On the global influence of conjugate points

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Abstract of the Dissertation

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Let M be a complete riemannian manifold of dimension n.

A new criterion is given for the occurence of conjugate points along a unit speed geodesic $c: [0, L] \to M$. Writing $Ric: SM \to R$ for the Ricci curvature function on the unit tangent bundle of M (so that Ric is $n-1$ when M is the standard n-dimensional sphere), the criterion is that $c(0)$ has a conjugate point along the geodesic c whenever

$$\int_0^L Ric(c'(t))\, dt \ \geq\ \pi(n-1)^{1/2} \sqrt{\max_{t\epsilon[0,L]}(0, Ric(c'(t)))}\ .$$

Birkhoff's Ergodic Theorem can now be used to give us

$$\int\limits_M Scal \ \leq\ \frac{\pi(n-1)^{1/2}n}{vol(S^{n-1}, can)} \sqrt{sup(0, Ric)} \int\limits_{SM} \underline{\psi}$$

whenever M is of finite volume, with Ricci curvature bounded above.

Here $\underline{\psi}: SM \to [0,\infty]$ is defined by

$$\underline{\psi}(v) = \liminf_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \text{ where } c_v(t) = exp(tv) \end{pmatrix}$$

and integration on the unit tangent bundle SM is with respect to the Liouville measure.

The above inequality generalizes the inequality

$$\int_M Scal \leq 0$$

of L.W. Green, which he proved for M compact and without conjugate points.

It is also shown that if M is a compact, connected n-dimensional riemannian manifold without conjugate points, and M has a nilpotent fundamental group, and M has an isometry group of at least dimension $n-2$, then M is a flat n-dimensional torus. The case where $n = 2$ was proved by E. Hopf.

To My Parents

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ACKNOWLEDGMENTS

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Chapter 1.

Introduction

1. *Unanswered questions*

During the 1970's and 1980's, great progress was made in the study of complete riemannian manifolds of nonpositive curvature. A larger class of complete riemannian manifolds are those without conjugate points. Very little is known about them.

Many geometers (M. Anderson, W. Ballman, M. Brin, K. Burns, C. Croke, G. Knieper, V. Schroeder) have focussed their attention on manifolds without conjugate points during the mid 1980's. Motivating almost all of the questions being asked, and occasionally answered, is the desire to know just how different being without conjugate points is from having nonpositive curvature.

The two major unanswered questions at the present time are

Question 1. *Given an n-dimensional manifold with a complete riemannian metric without conjugate points, does the manifold admit a complete riemannian metric of nonpositive curvature?*

Question 2. *Given an n-dimensional torus with a riemannian metric without conjugate points, is it flat?*

Both questions are open for n greater than 2.

This thesis has resulted from unsuccesfully attempting to answer the second question.

2. Some history

Complete riemannian manifolds without conjugate points had been studied before the mid 1980's, and several of their properties had been established. The following five results summarize almost all that was known before the surge of recent activity [C] [H] [Gr1] [Gu] [O'S].

Theorem (Cartan-Hadamard 1928). *Complete riemannian manifolds without conjugate points are smoothly covered by Euclidean space.*

In fact, letting M denote the riemannian manifold, $exp_p: TM_p \to M$ is a smooth covering for all points p of M.

Theorem (E. Hopf 1948). *Any riemannian metric without conjugate points on a 2-dimensional torus is flat.*

Theorem (L.W. Green 1958).

(1) For M a compact riemannian manifold without conjugate points,

$$\int_M Scal \leq 0 .$$

(2) Equality occurs precisely when M is flat.

Theorem (J. Gulliver 1974). *There exist compact riemannian manifolds without conjugate points which possess regions of positive scalar curvature.*

Theorem (J.J. O'Sullivan 1974). *Any Killing field on a compact riemannian manifold without conjugate points is parallel.*

In fact, any Killing field X on a complete riemannian manifold without conjugate points is parallel, provided that $||X||$ has an upper bound which is attained.

Cartan actually stated the Cartan-Hadamard theorem as a nonpositive curvature result, but the proof is applicable to the no conjugate point case (Hadamard proved the 2 dimensional case).

Hopf's theorem was the first major result that required techniques unnecessary for dealing with nonpositive curvature. The nonpositive curvature analogue of his theorem can be proved by the Gauss-Bonnet theorem of the last century.

Green's generalization of Hopf's result resulted from an examination of Hopf's techniques for surfaces, and then creating the necesary generalizations for dealing with higher dimensional spaces.

3. An overview

As mentioned before, this thesis arose from an attempt to prove or disprove the famous

Hopf Conjecture. *Any riemannian metric without conjugate points on an n dimensional torus is flat.*

A natural simplification of the problem is to assume some symmetry. Along these lines, John J. O'Sullivan has shown

Theorem (J.J. O'Sullivan 1974). *Any homogeneous riemannian metric without conjugate points on an n-dimensional torus is flat.*

In fact, O'Sullivan showed more [O'S]. A homogeneous compact connected riemannian manifold without conjugate points of dimension n, must be a flat n-dimensional torus (this result is an immediate consequence of O'Sullivan's Killing field theorem, together with the well known fact that the only homogeneous flat compact connected riemmanian manifolds are flat tori).

In chapter 6, it is established that the following is true

Theorem. *Any riemannian metric without conjugate points on an n-dimensional torus, that has an isometry group of at least dimension $n-2$, is flat.*

In fact, we have that if M is a compact, connected n-dimensional riemannian manifold without conjugate points, and the fundamental group of M is nilpotent, and the dimension of the isometry group of M is at least $n-2$, then M is a flat torus. The case where $n=2$ was proved by E. Hopf [H].

Hopf's proof of the case where $n=2$ inspired L.W. Green to show that for M a compact riemannian manifold without conjugate points,

$$\int_M \mathrm{Scal} \leq 0 \ ,$$

with equality occuring precisely when M is flat [Gr1].

In Chapter 5, it is established that the following generalization of Green's inequality statement holds

Theorem. *For M a compact riemannian manifold of dimension n,*

$$\int_M Scal \leq \frac{\pi(n-1)^{1/2}n}{vol(S^{n-1}, can)} \sqrt{max(0, supRic)} \int_{SM} \underline{\psi} \ .$$

Here SM is the unit tangent bundle with the induced Liouville measure, $Ric: SM \to R$ is the Ricci curvature, and $\underline{\psi}: SM \to R$ is defined by

$$\underline{\psi}(v) = \liminf_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c(0) \text{ along } c: [0,T] \to M \\ \text{where } c(t) = exp(tv) \end{pmatrix}.$$

Note that if M is a compact riemannian manifold of constant sectional curvature 1, we have that $Scal = n(n-1)$, $Ric = (n-1)$, $\underline{\psi} = 1/\pi$, and $vol(SM) = vol(M)vol(S^{n-1}, can)$, so that equality occurs in the above.

At present, a generalization of L.W. Green's equality statement does not exist. A plausible conjecture would be that equality occurs precisely when M has constant sectional curvature.

The above inequality can be generalized further. For example, it's true if M is of finite volume with Ricci curvature bounded above. Such improvements are discussed in Chapter 5.

In arriving at the above generalization of Green's inequality, various results of independent interest were stumbled upon. They are discussed in chapters 3 and 4.

Chapter 2.

Preliminaries

1. *Conjugate points along geodesics of a riemannian manifold*

We refer to [C-E] as a basic reference.

Let M be a riemannian manifold, and let $c: [0, L] \to M$ be a unit speed geodesic on M. $c(\tau)$ is said to be a conjugate point of $c(0)$ along c (where $0 < \tau \leq L$) if

$$exp_{c(0)}: TM_{c(0)} \to M$$

is singular at $\tau c'(0)$.

The multiplicity of the conjugate point is defined to be the dimension of the nullspace of

$$dexp_{c(0)}(\tau c'(0)): (TM_{c(0)})_{\tau c'(0)} \to TM_{c(\tau)} .$$

It can be any integer ranging from 1 to $n - 1$.

To study conjugate points, two extremely useful tools are Jacobi fields and the Index form.

$J: [0, L] \to TM$ is a Jacobi field along c if

$$\frac{D^2 J}{dt^2} + R(J, c')c' = 0 .$$

It is a smooth vector field over c.

$c(\tau)$ is conjugate to $c(0)$ precisely when there is a Jacobi field J along c, other than the zero field, that vanishes at 0 and τ. The multiplicity of $c(\tau)$ is then the dimension of the vector space generated by all such Jacobi fields.

The index form associated to c, I_c, is defined to be

$$I_c(V,W) = \int_0^L < \frac{DV}{dt}, \frac{DW}{dt} > - < R(V,c'(t))c'(t), W > dt$$

where V, W are continuous piecewise smooth vector fields on c that are orthogonal to c, and vanish at 0 and L. Such vector fields will be called admissible (or c-admissible, whenever there might be confusion).

The index form is bilinear and symmetric.

There exists a conjugate point $c(\tau)$ to $c(0)$ along c, $0 < \tau \leq L$, precisely when there exists an admissible vector field V on c, where V is not the zero field, for which $I_c(V,V) \leq 0$. If the only conjugate point to $c(0)$ along c is $c(L)$, then the admissible V for which $I_c(V,V) \leq 0$ are precisely the Jacobi fields vanishing at 0 and L.

The following result will be used repeatedly

The Morse Index Theorem. *The number of conjugate points to $c(0)$ along $c|_{[0,L)}$, counted according to multiplicity, is equal to the dimension of a maximal subspace of admissible fields for which I_c is negative definite.*

The counting of conjugate points will rely heavily on the above statement. In this thesis, counting of conjugate points will usually not be according to multiplicity.

Lemma 1. *If* $0 < T_1 < T_2 \leq L$ *and it is known that*

$$c(0) \textit{ has a conjugate point along } c|_{[0,T_1]},$$

$$\textit{and } c(T_1) \textit{ has a conjugate point along } c|_{[T_1,T_2]},$$

then it follows that $c(0)$ *has a conjugate point* $c(T)$ *along* c*, where*

$$T_1 < T \leq T_2 \quad .$$

Lemma 2. *If* $0 < T_1 < T_2 < \ldots < T_k \leq L$ *and it is known that*

$$c(0) \textit{ has a conjugate point along } c|_{[0,T_1]}$$

$$c(T_1) \textit{ has a conjugate point along } c|_{[T_1,T_2]}$$

$$\ldots$$

$$c(T_{k-1}) \textit{ has a conjugate point along } c|_{[T_{k-1},T_k]}$$

then it follows that $c(0)$ *has at least* k *conjugate points along* c*.*

In Lemma 3, conjugate points are counted according to multiplicity.

Lemma 3. *If* $0 < T_1 < T_2 < \ldots < T_k \leq L$ *and it is known that*

$$c(0) \textit{ has } \alpha_1 \textit{ conjugate points along } c|_{[0,T_1]}$$

$$c(T_1) \textit{ has } \alpha_2 \textit{ conjugate points along } c|_{[T_1,T_2]}$$

...

$$c(T_{k-1}) \textit{ has } \alpha_k \textit{ conjugate points along } c|_{[T_{k-1},T_k]}$$

then it follows that

$$\sum_{i=1}^{k} \alpha_i - k(n-1) \leq \alpha \leq \sum_{i=1}^{k} \alpha_i + k(n-1) \ ,$$

where α is the number of points conjugate to $c(0)$ along $c|_{[0,T_k]}$.

2. The geodesic flow and Birkhoff's Ergodic Theorem

We refer to [P] as a basic reference.

Let M be a complete riemannian manifold. Let SM be its unit tangent bundle.

The geodesic flow $G: SM \times R \to SM$ on SM is defined by $G(v,t) = c'(t)$, where $c(s) = exp(sv)$. $G(v,t)$ will often be written as $G_t(v)$.

The unit tangent bundle possesses a Borel measure determined by the riemannian structure of its base manifold. It's called the Liouville measure, and the geodesic flow is invariant with respect to it.

Birkhoff's Ergodic Theorem applies to measure spaces with a measure invariant flow. In our case, we have

Birkhoff's Ergodic Theorem. *Let M be a complete riemannian manifold, let $G: SM \times R \to SM$ be the geodesic flow, and let $f: SM \to R$ be a function whose positive or negative part is integrable with respect to the Liouville measure μ. Then*

(1) The following limit exists for almost all v in SM

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(G_t v)\,dt\ .$$

(2) If A is a flow invariant subset of SM of finite measure, then

$$\int_A f(v)\,d\mu(v) \;=\; \int_A \lim_{T\to\infty} \frac{1}{T}\int_0^T f(G_t v)\,dt\,d\mu(v)\ .$$

Letting α be any real number, the above statement is also true when

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(G_t v)\, dt$$

is replaced by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(G_{k\alpha} v) \ .$$

3. Proofs of lemmas

Proof of Lemma 1.

Without loss of generality $T_2 < L$.

Pick $\epsilon_2 > 0$ $(\epsilon_2 < L - T_2)$.

Then there exists $\epsilon_1 > 0$ for which $c(T_1 + \epsilon_1)$ has a conjugate point along $c|_{[T_1+\epsilon_1, T_2+\epsilon_2)}$.

Take a maximal subspace of $c|_{[0,T_1+\epsilon_1]}$-admissible vector fields for which the index form is negative definite, and extend it to a subspace of $c|_{[0,T_2+\epsilon_2]}$-admissible vector fields by taking the vector fields to be zero outside of $[0, T_1 + \epsilon_1]$. Call the resulting subspace W_1.

Now take a maximal subspace of $c|_{[T_1+\epsilon_1,T_2+\epsilon_2]}$-admissible vector fields for which the index form is negative definite, and extend it in a similar way to give W_2.

By the Morse Index Theorem $dim(W_2) > 0$, so that $dim(W_1 + W_2) > dim(W_1)$.

Applying the Morse Index Theorem again, we see that $c(0)$ has a conjugate point $c(T)$ along c, where $T_1 + \epsilon_1 \leq T < T_2 + \epsilon_2$.

Now, there exists S satisfying $T_1 < S < T_2$, for which $c(0)$ has no conjugate points $c(\tau)$ satisfying $T_1 < \tau < S$.

This tells us that $c(0)$ has a conjugate point $c(T)$ along c, where $S \leq T < T_2 + \epsilon_2$.

This is true for any positive ϵ_2, and $S > T_1$, so that $c(0)$ has a conjugate point $c(T)$ along c, where $T_1 < T \leq T_2$.

Proof of Lemma 2.

By repeated application of Lemma 1, there exist conjugate points

$$c(\tau_1), c(\tau_2), \ldots, c(\tau_k)$$

to $c(0)$ along c, satisfying

$$0 < \tau_1 \leq T_1 \leq \tau_2 \leq T_2 < \ldots < \tau_k \leq T_k \ .$$

Proof of Lemma 3. Define T_0 to be 0. Note that conjugate points are counted according to multiplicity in this proof.

Take a maximal subspace of $c|_{[T_{i-1},T_i]}$-admissible vector fields for which the index form is negative definite. Call it $\underline{W_i}$. Extend it to a subspace of $c|_{[0,T_k]}$-admissible vector fields by taking the vector fields of $\underline{W_i}$ to be zero outside of $[T_{i-1}, T_i]$. Call this subspace W_i. Let W be the direct sum of the W_i's.

Since $c(T_{i-1})$ has at least $\alpha_i - (n-1)$ conjugate points along $c|_{[T_{i-1},T_i)}$, we have that $dim(W_i) \geq \alpha_i - (n-1)$, and so

$$dim(W) \geq \sum_{i=1}^{k} \alpha_i - k(n-1) \ .$$

Since $\alpha \geq dim(W)$, we have

$$\sum_{i=1}^{k} \alpha_i - k(n-1) \leq \alpha \ .$$

Since $c(T_{i-1})$ has at most α_i conjugate points along $c|_{[T_{i-1},T_i)}$, we have that $dim(W_i) \leq \alpha_i$, and so

$$dim(W) \leq \sum_{i=1}^{k} \alpha_i \ .$$

Now extend W to a maximal subspace of $c|_{[0,T_k]}$-admissible vector fields for which the index form is negative definite. Call this space X, and note that

$$\alpha \leq dimX + (n-1)$$

.

Let W' be the orthogonal complement of W in X (with respect to I). By showing that $dimW' \leq (k-1)(n-1)$, we are done, since then

$$\begin{aligned} \alpha &\leq dimX + (n-1) \\ &= dimW + dimW' + (n-1) \\ &\leq \sum_{i=1}^{k} \alpha_i + (k-1)(n-1) + (n-1) \ . \end{aligned}$$

Suppose that $dimW' > (k-1)(n-1)$.

Then there exists a non-trivial vector field V in W' for which

$$V(T_1) = 0, V(T_2) = 0, \ldots, V(T_{k-1}) = 0 \ .$$

For some j from 1 to k, $V_j = V|_{[T_{j-1},T_j]}$ is a non-trivial vector field for which $I(V_j, V_j) < 0$, and $I(V_j, \underline{W_j}) = 0$ (since $I(V, W) = 0$). This contradicts the maximal property of $\underline{W_j}$.

Chapter 3.

Curvature criteria for conjugate points along geodesics

1. *Introduction and Main Theorems*

Differential geometry and the Sturm-Liouville theory of second order differential equations are not disjoint areas of mathematics. The following two theorems illustrate this [L] [M].

Theorem 1 (Sturm). *Consider the second order differential equation $x'' + Fx = 0$, where F is a continuous function defined on $[0, L]$.*

Let $z: [0, L] \to R$ be a solution for which $z(0) = 0$ and $z'(0) \neq 0$. If

$$F(t) \geq \frac{\pi^2}{L^2} \quad \textit{for all } t \textit{ in } [0, L]$$

then $z(T) = 0$ for some T in $(0, L]$.

Furthermore, if the smallest such T is L, then $F(t) = \pi^2/L^2$ for all t in $[0, L]$.

Theorem 2 (Myers 1941). *Let $c: [0, L] \to M$ be a unit speed geodesic on a riemannian manifold of dimension n. If*

$$Ric(c'(t)) \geq (n-1)\frac{\pi^2}{L^2} \quad \textit{for all } t \textit{ in } [0, L]$$

then $c(0)$ has a conjugate point $c(T)$ along c, for some T in $(0, L]$.

Furthermore, if the smallest such T is L, then $K(\sigma) = \pi^2/L^2$ for all tangent two planes σ containing a tangent vector to γ.

The following analogues are also true.

Theorem 3. *Consider the second order differential equation $x'' + Fx = 0$, where F is a continuous function defined on $[0, L]$.*

Let $z: [0, L] \to R$ be a solution for which $z(0) = 0$ and $z'(0) \neq 0$. If

$$\int_0^L F(t)\, dt \;\geq\; \pi \sqrt{max(0, \max_{t\epsilon[0,L]} F(t)}$$

and F is not identically zero, then $z(T) = 0$ for some T in $(0, L]$.

Furthermore, if the smallest such T is L, then $F(t) = \pi^2/L^2$ for all t in $[0, L]$.

Theorem 4. *Let $c: [0, L] \to M$ be a unit speed geodesic on a riemannian manifold of dimension n. If*

$$\int_0^L Ric(c'(t))\, dt \;\geq\; \pi(n-1)^{1/2} \sqrt{max(0, \max_{t\epsilon[0,L]} Ric(c'(t))}$$

and $Ric(c')$ is not identically zero, then $c(0)$ has a conjugate point $c(T)$ along c, for some T in $(0, L]$.

Furthermore, if the smallest such T is L, then $K(\sigma) = \pi^2/L^2$ for all tangent two planes σ containing a tangent vector to c.

It should be noted that the curvature condition in Theorem 4 cannot be replaced by

$$\int_0^L Ric(c'(t))\,dt \;\geq\; (n-1)\frac{\pi^2}{L}$$

(a sufficiently long geodesic beginning from the vertex of the paraboloid $z = x^2 + y^2$ demonstrates this). However, the curvature condition

$$\int_0^L Ric(c'(t))(1 - cos\left(\frac{2\pi t}{L}\right))\,dt \;\geq\; (n-1)\frac{\pi^2}{L}$$

is a valid replacement, as shown by L.W. Green in 1963 [Gr2]. At present, this is the strongest generalization of Myer's criterion.

As an immediate corollary of Theorem 4, we have the following supplement to a result of Ambrose [Am].

Corollary. *Let* $c\colon[0,\infty) \to M$ *be a unit speed geodesic on a riemannian manifold of dimension* n*, which gives rise to no conjugate points of* $c(0)$*. Then*

$$\limsup_{T\to\infty} \int_0^T Ric(c'(t))\,dt \;\leq\; \pi(n-1)^{1/2}\sqrt{max(0, \sup_{t\in[0,\infty)} Ric(c'(t))}$$

Ambrose has shown that with the same hypotheses

$$\lim_{T\to\infty} \int_0^T Ric(c'(t))\,dt \quad \text{is not } +\infty \;.$$

2. Proofs

Proof of Theorem 3.

Suppose that

$$\int_0^L F(t)\,dt \;\geq\; \pi\sqrt{\max(0, \max_{t\in[0,L]} F(t)}$$

and that F is not identically zero.

To prove the existence of T in $(0, L]$ satisfying $z(T) = 0$, it suffices to find a continuous piecewise differentiable function $\phi\colon [0, L] \to R$ such that $\phi(0) = \phi(L) = 0$ (and ϕ not identically zero) for which

$$\int_0^L (\phi'(t))^2 - F(t)(\phi(t))^2\,dt \;\leq\; 0\ .$$

Letting

$$\beta = \max_{t\in[0,L]} F(t)$$

we have that β is positive.

Let

$$y = \frac{\pi}{2\sqrt{\beta}}$$

so that

$$0 < y \leq \frac{L}{2}\ .$$

Define $v\colon [0, L] \to R$ by

$$v(t) = \begin{cases} sin(\pi t/2y), & \text{if } 0 \leq t \leq y\ ; \\ 1, & \text{if } y \leq t \leq L - y\ ; \\ sin(\pi(L-t)/2y) & \text{if } L - y \leq t \leq L\ . \end{cases}$$

We now have

$$
\begin{aligned}
\int_0^L v'(t)^2 - F(t)v(t)^2\,dt = \quad & \int_0^y v'(t)^2 + (1 - v(t)^2)F(t)\,dt \\
& + \int_{L-y}^L v'(t)^2 + (1 - v(t)^2)F(t)\,dt \\
& - \int_0^L F(t)\,dt \\
\leq \quad & \int_0^y v'(t)^2 + (1 - v(t)^2)\beta\,dt \\
& + \int_{L-y}^L v'(t)^2 + (1 - v(t)^2)\beta\,dt \\
& - \pi\sqrt{\beta} \\
= 0.
\end{aligned}
$$

This means that there exists T in $(0, L]$ for which $z(T) = 0$.

We now move onto the equality statement.

Suppose that the smallest such T is L.

Since

$$\int_0^L v'(t)^2 - F(t)v(t)\,dt \;\leq\; 0$$

we have that v is a solution of $x'' + Fx = 0$. This means that v is a C^2 function, so that $y = L/2$, giving us $v(t) = sin(\pi t/2y)$. The fact that v is a solution of $x'' = Fx = 0$ now tells us that $F(t) = \beta$ for all t in $[0, L]$, so that $z(t) = Asin(t/\sqrt{\beta})$, where A is a nonzero constant. Since the smallest T for which $z(T) = 0$ is L, it must be that $\pi/\sqrt{\beta} = L$, and so $F(t) = \pi^2/L^2$ for all t in $[0, L]$.

Proof of Theorem 4.

Suppose that

$$\int_0^L Ric(c'(t))\,dt \;\geq\; \pi(n-1)^{1/2}\sqrt{max(0, \max_{t\epsilon[0,L]} Ric(c'(t))}$$

and $Ric(c')$ is not identically zero.

To show that $c(0)$ has a conjugate point along c, it suffices to find an admissible vector field $W: [0, L] \to TM$ on c that is not identically zero, and for which $I_c(W, W) \leq 0$.

Letting

$$\beta = \max_{t\epsilon[0,L]} Ric(c'(t))$$

we have that β is positive.

Let

$$y = \frac{\pi}{2}\sqrt{\frac{n-1}{\beta}} \quad \text{so that} \quad 0 < y \leq \frac{L}{2} \ .$$

Define $v: [0, L] \to R$ by

$$v(t) = \begin{cases} sin(\pi t/2y), & \text{if } 0 \leq t \leq y \ ; \\ 1, & \text{if } y \leq t \leq L-y \ ; \\ sin(\pi(L-t)/2y) & \text{if } L-y \leq t \leq L \ . \end{cases}$$

Let E be a parallel unit vector field on c that is orthogonal to c.

Let $V = vE$. Then

$$\begin{aligned} I(V,V) &= \int_0^L < \frac{DV}{dt}, \frac{DV}{dt} > - < R(V, c'(t))c'(t), V > \, dt \\ &= \int_0^y v'(t)^2 + (1 - v(t)^2) < R(E, c'(t))c'(t), E > \, dt \\ &\quad + \int_{L-y}^L v'(t)^2 + (1 - v(t)^2) < R(E, c'(t))c'(t), E > \, dt \\ &\quad - \int_0^L < R(E, c'(t))c'(t), E > \, dt \ . \end{aligned}$$

Let $E_1, \ldots, E_{n-1}$ be mutually orthogonal parallel unit vector fields on c that are orthogonal to c. Letting $V_i = vE_i$ for $i = 1$ to $n-1$, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} I(V_i, V_i) = & \int_0^y (n-1)v'(t)^2 + (1 - v(t)^2)Ric(c'(t))\,dt \\
& + \int_{L-y}^{L} (n-1)v'(t)^2 + (1 - v(t)^2)Ric(c'(t))\,dt \\
& - \int_0^L Ric(c'(t))\,dt \\
\leq & \int_0^y (n-1)v'(t)^2 + (1 - v(t)^2)\beta\,dt \\
& + \int_{L-y}^{L} (n-1)v'(t)^2 + (1 - v(t)^2)\beta\,dt \\
& - \pi(n-1)^{1/2}\sqrt{\beta} \\
= & \; 0.
\end{aligned}$$

We now move onto the equality condition.

Suppose the first conjugate point to $c(0)$ along c is $c(L)$. Since

$$\sum_{i=1}^{n-1} I(V_i, V_i) \leq 0 \text{ and } I(V_i, V_i) \geq 0 \text{ for } i = 1 \text{ to } n-1,$$

each V_i is a Jacobi field. This means that v is C^∞, so that $y = L/2$, giving us $v(t) = sin(\pi t/2y)$. The fact that $V = vE$ is a Jacobi field now tells us that

$$< R(c'(t), E)E, c'(t) >= \left(\frac{\pi}{2y}\right)^2$$

for all t in $[0, L]$. Using $y = L/2$ one more time, we can conclude this proof with

$$\left(\frac{\pi}{2y}\right)^2 = \frac{\pi^2}{L^2}\,.$$

Chapter 4.

The density of conjugate points along a geodesic

1. Definitions and examples

Let M be a complete riemannian manifold.

Any unit tangent vector v on M determines a unit speed geodesic $c_v: [0, \infty) \to M$ via $c_v'(0) = v$. It then determines two elements of $[0, +\infty]$, namely

$$\underline{\psi}(v) = \liminf_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]} \end{pmatrix}$$

$$\overline{\psi}(v) = \limsup_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]} \end{pmatrix}$$

Examples.

(1) Let M be an n-sphere of constant curvature K. For all unit vectors v

$$\underline{\psi}(v) = \overline{\psi}(v) = \frac{\sqrt{K}}{\pi} .$$

(2) Let M be a complete manifold of non-positive curvature (or, more generally, without conjugate points). For all unit vectors v

$$\underline{\psi}(v) = \overline{\psi}(v) = 0 .$$

(3) Let M be n-dimensional complex projective space with the Fubini-Study metric (so that the Hopf map from the unit $(2n+1)$-sphere is a riemannian submersion). For all unit vectors v

$$\underline{\psi}(v) = \overline{\psi}(v) = \frac{2}{\pi} .$$

(4) Let M be the paraboloid of revolution $z = x^2 + y^2$.
For all unit vectors v

$$\underline{\psi}(v) = \overline{\psi}(v) = 0 .$$

2. Propositions

Proposition 1. Let M be a complete riemannian manifold of dimension n. Let v be a unit vector on M. Let $c_v\colon [0,\infty) \to M$ be the geodesic given by $c_v'(0) = v$.

(1) If there are no conjugate points of $c_v(0)$ along c_v, then

$$\underline{\psi}(v) = \overline{\psi}(v) = 0 \; .$$

(2) If $\underline{\psi}(v) \neq 0$, then there an infinite number of conjugate points to $c_v(0)$ along c_v.

(3) If $K(\sigma) \geq \alpha$, for all tangent two planes σ containing a tangent vector to c_v, then

$$\underline{\psi}(v) \geq \frac{\sqrt{\alpha}}{\pi} \; .$$

(4) If $K(\sigma) \leq \beta$, where β is positive, for all tangent two planes σ containing a tangent vector to c_v, then

$$\overline{\psi}(v) \;\leq\; (n-1)\frac{\sqrt{\beta}}{\pi} \; .$$

(5) If $Ric(c_v'(t)) \geq (n-1)\alpha$, where α is positive, for all $t \geq 0$, then

$$\underline{\psi}(v) \geq \frac{\sqrt{\alpha}}{\pi} \; .$$

(6) If β is a positive constant for which $Ric(c_v'(t)) \leq \beta$, for all $t \geq 0$, then

$$\underline{\psi}(v) \;\geq\; \frac{1}{\pi\sqrt{\beta(n-1)}} \liminf_{T\to\infty} \frac{1}{T} \int_0^T Ric(c_v'(t))\, dt \; .$$

Proposition 2. Let M be a finite volume complete riemannian manifold of dimension n.

(1) The set of unit vectors v for which $\underline{\psi}(v) = 0$ and the set of those for which $c_v(0)$ has no conjugate points along c_v differ by a set of measure zero.

Equivalently, the set of unit vectors v for which $\underline{\psi}(v) \geq 0$ and the set of those for which $c_v(0)$ has a conjugate point along c_v differ by a set of measure zero.

(2) Suppose that the unit tangent vector w has the property that there exists a conjugate point to $c_w(0)$ along c_w.

Then there exists an open neighborhood U of w, for which $\underline{\psi}(v) > 0$ for almost all v in U.

(3) The condition "$\underline{\psi}(v) = 0$ for almost all unit vectors v" is eqivalent to the condition "M has no conjugate points".

With regard to $\underline{\psi}$ and $\overline{\psi}$, it is natural to ask whether they are really different. To show that they are the same, it suffices to show that $\underline{\psi} \geq \overline{\psi}$.

Along these lines, it is true that $(n-1)\underline{\psi} \geq \overline{\psi}$ almost everywhere, for M a complete riemannian manifold of dimension n. This follows from

Proposition 3. Let M be a complete riemannian manifold. Then the following is a well defined element of $[0, +\infty]$ for almost all v in SM

$$\lim_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \\ \text{counted according to multiplicity} \end{pmatrix}.$$

3. Proofs

Firstly, some notation.

For v a unit tangent vector of a riemannian manifold M, let $c_v: [0, \infty) \to M$ be the geodesic $c_v(t) = exp(tv)$.

Z denotes the set of unit vectors v for which $c_v(0)$ has no conjugate points along c_v.

Z' denotes the set of unit vectors v for which c_v has finitely many conjugate points along c_v.

Z'' denotes the set of unit vectors v for which $\overline{\psi}(v) = 0$.

Z''' denotes the set of unit vectors v for which $\underline{\psi}(v) = 0$.

We have $Z \subseteq Z' \subseteq Z'' \subseteq Z'''$.

Note that Z', Z'', Z''' are each invariant with respect to the geodesic flow.

μ will denote the Liouville measure on SM.

Proof of Proposition 1.

(1) This is clear from the definition.

(2) This is clear from the definition.

(3) This is a consequence of Rauch's comparison theorem.

(4) This is a consequence of Rauch's comparison theorem.

(5) This is a consequence of Myer's criterion for conjugate points.

(6) This is a consequence of Theorem 4 of Chapter 3.

Proof of Proposition 2.

(1) Equivalence is clear.

We are required to show that Z and Z''' differ by a set of measure zero. It suffices to show that $\mu(SM - Z''') \geq \mu(SM - Z)$.

For each positive integer j, define $f_j: SM \to R$ by

$$f_j = \begin{cases} 1, & \text{if } c_v(0) \text{ has a conjugate point along } c_v|_{[0,j]}; \\ 0, & \text{otherwise.} \end{cases}$$

Also, define $f: SM \to R$ by

$$f = \begin{cases} 1, & \text{if } c_v(0) \text{ has a conjugate point along } c_v; \\ 0, & \text{otherwise.} \end{cases}$$

Then f_j approaches f from below, as j goes to infinity. It follows that

$$\lim_{j\to\infty} \int_{SM} f_j = \int_{SM} f \ .$$

By Lemma 2 of Chapter 2,

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{f_j(G_{jk}v)}{jN} \leq \underline{\psi}(v)$$

for all unit vectors v for which the above limit exists. For v in Z''', we then have

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{f_j(G_{jk}v)}{N} = 0 \ .$$

We can now conclude that

$$
\begin{aligned}
\mu(SM - Z) &= \int\limits_{SM} f \\
&= \lim_{j\to\infty} \int\limits_{SM} f_j \\
&= \lim_{j\to\infty} \int\limits_{SM} \lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{f_j(G_{jk}v)}{N} \; d\mu(v) \\
&= \lim_{j\to\infty} \int\limits_{SM-Z'''} \lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{f_j(G_{jk}v)}{N} \; d\mu(v) \\
&\leq \int\limits_{SM-Z'''} 1 \\
&= \mu(SM - Z''') \,.
\end{aligned}
$$

(2) This is an immediate consequence of (1).

(3) M having no conjugate points certainly implies $\underline{\psi} = 0$ almost everywhere (in fact, everywhere !).

If $\underline{\psi} = 0$ almost everywhere, then almost all unit vectors v have no conjugate points along c_v, by Proposition 2(1). This means that the set of unit vectors v for which c_v has no conjugate points is dense. It's also closed, and so must be SM, from which it follows that M has no conjugate points.

Proof of Proposition 3.

For v a unit tangent vector, define

$$\underline{\phi}(v) = \liminf_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \\ \text{counted according to multiplicity} \end{pmatrix}$$

$$\overline{\phi}(v) = \limsup_{T\to\infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \\ \text{counted according to multiplicity} \end{pmatrix}$$

It suffices to show that $\overline{\phi}(v) \leq \underline{\phi}(v)$ for almost all v.

For j a positive integer, define $g_j\colon SM \to R$ by

$$g_j(v) = \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,j]}, \\ \text{counted according to multiplicity} \end{pmatrix}$$

Define $\hat{g}_j$ by

$$\hat{g}_j = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} g_j(G_{jk}v) .$$

By Birkhoff's Ergodic Theorem, this is well defined almost everywhere.

Whenever $\hat{g}_j(v)$ exists, we have

$$\underline{\phi}(v) \geq \frac{1}{j}\hat{g}_j(v) - \frac{(n-1)}{j}$$
$$\overline{\phi}(v) \leq \frac{1}{j}\hat{g}_j(v) + \frac{(n-1)}{j}$$

by Lemma 3 of Chapter 2.

For almost all v, we then have

$$\overline{\phi}(v) \leq \underline{\phi}(v) + \frac{2(n-1)}{j} .$$

Letting j go to infinity, we are done.

Chapter 5.

Generalizations of Green's inequality for the total scalar curvature

1. *Introduction and Main Theorems*

In 1958, L.W. Green published a curvature inequality for compact riemannian manifolds without conjugate points [Grl]. For M a riemannian manifold as above, he showed that $\int_M Scal \leq 0$.

By Ambrose's criterion for conjugate points, and an observation of A. Avez regarding the use of Birkhoff's Ergodic Theorem, Green's inequality can be quickly derived [Am] [Av]. With the use of a new criterion for conjugate points (Theorem 4 of Chapter 3) we have the following generalization of Green's inequality

Theorem 1. *Let M be a finite volume complete riemannian manifold of dimension n with Ricci curvature bounded above. Then*

$$\int_M Scal \leq \frac{\pi(n-1)^{1/2}n}{vol(S^{n-1}, can)} \sqrt{max(0, supRic)} \int_{SM} \underline{\psi} \ .$$

In the above theorem, SM is the unit tangent bundle with the induced Liouville measure, $Ric: SM \to R$ is the Ricci curvature function, and $\underline{\psi}: SM \to [0, \infty]$ is defined, as in Chapter 4, by

$$\underline{\psi}(v) = \liminf_{T \to \infty} \frac{1}{T} \begin{pmatrix} \text{the number of points conjugate to} \\ c_v(0) \text{ along } c_v|_{[0,T]}, \text{ where } c_v(t) = exp(tv) \end{pmatrix} .$$

For M the standard n-sphere of constant sectional curvature 1, $\underline{\psi} = 1/\pi$, $Ric = n-1$, $Scal = n(n-1)$, and $vol(SM) = vol(M) vol(S^{n-1}, can)$. The standard n-sphere shows that the above generalization of Green's inequality is sharp.

By considering the integral of the Ricci curvature, instead of the scalar curvature, Theorem 1 can be slightly strengthened to

Theorem 2. *Let M be a finite volume complete riemannian manifold of dimension n.*

(1) If Ric has an integrable positive or negative part then

$$\int_Z Ric \leq 0 .$$

(2) If Ric is bounded above then

$$\int_{SM-Z} Ric \leq \pi(n-1)^{1/2} \sqrt{max(0, supRic)} \int_{SM} \underline{\psi} .$$

In the above theorem, Z (also defined in Chapter 4) is the set of unit tangent vectors v for which $c(0)$ has no conjugate points along $c: [0, \infty) \to M$, where $c(t) = exp(tv)$.

2. *Proofs*

c_v, Z, Z', Z'', Z''', and μ will be as in Chapter 4.

Proof of Theorem 2(1).

By Proposition 2(1) of Chapter 4, it suffices to prove

$$\int_{Z'} Ric \leq 0 .$$

By Birkhoff's Ergodic Theorem,

$$\int_{Z'} Ric = \int_{Z'} \lim_{T\to\infty} \frac{1}{T}\int_0^T Ric(G_t v)\,dt\,d\mu(v) .$$

By Ambrose's criterion for conjugate points, and Lemma 2 of Chapter 2, we have that for v in Z',

$$\liminf_{T\to\infty} \int_0^T Ric(c_v'(t))\,dt < +\infty$$

from which it follows that

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T Ric(c_v'(t))\,dt \leq 0 .$$

In terms of the geodesic flow,

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T Ric(G_t v)\,dt \leq 0 .$$

Combining this with

$$\int_{Z'} Ric = \int_{Z'} \lim_{T\to\infty} \frac{1}{T}\int_0^T Ric(G_t v)\,dt\,d\mu(v)$$

we are done.

Proof of Theorem 2(2).

By Proposition 2(1) of Chapter 4, it suffices to prove

$$\int_{SM-Z'''} Ric \;\leq\; \pi(n-1)^{1/2}\sqrt{max(0, supRic)} \int_{SM} \underline{\psi} \;.$$

Let β be any positive upper bound to the Ricci curvature. Birkhoff's Ergodic Theorem and Proposition 1(6) of Chapter 4 now gives

$$\begin{aligned}\int_{SM-Z'''} Ric &\leq \int_{SM-Z'''} \lim_{T\to\infty} \frac{1}{T}\int_0^T Ric(G_t v)\, dt\, d\mu(v)\\ &= \int_{SM-Z'''} \liminf_{T\to\infty} \frac{1}{T}\int_0^T Ric(G_t v)\, dt\, d\mu(v)\\ &\leq \int_{SM-Z'''} \pi\sqrt{\beta(n-1)}\underline{\psi}\, d\mu(v)\\ &= \pi\sqrt{\beta(n-1)}\int_{SM} \underline{\psi} \;.\end{aligned}$$

Proof of Theorem 1.

By Theorem 2

$$\int_{SM} Ric \;\leq\; \pi(n-1)^{1/2}\sqrt{max(0, supRic)} \int_{SM} \underline{\psi}$$

Using

$$\int_{SM} Ric \;=\; \frac{vol(S^{n-1}, can)}{n} \int_{M} Scal$$

we are done.

Chapter 6.

A codimension 2 version of Hopf's Theorem

1. Introduction and Main Theorem

As an immediate consequence of the Gauss-Bonnet theorem, we have that a 2-dimensional torus with a riemannian metric of nonpositive curvature is flat. This fact has a well known generalization, namely that a 2-dimensional torus with a riemannian metric without conjugate points is flat [H].

Many geometers have unsuccessfully attempted to prove the more general

Conjecture (E. Hopf). *Any riemannian metric without conjugate points on an n-dimensional torus is flat.*

A more modest result would be to establish that any sufficiently symmetric riemannian metric without conjugate points on the n-dimensional torus is flat. Along these lines, it is known that a homogeneous n-dimensional compact connected riemannian manifold without conjugate points is a flat n-dimensional torus [O'S]. We will show

Theorem. *An n-dimensional compact connected riemannian manifold without conjugate points, with a nilpotent fundamental group, is a flat n-dimensional torus provided the dimension of its isometry group is at least* $n-2$.

It should be noted that the nonpositive curvature analogue of the Hopf conjecture has been shown to be true [W]. In fact, it was shown that an n-dimensional compact connected riemannian manifold of nonpositive curvature, with a nilpotent fundamental group, is a flat n-dimensional torus. It should also be noted that the Hopf conjecture is unresolved for all n greater than 2.

2. Basic facts and Technical Lemmas

For studying isometry groups of compact riemannian manifolds without conjugate points, some very important results are due to J.J. O'Sullivan [O'S]. In particular, he showed that for all compact riemannian manifolds without conjugate points, the Killing fields are parallel. Two immediate consequences are that the identity component of the isometry group is a toral group, and that the isotropy group of any point is finite.

The following lemmas will be needed.

Lemma 1. *If M is a compact riemannian manifold without conjugate points, and all its geodesic loops are smoothly closed, then the identity component of its isometry group acts freely.*

Lemma 2. *If M is a complete riemannian manifold without conjugate points, and G is a compact Lie group that acts freely by isometries, then M/G, with the induced riemannian metric, is a compact riemannian manifold without conjugate points.*

3. *Proofs*

We write $I(M)$ for the isometry group of M, and $I(M)_o$ for its identity component. Given p in M and g in $I(M)$, g sends p to the point $g \cdot p$.

Proof of Lemma 1.

It suffices to prove that all S^1 subgroups of $I(M)_o$ act freely on M, since the isotropy group of any point is finite.

Suppose that this is not true. Then it is possible to find X in the Lie algebra of $I(M)_o$ and p_o a point of M for which $exp(X)$ is the identity of $I(M)_o$, $exp(tX)$ is not the identity for t in the open interval $(0,1)$, and $exp(t_oX) \cdot p_o = p_o$ for some t_o in $(0,1)$. Since the isotropy group of p_o is finite, $exp(\frac{1}{k}X) \cdot p_o = p_0$ for some integer k larger than 1.

Let p be a point of M. Note that $\alpha: [0,1] \to M$, where $\alpha(t) = exp(tX) \cdot p$ is a smoothly closed geodesic. It represents some nontrivial element a of $\pi_1(M,p)$. Let $\phi: \pi_1(M,p) \to \pi_1(M,p_o)$ be the isomorphism provided by a curve from p to p_o. Then ϕa is represented by $\alpha_o: [0,1] \to M$, where $\alpha_o(t) = exp(tX) \cdot p_o$, so that $\phi a = c^k$, where c is represented by $\gamma: [0,1] \to M$ where $\gamma(t) = exp(\frac{t}{k}X) \cdot p_o$. This tells us that $a = b^k$ for some b. Let $\beta: [0,1] \to M$ be the geodesic loop, based at p, that represents b. The fact that it is smoothly closed tells us that $\beta(t) = \alpha(\frac{t}{k})$. Taking $t = 1$, we obtain $exp(\frac{1}{k}X) \cdot p = p$.

This holds for all p, which contradicts the fact that $I(M)_o$ acts effectively on M.

Proof of Lemma 2.

Let $\underline{c}: [0, L] \to M/G$ be a unit speed geodesic, and let $\underline{J}$ be a Jacobi field of $\underline{c}$ which vanishes at 0.

Now let c be a horizontal lift of $\underline{c}$ to M, and let J be the Jacobi field of c that vanishes at 0 and has

$$\frac{DJ}{dt}(0) \text{ being a horizontal lift of } \frac{D\underline{J}}{dt}(0) .$$

Writing $\pi: M \to M/G$ for the riemannian submersion, we can conclude that $\pi_* J = \underline{J}$.

To finish the proof, it suffices to show that J is horizontal. To do that, it suffices to show $< J(t), X(c(t)) > = 0$ for all t in $[0, L]$, for X any Killing field induced by the action of G on M.

This is clear, since

$$< J, X \circ c > (0) = 0$$

$$< J, X \circ c >' (0) = 0$$

$$\text{and} \quad < J, X \circ c >'' (t) = 0 \text{ for all } t.$$

The last two equalities follow from the fact that X is parallel.

Proof of Theorem. Let M be an n-dimensional compact riemannian manifold that is without conjugate points, has a nilpotent fundamental group, and has an isometry group of dimension at least $n-2$. By a result of Croke and Schroeder, we have that $\pi_1(M)$ is abelian [C-S]. By a result of O'Sullivan, all geodesic loops on M are smoothly closed [O'S]. By Lemma 1, $I(M)_o$ acts freely on M.

Let G be an $n-2$ dimensional toral subgroup of $I(M)_o$. Since G acts freely, we have a riemannian submersion $\pi: M \to M/G$. From the exact sequence

$$\pi_1(M) \to \pi_1(M/G) \to \pi_0(G)$$

and the fact that $\pi_1(M)$ is abelian, it follows that $\pi_1(M/G)$ is abelian. The classification of compact surfaces tells us that M/G is either a sphere, a projective plane, or a torus. By Lemma 2, M/G has no conjugate points, and thus must be a torus by the Cartan-Hadamard theorem. Hopf's theorem now tells us that M/G is a flat torus.

Let σ be a tangent 2-plane in M spanned by $H_1 + V_1$ and $H_2 + V_2$, where H_1, H_2 are horizontal, and V_1,V_2 are vertical. Extend H_1 and H_2 locally, and extend V_1 and V_2 to parallel fields (they extend to Killing fields, and Killing fields are parallel). Then

$$< R(H_1 + V_1, H_2 + V_2)H_2 + V_2, H_1 + V_1 > = < R(H_1, H_2)H_2, H_1 >$$

since V_1 and V_2 are parallel. The last expression is nonpositive, since riemannian submersions are curvature nondecreasing on horizontal planes, and M/G is flat.

By a result of Wolf, we have that M is a flat n-dimensional torus [W].

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