SHARP ESTIMATE AND DIRAC OPERATOR

A Dissertation presented

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Marcelo Llarull

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Sharp Estimate and Dirac Operator

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Marcelo Llarull

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This work established a global conservation phenomenon for the scalar curvature function on a Riemannian manifold.

A classical result occurs in dimension 2 and it is given by the Gauss-Bonnet Theorem, which states that the average of the scalar curvature is a constant depending only on the topology of the surface.

In higher dimension, any conservation phenomenon for the scalar curvature is a weak measure of the Riemannian structure. Using Dirac operator methods, certain sharp results are established.
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To my parents,

Mira Raiden and Marcelo Llarull
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Introduction

The main result of this work could be characterized as establishing a global conservation phenomenon for the scalar curvature function on a Riemannian manifold.

A classical example of this phenomenon is given by the Gauss-Bornet Theorem in dimension 2, which states that the average of the scalar curvature is a constant depending only on the topology of the surface.

In high dimensions any conservation phenomenon for the scalar curvature must be far more delicate since the scalar curvature is a rather weak measure of the Riemannian structure. Nevertheless, using Dirac operator methods we do succeed in establishing certain sharp results.

We begin by examining the basic problem (originally posed by M. Gromov) of studying perturbations of the canonical metric $g$ on the $n$-sphere with normalized scalar curvature $\tilde{\kappa} = 1$. The normalized scalar curvature is defined to be $\tilde{\kappa} = \kappa / n(n-1)$ where $\kappa$ is the usual scalar curvature and $n =$ dimension of the manifold.

**Theorem 1.2.** Let $g$ be any Riemannian metric on $S^n$ with the property that $g \geq g_0$. Then either there exists some $x \in S$ with $\kappa_g(x) < n(n-1)$, or $g = g_0$.

This situation can be extended in the following way. A map $f : M \to N$ between Riemannian manifolds is said to be $\epsilon$-constructing if $\|f_* v\| \leq \epsilon \|v\|$ for all tangent vectors $v$ on $M$. 

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Theorem 2.1. Let $M$ be a compact Riemannian spin manifold of dimension $n$. Suppose there exists a $1$-contracting map $f : (M,g) \to (S^n, g_0)$ of non-zero degree. Then either there exists $x \in M$ with $\tilde{\kappa}(x) < 1$ or $M = S^n$ and $f$ is an isometry.

Note that the result is sharp since the identity $\text{Id} : (S^n, g_0) \to (S^n, g_0)$ is $1$-contracting and $\tilde{\kappa} = 1$.

Theorem 1 has the following immediate consequence:

Corollary 2.2. Let $M$ be compact Riemann spin manifold of dimension $n$ with $\tilde{\kappa} \geq a^2$. Then $M$ admits no $c$-contracting maps $f : M \to S^n$ of non-zero degree for any $c > 1/a$.

These results can be generalized in the following way. We define the $A$-degree of a map $f : M \to N$ between compact differentiable manifolds to be $[f^* w_n, \hat{A}[M]][M]$ where $w_n$ is the generator of $H^n(S^n, \mathbb{Z})$ and $A$ is the total $\hat{A}$-class of $M$.

Theorem 3.1. Let $(M,g)$ be a compact Riemannian spin manifold of dimension $n + 4k$ which admits a $1$-contracting map $f : (M,g) \to (S^n, g_0)$ of non-zero $\hat{A}$-degree. Then there exists a point $x \in M$ where $\tilde{\kappa}(x) < n(n-1)/(n+4k)(n+4k-1)$ or $f$ is an isometric submersion.

Theorem 2 has a corollary analogous to Corollary 1 above.

All of the results above continue to hold under a weaker hypothesis on the map $f$. We shall define a smooth map $f : M \to N$ between Riemannian manifolds to be $(\varepsilon, \Lambda^k)$-contracting if $\|f^* \alpha\| \leq \varepsilon \|\alpha\|$ for all $\alpha \in \Lambda^k(N)$.

Note that "$1$-contracting" means $(1, \Lambda^1)$-contracting.
Theorem 4.1. The statement of Theorem 1 and 2 continue to hold
if the hypothesis that $f$ be $1$-contracting is replaced by the
hypothesis that $f$ be $(1, \Lambda^2)$-contracting.

Remark. The hypothesis can not be weakened further, that is, there
are counterexamples that show that the theorem is false for $(1, \Lambda^m)$-
contracting maps with $m \geq 3$. 
Chapter 0

In this section we shall recall some basic definitions and results for Dirac operators on a spin compact Riemannian manifold.

Some Definitions and Notations

Let $(M,g)$ be a spin compact Riemannian manifold with metric $g$. Let $(S^n,g_0)$ be the unit sphere in $\mathbb{R}^{n+1}$ with the standard metric $g_0$. Given a map $f$ between two compact manifolds, the degree of $f$ is defined as

$$\text{deg}(f) = \sum_{p \in f^{-1}(q)} \text{sign}(\det f_*)_p,$$

where $q$ is a regular point. A map $f : M \to N$ is said to be $\epsilon$-contracting if

$$\|f_* v\| \leq \epsilon \|v\|,$$

for all tangent vectors $v$ in $M$. A map $f : M \to N$ is said to be $(\epsilon,A^k)$-contracting if

$$\|f^* \varphi\| \leq \epsilon \|\varphi\|$$

for all $k$-forms $\varphi \in \Lambda^k(N)$. The normalized scalar curvature of a manifold $M$ of dimension $n$ is defined to be

$$\tilde{\kappa} = \frac{\kappa}{n(n-1)},$$

where $\kappa$ is the usual scalar curvature.

Spin Structure

A spin manifold is an oriented manifold with a spin structure on its tangent bundle. Let $E$ be an oriented vector bundle, a spin
structure on $E$ is a 2-sheeted covering

$$\xi : P_{\text{Spin}_n}(E) \to P_{\text{SO}_n}(E)$$

such that $\xi(p \cdot g) = \xi(p) \cdot \xi_0(g)$ for all $p \in P_{\text{Spin}_n}(E)$ and $g \in \text{Spin}_n$, where

$$\xi_0 : \text{Spin}_n \to \text{SO}_n$$

is the universal covering homomorphism with kernel $\mathbb{Z}_2$, and $P_{\text{Spin}_n}(E)$ and $P_{\text{SO}_n}(E)$ are principal $\text{Spin}_n$- and $\text{SO}_n$- bundle respectively.

Note that a manifold $M$ is spin if the first and second Whitney classes of $M$, $\omega_1$ and $\omega_2$ are both zero.

A real Spinor bundle of $E$ is a bundle of the form

$$S(E) = P_{\text{Spin}_n}(E) \times_{\lambda} V$$

where $V$ is a left module for the Clifford algebra $\mathbb{C}l(\mathbb{R}^n) = \mathbb{C}l_n$ and $\lambda : \text{Spin}_n \to \text{SO}(V)$ is a representation by left multiplication of elements of $\text{Spin}_n \subseteq \mathbb{C}l(\mathbb{R}^n) = \mathbb{C}l_n$.

A complex spinor bundle of $E$ is the bundle $S_C(E) = P_{\text{Spin}_n}(E) \times_{\lambda} V_C$ where $V_C$ is a complex left module for the Clifford algebra $\mathbb{C}l(\mathbb{R}^n) \otimes \mathbb{C} = \mathbb{C}l_n$. The Clifford algebra $\mathbb{C}l_n(V)$ is generated by $V^n$ subject to the relations $v \cdot v = -\|v\|^2$ for all $v \in V^n$. The automorphism $\alpha : \mathbb{C}l_n \to \mathbb{C}l_n$ that extends the map $\alpha(v) = -v$ gives rise to a decomposition

$$\mathbb{C}l_n = \mathbb{C}l_n^0 \oplus \mathbb{C}l_n^1,$$
where \( \mathcal{C}_n^i = \{ \varphi \in \mathcal{C}_n : \alpha(\varphi) = (-1)^i \varphi \} \) are the eigenspaces of \( \alpha \).

\[ \text{Spin}_n = \text{Pin}_n \cap \mathcal{C}^0_n, \]

where \( \text{Pin}_n \) is defined as the subgroup of \( \mathcal{C}_n - \{0\} \) generated by the elements \( v \), with \( \|v\| \neq 0 \). Given a manifold \( M \), \( \mathcal{C}(M) \) will be the Clifford bundle of \( M \), which is the bundle over \( M \) whose fibre at a point \( p \in M \) is the Clifford algebra \( \mathcal{C}(T_p M) \) of the tangent space at \( p \).

Note that \( T(M) \subseteq \mathcal{C}(M) \). We extend the metric and the connection of \( M \) to \( \mathcal{C}(M) \) with the connection \( \nabla \) preserving the metric and such that

\[ \nabla(\varphi \cdot \psi) = (\nabla_\varphi \cdot \psi) + \psi \cdot (\nabla_\psi \cdot \varphi) \]

for all sections \( \varphi \) and \( \psi \in \Gamma(\mathcal{C}(M)) \). Let us consider the following complex bundle over \( (M, g) \), where \( M \) is Spin compact 2n-dimensional Riemannian manifold

\[ = \text{P}_{\text{Spin}_{2n}} (M) \times_{\lambda} \mathcal{C}_{2n} \]

with the induced connection, where \( \lambda \) is the representation by left multiplication.

We introduce a \( \mathbb{Z}_2 \)-grading on \( S \). Fix \( p \in M \) and choose local pointwise orthonormal tangent vector fields around \( p[e_1, \ldots, e_{2n}] \) such that \( (\nabla_{e_k})_p = 0 \). Let \( \omega \) be the oriented "volume element"

\[ \omega = i^n e_1 \wedge \ldots \wedge e_{2n}, \]

where \( \cdot \) denotes Clifford multiplication. This is a globally defined section of \( \mathcal{C}(M) \) with the following properties:
i) \( \nabla_\omega = 0 \)

ii) \( \omega^2 = 1 \)

iii) \( \omega e = -e \omega \) for any \( e \in \text{TM} \).

Then \( S \) has the decomposition

\[
S = S^+ \oplus S^-
\]

into the +1 and -1 eigenvalues of Clifford multiplication by \( \omega \).

For any \( e \in \text{TM} \),

\[
e \cdot S^+ \subseteq S^- \quad \text{and} \quad e \cdot S^- \subseteq S^+
\]

over \( (S^{2n}, g_0) \) we can carry out the same construction to get the bundle

\[
E^+_0 = P_{\text{Spin}_{2n}}(S^{2n}) \times_{\lambda} \mathbb{C}^\ell_{2n}
\]

with the induced metric and connections from \( (S^{2n}, g_0) \). Fix \( x \in S^{2n} \) and choose local pointwise orthonormal tangent vector fields around \( x, \{e_1, \ldots, e_{2n}\} \) such that \( (\nabla_\epsilon) = 0 \). Let \( \omega_0 \) be the "volume element"

\[
\omega_0 = i^n e_1 \ldots e_{2n}
\]

As before, \( \omega_0 \) gives the splitting

\[
E_0 = E^+_0 \oplus E^-_0
\]

into the +1 and -1 eigenspaces of \( \omega_0 \).

Suppose that \( f : (M^{2n}, g) \rightarrow (S^{2n}, g_0) \) is a map of non-zero degree.

We can consider the pull-back bundle \( f^*E_0 = E \) over \( (M, g) \). The pull-back
bundle $E$ has also a splitting $E = E^+ \otimes E^- = f^*E_0^+ \otimes f^*E_0^-$. Now we consider the tensor product bundle $S \otimes E$ over $M$ with the tensor product metric and connection. And

$$S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E).$$

We consider the Dirac operator of

$$D : \Gamma(S) \to \Gamma(S)$$

which in terms of an orthonormal basis of tangent vectors at $p$ is given by

$$D = \sum_{k=1}^{2n} e_k \nabla e_k.$$

Moreover, we can consider the twisted Dirac operator $D_E$ on $S \otimes E$. $D_E$ on simple elements $\varphi \otimes v \in \Gamma(S \otimes E)$ is defined by

$$D_E(\varphi \otimes v) = \sum_{k} (e_k \nabla e_k \varphi) \otimes v + \sum_{k} (e_k \varphi) \otimes (\nabla e_k v).$$

This first order operator $D_E$ preserves $E^+$, i.e.

$$S \otimes E = S \otimes E^+ \oplus S \otimes E^-$$

and

$$D_E(S \otimes E^+) \subseteq S \otimes E^+.$$ 

In fact, $E^+ = f^*(E_0^+) = f^*(\{v \in E_0 : \omega_0 v = v\}) = \{v \in E : f^* \omega_0 v = v\}$ so if $\varphi \otimes v \in S \otimes E^+$,

then

$$D_E(\varphi \otimes v) = \sum_{i} e_i \nabla_i (\varphi \otimes v) = \sum_{i} e_i (\nabla_i \varphi \otimes v + \varphi \nabla_i v) = \sum_{i} e_i (\nabla_i \varphi \otimes v + \varphi \nabla_i v) = \sum_{i} e_i \nabla_i (\varphi \otimes v)$$
where $V_i = V_{e_i}$.

Since $v \in S^+$, $V_i v \in E^+$ because $V_i v = V_i (w,v) = (V_i w)v + w(V_i v)$ and $(V_i w) = 0$. Therefore, $D_E (\sigma \otimes v) \in S \otimes E^+$. Since any element of $S \otimes E^+$ is the sum of simple elements of the form $\sigma \otimes v$, we can write

$$D_E^+ = D_e \bigg|_{S \otimes E^+}.$$

Furthermore, since $e . S^+ \subseteq S^+$, then

$$D_E^+ = D_E^+ \otimes D_E^-$$

and

$$D_E^{\pm}: S^\pm \otimes E^+ \to S^\mp \otimes E^+.$$

**Bochner–Lichnerowicz–Weitzenböck Formula**

We now recall the fundamental B-L-W formula for the twisted Dirac operator $D_E$ of the bundle $S \otimes E$ over $M$, see [IM]

$$D_E^2 = \nabla^* \nabla + \frac{1}{4} \kappa + R^E.$$

$\nabla^* : \Gamma (S \otimes E) \to \Gamma (S \otimes E)$ is defined in terms of a local basis of point-wise orthonormal tangent vector fields by $\nabla^* = \sum_{i,j} g(R_{e_i, e_j, e_i, e_j})$ is the scalar curvature of $M$, $g$ is the Riemannian metric and $R$ the curvature tensor of $M$. $R^E$ is defined in simple elements $\sigma \otimes v \in \Gamma (S \otimes E)$ by $R^E (\sigma \otimes v) = \frac{1}{2} \sum_{i,j} g(e_i, e_j, e_i, e_j) \otimes (R^E e_i e_j \sigma)$ where $R^E$ denotes the curvature tensor of $E$.

Note that $R^E$ depends linearly on the curvature tensor $R^E$ of $E$.

For a more detailed description see [IM].
Chapter 1

Case of $S^n$

We start this section by examining the basic problem, originally posed by M. Gromov [Gr] of studying perturbations of the canonical metric $g_0$ on the n-sphere with normalized curvature 1.

Theorem 1.0. Let $g$ be any Riemannian metric on $S^n$ with the property that $g \geq g_0$. Then either there exist some $x \in S^n$ with $\tilde{\kappa}_g(x) < 1$, or $g \equiv g_0$.

Example 1.1. $(S^n, g) \xrightarrow{\text{Id}} (S^n, g_0)$,

where $g = (1+\varepsilon) g_0$, $\varepsilon > 0$, then $g \geq g_0$. In this case $\tilde{\kappa}_g = \frac{1}{(1+\varepsilon)} < 1$ for all $x$.

This result is also true when the map between the spheres is of non-zero degree and not necessarily the id.

Theorem 1.2. Let $f : (S^n, g) \rightarrow (S^n, g_0)$ be a map of non-zero degree. Suppose that $f$ is 1-contracting. Then either there exists some $x \in S^n$ with $\tilde{\kappa}_g(x) < 1$, or $f$ is an isometry.

Remark 1.3. This result is sharp since the identity $\text{Id} : (S^n, g) \rightarrow (S^n, g_0)$ is 1-contracting and $\tilde{\kappa}_g = 1$.

Theorem 1.2 has the following immediate consequence:

Corollary 1.4. If $(S^n, g)$ has $\tilde{\kappa}_g \geq 1$, there exists a constant $c$, such that there exists no $c$-contracting maps $f : (S^n, g) \rightarrow (S^n, g_0)$.
of non-zero degree.

When we go through the proofs of these results, one realizes that \( (S^n, g) \) can be replaced by any compact Spin manifold. We shall leave the proofs as part of the more general situation studied in Chapter 2.
Chapter 2

Results for a Compact Spin Manifold

The results in Chapter 1 can be extended in the following way.

Theorem 2.1. Let $M$ be a compact Riemannian Spin manifold of dimension $n$. Suppose there exists a $1$-contracting map $f : (M, g) \rightarrow (S^n, g_0)$ of non-zero degree. Then either there exists $x \in M$ with $\tilde{\omega}_g(x) < 1$ or $M = S^n$ and $f$ is an isometry.

As an immediate consequence, we have

Corollary 2.2. Let $M$ be a compact Riemannian Spin manifold of dimension $n$ with $\tilde{\omega} \geq \tilde{a}^2$. Then $M$ admits no $c$-contracting maps $f : M \rightarrow S^n$ of non-zero degree for any $c > \frac{1}{\tilde{a}}$.

Proof. Consider $\delta : [M, S^n] \rightarrow \mathbb{R}$

$$f \mapsto \delta_f = \max_{\|v\|=1} \|f_*v\|.$$ 

If $\tilde{\omega} > \tilde{a}^2$, then $\delta_f > \frac{1}{\tilde{a}^2}$.

Theorem 2.1 will be proved by contradiction. The idea is the following. Consider a twisted spinor bundle $S \otimes E^+$ over $M$ and its Dirac operator $D_{E^+}$.

Using the Atiyah-Singer Index Theorem we will show that Index $\left(D_{E^+}\right) \neq 0$.

Assuming that $\tilde{\omega}_g \geq 1$ all over $M$ and considering the B-L-W formula for $D_{E^+}$ we will show that Index $\left(D_{E^+}\right) = 0$. The key point
of the proof is choosing the appropriate coefficient bundle $E$. This method has been used by Gromov and Lawson, see [GL]. But in their work the choice of the coefficient bundle was not essential for their results.

Proof of Theorem 2.1. The proof will be done first for the even dimensional case. Let $M$ be a compact Spin $2m$-dimensional Riemannian manifold with metric $g$. Let $S^{2n}$ be the unit $2n$-sphere with standard metric $g_0$. Let $f : M \rightarrow S^{2n}$ be a 1-contracting map of non-zero degree.

By contradiction, assume that $\lambda_\varphi \geq 1$ all over $M$. We consider the twisted vector bundle $S \otimes E^+$ over $M$ and its Dirac operator $D$. As we did in Chapter 0, recall that $D = D_E[S \otimes E^+]$. Fix $p \in M$.

Let $\{e_1, \ldots, e_{2n}\}$ be a $g$-orthonormal tangent frame near $p \in M$ such that $(\nabla e_\alpha)^p = 0$ for each $\alpha$. Let $\{\epsilon_1, \ldots, \epsilon_{2n}\}$ be a $g$-orthonormal tangent frame near $f(p) \in S^{2n}$ such that $(\nabla \epsilon_\alpha)^{f(p)} = 0$ for each $\alpha$.

Since $f$ has non-zero degree, $f^*_p$ can be simultaneously diagonalized with respect to the bases $\{e_1, \ldots, e_{2n}\}$ and $\{\epsilon_1, \ldots, \epsilon_{2n}\}$. Therefore, we can find positive scalar $[\lambda_1]_{i=1}^2$ such that

$$\epsilon_j = \lambda_j f^*_p e_j.$$

Note that $\lambda_j \geq 1$ since $f$ is 1-contracting and

$$1 = g_0(\epsilon_j, \epsilon_j) = g_0(\lambda_j f^*_p e_j, \lambda_j f^*_p e_j) = \lambda_j^2 g_0(f^*_p e_j, f^*_p e_j).$$

(2.3)  \hspace{1cm} 1 = \lambda_j^2 g_0(f^*_p e_j, f^*_p e_j) \leq \lambda_j^2 g(e_j, e_j) = \lambda_j^2$$
Considering the inner product $\langle , \rangle$ on the space $\Gamma(S\otimes E)$ of cross-sections defined by

\[ \langle \phi, \psi \rangle = \int_M g_x(\phi, \psi) \quad \forall \phi, \psi \in \Gamma(S\otimes E), \]

we can write the B-L-W formula as

\[ \langle D_B^2 \phi, \phi \rangle = \langle \nabla^2 \phi, \phi \rangle + \frac{1}{4} \langle \phi, \phi \rangle + \langle R^E \phi, \phi \rangle \]

\[ = \langle \nabla \phi, \nabla \phi \rangle + \frac{1}{4} \langle \phi, \phi \rangle + \langle R^E \phi, \phi \rangle \]

\[ \therefore \quad \langle D_B^2 \phi, \phi \rangle \geq \frac{1}{4} \| \phi \|^2 + \langle R^E \phi, \phi \rangle. \]

In order to establish the result, we must look at the term $\langle R^E \phi, \phi \rangle$ in more detail.

On simple elements $\sigma \otimes v \in \Gamma(S\otimes E)$, $R^E$ is defined by

\[ R^E(\sigma \otimes v) = \frac{1}{2} \sum_{i,j=1}^{2n} (e_i \otimes e_j) \otimes (R^E e_i \otimes e_j) \cdot R^E e_i \otimes e_j. \]

More explicitly, see [LM].

\[ R^E e_i e_j = R^E_{e_i e_j} = R^E_{f_{e_i} e_j} = \frac{1}{4} \sum_{k, \ell} g_0 (R^E_{f_{e_i} e_j} e_k, e_\ell) e_k e_\ell \]

where $R^E$ is the curvature tensor on $S^{2n}$. Therefore,

\[ R^E e_i e_j = \frac{1}{4} \sum_{k, \ell=1}^{2n} \left[ g_0(f_{e_i} e_j, e_k) - g_0(f_{e_j} e_i, e_k) - g_0(f_{e_i} e_j, e_\ell) + g_0(f_{e_j} e_i, e_\ell) \right] e_k e_\ell \]

\[ = \frac{1}{4} \sum_{k, \ell=1}^{2n} \left[ g_0(\frac{e_i}{\lambda_j}, e_k) - g_0(\frac{e_j}{\lambda_i}, e_\ell) - g_0(\frac{e_i}{\lambda_1}, e_k) + g_0(\frac{e_j}{\lambda_1}, e_\ell) \right] e_k e_\ell \]

\[ = \frac{1}{4} \sum_{k, \ell=1}^{2n} \left[ \frac{1}{\lambda_i \lambda_j} \delta_{i \ell} \delta_{jk} - \frac{1}{\lambda_i \lambda_j} \delta_{j \ell} \delta_{ik} \right] e_k e_\ell \]
\[
= \frac{1}{4} \left[ \frac{1}{\lambda_i \lambda_j} \varepsilon_j \varepsilon_i - \frac{1}{\lambda_i \lambda_j} \varepsilon_i \varepsilon_j \right]
\]
\[
= \frac{1}{4} \frac{1}{\lambda_i \lambda_j} 2 \varepsilon_j \varepsilon_i
\]
\[
\therefore \quad R^E_{e_i e_j} = \frac{1}{2} \frac{1}{\lambda_i \lambda_j} \varepsilon_j \varepsilon_i.
\]

Let \(\{\sigma_\alpha\}_{\alpha=1}^{2n}\) be a basis for \(S\) and \(\{v_\beta\}_{\beta=1}^{2n}\) be a basis for \(E_0\). Then if \(\phi \in S \otimes E\),
\[
\phi = \sum_{\alpha_1 \beta} a_{\alpha \beta} \sigma_\alpha \otimes v_\beta
\]
and
\[
(2.4) \quad \langle k^E \phi, \phi \rangle = \langle k^E (\sum a_{\alpha \beta} \sigma_\alpha \otimes v_\beta), \sum a_{\alpha \beta} \sigma_\alpha \otimes v_\beta \rangle
\]
\[
= \frac{1}{2} i+j \sum_{\alpha \beta} a_{\alpha \beta} \varepsilon_j \varepsilon_i \sigma_\alpha \otimes \frac{1}{\lambda_i \lambda_j} \frac{1}{2} \varepsilon_j \varepsilon_i v_\beta, \sum_{\alpha \beta} a_{\alpha \beta} \sigma_\alpha \otimes v_\beta
\]
\[
= \frac{1}{4} i+j \sum_{\alpha \beta} a_{\alpha \beta} \varepsilon_j \varepsilon_i \sigma_\alpha \otimes \frac{1}{\lambda_i \lambda_j} \varepsilon_j \varepsilon_i v_\beta, v_\beta
\].

This suggests choosing the bases \(\{\sigma_\alpha\}\) and \(\{v_\beta\}\) "invariant" by \(e_i e_j\) and \(\varepsilon_j \varepsilon_i\) respectively.

Consider the following bases \(\{e_i^1, \ldots, e_i^k\}\) for \(S\) and \(\{e_j^1, \ldots, e_j^1\}\) for \(E_0\), where \(\sigma \in S\) with \(\|\sigma\| = 1\) and \(v \in E_0\) with \(\|v\| = 1\). For each fixed pair \((i, j)\), \(e_i e_j \cdot : S \to S\) permutes the basis for \(S\) (up to sign); and so does \(\varepsilon_j \varepsilon_i \cdot : E_0 \to E_0\) for the basis of \(E_0\).

Moreover, since \((e_i e_j)^2 = -1\), if \(\sigma_\alpha\) is any element of \(\{e_i^1, \ldots, e_i^k\}\), then so is \(\pm e_i e_j \sigma_\alpha\) and the subspace generated by \(\{\sigma_\alpha, e_i e_j \sigma_\alpha\}\) is invariant under \(e_i e_j\). Analogous considerations are true for \(E_0\) and
the basis \( \{ \epsilon_{j_1}, \ldots, \epsilon_{j_s} \} \) since \((\epsilon_{j_1} \epsilon_{j_1})^2 = -1\). Therefore, if \( \{ \sigma_{j} \} \)
represents \( \{ \epsilon_{j_1}, \ldots, \epsilon_{j_k} \} \) and \( \{ \nu_{\beta} \} \) represents \( \{ \epsilon_{j_1}, \ldots, \epsilon_{j_s} \} \), then
the above sum can be bounded as follows.

For each pair \((i, j)\), the sum
\[
\sum_{\alpha} a_{\alpha \beta} e_{i} \epsilon_{j} \sigma \otimes \epsilon_{i} \epsilon_{j} \nu_{\beta} \sum_{k, \ell} a_{k \ell} \sigma_{k} \otimes \nu_{\ell}
\]
can be rewritten in a 4-term sum. Each pair \( \{ \sigma, e_{i} \epsilon_{j} \sigma \} \) and
\( \{ \nu_{\beta}, e_{j} \epsilon_{i} \nu_{\beta} \} \) will give the following four orthogonal basis elements
for \( S \otimes E : \)
\[
\sigma_{\alpha} \otimes \nu_{\beta}, \sigma_{\alpha} \otimes \epsilon_{j} \epsilon_{i} \nu_{\beta}, e_{i} \epsilon_{j} \sigma \otimes \nu_{\beta}
\text{ and } e_{i} \epsilon_{j} \sigma \otimes \epsilon_{i} \epsilon_{j} \nu_{\beta}
\]
\[
\sum_{\alpha} a_{\alpha \beta} e_{i} \epsilon_{j} \sigma \otimes \epsilon_{j} \epsilon_{i} \nu_{\beta} + a_{\alpha \beta} e_{i} \epsilon_{j} \sigma \otimes (\epsilon_{j} \epsilon_{i})^2 \nu_{\beta}
\]
\[
+ a_{\alpha \beta}' (e_{i} \epsilon_{j})^2 \sigma \otimes \epsilon_{j} \epsilon_{i} \nu_{\beta} + a_{\alpha \beta} (e_{i} \epsilon_{j})^2 \sigma \otimes (\epsilon_{j} \epsilon_{i})^2 \nu_{\beta}
\]
\[
\sum_{k, \ell} (a_{k \ell} \sigma_{k} \otimes \nu_{\ell} + a_{k \ell} \sigma_{k} \otimes \epsilon_{j} \epsilon_{i} \nu_{\ell} + a_{k \ell} \sigma_{k} \otimes \epsilon_{j} \epsilon_{i} \nu_{\ell})
\]
\[
= a_{\alpha \beta} a_{\alpha \beta}' - a_{\alpha \beta} a_{\alpha \beta}' - a_{\alpha \beta} a_{\alpha \beta}' + a_{\alpha \beta} a_{\alpha \beta}'
\]
\[
= 2a_{\alpha \beta} a_{\alpha \beta}' - 2a_{\alpha \beta} a_{\alpha \beta}'
\]
since
\[
2a_{\alpha \beta} a_{\alpha \beta}' \geq -(a_{\alpha \beta} + a_{\alpha \beta}')^2
\]
and
\[-2a_{\alpha \beta} a_{\alpha \beta}' \geq -(a_{\alpha \beta} + a_{\alpha \beta}')^2.
\]
We get that each four-term sum for fixed \((\alpha, \beta)\) is
\[
\geq -(a_{\alpha \beta} + a_{\alpha \beta} + a_{\alpha \beta} + a_{\alpha \beta}).
\]
Summing over $\alpha$ and $\beta$, we obtain that
\[
\frac{1}{4} \sum_{i \neq j} \sum_{k, \ell} \sum_{\alpha \beta} \frac{a_{\alpha \beta}}{\lambda_i \lambda_j} \langle e_i, e_j, \frac{\sigma_k}{\lambda_i}, \langle e_i, e_j, \frac{\sigma_k}{\lambda_i}, \frac{\sigma_k}{\lambda_j} \rangle \rangle \geq - \frac{1}{4} \sum_{i \neq j} \frac{1}{\lambda_i \lambda_j} \|\phi\|^2.
\]

Thus,
\[
\langle R^E \phi, \phi \rangle \geq - \frac{1}{4} \sum_{i \neq j} \frac{1}{\lambda_i \lambda_j} \|\phi\|^2.
\]

Recall that $\lambda_i \geq 1$, (2.3)
\[
\therefore \quad \langle R^E \phi, \phi \rangle \geq - \frac{1}{4} \sum_{i \neq j} \|\phi\|^2 \geq - \frac{1}{4} 2n(2n-1)\|\phi\|^2.
\]

Consequently,
\[
\langle D^2_{E^+} \phi, \phi \rangle \geq \frac{1}{4} \|\phi\|^2 - \frac{1}{4} 2n(2n-1)\|\phi\|^2
\]
\[
\langle D^2_{E^-} \phi, \phi \rangle \geq \frac{1}{4} 2n(2n-1)(\tilde{\kappa}-1)\|\phi\|^2
\]

where $\tilde{\kappa} = \frac{\kappa}{2n(2n-1)}$ is the normalized scalar curvature of $M$. Therefore, if $\tilde{\kappa} > 1$, then $\ker D^2_{E} = 0$. But $\ker D^2_{E} = \ker D_{E^+}$, see [GLl].

Since $D_{E} : S \otimes E^+ \oplus S \otimes E^- \to S \otimes E^+ \oplus S \otimes E^-$ preserves the direct sum,
\[
0 = \ker D_{E} \big|_{S \otimes E^+} = \ker D^+_{E^+}.
\]

We also have that $D^+_{E^+} = D^+_E \oplus D^-_{E^+}$.
where \( D^+ : \mathfrak{S}^+ \otimes E^+ \to \mathfrak{S}^+ \otimes E^+ \).

Since \( 0 = \ker D^+_+ = \ker D^+_+ \otimes \ker D^-_+ \),

we get that \( \ker D^+_+ = \ker D^-_+ = 0 \).

The index of \( D^+ \) is given by

\[
\text{Index}(D^+) = \dim(\ker D^+_+) - \dim(\ker D^-_+) = 0.
\]

However, this index can also be found by using the Atiyah-Singer Index Theorem.

\[
\text{Index}(D^+) = [\text{ch } E^+ \hat{A}(M)] [M],
\]

where \( \text{ch } E^+ \) is the Chern character of \( E^+ \) and \( \hat{A} \) is the total \( \hat{A} \)-class of \( M \).

Recall that \( \text{ch } E^+ = \dim E^+ + \text{ch}_1(E^+) + \cdots + \text{ch}_n(E^+) \) where \( \text{ch}_i = \text{i}^{\text{th}} \) symmetric polynomial in the Chern class \( c_i \), with \( \text{ch}_i \in H^{2i}(M) \), see [Hi].

Remember that \( E^+ \) is the pull-back bundle of \( E_0^+ \) through \( f^* \).

On \( S^{2n} \) is the Chern character of a vector bundle is given by

\[
\text{ch } E_0^+ = \dim E_0^+ + \frac{1}{(n-1)!} c_n(E_0^+).
\]

Therefore on the pull-back

\[
\text{ch}(E^+) = \dim E_0^+ + \frac{1}{(n-1)!} f^* c_n(E_0^+)
\]

\[
\text{ch}(E^+) = 2^{2n-1} + \frac{1}{(n-1)!} f^* c_n(E_0^+).
\]

Applying the B-L-W type formula, Atiyah, Hitchin, Lichnerowicz and Singer showed that a compact spin manifold \( M \) with \( \kappa > 0 \) must
have \( \hat{\mathbf{A}}(M) = 1 \). In our case we are under the assumption that 
\( \tilde{\nu} \geq 1 \) and so \( \nu > 0 \) and \( \hat{\mathbf{A}}(M) = 1 \). Consequently,

\[
\text{Index}(D_{E^+_g}) = \left\{ (2^{2n-1} + \frac{1}{(n-1)!} f^* c_n(E^+_0)) \hat{\mathbf{A}}[M] \right\} [M]
\]

\[
\text{Index}(D_{E^+_g}) = \frac{1}{(n+1)!} f^* c_n(E^+_0)[M]
\]
\[
= \frac{1}{(n+1)!} \int_{M} f^* c_n(E^+_0)
\]
\[
= \frac{1}{(n+1)!} \text{(degree \( f \))} \int_{S^{2n}} c_n(E^+_0)
\]

Claim 2.6. \( c_n(E^+_0) \neq 0 \) \quad [See Appendix 1]

Therefore, if \( \text{Index}(D_{E^+_g}) = 0 \), then \( \text{degree}(f) = 0 \); which is a contradiction.

If \( \tilde{\nu}_g = 1 \), then \( f \) is an isometry. In fact, if \( \tilde{\nu}_g = 1 \), or equivalently \( \nu_g = 2n(2n-1) \), then inequality (2.5) gives

\[
\langle D^2_{E} \phi, \phi \rangle \geq \frac{1}{4} \lambda \| \phi \|^2 - \frac{1}{4} \sum_{i \neq j} \frac{2n}{\lambda_i \lambda_j} \| \phi \|^2
\]
\[
\langle D^2_{E} \phi, \phi \rangle \geq \frac{1}{4} \| \phi \|^2 (2n(2n-1) - \sum_{i \neq j} \frac{1}{\lambda_i \lambda_j})
\]
\[
\geq \frac{1}{4} \| \phi \|^2 \left[ 2n - \sum_{i \neq j} \left( 1 - \frac{1}{\lambda_i \lambda_j} \right) \right].
\]

Since \( \text{Index}(D_{E^+_g}) \neq 0 \), \( \ker D_{E^+_g} \neq 0 \), there exists \( 0 \neq \phi \in \Gamma(S^0E) \) such that \( D\phi = 0 \), so

(2.7) \quad 0 \geq \frac{1}{4} \| \phi \|^2 \left[ \sum_{i \neq j} \left( 1 - \frac{1}{\lambda_i \lambda_j} \right) \right].

Recall that each \( \lambda_i \geq 1 \), so \( 1 - \frac{1}{\lambda_i \lambda_j} \geq 0 \)

and \( 1 - \frac{1}{\lambda_i \lambda_j} = 0 \quad \forall i \neq j \)
equivalently,

\[ \lambda_i \lambda_j = 1. \]

Therefore, \( \lambda_i = 1 \) for all \( 1 \leq i \leq 2n \)

and \( f \) is an isometry.

**Odd-dimensional Case**

Let \( M^{2n-1} \) be a compact spin manifold of dimension \( 2n-1 \), with Riemannian metric \( g \). Let \( S_r^{2n-1} \) be \((2n-1)\)-sphere of radius \( r \) with the standard metric \( g_0 \).

Let \( \tilde{\kappa} \) denote the normalized scalar curvature of \( M \).

Let \( f : M \to S^{2n-1} \) be a \( 1 \)-contracting map of non-zero degree.

We want to show that there exists \( x \in M \) where \( \tilde{\kappa}_g(x) < 1 \).

Consider

\[ M \times S_r^1 \xrightarrow{f \times \frac{1}{r} \text{id}} S^{2n-1} \times S^1 \xrightarrow{h} S^{2n-1} \times S^1 \simeq S^{2n} \]

where \( S_r^1 \) is the one dimensional sphere of radius \( r \), \( f \times \frac{1}{r} \text{id} \)

is defined as \( (f \times \frac{1}{r} \text{id})(p,t) = (f(p), \frac{t}{r}) \) \( \forall (p,t) \in M \times S^1 \), and where \( h \) is a \( 1 \)-contracting map into the smash product of non-zero degree.

Let us consider the following metrics.

On \( M \times S_r^1 \), \( g + ds^2 \) where \( ds^2 \) is the standard metric on \( S_r^1 \).

On \( S^{2n-1} \times S^1 \), \( g_0 + ds^2 \) where \( ds^2 \) is the standard metric on \( S^1 \).

And on \( S^{2n} \), \( -g \) is the standard metric of the unit sphere \( S^{2n} \).
The composite map \( \tilde{f} = h_*(f \cdot \frac{1}{r} \text{id}) \) is of non-zero degree from \( M^{2n-1} \times S^1 + S^{2n} \). It is also 1-contracting,

\[
\| \tilde{f}_*(v, t) \| = \| h_*(f_* v, \frac{1}{r} t) \| \leq \| f_* v \| + \| \frac{1}{r} t \| \leq \| v \| + \frac{1}{r} \| t \| \\
\leq \| v \| + \| t \| .
\]

We assume \( r > 1 \).

We can now apply the same method we used for the even-dimensional case. Construct complex vector bundles \( S \) over \( M^{2n-1} \times S^1 \) and \( E_0 \) over \( S^{2n} \) and consider the bundle

\[ S \otimes f_* E_0^+ \]

over \( M^{2n-1} \times S^1 \).

Choose a basis \( \{ e_1, \ldots, e_{2n-1}, e_{2n} \} \) of \( (g + ds^2) \)-orthonormal adapted tangent vectors around \( x \in M^{2n-1} \times S^1 \) such that \( \langle e_k x \rangle = 0 \) for each \( k \) and such that \( e_{2n} \) is tangent to \( S^1 \) and \( e_1, \ldots, e_{2n-1} \) are tangent to \( M^{2n-1} \). As before choose \( \tilde{g} \)-orthonormal basis \( \{ \varepsilon_1, \ldots, \varepsilon_{2n} \} \) of \( S^{2n} \) around \( \tilde{f}(x) \). Therefore, we can find positive scalar \( \{ \lambda_1 \}_{i=1}^{2n} \) such that \( \varepsilon_j = \lambda_j \tilde{f}_* e_j \).

Then we have that

\[
1 = \tilde{g}(\varepsilon_j, \varepsilon_j) = \tilde{g}(\lambda_j \tilde{f}_* e_j, \lambda_j \tilde{f}_* e_j) = \lambda_j^2 \tilde{g}(\tilde{f}_* e_j, \tilde{f}_* e_j) = \lambda_j^2 g_0(\tilde{f}_* e_j, \tilde{f}_* e_j) = \lambda_j^2 g(e_j, e_j) = \lambda_j^2
\]

so for \( 1 \leq j \leq 2n - 1 \)

\[
1 = \lambda_j^2 \tilde{g}(\tilde{f}_* e_j, \tilde{f}_* e_j) \leq \lambda_j^2 g_0(\tilde{f}_* e_j, f_* e_j) \leq \lambda_j^2 g(e_j, e_j) = \lambda_j^2
\]

\[ 1 \leq \lambda_j^2 \]
and for $j = 2n$

$$1 = \lambda_{2n}^2 g(\tilde{f}_* e_{2n}, \tilde{f}_* e_{2n}) \leq \lambda_{2n}^2 \frac{e_{2n}}{r} \frac{e_{2n}}{r}$$

$$1 \leq \frac{\lambda_{2n}^2}{r^2}$$

$$\lambda_{2n}^2 \geq r^2.$$ 

In the B-L-W formula for the twisted bundle $S \otimes E$ and its Dirac operator $D_E$,

$$(2.8) \quad D_E^2 = \nabla^* \nabla + \frac{1}{4} \kappa + R^E,$$

the curvature term $R^E$ (2.4) can be bounded as follows, by separating the terms coming from the $S^1 \times \mathbb{R}$ factor

$$\langle R^E \sigma, \phi \rangle = \frac{1}{4} \sum_{1 \neq j, k, \ell} 2n-1 \sum_{\alpha, \beta} \sum_{\lambda, \lambda_j} \frac{a_{\alpha} a_{\beta} a_{k\ell}}{\lambda \lambda_j} \langle \epsilon_i, \epsilon_j, \sigma_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_j, \epsilon_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_j, v_{\alpha \beta}^{\sigma \lambda_j} \rangle$$

$$+ \frac{1}{4} \sum_{i=1}^{2n-1} \sum_{k, \ell} \sum_{\alpha, \beta} \sum_{\lambda, \lambda_j} \frac{a_{\alpha} a_{\beta} a_{k\ell}}{\lambda \lambda_j} \langle \epsilon_i, \epsilon_{2n}, \sigma_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_{2n}, \epsilon_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_{2n}, v_{\alpha \beta}^{\sigma \lambda_j} \rangle$$

$$+ \frac{1}{4} \sum_{i=1}^{2n-1} \sum_{k, \ell} \sum_{\alpha, \beta} \sum_{\lambda, \lambda_j} \frac{a_{\alpha} a_{\beta} a_{k\ell}}{\lambda \lambda_j} \langle \epsilon_i, \epsilon_{2n}, \sigma_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_{2n}, v_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_{2n}, v_{\alpha \beta}^{\sigma \lambda_j} \rangle.$$

Therefore,

$$\langle R^E \sigma, \phi \rangle \geq \frac{1}{4} \sum_{1 \neq j, k, \ell} \sum_{\alpha, \beta} \sum_{\lambda, \lambda_j} \frac{a_{\alpha} a_{\beta} a_{k\ell}}{\lambda \lambda_j} \langle \sigma_{\alpha \beta}^{\sigma \lambda_j} \rangle < \epsilon_i, \epsilon_{2n}, v_{\alpha \beta}^{\sigma \lambda_j} \rangle$$

$$- 2 \left[ \frac{1}{4} \sum_{i=1}^{2n-1} \sum_{k, \ell} \sum_{\alpha, \beta} a_{\alpha} a_{k\ell} \sigma_{\alpha \beta}^{\sigma \lambda_j} < \epsilon_i, \epsilon_{2n}, v_{\alpha \beta}^{\sigma \lambda_j} \rangle \right]$$
\[ \langle R^E \phi, \phi \rangle \geq -\frac{1}{4} \sum_{i \neq j} 2^{n-1} \frac{1}{\| \phi \|^2} - 2 \frac{1}{4 r} \sum_{i=1}^{2n-1} \frac{2n-1}{\| \phi \|^2} \]

\[ \langle R^E \phi, \phi \rangle \geq -\frac{1}{4} (2n-1) (2n-2) \| \phi \|^2 - \frac{1}{2 r} (2n-1) \| \phi \|^2 \]

Note that in (2.8) \( \kappa \) is the unnormalized scalar curvature of \( M^{2n-1} \times S^1_r \)
which is equal to the unnormalized scalar curvature of \( M^{2n-1} \).

And

normalized scalar curvature of \( M^{2n-1} = \frac{\kappa}{(2n-1) (2n-2)} = \bar{\kappa} \).

Consequently,

\[ \langle D^2 \phi, \phi \rangle \geq \left[ \frac{1}{4} \kappa - \frac{1}{4} (2n-1) (2n-2) - \frac{2n-1}{2 r} \right] \| \phi \|^2 \]

\[ \langle D^2 \phi, \phi \rangle \geq \frac{1}{4} (2n-1) (2n-2) \left[ \bar{\kappa} - 1 - \frac{2}{(2n-2) r} \right] \| \phi \|^2. \]

As before, if \( \bar{\kappa} \equiv 1 \), since \( f \) is a 1-contracting map, \( f \) is an
isometry (see (2.7)).

If \( \bar{\kappa} > 1 \), since the last inequality is valid for all \( r > 0 \),
then

\[ \ker D^2_E = \ker D^*_E = 0. \]

And \( \ker (D^+_E) = 0 \), hence Index \( (D^+_E) = 0 \). But the Atiyah-Singer
Index Theorem gives

\[ \text{Index}(D^+_E) \neq 0 \quad \text{as before (2.6)}. \]
Chapter 3

Manifold of Dimension n + 4k

The notion of degree of a map between two compact manifolds can be extended to the case where the dimensions are not the same.

The \( \hat{\Lambda} \)-degree of a map \( f : M \to N \) is defined to be

\[
\hat{\Lambda}-\text{degree}(f) = \{f^* \omega_n \hat{\Lambda}[M]\} [M],
\]

where \( \omega_n \) is the generator of \( H^n(N, \mathbb{Z}) \) and \( \hat{\Lambda}[M] \) is the total \( \hat{\Lambda} \)-class of \( M \).

Now Theorem 2.1 can be generalized in the following:

**Theorem 3.1.** Let \( (M^{n+4k}, g) \) be a compact spin Riemannian manifold of \( \dim n + 4k \) which admits a 1-contracting map \( f : (M^{n+4k}, g) \to (S^n, g_0) \) of non-zero \( \hat{\Lambda} \)-degree. Then there exists a point \( x \in M \) where

\[
\hat{\kappa}_g(x) < \frac{n(n-1)}{(n+4k)(n+4k-1)} \quad \text{or} \quad f \text{ is an isometric submersion}
\]

**Corollary 3.2.** If \( M^{n+4k} \) has \( \hat{\kappa} \geq \frac{n(n+1)}{(n+4k)(n+4k-1)} \), then there exists no \( c \)-contracting map \( f : M \to S^n \) of non-zero \( \hat{\Lambda} \)-degree with \( c < 1 \).

**Proof of Theorem 3.1.** We proceed as in the proof of Theorem 2.

Suppose \( n \) is even. Let \( \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+4k}\} \) be a basis of \( g \)-orthogonal tangent vectors near \( p \in M \).

Let \( \{e_1, \ldots, e_n\} \) be a \( g_0 \)-orthogonal basis of tangent vectors near \( f(p) \in S^n \). Since \( f \) has \( \hat{\Lambda} \)-degree \( \neq 0 \), we can find scalar \( \lambda_j \)
such that

\[ \lambda_j f^* e_j = e_j \quad \text{with} \quad \lambda_j \geq 1 \quad 1 \leq j \leq n \]

\[ \therefore \quad f^* p = \begin{bmatrix}
\frac{1}{\lambda_1} & & & \\
& \ddots & & \\
& & \ddots & \\
0 & & & \frac{1}{\lambda_n} \\
\end{bmatrix}
\]

Then the term \( \langle D^2 \phi, \phi \rangle \) in the B-L-W-formula can be bounded as follows.

\[ \langle R^E \phi, \phi \rangle = \frac{1}{4} \sum_{i \neq j} 2^n \sum_{k, \ell} \frac{a_i a_j}{\lambda_i \lambda_j} \langle e_i, e_j, \sigma_k, \sigma_{\ell} \rangle \langle e_i, e_j, \sigma_k, \sigma_{\ell} \rangle. \]

Fixing each pair \((i, j)\) we use the same bound as in the proof of Theorem 2.1.

\[ \therefore \quad \langle R^E \phi, \phi \rangle \geq -\frac{1}{4} \sum_{i \neq j} \frac{1}{\lambda_i \lambda_j} ||\phi||^2 \]

\[ \therefore \quad \langle R^E \phi, \phi \rangle \geq -\frac{1}{4} n(n-1)||\phi||^2, \quad \text{since} \quad \lambda_i \geq 1 \]

\[ \therefore \quad \langle D^2 \phi, \phi \rangle \geq \frac{1}{4} n ||\phi||^2 - \frac{1}{4} n(n-1)||\phi||^2 \]

where \( \kappa \) is the unnormalized scalar curvature of \( M^{n+4k} \). The normalized scalar curvature of \( M^{n+4k} \) is

\[ \tilde{\kappa} = \frac{\kappa}{(n+4k)(n+4k-1)} \]
\[ \langle \mathcal{D}^2 \phi, \phi \rangle \geq \frac{1}{4} \frac{(n+4k)(n+4k-1)}{(n+4k)(n+4k-1)} \left( \frac{n(n-1)}{(n+4k)(n+4k-1)} \right) \| \phi \|^2. \]

If \( \tilde{n} \geq \frac{n(n-1)}{(n+4k)(n+4k-1)} \) and \( \tilde{n} \neq \frac{n(n-1)}{(n+4k)(n+4k-1)} \), then as before, we conclude that \( \text{Index}(D_{E'}) = 0 \) by using the above inequality.

But Atiyah-Singer Index Theorem gives

\[
\text{Index}(D_{E'}) = \{ \text{ch } E^+ \hat{A}(M) \} [M] \\
= \frac{1}{(n/2+1)!} \frac{n}{2} f^* c(E^+_0) [M] \\
= \frac{1}{(n/2+1)!} \int_M c_n(E^+ \hat{A}) [M] 
\]

\( c_n(E^+_0) = c \omega_n \) where \( 0 \neq c \) is a constant and \( \omega_n \) is a generator of \( H^n(S^n, \mathbb{Z}) \).

Therefore,

\[
0 = \text{Index}(D_{E'}) = \frac{c}{(n/2+1)!} \int_M f^* \omega_n \hat{A} [M] \\
= \frac{c}{(n/2+1)!} \hat{A}\text{-degree}(f)
\]

what contradicts the hypothesis that \( \hat{A}\text{-degree}(f) \neq 0 \).

Notice that the condition on the dimension of the manifold \( M \) to be \( n + 4k \) is because we need the hypothesis \( \hat{A}\text{-degree}(f) = [f^* \omega_n \hat{A}(M)] [M] \neq 0 \). And \( \hat{A}(M) \) has non-vanishing components only
in dim 4k.

If \[ \tilde{\kappa} = \frac{n(n-1)}{(n+4k)(n+4k-1)} \], then \( f \) is an isometric submersion.

In fact, if \( \kappa \cong n(n-1) \), then the inequality gives

\[ \langle D^2 \phi, \phi \rangle \geq \frac{1}{4} \|\phi\|^2 \left[ n(n-1) - \sum_{i \neq j} \frac{n}{\lambda_i \lambda_j} \right]. \]

Take a harmonic spinor \( 0 \neq \phi \in S \otimes E \), so

\[ 0 \geq \frac{1}{4} \|\phi\|^2 \left[ n(n-1) - \sum_{i \neq j} \frac{n}{\lambda_i \lambda_j} \right]. \]

\[ 0 \geq \frac{1}{4} \|\phi\|^2 \left[ \sum_{i \neq j} (1 - \frac{1}{\lambda_i \lambda_j}) \right] \geq 0. \]

So, \( \lambda_i \lambda_j = 1 \) and consequently \( \lambda_i = 1 \) for \( 1 \leq i \leq n \)

and \( f \) is an isometric submersion. In particular, \( f : M^{n+4k} \to S^n \)

is a fiber bundle map.
Chapter 4

Weaker Hypotheses

In this section we analyze how these results change when we modify the hypothesis of \( f \) being \( 1 \)-contracting. Recall that a map \( f : M \to N \) between Riemannian manifolds is \( (\varepsilon, \Lambda^k) \)-contracting if

\[
\|f^*\alpha\| \leq \varepsilon \|\alpha\| \quad \forall \alpha \in \Lambda^k(N).
\]

Note that "\( 1 \)-contracting" means \( (1, \Lambda^1) \)-contracting. We have the following immediate result.

**Theorem 4.1.** The statements of Theorem 2 and Theorem 3 continue to hold if the hypothesis that \( f \) be \( 1 \)-contracting is replaced by the hypothesis that \( f \) be \( (1, \Lambda^2) \)-contracting.

**Proof.** It follows through the proofs of Theorem 2.1 and 3.1. We only need to point out that \( \{\lambda_i\}_{i=1}^n \) satisfy

\[
1 = \|\psi_i^* \psi_j \|_{g_0} = \|f_i^* e_i \wedge f_j^* e_j \|_{g_0} = \lambda_i \lambda_j \|f_i^* e_i \wedge f_j^* e_j \|_{g_0}
\]

\[
\therefore 1 = \lambda_i \lambda_j \|f^*(e_i \wedge e_j) \|_{g_0} \leq \lambda_i \lambda_j \|e_i \wedge e_j \|_{g} = \lambda_i \lambda_j
\]

\[
\therefore \lambda_i \lambda_j \geq 1.
\]

But this is what we need in those proofs rather than asking that each \( \lambda_i \geq 1 \) as we had originally.

Now we ask the question: Are these same results true when the map \( f \) is \( (1, \Lambda^k) \)-contracting for \( 3 \leq k \leq n \)? The answer is no.
The hypothesis on $f$ cannot be further weakened in that sense.

The following construction provides counterexamples:

**Counterexample 4.2.** Consider the connected sum of two $S^n$ unit spheres with $n \geq 3$

Using the Gromov-Lawson construction, see [CL] we can find a metric on this space with scalar curvature $\geq c^2$ uniformly. Renormalizing if necessary, we can assume that $c^2 \geq 1$.

Consider now the connected sum of $m$ $S^n$'s with metrics $g_m$, with $\kappa_{g_m} \geq 1$. Notice that $\text{vol}_{g_m} \to \infty$ when $m \to \infty$, i.e. the volume can be made as large as we want by taking enough terms in the connected sum.

Set $\tilde{g}_m = [\frac{A}{\text{vol}_{g_m}}]^{2/n} g_m$, then the changes in the scalar curvature and volume are as follows:

$$\tilde{\kappa}_{\tilde{g}_m} = \left[ \frac{A}{\text{vol}_{g_m}} \right]^{2/n} \kappa_{g_m} \geq \left[ \frac{A}{2/n} \right]^{2/n}$$

$$\text{vol}_{\tilde{g}_m} = \left[ \frac{A}{\text{vol}_{g_m}} \right]^{1/n} m \text{ vol}_{g_m} \equiv A \text{ independent of } m.$$
Notice that the scalar curvature can be as large as we want and that the volume remain unchanged.

Now send the space

\[ M = \]

into the standard \( S^n \) in the obvious way, choosing \( A = \text{volume of } S^n \).

Since the volumes are the same (4.4), according to Moser's Theorem, there exists a map

\[ f : (M, g) \to (S^n, g_0) \]

volume preserving and therefore, \((1, \Lambda^n)\)-contracting. But \( \frac{\Lambda}{g} \gg 1 \)

(4.2).

For counterexamples of maps \((1, \Lambda^k)\)-contracting with \(3 \leq k < n\), we take the construction above for \( n = 3 \)

\[ f : M \to S^3 \quad (1, \Lambda^3)\)-contracting.

The map \( h : M \times S^p_r \to S^{p+3} \) got as

\[ M \times S^p_r \xrightarrow{\text{id} \times \frac{1}{r} \text{id}} M \times S^p \xrightarrow{f \text{id}} S^3 \times S^p \to S^3 \cdot S^p \simeq S^{p+3} \]

is \((1, \Lambda^3)\)-contracting for \( r \) large enough, see page 21.
Lemma 4.5. If \( f \) is \((1,\lambda^P)\)-contracting, then \( f \) is \((1,\lambda^{P+k})\)-contracting \( \forall k \geq 0 \).

Proof. We only have to notice that

\[
|\lambda_{i_1} \cdots \lambda_{i_p}| \leq 1 \quad \text{for any } p\text{-tuple with all }
\lambda_{i_j} \text{ different, then}
\]

\[
|\lambda_{i_1} \cdots \lambda_{i_{p+1}}| \leq 1.
\]

But

\[
|\lambda_{j_1} \cdots \lambda_{j_{p+1}}| \leq |\lambda_{j_s}| \quad \text{for any } j_s
\]

\[
\therefore \quad |\lambda_{j_1} \cdots \lambda_{j_{p+1}}| \leq \min |\lambda_i| \leq 1.
\]

By induction, it is true for all \( k \geq 1 \).

Therefore, the example above gives also counterexamples for \((1,\lambda^k)\)-contracting maps with \( 3 \leq k \leq n \).

Remark 4.6. The Gromov-Lawson construction of a metric with \( \tilde{u} \geq C^2 \) can only be done in dimension \( \geq 3 \).

In the manifolds of dimension \( \geq 3 \), the scalar curvature has no control on the volume of the manifold. In dimension 2, however, the scalar curvature which twice the sectional curvature, determines the volume completely. In fact, Gauss-Bonnet says that

\[
\int_M \chi = \frac{1}{2\pi} \int_M \kappa_0
\]
where $\chi$ is the Euler characteristic of the surface $M$, $\kappa$ is the sectional curvature and $\omega$ is the volume element of $M$.

**Counterexample 4.7.** Finally, we will describe an explicit counterexample for the case of dimension 3 and a $(1, \lambda)$-contracting map. This counterexample was pointed out to me by M. Katz.

In $\mathbb{R}^4$ we consider the manifold that we get by rotating a curve $\alpha_m$ about the $x_1$ axis.

We can assume that $\alpha_m$ is parametrized by arc-length $s$.

Consider the map $f_m$ into $S^3$ that sends the curve $\alpha_m$ into the half circle that when rotated about $x_1$ gives the sphere $S^3$ and each $S^2$. 
orthogonal to $x_1$ in $M^3$ into and $S^2$ orthogonal to $x_1$ in $S^3$.

We provide $M^3_m$ with metric $q_m$ such that $\tilde{\nu}q_m \geq 1$. For a point $x \in M^3$ we have that

$$df_m : T_xM^3 \mapsto T_{f_m(x)}S^3$$

$$e_1 \mapsto \frac{1}{f_m} e_1$$

$$e_j \mapsto c(s)e_j \quad j = 1, 2$$

where $f_m$ is the function length of $\alpha_m$ and $c(s)$ some bounded functions of $s$, say $a \leq c(s) \leq b$. These constants $a$ and $b$ are independent of $m$.

Thus, $f_m$ is $(\frac{b^2}{f_m}, \Lambda^3)$-contracting.

By choosing $m$ large enough, $\frac{b^2}{f_m}$ can be made $\leq 1$ (and the volume of $M$ as big as we want). Consequently, $f_m$ is $(1, \Lambda^3)$-contracting and $\tilde{\nu}q_m > 1$.

**Spin Hypothesis**

Currently I am investigating the extent to which the spin hypothesis is necessary to my results on scalar curvature. I am also trying to refine these results by considering the more delicate mod-2 invariants related to the Dirac operator via the $Cl_k$-Index Theorem.
Appendix 1

Proof of Claim 2.6

Recall that

$$E_0 = \mathbb{P}_{\text{Spin}_{2n}}(S^{2n}) \times \mathfrak{c}^2_{2n}.$$  

$\mathfrak{c}^2_{2n}$ acts on $E_0$ on the right since right Clifford multiplication commutes with $\lambda$. $E_0$ has a decomposition into irreducible modules under left multiplicaition that comes from the decomposition of $\mathfrak{c}^2_{2n}$ into irreducible modules.

$$E_0 = \mathfrak{g} \oplus \ldots \oplus \mathfrak{g}$$

$E_0$ is the direct sum of $2^n$ copies of $\mathfrak{g}$, where

$$\mathfrak{g} \cong \mathbb{P}_{\text{Spin}_{2n}}(S^{2n}) \times \Delta^2_{2n},$$

and $\Delta$ is the fundamental representation of $\text{Spin}_{2n}$ into $U_{2^n}$.

The splitting $E_0 = E_0^+ \oplus E_0^-$ gives rise to the splitting

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-,$$

where $\dim(\mathfrak{g}^+) = 2^{n-1}$. Therefore,

$$E_0^+ = \mathfrak{g}^+ \oplus \ldots \oplus \mathfrak{g}^+ \quad (2^n \text{ terms}).$$

Each $\mathfrak{g}^+ = \mathbb{P}_{\text{Spin}_{2n}}(S^{2n}) \times \Delta^2_{2n-1}$ with

$$\Delta^+ : \text{Spin}_{2n} \rightarrow U_{2^{n-1}}.$$
For a detailed description, see [Hu], [Hi] and [Pa].

Now the Chern character of $E_0^+$ is given by

$$\text{ch}(E_0^+) = \text{ch}(g^+ \otimes \cdots \otimes g^+) = 2^n \text{ch}(g^+).$$

Over $S^{2n}$, $\text{ch}(S^+) = \text{dim}(g^+) + \frac{1}{(n-1)!} c_n(S^+)$.

To prove the claim, it is enough to show that $c_n(g^+) \neq 0$.

Let $T^n$ be a maximal torus of $\text{Spin}_{2n}$

$$T^n \longrightarrow \text{Spin}_{2n}.$$

On the classifying spaces we have

$$\rho : B T^n \longrightarrow B \text{Spin}_{2n}.$$

Recall that $B T^n = \text{CP}(\infty) \times \cdots \times \text{CP}(\infty)$, therefore,

$$H^*(B T^n, \mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_n]$$

$$H^{**}(B T^n, \mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_n]$$ (the ring of formal power series)

where $x_1 \in H^2(B T^n, \mathbb{Q})$

Let $W = W(\text{Spin}_{2n}, T^n)$ be the Weyl group of automorphisms of $T^n$ which can be extended to automorphisms of $\text{Spin}_{2n}$. The map

$$\rho^* : H^*(B \text{Spin}_{2n}, \mathbb{Q}) \rightarrow H^*(B T^n, \mathbb{Q})$$

is injective. The image consists of those elements in $H^*(B T^n, \mathbb{Q})$ which are invariant under the action of $W$. 
We have

\[ T^1(e_1, \ldots, e_{2n}) = T \xrightarrow{\xi_0} \operatorname{Spin}_{2n} \]
\[ \xi_0 \omega \rightarrow T(e_1, \ldots, e_{2n}) = T \xrightarrow{\xi_0} \operatorname{SO}(2n) \]

where \( \xi_0 \) is the covering homomorphism and \( \omega \) is defined as follows.

Let \( \omega_j : S^1 = \mathbb{R}/\mathbb{Z} \to \operatorname{Spin}_{2n} \) be the homomorphism given by

\[ \omega_j(\theta) = \cos 2\pi \theta + e_2 j - 1 \cdot e_j \sin 2\pi \theta \quad \text{for } 1 \leq j \leq n. \]

Note that \( \xi_0 \omega_j(\theta) = \text{diag}(0, \ldots, 0, 2\theta, 0, \ldots, 0) \) for all \( \theta \in \mathbb{R}/\mathbb{Z} \).

Let \( \omega : T^n \to \operatorname{Spin}_{2n} \) be defined by

\[ \omega(\theta_1, \ldots, \theta_n) = \omega_1(\theta_1) \cdots \omega_n(\theta_n) \]

for \( (\theta_1, \ldots, \theta_n) \in T^n \).

Then the Weyl group consists of the \( 2^n n! \) permutations of the indexes of \( (\theta_1, \ldots, \theta_n) \) compose with \( (\theta_1, \ldots, \theta_n) \rightarrow (\epsilon_1 \theta_1, \ldots, \epsilon_n \theta_n) \) with \( \epsilon_1 = \pm 1 \) and \( \epsilon_1, \ldots, \epsilon_n = 1 \). See [Hu].

\( \text{ch}(\mathcal{Z}) \) lies in \( H^*(B \operatorname{Spin}_{2n}, \mathbb{Q}) \to H^*(BT^n, \mathbb{Q}) \).
Since we want $\text{ch}(\mathcal{G}^+)$ we must consider $(\Theta_1, \ldots, \Theta_n) \to (\pm \Theta_1, \ldots, \pm \Theta_n)$ with $\varepsilon_1, \ldots, \varepsilon_n = 1$ and with even number of pluses.

$$
\text{ch}(\mathcal{G}^+) = \sum_{\text{even } \# \text{ of } +'s} \frac{1}{2} e^{\sum_{i=1}^{n} \pm \tilde{x}_i} \in H^*(S^{2n})
$$

where $\tilde{x}_i = \sigma x_i$ for $\sigma : T^n \to T$.

But from $\text{ch}(\mathcal{G}^+)$ we only need to calculate the terms that belongs to $H^{2n}(S^{2n})$. That term will come from

$$
\sum_{\text{even } \# \text{ of } +'s} \frac{1}{n!} \frac{1}{2^n} (\pm \tilde{x}_1, \pm \tilde{x}_2, \ldots, \pm \tilde{x}_n)^n.
$$

Since to Pontrjagin classes over the sphere are all zero, the only term that remains is $\frac{1}{2^n} \tilde{x}_1, \ldots, \tilde{x}_n$, which is the Euler class of $S^{2n}$ and therefore $\neq 0$. 
References


