{\tau}_t = \hat{\omega} - \omega_0 = d\beta$ and if $\epsilon \leq 1/2$ then $\|\beta(x)\| \leq \frac{1}{2}r(x)$.

Solving the equation $u_t \lrcorner \tau_t = -\beta$ we obtain a family of vector fields u_t , $t \in [0,1]$. Then we have that $u_t \lrcorner \omega_0 = \beta - (u_t \lrcorner t(\hat{\omega} - \omega_0)$. Therefore

$$\begin{aligned} \|u_t\| &= \|\beta - (u_t \lrcorner t(\hat{\omega} - \omega_0)\| \\ &\leq \|\beta\| + \epsilon\|u_t\| \ , \ \forall t \in [0,1]. \end{aligned}$$

Thus $(1-\epsilon)\|u_t\| \leq \|\beta\|$, this implies that $\|u_t\| \leq (\frac{1}{1-\epsilon})\|\beta\| \leq 2\|\beta\|$ since $\epsilon \leq 1/2$. Hence $\|u_t\| = O(r)$. Consequently the vector field u_t that Moser's method provides is complete, thus it can be integrated to a family of diffeomorphisms ψ_t such that $\psi_t^* \tau_t = \omega_0$.

Let $\Phi_2 = \psi_1^{-1} \circ \tilde{\varphi}$. Since $\varphi|_Q = \tilde{\varphi}|_Q$, φ takes Q into the linear subspace Q_0. It is easy to check that the family of vector fields u_t is tangent to $Q_0 = \mathbb{R}^k$, therefore each ψ_t takes Q_0 into itself. Hence Φ_2 is a C^∞-diffeomorphism which pushes forward ω_ρ into ω_0 and takes the isotropic submanifold Q into the linear subspace Q_0.

This completes the proof of Theorem 4.1.1.

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