THE WILLMORE PROBLEM

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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The Willmore problem is discussed and the literature surveyed. The first and second variations of the Willmore integral in $\mathbb{R}^3$ and S^3 are calculated. Lawson's minimal surfaces in S^3 are shown to be branched coverings of a square torus or Riemann sphere. Therefore, their metric satisfies a hyperbolic-sine-Gordon equation. First and second generation solutions to this equation are produced via a Bäcklund transformation and Permutability Theorem.

In memory of

my grandfather

Candelario Hernandez

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Introduction

The total mean curvature of a surface Σ in $\mathbb{R}^3$ is given by the integral

$$\int_\Sigma H^2 d\Sigma$$

where H is the mean curvature of the surface. This integral represents the amount of work needed to deform a flat piece of elastic material into the shape of the surface Σ. It is invariant under conformal transformations of $\mathbb{R}^3$.

The Willmore problem is to find, for each genus, the surface in $\mathbb{R}^3$ of minimum total mean curvature among all surfaces in $\mathbb{R}^3$ of that genus. The absolute minimum, among all surfaces in $\mathbb{R}^3$, is achieved by round spheres. In the case of genus 1, T. J. Willmore conjectured that $\int_\Sigma H^2 d\Sigma \geqq 2\pi^2$ with equality for surfaces conformally equivalent to a certain torus of revolution. The generalized Willmore conjecture states that the solution to the Willmore problem for each genus g is a surface in $\mathbb{R}^3$ conformally equivalent, via stereographic projection, to Lawson's minimal surface $\xi_{g,1}$ in S^3.

In this Thesis, after a historical survey in Chapter 1, we record some work we have done on various approaches to the problem.

In Chapter 2 we apply calculus of variations to the Willmore problem. We calculate the first and second variation of total mean curvature with the aim of showing the second variation is non-negative on the standard torus. The Euler-Lagrange equation is the standard equation for non-linear elasticity, and was already known in the 1920's.

The second variation formula we obtain proves to be unwieldy and we can only apply it to a special case on the standard torus.

In Chapter 3 we transfer the problem to S^3 via stereographic projection. There the appropriate integral to study is $\int_\Sigma (H^2+1)\,d\Sigma$ for a surface Σ in S^3. Again we calculate the first and second variations. These were already calculated in a different manner by J. Weiner. Minimal surfaces in S^3 are trivial solutions to the Euler-Lagrange equations. This and the fact that $\xi_{g,1}$ is a natural generalization of the Clifford torus $\xi_{1,1}$, which corresponds to a standard torus in $\mathbb{R}^3$ via stereographic projection, leads to the generalized Willmore conjecture. The second variation is shown to be non-negative on the Clifford torus.

Since the Willmore integral $\int_\Sigma (H^2+1)\,d\Sigma$ for a minimal surface in S^3 is just the area, we shift our attention in Chapter 4 to the problem of finding the metric on Lawson's minimal surfaces ξ_{mn}. By taking quotients of these surfaces by certain groups of rotations, we exhibit them as branched coverings of either a square torus or the Riemann sphere. We then show that the Gauss curvature equation leads to a hyperbolic-sine-Gordon equation for the metric, via the study of a certain holomorphic quadratic differential on minimal surfaces in S^3.

In Chapter 5 we record some of the work we have done in applying Backlund transformations to the sinh-Gordon equation to generate families of solutions. We show that the sinh-Gordon equation is invariant under a certain Bäcklund transformation, and use this to prove a Permutability Theorem. This theorem allows us to generate families of solutions by a purely algebraic process. We write down two first-generation families and two second-generation families, which we are currently in the process of studying.

In Appendices A,B,C we give some background material for the main body. In Appendix D we show directly that $\int_\Sigma H^2 d\Sigma \geq 2\pi^2$ for certain tori in $\mathbb{R}^4$.

Chapter 1

Historical Survey

Let M be a closed two-dimensional smooth manifold and $f : M \to \mathbb{R}^3$ a smooth immersion. The Willmore integral or total mean curvature of f is the integral

$$\mathcal{W}(f) = \int_M H^2 ds$$

where H is the mean curvature of the immersion. The general Willmore problem may be divided into three parts [11]. For each M:

(1) Determine $\mathcal{W}(M) = \inf\{\mathcal{W}(f)\}$ where f ranges over all smooth immersions of M into $\mathbb{R}^3$.

(2) Classify all f for which $\mathcal{W}(f)$ equals the minimum value $\mathcal{W}(M)$.

(3) Determine all critical points f of $\mathcal{W}$ and the corresponding value $\mathcal{W}(f)$.

The integral $\int H^2 ds$ was proposed by Sophie Germain in 1810 as the "virtual work" in her study of vibrating curved plates [11]. Early in this century the total mean curvature and its properties (conformal invariance, Euler equation, critical points) were studied by Schadow, Thomsen, and Blascke.

In recent years, the problem of minimum total mean curvature was proposed by Willmore [18], and he gave the answer to the question of the minimum among all surfaces in $\mathbb{R}^3$: $\mathcal{W}(f) \geq 4\pi$, and $\mathcal{W}(f) = 4\pi$ if and only if f is an embedding of S^2 as a standard round sphere. Willmore posed the problem of finding, for each g, $\mathcal{W}_g = \inf \int_\Sigma H^2 ds$

among all surfaces $\sum$ of genus g in $\mathbb{R}^3$. He considered the problem for tori in $\mathbb{R}^3$ and showed that if M is a torus embedded in $\mathbb{R}^3$ as a "tube" of constant circular cross-section, then $\int_M H^2 ds \geqq 2\pi^2$, with equality if and only if the torus is embedded as a surface of revolution, the ratio of the radii being $1 : \sqrt{2}$. (Shiohama and Takagi [13] obtained an equivalent result). Based on this, Willmore made the still unconfirmed conjecture that now carries his name.

<u>Willmore Conjecture</u>. $\int H^2 ds \geqq 2\pi^2$ for all tori immersed in $\mathbb{R}^3$, with equality only for the circular torus $1 : \sqrt{2}$, up to conformal transformations of $\mathbb{R}^3$.

The fundamental property of the Willmore integral is its conformal invariance: If $g : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$ is a conformal transformation and $f : M \to \mathbb{R}^3$ an immersion of a surface M, then $\mathcal{W}(g \circ f) = \mathcal{W}(f)$. This was established by Thomsen [15] in the form $(\bar{H}^2 - \bar{K})\overline{ds} = (H^2 - K)dS$ where a bar denotes the quantities after the conformal transformation, and was "rediscovered" by White [17].

More generally, for immersions $f : M \to N$ of a 2-manifold M into a smooth Riemannian manifold, the Willmore integral may be defined as $\mathcal{W}(f) = \int_m (H^2 + k) dS$ where k is the sectional curvature of N on planes tangent to $f(M)$. This integral is invariant under conformal changes in the metric on N, [16]. In particular, for $f : M \to S^3$, $\mathcal{W}(f) = \int_M (H^2+1) dS$ and if the immersion is minimal, $\mathcal{W}(f) = \text{area}(f)$.

The Euler-Language equation for the Willmore integral is

$$\Delta H + 2H(H^2 - K) = 0,$$

where K is the Gauss curvature. Immersions $f : M \to \mathbb{R}^3$ which satisfy this equation are called Willmore surfaces or static surfaces: surfaces which tend to retain their shape. Willmore [20] lectured on this equation in the 60's, thinking it was new. Voss informed him that he knew of it in the 50's, but did not publish it. Later they both found out this equation appears in Blaschke's text [1] and is attributed to Thomsen. Blaschke and Thomsen proved that stereographic projections of compact minimal surfaces in S^3 are always Willmore surfaces. Thomsen attributes the Euler equation to Schadow, but some form of it may have been known to Germain or Poisson [6]. Poisson was aware of the equivalence between the functionals $W(f)$ and $\overline{W}(f) = \int_m (k_1^2+k_2^2)\,dS$ (k_1, k_2 are the principal curvatures) for the variational problem, this before the Gauss-Bonnet Theorem [11].

J. L. Weiner [16] showed the following: Let $f : M \to S^3$ be an immersion of a closed orientable surface, with G = determinant of the second fundamental form relative to S^3. Let $\sigma : S^3 \to \mathbb{R}^3$ be stereographic projection. Then f satisfies (1) $\Delta H + 2H(H^2-G) = 0$ if and only if $\sigma \circ f$ satisfies (2) $\Delta H + 2H(H^2-K) = 0$. Therefore, because any minimal immersion satisfies (1), and because of Lawson's construction [8] of minimal surfaces of arbitrary genus in S^3:

<u>Theorem</u>. There exist Willmore surfaces of arbitrary genus in $\mathbb{R}^3$.

Lawson's minimal surface $\xi_{1,1}$ is the Clifford torus, which under stereographic projection maps to the circular torus $1 : \sqrt{2}$ of Willmore's conjecture. The conjectured value $W_1 = 2\pi^2$ is the area of the Clifford torus, which is isometric to a square of side $\pi\sqrt{2}$. Kusner [6] has generalized Willmore's conjecture.

Conjecture. For each genus g, the minimum value W_g is the area a(g) of Lawson's minimal surface $E_{g,1}$ in S^3. The minimizing surface in $\mathbb{R}^3$ is a stereographic image of $E_{g,1}$.

Kusner [5] showed that area$(E_{mn}) < 4\pi(n+1)$ and that area $(E_{mn}) \to 4\pi(n+1)$ as $m \to \infty$, where E_{mn} is Lawson's genus mn minimal surface in S^3. Therefore, $a(g) < 8\pi$ and $a(g) \to 8\pi$ as $g \to \infty$. Using their concept of conformal volume (more below), Li and Yau [9] showed: If $\psi : M \to \mathbb{R}^m$ is an immersion of a compact surface, and if there is a point $p \in \mathbb{R}^n$ such that $\psi^{-1}(p) = \{x_1,\ldots,x_k\}$ where the x_i's are distinct points in M, then $\int_M H^2 \geq 4k\pi$. In particular, if an immersion $\psi : M \to \mathbb{R}^n$ has the property that $\int_M H^2 < 8\pi$, then ψ must be an embedding. Therefore, $W_g \leq a(g) < 8\pi$, and any Willmore-minimizing surface in $\mathbb{R}^3$ must be embedded: if M is an image under stereographic projection of $E_{g,1}$, then by the conformal invariance of W and the minimality of $E_{g,1}$:

$$W_g \leq W(M) = W(E_{g,1}) = \text{area}(E_{g,1}) = a(g) < 8\pi.$$

Using methods of geometric measure theory, Leon Simon proved the existence of Willmore surfaces which achieve the minimum W_g. Collecting the above results:

Theorem. For each $g \geq 0$ there exists an embedding $f_g : M_g \to \mathbb{R}^3$ of a surface of genus g with $W(f_g) = W_g$. In addition, $W_0 = 4\pi$, $W_g > 4\pi$ if $g > 0$, and $W_g \leq a(g) < 8\pi$.

In [5], Kusner estimates the infimum of W for each regular homotopy class of immersed surfaces in $\mathbb{R}^3$. His main theorem is: The infimum $W_{[M]}$ for W over any regular homotopy class $[M]$ of compact immersed surfaces M in $\mathbb{R}^3$ satisfies $W_{[M]} < 20\pi$. (He gives specific estimates for each homotopy class.) In particular, the infimum of W among compact immersed surfaces of a given topological type M is strictly less than 8π if M is orientable, 12π if M is non-orientable with even Euler number, 16π if M is non-orientable with odd Euler number. W. Kuhnel and U. Pinkall [4] obtain $\leq$ inequalities above.

The Willmore problem has been solved in the case of $\mathbb{R}\mathbb{P}^2$ and S^2. For S^2, as noted above, Willmore showed $W(S^2) = 4\pi$ and the minimizing surfaces are round spheres. Robert Bryant [2] completely classified all Willmore immersions $f : S^2 \to S^3$. He showed that all the critical values of W on spherical immersions are non-negative multiples of 4π. For $\mathbb{R}\mathbb{P}^2$, the theorem of Li and Yau quoted above shows that $W(\mathbb{R}\mathbb{P}^2) \geq 12\pi$, since any immersed projective plane in $\mathbb{R}^3$ must have a triple point. Bryant and Kusner have independently found explicit immersions $f : \mathbb{R}\mathbb{P}^2 \to \mathbb{R}^3$ with $W(f) = 12\pi$. Therefore, $W(\mathbb{R}\mathbb{P}^2) = 12\pi$. Bryant classified all minimizing $\mathbb{R}\mathbb{P}^2$'s in $\mathbb{R}^3$.

In 1982, Li and Yau [9] introduced a new conformal invariant called conformal volume. They defined it as follows. Let M be an m-dimensional compact manifold which admits a conformal map ϕ into the n-dimensional unit sphere S^n. Let G be the group of conformal diffeomorphisms of S^n, and dV_g the volume form induced on M by $g \circ \phi$, where $g \in G$. Then the n-conformal volume of M is defined to be

$$V_c(n,M) = \inf_{\phi} \sup_{g \in G} \int_M dV_g$$

where ϕ runs over all non-degenerate conformal mappings of M into S^n. Since $V_c(n,M) \geq V_c(n+1,M)$, the conformal volume of M is defined to be

$$V_c(M) = \lim_{n \to \infty} V_c(n,M).$$

Li and Yau showed that if M is a compact surface without boundary in $\mathbb{R}^n$, then $\int_M H^2 \geq V_c(n,M)$. Furthermore, equality implies M is the image of some minimal surface in S^n under some stereographic projection. They conjecture that if M can be conformally embedded as a minimal surface in S^3, then $\mathcal{W}(M)$ is not less than the area of this minimal surface.

Some progress has been made on the original Willmore problem on the torus. Applying their work on conformal volume to the torus, Li and Yau proved the following: Suppose M is a surface of genus 1 in $\mathbb{R}^n$ that is conformally equivalent to a flat torus $\mathbb{R}^2/\Gamma$ with lattice Γ generated by $\{(1,0)\,(x,y)\}$ where $0 \leqq x \leqq \frac{1}{2}$ and $\sqrt{1-x^2} \leqq y \leqq 1$. Then $\int_M H^2 \geq 2\pi^2$. Equality implies M must be conformally equivalent to the square torus and is the image of a stereographic projection of a minimal torus in S^3.

Let γ be a regular closed curve with geodesic curvature k in the hyperbolic plane P, where P is represented by the upper half-plane above the x-axis. Bryant and Ulrich Pinkall independently observed that if f is the immersion of the torus obtained by revolving γ about the x-axis, then $\mathcal{W}(f) = \frac{1}{2}\int_\gamma k^2 ds$. Langer and Singer [7] showed that $\int_\gamma k^2 ds \geq 4\sqrt{-G}$ with equality precisely for the circle of radius $\frac{\sinh^{-1}(1)}{\sqrt{-G}}$ (where the hyperbolic plane is of curvature $G < 0$). Combining the two results, they concluded that $\mathcal{W}(f) \geq 2\pi^2$ for all tori of revolution, with equality for the circular torus $1:\sqrt{2}$. Willmore [19] states that a Professor Hombu had proved this.

Pinkall [10] found the first example of compact embedded Willmore surfaces which are not stereographic projections of minimal surfaces in S^3, as follows. Let γ be a closed curve in S^2 with curvature function k. Let $\pi : S^3 \to S^2$ be the Hopf fibration. The inverse image of γ under π is an immersed torus f_γ in S^3, called a Hopf torus, and k is also the mean curvature of f_γ. Therefore, $\mathcal{W}(f_\gamma) = \int_{T^2} (1+k^2)$. The immersion f_γ is a Willmore surface if and only if γ is an extremal curve of $\oint_\gamma (1+k^2)$. Langer and Singer have shown that there are infinitely many simple closed curves on S^2 that are critical points for $\oint(1+k^2)$. Therefore, there are infinitely many embedded Hopf tori that are critical points for the Willmore integral. The stereographic projections of these tori are embedded Willmore tori in $\mathbb{R}^3$. All of these are unstable critical points for $\mathcal{W}$.

H. Karcher, Pinkall, and I. Sterling [3] have constructed new examples of compact embedded minimal surfaces in S^3, of varying genus, and therefore new Willmore surfaces. Computer estimates show these do not minimize the Willmore integral.

Chapter 2

Variation Formulas for Surfaces in $\mathbb{R}^3$

In this chapter we establish the first and second variation formulas for total mean curvature for surfaces in $\mathbb{R}^3$.

Let Σ be a compact orientable 2-dimensional C^∞ manifold, f a C^∞ immersion of Σ into Euclidean space, and $F : \Sigma \times (-1,1) \to \mathbb{R}^3$ a C^∞ variation of f. For each $t \in (-1,1)$, the map $f_t : \Sigma \to \mathbb{R}^3$ defined by $f_t(p) = F(p,t)$ for each $p \in \Sigma$ is a C^∞ immersion of Σ, and $f_o = f$.

Given local coordinates (x^1,x^2) defined on some open set of Σ, we may consider F to be a smooth map defined on an open set $U \times (-1,1) \subset \mathbb{R}^2 \times (-1,1)$:

$$F : (x^1,x^2,t) \in U \times (-1,1) \mapsto F(x^1,x^2,t) \in \mathbb{R}^3.$$

We may also consider all vector fields along F and all functions on $\Sigma \times (-1,1)$ as smooth maps defined on $U \times (-1,1)$, for the purpose of local computations. We suppress the dependence on (x^1,x^2,t) in our notation, and we can freely interchange ordinary derivatives with respect to x^1,x^2, and t.

There are four basic C^∞ vector fields in $\mathbb{R}^3$ defined along F.

a) The coordinate vector fields $F_i = \dfrac{\partial F}{\partial x^i}$, for $i = 1,2$.

These are tangent to the surfaces $\sum_t = f_t(\sum)$, and are only locally defined.

b) The variation vector field V of F : $V_t = \frac{\partial F}{\partial t}$, defined on all of $\sum \times (-1,1)$.

c) A unit normal vector field n, which is normal to the surfaces $\sum_t$. It is defined on all of $\sum \times (-1,1)$.

Let $\langle , \rangle$ denote the standard Euclidean metric on $\mathbb{R}^3$. Since V and n are globally defined, there exists a C^∞ function $\phi : \sum \times (-1,1) \to \mathbb{R}$ defined by $\phi = \langle n,V \rangle$. At each point $p \in F(U\times(-1,1))$ the set $\{n,F_1,F_2\}$ is a basis for $T_p\mathbb{R}^3$, and this basis depends smoothly on (x^1,x^2,t). Also, $\langle n,F_1 \rangle = \langle n,F_2 \rangle = 0$. Therefore, there exist C^∞ functions $\psi^i : U \times (-1,1) \to \mathbb{R}$ for $i = 1,2$ such that $V = \phi n + \psi^i F_i$ on $U \times (-1,1)$.

Finally we have the usual locally defined coefficients and symbols g_{ij}, g^{ij}, h_{ij}, h^j_i, Γ^k_{ij}, and the globally defined mean curvature H and Gauss curvature K. For local coordinate computations we consider all of these as smooth functions of (x^1,x^2,t).

The first step is to calculate the first variation of the volume form $d\sum$.

<u>Lemma 2.1</u>

$$\frac{\partial g_{ij}}{\partial t} = -2\phi h_{ij} + (\psi^k_i g_{kj}+\psi^k_j g_{ki}) + \psi^k(\Gamma^\ell_{ik} g_{\ell j}+\Gamma^\ell_{jk} g_{\ell i}).$$

Proof. Using $g_{ij} = \langle F_i, F_j\rangle$, $V = F_t = \phi n + \psi^k F_k$, $\langle n, F_i\rangle = 0$, and Weingarten's formulas we calculate:

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}\langle F_i, F_j\rangle = \langle F_{it}, F_j\rangle + \langle F_i, F_{jt}\rangle = \langle V_i, F_j\rangle + \langle F_i, V_j\rangle$$

$$= \langle \phi_i n + \phi n_i + \psi_i^k F_k + \psi^k F_{ki}, F_j\rangle + \langle F_i, \phi_j n + \phi n_j + \psi_j^k F_k + \psi^k F_{kj}\rangle$$

$$= \phi(\langle n_i, F_j\rangle + \langle n_j, F_i\rangle) + (\psi_i^k\langle F_k, F_j\rangle + \psi_j^k\langle F_i, F_k\rangle) + \psi^k(\langle F_{ki}, F_j\rangle + \langle F_{kj}, F_i\rangle)$$

$$= -2\phi h_{ij} + (\psi_i^k g_{kj} + \psi_j^k g_{ki}) + \psi^k(\Gamma_{ik}^{\ell} g_{\ell s} + \Gamma_{jk}^{\ell} g_{\ell i})$$

QED

Let G be the matrix whose ij^{th} entry is g_{ij}, and let $g = \det G$. In the next lemma we use the standard formula $\frac{\partial g}{\partial t} = g\,\mathrm{tr}(G^{-1}\frac{\partial G}{\partial t})$ and the notation V^T for the component of V which is tangent to the surfaces $\sum_t$. The C^∞ vector field V^T is globally defined by $V^T = V - \langle V, n\rangle n$ and is given locally by $V^T = \psi^k F_k$.

Lemma 2.2

$$\frac{\partial\sqrt{g}}{\partial t} = (-2\phi H + \mathrm{div}(V^T))\sqrt{g}$$

Proof

$$\frac{\partial}{\partial t}\sqrt{g} = \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial t} = \frac{1}{2}\sqrt{g}\ \mathrm{tr}(G^{-1}\frac{\partial G}{\partial t}) = \frac{1}{2}\sqrt{g}\ g^{ij}\frac{\partial g_{ij}}{\partial t}$$

$$= \frac{1}{2}\sqrt{g}[-2\phi g^{ij}h_{ij} + (\psi_i^k g^{ij} g_{kj} + \psi_j^k g^{ij} g_{ki}) + \psi^k(\Gamma_{ik}^{\ell} g^{ij} g_{\ell j} + \Gamma_{jk}^{\ell} g^{ij} g_{\ell i})]$$

$$= \frac{1}{2}\sqrt{g}[-2\phi h_i^i + 2\psi_i^i + 2\psi^k\Gamma_{ik}^i] = [-\phi h_i^i + (\psi_i^i + \psi^j\Gamma_{ij}^j)]\sqrt{g} = [-2\phi H + \mathrm{div}(V^T)]\sqrt{g}$$

QED

The local coordinate expression for the volume form $d\Sigma$ is $\sqrt{g}\, dx^1 dx^2$. From Lemma 2.2 we see that the "first variation" of $d\Sigma$ is expressed globally by

$$\frac{\partial}{\partial t}(d\Sigma) = [-2\phi H + \mathrm{div}(V^T)]\,d\Sigma.$$

In passing we note that the area of the immersion f_t of Σ is $A(t) = \int_\Sigma d\Sigma$ and thus, by Lemma 2.2 and the Divergence Theorem,

$$A'(t) = \int_\Sigma -2\phi H d\Sigma \qquad \text{for } t \in (-1,1).$$

The next step is to calculate the first variation of the coefficients of the second fundamental form.

<u>Lemma 2.3</u>

$$\frac{\partial n}{\partial t} = -(g^{ij}\phi_j + \psi^j h_j^i)F_i.$$

<u>Proof</u>. Define C^∞ functions $\mu : U \times (-1,1) \to \mathbb{R}$ and $\nu^i : U \times (-1,1) \to \mathbb{R}$ for $i = 1,2$ by $\frac{\partial n}{\partial t} = \mu n + \nu^i F_i$. Since $\langle n,n\rangle = 1$ and $\langle n,F_i\rangle = 0$: $\mu = \langle\frac{\partial n}{\partial t},n\rangle = \frac{1}{2}\frac{\partial}{\partial t}\langle n,n\rangle = 0$.

Also,

$$\begin{aligned}
\nu^i &= \nu^j \delta_j^i \\
&= \nu^j g_{jk} g^{ki} \\
&= \langle \nu^j F_j, F_k\rangle g^{ki} \\
&= \langle \frac{\partial n}{\partial t}, F_k\rangle g^{ki} \\
&= [\frac{\partial}{\partial t}\langle n,F_k\rangle - \langle n,F_{kt}\rangle] g^{ki} \\
&= -\langle n,V_k\rangle g^{ki}
\end{aligned}$$

$$= -\langle n, \phi_k n + \phi n_k + \psi^j_k F_j + \psi^j F_{jk} \rangle g^{ki}$$

$$= -[\phi_k \langle n,n \rangle + \psi^j \langle n, F_{jk} \rangle] g^{ki}$$

$$= -(\phi_k + \psi^j h_{jk}) g^{ki}$$

$$= -(g^{ki}\phi_k + \psi^j h^i_j)$$

$$\therefore \qquad \nu^i = -(g^{ji}\phi_j + \psi^j h^i_j)$$

QED

In this proof we derived the formula $\langle n, V_i \rangle = \phi_i + \psi^j h_{ji}$ which we use in the next lemma.

<u>Lemma 2.4</u>

$$\frac{\partial h_{ij}}{\partial t} = (\phi_{ij} - \phi_k \Gamma^k_{ij}) - \phi h^k_i h_{kj} + (\psi^k_i h_{kj} + \psi^k_j h_{ki}) + \psi^k \frac{\partial h_{ij}}{\partial x^k} .$$

<u>Proof</u>. $\displaystyle \frac{\partial h_{ij}}{\partial t} = \frac{\partial}{\partial t}\langle F_{ij}, n \rangle$

$$= \langle F_{ijt}, n \rangle + \langle F_{ij}, \frac{\partial n}{\partial t} \rangle$$

$$= \langle V_{ij}, n \rangle + \langle F_{ij}, -(g^{k\ell}\phi_k + \psi^k h^\ell_k) F_\ell \rangle$$

$$= \langle V_{ij}, n \rangle - (g^{k\ell}\phi_k + \psi^k h^\ell_k)\Gamma^m_{ij} g_{m\ell}$$

$$= \langle V_{ij}, n \rangle - \phi_m \Gamma^m_{ij} - \psi^k \Gamma^m_{ij} h_{mk} .$$

$$\langle V_{ij}, n \rangle = \frac{\partial}{\partial x^j}\langle V_i, n \rangle - \langle V_i, n_j \rangle$$

$$= \frac{\partial}{\partial x^j}(\phi_i + \psi^k h_{ki}) - \langle \phi_i n + \phi n_i + \psi^k_i F_k + \psi^k F_{ki}, -h^\ell_j F_\ell \rangle$$

$$= \phi_{ij} + \psi^k_j h_{ki} + \psi^k \frac{\partial h_{ki}}{\partial x^j} - \phi h^\ell_j h_{\ell i} + \psi^k_i h^\ell_j g_{k\ell} + \psi^k h^\ell_j \Gamma^m_{ki} g_{m\ell}$$

$$= \phi_{ij} - \phi h_{jk} g^{k\ell} h_{\ell i} + (\psi^k_j h_{ki} + \psi^k_i h_{kj}) + \psi^k \left(\frac{\partial h_{ki}}{\partial x^j} + \Gamma^m_{ki} h_{mj}\right)$$

$$= \phi_{ij} - \phi h^k_i h_{kj} + (\psi^k_j h_{ki} + \psi^k_i h_{kj}) + \psi^k \left(\frac{\partial h_{ij}}{\partial x^k} + \Gamma^m_{ij} h_{mk}\right) .$$

The Codazzi-Mainardi equations were used in the last step. The lemma follows from the expressions for $\frac{\partial h_{ij}}{\partial t}$ and $<V_{ij}, n>$.

QED

The next few lemmas are preparation for the calculation of the first variation of the mean curvature H.

Lemma 2.5

$$\frac{\partial g^{ij}}{\partial t} = 2\phi g^{ik}h_k^j - (g^{ik}\psi_k^j + g^{jk}\psi_k^i) - \psi^k(g^{i\ell}\Gamma_{\ell k}^j + g^{j\ell}\Gamma_{\ell k}^i).$$

Proof. The definition $g^{ij}g_{jk} = \delta_k^i$ implies

$\frac{\partial g^{ij}}{\partial t} = -g^{ik}\frac{\partial g_{k\ell}}{\partial t}g^{\ell j}$. Applying Lemma 2.1,

$$\frac{\partial g^{ij}}{\partial t} = 2\phi h_{k\ell}g^{ik}g^{\ell j} - (\psi_k^m g_{m\ell}g^{ik}g^{\ell j} + \psi_\ell^m g_{mk}g^{ik}g^{\ell j})$$

$$- \psi^m(\Gamma_{km}^n g_{n\ell}g^{ik}g^{\ell j} + \Gamma_{\ell m}^n g_{nk}g^{ik}g^{\ell j})$$

$$= 2\phi h_k^j g^{ik} - (\psi_k^m \delta_m^j g^{ik} + \psi_\ell^m \delta_m^i g^{\ell j}) - \psi^m(\Gamma_{km}^n \delta_n^j g^{ik} + \Gamma_{\ell m}^n \delta_n^i g^{\ell j})$$

$$= 2\phi g^{ik}h_k^j - (g^{ik}\psi_k^j + g^{\ell j}\psi_\ell^i) - \psi^m(\Gamma_{km}^j g^{ik} + \Gamma_{\ell m}^i g^{\ell j})$$

$$= 2\phi g^{ik}h_k^j - (g^{ik}\psi_k^j + g^{jk}\psi_k^i) - \psi^k(g^{i\ell}\Gamma_{\ell k}^j + g^{j\ell}\Gamma_{\ell k}^i)$$

QED

Lemma 2.6

$$2<\nabla H, V^T> = \psi^k(g^{ij}\frac{\partial h_{ij}}{\partial x^k} - 2h_i^j\Gamma_{jk}^i)$$

Proof.

$$2<\nabla H, V^T> = 2<g^{ij}H_iF_j, \psi^k F_k> = 2H_i\psi^k g^{ij}g_{jk} = 2H_k\psi^k.$$

Therefore,

$$2\langle \nabla H, V^T\rangle = \psi^k \frac{\partial}{\partial x^k}(g^{ij}h_{ij})$$

$$= \psi^k (g^{ij} \frac{\partial h_{ij}}{\partial x^k} + h_{ij} \frac{\partial g^{ij}}{\partial x^k})$$

$$= \psi^k [g^{ij} \frac{\partial h_{ij}}{\partial x^k} + h_{ij}(-\Gamma^i_{\ell k} g^{\ell j} - \Gamma^j_{\ell k} g^{\ell i})]$$

$$= \psi^k [g^{ij} \frac{\partial h_{ij}}{\partial x^k} - \Gamma^i_{\ell k} h^{\ell}_i - \Gamma^j_{\ell k} h^{\ell}_j]$$

$$= \psi^k [g^{ij} \frac{\partial h_{ij}}{\partial x^k} - 2\Gamma^j_{ik} h^i_j]$$

QED

In the next lemma, S is the square of the length of the second fundamental form.

<u>Lemma 2.7</u>

$$S = h^j_i h^i_j = 4H^2 - 2K.$$

<u>Proof</u>. $S = \langle h_{ij}dx^i dx^j, h_{k\ell}dx^k dx^\ell\rangle = h_{ij}h_{k\ell}g^{ik}g^{j\ell} = h^\ell_i h^i_\ell$.

On the other hand, since the matrix $A = (h^j_i)$ represents the Weingarten map in the basis $\{F_1, F_2\}$ and the principal curvatures λ_1, λ_2 are the eigenvalues of the Weingarten map, we can calculate as follows.

$$h^\ell_i h^i_\ell = \mathrm{trace}(A^2)$$

$$= \lambda_1^2 + \lambda_2^2$$

$$= (\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2$$

$$= (2H)^2 - 2K$$

QED

Lemma 2.8

$$\frac{\partial H}{\partial t} = \frac{1}{2}\Delta\phi + (2H^2-K)\phi + \langle\nabla H, V^T\rangle$$

Proof. From $2H = g^{ij}h_{ij}$ and Lemmas 2.4-2.7,

$$2\frac{\partial H}{\partial t} = \frac{\partial g^{ij}}{\partial t}h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t}$$

$$= 2\phi g^{ik}h_{ij}h^j_k - (g^{ik}h_{ij}\psi^j_k + g^{jk}h_{ij}\psi^i_k) - \psi^k(g^{i\ell}h_{ij}\Gamma^j_{\ell k} + g^{j\ell}h_{ij}\Gamma^i_{\ell k})$$

$$+ g^{ij}(\phi_{ij} - \phi_k\Gamma^k_{ij}) - \phi h^k_i g^{ij}h_{kj} + (\psi^k_i g^{ij}h_{kj} + \psi^k_j g^{ij}h_{ki}) + \psi^k g^{ij}\frac{\partial h_{ij}}{\partial x^k}$$

$$= 2\phi h^k_j h^j_k - (h^k_j\psi^j_k + h^k_i\psi^i_k) - \psi^k(h^\ell_j\Gamma^j_{\ell k} + h^\ell_i\Gamma^i_{\ell k})$$

$$+ g^{ij}(\phi_{ij} - \phi_k\Gamma^k_{ij}) - \phi h^k_i h^i_k + (h^i_k\psi^k_i + h^j_k\psi^k_j) + \psi^k g^{ij}\frac{\partial h_{ij}}{\partial x^k}$$

$$= g^{ij}(\phi_{ij} - \phi_k\Gamma^k_{ij}) + \phi h^j_i h^i_j + \psi^k\left(g^{ij}\frac{\partial h_{ij}}{\partial x^k} - 2h^j_i\Gamma^i_{jk}\right)$$

$$= \Delta\phi + (4H^2-2K)\phi + 2\langle\nabla H, V^T\rangle.$$

QED

Now we can calculate the first variation of total mean curvature.

For each $t \in (-1,1)$, let $\mathcal{H}(t)$ denote the total mean curvature of the immersion f_t of Σ:

$$\mathcal{H}(t) = \int_\Sigma H^2 d\Sigma.$$

Proposition 2.1

$$\mathcal{H}'(t) = \int_\Sigma \phi[\Delta H + 2H(H^2-K)]d\Sigma \qquad \text{for all } t \in (-1,1).$$

Proof. By Lemma 2.2 and 2.8,

$$\frac{\partial}{\partial t}(H^2\sqrt{g}) = 2H\frac{\partial H}{\partial t}\sqrt{g} + H^2\frac{\partial\sqrt{g}}{\partial t}$$

$$= [H\Delta\phi + 2H(2H^2-K)\phi + 2H\langle\nabla H, V^T\rangle - 2\phi H^3 + H^2\mathrm{div}(V^T)]\sqrt{g}$$

$$= [H\Delta\phi + 2H(H^2-K)\phi + \langle\nabla(H^2), V^T\rangle + H^2\mathrm{div}(V^T)]\sqrt{g}\ .$$

Integrating, and applying

$$\int_\Sigma (H\Delta\phi)\,d\Sigma = \int_\Sigma (\phi\Delta H)\,d\Sigma$$

$$\int_\Sigma \langle\nabla H^2, V^T\rangle d\Sigma = -\int_\Sigma H^2\mathrm{div}(V^T)\,d\Sigma$$

we obtain

$$\mathcal{H}'(t) = \int_\Sigma \frac{\partial}{\partial t}(H^2 d\Sigma) = \int_\Sigma [\phi\Delta H + 2H(H^2-K)\phi]\,d\Sigma$$

$$= \int_\Sigma \phi[\Delta H + 2H(H^2-K)]\,d\Sigma$$

QED

Thus, $\Delta H + 2H(2H^2-K) = 0$ is a necessary and sufficient condition for an immersion f of Σ to be stationary with respect to total mean curvature.

We proceed to calculate the second variation of total mean curvature under the assumptions (a) $\Delta H + 2H(H^2-K) = 0$ at $t = 0$ and (b) $V^T = 0$ at $t = 0$, that is, the variation vector field of F is normal to the surface $f(\Sigma)$. This is equivalent to assuming $\psi^k = 0$ at $t = 0$, and therefore, the formulas in Lemmas 2.1-2.8 simplify considerably. We use the notation $\delta = \frac{\partial}{\partial t}\Big|_{t=0}$, $\partial_i = \frac{\partial}{\partial x^i} = F_i$.

(1)' $\delta g_{ij} = -2\phi h_{ij}$ (2)' $\delta\sqrt{g} = -2\phi H\sqrt{g}$

(3)' $\delta n = -g^{ij}\phi_j\partial_i$ (4)' $\delta h_{ij} = (\phi_{ij}-\phi_k\Gamma^k_{ij}) - \phi h^k_i h_{kj}$

(5)' $\delta g^{ij} = 2\phi g^{ik}h^j_k$ (6)' $\delta H = \frac{1}{2}\Delta\phi + (2H^2-K)\phi$.

Observe that, because of assumption (a), $H''(0) = \int_\Sigma \phi\delta[\Delta H+2H(H^2-K)]d\Sigma$.

Lemma 2.9

$\delta(\Delta H) = \Delta(\delta H) + 2\mathrm{div}(\phi A(\nabla H)) - \langle\nabla\phi,\nabla H^2\rangle - 2\phi|\nabla H|^2$.

Proof. $\Delta H = \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\ H_j) = \frac{1}{2\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\ \partial_j h^k_k)$.

Therefore,

$$\delta(\Delta H) = \delta(\tfrac{1}{2}g^{-1/2})\partial_i(g^{ij}\sqrt{g}\ \partial_j h^k_k) + \frac{1}{2\sqrt{g}}\partial_i[\delta(g^{ij}\sqrt{g}\ \partial_j h^k_k)]$$

$$= \phi H\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\ \partial_j h^k_k) + \frac{1}{2\sqrt{g}}\partial_i[g^{ij}\sqrt{g}\ \partial_j(\delta h^k_k)] + \frac{1}{2\sqrt{g}}\partial_i[(\partial_j h^k_k)\delta(g^{ij}\sqrt{g})]$$

$$= 2\phi H\Delta H + \Delta(\delta H) + \frac{1}{\sqrt{g}}\partial_i[H_j(-2\phi Hg^{ij}\sqrt{g}+2\phi g^{i\ell}h^j_\ell\sqrt{g})]$$

The third term gives

$$\frac{1}{\sqrt{g}}\partial_i[-2\phi H_j(Hg^{ij}-g^{i\ell}h^j_\ell)\sqrt{g}]$$

$$= 2\mathrm{div}[\phi H_j(g^{i\ell}h^j_\ell-Hg^{ij})\partial_i]$$

$$= 2\mathrm{div}[\phi g^{i\ell}h^j_\ell H_j\partial_i] - 2\mathrm{div}[\phi g^{ij}H_j H\partial_i]$$

$$= 2\mathrm{div}(\phi A(\nabla H)) - 2\mathrm{div}(\phi H\nabla H).$$

The second divergence term gives

$$2\operatorname{div}(\phi H \nabla H) = \operatorname{div}(\phi \nabla H^2)$$

$$= \langle \nabla\phi, \nabla H^2 \rangle + \phi \Delta H^2$$

$$= \langle \nabla\phi, \nabla H^2 \rangle + 2\phi H \nabla H + 2\phi |\nabla H|^2.$$

The lemma follows.

QED

We calculate $\delta[2H(H^2-K)]$ by an indirect route. Observe that $H^2 - K = (2H^2-K) - H^2 = \frac{1}{2}S - H^2$ and therefore,

$$\delta[2H(H^2-K)] = \delta[HS-2H^3] = H\delta S + (S-6H^2)\delta H.$$

<u>Lemma 2.10</u>

$$\delta S = 2\langle B, \text{Hess } \phi \rangle + 4\phi H(4H^2-3K).$$

<u>Proof</u>.

$$\delta h_i^j = \delta(h_{ik}g^{kj}) = g^{kj}(\phi_{ik} - \Gamma_{ik}^{\ell}\phi_\ell - \phi h_i^\ell h_{\ell k}) + h_{ik}(2\phi g^{k\ell} h_\ell^j)$$

$$= g^{kj}(\phi_{ik} - \Gamma_{ik}^{\ell}\phi_\ell) - \phi h_i^\ell h_\ell^j + 2\phi h_i^\ell h_\ell^j$$

$$= g^{kj}(\phi_{ik} - \Gamma_{ik}^{\ell}\phi_\ell) + \phi h_i^\ell h_\ell^j.$$

Therefore,

$$\delta S = \delta(h_i^j h_j^i) = 2h_j^i g^{kj}(\phi_{ik} - \Gamma_{ik}^{\ell}\phi_\ell) + 2\phi h_j^i h_i^\ell h_\ell^j.$$

Note that

$$h_j^i h_i^\ell h_\ell^j = \operatorname{trace}(A^3) = \lambda_1^3 + \lambda_2^3 = (\lambda_1+\lambda_2)[(\lambda_1+\lambda_2)^2 - 3\lambda_1\lambda_2]$$

$$= 2H(4H^2-3K)$$

and also

$$<B, \text{Hess } \phi> = <h_{\ell j} dx^{\ell} dx^{j},\ (\phi_{ik} - \phi_{\ell} \Gamma^{\ell}_{ik}) dx^{i} dx^{k}>$$

$$= h_{\ell j} (\phi_{ik} - \Gamma^{\ell}_{ik} \phi_{\ell}) g^{\ell i} g^{jk}$$

$$= h^{i}_{j} g^{jk} (\phi_{ik} - \Gamma^{\ell}_{ik} \phi_{\ell}).$$

QED

Using Lemma 2.10 and the previous observation, we obtain:

Lemma 2.11

$$\delta[2H(H^2-K)] = 2H<B, \text{Hess } \phi> - (H^2+K)\Delta\phi + \phi(12H^4 - 14H^2K + 2K^2).$$

Combining Lemmas 2.9 and 2.11 we can write

$$\begin{aligned} \mathcal{H}''(0) = & \int [2\phi H <B, \text{Hess } \phi> + \phi^2(12H^4 - 14H^2K + 2K^2)] \\ & + \int [\phi\Delta(\delta H) - \phi(H^2+K)\Delta\phi] \\ & + \int 2\phi \operatorname{div}(\phi A(\nabla H)) \\ & + \int [-\phi <\nabla\phi, \nabla H^2> - 2\phi^2 |\nabla H|^2]. \end{aligned}$$

We treat the last three integrals separately.

$$\begin{aligned} & \int [\phi\Delta(\delta H) - \phi(H^2+K)\Delta\phi] \\ & = \int [\delta H - \phi(H^2+K)]\Delta\phi \\ & = \int [\tfrac{1}{2}\Delta\phi + (H^2-2K)\phi]\Delta\phi \\ & = \int [\tfrac{1}{2}(\Delta\phi)^2 + (H^2-2K)\phi\Delta\phi] \\ & = \int [\tfrac{1}{2}\phi\Delta^2\phi + (H^2-2K)\phi\Delta\phi \\ & = \int \phi[\tfrac{1}{2}\Delta^2\phi + (H^2-2K)\Delta\phi]. \end{aligned}$$

Next $\int 2\phi \operatorname{div}(\phi A(\nabla H)) = \int \phi^2 \operatorname{div}(A(\nabla H))$.

This follows from observing

$2\phi \operatorname{div}(\phi X) - \phi^2 \operatorname{div} X = \operatorname{div}(\phi^2 X)$ for vector fields X.

Finally

$$\int [-\phi\langle\nabla\phi, \nabla H^2\rangle - 2\phi^2|\nabla H|^2]$$

$$= \int [\tfrac{1}{2}\phi^2 \Delta H^2 - 2\phi^2 |\nabla H|^2]$$

$$= \int \phi^2 [H\Delta H + |\nabla H|^2 - 2|\nabla H|^2]$$

$$= \int \phi^2 [-2H^2(H^2-K) - |\nabla H|^2].$$

Here we used $\Delta H + 2H(H^2-K) = 0$.

Combining these integrals yields the second variation formula for total mean curvature.

Proposition 2.2

$$\mathcal{H}''(0) = \int_\Sigma \phi L(\phi)\, d\Sigma, \quad \text{where}$$

$$L(\phi) = \tfrac{1}{2}\Delta^2\phi + 2H\langle B, \text{Hess}\ \phi\rangle + (H^2-2K)\Delta\phi$$

$$+ \phi[\operatorname{div}(A(\nabla H)) - |\nabla H|^2 + 2(5H^2-K)(H^2-K)].$$

More work is needed to put the second variation formula into a more manageable form. However, we were able to analyze a special case on the torus.

The standard embedding of the torus $S^1 \times S^1$ in $\mathbb{R}^3$ is obtained by rotating a circle of radius $1/\sqrt{2}$ around an axis 1 unit from its center. In local coordinates the embedding $f : S^1 \times S^1 \to \mathbb{R}^3$ is given by

$$f(\alpha,\beta) = ((1+\frac{1}{\sqrt{2}}\cos\beta)\cos\alpha, (1+\frac{1}{\sqrt{2}}\cos\beta)\sin\alpha, \frac{1}{\sqrt{2}}\sin\beta).$$

This embedding is stationary with respect to total mean curvature, i.e. the first variation is identically zero.

We have calculated the second variation in the special case of a variation of constant cross-section, i.e. $\phi = \phi(\alpha)$. In this case

$$H''(0) = 13\pi[\frac{8}{13}\int_0^{2\pi}\phi^2 d\alpha + \frac{5}{13}\int_0^{2\pi}(\phi_{\alpha\alpha})^2 d\alpha - \int_0^{2\pi}(\phi_\alpha)^2 d\alpha].$$

Using Fourier series we have shown that this is non-negative, and is zero if and only if $\phi(\alpha) = A\cos\alpha + B\sin\alpha$.

Chapter 3

The Willmore Integral for Surfaces in S^3

In this chapter we calculate the first and second variation formulas for the Willmore integral

$$\mathcal{H} = \int_{\Sigma} (H^2+1)\, d\Sigma$$

for surfaces in S^3. These were calculated by J. Weiner using a different method.

We set up very similar machinery to that in the previous chapter. Let $f : \Sigma \to S^3 \subset \mathbb{R}^4$ be an immersion and $F : \Sigma \times (-1,1) \to S^3 \subset \mathbb{R}^4$ a variation of f, both thought of of as $\mathbb{R}^4$-valued. As before, we have several fields (now in $\mathbb{R}^4$) defined along F: The coordinate vector fields F_i, the variation vector field V, a unit normal vector field n. These are all tangent to S^3. In addition, we let N be the inward-pointing unit normal vector field to S^3 in $\mathbb{R}^4$. Note that $N = -F$ when restricted to $F(\Sigma)$. Finally, we may write $V = \phi n + \psi^i F_i$ as before.

The formulas for $\frac{\partial g_{ij}}{\partial t}$, $\frac{\partial g^{ij}}{\partial t}$, $\frac{\partial \sqrt{g}}{\partial t}$ are as before. On the other hand:

<u>Lemma 3.1</u>

$$\frac{\partial n}{\partial t} = -(g^{ij}\phi_j + \psi^j h_j^i) F_i + \phi N$$

<u>Proof</u>. The vector field n is determined up to sign by the relations: $\langle n,n\rangle = 1$, $\langle n,F_i\rangle = 0$, $\langle n,N\rangle = 0$.

Define (at least locally) C^∞ functions λ, μ, ν^i on Σ by $\frac{\partial n}{\partial t} = \lambda N + \mu n + \nu^i F_i$. Then:

$$\lambda = \langle \frac{\partial n}{\partial t}, N \rangle = -\langle n, \frac{\partial N}{\partial t} \rangle = \langle n, F_t \rangle = \langle n, V \rangle = \phi,$$

$$\mu = \langle \frac{\partial n}{\partial t}, n \rangle = \frac{1}{2} \frac{\partial}{\partial t} \langle n, n \rangle = 0,$$

$$\nu^i = \langle \nu^j F_j, F_k \rangle g^{ki} = \langle \frac{\partial n}{\partial t} - \phi N, F_k \rangle g^{ki} = \langle \frac{\partial n}{\partial t}, F_k \rangle g^{ki}$$

$$= -(g^{ij}\phi_j + \psi^j h_j^i), \text{ as before.}$$

QED

Let K be the Gaussian curvature and G the "relative" curvature. They are related by $G = K - 1$. Lemma 3.2 and the fact that $S = 4H^2 - 2G = 4H^2 - 2K + 2$ lead to a slightly different formula for $\frac{\partial H}{\partial t}$.

Lemma 3.2

$$\frac{\partial H}{\partial t} = \frac{1}{2}\Delta\phi + (2H^2 - K + 2)\phi + \langle \nabla H, V^T \rangle.$$

The proof is so similar to that of Lemma 2.8 that we omit it.

Now we are ready to derive the variation formulas for the Willmore integral.

Proposition 3.1

$$\mathcal{H}'(t) = \int_\Sigma \phi[\Delta H + 2H(H^2 - K + 1)]\, d\Sigma.$$

Proof. $$\frac{\partial}{\partial t}[(H^2+1)\sqrt{g}] = 2H \frac{\partial H}{\partial t}\sqrt{g} + (H^2+1)\frac{\partial \sqrt{g}}{\partial t}$$

$$= [2H \frac{\partial H}{\partial t} + (H^2+1)(-2\phi H + \mathrm{div}(V^T))]\sqrt{g}$$

$$= [H\Delta\phi + 2\phi H(H^2-K+1) + \langle\nabla(H^2+1), V^T\rangle + (H^2+1)\,\mathrm{div}(V^T)]\sqrt{g}$$

$$= [H\Delta\phi + 2\phi H(H^2-K+1) + \mathrm{div}((H^2+1)\ V^T)]\sqrt{g}\ .$$

Integrating:

$$H'(t) = \int \frac{\partial}{\partial t}[H^2+1)\,d\Sigma]$$

$$= \int [H\Delta\phi + 2\phi H(H^2-K+1)]\,d\Sigma$$

$$= \int [\phi\Delta H + 2\phi H(H^2-K+1)]\,d\Sigma.$$

QED

Therefore, $\Delta H + 2H(H^2-K+1) = 0$ is a necessary and sufficient condition for an immersion f of Σ to be stationary with respect to the Willmore integral.

Corollary (Weiner [16]). A minimal immersion $f : \Sigma \to S^3$ is stationary with respect to the Willmore integral.

We calculate the second variation assuming that f is a minimal immersion. Thus, at $t = 0$: $H = H_j = 0$ and $\Delta H + 2H(H^2-K+1) = 0$.

Lemma 3.3

$$\delta(\Delta H) = \Delta(\delta H).$$

Proof. $\delta(\Delta H) = \delta[\frac{1}{\sqrt{g}}\ \partial_i(g^{ij}\sqrt{g}\ H_j)]$

$$= \delta(\frac{1}{\sqrt{g}})\partial_i(g^{ij}\sqrt{g}\ H_j) + \frac{1}{\sqrt{g}}\partial_i[\delta(g^{ij}\sqrt{g})H_j + g^{ij}\sqrt{g}(\delta H)_j]$$

$$= 0 + 0 + \frac{1}{\sqrt{g}}\partial_i[g^{ij}\sqrt{g}(\delta H)_j] = \Delta(\delta H).$$

QED

Proposition 3.2

$$H''(0) = \frac{1}{2}\int_\Sigma \phi L(\phi)\, d\Sigma, \text{ where } L = (\Delta - 2K + 2)(\Delta - 2K + 4).$$

Proof. $H''(0) = \int_\Sigma \phi\,\delta[\Delta H + 2H(H^2 - K + 1)]\, d\Sigma$.

Since H = 0 at t = 0:

$$\begin{aligned}
\delta[\Delta H + 2H(H^2 - K + 1)] &= \Delta(\delta H) + 2(-K+1)(\delta H) \\
&= (\Delta - 2K + 2)(\delta H) \\
&= (\Delta - 2K + 2)(\tfrac{1}{2}\Delta\phi + (2-K)\phi) \\
&= \tfrac{1}{2}(\Delta - 2K + 2)(\Delta - 2K + 4)\phi
\end{aligned}$$

QED

Weiner [16] has applied this to the Clifford minimal embedding of the torus in S^3. This embedding can be expressed in local coordinates by

$$f(\alpha,\beta) = \frac{1}{\sqrt{2}}(\cos\alpha, \sin\alpha, \cos\beta, \sin\beta).$$

The Clifford torus is flat, i.e. K = 0, and therefore

$$L = \frac{1}{2}(\Delta + 2)(\Delta + 4).$$

The eigenvalues of the Laplacian on the Clifford torus are $\lambda_{k\ell} = -2(k^2+\ell^2)$ for $k, \ell \geq 0$. Therefore, those of L are $\mu_{k\ell} = 2(k^2+\ell^2-2)(k^2+\ell^2-1) \geq 0$. This implies that the second variation is non-negative for the Clifford torus.

Chapter 4

Quotients of Lawson's Minimal Surfaces

In this chapter we study quotients of Lawson's minimal surfaces $\mathcal{E}_{mn}$ in the sphere S^3 by groups of rotations. The main goal is to show that the metric on these surfaces satisfies a hyperbolic-sine-Gordon equation.

We begin with a review of the construction of the surfaces $\mathcal{E}_{mn}$. Consider $\mathbb{R}^4$ as $\mathbb{C}^2$ and S^3 as the unit sphere in $\mathbb{C}^2$: $\mathbb{R}^4 \cong \mathbb{C}^2 = \{(z,w) : z \in \mathbb{C}, w \in \mathbb{C}\}$ and $S^3 = \{(z,w)\} \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$. On S^3 distinguish two great circles $C_1 = \{(z,w) : |z| = 1\}$ and $C_2 = \{(z,w) : |w| = 1\}$. For each pair of non-negative integers m,n choose $2(m+1)$ equally spaced points $P_0,\ldots,P_{2m+1}$ on C_1 and $2(n+1)$ equally spaced points $Q_0,\ldots,Q_{2n+1}$ on C_2. Join each point P_k to each point Q_ℓ with a great circle segment in S^3. In this way obtain a geodesic lattice $\mathcal{L}_{m,n}$ in S^3 which divides it into $4(m+1)(n+1)$ congruent cells, each cell within a frame Γ_{mn} consisting of four geodesic segments. Choosing one of these cells arbitrarily, solve the Plateau problem in S^3 for that cell with respect to its frame Γ_{mn} to obtain a unique smooth embedded surface M_{mn} of least area having Γ_{mn} as its boundary. Finally, reflect M_{mn} throughout S^3 by geodesic reflection in the segments of $\mathcal{L}_{mn}$. This produces a smooth orientable compact minimal surface $\mathcal{E}_{mn}$ embedded in S^3. The surface $\mathcal{E}_{mn}$ is of genus mn and consists of $2(m+1)(n+1)$ congruent copies of M_{mn}, one copy in every other cell of $\mathcal{L}_{mn}$ in a checkerboard pattern. The surfaces $\mathcal{E}_{m0}$ and $\mathcal{E}_{0n}$

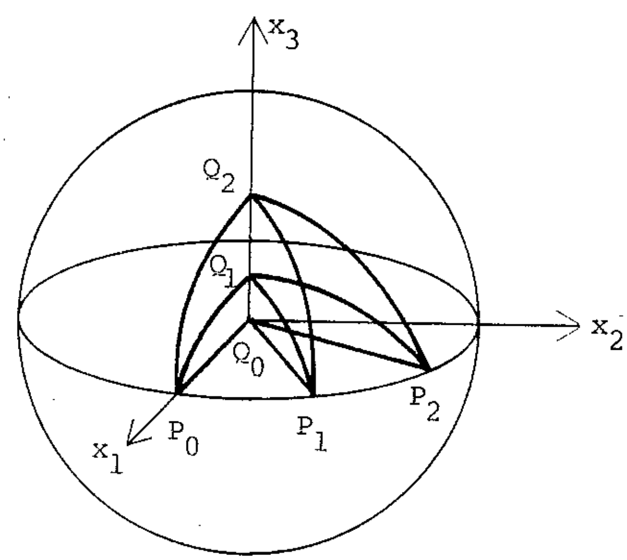
are geodesic 2-spheres, and E_{11} is the Clifford torus. Henceforth we assume $m,n \geqq 1$.

Suppose the vertices of L_{mn} are $P_k = (\exp(\frac{k\pi}{m+1}i),0)$ for $k = 0$ to $2m + 1$, and $Q_\ell = (0,\exp(\frac{\ell\pi}{n+1}i))$ for $\ell = 0$ to $2n + 1$. We can choose coordinate axes x_1, x_2, x_3 in $\mathbb{R}^3$ so that under stereographic projection $\sigma_1 : S^3 \to \mathbb{R}^3$ from $(0,-1) \in \mathbb{C}^2$ the circle C_1 is mapped to the circle $x_1^2 + x_2^2 = 1$ and the circle C_2 is mapped to the x_3-axis.

Then $\sigma_1(P_k) = (\cos(\frac{k\pi}{m+1}), \sin(\frac{k\pi}{m+1}),0)$ and $\sigma_1(Q_\ell) = (0,0,\dfrac{\sin(\frac{\ell\pi}{n+1})}{1+\cos(\frac{\ell\pi}{n+1})})$, $\sigma_1(Q_{n+1}) = \infty$. This gives us the picture of L_{mn} partially shown in Figure (1). On the other hand, we can choose coordinates axes y_1, y_2, y_3 in $\mathbb{R}^3$ so that stereographic projection $\sigma_2 : S^3 \to \mathbb{R}^3$ from $(-1,0) \in \mathbb{C}^2$ maps C_1 to the y_3-axis and C_2 to the circle $y_1^2 + y_2^2 = 1$.

Then $\sigma_2(P_k) = (0,0,\dfrac{\sin(\frac{k\pi}{m+1})}{1+\cos(\frac{k\pi}{m+1})})$ and $\sigma_2(Q_\ell) = (\cos(\frac{\ell\pi}{n+1}), \sin(\frac{\ell\pi}{n+2}),0)$.

This interchanges the vertices in Figure (1).



Now, a reflection in one of the great circles of L_{mn} maps the surface E_{mn} onto itself, by construction. And, the product of reflections in intersecting great circles is a rotation through twice the angle between them. Therefore, the group $\mathbb{Z}_{m+1} \times \mathbb{Z}_{n+1}$ acts on E_{mn} by orientation-preserving isometries. The generators α, β for this group are defined by

$$\alpha : E_{mn} \to E_{mn}, \ \alpha(z,w) = (z \exp(\tfrac{2\pi}{m+1} i), w)$$

$$\beta : E_{mn} \to E_{mn}, \ \beta(z,w) = (z, \ w \exp(\tfrac{2\pi}{n+1} i)).$$

The generator α corresponds precisely to a rotation through $\frac{2\pi}{m+1}$ around the x_3-axis in figure (1), and β corresponds to a rotation through $\frac{2\pi}{n+1}$ around the y_3-axis. Furthermore, if p divides $m+1$ and q divides $n+1$ then $\mathbb{Z}_p \times \mathbb{Z}_q$ also acts on E_{mn} by orientation-preserving isometries, the generators corresponding to rotations of $2\pi/p$ around the x_3-axis and $2\pi/q$ around the y_3-axis. These observations lead us to the following theorem.

<u>Theorem 4.1</u>. The surface E_{mn}, regarded as a Riemann surface, is a pq-sheeted branched covering of a compact Riemann surface $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ of genus $(\frac{m+1}{p} - 1)(\frac{n+1}{q} - 1)$. The covering branches at the points P_k (with branch number $q-1$) and Q_ℓ (with branch number $p-1$).

<u>Proof</u>. In general, if Σ is a compact Riemann surface and $\Gamma \subset \mathrm{Aut}(\Sigma)$ is a finite group acting on Σ by orientation-preserving conformal diffeomorphisms of Σ, then the quotient Σ/Γ has a conformal structure

which makes it a compact Riemann surface. The projection is then a finite-sheeted ramified covering map.

In the present situation, let $E^\circ_{mn} = E_{mn} - \{P_0,\ldots,P_{2m+1},Q_0,\ldots,Q_{2n+1}\}$ be the open set where $\mathbb{Z}_p \times \mathbb{Z}_q$ acts freely. The orbit of any point $(z,w) \in E^\circ_{mn}$ under the $\mathbb{Z}_p \times \mathbb{Z}_q$ action consists of the pq points $(z \exp(\frac{2\pi k}{p}i), w \exp(\frac{2\pi \ell}{q}i))$ for $k = 0$ to $p - 1$ and $\ell = 0$ to $q - 1$. The orbit of any M_{mn} consists of pq congruent copies of M_{mn}. Any point $x \in E^\circ_{mn}$ has a neighborhood $U \subset E^\circ_{mn}$ whose orbit is, likewise, pq copies of U. (Specifically, M°_{mn} , the interior of M_{mn}, if x is not on the lattice L_{mn}, and two adjacent copies of M°_{mn} along with the geodesic segment in between minus endpoints otherwise, for example). Let z be a local coordinate in U, and transfer it throughout E_{mn} by the $\mathbb{Z}_{m+1} \times \mathbb{Z}_{n+1}$ action. Define a local coordinate w near any point y in $E^\circ_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$, where $\pi : E_{mn} \to E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ is the natural projection, simply by $w(y) = z(\pi^{-1}y)$. That this is well-defined follows from the invariance of z under the $\mathbb{Z}_{m+1} \times \mathbb{Z}_{n+1}$ action.

On the other hand, any P_k has an orbit consisting of p of the P_ℓ's, is fixed by $\mathbb{Z}_q$, and has a neighborhood $U \subset E_{mn}$ whose orbit consists of p copies of U. Let z be a local coordinate in U such that $z(P_k) = 0$ and the generator β in this coordinate is $\beta(z) = z \exp(\frac{2\pi}{n+1}i)$. Transfer this coordinate throughout E_{mn} by the $\mathbb{Z}_{m+1}$ action. Define a local coordinate w near $\pi(P_k)$ by: $w(y) = [z(\pi^{-1}y)]^q$. That this is well-defined follows from noting that $\pi^{-1}y$ consists of pq points whose coordinates, by construction, are of the form $z_0 \exp(\frac{2\pi k}{q}i)$.

Analogously, near Q_k and $\pi(Q_k)$ define w,z by $w(y) = [z(\pi^{-1}y)]^p$. We have thus shown that the projection π may be expressed in local coordinates by $w = z$ (local homeomorphism away from fixed points), $w = z^q$ (branch point with branch number $q - 1$ at P_k), or $w = z^p$ (branch point with branch number $p - 1$ at Q_k). Furthermore, if y is any point of $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ and $b(x)$ is the branch number of π at $x \in E_{mn}$, then $\sum_{x\in\pi^{-1}y} (b(x)+1) = pq$. Thus π is pq-sheeted with the required branch points.

A fundamental domain for the action of $\mathbb{Z}_p \times \mathbb{Z}_q$ on E_{mn} consists of the $2(m+1)(n+1)/pq$ copies of M_{mn} contained in the cells bounded by the geodesic segments joining the points $Q_0,\ldots,Q_{2(n+1)/q}$ to $P_0,\ldots,P_{2(m+1)/p}$. Triangulating each copy of M_{mn} by inserting an edge between its Q-vertices, we obtain a triangulation of $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ which has $2(m+1)/p + 2(n+1)q$ vertices, $4(m+1)(n+1)/pq$ faces, and $6(m+1)(n+1)/pq$ edges. Applying $2 - 2g = V - E + F$, the required genus g follows.

QED

Observe that $E_{mn}/\mathbb{Z}_{m+1} \times \mathbb{Z}_{n+1}$ is the Riemann sphere, and if m and n are odd, $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$ is a torus. We will show it is a square torus by considering its group of symmetries.

We begin by examining the group G_{mn} of symmetries of E_{mn}. Let S_1 be the geodesic 2-sphere determined by P_0, P_1 and the midpoint of Q_0Q_1

on the great circle C_2, and S_2 the 2-sphere determined by Q_0, Q_1 and the midpoint of P_0P_1 on C_1. The following orthogonal transformations of $\mathbb{R}^4$ are symmetries of ξ_{mn} by construction. Indices on P_k and Q_ℓ are written mod(2m+2) and mod(2n+2), respectively.

(i) **a** : Geodesic reflection in the great circle Q_0P_0.
On vertices, **a** is defined by $\mathbf{a}(P_k) = P_{-k}$, $\mathbf{a}(Q_\ell) = Q_{-\ell}$.
As a map on $\mathbb{C}^2$, it is defined by $\mathbf{a}(z,w) = (\bar{z},\bar{w})$. As an element of O(4), it is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(ii) **b** : Geodesic reflection in Q_0P_1. On vertices, $\mathbf{b}(P_k) = P_{2-k}$ and $\mathbf{b}(Q_\ell) = Q_{-\ell}$. On $\mathbb{C}^2$, $\mathbf{b}(z,w) = (\bar{z}\exp(\frac{2\pi}{m+1}i), \bar{w})$. In O(4),

$$\mathbf{b} = \begin{pmatrix} \cos(\frac{2\pi}{m+1}) & \sin(\frac{2\pi}{m+1}) & 0 & 0 \\ \cos(\frac{2\pi}{m+1}) & -\cos(\frac{2\pi}{m+1}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(iii) **c** : Geodesic reflection in the 2-sphere S_1. On vertices, $\mathbf{c}(P_k) = P_k$ and $\mathbf{c}(Q_\ell) = Q_{1-\ell}$. On $\mathbb{C}^2$, $\mathbf{c}(z,w) = (z, \bar{w}\exp(\frac{\pi}{n+1}i))$.
In O(4),

$$\mathbf{c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{n+1}) & \sin(\frac{\pi}{n+1}) \\ 0 & 0 & \sin(\frac{\pi}{n+1}) & -\cos(\frac{\pi}{n+1}) \end{pmatrix}$$

(iv) $\mathbf{d}$: Geodesic reflection in S_2. On vertices, $\mathbf{d}(P_k) = P_{1-k}$, $\mathbf{d}(Q_\ell) = Q_\ell$. On $\mathbb{C}^2$, $\mathbf{d}(z,w) = (\bar{z}\exp(\frac{\pi}{m+1}i),w)$. In $O(4)$,

$$\mathbf{d} = \begin{pmatrix} \cos(\frac{\pi}{m+1}) & \sin(\frac{\pi}{m+1}) & 0 & 0 \\ \sin(\frac{\pi}{m+1}) & -\cos(\frac{\pi}{m+1}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(v) If $m = n$, then the angles between the pairs of vertices are equal and there is a symmetry $\mathbf{e}$ which interchanges them: $\mathbf{e}$ is reflection in the plane $z = w$. On vertices, $\mathbf{e}(P_k) = Q_k$, $\mathbf{e}(Q_\ell) = P_\ell$. On $\mathbb{C}^2$, $\mathbf{e}(z,w) = (w,z)$. In $O(4)$,

$$\mathbf{e} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Lemma 4.1. The symmetries $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ (and $\mathbf{e}$ if $m = n$) generate the group G_{mn} of symmetries of ξ_{mn}.

Proof. Any symmetry **S** of E_{mn}, being an element of O(4), is completely determined by its action on the vertices Q_0, P_0, Q_1, P_1, since these are linearly independent in $\mathbb{R}^4$. Assuming there is a copy of M_{mn} in the cell $Q_0P_0Q_1P_1$, then there is a copy of M_{mn} in each cell $Q_{2k}P_{2\ell}Q_{2k+1}P_{2\ell+1}$ and $Q_{2k-1}P_{2\ell-1}Q_{2k}P_{2\ell}$. For convenience, define maps **A** = **cda** and **B** = **bdc**. These are "diagonal" symmetries of the checkerboard pattern of E_{mn} : **A** is "one over, one up," **B** is "one-over, one down." On vertices, $\mathbf{A}(P_k) = P_{k+1}$, $\mathbf{A}(Q_\ell) = Q_{\ell+1}$, $\mathbf{B}(P_k) = P_{k+1}$, $\mathbf{B}(Q_\ell) = Q_{\ell-1}$.

Now **S** maps $Q_0P_0Q_1P_1$ to some $Q_{2k}P_{2\ell}Q_{2k+1}P_{2\ell+1}$ or $Q_{2k-1}P_{2\ell-1}Q_{2k}P_{2\ell}$. This is accomplished by either $\phi = \mathbf{A}^{k+\ell}\mathbf{B}^{\ell-k}$ or $\psi = \mathbf{A}^{k+\ell-1}\mathbf{B}^{\ell-k}$, respectively. Therefore $\phi^{-1}\mathbf{S}$ or $\psi^{-1}\mathbf{S}$ is a symmetry of E_{mn} which maps the cell $Q_0P_0Q_1P_1$ to itself and thus must be a symmetry of M_{mn}. By uniqueness, such a symmetry must be a symmetry of the frame $Q_0P_0Q_1P_1$. But this frame has at most 8 symmetries:

(i) identity : $Q_0P_0Q_1P_1 \to Q_0P_0Q_1P_1$

(ii) **c** : $Q_0P_0Q_1P_1 \to Q_1P_0Q_0P_1$

(iii) **d** : $Q_0P_0Q_1P_1 \to Q_0P_1Q_1P_0$

(iv) **cd** : $Q_0P_0Q_1P_1 \to Q_1P_1Q_0P_0$ (rotation by π)

(and if m = n)

(v) **e** : $Q_0P_0Q_1P_1 \to P_0Q_0P_1Q_1$

(vi) **ce** : $Q_0P_0Q_1P_1 \to P_0Q_1P_1Q_0$

(vii) **de** : $Q_0P_0Q_1P_1 \to P_1Q_0P_0Q_1$

(viii) **cde** : $Q_0P_0Q_1P_1 \to P_1Q_1P_0Q_0$

In each case, S is a product of powers of **a,b,c,d,e**.

QED

Let H_m be the subgroup of G_{mm} generated by **a,b,c,d**.

Theorem 4.2. The group of symmetries of $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ is:
(i) $G_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ if $m \neq n$, (ii) $G_{mn}/\mathbb{Z}_p \times \mathbb{Z}_p$ if $m = n$ and $p = q$ and (iii) $H_m/\mathbb{Z}_p \times \mathbb{Z}_q$ if $m = n$ and $p \neq q$.

Proof. If Γ is the group of symmetries of a space X, and G is a subgroup of Γ with normalizer N(G), then N(G)/G is the group of symmetries of X/G. Let $\Theta = \alpha^{\frac{m+1}{p}}$ and $\gamma = \beta^{\frac{n+1}{q}}$ be the generators of $\mathbb{Z}_p \times \mathbb{Z}_q$. The relations $\mathbf{a}\Theta\mathbf{a} = \Theta^{-1}$, $\mathbf{b}\Theta\mathbf{b} = \Theta^{-1}$, $\mathbf{c}\Theta\mathbf{c} = \Theta$, $\mathbf{d}\Theta\mathbf{d} = \Theta^{-1}$, $\mathbf{a}\gamma\mathbf{a} = \gamma^{-1}$, $\mathbf{b}\gamma\mathbf{b} = \gamma^{-1}$, $\mathbf{c}\gamma\mathbf{c} = \gamma^{-1}$, $\mathbf{d}\gamma\mathbf{d} = \gamma$ show that, if $m \neq n$, G_{mn} is the normalizer of $\mathbb{Z}_p \times \mathbb{Z}_q$. If $m = n$, then in addition to the above relations we have $\mathbf{e}\alpha\mathbf{e} = \beta$ and $\mathbf{e}\beta\mathbf{e} = \alpha$, which imply that G_{mm} normalizes $\mathbb{Z}_p \times \mathbb{Z}_q$ if and only if $p = q$. If $p \neq q$, then H_m is the normalizer.

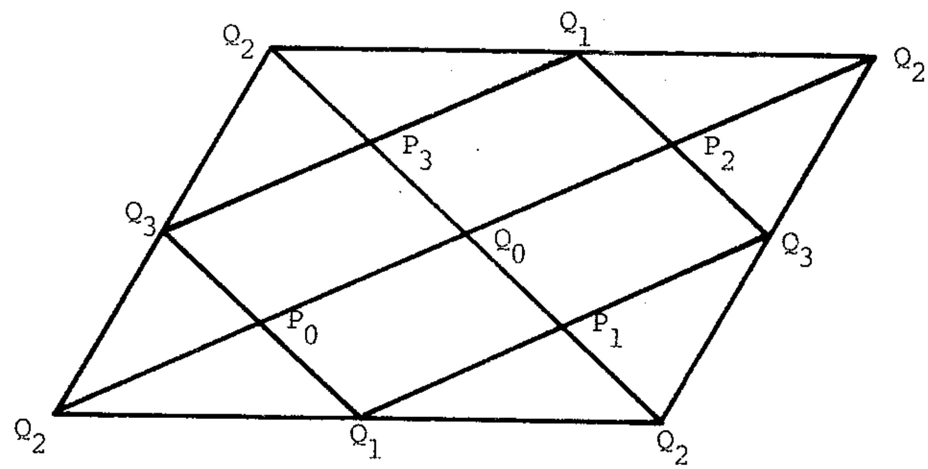
QED

Corollary 4.1. The Riemann surface $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$ is conformally equivalent to a square torus.

Proof. Any torus is conformally equivalent to $\mathbb{C}/\Gamma$ where Γ is some lattice in the plane. Any symmetry of the torus is a conformal or anticonformal automorphism and therefore corresponds to a transformation $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$ of the plane which fixes the lattice Γ.

fixed point sets of the symmetry map to fixed point sets of the corresponding transformation and thus to lines or points.

We map a fundamental domain of $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$ (for example, the 8 copies of M_{mn} contained in the cells with vertices P_0 through P_4 and Q_0 through Q_4) to a fundamental domain for $\mathbb{C}/\Gamma$ in the plane. The geodesics Q_kP_ℓ are mapped to lines and the copies of M_{mn} they bound are mapped to congruent parallelograms. This gives us the picture of the fundamental domain in Figure 2. Note that $P_0 \sim P_4$, $Q_0 \sim Q_4$.



The reflection $\tilde{\mathbf{c}} = \mathbf{c} \pmod{\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}}$ fixes each P_k and interchanges $Q_0 \leftrightarrow Q_1$, $Q_2 \leftrightarrow Q_3$. The corresponding map in the plane is a reflection in the line P_0P_1 which must interchange Q_0 and Q_1, this forces the lattice to be rectangular. Similarly, the symmetry $\tilde{\mathbf{a}}$ fixes P_0, Q_0, P_2, Q_2 and interchanges $P_1 \leftrightarrow P_3$, $Q_1 \leftrightarrow Q_3$. The corresponding map in the plane is a reflection in the line P_0Q_0 which must interchange $P_1 \leftrightarrow P_3$. Therefore, the fundamental domain is a square.

QED

The remainder of this chapter is dedicated to showing that the Gauss curvature equation on E_{mn} leads to a hyperbolic sine-Gordon equation for the metric on E_{mn}.

Let h_{ij} be the coefficients of the second fundamental form of E_{mn} in S^3, let $\phi = \frac{1}{2}(h_{11} - ih_{12})$, and write the metric on E_{mn} as $ds^2 = 2F|dz|^2$. Because E_{mn} is minimal, the differential form $\omega = \phi\ dz^2$ is holomorphic and the Gauss curvature equation takes the form $F^2(1-K) = |\phi|^2$ where K is the Gauss curvature. See Appendix C for details.

We begin by analyzing the nature of ω on E_{mn}.

Lemma 4.2. The differential form ω has zeroes of order $n - 1$ at each P_k, and order $m - 1$ at each Q_ℓ. It has no other zeroes.

Proof. At each P_k, the geodesics $Q_0P_k, \ldots, Q_{2n+2}P_k$ divide the surface into $2n + 2$ wedge-like regions. If d is the degree of the zero of ω at P_k, then $2d + 4 = 2n + 2$ gives $d = n - 1$ (see Appendix C). A similar argument holds at Q_ℓ.

The degree of a meromorphic differential on a surface of genus mn is $4mn - 4$ = # zeroes − # poles, counting multiplicity. But ω has no poles, and the sum of the degrees of the above zeroes is $(2m+2)(n-1) + (2n+2)(m-1) = 4mn - 4$. Therefore, ω can have no other zeroes.

QED

Of course if $m = 1$ or $n = 1$, "a zero of order zero" is no zero at all.

Next we look at power series expansions of ϕ at P_k and Q_ℓ.

Lemma 4.3. At $P_0,\ldots,P_{2m+1}$, ϕ has a power series expansion $\phi(z) = \sum_{k=1}^{\infty} a_k z^{(n+1)k-2}$, where $a_1 \neq 0$. Similarly, at $Q_0,\ldots,Q_{2n+1}$, $\phi(z) = \sum_{k=1}^{\infty} b_k z^{(m+1)k-2}$ where $b_1 \neq 0$.

Proof. Since ω depends on the first and second fundamental forms, it is invariant under symmetries of E_{mn}. Suppose that at $p \in E_{mn}$, ω is invariant under rotation by ${}^{2\pi}/\ell$. Choose a coordinate z centered at p so that this rotation is multiplication by $\lambda = \exp(\frac{2\pi}{\ell} i)$. Then invariance requires

$$\phi(\lambda z)[d(\lambda z)]^2 = \phi(z)dz^2$$

which gives
$$\lambda^2\phi(\lambda z) = \phi(z).$$

Writing $\phi(z) = \sum_{j=0}^{\infty} c_j z^j$, this becomes

$$\sum_{j=0}^{\infty} c_j \lambda^{j+2} z^j = \sum_{j=0}^{\infty} c_j z^j$$

and therefore, $c_j\lambda^{j+2} = c_j$ for $j = 0$ to ∞. The only possible non-zero coefficients c_j are those for which $\lambda^{j+2} = 1$, which implies $j = k\ell - 2$ for $k = 1$ to ∞. Setting $a_k = c_{k\ell-2}$, we arrive at the expansion

$$\phi(z) = \sum_{k=1}^{\infty} a_k z^{k\ell-2}.$$

The first term is $a_1 z^{\ell-2}$ and so if ω has a zero of order $\ell - 2$, then $a_1 \neq 0$.

The required expansions follow by noting that at $P_0,\ldots,P_{2m+1}$, $\ell = n + 1$, and at $Q_0,\ldots,Q_{2n+1}$, $\ell = m + 1$.

QED

The differential ω projects to a differential $\tilde{\omega}$ on $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ such that $\pi^*\tilde{\omega} = \omega$. Recall that we can choose coordinates so that the projection π is given by $w = z$ away from branch points, $w = z^q$ at P_k and $w = z^p$ at Q_ℓ. We will use these coordinates and the previous lemma to prove the following.

Lemma 4.4. The differential $\tilde{\omega}$ has simple poles at the 4 branch points of $E_{mn}/\mathbb{Z}_{m+1} \times \mathbb{Z}_{n+1}$, and is holomorphic on $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ if $p < m + 1$ and $q < n + 1$.

Proof. We work at Q_ℓ. Write $\tilde{\omega} = \tilde{\phi}(w)\,dw^2$, $\omega = \phi(z)\,dz^2$, where $w = z^p$. Then $\pi^*\tilde{\omega} = \omega$ becomes

$$\tilde{\phi}(z^p)\,[d(z^p)]^2 = \phi(z)\,dz^2.$$

This leads to

$$\begin{aligned} p^2\tilde{\phi}(z^p) &= z^{2-2p}\phi(z) \\ &= z^{2-2p}\sum_{k=1}^{\infty} a_k z^{(m+1)k-2} \\ &= \sum_{k=1}^{\infty} a_k z^{p(\bar{p}k-2)} \end{aligned}$$

where we have written $m + 1 = p\bar{p}$.

Therefore

$$p^2\tilde{\phi}(w) = \sum_{k=1}^{\infty} a_k w^{\bar{p}k-2}$$

where $a_1 \neq 0$. Thus $\tilde{\phi}$ has a simple pole at Q_ℓ if $p = m + 1$, but none otherwise. An entirely analogous calculation holds at the branch points P_k.

QED

<u>Theorem 4.3</u>. For m,n odd, the metric $ds^2 = 2F|dz|^2$ on the surface E_{mn} satisfies a hyperbolic-sine-Gordon equation

$$\partial\bar{\partial}v = -2c \sinh v$$

where $F = ce^v$, $c > 0$. For all m,n, the metric satisfies

$$v_{\eta\xi} = -2c \sinh v$$

where $F = \frac{c}{|z^4-1|} e^v$, $c > 0$, and η, ξ are variables defined below.

In each case, the Gauss curvature $K = 1 - e^{-2v}$.

<u>Proof</u> (m,n odd). By the previous lemma, the differential $\tilde{\omega}$ on the torus $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$ is holomorphic. Any holomorphic quadratic differential on a torus is of the form: αdw^2, $\alpha \in \mathbb{C}$, where w is a local coordinate. Therefore, $|\tilde{\phi}|$ = constant $c > 0$. Except at the branch points, we may choose local coordintes so that the projection π is the identity, and hence $|\phi| = |\tilde{\phi}| = c$. The Gauss curvature equation then becomes

$$F^2(1-K) = c^2$$

or

$$F^2 + F\partial\bar{\partial} \log F = c^2.$$

Upon substituting $F = ce^v$, this becomes

$$ce^{2v} + e^v\partial\bar{\partial}v = c$$

which is equivalent to

$$\partial\bar{\partial}v = -2c \sinh v.$$

In general, on the Riemann sphere $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$, the quadratic differential $\tilde{\omega}$ has four simple poles and therefore must be of the form

$$\tilde{\omega} = \frac{\alpha}{z^4-1}\, dz^2, \quad \alpha \in \mathbb{C}.$$

As above, by the Gauss curvature equation we obtain

$$F^2 + F\partial\bar{\partial} \log F = \frac{c^2}{|z^4-1|^2} .$$

We make the substitution $F = \frac{c}{|z^4-1|}\, e^v$ and obtain

$$|z^4-1|\partial\bar{\partial}v = c(e^{-v}-e^v),$$

since $\partial\bar{\partial} \log|z^4-1| = 0$. Now we make the change of independent variable

$$\xi = \int \frac{dz}{\sqrt{z^4-1}} \quad \text{and} \quad \eta = \int \frac{d\bar{z}}{\sqrt{\bar{z}^4-1}}$$

and note

$$v_{\eta\xi} = |z^4-1|\,\partial\bar{\partial}v.$$

The sinh-Gordon equation follows.

Finally, either

$$K = -\frac{1}{F}\partial\bar{\partial} \log F = -\frac{1}{ce^v}\partial\bar{\partial}v = \frac{2 \sinh v}{e^v} = 1 - e^{-2v}$$

or

$$K = -\frac{|z^4-1|}{ce^v}\partial\bar{\partial}v = -\frac{v_{\eta\xi}}{ce^v} = \frac{2 \sinh v}{e^v} = 1 - e^{-2v}.$$

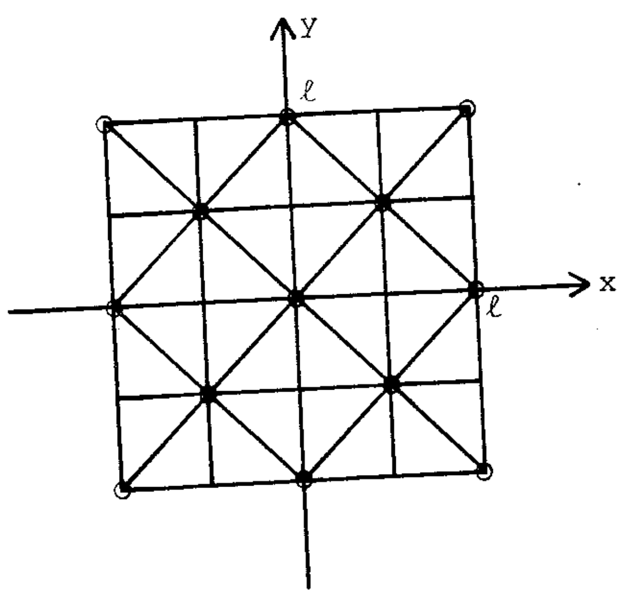
QED

At the branch points, the projection π is no longer the identity, the change of coordinates are singular $(\frac{dw}{dz} = 0)$, and we expect our solutions to have singularities, which we will investigate in the next chapter. This also follows from $K = 1 - e^{-2v}$. Since $K \to 1$ at the branch points, $v \to \infty$ there.

On the torus $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$, writing $z = x + iy$, the sinh-Gordon equation becomes

$$v_{xx} + v_{yy} = -8c \sinh v.$$

Thus, we are looking for real-valued, periodic solutions with the symmetries illustrated in the fundamental domain. The group of symmetries is generated by reflections in the lines shown.



$v(x+\ell,y) = v(x,y+\ell) = v(x,y)$

$v(x,y) = v(y,x)$

$v(-x,y) = v(x,-y) = v(x,y)$

These suffice to generate all the symmetries.

The solution v will be ∞ at the points indicated by o, these are the branch points. In addition, v is determined by its values in any of the triangles. Any "diamond" corresponds to an M_{mn}.

Chapter 5

The Hyperbolic Sine-Gordon Equation

In this chapter we prove a "Permutability Theorem" which allows us to generate solutions to the sinh-Gordon equation algebraically. We obtain first and second-generation solutions. In addition, we examine the behavior of the metric on E_{mn} at the branch points P_k and Q_ℓ.

The Backlund transformation

$$u_s = v_s - 4c\beta\sinh\left(\frac{u+v}{2}\right)$$

$$u_t = -v_t + \frac{2}{\beta}\sinh\left(\frac{u-v}{2}\right)$$

leaves the differential equation $u_{st} = -2c \sinh u$ invariant. In other words, if u and v are related by this transformation, then u is a solution if and only if v is. More precisely, we state the following proposition.

Proposition 5.1. Suppose that

(a) $$u_{st} = -2c \sinh u$$

and consider the system

$$(\mathbb{B})\quad \begin{aligned} u_s &= \mathbb{B}_1(\beta,s,t) = v_s - 4c\beta\sinh\left(\frac{u+v}{2}\right) \\ u_t &= \mathbb{B}_2(\beta,s,t) = -v_t + \frac{2}{\beta}\sinh\left(\frac{u-v}{2}\right) \end{aligned}$$

where β is an arbitrary constant. Then the integrability condition

$$\frac{\partial \mathbb{B}_1}{\partial t} = \frac{\partial \mathbb{B}_2}{\partial s}$$

is equivalent to the condition

(b) $v_{st} = -2c \sinh v.$

Conversely, if (b) holds, then the integrability condition for the system

$$(\mathbb{B}') \quad \begin{aligned} v_s &= \mathbb{B}_1'(\beta,s,t) = u_s + 4c\beta \sinh(\tfrac{u+v}{2}) \\ v_t &= \mathbb{B}_2'(\beta,s,t) = -u_t + \tfrac{2}{\beta} \sinh(\tfrac{u-v}{2}) \end{aligned}$$

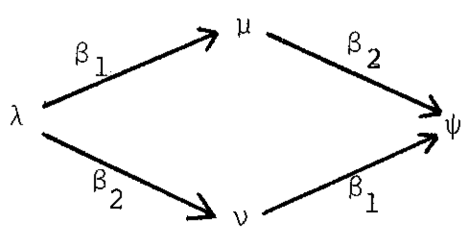
is equivalent to (a). The proof is a simple calculation.

We say that "u generates v, via β" if they are related by the transformation ($\mathbb{B}$) (or equivalently, ($\mathbb{B}'$)). The usefulness of this transformation is that we can generate solutions by a purely algebraic process, as follows:

Theorem 5.1 (Permutability Theorem). Suppose that λ generates μ and ν via β_1 and β_2, respectively. Then μ and ν generate a solution ψ via β_2 and β_1, respectively, given by

$$\psi = 4 \tanh^{-1}\left(\frac{\beta_1+\beta_2}{\beta_1-\beta_2} \tanh \frac{\mu-\nu}{4}\right) + \lambda.$$

Schematically,



Proof. Applying the two transformations

$$\lambda \xrightarrow{\beta_1} \mu \xrightarrow{\beta_2} \psi$$

we obtain

$$\psi_s = \lambda_s + 4c\beta_1\sinh(\frac{\mu+\lambda}{2}) + 4c\beta_2\sinh(\frac{\psi+\mu}{2})$$

$$\psi_t = \lambda_t - \frac{2}{\beta_1}\sinh(\frac{\lambda-\mu}{2}) + \frac{2}{\beta_2}\sinh(\frac{\mu-\psi}{2})$$

Similarly, from $\lambda \xrightarrow{\beta_2} \nu \xrightarrow{\beta_1} \psi$ we obtain

$$\psi_s = \lambda_s + 4c\beta_2\sinh(\frac{\lambda+\nu}{2}) + 4c\beta_1\sinh(\frac{\psi+\nu}{2})$$

$$\psi_t = \lambda_t - \frac{2}{\beta_2}\sinh(\frac{\lambda-\nu}{2}) + \frac{2}{\beta_1}\sinh(\frac{\nu-\psi}{2}) \quad .$$

Subtracting and rearranging, via an identity, gives

$$0 = \cosh(\frac{\lambda+\mu+\nu+\psi}{4})\,[\beta_1\sinh(\frac{\lambda+\mu-\nu-\psi}{4}) + \beta_2\sinh(\frac{\mu+\psi-\lambda-\nu}{4})]$$

$$0 = \cosh(\frac{\mu+\nu-\lambda-\psi}{4})\,[\beta_1\sinh(\frac{\lambda+\mu-\nu-\psi}{4}) + \beta_2\sinh(\frac{\mu+\psi-\lambda-\nu}{4})] \quad .$$

This in turn requires

$$\beta_1\sinh(\frac{\lambda-\psi}{4} + \frac{\mu-\nu}{4}) = \beta_2\sinh(\frac{\lambda-\psi}{4} - \frac{\mu-\nu}{4}) .$$

Expanding both sides, dividing by $\cosh(\frac{\lambda-\psi}{4})\cosh(\frac{\mu-\nu}{4})$, and rearranging leads to

$$(\beta_1-\beta_2)\tanh(\frac{\lambda-\psi}{4}) = -(\beta_1+\beta_2)\tanh(\frac{\mu-\nu}{4}) .$$

Solving for ψ gives the desired result.

QED

If we seek solutions of the form $u(s,t) = f(s) + g(t)$, then $u_{st} = 0 = \sinh u$ and hence we have a family of constant solutions

$$u = k\pi i, \; k \in \mathbb{Z}$$

of $u_{st} = -2c \sinh u$, for any c. We will use these to obtain a pair of families of non-trivial solutions.

Substituting $u = k\pi i$ in the Bäcklund transformation, we arrive at the system

$$v_s = 4c\beta \sinh\left(\frac{v}{2} + \frac{k\pi i}{2}\right)$$

$$v_t = -\frac{2}{\beta} \sinh\left(\frac{v}{2} - \frac{k\pi i}{2}\right) \quad .$$

There are two distinct cases: $k = 0$ and $k = 1$.

$k = 0$: The system becomes

$$v_s = 4c\beta \sinh \frac{v}{2}$$

$$v_t = -\frac{2}{\beta} \sinh \frac{v}{2} \quad .$$

Let $V(s) = \frac{1}{2} v(s,0)$. Then $\frac{dV}{ds} = \frac{1}{2} v_s = 2c\beta \sinh V$

and $\frac{dV}{\sinh V} = 2c\beta \, ds$

$$\ell n(\tanh \tfrac{1}{2} V) = 2c\beta s + a_0$$

$$V = 2 \tanh^{-1}(ae^{2c\beta s}) .$$

Therefore,

$$v(s,0) = 4 \tanh^{-1}(ae^{2c\beta s}) \qquad (*)$$

Now, for each s, define $W(t) = \frac{1}{2} v(s,t)$.

Then $$\frac{dW}{dt} = -\frac{1}{\beta}\sinh W$$

$$\frac{dW}{\sinh W} = -\frac{1}{\beta}\,dt$$

$$\ell n(\tanh \tfrac{1}{2} W) = -\frac{1}{\beta} t + b_0$$

$$W = 2\tan^{-1}(be^{-\frac{1}{\beta}t}).$$

Therefore, $$v(s,t) = 4\tanh^{-1}(be^{-\frac{1}{\beta}t}).$$

This gives $$v(s,0) = 4\tanh^{-1}b$$

which on comparison with (*), tells us that $b = ae^{2c\beta s}$. Therefore, we have obtained a (formal) solution

$$v(s,t) = 4\tanh^{-1}(ae2c\beta s - \tfrac{1}{\beta}t)$$

which we may rewrite as

$$v(s,t) = 4\tanh^{-1}(\exp(2c\beta s - \tfrac{1}{\beta}t + a)).$$

A short calculation shows that this indeed satisfies $v_{st} = -2c\sinh v$.

$k = 1$: The system becomes

$$v_s = 4c\beta i \cosh\frac{v}{2}$$

$$v_t = \frac{2}{\beta} i \cosh\frac{v}{2}$$

Letting $V(s) = \frac{1}{2}v(s,0)$ we obtain

$$\frac{dV}{\cosh V} = 2c\beta i\,ds$$

which leads to $v(s,0) = 2 \sinh^{-1}(\tan(2c\beta is+b_0))$. Defining $W(t) = \frac{1}{2} v(s,t)$ for each s leads to

$$\frac{dW}{\cosh W} = \frac{1}{\beta} i\, dt$$

which gives us $v(s,t) = 2 \sinh^{-1}(\tan(\frac{1}{\beta} it+a_0))$. On comparison with the above $v(s,0)$, we have

$$v(s,t) = 2 \sinh^{-1}(\tan(2c\beta is+ \frac{1}{\beta} it+b_0))$$

which we rewrite as

$$v(s,t) = 2 \tanh^{-1}(\sin i(2c\beta s+ \frac{1}{\beta} t+b)).$$

Direct verification shows that this satisfies

$$v_{st} = -2c \sinh v.$$

We summarize these results in:

Proposition 5.2. The trivial solution $u = 0$ to the sinh-Gordon equation generates the solution

(A) $v(s,t) = 4 \tanh^{-1}(\exp(2c\beta s- \frac{1}{\beta} t+a))$ via the parameter β.

The solution $u = \pi i$ generates

(B) $v(s,t) = 2 \tanh^{-1}(\sin i(2c\beta s+ \frac{1}{\beta} t+a))$ via the parameter β.

Next we will obtain real, periodic solutions from (B) by requiring

$$\mathrm{Re}(2c\beta s + \frac{1}{\beta} t + a) = 0.$$

Substituting $c = c_1 + ic_2$, $\beta = b_1 + ib_2$, $s = x + iy$, $t = x - iy$, $a = a_1 + ia_2$, this requirement leads to the system

$$2(c_1b_1 - c_2b_2) + \frac{b_2}{b_1^2+b_2^2} = 0$$

$$-2(c_1b_2 + c_2b_1) - \frac{b_2}{b_1^2+b_2^2} = 0$$

which reduces to the requirements

$$2c_1(b_1^2+b_2^2) = -1$$

$$c_2 = 0$$

$$a_1 = 0 \quad .$$

The solution (B) then becomes

$$\text{(B')} \qquad v(x,y) = 2\tanh^{-1}\left(\sin\left(\frac{2b_2}{b_1^2+b_2^2}x + \frac{2b_1}{b_1^2+b_2^2}y - a_2\right)\right)$$

and satisfies the equation

$$v_{xx} + v_{yy} = \frac{4}{b_1^2+b_2^2}\sinh v.$$

The solution (B') has singularities wherever $\sin X = \pm 1$, that is, on the lines

$$\frac{2b_2}{b_1^2+b_2^2}x + \frac{2b_1}{b_1^2+b_2^2}y - a_2 = \frac{\pi}{2} + k \ , \ k \in \mathbb{Z}.$$

Note that for each value of $b_1^2 + b_2^2$ we have a family of solutions in one-to-one correspondence with S^1. We make this explicit by substituting

$$b_1 = \frac{2\cos\Theta}{\lambda},\ b_2 = \frac{2\sin\Theta}{\lambda},\ b_1^2 + b_2^2 = \frac{4}{\lambda^2}$$

and summarizing our results in

Proposition 5.3. The equation $v_{xx} + v_{yy} = \lambda^2 \sinh v$

has real, periodic solutions

(B') $\quad v(x,y) = 2\tanh^{-1}(\sin(\lambda x \sin\Theta + \lambda y \cos\Theta + a))$

which have singularities on the lines

$$\lambda x \sin\Theta + \lambda y \cos\Theta + a = \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z}).$$

These solutions are generated by πi via $\beta = \frac{2}{\lambda} e^{i\Theta}$.

The family of solutoins (A) cannot be made both real and periodic. However, by similar considerations to those above, we can state this proposition:

Proposition 5.4. The equation $v_{xx} + v_{yy} = -\lambda^2 \sinh v$
has periodic solutions

(A') $\quad v(x,y) = 4\tanh^{-1}(a\, e^{i\lambda(x\sin\Theta + y\cos\Theta)}). \qquad (a \in \mathbb{C})$

These are generated by the trivial solution $v = 0$ via $\beta = \frac{2}{\lambda} e^{i}$, and have singularities on the lines

$$\lambda(x \sin\Theta + y\cos\Theta) = \pm i \log a \quad (\text{if } |a| = 1).$$

The Permutability Theorem can be used to generate more families of solutions to the sinh-Gordon equation. This work is still in progress.

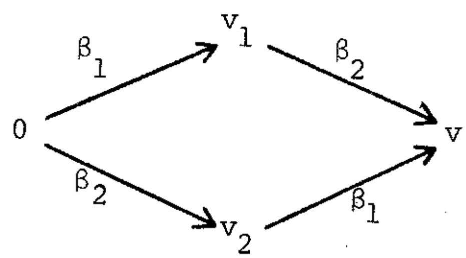
We include two families of second-generation solutions.

Let $\quad L_k(x,y) = x \sin \Theta_k + y \cos \Theta_k \quad , \quad k = 1,2,$

$$v_k(x,y) = 4 \tanh^{-1}[a_k e^{i\lambda L_k(x,y)}] \quad , \quad a_k \in \mathbb{C},$$

and $\quad \beta_k = \frac{2}{\lambda} e^{i\Theta_k}.$

Using the Bianchi diagram



we arrive at the family of solutions (A")

$$v(x,y) = 4 \tanh^{-1}[-i \cot(\frac{\Theta_1 - \Theta_2}{2}) \frac{a_1 e^{i\lambda L_1(x,y)} - a_2 e^{i\lambda L_2(x,y)}}{1 - a_1 a_2 e^{i\lambda(L_1(x,y)+L_2(x,y))}}]$$

to the equation $v_{xx} + v_{yy} = -\lambda^2 \sinh v$.

Similarly, using the family (B') we derive a family (B")

$$v(x,y) = 4 \tanh^{-1}[-i \cot(\frac{\Theta_1 - \Theta_2}{2}) \frac{\sin \lambda L_1(x,y) - \sin \lambda L_2(x,y)}{1+\cos \lambda(L_1(x,y)+L_2(x,y))}]$$

which satisfies $v_{xx} + v_{yy} = \lambda^2 \sinh v$.

We conclude by studying the behavior of the metric on $\mathcal{E}_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$ near the branch points P_ℓ. Let $p = \frac{m+1}{2}$ and $q = \frac{n+1}{2}$.

Choose a local coordinate z on $\mathcal{E}_{mn}$ near P_ℓ and a coordinate w on

$E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ near $\pi(p)$ as in the proof of Theorem 4.1. That is, rotation by $\frac{2\pi}{n+1}$ around P_ℓ is given by $z \mapsto ze^{\frac{2\pi i}{n+1}} = ze^{\frac{\pi i}{q}}$, and $w = z^q$. The metric on E_{mn} is $ds^2 = f(z)|dz|^2$, on $E_{mn}/\mathbb{Z}_p \times \mathbb{Z}_q$ its $\tilde{ds}^2 = g(w)|dw|^2$, where $ds^2 = \pi^*\tilde{ds}^2$.

The metric on E_{mn} is invariant under the rotation $z \mapsto ze^{\pi i/q}$. Writing $\Theta = ze^{\pi i/q}$,

then $$f(\Theta(z))\left|\frac{d\Theta}{dz}\,dz\right|^2 = f(z)|dz|^2$$

gives $$f(ze^{\pi i/q})\left|e^{\pi i/q}dz\right|^2 = f(z)|dz|^2$$

which leads to

$$f(ze^{\pi i/q}) = f(z). \qquad (*)$$

If we write $f(z) = \sum_{j,k=0}^{\infty} b_{jk}z^j\bar{z}^k$, where $b_{jk} = \overline{b_{kj}}$ since f is real, and $b_{00} \neq 0$, then (*) translates to

$$\sum_{j,k=0}^{\infty} b_{jk}z^j\bar{z}^k e^{\frac{\pi i}{q}(j-k)} = \sum_{j,k=0}^{\infty} b_{jk}z^j\bar{z}^k$$

and therefore $b_{jk} = b_{jk}e^{\frac{\pi i}{q}(j-k)}$ for $j,k = 0$ to ∞. If $b_{jk} \neq 0$, then we must have

$$\frac{\pi}{q}(j-k) = 2N\pi, \qquad N \in \mathbb{Z}.$$

This means $\quad j \equiv k \pmod{2q}$.

We may replace j by $r + 2qj$, k by $r + 2qk$, and b_{jk} by $a_{rjk} = b_{r+2qj,r+2qk}$ where $0 \leq r \leq 2q - 1$, and $j,k = 0$ to ∞. Then

$$f(z) = \sum_{rjk} a_{rjk} |z|^{2r} z^{2qj} \bar{z}^{2qk},$$

or

$$f(z) = \sum_{rjk} a_{rjk} |z|^{2r} z^{(n+1)j} \bar{z}^{(n+1)k}.$$

Now the requirement $ds^2 = \pi^* \tilde{ds}^2$ means

$$g(w(z)) \left|\frac{dw}{dz} dz\right|^2 = f(z) |dz|^2$$

$$g(z^q) |qz^{q-1} dz|^2 = f(z) |dz|^2.$$

Then
$$q(z^q) = \frac{1}{q^2} \frac{1}{|z|^{2(q-1)}} f(z)$$

$$= \frac{1}{q^2} \frac{1}{|z|^{n-1}} \sum_{rjk} a_{rjk} |z|^{2r} z^{(n+1)j} \bar{z}^{(n+1)k}.$$

Therefore

$$g(w) = \frac{1}{q^2} \frac{1}{|w|^{2(n-1)/(n+1)}} \sum_{rjk} a_{rjk} |w|^{\frac{4r}{n+1}} w^{2j} \bar{w}^{2k},$$

where $a_{ooo} \neq 0$, $a_{rjk} = \bar{a}_{rkj}$. Thus, near the branch points P_ℓ, the metric on $E_{mn}/\mathbb{Z}_{\frac{m+1}{2}} \times \mathbb{Z}_{\frac{n+1}{2}}$ has a singularity of the type

$$\frac{1}{|w|^{2(n-1)/(n+1)}} \qquad \text{for } n > 1.$$

The behavior at the branch points Q_ℓ is analogous, with m replacing n.

Appendix A

Basic Operators On Manifolds

Let M be a Riemannian manifold with metric $\langle , \rangle$. If $f : M \to \mathbb{R}$ is a smooth function on M, the gradient of f, ∇f, is defined to be the unique vector field on M satisfying

$$\langle \nabla f, Y \rangle = Y(f) = df(Y)$$

for all smooth vector fields Y on M. If $g : M \to \mathbb{R}$ is another smooth function on M, then

$$\nabla(fg) = f\nabla g + g\nabla f.$$

The expression for the gradient in local coordinates is

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i} = g^{ij} f_j \partial_i .$$

The divergence of a vector field X on M is a smooth function div X which may be defined pointwise as the trace of the linear map $Y \to \nabla_Y X$:

$$(\mathrm{div}\ X)(p) = \mathrm{trace}(Y \to \nabla_Y X)$$

for $p \in M$ and $Y \in T_pM$. If f is a smooth function on M then

$$\mathrm{div}(fX) = \langle \nabla f, X \rangle + f\ \mathrm{div}\ X.$$

Two local coordinate expressions for the divergence of $X = a^i \frac{\partial}{\partial x^i}$ are

$$\mathrm{div}(a^i \frac{\partial}{\partial x^i}) = \frac{\partial a^i}{\partial x^i} + a^j \Gamma^i_{ij}$$

and
$$\mathrm{div}(a^i \frac{\partial}{\partial x^i}) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(a^i \sqrt{g}).$$

The Laplacian of a smooth function f on M is defined to be the smooth function

$$\Delta f = \mathrm{div}(\nabla f).$$

If g is another such function,

$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle\nabla f, \nabla g\rangle.$$

Local coordinate expressions for the Laplacian are

$$\Delta f = g^{ij}(f_{ij} - f_k \Gamma^k_{ij})$$

and
$$\Delta f = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g}\, f_j).$$

The Hessian of a function f on M is defined to be the 2-form

$$\mathrm{Hess}\, f = \nabla df.$$

It satisfies the identity

$$\mathrm{Hess}(fg) = f\,\mathrm{Hess}\, g + g\,\mathrm{Hess}\, f + 2df \otimes dg.$$

In local coordinates

$$\text{Hess } f = (f_{ij} - f_k \Gamma^{i}_{ij})\, dx^i dx^j.$$

Now suppose that M is a compact manifold without boundary. The fundamental integration identifies involving these operators are the Divergence Theorem

$$\int_M (\text{div } X)\, dV = 0$$

and Green's formulas, which follow from the Divergence Theorem:

$$\int_M \langle \nabla f, \nabla g \rangle dV = -\int_M (f \Delta g)\, dV$$

$$\int_M (f \Delta g)\, dV = \int_M (g \Delta f)\, dV.$$

Here dV is the volume form on M.

Integrating the identity $\Delta(f^2) = 2f\Delta f + 2|\nabla f|^2$ over M and applying the Divergence Theorem, we obtain $\int f\Delta f = -\int |\nabla f|^2$. If we define an inner product (,) on $C^\infty(M)$ by $(f,g) = \int fg$, then we can write

$$(\Delta f, f) = -\int_M |\nabla f|^2 dV,$$

showing that Δ is a negative operator on M.

Appendix B

Surfaces In $\mathbb{R}^3$, Some Identities

In this appendix we review the basic geometry of surfaces in $\mathbb{R}^3$. Let $f : \sum \to \mathbb{R}^3$ be an immersion of a 2-manifold as a surface in $\mathbb{R}^3$. In terms of local coordinates (x^1,x^2), we will write the coordinate vector fields as $\frac{\partial}{\partial x^i}$ or ∂_i when thought of as derivations or vector fields on $\sum$, and as $\frac{\partial f}{\partial x^i}$ or f_i when thought of as vector fields in $\mathbb{R}^3$. If $\langle , \rangle$ is the standard Riemannian metric on $\mathbb{R}^3$, the coefficients g_{ij} of the induced metric on $\sum$ are defined by $g_{ij} = \langle f_i, f_j \rangle$. From the fact that the metric tensor is parallel with respect to the induced Riemannian connection we have

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^{\ell}_{ki} g_{\ell j} + \Gamma^{\ell}_{kj} g_{\ell i} .$$

The symbols g^{ij} are defined by the relations $g^{ij} g_{jk} = \delta^i_k$ so that in matrix terms $(g^{ij}) = (g_{ij})^{-1}$. Also, $g = \det(g_{ij})$.

Let ∇ be the standard Riemannian connection on $\mathbb{R}^3$, and n a unit normal vector field at least locally defined on $\sum$. If X,Y are tangent vector fields on $\sum$, the vector field $\nabla_X Y$ can be separated into components tangent and normal to $\sum$ by writing

$$\nabla_X Y = \tilde{\nabla}_X Y + B(X,Y)n.$$

In this equation $\tilde{\nabla}$ is the induced connection on $\sum$ and B is a symmetric 2-form called the second fundamental form of $\sum$.

Let us write $B = h_{ij}dx^i \otimes dx^j$ where dx^i are the dual 1-forms to the coordinate vector fields. Recalling the definition of the Christoffel symbols, $\tilde{\nabla}_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, the equation $\nabla_{\partial_i}\partial_j = \tilde{\nabla}_{\partial_i}\nabla\partial_j + B(\partial_i,\partial_i)n$ leads immediately to <u>Gauss' formulas</u>:

$$f_{ij} = \Gamma^k_{ij}\partial_k + h_{ij}n.$$

These in turn give us the relations $h_{ij} = \langle f_{ij}, n\rangle$ and $\langle f_{ij}, f_k\rangle = \Gamma^\ell_{ij}g_{\ell k}$.

The Weingarten map A is defined by $A(X) = -\nabla_X n$ for vectors X tangent to $\sum$. At each point $p \in \sum$, it is a linear transformation of the tangent space $T_p\sum$. For tangent vector fields X,Y on $\sum$ the fact that $\langle n,Y\rangle = 0$ leads to $0 = \nabla_X\langle n,Y\rangle = \langle\nabla_X n,Y\rangle + \langle n,\nabla_X Y\rangle$ and therefore, $B(X,Y) = \langle A(X),Y\rangle$. Since B is symmetric, A is self-adjoint.

Let (h^j_i) be the matrix representing A in the basis $\{\partial_1,\partial_2\}$ of T_pM. Then on one hand, $A(\partial_i) = -\nabla_{\partial_i}n = -n_i$, and on the other, $A(\partial_i) = h^j_i\partial_j$. Therefore, we have Weingarten's equations

$$n_i = -h^j_i\partial_j.$$

Furthermore, $h_{ij} = B(\partial_i, \partial_j) = \langle A(\partial_i), \partial_j\rangle = \langle h_i^k \partial_k, \partial_j\rangle = h_i^k g_{kj}$ gives us the relations

$$h_i^j = g^{jk} h_{ki}.$$

Differentiating Gauss' formulas we obtain $\langle f_{ijk}, n\rangle = \Gamma_{ij}^{\ell} h_{\ell k} + \dfrac{\partial h_{ij}}{\partial x^k}$. On the other hand, $\langle f_{ijk}, n\rangle = \langle f_{ikj}, n\rangle = \Gamma_{ik}^{\ell} h_{\ell j} + \dfrac{\partial h_{ik}}{\partial x^j}$. Therefore, we have the Codazzi-Mainardi equations

$$\frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} = \Gamma_{ik}^{\ell} h_{\ell j} - \Gamma_{ij}^{\ell} h_{\ell k}.$$

The eigenvalues λ_1 and λ_2 of the Weingarten map are called the principal curvatures. The mean curvature H is the average of the principal curvatures, or half the trace:

$$2H = \lambda_1 + \lambda_2 = \text{trace } A = h_i^i = g^{ij} h_{ji}.$$

It is an extrinsic curvature measure in the sense that it depends on the immersion. The Gauss curvature K is the product of the principal curvatures, or the determinant of the Weingarten map:

$$K = \lambda_1 \lambda_2 = \det(h_i^j).$$

It is an intrinsic curvature measure in the sense that it depends only on the metric.

Now we prove some identities found in the course of our work. Here ϕ is a smooth function on Σ.

(a) $\operatorname{div}(A(\nabla\phi)) = \langle B, \operatorname{Hess}\phi\rangle + 2\langle\nabla H, \nabla\phi\rangle$.

(b) $\langle B, \operatorname{Hess}\phi\rangle - 2H\Delta\phi = \operatorname{div}(A(\nabla\phi)) - 2\operatorname{div}(H\nabla\phi)$.

(c) If Σ is compact without boundary,

$$\int_\Sigma \langle B, \operatorname{Hess}\phi\rangle d\Sigma = \int_\Sigma 2H\Delta\phi\, d\Sigma.$$

Proof. The identity (c) follows immediately from (b), which in turn follows from (a) by noting $\langle\nabla H, \nabla\phi\rangle = \operatorname{div}(H\nabla\phi) - H\Delta\phi$. To prove (a): The Codazzi-Mainardi equations imply

$$\partial_i h^k_k - \partial_k h^k_i = h^\ell_i \Gamma^k_{\ell k} - h^\ell_k \Gamma^k_{\ell i}.$$

Therefore,

$$\begin{aligned} 2\langle\nabla H, \nabla\phi\rangle &= 2g^{ij} H_i \phi_j \\ &= g^{ij}(\partial_i h^k_k)\phi_j \\ &= g^{ij}\phi_j(\partial_k h^k_i + h^\ell_i \Gamma^k_{\ell k} - h^\ell_k \Gamma^k_{\ell i}) \end{aligned}$$

$$2\langle\nabla H, \nabla\phi\rangle = g^{ij}\phi_j(\partial_k h^k_i + h^\ell_i \Gamma^k_{\ell k}) - g^{\ell j}\phi_j h^k_i \Gamma^i_{k\ell}.$$

Also, $$\begin{aligned} \langle B, \operatorname{Hess}\phi\rangle &= \langle h_{ij} dx^i dx^j, (\phi_{k\ell} - \phi_m \Gamma^m_{k\ell}) dx^k dx^\ell\rangle \\ &= h^k_j g^{j\ell}(\phi_{k\ell} - \phi_m \Gamma^m_{k\ell}) \end{aligned}$$

$$\langle B, \operatorname{Hess}\phi\rangle = h^k_i g^{ij}\phi_{kj} - h^k_i g^{i\ell}\phi_j \Gamma^j_{k\ell}.$$

From $A(\nabla\phi) = h_i^k g^{ij}\phi_j \partial_k$ we calculate:

$$\begin{aligned}
\operatorname{div}(A\nabla\phi) &= \partial_k(h_i^k g^{ij}\phi_j) + h_i^\ell g^{ij}\phi_j \Gamma_{\ell k}^k \\
&= h_i^k g^{ij}\phi_{jk} + h_i^k\phi_j(\partial_k g^{ij}) + g^{ij}\phi_j(\partial_k h_i^k) + h_i^\ell g^{ij}\phi_j\Gamma_{\ell k}^k \\
&= h_i^k g^{ij}\phi_{jk} + h_i^k\phi_j(-\Gamma_{\ell k}^j g^{i\ell} - \Gamma_{\ell k}^i g^{j\ell}) + g^{ij}\phi_j(\partial_k h_i^k) \\
&\qquad + g^{ij}\phi_j h_i^\ell \Gamma_{\ell k}^k \\
&= \langle B, \text{Hess } \phi\rangle + 2\langle\nabla H, \nabla\phi\rangle.
\end{aligned}$$

QED

Appendix C

Surfaces in S^3

Here we present a treatment of surfaces in S^3 as immersions of Riemann surfaces, following Lawson [8].

Let $S^3 = \{(y_1, y_2, y_3, y_4, \in \mathbb{R}^4 : \sum y_i^2 = 1\}$ with the induced metric $<,>$. Let $\psi : R \to S^3$ be a conformal immersion of a Riemann surface R. If $z = x_1 + ix_2$ is a local complex coordinate on R and ds^2 the metric induced on R by ψ from $<,>$, we may write

$$ds^2 = 2F|dz|^2 = 2F(dx_1^2+dx_2^2).$$

We note that $2F = <\psi_1, \psi_1> = <\psi_2, \psi_2>$ and $<\psi_1, \psi_2> = 0$.

Let $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $\overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$, and $\partial\overline{\partial} = \frac{1}{4}(\partial_1^2 + \partial_2^2)$. Then the Gauss curvature of the metric induced by ψ is

$$K = -\frac{1}{F}\partial\overline{\partial} \log F.$$

Let n be a unit vector field normal to R but tangent to S^3. Then the coefficients of the second fundamental form of R in S^3 are given by $h_{ij} = <\psi_{ij}, n>$. Since $\psi \wedge \psi_1 \wedge \psi_2$ is perpendicular to ψ_1 and ψ_2 it is normal to R. Since it is perpendicular to ψ, it is tangent to S^3. And since

$$\|\psi\wedge\psi_1\wedge\psi_2\|^2 = \begin{vmatrix} <\psi,\psi> & <\psi,\psi_1> & <\psi,\psi_2> \\ <\psi_1,\psi> & <\psi_1,\psi_1> & <\psi_1,\psi_2> \\ <\psi_2,\psi> & <\psi_2,\psi_1> & <\psi_2,\psi_2> \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2F & 0 \\ 0 & 0 & 2F \end{vmatrix} = 4F^2$$

We may write $n = \frac{1}{2F} \psi \wedge \psi_1 \wedge \psi_2$ and therefore

$$h_{ij} = \frac{1}{2F} \psi \wedge \psi_1 \wedge \psi_2 \wedge \psi_{ij}.$$

A simple calculation shows that $\frac{1}{iF} \psi \wedge \partial\psi \wedge \bar{\partial}\psi = \frac{1}{2F} \psi \wedge \psi_1 \wedge \psi_2$ and therefore the second fundamental form may be written as

$$B(X,Y) = \frac{1}{iF} \psi \wedge \partial\psi \wedge \bar{\partial}\psi \wedge \nabla_x \nabla_y \psi.$$

The second fundamental form, metric, and Weingarten map satisfy

$$G = \det(h_i^j) = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{h_{11}h_{22} - h_{12}^2}{4F^2}.$$

Now recalling that for surfaces in the sphere, $G = K - 1$, we obtain the <u>Gauss curvature equation</u>:

$$4F^2(1-K) = h_{12}^2 - h_{11}h_{22}.$$

The immersion ψ is called minimal if the mean curvature vanishes, i.e. $h_1^1 + h_2^2 = 0$. Since $h_i^j = h_{ij}/2F$, this implies $h_{11} + h_{22} = 0$. This is equivalent to the equation

$$\partial\bar{\partial}\psi = -F\psi.$$

To see this, note that

$$h_{11} + h_{22} = \frac{1}{2F} \psi \wedge \psi_1 \wedge \psi_2 \wedge (\psi_{11} + \psi_{22})$$

$$= \frac{4}{iF} \psi \wedge \partial\psi \wedge \bar{\partial}\psi \wedge \partial\bar{\partial}\psi$$

and therefore $h_{11} + h_{22} = 0$ if and only if $\partial\bar{\partial}\psi$ is a linear combination of ψ, $\partial\psi$, and $\bar{\partial}\psi$. Now we apply the following lemma.

Lemma $\langle \partial^k\psi, \partial^\ell\psi \rangle = \langle \bar{\partial}^k\psi, \bar{\partial}^\ell\psi \rangle = 0$ for $1 \leqq k + \ell \leqq 3$.

$$\langle \partial\bar{\partial}\psi, \partial\psi \rangle = \langle \partial\bar{\partial}\psi, \bar{\partial}\psi \rangle = 0$$

$$\langle \partial\psi, \bar{\partial}\psi \rangle = -\langle \partial\bar{\partial}\psi, \psi \rangle = F$$

Proof. Using $\langle \psi_1, \psi_2 \rangle = 0$ and $\langle \psi_1, \psi_1 \rangle = \langle \psi_2, \psi_2 \rangle = F$ and simple calculations give

$$\langle \partial\psi, \partial\psi \rangle = \langle \bar{\partial}\psi, \bar{\partial}\psi \rangle = 0, \ \langle \partial\psi, \bar{\partial}\psi \rangle = F.$$

Differentiating the identity $\langle \psi, \psi \rangle = 1$ immediately gives $\langle \partial\psi, \psi \rangle = \langle \bar{\partial}\psi, \psi \rangle = 0$. The others are obtained by applying these and repeated differentiations.

QED

If we write $\partial\bar{\partial}\psi = \lambda\psi + \mu\partial\psi + \nu\bar{\partial}\psi$, applying the above identities leads to $\lambda = -F$, $\mu = \nu = 0$. If on the other hand we assume $\partial\bar{\partial}\psi = -F\psi$, then from the expression for $h_{11} + h_{22}$ we immediately see that ψ is minimal.

Lemma. Let $\phi = \frac{1}{2}(h_{11} - ih_{12})$. If ψ is minimal, the differential form $\omega = \phi dz^2$ is holomorphic.

Proof. A short calculation shows that $\phi = \frac{1}{iF}\, \psi \wedge \partial\psi \wedge \bar{\partial}\psi \wedge \partial^2\psi$. Therefore,

$$\phi^2 = -\frac{1}{F^2}(\psi\wedge\partial\psi\wedge\bar{\partial}\psi\wedge\partial^2\psi)^2$$

$$= -\frac{1}{F^2}\begin{vmatrix} \langle\psi,\psi\rangle & \langle\psi,\partial\psi\rangle & \langle\psi,\bar{\partial}\psi\rangle & \langle\psi,\partial^2\psi\rangle \\ \langle\partial\psi,\psi\rangle & \langle\partial\psi,\partial\psi\rangle & \langle\partial\psi,\bar{\partial}\psi\rangle & \langle\partial\psi,\partial^2\psi\rangle \\ \langle\bar{\partial}\psi,\psi\rangle & \langle\bar{\partial}\psi,\partial\psi\rangle & \langle\bar{\partial}\psi,\bar{\partial}\psi\rangle & \langle\bar{\partial}\psi,\partial^2\psi\rangle \\ \langle\partial^2\psi,\psi\rangle & \langle\partial^2\psi,\partial\psi\rangle & \langle\partial^2\psi,\bar{\partial}\psi\rangle & \langle\partial^2\psi,\partial^2\psi\rangle \end{vmatrix}$$

$\phi^2 = \langle\partial^2\psi,\partial^2\psi\rangle$, by the previous lemma. Now ϕ is holomorphic if and only if $\bar{\partial}\phi = 0$, this is equivalent to the Cauchy-Riemann equations.

$$\begin{aligned} \bar{\partial}\phi^2 &= \bar{\partial}\langle\partial^2\psi,\partial^2\psi\rangle \\ &= 2\langle\bar{\partial}\partial^2\psi,\partial^2\psi\rangle \\ &= 2\langle\partial(\partial\bar{\partial}\psi),\partial^2\psi\rangle \\ &= -2\langle\partial(F\psi),\partial^2\psi\rangle \\ &= -2F\langle\partial\psi,\partial^2\psi\rangle - 2(\partial F)\langle\psi,\partial^2\psi\rangle \\ &= 0 \end{aligned}$$

QED

If ψ is minimal, the Gauss curvature equation becomes $4F^2(1-K) = (h_{11}^2+h_{12}^2)$ and thus

$$F^2(1-K) = |\phi|^2.$$

Therefore,

Proposition. The Gauss curvature of a minimal surface in S^3 satisfies $K \leq 1$, and $K = 1$ precisely at the isolated zeroes of the holomorphic differential ω.

A non-zero meromorphic quadratic differential on a compact Riemann surface of genus g has degree

$$4g - 4 = \#\text{ zeroes} - \#\text{ poles},$$

counting multiplicity. Thus on the sphere (g=0), it must have poles. Since ω is holomorphic, it must be identically zero. Then $K = 1$ identically and the immersion is totally geodesic. If $g \geq 1$, ω has $4g - 4$ zeroes since it has no poles. On the torus (g=1), ω has no zeroes and hence $K < 1$.

Now we'll give a geometric interpretation of the order of the zeroes of ω. First, some vocabulary.

The k-jet of ψ at p is the linear subspace of $\mathbb{R}^4$ spanned by the derivatives of ψ up to and including the k^{th} order ones.

$$\text{k-jet of } \psi \text{ at } p = L\{\frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} : 0 \leq i + j \leq k\}.$$

Thus, the 1-jet of ψ at p is the 3-dimensional subspace of $\mathbb{R}^4$ spanned by $\psi(p)$, $\psi_1(p)$, $\psi_2(p)$. The 2-jet is spanned by ψ, ψ_1, ψ_2, ψ_{11}, ψ_{12}, ψ_{22} at p. Call the 1-jet at P, P_p.

Let S_p denote the geodesic 2-sphere which is tangent to the immersed surface at $\psi(p)$. Note that $S_p = P_p \cap S^3$. ($P_p \cap S^3$ is the intersection of a hyperplane through the origin with S^3, and therefore is a geodesic 2-sphere.

Its tangent space at p is spanned by ψ_1 and ψ_2, i.e. the same as the immersed surface. So it must be S_p.) The <u>order of contact</u> $\mathcal{O}_p$ of ψ with S_p at p is the largest integer k such that P_p contains the k-jet of ψ at p. Since P_p is the 1-jet, $\mathcal{O}_p \geqq 1$. The <u>degree of spherical flatness</u> of ψ at p is $d_p = \mathcal{O}_p - 1$.

Now, since $\phi = h_{11} - ih_{12}$ and $h_{11} = -h_{22}$, ω is zero at p if and only if the second fundamental form vanishes at p. And this occurs if and only if for each pair i,j:

$$0 = h_{ij}(p) = \frac{1}{2F}\psi(p) \wedge \psi_1(p) \wedge \psi_2(p) \wedge \psi_{ij}(p).$$

$\iff$ $\psi(p)$, $\psi_1(p)$, $\psi_2(p)$, $\psi_{ij}(p)$ are linearly dependent for each pair i,j.

$\iff$ The 2-jet of ψ at p is contained in the 1-jet P_p.

Therefore, ω has a zero at p if and only if the 2-jet of ψ at p is contained in P_p. More generally, the order of the zero is k if and only if the (k+1)-jet is contained in P_p, i.e., if and only if the order of contact is k + 1.

This gives us the desired geometric interpretation: The order of the zero of ω at p is precisely the degree of spherical flatness of ψ at p.

The degree d_p can be measured at any point p of the surface as follows. Small neighborhoods of p on the surface are divided by S_p like a pie into $2d_p + 4$ wedge-like regions. The surface crosses from above to below S_p from wedge to wedge producing a pattern of + and −. Generically, graphing a minimal surface in such a manner over its tangent geodesic 2-sphere produces a saddle, and the pattern is $\begin{array}{c|c} + & - \\ \hline - & + \end{array}$. Here, $d_p = 0$, ω does not vanish and $K < 1$. At points where there are more than four wedges (if $g > 1$ such points must occur), $d_p > 0$, ω has a zero and $K = 1$.

Appendix D

Tori in $\mathbb{R}^4$

Here we record a calculation of the minimum total mean curvature of a special class of tori in $\mathbb{R}^4$. Given embeddings $\gamma_i : S^1 \to \mathbb{R}^2$, $i = 1,2$, we will show that $\int H^2 dS \geq 2\pi^2$ for the product $f = \gamma_1 \times \gamma_2 : S^1 \times S^1 \to \mathbb{R}^4$, with equality if and only if γ_1 and γ_2 are circles of equal length.

We suppose that γ_1 and γ_2 are parametized by arc-length s_1 and s_2, respectively. Then $\left\|\frac{d\gamma_i}{ds_i}\right\| = 1$ and $\frac{d^2\gamma_i}{ds_i^2} = \varkappa_i \eta_i$ where $\varkappa_i$ is the curvature of γ_i and η_i is a unit normal field to γ_i in $\mathbb{R}^2$.

Writing vectors in $\mathbb{R}^4$ as (v_1,v_2) where $v_i \in \mathbb{R}^2$, and $\cdot$ for the standard innerproduct in $\mathbb{R}^4$, the coefficients of the metric on $\gamma_1 \times \gamma_2$ in the coordinates (s_1,s_2) are

$$\begin{pmatrix} \frac{\partial f}{\partial s_1}\cdot\frac{\partial f}{\partial s_1} & \frac{\partial f}{\partial s_1}\cdot\frac{\partial f}{\partial s_2} \\ \frac{\partial f}{\partial s_2}\cdot\frac{\partial f}{\partial s_1} & \frac{\partial f}{\partial s_2}\cdot\frac{\partial f}{\partial s_2} \end{pmatrix} = \begin{pmatrix} (\frac{d\gamma_1}{ds_1},0)\cdot(\frac{d\gamma_1}{ds_1},0) & (\frac{d\gamma_1}{ds_1},0)\cdot(0,\frac{d\gamma_2}{ds_2}) \\ (0,\frac{d\gamma_2}{ds_2})\cdot(\frac{d\gamma_1}{ds_1},0) & (0,\frac{d\gamma_2}{ds_2})\cdot(0,\frac{d\gamma_2}{ds_2}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The coefficients of the second fundamental form in direction $\xi_1 = (n_1,0)$ are

$$\begin{pmatrix} \frac{\partial^2 f}{\partial s_1^2}\cdot\xi_1 & \frac{\partial^2 f}{\partial s_1 \partial s_2}\cdot\xi_1 \\ \frac{\partial^2 f}{\partial s_2 \partial s_1}\cdot\xi_1 & \frac{\partial^2 f}{\partial s_2^2}\cdot\xi_1 \end{pmatrix} = \begin{pmatrix} \frac{d^2\gamma_1}{ds_1^2}\cdot\xi_1 & 0 \\ 0 & \frac{d^2\gamma_2}{ds_2^2}\cdot\xi_1 \end{pmatrix} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and those in direction $\xi_2 = (0,n_2)$ are

$$\begin{pmatrix} 0 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

The Weingarten map in direction ξ_1 is then given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and in direction ξ_2 it is given by

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

The mean curvature normal is then

$$\vec{H} = \frac{1}{2}\sum_{i=1}^{2}(\text{trace } A_i)\xi_i = \frac{1}{2}(\kappa_1\xi_1+\kappa_2\xi_2)$$

and finally the mean curvature is given by

$$H^2 = \frac{1}{4}(\kappa_1^2+\kappa_2^2).$$

Let ℓ_i be the length of γ_i. Then the total mean curvature is

$$\int H^2 ds = \int_0^{\ell_1}\int_0^{\ell_2} \frac{1}{4}(\kappa_1^2+\kappa_2^2)\,ds_1 ds_2 = \frac{\ell_2}{4}\int_0^{\ell_1}\kappa_1^2 ds_1 + \frac{\ell_1}{4}\int_0^{\ell_2}\kappa_2^2 ds_2.$$

By the Cauchy-Schwarz inequality,

$\int_0^{\ell_i}\kappa_i^2 ds_i \geq \frac{1}{\ell_i}[\int_0^{\ell_i}\kappa_i ds_i]^2$ with equality if and only if κ_i is constant.

By Fenchel's Theorem, $\int_0^{\ell_i}\kappa_i ds_i \geq 2\pi$ with equality if and only if γ_i is a plane convex curve. Therefore,

$$\int H^2 ds \geq \frac{1}{4}\frac{\ell_2}{\ell_1}\left[\int_0^{\ell_1} \kappa_1^2 ds_1\right]^2 + \frac{1}{4}\frac{\ell_1}{\ell_2}\left[\int_0^{\ell_2} \kappa_2^2 ds_2\right]^2$$

$$\geq \frac{1}{4}\frac{\ell_2}{\ell_1}(2\pi)^2 + \frac{1}{4}\frac{\ell_1}{\ell_2}(2\pi)^2$$

$$= \left(\frac{\ell_2}{\ell_1} + \frac{\ell_1}{\ell_2}\right)\pi^2$$

$\geq 2\pi^2$, with equality if and only if γ_1 and γ_2 are circles of equal length.

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