h-Cobordisms over certain nonpositively curved spaces

A dissertation presented

by

Bizhong Hu

to

The Graduate School

in Partial Fullfillment of the Requirements

for the degree of

Doctor of Philosophy

in

Department of Mathematics

State University of New York

at Stony Brook

May 1989

STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

THE GRADUATE SCHOOL

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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We prove in particular that for a compact riemannian space with $K \leq 0$ such that there is no totally geodesic $\mathbb{R} \times [a, b]$ immersed in it, the Whitehead group of its fundamental group is zero. To do so we use the ordinary control theory and the foliated version, an analysis of periodic geodesics in a space with $K \leq 0$, and the weakly Anosov phenomenon.

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Acknowledegements

This paper is the result of efforts of mine and my teacher Professor Lowell Jones, to him I owe my sincere thank. Also I feel that the experiences I gained during the period of about one year when he guided me in discussing ideas in geometric topology are particularly precious to me. My ability in studying mathematics seems greatly improved since then.

1. Introduction

Suppose W is a compact C^{∞} manifold with boundaries M and N such that $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences, i.e., W is an h-cobordism. Suppose $\dim(M) \geq 5$. Then there is unique obstruction τ, called Whitehead torsion, for W being a product, $\tau \in Wh(\Gamma)$, $\Gamma = \pi_1(M)$, where $Wh(\Gamma)$ is the Whitehead group. In fact any element of $Wh(\Gamma)$ is the Whitehead torsion of an h-cobordism. This is the handle-moving theory of S. Smale and others, leading to the solution of the higher Poincaré conjecture. It is a conjecture that for any compact aspherical manifold, the Whitehead group of the fundamental group is zero. $Wh(\Gamma)$ is in general hard to compute. But if all h-cobordisms over M are products, $Wh(\Gamma)$ is zero. Another theory, called control theory, has been available, which measures h-cobordisms by real numbers and tells when an h-cobordism is product. A theory along these lines is developed by T. Farrell and L. Jones in [FJ1]. They suppose M is a riemannian space with $K < 0$ and use the geodesic flow which is Anosov to change W and then apply their foliated control theory to conclude that W has a pruduct structure. So for any compact riemannian space with $K < 0$, M, $Wh(\Gamma) = 0$, $\Gamma = \pi_1(M)$.

Our general purpose is to use the above theory and use sharper tools developed by Gromov in [G1] to say something about the Whitehead groups of hyperbolic groups and this paper is a step in that direction. The paper contains the following results.

1.1. Theorem. If M is a compact riemannian space with $K \leq 0$ such that there is no totally geodesic $\mathbb{R} \times [a, b]$ immersed in M, then $Wh(\Gamma) = 0$, $\Gamma = \pi_1(M)$.

1.2. Corollary. Suppose M is as above and $\dim(M) \geq 5$. Then for any compact C^∞ manifold N h-cobordant to M, N is diffeomorphic to M.

We will work on a more general geometric model, i.e., we will consider a compact topological manifold with a "$K \leq 0$" geometric structure. The concept of "space with $K \leq 0$" is defined by M. Gromov by taking out several properties of riemannian space with $K \leqslant 0$ as axioms. This concept becomes important after he presents several kinds of examples and the meaning of "$K \leqslant 0$" in PL geometry. See 4.1 for definition.

1.3. Theorem. Suppose M is a compact manifold with a $K \leqslant 0$ geometric structure suth that there is no totally geodesic $\mathbb{R} \times [a, b]$ immersed in M. Assume the geodesic flow has a transversal PL or C^∞ structure, or assume that 3.4 is true. Then $Wh(\Gamma) = 0$, $\Gamma = \pi_1(M)$.

1.4. Corollary. Suppose M is as above and $\dim(M) \geqslant 5$. Then for any compact manifold N h-cobordant to M, N is homeomorphic to M.

1.5. Remarks.

(1). By [G1], if a compact space with $K \leqslant 0$ has no immersed totally geodesic $\mathbb{R}^2$, its fundamental group is hyperbolic. In particular, the Γ of 1.1 or 1.3 is a hyperbolic group.

(2). For a compact space with $K \leqslant 0$ and with hyperbolic fundamental group Γ, its geodesic flow as a foliation is precisely Gromov's flow $G(\Gamma)$ in [G1] if and only if there is no totally geodesic $\mathbb{R} \times [a, b]$ immersed in it.

(3). The additional assumption in 1.3 is unnecessary if the following is true.

<u>True or not:</u> If $A_1, \cdots A_k$ are submanifolds in $\mathbb{R}^n$ containing the point 0, then there is set $A \subset \mathbb{R}^n$ seperating 0 and ∞ such that A is covered by finitely many compact submanifolds of dimensions $\leqslant n-1$ and that for $1 \leqslant i \leqslant k$, $A \cap A_i$ is covered by finitely many compact submanifolds of dimensions $\leqslant \dim(A_i) - 1$.

(4). There is a way to get stronger results. The method is explained in detail in [FJ1]. Suppose M and Γ are as in 1.1 or 1.3. Instead of considering M, we can consider $M \times S^5$, getting rid of the dimension restriction in the control theory. If consider $M \times S^5 \times T^k$, $k = 0, 1, 2, \cdots$, we can imply that

$Wh(\Gamma \oplus \bigoplus_k Z) = 0$, $k = 0, 1, 2, \cdots$,

which then imply the following vanishing of algebraic K-groups:

$\widetilde{K}_{-k}(\mathbb{Z}\Gamma) = 0,\ k = 0, 1, 2, \cdots$.

Here is an outline of the paper. Chpater 2 describes the ordinary control theory and chapter 3 the foliated version. Chapter 4 contains the definition of space with $K \lesssim 0$, an analysis of periodic geodesics, presentation of a phenomenon that will lead to the weakly Anosov phemamenon, and the construction of crossing $\mathbb{R}$. Chapter 5 contains the study of the collection of all geodesics as a space, and the weakly Anosov flow. Chapter 6 contains the asymptotic lifting of an h-cobordism over a space with $K \lesssim 0$, the decomposition of the geodesic flow of a space with $K \lesssim 0$ crossed by $\mathbb{R}$, a dynamical behavior in a weakly Anosov flow, and the proof of theorems 1.1 and 1.3.

2. A controled h-cobordism theorem

Suppose W is a manifold with two boundaries M and N, and suppose the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ are homotopy equivalences. Then there are

$W \xrightarrow{p_t} W$, $0 \leqslant t \leqslant 1$,

$W \xrightarrow{q_t} W$, $0 \leqslant t \leqslant 1$,

strong deformations of W to M and to N. Set $W \xrightarrow{p = p_1} M$. Also suppose that everything is trivial outside a compact set of M. That is, there must be compact set $A \subset M$ such that

$p^{-1}(M-A) = (M-A) \times [0, 1]$,

$(M-A) \times [0, 1] \xrightarrow{p_t = Id \times (1-t) Id} (M-A) \times [0, 1]$,

$(M-A) \times [0, 1] \xrightarrow{q_t = Id \times [t + (1-t) Id]} (M-A) \times [0, 1]$.

W is called an h-cobordism. The following curves in M are called associated curves of the h-cobordism:

$p \circ p_t(x)$, $0 \leqslant t \leqslant 1$, $x \in W$,

$p \circ q_t(x)$, $0 \leqslant t \leqslant 1$, $x \in W$.

Suppose $B \subset M$, $B \times [0, 1] \xrightarrow{P} W$ is an embedding such that

$P|_{B \times 0} = Id$,

$P(B \times 1) \subset N$,

$P[B \times (0, 1)] \subset W^{\circ}$.

P is called a product structure for W over B. The following curves in M are called associated curves of the product structure:

$p[P(x, t)]$, $0 \leqslant t \leqslant 1$, $x \in B$.

Let d denote a metric of M. The maximum of the diameters of associated curves of W is called the diameter of W. In the same way we have the diameter of P. The following theorem belongs to Chapman, Ferry and Quinn. Refer to for example [CF] .

2.1. Theorem.

(1). Suppose M is a manifold, $\dim(M) \geqslant 5$, d is a metric on M, A, $B \subset M$ both compact, U is a neighborhood of B in M. Then given $\varepsilon > 0$, there is $\delta > 0$ such that the following is true. For any h-cobordism W over M, with product structure P over A and $\operatorname{diam}(W) \leqslant \delta$, $\operatorname{diam}(P) \leqslant \delta$, there is a product structure Q for W over $A \cup B$ such that $Q|_{A-U} = P$, $\operatorname{diam}(Q) \leqslant \varepsilon$.

(2). Suppose $E \xrightarrow{\pi} M$ is a fiber bundle, $\dim(E) \geqslant 5$, the fiber F is a compact manifold such that $Wh[\pi_1(F) \oplus \bigoplus_k Z] = 0$, $k = 0, 1, 2, \cdots$. Substitute A, B, U by $\pi^{-1}(A)$, $\pi^{-1}(B)$, $\pi^{-1}(U)$, consider h-cobordisms over E, measure diameters in M. Then (1) is still true.

3. A foliated control theorem

<u>3.1. Definition.</u> Suppose M is a manifold with a 1 - dimensional foliation structure and a metric. $\alpha(t)$, $0 \leqslant t \leqslant 1$ is a curve in M. ℓ and a are positive numbers and I is a leaf segment of the foliation suth that length(I) $< \ell$ and $d[\alpha(t), I] \leqslant a$, $0 \leqslant t \leqslant 1$. Then we say that the curve has diameter $\leqslant (\ell, a)$. If there is a collection of curves such that every curve has diameter $\leqslant (\ell, a)$, then we say that the collection has diameter $\leqslant (\ell, a)$. If W is an h-cobordism and P is a product for W, then their diameters are those of their respective associated curves.

<u>3.2. Definition.</u> Suppose M is a n-manifold with a 1-dimensional foliation structure. A split open set in M by definition is an open set in M split into product $\mathbb{R}^{n-1} \times \mathbb{R}$ M such that every fiber $x \times \mathbb{R}$, $x \in \mathbb{R}^{n-1}$ is contained in a single leaf of the foliation. If M is covered by split open sets $\{ \mathbb{R}^{n-1} \times \mathbb{R} \xrightarrow{\phi_i} M \}$ such that for any $\phi_i(\mathbb{R}^{n-1} \times \mathbb{R}) \cap \phi_j(\mathbb{R}^{n-1} \times \mathbb{R}) \neq \emptyset$, the induced homeomorphism between open sets of $\mathbb{R}^{n-1}$, $U \xrightarrow{\phi_{ij}} V$ is PL, then the foliation is said to have a

transversal PL structure. Transversal C^{∞} structure is defined in the same way.

The following theorem belongs to Farrell and Jones. We refer the reader to [FJ1] for a proof.

<u>3.3. Theorem.</u> $n \geq 5$. M is a n-manifold with a metric and a 1-dimensional foliation structure. Suppose the foliation has a transversal PL or C^{∞} structure. W is an h-cobordism over M with diameter $\leq (\ell, a)$. $A \subset M$ is a compact set such that any leaf intersecting A has length $> C \cdot \ell$. Then W has a product P over A with diameter $\leq (C \cdot \ell, b)$. Where $C = C(n)$ depends only on n, b does not depend on W, and $\lim_{a \to 0} b = 0$.

<u>3.4. Conjecture.</u> 3.3 is true without assuming that the foliation has a transversal PL or C^{∞} structure.

4. Spaces with $K \leq 0$

4.1

4.1.1. Definition. Let M be a topological manifold. Suppose for each curve α in M there is number $L(\alpha)$ such that $L(\alpha) \geq 0$, $L(\alpha) = 0$ only when α is a point, $L(\alpha) = L(\alpha^{-1})$, and $L(\alpha * \beta) = L(\alpha) + L(\beta)$. Then call M with L a length space. Define distance as the minimum of lengths to get a metric of M. A geodesic is a local isometry $\mathbb{R} \longrightarrow M$. We will assume that the following are true. Any two points can be joined by a geodesic realizing distance. For any $x \in M$, $\varepsilon > 0$ small, the ε- neighborhood is homeomorphic to E^n. Any geodesic segment can be extended in a unique way. Note: this last assumption is not made in the definition by Gromov, it can be violated in PL geometry, but we need it here.

4.1.2. Definition. A triangle in a length space is composed of three points and three geodesic segments connecting them. We express a triangle by $\Delta = (x_1, x_2, x_3)$, or, more precisely,

$\Delta = (x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3,)$,

where (x_1, x_2, x_3) are three points,

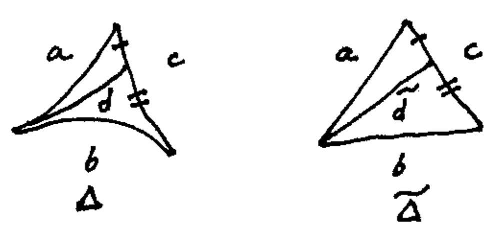
$(\alpha_1, \alpha_2, \alpha_3)$ are unoriented geodesic segments

connecting x_2 and x_3, x_3 and x_1, x_1 and x_2.

Denote $|\Delta| = (L(\alpha_1), L(\alpha_2), L(\alpha_3))$.

Two triangles Δ and $\tilde{\Delta}$ in two length spaces are equivalent if $|\Delta| = |\tilde{\Delta}|$.

<u>4.1.3. Definition.</u> Suppose Δ and $\tilde{\Delta}$ are equivalent triangles in length spaces M and N. We say that Δ is thinner than $\tilde{\Delta}$ if the following is always true.



$d \leq \tilde{d}$.

<u>4.1.4. Definition.</u> Denote the simply connected 2 - riemannian space of constant curvature $K \equiv C \leq 0$ by $\mathbb{H}$. Suppose M is a length space such that any small triangle in M is thinner than its equivalent triangle in $\mathbb{H}$, then M is said to be a space with

$K \leqslant C$. Call a simply connected space with $K \leqslant 0$ a Hadamard space.

4.1.5. Theorem. Suppose X is a Hadamard space. Then for any two geodesics $\alpha(t)$ and $\beta(t)$, $f(t) = d[\,\alpha(t), \beta(t)\,]$, $t \in \mathbb{R}$, is convex. See [G1].

4.1.6. Remarks. For a Hadamard space X, any geodesic $\mathbb{R} \longrightarrow X$ is an isometry, two points can be joined by a unique geodesic. These are seen by 4.1.5. Two geodesics $\alpha(t)$ and $\beta(t)$ are said to be asymptotic if $d[\,\alpha(t), \beta(t)\,]$, $t \geqslant 0$ is bounded. For any geodesic and any point, there is unique geodesic from the point which is asymptotic to the geodesic given. The existance can be proved directly and the uniqueness by 4.1.5. Asymptotic relation is an equivalence relation. Equivalent classes form ∂X. $\overline{X} = X \cup \partial X$. For any $x \in X$, define Σ_x as the collection of sets in $\overline{X}$ each of which contains a neighborhood of x in X. For any $z \in \partial X$, take $x \in X$, $y \in S_x(1)$ is the point between x and z, take a neighborhood S of y in $S_x(1)$, consider those those geodesic rays from x through S, take a set A in $\overline{X}$ containing all of them, define $\sum_z = \{\ A\ \}$. The following are true for $\{ \Sigma_x , x \in \bar{X} \}$.

(1). $A \in \Sigma_x \Longrightarrow x \in A$.

(2). $A, B \in \Sigma_x \Rightarrow A \cap B \in \Sigma_x$.

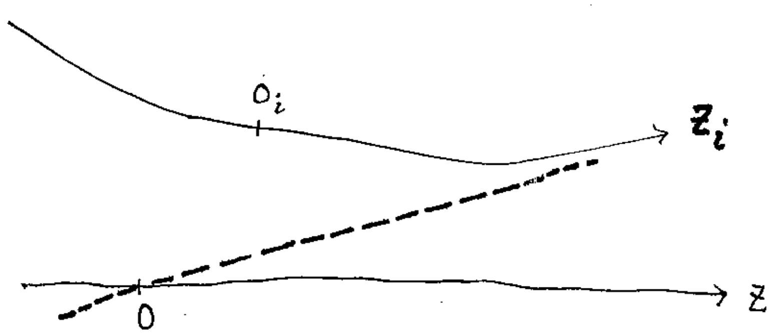
(3). $A \subset B, A \in \Sigma_x \Longrightarrow B \in \Sigma_x$.

(4). $\forall A \in \Sigma_x, \exists U \subset A, U \in \Sigma_x, \forall y \in U, U \in \Sigma_y$.

That (2) is true for a point in ∂X requires a proof, which can be done with the help of 4.1.5. We get a topology to $\bar{X}$ such that the above sets are all neighborhoods. One can show that for any $x \in X$, $\varepsilon > 0$, the $E_x(\varepsilon)$ is naturally homeomorphic to $\bar{X}$.

4.1.7. Proposition. Suppose X is a Hadamard space, $\alpha_i(t)$, $i = 1, 2, \cdots$, $\alpha(t)$ are geodesics, then $\alpha_i(t) \longrightarrow \alpha(t)$ pointwise if and only if $\alpha_i(0) \rightarrow \alpha(0)$ and $\alpha_i(+\infty) \longrightarrow \alpha(+\infty)$.

The proof is done with 4.1.5, 4.1.6 and the following picture.



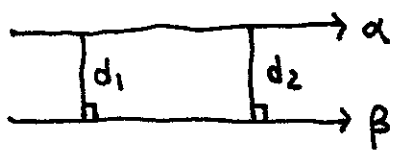
4.2

4.2.1. Lemma. Suppose X is a Hadamard space. α and β are geodesics such that d [$\alpha(t), \beta(t)$], $t \in \mathbb{R}$ is bounded. Then α and β span a totally geodesic $\mathbb{R} \times [a,b]$.

Proof. Here is a proof by the following five steps, where both 4.1.4 and 4.1.5 are used.

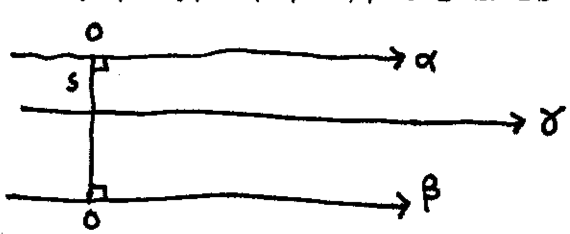
(1). Denote a = d (Im α, Im β), then there is parameterization such that d [$\alpha(t)$, $\beta(t)$] = a, $t \in \mathbb{R}$.

Hint:



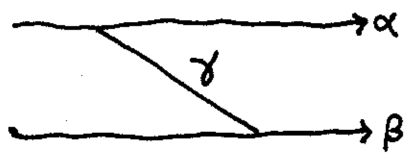
Compare d_1 and d_2 .

(2). Denote the unit speed geodesic from $\alpha(t)$ to β (t) as H(t, s), $t \in \mathbb{R}$, $s \in [0, a]$. Then for any $s \in [0, a]$, H(t, s), $t \in \mathbb{R}$ is a geodesic. Hint:

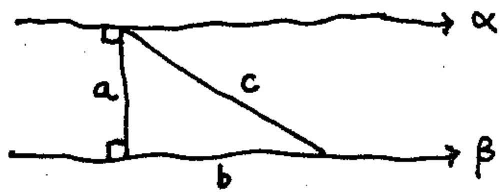


Consider the geodesic γ .

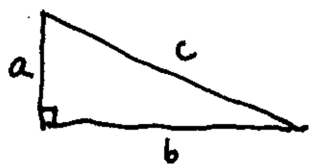
(3). Consider any geodesic segment γ from a point in α to one in β .



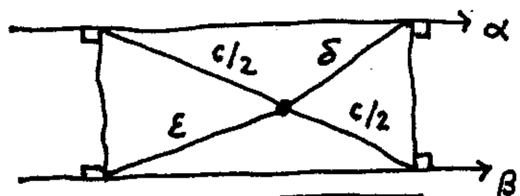
Denote the following triangle as Δ .



Then Δ is equivalent to the following triangle $\widetilde{\Delta}$ in $\mathbb{R}^2$.



Furthermore, the following two geodesic segments ε and δ form a geodesic segment $\varepsilon \cup \delta$ and $L(\varepsilon) = L(\delta)$.



(4). $c = \sqrt{a^2 + b^2}$.

(5). The middle point of γ is in H. In fact the whole γ is in H. We are done.

Suppose M is a space with $K \leqslant 0$. $X \longrightarrow X/\Gamma = M$ is the universal covering. If $\alpha(t)$, $0 \leqslant t \leqslant 1$ is a closed curve at x in M, $\widetilde{\alpha}(t)$, $0 \leqslant t \leqslant 1$ is a lifting , from x to f(x), where $f \in \Gamma$, then

$$\bigcup_{k=-\infty}^{+\infty} f^k(\widetilde{\alpha})$$

is a lifting of $\bigcup_{\mathbb{Z}} \alpha$. Now suppose $\alpha(t)$, $t \in \mathbb{R}$ is a periodic geodesic at x, with period a, $\widetilde{\alpha}(t)$, $0 \leq t \leq a$ is a lifting of $\alpha(t)$, $0 \leq t \leq a$, from x to f(x), $f \in \Gamma$, then

$$\bigcup_{k=-\infty}^{+\infty} f^k(\widetilde{\alpha})$$

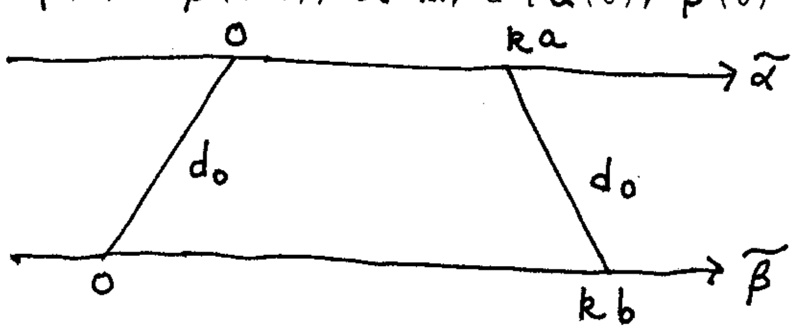
is a lifting of $\alpha(t)$, $t \in \mathbb{R}$, in particular, it must be a geodesic in X. If $\alpha(t)$, $0 \leq t \leq 1$ and $\beta(t)$, $0 \leq t \leq 1$ are closed curves in M, H is a homotopy from α to β, $\widetilde{H}$ is a lifting of H, a homotopy from $\widetilde{\alpha}(t)$, $0 \leq t \leq 1$ to $\widetilde{\beta}(t)$, $0 \leq t \leq 1$, then there is unique $f \in \Gamma$ such that

$$\bigcup_{k=-\infty}^{+\infty} f^k(\widetilde{H})$$

is a homotopy between the following two curves:

$$\bigcup_{k=-\infty}^{+\infty} f^k(\widetilde{\alpha}) , \quad \bigcup_{k=-\infty}^{+\infty} f^k(\widetilde{\beta}) .$$

Now suppose $\alpha(t), t \in \mathbb{R}$ and $\beta(t)$, $t \in \mathbb{R}$ are periodic geodesics in M, with periods a and b, that are homotopic, with $\alpha(0)$ going along curve γ to $\beta(0)$, $L(\gamma) = L$. The above discussions indicate that there are liftings $\widetilde{\alpha}(t)$ and $\widetilde{\beta}(t)$ of $\alpha(t)$ and $\beta(t)$, $f \in \Gamma$, $f\,\widetilde{\alpha}(t) = \widetilde{\alpha}(t+a)$, $f\,\widetilde{\beta}(t) = \widetilde{\beta}(t+b)$, $t \in \mathbb{R}$, $d\,[\,\widetilde{\alpha}(0), \widetilde{\beta}(0)\,] = d_0 \leq L$.



$| ka - kb | \leq 2d_0$, $k \in \mathbb{Z}$.

This means that a = b. And then $d[\tilde{\alpha}(t), \tilde{\beta}(t)]$, $t \in \mathbb{R}$ is bounded. By 4.2.1, $\tilde{\alpha}$ and $\tilde{\beta}$ bound a totally geodesic $\mathbb{R} \times [0, \ell] \subset X$. We may assume that $\tilde{\alpha}(t) = (t, 0)$, $\tilde{\beta}(t) = (t, \ell)$, $t \in \mathbb{R}$. One can show that $f(t, s) = (t+a, s)$. We have porved the following fact.

4.2.2. Proposition. Suppose M is a space with $K \leqslant 0$. $\alpha(t)$, $0 \leqslant t \leqslant a$ and $\beta(t)$, $0 \leqslant t \leqslant b$ are homotopic periodic geodesics. Then a = b, α and β span a totally geodesic $S^1(a) \times [0, \ell]$.

Suppose M is a space with $K \leqslant 0$. $X \longrightarrow X/\Gamma = M$ is the universal covering. α and β are closed curves at x and at y in M. Suppose $\tilde{\alpha}$ and $\tilde{\beta}$ are liftings of α and β, from $\tilde{x}$ to $f(\tilde{x})$ and $\tilde{y}$ to $f(\tilde{y})$, where $f \in \Gamma$. Take a curve $\tilde{\gamma}$ from $\tilde{x}$ to $\tilde{y}$. Downstairs we have curve γ from x to y. $\tilde{\alpha} \cup \tilde{\beta} \cup \tilde{\gamma} \cup f\tilde{\gamma}$ is a closed curve in X, which is simply connected, and thus can be extended. Go back to M and we get a homotopy from α to β, with x going along γ to y. Now consider a connected set $A \subset M$ over which the universal covering is trivial, the collection ΓA is discrete. Also consider a constant C. Remember that in X any two points are joined by a unique geodesic segment realizing distance. We see the following is true.

4.2.3. Proposition. Suppose M is a space with $K \leqslant 0$, $A \subset M$ is a connected set over which the universal covering is trivial, and C is a constant. Then the following is a finite set:

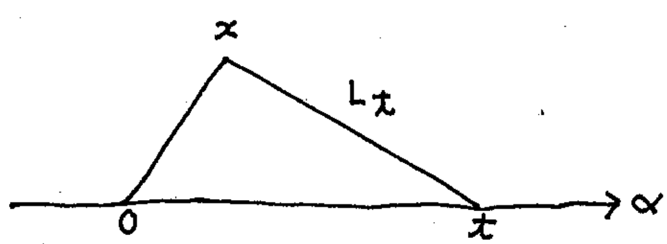
$$\left\{ \begin{array}{l} \text{Homotopy classes of periodic geodesics which interset} \\ \text{A and have periods} \leqslant C \end{array} \right\}.$$

4.2.4. Corollary. Suppose M is a compact space with $K \leqslant 0$ such that there is no totally geodesic $S^1(a) \times [0, \ell]$ immersed in M. Then for any constant C, the following is a finite set:

{ Periodic geodesics with periods $\leqslant$ C }.

4.3

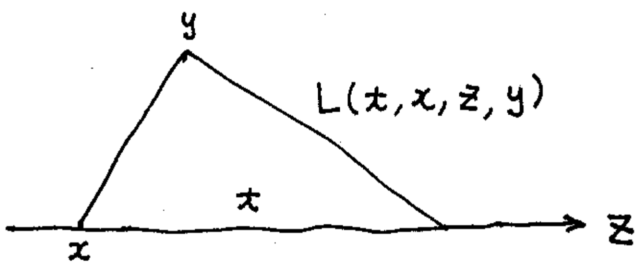
Suppose X is a Hadamard space. Recall the Busemann function $B = B(\alpha, x)$, where α should be a geodesic and $x \in X$.



$$L_t = t + B + \varepsilon, \quad \lim_{t \to +\infty} \varepsilon = 0.$$

Here is a proof that the Busemann is continuous. Suppose X is a topological space, $X \xrightarrow{f} R$ is a map, $\forall x \in X$, $\varepsilon > 0$, there is a neighborhood U of x such that $f(U) \leqslant f(x) + \varepsilon$, then say that f is (+)-continuous. Suppose X is a topological space, $\mathbb{R} \times X \xrightarrow{f(t,x)} \mathbb{R}$ is a continuous map, when $t \longrightarrow +\infty$, $f(t, x)$ monotonically decreases to a finite number, $x \in X$, then $\lim_{t \to +\infty} f(t, x)$ is (+)-continuous. Proof left to the reader. Now suppose X is a Hadamard space. We have the Busemann function

$X \times \partial X \times X \xrightarrow{B(x,z,y)} \mathbb{R}$,



$L(t,x,z,y) = t + B(x,z,y) + \varepsilon$, $\lim_{t \to +\infty} \varepsilon = 0$.

In fact we know that when $t \longrightarrow +\infty$, $L(t,x,z,y) - t$ monotonically decreases to $B(x,z,y)$. So $B(x,z,y)$ is (+)-continuous. But we have the following relation

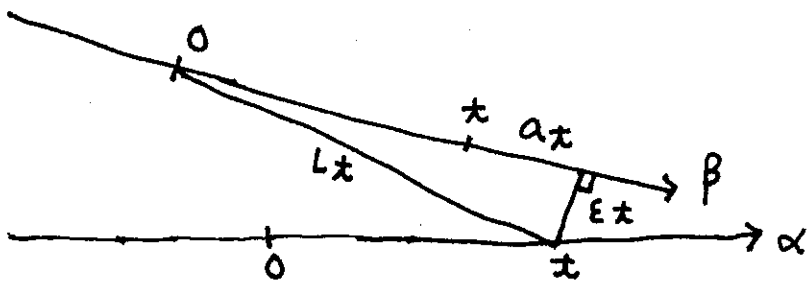
$B(x,z,y) = - B(y,z,x)$.

So $- B(x,z,y)$ is also (+)-continuous. So $B(x,z,y)$ is continuous.

<u>4.3.1. Theorem.</u> Suppose X is a Hadamard space, α and β are asymptotic geodesics such that $d(\mathrm{Im}\,\alpha, \mathrm{Im}\,\beta) = 0$. Then

$$\lim_{t\to+\infty} d\{\alpha(t), \beta[t + B(\alpha, \beta(0))]\} = 0.$$

<u>Proof.</u>



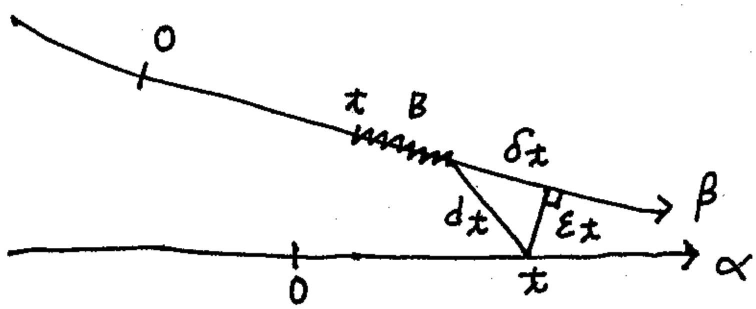
$|L_t - (t + a_t)| \leqslant \varepsilon_t,$

$\varepsilon_t \longrightarrow 0,$

$L_t - t - a_t \longrightarrow 0,$

$L_t - t \longrightarrow B = B(\alpha, \beta(0)),$

$a_t \longrightarrow B.$



$d_t \leqslant \varepsilon_t + \delta_t,$

$\varepsilon_t \longrightarrow 0,$

$\delta_t = |a_t - B| \longrightarrow 0,$

$d_t \longrightarrow 0.$

4.3.2. Corollary. In 4.3.1, with parameterization by horospheres, i.e., levels of the Busemann function, $\lim_{t \to +\infty} d\,[\alpha(t), \beta(t)] = 0.$

4.3.3. Theorem. Suppose X is a Hadamard space such that there is compact set whose isometric transitions cover the whole X, and that there is no totally geodesic $\mathbb{R} \times [a, b]$ in X. Then for any two asymptotic geodesics α and β, $d\,(\mathrm{Im}\alpha, \mathrm{Im}\beta) = 0$.

Proof. The idea of the following proof comes from [EO], 4.13.

First, take any geodesic $\alpha(t)$. Take $(t_k, k \in \mathbb{N}) \subset \mathbb{R}$ such that $\lim_{k \to +\infty} t_k = +\infty$. Take for each $k \geqslant 1$ f_k, an isometry of X such that $\{ f_k \alpha(t_k), k \geqslant 1 \}$ is in a compact set. We may assume that $f_k \alpha(t_k) \longrightarrow x$. We may also assume that $f_k \alpha(+\infty) \longrightarrow z$. That is, by 4.1.7, the geodesics $f_k \alpha(t + t_k)$, $k \geqslant 1$, convergence pointwise to a geodesic $\hat{\alpha}(t)$.

Now suppose α and β are geodesics such that $d\,[\alpha(t), \beta(t)] \leqslant C$, $t \geqslant 0$. We may assume that there are $(t_k, k \geqslant 1) \subset \mathbb{R}$, $t_k \longrightarrow +\infty$, isometries $(f_k, k \geqslant 1)$ of X, and geodesics $\hat{\alpha}(t)$ and $\hat{\beta}(t)$ such that

$$\lim_{k \to +\infty} f_k \alpha(t + t_k) = \hat{\alpha}(t),$$
$$\lim_{k \to +\infty} f_k \beta(t + t_k) = \hat{\beta}(t).$$

Since for any $t \in \mathbb{R}$, when k is large, we have

$$d[f_k \alpha(t + t_k), f_k \alpha(t + t_k)] \leqslant C,$$

we must also have

$$d [\hat{\alpha}(t), \hat{\beta}(t)] \leqslant C, \ t \in \mathbb{R}.$$

By this result, lemma 4.2.1, and our hypothesis, $\hat{\alpha} = \hat{\beta}$ up to parameterization. We may suppose that $\hat{\alpha}(t) = \hat{\beta}(t)$, $t \in \mathbb{R}$. Since

$$d [\alpha(t_k), \beta(t_k)] = d [f_k \alpha(t_k), f_k \beta(t_k)],$$
$$\lim_{k \to +\infty} d [\alpha(t_k), \beta(t_k)] = d [\hat{\alpha}(0), \hat{\beta}(0)] = 0.$$

This completes the proof.

4.4

Suppose M is a riemannian space, $R \xrightarrow{f} R$ is a positive C^{∞} function. Then $M \times \mathbb{R}$ can be given riemannian structure $f(t)^2 ds^2 + dt^2$. The sectional curvature can be figured out explicitly.Computations are left to the reader. Denote the direction of $\mathbb{R}$ in $M \times \mathbb{R}$ by ν . If Σ is a plane of M with curvature K, then in $M \times \mathbb{R}$,

$$K(\Sigma) = \frac{1}{f^2} K - \left(\frac{f'}{f}\right)^2.$$

If $\Sigma = \mathbb{R}X + \mathbb{R}\nu$, where X is a vector of M, then

$K(\Sigma) = -f''/f.$

In general, if Σ is a plane with basis $(X + a\nu)$ and Y, where X and Y are linearly independent vectors of M, with curvature K, then

$$K(\Sigma) = \frac{f^2|X|^2}{f^2|X|^2 + a^2}\left[\frac{1}{f^2}K - \left(\frac{f'}{f}\right)^2\right] + \frac{a^2}{f^2|X|^2 + a^2}\left(-\frac{f''}{f}\right).$$

4.4.1. Corollary. Let $f(t) = (e^t + e^{-t})/2$. Consider $\mathbb{R}^2 \times \mathbb{R}$ with $f(t)^2 ds^2 + dt^2$.

For $\mathbb{R}^2 \times \mathbb{R}$, $K \leqslant 0$.

For $\mathbb{R}^2 \times ((-\infty, -a] \cup [a, +\infty))$, $a > 0$, $K \leqslant -\left(\frac{e^a - e^{-a}}{e^a + e^{-a}}\right)^2$.

Suppose X is a Hadamard space. $f(t) = (e^t + e^{-t})/2$. Consider $X \times \mathbb{R}$. For any curve in $X \times \mathbb{R}$, $\alpha(t) \times \beta(t)$, $0 \leqslant t \leqslant 1$, define its length by the following formula, which can be understood as the maximum of discrete sums.

$$L = \int_0^1 \sqrt{\{f[\beta(t)]\}^2 \cdot |\alpha'(t)|^2 + |\beta'(t)|^2}\ dt.$$

4.4.2. Proposition.

(1). $X \times \mathbb{R}$ is a Hadamard space.

(2). $X \times ((-\infty, -a]\ \ [a, +\infty))$, $a > 0$, is a space with $K \leqslant -\left(\frac{e^a - e^{-a}}{e^a + e^{-a}}\right)^2$.

(3). $X = X \times 0 \subset X \times \mathbb{R}$ is totally geodesic.

(4). $x \times R$ is a geodesic, $x \in X$.

(5). For any isometry σ of X, $\sigma \times Id$ is isometry.

(6). If $\alpha \times \beta$ is geodesic of $X \times \mathbb{R}$, Im α must be geodesic of X.

(7). For any geodesic α of X, $\alpha \times R$ is totally geodesic and is isometric to the standard $\mathbb{H}^2$.

<u>Proof.</u> Proofs of (3)—(7) are direct and left to the reader. We will use 4.1.4 to prove (1) and (2). Here is the trick to do so: We try to compare $X \times \mathbb{R}$ with $\mathbb{R}^2 \times \mathbb{R}$ and find that $X \times \mathbb{R}$ is in some sense more negatively curved that $\mathbb{R}^2 \times \mathbb{R}$. Details are in the next lemma 4.4.3. Now $\mathbb{R}^2 \times \mathbb{R}$ is riemannian and, by 4.4.1, satisfies (1) and (2). So $X \times \mathbb{R}$ satisfies (1) and (2). Done.

Suppose X is a Hadamard space. We try to compare $X \times \mathbb{R}$ with $\mathbb{R}^2 \times \mathbb{R}$. First, suppose $(x \times a, y \times b)$ is a geodesic segment in $X \times \mathbb{R}$, (x, y) is the geodesic segment in X, take a geodesic segment $(\tilde{x}, \tilde{y})$ in $\mathbb{R}^2$ such that $d(x,y) = d(\tilde{x}, \tilde{y})$, then we get a geodesic segment $(\tilde{x} \times a, \tilde{y} \times b)$ in $\mathbb{R}^2 \times \mathbb{R}$. Since $d(x, y) = d(\tilde{x}, \tilde{y})$, there is natural isometry $(x, y) \xrightarrow{\sigma} (\tilde{x}, \tilde{y})$, we get an isometry $(x, y) \times R \xrightarrow{\sigma \times Id} (x, y) \times \mathbb{R}$. In particular, $d(x \times a, y \times b) = d(\tilde{x} \times a, \tilde{y} \times b)$.

Second, we show that in $\mathbb{R} \times \mathbb{R}$, and thus in any $X \times \mathbb{R}$, $d(x \times a, y \times b)$ increases when $d(x, y)$ increases. Suppose

$x \times a$, $y \times b$, $x_0 \times a$, $y_0 \times b$ are four points in $\mathbb{R} \times \mathbb{R}$, $d(x_0, y_0) = d_0 \leq d = d(x, y)$. Suppose $\alpha(t) \times \beta(t)$, $0 \leq t \leq 1$ is a geodesic from $x \times a$ to $y \times b$, after a new parameterization,we may suppose that $|\alpha'(t)| = d$, $0 \leq t \leq 1$. Take a curve $\alpha_0(t)$, $0 \leq t \leq 1$ from x_0 to y_0, with $|\alpha_0'(t)| = d_0$, $0 \leq t \leq 1$, then $\alpha_0(t) \times \beta(t)$, $0 \leq t \leq 1$ is a curve from $x_0 \times a$ to $y_0 \times b$.

$$d(x \times a, y \times b) = \int_0^1 \sqrt{\{f[\beta(t)]\}^2 \cdot d^2 + |\beta'(t)|^2}\, dt,$$

$$d(x_0 \times a, y_0 \times b) \leq \int_0^1 \sqrt{\{f[\beta(t)]\}^2 \cdot d_0^2 + |\beta'(t)|^2}\, dt,$$

$d_0 \leq d$.

That is $d(x_0 \times a, y_0 \times b) \leq d(x \times a, y \times b)$.

Third, remember that any triangle in X is thinner than its equivalent in $\mathbb{R}^2$. With these three points, we see the following is true.

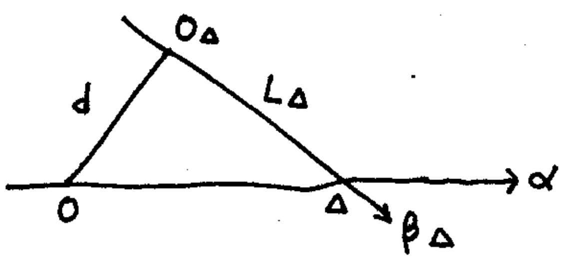
4.4.3. Lemma. Suppose X is a Hadamard space. Take any triangle $\Delta = (x \times a, y \times b, z \times c)$ in $X \times \mathbb{R}$. $\Delta_0 = (x, y, z)$ is the triangle in X. Take a triangle $\tilde{\Delta}_0 = (\tilde{x}, \tilde{y}, \tilde{z})$ in $\mathbb{R}^2$ equivalent to Δ_0 . We get a triangle $\tilde{\Delta} = (\tilde{x} \times a, \tilde{y} \times b, \tilde{z} \times c)$ in $\mathbb{R}^2 \times \mathbb{R}$. In fact Δ and $\tilde{\Delta}$ are equivalent and Δ is thinner than $\tilde{\Delta}$.

4.5

<u>4.5.1. Lemma.</u>

(1). $\mathbb{H}$ denotes the simply connected 2-riemannian space of constant curvature $K \equiv C \leqslant 0$. For any $(B, d) \in \mathbb{R} \times [0, +\infty)$, there is a pair of geodesics (α, β) in $\mathbb{H}$, unique up to isometries, such that $B(\alpha, \beta(0)) = B$ and $d[\alpha(0), \beta(0)] = d$.

(2). Suppose $\mathbb{R} \xrightarrow{L_t} \mathbb{R}$ is a continuous map such that $\lim_{t \to +\infty} (L_t - t) = B$, $d \in [0, +\infty)$. Consider the following in $\mathbb{H}$.



Then, when $\Delta \to +\infty$, $\beta_\Delta \to \beta$, β is a geodesic, α and β are asymptotic, $B(\alpha, \beta(0)) = B$, $d[\alpha(0), \beta(0)] = d$.

(3). Suppose X is a simply connected space with $K \leqslant C$. $\alpha(t)$ and $\beta(t)$ are asymptotic geodesics in X, $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$ are asymptotic geodesics in $\mathbb{H}$ such that $B(\alpha, \beta(0)) = B(\tilde{\alpha}, \tilde{\beta}(0))$, $d[\alpha(0), \beta(0)] = d[\tilde{\alpha}(0), \tilde{\beta}(0)]$. Then

$$d[\alpha(t), \beta(t)] \geqslant d[\tilde{\alpha}(t), \tilde{\beta}(t)], \ t \leqslant 0,$$

$$d[\alpha(t), \beta(t)] \leqslant d[\tilde{\alpha}(t), \tilde{\beta}(t)], \ t \geqslant 0.$$

Suppose X is a Hadamard space, $\gamma(t) = \alpha(t) \times \beta(t)$, $t \in \mathbb{R}$ is a geodesic in $X \times \mathbb{R}$, with arbitrary parameterization. By 4.4.2, $\alpha(t)$ $t \in \mathbb{R}$ is a geodesic in X, X is totally geodesic in $X \times \mathbb{R}$,

$d[\gamma(t), \alpha(t)] = |\beta(t)| = d[\gamma(t), X] = d[\gamma(t), \mathrm{Im}\,\alpha]$, $t \in \mathbb{R}$, $\alpha \times \mathbb{R}$ is totally geodesic in $X \times \mathbb{R}$ and is $\mathbb{H}^2$. We conclude that $\beta(+\infty) = 0, \pm\infty$. By these facts, 4.4.2.(2) and 4.5.1, we see that the following is true.

<u>4.5.2. Theorem.</u> If X is a Hadamard space such that any two asymptotic geodesics have zero distance as sets, then $X \times \mathbb{R}$ has the same property.

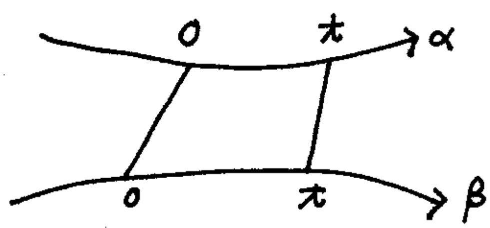
5. The space of geodesics

<u>5.1. Definition.</u> Suppose X is a Hadamard space. Define $G(X) = \{ \mathbb{R} \longrightarrow X \text{ isometry} \}$. The following metric to $G(X)$ appears in [G1] , 8.3.

$$d(\alpha, \beta) = \int_{-\infty}^{+\infty} d[\alpha(t), \beta(t)] \cdot e^{-|t|}\, dt.$$

We will call this the geodesic metric. $\mathbb{R}$ naturally acts on $G(X)$, defining the geodesic flow. For a geodesic $\alpha(t)$, $\tau \in \mathbb{R}$, the action produces a geodesic $\alpha(t + \tau)$. The distance between these two geodesics is $2|\tau|$.

5.2. Consider two geodesics α and β in a Hadamard space.



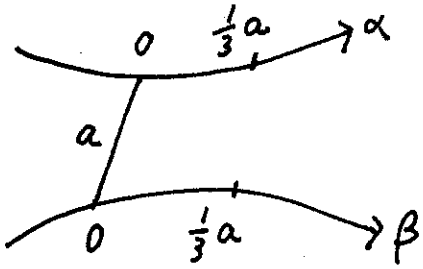
We have the following :

$$d[\alpha(0), \beta(0)] - 2|t| \leqslant d[\alpha(t), \beta(t)] \leqslant d[\alpha(0), \beta(0)] + 2|t|.$$

5.3. As a consequence of 5.2,

$$2d[\alpha(0), \alpha(0)] - 4 \leqslant d(\alpha, \beta) \leqslant 2d[\alpha(0), \beta(0)] + 4.$$

5.4. Still consider 5.2. We compute as follows.



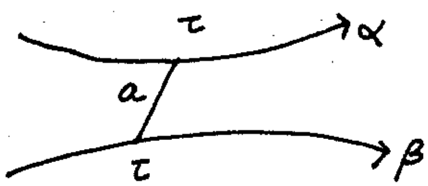
$$\int_{-\infty}^{+\infty} d[\alpha(t),\ \beta(t)]\cdot e^{-|t|}\,dt$$

$$\geqslant \int_{0}^{\frac{1}{3}a} d[\alpha(t),\ \beta(t)]\cdot e^{-|t|}\,dt$$

$$\geqslant \int_{0}^{\frac{1}{3}a} (a - 2/3\ a)\cdot e^{-t}\,dt$$

$$= 1/3\ a\cdot(1 - e^{-\frac{1}{3}a}).$$

In fact the following is also true.

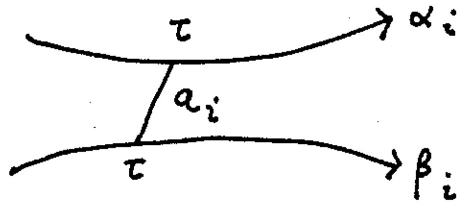


$$\int_{-\infty}^{+\infty} d[\alpha(t),\ \beta(t)]\cdot e^{-|t|}\,dt \geqslant 1/3\ a\cdot(1 - e^{-\frac{1}{3}a})\cdot e^{-|\tau|}.$$

<u>5.5. Theorem.</u> The obvious 1-1 correspondence $G(X) \longleftrightarrow X \times \partial X$ is a homeomorphism.

<u>Proof.</u> It is enough to show that for two sequences of geodesics $\{\alpha_i(t)\}$ and $\{\beta_i(t)\}$, $\lim\limits_{i\to+\infty} d[\alpha_i(t),\ \beta_i(t)] = 0$, $t \in \mathbb{R}$, if and only if $\lim\limits_{i\to+\infty} d(\alpha_i,\ \beta_i) = 0$. See 4.1.7.

First,



$d(\alpha_i, \beta_i) \geq 1/3\, a_i (1 - e^{-\frac{1}{3} a_i}) \cdot e^{-|\tau|}$, by 5.4. This formula shows that if $\lim_{i \to +\infty} d(\alpha_i, \beta_i) = 0$, then $\lim_{i \to +\infty} a_i = 0$, $\forall \tau \in \mathbb{R}$.

On the other hand, suppose $\lim_{i \to +\infty} d[\alpha_i(t), \beta_i(t)] = 0$, $t \in \mathbb{R}$. Then $\lim_{i \to +\infty} d[\alpha_i(0), \beta_i(0)] = 0$, and hence $d[\alpha_i(0), \beta_i(0)] \leq C$, $\forall$ i.

$d[\alpha_i(t), \beta_i(t)] \leq C + 2|t|$, $t \in \mathbb{R}$, by 5.2.

$$\int_{-\infty}^{+\infty} (C + 2|t|) \cdot e^{-|t|}\, dt < +\infty .$$

According to the Lebesque theorem in integration theory,

$$\lim_{i \to +\infty} \int_{-\infty}^{+\infty} d[\alpha_i(t), \beta_i(t)] \cdot e^{-|t|}\, dt$$

$$= \int_{-\infty}^{+\infty} \lim_{i \to +\infty} d[\alpha_i(t), \beta_i(t)] \cdot e^{-|t|}\, dt$$

$$= 0.$$

5.6. Suppose $X \xrightarrow{\pi} X/\Gamma = M$ is a universal covering of manifolds. Suppose also that X has a metric d such that elements of Γ are all isometries. Consider $d(\Gamma x, \Gamma y)$ for $x, y \in X$. One sees that this is a metric for M. In fact for $x, y \in X$, $d(\Gamma x, \Gamma y)$ is reached, because $d(\Gamma x, \Gamma y) = d(x, \Gamma y)$, Γy is discrete. $X \longrightarrow M$ is local isometry. In fact for any $x \in X$, there is open neighborhood U of x so that $d(U, fU) > \mathrm{diam}(U)$, $\forall f \in \Gamma - e$, and thus $U \longrightarrow \pi(U)$ is isometry.

5.7. Suppose M is a space with $K \leq 0$. Define $G(M) = \{ \mathbb{R} \longrightarrow M$ local isometry $\}$. Suppose $X \longrightarrow X/\Gamma = M$ is the univeral covering. 5.5 indicates that $G(X) \longrightarrow G(X)/\Gamma = G(M)$ is a universal covering. Since $G(X)$ has the geodesic metric, for which elements of Γ are obviously isometries, we get a metric for $G(M)$ by 5.6. We call this the geodesic metric of $G(M)$.

<u>5.8. Corollary.</u> Suppose M is a space with $K \leq 0$. Consider $G(M)$ with the geodesic metric. $\mathbb{R}$ naturally acts on $G(M)$, defining the geodesic flow. $G(M) \longrightarrow M$ is a fiber bundle with fiber S^{n-1}, $n = \dim(M)$.

<u>5.9. Corollary.</u> Suppose M is a space with $K \leq 0$ and X is the universal covering. $\alpha(t)$, $\beta(t)$ are geodesics in M and $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ are liftings of $\alpha(t)$, $\beta(t)$. Then $d(\alpha, \beta) \leq d(\tilde{\alpha}, \tilde{\beta})$.

<u>5.10. Proposition.</u> Suppose M is a space with $K \leq 0$. $\alpha(t)$, $\beta(t)$ are geodesics. Then

$$2\,d[\alpha(0),\beta(0)] - 4 \leq d(\alpha,\beta) \leq 2\,d[\alpha(0),\beta(0)] + 4.$$

<u>Proof.</u> Let $X \longrightarrow M$ be the universal covering. Take liftings $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ of $\alpha(t)$, $\beta(t)$, such that $d(\alpha,\beta) = d(\tilde{\alpha},\tilde{\beta})$. By 5.3,

$$2\,d[\tilde{\alpha}(0),\tilde{\beta}(0)] - 4 \leq d(\tilde{\alpha},\tilde{\beta}).$$

Note that $d[\alpha(0),\beta(0)] \leq d[\tilde{\alpha}(0),\tilde{\beta}(0)]$. So

$$2\,d[\alpha(0),\beta(0)] - 4 \leq d(\alpha,\beta).$$

On the other hand, take liftings $\tilde{\tilde{\alpha}}(t)$, $\tilde{\tilde{\beta}}(t)$ of $\alpha(t)$, $\beta(t)$, such that $d[\alpha(0),\beta(0)] = d[\tilde{\tilde{\alpha}}(0),\tilde{\tilde{\beta}}(0)]$. By 5.3,

$$d(\tilde{\tilde{\alpha}},\tilde{\tilde{\beta}}) \leq 2\,d[\tilde{\tilde{\alpha}}(0),\tilde{\tilde{\beta}}(0)] + 4.$$

By 5.9, $d(\alpha,\beta) \leq d(\tilde{\tilde{\alpha}},\tilde{\tilde{\beta}})$. So

$$d(\alpha,\beta) \leq 2\,d[\alpha(0),\beta(0)] + 4.$$

<u>5.11. Theorem.</u> Suppose X is a Hadamard spcae, and $\alpha(t)$, $\beta(t)$ are geodesics. Then $\lim\limits_{t\to+\infty} d[\alpha(t),\beta(t)] = 0$ if and only if in G(X) $\lim\limits_{\tau\to+\infty} d(\tau\cdot\alpha,\ \tau\cdot\beta) = 0$.

<u>Proof.</u> Denote $d_t = d[\alpha(t),\beta(t)]$. By 5.4,

$$\int_{-\infty}^{+\infty} d_{\tau+t}\cdot e^{-|t|}\,dt \geq 1/3\; d_\tau\cdot(1 - e^{-1/3\, d_\tau}).$$

One sees that if

$$\lim_{\tau\to+\infty}\int_{-\infty}^{+\infty} d_{\tau+t}\cdot e^{-|t|}\,dt = 0,$$

then $\lim_{\tau \to +\infty} d_\tau = 0$.

Now suppose $\lim_{t \to +\infty} d_t = 0$. Then $d_t \leqslant C$, $t \geqslant 0$. By 5.2, $d_{\tau+t} \leqslant d_\tau + 2|t|$. So $d_{\tau+t} \leqslant C + 2|t|$, $\tau \geqslant 0$, $t \in \mathbb{R}$. Note that

$$\int_{-\infty}^{+\infty} (C + 2|t|)\cdot e^{-|t|}\, dt < +\infty .$$

So, by the Lebesque theorem,

$$\lim_{\tau \to +\infty} \int_{-\infty}^{+\infty} d_{\tau+t}\cdot e^{-|t|}\, dt$$

$$= \int_{-\infty}^{+\infty} \lim_{\tau \to +\infty} d_{\tau+t}\cdot e^{-|t|}\, dt$$

$= 0$.

This completes the proof.

5.12. Definition. For a Hadamard space X, we know by 4.3.1 and 5.11 that the following two conditions are equivalent.

(1). For any two asymptotic geodesics α and β, $d(\mathrm{Im}\,\alpha, \mathrm{Im}\,\beta) = 0$.

(2). For any two asymptotic geodesics α and β, in G(X), $\lim_{\tau \to +\infty} d\{\tau\cdot\alpha,\ [\tau + B(\alpha, \beta(0))]\cdot\beta\} = 0$.

In fact, by 4.1.5 and by the appearance of the geodesic metric in 5.1, the term on the left hand side must monotonically decrease to zero.

Suppose M is a space with $K \leq 0$ such that its universal covering satisfies these conditions, then we say that the geodesic flow of M, G(M), is weakly Anosov.

<u>5.13. Corollary.</u> Suppose M is a compact space with $K \leq 0$ such that there is no totally geodesic $\mathbb{R} \times [a, b]$ immersed in it, then by 4.3.3, 4.5.2, and 5.12, the geodesic flows of both M and $M \times \mathbb{R}$ are weakly Anosov. Or suppose M is a space with $K \leq C < 0$, then by 4.5.1 and 5.12, the geodesic flow of M is weakly Anosov.

6. Changing h-cobordisms

6.1. Let M be a space with $K \leqslant 0$. Suppose $\alpha(t)$ is a curve in M and $\gamma(s)$ is a geodesic such that $\alpha(0) = \gamma(0)$. Consider the universal covering $X \longrightarrow X/\Gamma = M$. Lift $\alpha(t)$ and $\gamma(s)$ to $\tilde{\alpha}(t)$ and $\tilde{\gamma}(s)$ such that $\tilde{\alpha}(0) = \tilde{\gamma}(0)$. At $\tilde{\alpha}(t)$, draw the geodesic asymptotic to $\tilde{\gamma}(s)$. Express the result as $\tilde{\gamma}_{\tilde{\alpha}(t)}(s)$. For any two liftings, the results are the same up to an isometry in Γ. So we can go back to M to get only one result, denoted $\gamma_{\alpha(t)}(s)$.

6.2

Let M be a space with $K \leqslant 0$. W is an h-cobordism over M. By chapter 2, we have $W \xrightarrow{p_t} W$, $0 \leqslant t \leqslant 1$ and $W \xrightarrow{q_t} W$, $0 \leqslant t \leqslant 1$, strong deformations of W to M and to the other boundary, $W \xrightarrow{p=p_1} M$, and everything is trivial outside a compact set of M.

Consider

$\hat{W} = G(M) \times_M W$

$= \{ (\gamma, x) : \gamma \in G(M),\ x \in W,\ \gamma(0) = p(x) \}.$

$\hat{W}$ is a manifold with two boundaries, one $G(M)$.

Take a point in $\hat{W}$, that is, a geodesic γ in M and a point $x \in W$ such that $\gamma(0) = p(x)$. We produce the following two curves in $\hat{W}$, with notation from 6.1.

$$\left(\gamma_{p[p_t(x)]},\ p_t(x) \right),\quad 0 \leq t \leq 1.$$

$$\left(\gamma_{p[q_t(x)]},\ q_t(x) \right),\quad 0 \leq t \leq 1.$$

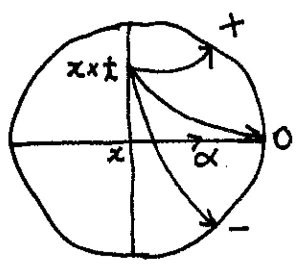
All the curves of the first kind form a strong deformation of $\hat{W}$ to $G(M)$ and those of the second kind form one to the other boundary of $\hat{W}$. And everything is trivial outside a compact set of $G(M)$. So $\hat{W}$ becomes an h-cobordism over $G(M)$. $\hat{W}$ is called the asymptotic lifting of W.

6.2.1. Corollary. If $\alpha(t)$, $0 \leq t \leq 1$ is an associated curve of W, then

$\{ \gamma_{\alpha(t)},\ 0 \leq t \leq 1 : \gamma \in G(M),\ \gamma(0) = \alpha(0) \}$

are associated curves of $\hat{W}$. If we go through all associated curves of W, then we get all associated curves of $\hat{W}$.

6.3. Suppose X is a Hadamard space. Consider $G(X \times \mathbb{R})$. Take $x \times t \in X \times \mathbb{R}$, take a geodesic α in X beginning at x. Then $\alpha \times \mathbb{R}$ is totally geodesic $\mathbb{H}^2$ and contains $x \times t$. So we can see geodesics through $x \times t$ that are in $\alpha \times \mathbb{R}$. Let α change. We will see all geodesics through $x \times t$.



We see that $G(X \times \mathbb{R}) = G^{+} \cup G^{0} \cup G^{-}$ where $G^{\pm}$ are bundles with fiber D^{n} and G^{0} is bundle with fiber S^{n-1}, $n = \dim(X)$. The decomposition respects isometries of X and the action of time. Also see 4.5.

6.4

Suppose X is a Hadamard space whose geodesic flow is weakly Anosov in the sense of 5.12. For a point $x \in X$ and a direction $z \in \partial X$, we denote the geodesic from x to z as $\gamma = \gamma(x, z)$. Suppose $\alpha(t)$, $0 \le t \le 1$ is a curve in X. Then

$\{\gamma(\alpha(t), z), 0 \le t \le 1 : z \in \partial X\}$

is a collection of curves in $G(X)$. We want to study the behavior of the these curves as time goes on. More precisely, we want to study the behavior of the diameter, with respect to the geodesic flow , of the following collection of curves in G(X),

$\{ \tau \cdot \gamma(\alpha(t), z),\ 0 \leq t \leq 1 : z \in \partial X \}$,

as $\tau \longrightarrow +\infty$.

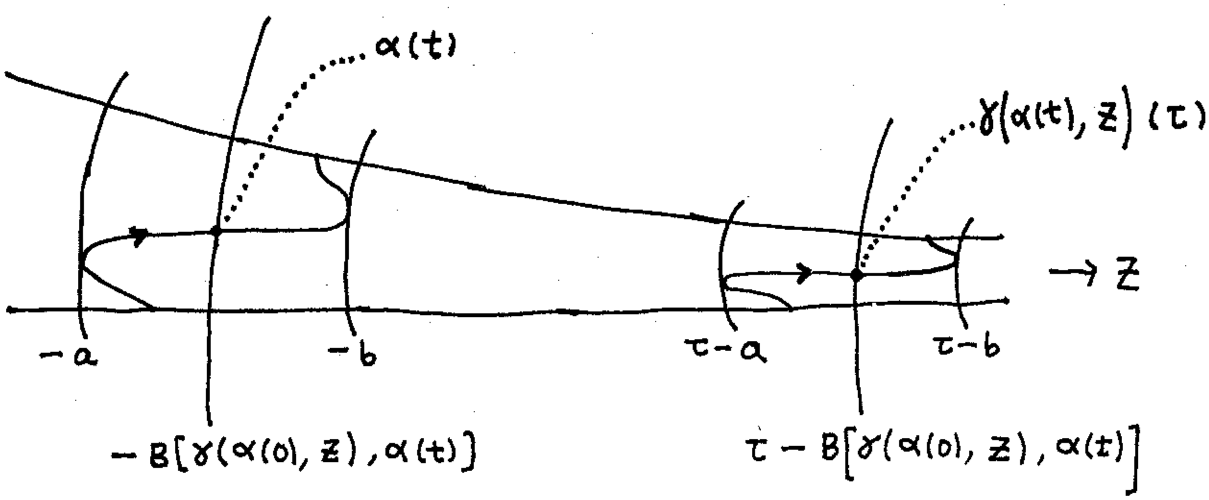
We set about doing so. Inspired by 5.12, we compare the following two expressions.

$\tau \cdot \gamma(\alpha(t), z)$,

$\{ \tau - B[\gamma(\alpha(0), z), \alpha(t)]\} \cdot \gamma(\alpha(0), z)$.

Consider $B[\gamma(\alpha(0), z), \alpha(t)]$, $0 \leq t \leq 1$. Let $a = a(z)$ and $b = b(z)$ be its maximum and minimum. Then the second expression is inside the following interval.

$I = [\tau - a, \tau - b] \cdot \gamma(\alpha(0), z)$.



By 5.1, length(I) = 2(a - b). Consider 2[a(z)- b(z)],

$z \in \partial X$. Let ℓ be its maximum.

On the other hand, denote the distance between the two expressions as f:

$\mathbb{R} \times [0, 1] \times \partial X \xrightarrow{f(\tau, t, z)} \mathbb{R}$.

By our assumption that 5.12 is satisfied, when $\tau \to +\infty$, f monotonically decreases to zero. Consider $f(\tau, t, z)$, $t \in [0, 1]$, $z \in \partial X$. Let $\varepsilon = \varepsilon(\tau)$ be its maximum.

6.4.1. Lemma. Suppose Y is a compact topological space, $\mathbb{R} \times Y \xrightarrow{f(\tau, y)} \mathbb{R}$ is continuous map, when $\tau \to +\infty$, $f(\tau, y)$ monotonically decreases to zero, $y \in Y$. $\varepsilon = \max\{f(\tau, y) : y \in Y\}$. Then $\lim_{\tau \to +\infty} \varepsilon = 0$.

Proof. Fix any $\delta > 0$. $\forall\, y \in Y$, $f(+\infty, y) = 0$, so there is $f(\tau(y), y) \leq \delta/2$. So there is a neighborhood $U(y)$ of y such that $f|_{\tau(y) \times U(y)} \leq \delta$. So $f|_{[\tau(y), +\infty) \times U(y)} \leq \delta$. Suppose $Y = U(y_1) \cup \cdots U(y_k)$. Let $\tau_0 = \max\{\tau(y_1), \cdots \tau(y_k)\}$. Then $f|_{[\tau_0, +\infty) \times Y} \leq \delta$.

6.4.2. Proposition. We have proved the following: Suppose X is a Hadamard space whose geodesic flow is weakly Anosov. $\alpha(t)$, $0 \leq t \leq 1$ is a curve in X. Then

$\{\tau \cdot \gamma(\alpha(t), z),\ 0 \leq t \leq 1 : z \in \partial X\}$

has diameter $\leq (\ell, \varepsilon)$, where ℓ is a constant and $\lim_{\tau \to +\infty} \varepsilon = 0$.

The following result is proved in the same way, plus 5.9 and 6.1. See 6.1 for the notation used.

<u>6.4.3. Proposition.</u> Suppose M is a space with $K \leq 0$ whose geodesic flow is weakly Anosov. P is a comapct topological space, $[0, 1] \times P \xrightarrow{\alpha(t,p)} M$ is a continuous map. Then in G(M)

$\{\tau \cdot \gamma_{\alpha(t,p)}, 0 \leq t \leq 1 : \gamma \in G(M), p \in P, \gamma(0) = \alpha(0, p)\}$

has diameter $\leq (\ell, \varepsilon)$, where ℓ is a constant and $\lim_{\tau \to +\infty} \varepsilon = 0$.

<u>6.5. Proof of theorems 1.1 and 1.3.</u>

Suppose M is a compact space with $K \leq 0$ such that there is no totally geodesic $\mathbb{R} \times [a, b]$ immersed in it. Take an h-cobordism W over $M \times \mathbb{R}$. By 6.2, we have h-cobordism $\hat{W}$ over $G(M \times \mathbb{R})$. $\hat{W}$ is lifting of W. If $\hat{W}$ is a product, then $\chi(S^n) \cdot \tau(W) = 0$, where $\chi(S^n) = 1 + (-1)^n$, $n = \dim(M)$. By 6.3, $G(M \times \mathbb{R}) = G^+ \cup G^0 \cup G^-$. If we in fact show that $\hat{W}$ has a product respecting this decomposition, then $\tau(W) = 0$.

By 5.13, $G(M \times \mathbb{R})$ is weakly Anosov. By 6.2.1 and 6.4.3, the diameter of the result of the time action τ of the associated curves in $G(M \times \mathbb{R})$ of $\hat{W}$ is bounded by (ℓ, ε), where ℓ is a constant and $\lim_{\tau \to +\infty} \varepsilon = 0$. If $\hat{W}$ after applying an action of time is a product, then $\hat{W}$ must be a product. So we can assume that $\hat{W}$ is (ℓ, ε) - controlled, where ℓ is a constant and ε can be as small as we like.

Note that periodic geodesics of $M \times \mathbb{R}$ are in $M \times 0 = M$. By 4.2.4, in M the number of periodic geodesics having periods $\leq C \cdot \ell$ is finite. $C = C(n)$ is the number in in 3.3. Take these periodic geodesics away from $G(M \times \mathbb{R})$. Apply 3.3 to see that we may assume $\hat{W}$ has a product structure P over $G(M \times \mathbb{R})|_{M \times [-a, a]}$ but away from those periodic geodesics taken out, with diameter bounded by $(C \cdot \ell, 1)$, where a can be as large as we like. Since S^1 has no torsion, P can be extended over those periodic geodesics taken away. In conclusion, we may assume that $\hat{W}$ has a product structure P over $G(M \times \mathbb{R})|_{M \times [-a, a]}$, the diameter of P at $G(M \times \mathbb{R})|_{M \times (\pm a)}$ and the diameter of $\hat{W}$ over $G(M \times \mathbb{R})|_{M \times ((-\infty, -a] \cup [a, +\infty))}$ have a bound A, where A is a constant and a can be as large as we like.

Consider for example $G(M \times \mathbb{R})|_{M \times [a, +\infty)}$. Consider

$G(M \times \mathbb{R})|_{M\times[a,+\infty)} \xrightarrow{\pi} M \times [a, +\infty)$. 5.10 says that $2\, d_o(\pi\times\pi) - 4 \leq d$, $d_o(\pi\times\pi) \leq 1/2\, d + 2$. So a set in $G(M \times R)|_{M\times[a,+\infty)}$ with diameter $\leq A$ is mapped to $M \times [a, +\infty)$ to be a set with diameter $\leq 1/2\, A + 2$. Suppose $\alpha(t) \times \beta(t)$, $0 \leq t \leq 1$ is a curve in $M \times [a, +\infty)$ with length L.

$$L = \int_0^1 \sqrt{\{f[\beta(t)]\}^2 \cdot |\alpha'(t)|^2 + |\beta'(t)|^2} \quad dt,$$

$$f(t) = (e^t + e^{-t})/2 \quad \geq t,\ t \geq 0,$$

$$L \geq \int_0^1 \sqrt{a^2 |\alpha'(t)|^2 + |\beta'(t)|^2 \quad dt}$$

$$= a \cdot \int_0^1 \sqrt{|\alpha'(t)|^2 + |\beta'(t)/a|^2} \quad dt.$$

Consider the following map

$$\begin{tikzcd} G(M \times \mathbb{R})|_{M\times[a,+\infty)} \arrow[r, "\pi"] \arrow[rr, bend right] & M \times [a, +\infty) \arrow[r, "Id \times \frac{1}{a} Id"] & M \times [1, +\infty). \end{tikzcd}$$

A set in $G(M \times \mathbb{R})|_{M\times[a,+\infty)}$ with diameter $\leq A$ is mapped to $M \times [1, +\infty)$, with product metric, to be a set with diameter $\leq 1/a\ (1/2\, A + 2)$.

Let a be large. Also note that the above map is a fiber bundle with fiber S^n. By 2.1, P can be extended. This completes the proof of 1.1 and 1.3.

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