SHARP ESTIMATE AND DIRAC OPERATOR

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Abstract of the Dissertation

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This work established a global conservation phenomenon for the scalar curvature function on a Riemannian manifold.

A classical result occurs in dimension 2 and it is given by the Gauss-Bonnet Theorem, which states that the average of the scalar curvature is a constant depending only on the topology of the surface.

In higher dimension, any conservation phenomenon for the scalar curvature is a weak measure of the Riemannian structure. Using Dirac operator methods, certain sharp results are established.

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To my parents,

Mira Raiden and Marcelo Llarull

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Introduction

The main result of this work could be characterized as establishing a global conservation phenomenon for the scalar curvature function on a Riemannian manifold.

A classical example of this phenomenon is given by the Gauss-Bonnet Theorem in dimension 2, which states that the average of the scalar curvature is a constant depending only on the topology of the surface.

In high dimensions any conservation phenomenon for the scalar curvature must be far more delicate since the scalar curvature is a rather weak measure of the Riemannian structure. Nevertheless, using Dirac operator methods we do succeed in establishing certain sharp results.

We begin by examining the basic problem (originally posed by M. Gromov) of studying perturbations of the canonical metric g on the n-sphere with normalized scalar curvature $\tilde{\kappa} = 1$. The normalized scalar curvature is defined to be $\tilde{\kappa} = \kappa/n(n-1)$ where κ is the usual scalar curvature and n = dimension of the manifold.

Theorem 1.2. Let g be any Riemannian metric on S^n with the property that $g \geq g_0$. Then either there exists some $x \in S$ with $\kappa_g(x) < n(n-1)$, or $g \equiv g_0$.

This situation can be extended in the following way. A map $f : M \to N$ between Riemannian manifolds is said to be ε-constructing if $\|f_* v\| \leq \varepsilon \|v\|$ for all tangent vectors v on M.

Theorem 2.1. Let M be a compact Riemannian spin manifold of dimension n. Suppose there exists a 1-contracting map $f : (M,g) \to (S^n, g_0)$ of non-zero degree. Then either there exists $x \in M$ with $\tilde{\kappa}(x) < 1$ or $M = S^n$ and f is an isometry.

Note that the result is sharp since the identity Id: $(S^n g_0) \to S^n, g_0)$ is 1-contracting and $\tilde{\kappa} = 1$.

Theorem 1 has the following immediate consequence:

Corollary 2.2. Let M be compact Riemann spin manifold of dimension n with $\tilde{\kappa} \geqq a^2$. Then M admits no c-contracting maps $f : M \to S^n$ of non-zero degree for any $c > 1/a$.

These results can be generalized in the following way. We define the A-degree of a map $f: M \to N$ between compact differentiable manifolds to be $\{f^*w_n.\hat{A}[M]\}[M]$ where w_n is the generator of $H^n(S^n, Z)$ and A is the total $\hat{A}$-class of M.

Theorem 3.1. Let (M,g) be a compact Riemannian spin manifold of dimension n + 4k which admits a 1-contracting map $f : (M,g) \to (S^n, g_0)$ of non-zero $\hat{A}$-degree. Then there exists a point $x \in M$ where $\tilde{\kappa}(x) < n(n-1)/(n+4k)(n+4k-1)$ or f is an isometric submersion.

Theorem 2 has a corollary analogous to Corollary 1 above.

All of the results above continue to hold under a weaker hypothesis on the map f. We shall define a smooth map $f : M \to N$ between Riemannian manifolds to be (ε, Λ^k)-contracting if $\|f^*\alpha\| \leq \varepsilon\|\alpha\|$ for all $\alpha \in \Lambda^k(N)$.

Note that "1-contracting" means $(1,\Lambda^1)$-contracting.

Theorem 4.1. The statement of Theorem 1 and 2 continue to hold if the hypothesis that f be 1-contracting is replaced by the hypothesis that f be $(1,\Lambda^2)$-contracting.

Remark. The hypothesis can not be weakened further, that is, there are counterexamples that show that the theorem is false for $(1,\Lambda^m)$-contracting maps with $m \geqq 3$.

Chapter 0

In this section we shall recall some basic definitions and results for Dirac operators on a spin compact Riemannian manifold.

Some Definitions and Notations

Let (M,g) be a spin compact Riemannian manifold with metric g. let (S^n,g_0) be the unit sphere in $\mathbb{R}^{n+1}$ with the standard metric g_0. Given a map f between two compact manifolds, the degree of f is defined as

$$\deg(f) = \sum_{p\in f^{-1}(q)} \operatorname{sign}(\det f_*)_p,$$

where q is a regular point. A map $f : M \to N$ is said to be ε-contracting if

$$\|f_* v\| \leqq \varepsilon \|v\|,$$

for all tangent vectors N in M. A map $f : M \to N$ is said to be (ε, Λ^k)-contracting if

$$\| f^* \varphi\| \leqq \varepsilon \ \|\varphi\|$$

for all k-forms $\varphi \in \Lambda^k(N)$. The normalized scalar curvature of a manifold M of dimension n is defined to be

$$\tilde{\kappa} = \frac{\kappa}{n(n-1)} \text{, where } \kappa \text{ is the usual scalar curvature.}$$

Spin Structure

A spin manifold is an oriented manifold with a spin structure on its tangent bundle. Let E be an oriented vector bundle, a spin

<u>structure on E</u> is a 2-sheeted covering

$$\xi : P_{Spin_n}(E) \to P_{SO_n}(E)$$

such that $\xi(p\cdot g) = \xi(p)\cdot\xi_0(g)$ for all $p \in P_{Spin_n}(E)$ and $g \in Spin_n$, where

$$\xi_0 : Spin_n \to SO_n$$

is the universal covering homomorphism with kernel $\mathbb{Z}_2$, and $P_{Spin_n}(E)$ and $P_{SO_n}(E)$ are principal $Spin_n$- and SO_n- bundle respectively. Note that a manifold M is spin if the first and second Whitney classes of M, ω_1 and ω_2 are both zero.

A <u>real Spinor bundle of E</u> is a bundle of the form

$$S(E) = P_{Spin}(E) x_\lambda V$$

where V is a left module for the Clifford algebra $C\ell(\mathbb{R}^n) = C\ell_n$ and $\lambda : Spin_n \to SO(v)$ is a representation by left multiplication of elements of $Spin_n \subseteq C\ell^\circ(\mathbb{R}^n) = C\ell^\circ_n$.

A complex spinor bundle of E is the bundle $S_{\mathbb{C}}(E) = P_{Spin_n}(E) x_\mu V_{\mathbb{C}}$ where $V_{\mathbb{C}}$ is a complex left module for the Clifford algebra $C\ell(R^n) \otimes \mathbb{C} = \mathbb{C}\ell_n$. The Clifford algebra $C\ell_n(V)$ is generated by V^n subject to the relations $v\cdot v = -\|v\|^2$ for all $v \in V^n$. The automorphism $\alpha : C\ell_n \to C\ell_n$ that extends the map $\alpha(v) = -v$ gives rise to a decomposition

$$C\ell_n = C\ell^\circ_n \oplus C\ell^1_n\ ,$$

where $C\ell_n^i = \{\varphi \in C\ell_n : \alpha(\varphi) = (-1)^i\varphi\}$ are the eigenspaces of α. $Spin_n = Pin_n \cap C\ell_n^\circ$, where Pin_n is defined as the subgroup of $C\ell_n - \{0\}$ generated by the elements v, with $\|v\| \neq 0$. Given a manifold M, $C\ell(M)$ will be the Clifford bundle of M, which is the bundle over M whose fibre at a point $p \in M$ is the Clifford algebra $C\ell(T_pM)$ of the tangent space at p.

Note that $T(M) \subseteq C\ell(M)$. We extend the metric and the connection of M to $C\ell(M)$ with the connection ∇ preserving the metric and such that

$$\nabla(\varphi\cdot\psi) = (\nabla_\varphi)\cdot\psi + \psi\cdot(\nabla_\psi)$$

for all sections φ and $\psi \in \Gamma(C\ (M))$. Let us consider the following complex bundle over (M,g), where M is Spin compact 2n-dimensional Riemannian manifold

$$= P_{Spin_{2n}}(M) \times_\lambda \mathbb{C}\ell_{2n}$$

with the induced connection, where λ is the representation by left multiplication.

We introduce a $\mathbb{Z}_2$-grading on S. Fix $p \in M$ and choose local pointwise orthonormal tangent vector fields around $p\{e_1,\ldots,e_{2n}\}$ such that $(\nabla_{e_k})_p = 0$. Let ω be the oriented "volume element" $\omega = i^n e_1,\ldots,e_{2n}$, where $\cdot$ denotes Clifford multiplication. This is a globally defined section of $\mathbb{C}\ell(M)$ with the following properties:

i) $\nabla\omega = 0$

ii) $\omega^2 = 1$

iii) $\omega e = -e\omega$ for any $e \in TM$.

Then S has the decomposition

$$S = S^+ \oplus S^-$$

into the +1 and -1 eigenvalues of Clifford multiplication by ω. For any $e \in TM$,

$$e.S^+ \subseteq S^- \quad \text{and} \quad e.S^- \subseteq S^+.$$

Over (S^{2n}, g_0) we can carry out the same construction to get the bundle

$$E_0 = P_{Spin_{2n}}(S^{2n}) \times_\lambda \mathbb{C}\ell_{2n} ,$$

with the induced metric and connections from (S^{2n}, g_0). Fix $x \in S^{2n}$ and choose local pointwise orthonormal tangent vector fields around x, $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ such that $(\nabla_{\varepsilon_k})_x = 0$. Let ω_0 be the "volume element"

$$\omega_0 = i^n \varepsilon_1, \ldots, \varepsilon_{2n}$$

As before, ω_0 gives the splitting

$$E_0 = E_0^+ \oplus E_0^-$$

into the +1 and -1 eigenspaces of ω_0.

Suppose that $f : (M^{2n}, g) \to (S^{2n}, g_0)$ is a map of non-zero degree. We can consider the pull-back bundle $f^*E_0 = E$ over (M,g). The pull-back bundle E has also a splitting $E = E^+ \oplus E^- = f^*E_0^+ \oplus f^*E_0^-$. Now we consider the tensor product bundle $S \otimes E$ over M with the tensor product metric and connection. And

$$S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E).$$

We consider the Dirac operator of

$$D : \Gamma(S) \to \Gamma(S)$$

which in terms of an orthonormal basis of tangent vectors at p is given by

$$D = \sum_{k=1}^{2n} e_k \cdot \nabla_{e_k} \cdot$$

Moreover, we can consider the <u>twisted Dirac operator</u> D_E on $S \otimes E$. D_E, on simple elements $\varphi \otimes v \in \Gamma(S \otimes E)$ is defined by

$$D_E(\varphi \otimes v) = \sum_k (e_k \nabla_{e_k} \varphi) \otimes v + \sum_k (e_k \varphi) \otimes (\nabla_{e_k} v).$$

This first order operator D_E preserves E^+, i.e.

$$S \otimes E = S \otimes E^+ \oplus S \otimes E^-$$

and

$$D_E(S \otimes E^+) \subseteq S \otimes E^+.$$

In fact, $E^+ = f^*(E_0^+) = f^*(\{v \in E_0 : \omega_0 v = v\}) = \{v \in E : f^* \omega_0 v = v\}$ so if $\sigma \otimes v \in S \otimes E^+$,

then $$D_E(\sigma \otimes v) = \sum_i e_i \nabla_{e_i} (\sigma \otimes v)$$
$$= \sum_i e_i (\nabla_i \sigma \otimes v + \sigma \otimes \nabla_i v) = (\sum e_i \nabla_i \sigma) \otimes v + \sum_i e_i v \otimes \nabla_i v)$$

where $\nabla_i = \nabla_{e_i}$.

Since $v \in E^+$, $\nabla_i v \in E^+$ because $\nabla_i v = \nabla_i(\omega, v) = (\nabla_i \omega)v + \omega(\nabla_i v)$ and $(\nabla_i \omega) = 0$. Therefore, $D_E(\sigma \otimes v) \in S \otimes E^+$. Since any element of $S \otimes E^+$ is the sum of simple elements of the form $\sigma \otimes v$, we can write

$$D_{E^+} = D_E\Big|_{S\otimes E^+}.$$

Furthermore, since $e \cdot S^{\pm} \subseteq S^{\mp}$, then

$$D_{E^+} = D^+_{E^+} \oplus D^-_{E^+}$$

and

$$D^{\pm}_{E^+} : S^{\pm} \otimes E^+ \to S^{\mp} \otimes E^+$$

Bochner-Lichnerowicz-Weitzenböck Formula

We now recall the fundamental B-L-W formula for the twisted Dirac operator D_E of the bundle $S \otimes E$ over M, see [LM]

$$D_E^2 = \nabla^*\nabla + \frac{1}{4}\kappa + R^E.$$

$\nabla^*\nabla : \Gamma(S\otimes E) \to \Gamma(S\otimes E)$ is defined in terms of a local basis of pointwise orthonormal tangent vector fields by $\nabla^*\nabla = -\sum_k \nabla_{e_k}\nabla_{e_k} - \nabla_{\nabla_{e_k}e_k}$ $\kappa = \sum_{i,j} g(R_{e_i}, e_j, e_j, e_i)$ is the scalar curvature of M, g is the Riemannian metric and R the curvature tensor of M. R^E is defined in simple elements $\sigma \otimes v \in \Gamma(S\otimes E)$ by $R^E(\sigma\otimes v) \equiv \frac{1}{2}\sum_{i,j}^{2n}(e_i e_j \sigma) \otimes (R^E e_i e_j \varphi)$ where R^E denotes the curvature tensor of E.

Note that R^E depends linearly on the curvature tensor R^E of E. For a more detailed description see [LM].

Chapter 1

Case of S^n

We start this section by examining the basic problem, originally posed by M. Gromov [Gr] of studying perturbations of the canonical metric g_0 on the n-sphere with normalized curvature 1.

Theorem 1.0. Let g be any Riemannian metric on S^n with the property that $g \geqq g_0$. Then either there exist some $x \in S^n$ with $\tilde{\varkappa}_g(x) < 1$, or $g \equiv g_0$.

Example 1.1. $(S^n, g) \xrightarrow{\text{Id}} (S^n, g_0)$,

where $g = (1+\varepsilon)\, g_0$, $\varepsilon > 0$, then $g \geqq g_0$. In this case $\tilde{\varkappa}_g = \frac{1}{(1+\varepsilon)} < 1$ for all x.

This result is also true when the map between the spheres is of non-zero degree and not necessarily the id.

Theorem 1.2. Let $f : (S^n, g) \to (S^n, g_0)$ be a map of non-zero degree. Suppose that f is 1-contracting. Then either there exists some $x \in S^n$ with $\tilde{\varkappa}_g(x) < 1$, or f is an isometry.

Remark 1.3. This result is sharp since the identity $\text{Id} : (S^n, g) \to (S^n, g_0)$ is 1-contracting and $\tilde{\varkappa}_{g_0} = 1$.

Theorem 1.2 has the following immediate consequence:

Corollary 1.4. If (S^n, g) has $\tilde{\varkappa}_g \geqq 1$, there exists a constant c, such that there exists no c-contracting maps $f : (S^n, g) \to (S^n, g_0)$

of non-zero degree.

When we go through the proofs of these results, one realizes that (S^n,g) can be replaced by any compact Spin manifold. We shall leave the proofs as part of the more general situation studied in Chapter 2.

Chapter 2

Results for a Compact Spin Manifold

The results in Chapter 1 can be extended in the following way.

Theorem 2.1. Let M be a compact Riemannian Spin manifold of dimension n. Suppose there exists a 1-contracting map $f : (M,g) \to (S^n, g_0)$ of non-zero degree. Then either there exists $x \in M$ with $\tilde{\kappa}_g(x) < 1$ or $M = S^n$ and f is an isometry.

As an immediate consequence, we have

Corollary 2.2. Let M be a compact Riemannian Spin manifold of dimension n with $\tilde{\kappa} \geq a^2$. Then M admits no c-contracting maps $f : M \to S^n$ of non-zero degree for any $c > \frac{1}{a}$.

Proof. Consider $\delta : [M, S^n] \to \mathbb{R}$

$$f \mapsto \delta_f = \max_{\|v\|=1} \|f_* v\|.$$

If $\tilde{\kappa} > a^2$, then $\delta_f > \frac{1}{a}$.

Theorem 2.1 will be proved by contradiction. The idea is the following. Consider a twisted spinor bundle $S \otimes E^+$ over M and its Dirac operator D_{E^+}.

Using the Atiyah-Singer Index Theorem we will show that Index $(D_{E^+}) \neq 0$.

Assuming that $\tilde{\kappa}_g \geq 1$ all over M and considering the B-L-W formula for D_{E^+} we will show that Index $(D_{E^+}) = 0$. The key point of the proof is choosing the appropriate coefficient bundle E. This method has been used by Gromov and Lawson, see [GL]. But in their work the choice of the coefficient bundle was not essential for their results.

Proof of Theorem 2.1. The proof will be done first for the even dimensional case. Let M be a compact Spin 2m-dimensional Riemannian manifold with metric g. Let S^{2n} be the unit 2n-sphere with standard metric g_0. Let $f : M \to S^{2n}$ be a 1-contracting map of non-zero degree.

By contradiction, assume that $\tilde{\kappa}_g \geq 1$ all over M. We consider the twisted vector bundle $S \otimes E^+$ over M and its Dirac operator D_{E^+} as we did in Chapter 0. Recall that $D_{E^+} = D_E|S \otimes E^+$. Fix $p \in M$. Let $\{e_1, \ldots, e_{2n}\}$ be a g-orthognormal tangent frame near $p \in M$ such that $(\nabla_{e_k})_p = 0$ for each κ. Let $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ be a g -orthonormal tangent frame near $f(p) \in S^{2n}$ such that $(\nabla_{\varepsilon_k})_{f(p)} = 0$ for each κ. Since f has non-zero degree, f_{*p} can be simultaneously diagonalized with respect to the bases $\{e_1, \ldots, e_{2n}\}$ and $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$. Therefore, we can find positive scalar $\{\lambda_i\}_{i=1}$ such that

$$\varepsilon_j = \lambda_j f_* e_j .$$

Note that $\lambda_j \geq 1$ since f is 1-contracting and

$$1 = g_0(\varepsilon_j, \varepsilon_j) = g_0(\lambda_j f_* e_j, \lambda_j f_* e_j) = \lambda_j^2 g_0(f_* e_j, f_* e_j) .$$

(2.3) $$1 = \lambda_j^2 g_0(f_* e_j, f_* e_j) \leq \lambda_j^2 g(e_j, e_j) = \lambda_j^2$$

Considering the inner product $<,>$ on the space $\Gamma(S\otimes E)$ of cross-sections defined by

$$<\phi,\psi> \equiv \int_M g_x(\phi,\psi) \qquad \forall\phi,\psi \in \Gamma(S\otimes E),$$

we can write the B-L-W formula as

$$<D_E^2\phi,\phi> = <\nabla^*\nabla\phi,\phi> + \frac{1}{4}\kappa<\phi,\phi> + <R^E\phi,\phi>$$

$$= <\nabla\phi,\nabla\phi> + \frac{1}{4}\kappa<\phi,\phi> + <R^E\phi,\phi>$$

$$\therefore \quad <D_E^2\phi,\phi> \geq \frac{1}{4}\kappa\|\phi\|^2 + <R^E\phi,\phi>.$$

In order to establish the result, we must look at the term $<R^E\phi,\phi>$ in more detail.

On simple elements $\sigma \otimes v \in \Gamma(S\otimes E)$, R^E is defined by

$$R^E(\sigma\otimes v) \equiv \frac{1}{2}\sum_{i,j=1}^{2n}(e_ie_j\sigma) \otimes (R^E_{e_ie_j}v).$$

More explicitly, see [LM].

$$R^E_{e_ie_j} = R^{E_0}_{f_*e_if_*e_j} = \frac{1}{4}\sum_{k,\ell} g_0(R^{E_0}_{f_*e_if_*e_j}\varepsilon_k,\varepsilon_\ell)\varepsilon_k\varepsilon_\ell,$$

where R^{E_0} is the curvature tensor on S^{2n}. Therefore,

$$R^E_{e_ie_j} = \frac{1}{4}\sum_{k,\ell=1}^{2n}[g_0(f_*e_i,\varepsilon_\ell)g_0(f_*e_j,\varepsilon_k) - g_0(f_*e_j,\varepsilon_\ell)g_0(f_*e_i,\varepsilon_k)]\varepsilon_k\varepsilon_\ell$$

$$= \frac{1}{4}\sum_{k,\ell=1}^{2n}[g_0(\frac{\varepsilon_i}{\lambda_i},\varepsilon_\ell)g(\frac{\varepsilon_j}{\lambda_j},\varepsilon_k) - g_0(\frac{\varepsilon_j}{\lambda_j},\varepsilon_\ell)g_0(\frac{\varepsilon_i}{\lambda_i},\varepsilon_k)]\varepsilon_k\varepsilon_\ell$$

$$= \frac{1}{4}\sum_{k,\ell=1}^{2n}[\frac{1}{\lambda_i\lambda_j}\delta_{i\ell}\delta_{jk} - \frac{1}{\lambda_i\lambda_j}\delta_{j\ell}\delta_{ik}]\varepsilon_k\varepsilon_\ell$$

$$= \frac{1}{4}\left[\frac{1}{\lambda_i\lambda_j}\varepsilon_j\varepsilon_i - \frac{1}{\lambda_i\lambda_j}\varepsilon_i\varepsilon_j\right]$$

$$= \frac{1}{4}\frac{1}{\lambda_i\lambda_j} 2\,\varepsilon_j\varepsilon_i$$

$$\therefore \qquad R^E_{e_ie_j} = \frac{1}{2}\frac{1}{\lambda_i\lambda_j}\varepsilon_j\varepsilon_i .$$

Let $\{\sigma_\alpha\}_{\alpha=1}^{2^{2n}}$ be a basis for S and $\{v_\beta\}_{\beta=1}^{2^{2n}}$ be a basis for E_0. Then if $\phi \in S \otimes E$,

$$\phi = \sum_{\alpha,\beta} a_{\alpha\beta}\sigma_\alpha \otimes v_\beta$$

and

$$\text{(2.4)}\quad \langle \mathcal{R}^E\phi,\phi\rangle = \langle \mathcal{R}^E\big(\sum_{\alpha,\beta} a_{\alpha\beta}\sigma_\alpha\otimes v_\beta\big), \sum_{k,\ell} a_{k\ell}\sigma_k \otimes v_\ell\rangle$$

$$= \langle \frac{1}{2}\sum_{i\neq j}\sum_{\alpha,\beta} a_{\alpha\beta} e_ie_j\sigma_\alpha \otimes \frac{1}{\lambda_i\lambda_j}\frac{1}{2}\varepsilon_j\varepsilon_i v_\beta , \sum_{k,\ell} a_{k\ell}\sigma_k \otimes v_\ell\rangle$$

$$= \frac{1}{4}\sum_{i\neq j}\sum_{k,\ell}\sum_{\alpha,\beta}\frac{a_{\alpha\beta}a_{k\ell}}{\lambda_i\lambda_j}\langle e_ie_j\sigma_\alpha,\sigma_k\rangle\langle\varepsilon_j\varepsilon_i v_\beta, v_\ell\rangle .$$

This suggests choosing the bases $\{\sigma_\alpha\}$ and $\{v_\beta\}$ "invariant" by e_ie_j and $\varepsilon_j\varepsilon_i$ respectively.

Consider the following bases $\{e_{i_1},\ldots,e_{i_k}\sigma\}$ for S and $\{\varepsilon_{j_1},\ldots,\varepsilon_{j_s}v\}$ for E_0, where $\sigma \in S$ with $\|\sigma\| = 1$ and $v \in E_0$ with $\|v\| = 1$. For each fixed pair (i,j), $e_ie_j\,\cdot : S \to S$ permutes the basis for S (up to sign); and so does $\varepsilon_j\varepsilon_i\,\cdot : E_0 \to E_0$ for the basis of E_0. Moreover, since $(e_ie_j)^2 = -1$, if σ_α is any element $\in \{e_{i_1},\ldots,e_i\sigma_j\}$, then so is $\pm\, e_ie_j\sigma_\alpha$ and the subspace generated by $\{\sigma_\alpha, e_ie_j\sigma_\alpha\}$ is invariant under e_ie_j. Analogous considerations are true for E_0 and the basis $\{\varepsilon_{j_1},\ldots,\varepsilon_{j_s}v\}$ since $(\varepsilon_j\varepsilon_i)^2 = -1$. Therefore, if $\{\sigma_\alpha\}$ represents $\{e_{i_1},\ldots,e_{i_k}\sigma\}$ and $\{v_\beta\}$ represents $\{\varepsilon_{j_1},\ldots,\varepsilon_{j_s}v\}$, then the above sum can be bounded as follows.

For each pair (i,j), the sum

$$<\sum_{\alpha\beta} a_{\alpha\beta}e_ie_j\sigma_\alpha \otimes \varepsilon_j\varepsilon_i v_\beta;\ \sum_{k,\ell} a_{k\ell}\sigma_k \otimes v_\ell>$$

can be rewritten in a 4-term sum. Each pair $\{\sigma_\alpha, e_ie_j\sigma_\alpha\}$ and $\{v_\beta, \varepsilon_j\varepsilon_i v_\beta\}$ will give the following four orthogonal basis elements for $S \otimes E$:

$\sigma_\alpha \otimes v_\beta$, $\sigma_\alpha \otimes \varepsilon_j\varepsilon_i v_\beta$, $e_ie_j\sigma_\alpha \otimes v_\beta$ and $e_ie_j\sigma_\alpha \otimes \varepsilon_j\varepsilon_i v_\beta$

$$\therefore\ < a_{\alpha\beta}e_ie_j\sigma_\alpha \otimes \varepsilon_j\varepsilon_i v_\beta + a_{\alpha\beta'}e_ie_j\sigma_\alpha \otimes (\varepsilon_j\varepsilon_i)^2 v_\beta$$

$$+ a_{\alpha'\beta}(e_ie_j)^2\sigma_\alpha \otimes \varepsilon_j\varepsilon_i v_\beta + a_{\alpha'\beta'}(e_ie_j)^2\sigma_\alpha \otimes (\varepsilon_j\varepsilon_i)^2 v_\beta,$$

$$\sum_{k,\ell}(a_{k\ell}\sigma_k\otimes v_\ell + a_{k\ell'}\sigma_k\otimes\varepsilon_j\varepsilon_i v_\ell + a_{k'\ell}e_ie_j\sigma_k\otimes v_\ell + a_{k'\ell'}e_ie_j\sigma_k\otimes\varepsilon_j\varepsilon_i v_\ell)>$$

$$= a_{\alpha\beta}a_{\alpha'\beta'} - a_{\alpha\beta'}a_{\alpha'\beta} - a_{\alpha'\beta}a_{\alpha\beta'} + a_{\alpha'\beta'}a_{\alpha\beta}$$

$$= 2a_{\alpha\beta}a_{\alpha'\beta'} - 2a_{\alpha\beta'}a_{\alpha'\beta}$$

since $\qquad 2a_{\alpha\beta}a_{\alpha'\beta'} \geq -(a^2_{\alpha\beta}+a^2_{\alpha'\beta})$

and $\qquad -2a_{\alpha'\beta}a_{\alpha\beta'} \geq -(a^2_{\alpha'\beta}+a^2_{\alpha\beta'})$.

We get that each four-term sum for fixed (α,β) is

$$\geq -(a^2_{\alpha\beta}+a_{\alpha'\beta}+a^2_{\alpha'\beta}+a^2_{\alpha\beta}).$$

Summing over α and β, we obtain that

$$\frac{1}{4}\sum_{i\neq j}\sum_{k,\ell}\sum_{\alpha\beta}\frac{a_{\alpha\beta}\,a_k}{\lambda_i\lambda_j}\langle e_ie_j\sigma_\alpha,\sigma_k\rangle\langle\varepsilon_j\varepsilon_i v_\beta,v_\ell\rangle$$

$$\geqq -\frac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}\|\phi\|^2 .$$

Thus,

$$(2.5)\qquad \langle R^E\phi,\phi\rangle \geqq -\frac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}\|\phi\|^2 .$$

Recall that $\lambda_i \geqq 1$, (2.3)

$$\therefore \qquad \langle R^E\phi,\phi\rangle \geqq -\frac{1}{4}\sum_{i\neq j}\|\phi\|^2$$

$$\geqq -\frac{1}{4}\,2n(2n-1)\|\phi\|^2 .$$

Consequently,

$$\langle D_E^2\phi,\phi\rangle \geqq \frac{1}{4}\varkappa\|\phi\|^2 - \frac{1}{4}\,2n(2n-1)\|\phi\|^2$$

$$\langle D_E^2\phi,\phi\rangle \geqq \frac{1}{4}\,2n(2n-1)(\tilde\varkappa-1)\|\phi\|^2$$

where $\tilde\varkappa = \frac{\varkappa}{2n(2n-1)}$ is the normalized scalar curvature of M. Therefore, if $\tilde\varkappa > 1$, then $\ker D_E^2 = 0$. But $\ker D_E^2 = \ker D_E$, see [GL1]. Since $D_E : S\otimes E^+ \oplus S\otimes E^- \to S\otimes E^+ \oplus S\otimes E^-$ preserves the direct sum,

$$0 = \ker D_E\big|_{S\otimes E^+} = \ker D_{E^+} .$$

We also have that $D_{E^+} = D^+_{E^+} \oplus D^-_{E^+}$, where $\quad D^{\pm}_{E^+}: S^{\pm} \otimes E^{+} \to S^{\mp} \otimes E^{+}.$

Since $\quad 0 = \ker D_{E^+} = \ker D^{+}_{E^+} \oplus \ker D^{-}_{E^+}$,

we get that $\quad \ker D^{+}_{E^+} = \ker D^{-}_{E^+} = 0.$

The index of D_{E^+} is given by

$$\mathrm{Index}(D_{E^+}) = \dim(\ker D^{+}_{E^+}) - \dim(\ker D^{-}_{E^+}) = 0.$$

However, this index can also be found by using the Atiyah-Singer Index Theorem.

$$\mathrm{Index}(D_{E^+}) = \{ch\ E^{+}\ \hat{\mathbb{A}}(M)\}[M],$$

where $ch\ E^{+}$ is the Chern character of E^{+} and $\hat{\mathbb{A}}$ is the total $\hat{A}$-class of M.

Recall that $ch\ E^{+} = \dim E^{+} + ch^{1}(E^{+}) + \ldots + ch^{n}(E^{+})$ where $ch^{i} = i^{th}$ symmetric polynomial in the Chern class c_i, with $ch^{i} \in H^{2i}(M)$, see [Hi].

Remember that E^{+} is the pull-back bundle of E_0^{+} through f^*. On S^{2n} is the Chern character of a vector bundle is given by

$$ch\ E_0^{+} = \dim E_0^{+} + \frac{1}{(n-1)!}\, c_n(E_0^{+}).$$

Therefore on the pull-back

$$ch(E^{+}) = \dim E_0^{+} + \frac{1}{(n-1)!}\, f^* c_n(E_0^{+})$$

$$ch(E^{+}) = 2^{2n-1} + \frac{1}{(n-1)!}\, f^* c_n(E_0^{+}).$$

Applying the B-L-W type formula, Atiyah, Hitchin, Lichnerowicz and Singer showed that a compact spin manifold M with $\kappa > 0$ must have $\hat{\mathbb{A}}(M) = 1$. In our case we are under the assumption that $\tilde{\kappa} > 1$ and so $\kappa > 0$ and $\hat{\mathbb{A}}(M) = 1$. Consequently,

$$\text{Index}(D_{E^+}) = \{(2^{2n-1} + \frac{1}{(n-1)!} f^*c_n(E_0^+))\hat{\mathbb{A}}[M]\}[M]$$

$$\text{Index}(D_{E^+}) = \frac{1}{(n+1)!} f^*c_n(E_0^+)[M]$$

$$= \frac{1}{(n+1)!}\int_M f^*c_n(E_0^+)$$

$$= \frac{1}{(n+1)!}(\text{degree } f) \int_{S^{2n}} c_n(E_0^+)$$

<u>Claim 2.6</u>. $c_n(E_0^+) \neq 0$ [See Appendix 1]

Therefore, if $\text{Index}(D_{E^+}) = 0$, then degree(f) = 0; which is a contradiction.

If $\tilde{\kappa}_g \equiv 1$, then f is an isometry. In fact, if $\tilde{\kappa}_g \equiv 1$, or equivalently $\kappa_g \equiv 2n(2n-1)$, then inequality (2.5) gives

$$\langle D_E^2\phi,\phi\rangle \geq \frac{1}{4}\kappa_g\|\phi\|^2 - \frac{1}{4}\sum_{i\neq j}^{2n}\frac{1}{\lambda_i\lambda_j}\|\phi\|^2$$

$$\langle D_E^2\phi,\phi\rangle \geq \frac{1}{4}\|\phi\|^2(2n(2n-1) - \sum_{i\neq j}^{2n}\frac{1}{\lambda_i\lambda_j})$$

$$\geq \frac{1}{4}\|\phi\|^2[\sum_{i\neq j}^{2n}(1-\frac{1}{\lambda_i\lambda_j})].$$

Since $\text{Index}(D_{E^+}) \neq 0$, $\ker D_E \neq 0$, there exists $0 \neq \varphi \in \Gamma(S\otimes E)$ such that $D\varphi = 0$, so

$$0 \geq \frac{1}{4}\|\phi\|^2[\sum_{i\neq j}^{2n}(1-\frac{1}{\lambda_i\lambda_j})] \cdot \tag{2.7}$$

Recall that each $\lambda_i \geq 1$, so $1 - \frac{1}{\lambda_i\lambda_j} \geq 0$

and $\quad 1 - \frac{1}{\lambda_i\lambda_j} \equiv 0 \qquad \forall i \neq j$

equivalently,

$$\lambda_i \lambda_j \equiv 1.$$

Therefore, $\lambda_i \equiv 1$ for all $1 \leq i \leq 2n$

and f is an isometry.

Odd-dimensional Case

Let M^{2n-1} be a compact spin manifold of dimension 2n-1, with Riemannian metric g. Let S_r^{2n-1} be (2n-1)-sphere of radius r with the standard metric g_0.

Let $\tilde{\kappa}$ denote the normalized scalar curvature of M.

Let $f : M \to S^{2n-1}$ be a 1-contracting map of non-zero degree. We want to show that there exists $x \in M$ where $\tilde{\kappa}_g(x) < 1$.

Consider

$$M \times S_r^1 \xrightarrow{f \times \frac{1}{r} id} S^{2n-1} \times S^1 \xrightarrow{h} S^{2n-1} \wedge S^1 \simeq S^{2n}$$

where S_r^1 is the one dimensional sphere of radius r, $f \times \frac{1}{r}$ id is defined as $(f \times \frac{1}{r} id)(p,t) = (f(p), \frac{t}{r})$ $\forall (p,t) \in M \times S^1$, and where h is a 1-contracting map into the smash product of non-zero degree.

Let us consider the following metrics.

On $M \times S_r^1$, $g + ds^2$ where ds^2 is the standard metric on S_r^1.

On $S^{2n-1} \times S^1$, $g_0 + ds^2$ where ds^2 is the standard metric on S^1.

And on S^{2n}, $\bar{g}$ is the standard metric of the unit sphere S^{2n}.

The compose map $\tilde{f} = h\circ(f\times\frac{1}{r}\ \mathrm{id})$ is of non-zero degree from $M^{2n-1} \times S^1 \to S^{2n}$. It is also 1-contracting,

$$\|\tilde{f}_*(v,t)\| = \|h_*(f_*v,\tfrac{1}{r}t)\| \leq \|f_*v\| + \|\tfrac{1}{r}t\| \leq \|v\| + \frac{1}{r}\|t\|$$

$$\leq \|v\| + \|t\|.$$

We assume $r > 1$.

We can now apply the same method we used for the even-dimensional case. Construct complex vector bundles S over $M^{2n-1} \times S^1$ and E_0 over S^{2n} and consider the bundle

$$S \otimes f^*E_0^+$$

over $M^{2n-1} \times S^1_r$.

Choose a basis $\{e_1,\ldots,e_{2n-1},e_{2n}\}$ of $(g+ds^2)$-orthonormal adapted tangent vectors around $x \in M^{2n-1} \times S^1_r$ such that $(\nabla_{e_k})_x = 0$ for each κ and such that e_{2n} is tangent to S^1_r and $e_1,\ldots,e_{2n-1}$ are tangent to M^{2n-1}. As before choose $\tilde{g}$-orthonormal basis $\{\varepsilon_1,\ldots,\varepsilon_{2n}\}$ of S^{2n} around $\tilde{f}(x)$. Therefore, we can find positive scalar $\{\lambda_i\}_{i=1}^{2n}$ such that $\varepsilon_j = \lambda_j\tilde{f}_*e_j$

Then we have that

$$1 = \bar{g}(\varepsilon_j,\varepsilon_j) = \bar{g}(\lambda_j\tilde{f}_*e_j,\lambda_j\tilde{f}_*e_j) = \lambda_j^2\bar{g}(\tilde{f}_*e_j,\tilde{f}_*e_j)$$

so for $1 \leq j \leq 2n - 1$

$$1 = \lambda_j^2\bar{g}(\tilde{f}_*e_j,\tilde{f}_*e_j) \leq \lambda_j^2 g_0(f_*e_j,f_*e_j) \leq \lambda_j^2 g(e_j,e_j) = \lambda_j^2$$

$$1 \leq \lambda_j^2$$

and for $j = 2n$

$$1 = \lambda_{2n}^2 \bar{g}(\tilde{f}_* e_{2n}, \tilde{f}_* e_{2n}) \leq \lambda_{2n}^2 ds^2 (\frac{e_{2n}}{r}, \frac{e_{2n}}{r})$$

$$1 \leq \frac{\lambda_{2n}^2}{r^2}$$

$$\lambda_{2n}^2 \geq r^2.$$

In the B-L-W formula for the twisted bundle $S \otimes E$ and its Dirac operator D_E,

$$(2.8) \qquad D_E^2 = \nabla^* \nabla + \frac{1}{4} \kappa + \mathcal{R}^E,$$

the curvature term $\mathcal{R}^E$ (2.4) can be bounded as follows, by separating the terms coming from the S_r^1 factor

$$\langle \mathcal{R}^E \phi, \phi \rangle = \frac{1}{4} \sum_{1 \neq j}^{2n-1} \sum_{k,\ell} \sum_{\alpha,\beta} \frac{a_{\alpha\beta} a_{k\ell}}{\lambda_i \lambda_j} \langle e_i e_j \sigma_\alpha, \sigma_k \rangle \langle \varepsilon_j \varepsilon_i v_\beta, v_\ell \rangle$$

$$+ \frac{1}{4} \sum_{i=1}^{2n-1} \sum_{k,\ell} \sum_{\alpha\beta} \frac{a_{\alpha\beta} a_{k\ell}}{\lambda_i \lambda_{2n}} \langle e_i e_{2n} \sigma_\alpha, \sigma_k \rangle \langle \varepsilon_{2n} \varepsilon_i v_\beta, v_\ell \rangle$$

$$+ \frac{1}{4} \sum_{j=1}^{2n} \sum_{k,\ell} \sum_{\alpha\beta} \frac{a_{\alpha\beta} a_{k\ell}}{\lambda_{2n} \lambda_j} \langle e_{2n} e_j \sigma_\alpha, \sigma_k \rangle \langle \varepsilon_j \varepsilon_{2n} v_\beta, v_\ell \rangle.$$

Therefore,

$$\langle \mathcal{R}^E \phi, \phi \rangle \geq -\frac{1}{4} \sum_{i \neq j}^{2n-1} \sum_{\alpha,\beta} \sum_{k,\ell} a_{\alpha\beta} a_{k\ell} \langle \sigma_\alpha, \sigma_{k\ell} \rangle \langle v_\beta, v_{e_\ell} \rangle$$

$$- 2 [\frac{1}{4} \frac{1}{r} \sum_{i=1}^{2n-1} \sum_{k,\ell} \sum_{\alpha\beta} a_{\alpha\beta} a_{k\ell} \langle \sigma_\alpha, \sigma_{k\ell} \rangle \langle v_\beta, v_\ell \rangle]$$

$$\langle R^E\phi,\phi\rangle \geq -\frac{1}{4}\sum_{i\neq j}^{2n-1}\|\phi\|^2 - 2\cdot\frac{1}{4r}\sum_{i=1}^{2n-1}\|\phi\|^2$$

$$\langle R^E\phi,\phi\rangle \geq -\frac{1}{4}(2n-1)(2n-2)\|\phi\|^2 - \frac{1}{2r}(2n-1)\|\phi\|^2$$

Note that in (2.8) $\varkappa$ is the unnormalized scalar curvature of $M^{2n-1}\times S^1_r$ which is equal to the unnormalized scalar curvature of M^{2n-1}.

And

$$\text{normalized scalar curvature of } M^{2n-1} = \frac{\varkappa}{(2n-1)(2n-2)} = \tilde{\varkappa}\ .$$

Consequently,

$$\langle D^2\phi,\phi\rangle \geq \left[\frac{1}{4}\varkappa - \frac{1}{4}(2n-1)(2n-2) - \frac{2n-1}{2r}\right]\|\phi\|^2$$

$$\langle D^2\phi,\phi\rangle \geq \frac{1}{4}(2n-1)(2n-2)\left[\tilde{\varkappa} - 1 - \frac{2}{(2n-2)r}\right]\|\phi\|^2.$$

As before, if $\tilde{\varkappa} \equiv 1$, since f is a 1-contracting map, f is an isometry (see (2.7).

If $\tilde{\varkappa} > 1$, since the last inequality is valid for all $r > 0$, then

$$\ker D^2_E = \ker D_E = 0.$$

And $\ker(D^+_{E^+}) = 0$, hence Index $(D_{E^+}) = 0$. But the Atiyah-Singer Index Theorem gives

$$\text{Index}(D_{E^+}) \neq 0 \qquad \text{as before (2.6).}$$

Chapter 3

Manifold of Dimension n+4k

The notion of degree of a map between two compact manifolds can be extended to the case where the dimensions are not the same.

The $\hat{A}$-degree of a map $f : M \to N$ is defined to be

$$\hat{A}\text{-degree}(f) = \{f^*\omega_n \hat{\mathbb{A}}[M]\}\ [M],$$

where ω_n is the generator of $H^n(N, \mathbb{Z})$ and $\hat{\mathbb{A}}[M]$ is the total $\hat{A}$-class of M.

Now Theorem 2.1 can be generalized in the following:

Theorem 3.1. Let (M^{n+4k}, g) be a compact spin Riemannian manifold of dim n + 4k which admits a 1-contracting map $f : (M^{n+4k}, g) \to (S^n, g_0)$ of non-zero $\hat{A}$-degree. Then there exists a point $x \in M$ where

$$\tilde{\varkappa}_g(x) < \frac{n(n-1)}{(n+4k)(n+4k-1)} \quad \text{or} \quad f \text{ is an isometric submersion}$$

Corollary 3.2. If M^{n+4k} has $\tilde{\varkappa} \geq \frac{n(n+1)}{(n+4k)(n+4k-1)}$, then there exists no c- contracting map $f : M \to S^n$ of non-zero $\hat{A}$-degree with $c < 1$.

Proof of Theorem 3.1. We proceed as in the proof of Theorem 2. Suppose n is even. Let $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+4k}\}$ be a basis of g-orthogonal tangent vectors near $p \in M$.

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be a g_0-orthogonal basis of tangent vectors near $f(p) \in S^n$. Since f has $\hat{A}$-degree $\neq 0$, we can find scalar λ_j

such that

$$\lambda_j f_* e_j = \varepsilon_j \quad \text{with} \quad \lambda_j \geqq 1 \qquad 1 \leqq j \leqq n$$

$\therefore$ at p

$$f_{*p} = \begin{bmatrix} \frac{1}{\lambda_1} & & & 0 & \\ & \ddots & & & \\ & & \ddots & & 0 \\ 0 & & & \frac{1}{\lambda_n} & \end{bmatrix}_p .$$

Then the term $\langle D^2\phi,\phi\rangle$ in the B-L-W-formula can be bounded as follows.

$$\langle \mathcal{R}^E\phi,\phi\rangle = \frac{1}{4} \sum_{i\neq j} \sum_{k,\ell}^{2^n} \sum_{\alpha,\beta}^{2^n} \frac{a_{\alpha\beta}\, a_{\varkappa}}{\lambda_i \lambda_j} \langle e_i e_j \sigma_\alpha, \sigma_k\rangle \langle \varepsilon_j \varepsilon_i v_\beta, v_\ell\rangle .$$

Fixing each pair (i,j) we use the same bound as in the proof of Theorem 2.1.

$$\therefore \quad \langle \mathcal{R}^E \phi,\phi\rangle \geqq -\frac{1}{4} \sum_{i\neq j} \frac{1}{\lambda_i \lambda_j} \|\phi\|^2$$

$$\langle \mathcal{R}^E\phi,\phi\rangle \geqq -\frac{1}{4} n(n-1)\|\phi\|^2, \quad \text{since} \quad \lambda_i \geqq 1$$

$$\therefore \quad \langle D_E^2\phi,\phi\rangle \geqq \frac{1}{4}\varkappa \|\phi\|^2 - \frac{1}{4} n(n-1)\|\phi\|^2$$

where $\varkappa$ is the unnormalized scalar curvature of M^{n+4k}. The normalized scalar curvature of M^{n+4k} is

$$\tilde{\varkappa} = \frac{\varkappa}{(n+4k)(n+4k-1)}$$

$$\therefore \quad \langle D^2\phi,\phi\rangle \geq \frac{1}{4}(n+4k)(n+4k-1)\left[\tilde{\kappa} - \frac{n(n-1)}{(n+4k)(n+4k-1)}\right]\|\phi\|^2 .$$

If $\tilde{\kappa} \geq \frac{n(n-1)}{(n+4k)(n+4k-1)}$ and $\tilde{\kappa} \not\equiv \frac{n(n-1)}{(n+4k)(n+4k-1)}$, then

as before, we condlude that $\mathrm{Index}(D_{E^+}) = 0$ by using the above inequality.

But Atiyah-Singer Index Theorem gives

$$\mathrm{Index}(D_{E^+}) = \{\mathrm{ch}\, E^+ \hat{\mathbb{A}}(M)\}[M]$$

$$= \frac{1}{(\frac{n}{2}+1)!} f^* c_{\frac{n}{2}}(E_0^+)\,[M]$$

$$= \frac{1}{(\frac{n}{2}+1)!} \int_M c_{\frac{n}{2}}(E^+)\hat{\mathbb{A}}[M]$$

$c_{\frac{n}{2}}(E_0^+) = c\omega_n$ where $0 \neq c$ is a constant and ω_n is a generator of $H^n(S^n,\mathbb{Z})$

Therefore,

$$0 = \mathrm{Index}(D_{E^+}) = \frac{c}{(\frac{n}{2}+1)!} \int_M f^*\omega_n \hat{\mathbb{A}}[M]$$

$$= \frac{c}{(\frac{n}{2}+1)!} \hat{A}\text{-degree}(f)$$

what contradicts the hypothesis that $\hat{A}$-degree(f) $\neq 0$.

Notice that the condition on the dimension of the manifold M to be n + 4k is because we need the hypothesis $\hat{A}$-degree(f) = $\{f^*\omega_n \hat{\mathbb{A}}(M)\}[M] \neq 0$. And $\hat{\mathbb{A}}(M)$ has non-vanishing components only in dim 4k.

If $\quad \tilde{\varkappa} \equiv \frac{n(n-1)}{(n+4k)(n+4k-1)}$, then f is an isometric submersion.

In fact, if $\varkappa \equiv n(n-1)$, then the inequality gives

$$\langle D^2\phi,\phi\rangle \geq \frac{1}{4}\|\phi\|^2[n(n-1) - \sum_{i=j}^{n} \frac{1}{\lambda_i\lambda_j}].$$

Take a harmonic spinor $0 \neq \phi \varepsilon S \otimes E$, so

$$0 \geq \frac{1}{4}\|\phi\|^2[n(n-1) - \sum_{i\neq j}^{n} \frac{1}{\lambda_i\lambda_j}]$$

$$0 \geq \frac{1}{4}\|\phi\|^2[\sum_{i\neq j}(1- \frac{1}{\lambda_i\lambda_j})] \geq 0.$$

So, $\qquad \lambda_i \lambda_j \equiv 1$ and consequently $\lambda_i \equiv 1$ for $1 \leq i \leq n$

and f is an isometric submersion. In particular, $f : M^{n+4k} \to S^n$ is a fiber bundle map.

Chapter 4

Weaker Hypotheses

In this section we analyze how these results change when we modify the hypothesis of f being 1-contracting. Recall that a map $f : M \to N$ between Riemannian manifolds is (ε, Λ^k)-contracting if

$$\|f^*\alpha\| \leq \varepsilon\|\alpha\| \qquad \forall \alpha \in \Lambda^k(N).$$

Note that "1-contracting" means $(1, \Lambda^1)$-contracting. We have the following immediate result.

Theorem 4.1. The statements of Theorem 2 and Theorem 3 continue to hold if the hypothesis that f be 1-contracting is replaced by the hypothesis that f be $(1, \Lambda^2)$-contracting.

Proof. It follows through the proofs of Theorem 2.1 and 3.1. We only need to point out that $[\lambda_i]_{i=1}^n$ satisfy

$$1 = \|\varepsilon_1 \wedge \varepsilon_j\|_{g_0} = \|\lambda_i f_* e_i \wedge \lambda_j f_* e_j\|_{g_0} = \lambda_i \lambda_j \|f_* e_i \wedge f_* e_j\|_{g_0}$$

$$\therefore \quad 1 = \lambda_i \lambda_j \|f^*(e_i \ e_j)\|_{g_0} \leq \lambda_i \lambda_j \|e_i \wedge e_j\|_g = \lambda_i \lambda_j$$

$$\therefore \qquad \lambda_i \lambda_j \geqq 1.$$

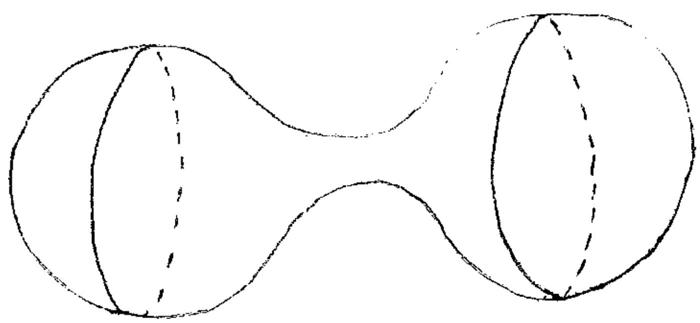
But this is what we need in those proofs rather than asking that each $\lambda_i \geqq 1$ as we had originally.

Now we ask the question: Are these same results true when the map f is $(1, \Lambda^k)$-contracting for $3 \leqq k \leqq n$? The answer is no.

The hypothesis on f cannot be further weakened in that sense.

The following construction provides counterexamples:

<u>Counterexample 4.2</u>. Consider the connected sum of two S^n unit spheres with $n \geqq 3$



Using the Gromov-Lawson construction, see [GL] we can find a metric on this space with scalar curvature $\geqq C^2$ uniformly. Renormalizing if necessary, we can assume that $C^2 \geqq 1$.

Consider now the connected sum of m S^n's with metrics g_m, with $\tilde{\varkappa}_{g_m} \geqq 1$. Notice that $vol_{g_m} \to \infty$ when $m \to \infty$, i.e. the volume can be made as large as we want by taking enough terms in the connected sum.

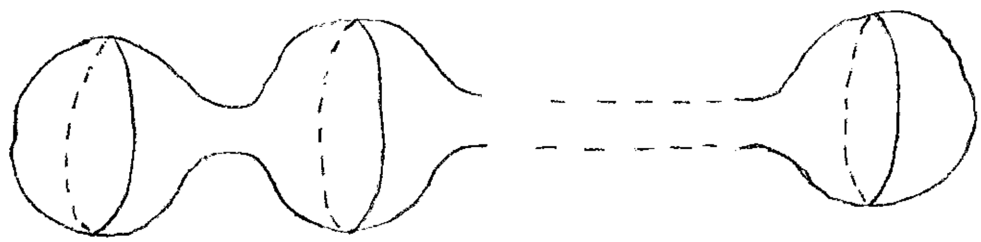
Set $\tilde{g}_m = [\frac{A}{vol_{g_m}}]^{2/n} g_m$, then the changes in the scalar curvature and volume are as follows:

(4.3) $$\tilde{\varkappa}_{\tilde{g}_m} = [\frac{vol_{g_m}}{A}]^{2/n} \tilde{\varkappa}_{g_m} \geqq [\frac{vol_{g_m}}{A^{2/n}}]^{2/n}$$

(4.4) $$vol_{\tilde{g}_m} = [[\frac{A}{vol_{g_m}}]^{1/n}]^m \, vol_{g_m} \equiv A \quad \text{independent of } m.$$

Notice that the scalar curvature can be as large as we want and that the volume remain unchanged.

Now send the space

M = 

into the standard S^n in the obvious way, choosing A = volume of S^n. Since the volumes are the same (4.4), according to Moser's Theorem, there exists a map

$$f : (M,g) \to (S^n, g_0)$$

volume preserving and therefore, $(1,\Lambda^n)$-contracting. But $\tilde{\kappa}_g >\!>\!> 1$ (4.2).

For counterexamples of maps $(1,\Lambda^k)$-contracting with $3 \leq \kappa < n$, we take the construction above for n = 3

$$f : M \to S^3 \ (1,\Lambda^3)\text{-contracting.}$$

The map $h : M \times S^P_r \to S^{P+3}$ got as

$$M \times S^P_r \xrightarrow{id \times \frac{1}{r} id} M \times S^P \xrightarrow{f \times id} S^3 \times S^P \to S^3 \wedge S^P \simeq S^{P+3}$$

is $(1,\Lambda^3)$-contracting for r large enough, see page 21.

Lemma 4.5. If f is $(1,\Lambda^{P})$-contracting, then f is $(1,\Lambda^{P+k})$-contracting $\forall k \geqq 0$

Proof. We only have to notice that

$$|\lambda_{i_1},\dots,\lambda_{ip}| \leqq 1 \quad \text{for any p-tuple with all}$$

λ_{i_j} different, then

$$|\lambda_{i_1},\dots,\lambda_{j_{p+1}}| \leqq 1.$$

But
$$|\lambda_{j_1},\dots,\lambda_{j_{p+1}}| \leqq |\lambda_{j_s}| \quad \text{for any } j_s$$

$$\therefore \quad |\lambda_{j_1},\dots,\lambda_{j_{p+1}}| \leqq \min|\lambda i| \leqq 1.$$

By induction, it is true for all $k \geqq 1$.

Therefore, the example above gives also counterexamples for $(1,\Lambda^{k})$-contracting maps with $3 \leqq k \leqq n$.

Remark 4.6. The Gromov-Lawson construction of a metric with $\tilde{\varkappa} \geqq C^2$ can only be done in dimension $\geqq 3$.

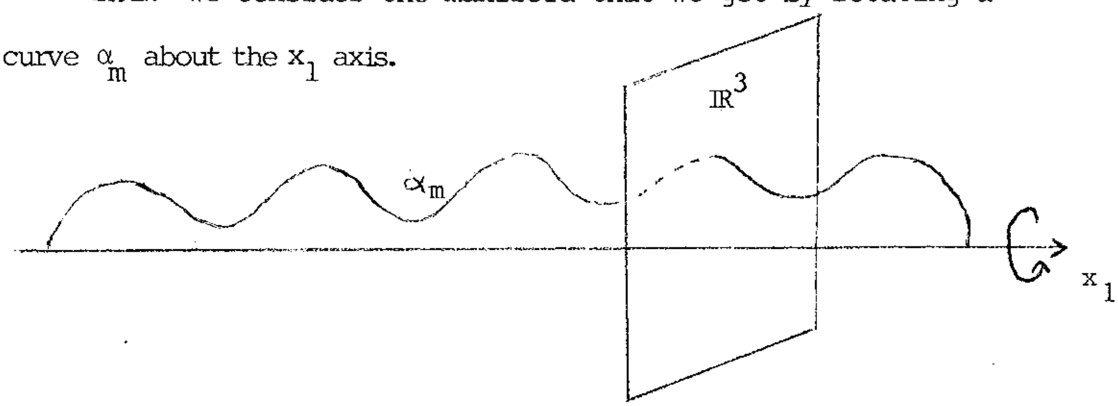
In the manifolds of dimension ≥ 3, the scalar curvature has no control on the volume of the manifold. In dimension 2, however, the scalar curvature which twice the sectional curvature, determines the volume completely. In fact, Gauss-Bonnet says that

$$\int_M \chi = \frac{1}{2\pi}\int_M \varkappa\omega$$

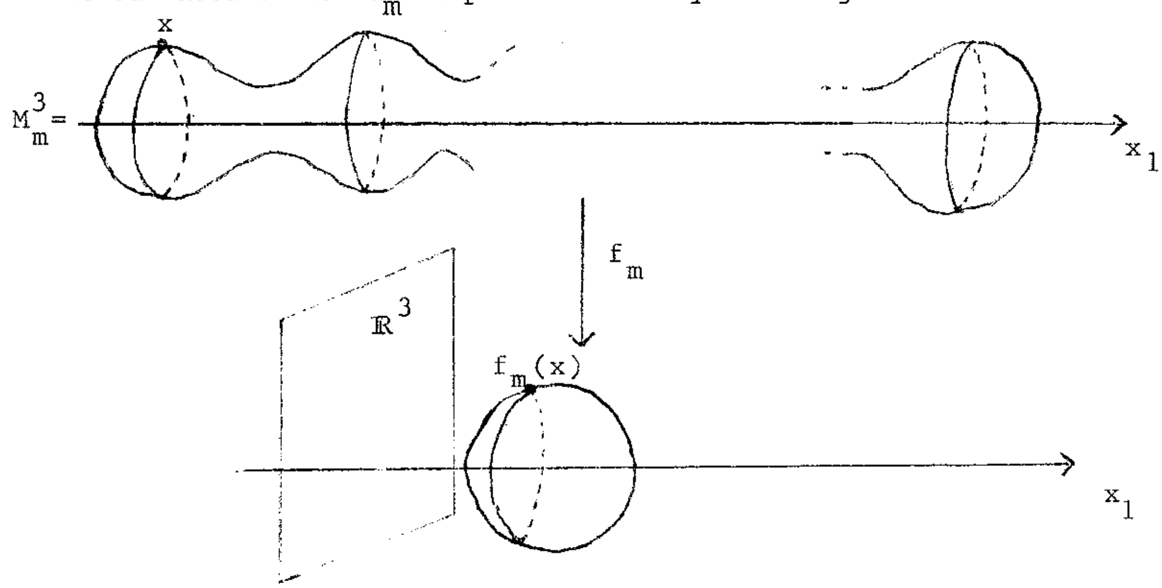
where χ is the Euler characteristic of the surface M, κ is the sectional curvature and ω is the volume element of M.

<u>Counterexample 4.7</u>. Finally, we will describe an explicit counterexample for the case of dimension 3 and a $(1,\Lambda^3)$-contracting map. This counterexample was pointed out to me by M. Katz.

In $\mathbb{R}^4$ we consider the manifold that we get by rotating a curve α_m about the x_1 axis.



We can assume that α_m is parametrized by arc-length s.



Consider the map f_m into S^3 that sends the curve α_m into the half circle that when rotated about x_1 gives the sphere S^3 and each S^2

orthogonal to x_1 in M^3 into and S^2 orthogonal to x_1 in S^3.

We provide M^3_m with metric g_m such that $\tilde{\kappa}_{g_m} \geq 1$. For a point $x \in M^3$ we have that

$$df_m : T_x M^3 \longmapsto T_{f_m(x)} S^3$$

$$e_1 \longmapsto \frac{1}{\ell_m} e_i$$

$$e_j \longmapsto c(s) e_j \qquad j = 1,2$$

where ℓ_m is the function length of α_m and $c(s)$ some bounded functions of s, say $a \leq c(s) \leq b$. These constants a and b are independent of m. Thus, f_m is $(\frac{b^2}{\ell_m}, \Lambda^3)$-contracting.

By choosing m large enough, $\frac{b^2}{\ell_m}$ can be made ≤ 1 (and the volume of M as big as we want). Consequently, f_m is $(1,\Lambda^3)$-contracting and $\tilde{\kappa}_{g_m} > 1$.

Spin Hypothesis

Currently I am investigating the extent to which the spin hypothesis is necessary to my results on scalar curvature. I am also trying to refine these results by considering the more delicate mod-2 invariants related to the Dirac operator via the Cl_k-Index Theorem.

Appendix 1

Proof of Claim 2.6

Recall that

$$E_0 = P_{Spin_{2n}}(S^{2n}) \times_\lambda \mathbb{C}\ell_{2n}.$$

$\mathbb{C}\ell_{2n}$ acts on E_0 on the right since right Clifford multiplication commutes with λ. E_0 has a decomposition into irreducible modules under left multiplicaiton that comes from the decomposition of $\mathbb{C}\ell_{2n}$ into irreducible modules.

$$E_0 = \$\oplus\ldots\oplus\$$$

E_0 is the direct sum of 2^n copies of $\$$, where

$$\$ \cong P_{Spin_{2n}}(S^{2n}) \times_\Delta \mathbb{C}^{2n},$$

and Δ is the fundamental representation of $Spin_{2n}$ into U_{2^n}.

The splitting $E_0 = E_0^+ \oplus E_0^+$ gives rise to the splitting

$$\$ = \$^+ \oplus \$^-,$$

where $\dim(\$^+) = 2^{n-1}$. Therefore,

$$E_0^+ = \$^+\oplus\ldots\oplus\$^+ \quad (2^n \text{ terms}).$$

Each $\$^+ \simeq P_{Spin_{2n}}(S^{2n}) \times_{\Delta^+} \mathbb{C}^{2^{n-1}}$ with

$$\Delta^+ : Spin_{2n} \to U_{2^{n-1}}.$$

For a detailed description, see [Hu], [Hi] and [Pa].

Now the Chern character of E_0^+ is given by

$$ch(E_0^+) = ch(\not{S}^+ \oplus \ldots \oplus \not{S}^+ = 2^n ch(\not{S}^+).$$

Over S^{2n} $ch(S^+) = \dim(\not{S}^+) + \frac{1}{(n-1)!} c_n(S^+)$.

To prove the claim, it is enough to show that $c_n(\not{S}^+) \neq 0$.

Let T^n be a maximal torus of $Spin_{2n}$

$$T^n \longrightarrow Spin_{2n}.$$

On the classifying spaces we have

$$\rho : BT^n \longrightarrow B\ Spin_{2n}.$$

Recall that $BT^n = CP(\infty) x \ldots x CP(\infty)$, therefore,

$$H^*(BT^n, Q) \simeq Q[x_1, \ldots, x_n]$$

$$H^{**}(BT^n, Q) \simeq Q[x_1, \ldots, x_n] \quad \text{(the ring of formal power series)}$$

where $x_i \in H^2(BT^n, Q)$

Let $W = W(Spin_{2n}, T^n)$ be the Weyl group of automorphisms of T^n which can be extended to automorphisms of $Spin_{2n}$. The map

$$\rho^* : H^*(B\ Spin_{2n}, Q) \to H^*(BT^n, Q)$$

is injective. The image consists of those elements in $H^*(BT^n, Q)$ which are invariant under the action of W.

We have

$$\begin{array}{ccccc} & & T^1(e_1,\ldots,e_{2n}) = T^1 & \text{———} & \text{Spin}_{2n} \\ & \overset{\omega}{\nearrow} & \downarrow \xi_0 & & \downarrow \xi_0 \\ T^n & & & & \\ & \underset{\xi_0\omega}{\searrow} & T(e_1,\ldots,e_{2n}) = T & \text{———} & SO(2n) \end{array}$$

where ξ_0 is the covering homomorphis and ω is defined as follows.

Let $\omega_j : S^1 = \mathbb{R}/_{\mathbb{Z}} \to \text{Spin}_{2n}$ be the homomorphism given by

$$\omega_j(\Theta) = \cos 2\pi\Theta + e_{2j-1}e_{2j}\sin 2\pi\Theta \qquad \text{for } 1 \leqq j \leqq n.$$

Note that $\xi_0\omega_j(\Theta) = \text{diag}(0,\ldots,0,2\Theta,0,\ldots,0)$ for all $\Theta \in \mathbb{R}/_{\mathbb{Z}}$.

Let $\omega : T^n \to \text{Spin}_{2n}$ be defined by

$$\omega(\Theta_1,\ldots,\Theta_n) = \omega_1(\Theta_1)\cdot\ldots\cdot\omega_n(\Theta_n)$$

$$\text{for } (\Theta_1,\ldots,\Theta_n) \in T^n.$$

Then the Weyl group consists of the $2^n n!$ permutations of the indexes of $(\Theta_1,\ldots,\Theta_n)$ compose with $(\Theta_1,\ldots,\Theta_n) \to (\varepsilon_1\Theta_1,\ldots,\varepsilon_n\Theta_n)$ with $\varepsilon_i = \pm 1$ and $\varepsilon_1,\ldots,\varepsilon_n = 1$. See [Hu].

$ch(\not{S})$ lies in $H^*(B\,\text{Spin}_{2n},\mathbb{Q}) \to H^*(BT^n,\mathbb{Q})$.

Since we want $ch(\not{S}^{+})$ we must consider $(\Theta_1,\ldots,\Theta_n) \to (\pm\Theta_1,\ldots,\pm\Theta_n)$ with $\varepsilon_1,\ldots,\varepsilon_n = 1$ and with even number of pluses.

$$ch(\not{S}^{+}) = \sum_{\substack{\text{even } \# \\ \text{of } +\text{'s}}} e^{\frac{1}{2}(\pm\tilde{x}_1 \pm\tilde{x}_2,\ldots,\pm\tilde{x}_n)} \in H^*(S^{2n})$$

where $\tilde{x}_i = \sigma^* x_i$ for $\sigma : T^n \to T$.

But from $ch(\not{S}^{+})$ we only need to calculate the terms that belongs to $H^{2n}(S^{2n})$. That term will come from $\sum_{\substack{\text{even } \# \\ \text{of } +\text{'s}}} \frac{1}{n!}\frac{1}{2^n}(\pm\tilde{x}_1 \pm\tilde{x}_2,\ldots,\pm\tilde{x}_n)^n$.

Since to Pontrjagin classes over the sphere are all zero, the only term that remains is $\frac{1}{2^n}\tilde{x}_1,\ldots,\tilde{x}_n$, which is the Euler class of S^{2n} and therefore $\neq 0$.

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