Foliations with Ehresmann Connections

by

Alan Stuart Horwitz

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Abstract of the Dissertation

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An Ehresmann connection for a foliation is a distribution which is complementary to the leaves and has properties similar to those of an Ehresmann connection on fiber bundles. In particular, a horizontal curve (ie. a curve which is tangent to the complementary distribution) can be lifted along any intersecting vertical curve (ie. curve in a leaf) to a horizontal path which depends only on the homotopy class of the vertical curve. Not every foliation has an Ehresmann connection, and even when one exists, not every distribution complementary to the foliation qualifies. The only examples of Ehresmann connections, described by Blumenthal and Hebda, are the orthogonal distributions to foliations that are totally geodesic or riemannian, or a product of these types.

This thesis begins with a discussion of flows with Ehresmann connections. The perturbed Hopf flow on S^3 is shown to be an example of such a foliation which is not of the types mentioned above.

Another major concern of this thesis is the growth of leaves in foliations

of compact manifolds with Ehresmann connections. Blumenthal and Hebda have shown that the universal covers of leaves are diffeomorphic, and they have observed, in the case of totally geodesic foliations, that this diffeomorphism is an isometry. In the thesis, two proofs are given to show that, in the riemannian case, the universal covers are quasi isometric. One proof uses a simple estimate to measure how much neighborhoods in a leaf grow as they are mapped to another leaf by an element of holonomy along a horizontal path. The estimate holds for certain non riemannian foliations. In the other proof, the riemannian foliation is lifted to a transversally parallelisable foliation of the orthonormal frame bundle and the lifted leaves are shown to be quasi isometric. From these proofs, it follows that the growth of a fundamental group of a compact leaf in a riemannian foliation places an upper bound on the growth of any other leaf. A finer estimate can be obtained by measuring the growth of an appropriate quotient of this fundamental group.

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INTRODUCTION

The motivation for this thesis was a paper [BH2] in which the following two theorems appear:

Theorem 1 *Let $\mathcal{F}$ be a smooth foliation of connected manifold M admitting an Ehresmann connection D. If $\mathcal{F}$ has a compact leaf L_0 with $H_D(L_0, p_0)$ (*see Chapter 4 for definition*) finite, then every leaf L is compact with $H_D(L, p)$ finite.*

Theorem 2 *Let F be a totally geodesic foliation of a connected, complete riemannian manifold M. If F has a leaf L_0, of finite volume with $H_D(L, p)$ finite, then every leaf has finite volume with $H_D(L, p)$ finite, where the Ehresmann connection is $D = (T\mathcal{F})^{\perp}$.*

A problem, proposed by my advisor, was to determine, for foliations of compact manifolds with Ehresmann connections, whether all leaves have polynomial growth when the finiteness condition in the hypothesis of each theorem is weakened to requiring that $H_D(L_0, p_0)$ have polynomial growth.

My best conclusion, which is stated as follows, falls short of a statement about all foliations with Ehresmann connections.

Corollary 4.24 *Let $D = (T\mathcal{F})^{\perp}$ be an Ehresmann connection and suppose that $\mathcal{F}$ is totally geodesic or that [2.3] holds. For any compact $L \in \mathcal{F}$, the growth type of $H_D(L)$ bounds the growth type of any other leaf from above.*

The preceeding corollary follows, indirectly, from Theorem 4.9, which states that the universal covers of the leaves have the same growth type. The proof of this theorem depends on results from Chapters 2 and 4 of the thesis. In Chapter 2, a variational argument is used to derive estimates for the energies of vertical paths in rectangles. From these estimates, a comparison between the lengths of vertical paths in rectangles is derived in Chapter 4, and used to show that a diffeomorphism between the universal covers of leaves in a foliation which satisfies condition [2.3] is a quasi isometry. The preceeding result holds for riemannian foliations since they satisfy [2.3].

A study of riemannian foliations from a different point of view, in Chapters 5 and 6, leads to an alternate proof of Corollary 4.24. A proof of a theorem of Molino, that the transverse orthonormal frame bundle of a riemannian foliation $\mathcal{F}$ has a transversally parallelisable foliation $\mathcal{F}_T$, obtained by lifting $\mathcal{F}$, is given in Chapter 5. Also in this chapter, transversally parallelisable foliations of compact manifolds are shown to have Ehresmann connections and leaves, all with the same growth type. In Chapter 6, a proof, different from that in [BH1], shows that a riemannian foliation $\mathcal{F}$ has an Ehresmann connection induced from that of its lift to the transversally parallelisable foliation, $\mathcal{F}_T$, of the transverse orthonormal frame bundle. The alternate proof of Corollary 4.24, for riemannian foliations, follows from observing that the lift $L_T \in \mathcal{F}_T$ of a leaf $L \in \mathcal{F}$ is a covering space of L with fundamental group $\pi_1(L_T) = K_D(L)$. (see Ch.4, p. 42 for definition)

Flows with Ehresmann connections are studied in Chapters 2 and 3. A sufficient condition for a flow to have an Ehresmann connection is discussed in Chapter 2. In Chapter 3, the perturbed Hopf flow on S^3 is shown to be an example of a foliation with an Ehresmann connection which is neither riemannian nor totally geodesic, nor a product of these types. The only examples of foliations with Ehresmann connections which are currently in the literature are the types mentioned above.

CHAPTER 1

In this chapter, we define an Ehresmann connection for a foliation and describe some examples of foliations with Ehresmann connections.

Let $\mathcal{F}$ be a codimension q foliation on a riemannian manifold M and consider a q dimensional distribution D which is transverse to $\mathcal{F}$, that is, $TM = T\mathcal{F} + D$. A curve is said to be horizontal if it is tangent to D and vertical if it lies in a leaf of $\mathcal{F}$.

Definition 1.1 *A rectangle $\delta(s,t)$ is a piecewise smooth map*

$$\delta : [a,b] \times [c,d] \to M$$

such that for each s, the path $t \to \delta(s,t)$ is vertical and for each t, the path $s \to \delta(s,t)$ is horizontal with respect to D.

For convenience, parameters s and t will often be chosen so that δ : $[0,1] \times [0,T] \to M$, for some $T > 0$. We call $\delta(s,0)$ the initial horizontal edge and $\delta(0,t)$ the initial vertical edge. Paths $\delta(s,T)$ and $\delta(1,t)$ are the terminal horizontal and vertical edges, respectively.

For a vertical path $\tau(t)$ and a horizontal path $\sigma(s)$ such that $\sigma(0) = \tau(0)$, we say that a rectangle $\delta(s,t)$ is determined by τ and σ provided $\delta(s,0) = \sigma(s)$ and $\delta(0,t) = \tau(t)$. From [BH1], τ and σ determine at most one rectangle, so δ is unique.

For any choice of D, a rectangle always exists, locally. One can see this as follows. The collection $\{U_i, \varphi_i\}$ of local submersion charts for $\mathcal{F}$ consists of open neighborhoods and locally defined submersions of rank q such that

$M = \cup_i U_i$ and $\mathcal{F}$ is determined on U_i by φ_i. Suppose that τ and σ lie in the domain of a local submersion chart $\{U, \varphi\}$. A unique rectangle δ in U is determined by letting, for each t', $\delta(s,t')$ be the unique horizontal path for which

$$\varphi\delta(s,t') = \varphi\sigma(s)$$

and

$$\delta(0,t') = \tau(t')$$

To show δ does not depend on $\{U, \varphi\}$, suppose that τ and σ lie in the domain of another local submersion chart $\{U', \varphi'\}$ and determine a rectangle δ' such that

$$\varphi'\delta'(s,t') = \varphi'\sigma(s)$$

and

$$\delta'(0,t') = \tau(t').$$

Claim that $\delta'(s,t') \equiv \delta(s,t')$. There is a local diffeomorphism $\gamma : \varphi'(U \cap U') \to \varphi(U \cap U')$ such that $\varphi = \gamma \circ \varphi'$, thus

$$\varphi\delta'(s,t') = \gamma \circ \varphi'\delta'(s,t') = \gamma \circ \varphi'\sigma(s)$$

agrees with $\varphi\sigma(s) = \gamma \circ \varphi'\sigma(s)$, so uniqueness implies that $\delta'(s,t') \equiv \delta(s,t')$.

A rectangle can always be constructed when one of τ or σ is sufficiently short. To see this when σ is short, cover τ with a finite collection $\{U_i, \varphi_i\}_{i=1}^n$ of submersion charts such that $U_i \cap U_{i-1} \neq \emptyset$. On U_1, a rectangle $\delta_1(s,t)$ is determined by $\tau|_{U_1}$ and $\sigma|_{U_1}$, and on U_i, a rectangle $\delta_i(s,t)$ is determined by $\tau|_{U_i}$ and the portion of the terminal horizontal edge of δ_{i-1} in $U_i \cap U_{i-1}$. A rectangle δ is contained in $\cup_{i=1}^n \delta_i$ which has all of $\tau(t)$, $t \in [0,1]$ as its initial vertical edge and a portion of $\sigma\mid_{U_1}$ as the initial horizontal edge. The argument is similar for constructing a rectangle when τ is short.

Let $L_0, L_1 \in \mathcal{F}$ such that $\sigma(0) \in L_0$ and $\sigma(1) \in L_1$. The horizontal path σ determines the germ of a local diffeomorphism from L_0 to L_1 as follows. Let us, first, suppose that σ lies entirely in the domain of a local submersion chart $\{U, \varphi\}$. For any vertical path τ in $U \cap L_0$ such that $\tau(0) = \sigma(0)$ and $\tau(1) = x$, there is a rectangle $\delta(s,t)$ with τ and σ as initial vertical and horizontal edges, respectively. An **element of holonomy** along σ is a map from $U \cap L_0$ onto $U \cap L_1$ which sends $x \mapsto \delta(1,1)$. Since, beginning at any $x \in U \cap L_0$, there is a unique horizontal path in U whose image under φ agrees with $\varphi\sigma$, we see that the element of holonomy along σ does not depend on the particular rectangle $\delta(s,t)$; if δ' is determined by σ and some other path τ in $U \cap L_0$ from $\sigma(0)$ to x, then δ' and δ share a terminal horizontal edge, that is $\delta'(s,1) \equiv \delta(s,1)$, so $\delta'(1,1) = \delta(1,1)$. An inverse map is provided by an element of holonomy along $\sigma(1-s)$, thus σ determines the germ of a local diffeomorphism from L_0 to L_1. Suppose, now, that σ lies in the domains of several local submersion charts, then a unique element of holonomy along σ can be assembled by cutting down the domains and piecing together elements of holonomy along small segments of σ. The element of holonomy maps an open neighborhood, $U \cap L_0$, of $\sigma(0)$, diffeomorphically onto an open neighborhood, $U \cap L_1$, of $\sigma(1)$, by sending $x \mapsto \delta(1,1)$, where δ is the rectangle determined by $\tau(t), t \in [0,1]$ and σ, for any path τ in $U \cap L_0$ joining $\sigma(0)$ and x.

Let τ be a vertical path with $\tau(0) = p$ and suppose $\tau(t)$ lies in a submersion chart $\{U, \varphi\}$ for $0 \leq t \leq \epsilon$. Let v_p be a horizontal vector at p, ie. $\mathrm{v}_p \in D_p$. At $\tau(\epsilon)$, there is a unique horizontal vector $\mathrm{v}_{\tau(\epsilon)} \in D_{\tau(\epsilon)}$ such that $d\varphi\mathbf{v}_{\tau(\epsilon)} = d\varphi\mathbf{v}_p$. For $\tau(t)$, $0 \le t \le \epsilon$ lying in some other submersion chart $\{U', \varphi'\}$ and $\mathbf{v}'_{\tau(\epsilon)} \in D_{\tau(\epsilon)}$ such that $d\varphi'\mathbf{v}'_{\tau(\epsilon)} = d\varphi'\mathbf{v}_p$, we see that $\mathbf{v}_{\tau(\epsilon)} = \mathbf{v}'_{\tau(\epsilon)}$, since $d\varphi\mathbf{v}'_{\tau(\epsilon)} = d\varphi \circ d\varphi'^{-1} \circ d\varphi'\mathbf{v}'_{\tau(\epsilon)} = d\varphi \circ d\varphi'^{-1} \circ d\varphi'\mathbf{v}_p = d\varphi\mathbf{v}_p$, where the map $d\varphi'^{-1}$ "lifts" vectors from $T_{\varphi'(p)}\varphi'(U')$ to $D_{\tau(\epsilon)}$. The linear map from D_p to $D_{\tau(\epsilon)}$ which sends $\mathbf{v}_p \mapsto \mathbf{v}_{\tau(\epsilon)}$ is called the **linear germinal holonomy** along $\tau\mid_U$, and the local picture extends to give a linear holonomy map along all of τ, from D_p to $D_{\tau(1)}$. Linear holonomy along vertical paths is used in Ch.6, although barely mentioned by name.

Definition 1.2 *A transverse distribution D is said to be an Ehresmann connection for a foliation $\mathcal{F}$ if for every vertical path τ and horizontal path σ with $\tau(0) = \sigma(0)$, there exists a rectangle $\delta(s,t)$ determined by τ and σ.*

The following is an example of a distribution $D \cap T\mathcal{F}$ which is not an Ehresmann connection for $\mathcal{F}$.

Example 1.3 *Consider the one dimensional foliation $\mathcal{F}$, (drawn in Fig. 1 with solid lines) on R^2 with the usual metric, and let the dotted lines represent the direction of $D = (T\mathcal{F})^{\perp}$. For leaves L_0 and L_1, let vertical path τ and horizontal path σ begin at $p \in L_0$ and end at points $r \in L_0$ and $q \in L_1$, respectively.. Claim that τ and σ do not determine a rectangle. For a rectangle to exist, its terminal vertical edge must lie in L_1, but this isn't possible since horizontal paths near r don't meet L_1. Thus, $D = (T\mathcal{F})^{\perp}$ is not an Ehresmann connection for $\mathcal{F}$.*

We, now, recall some examples of foliations with Ehresmann connections which appear in [BH1].

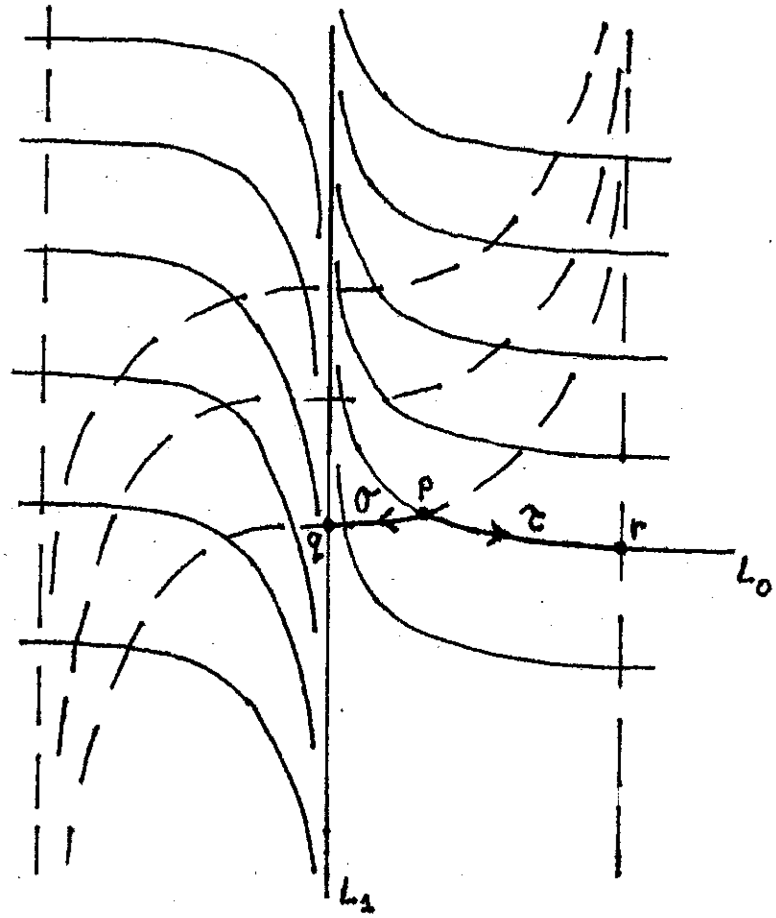

Figure 1.

Example 1.4 *The orthogonal distribution $D = (T\mathcal{F})^{\perp}$ is an Ehresmann connection for a riemannian foliation $\mathcal{F}$ on a manifold M, with a bundlelike metric.*

The explanation in [BH1] involves the following definition from [KN]. Given a manifold M of dimension n, we choose a fixed frame of n orthonormal tangent vectors to identify vectors in the tangent plane T_pM with points in R^n in the obvious way. The development of a curve c into $T_{c(0)}M$ is defined as a path $\underline{c}$ in $T_{c(0)}M$ obtained by integrating in R^n,

$$\underline{c}(s') = \int_0^{s'} P_{c(1-u)} \left(\frac{dc}{du}_{| \ u=l}\right) dl,$$

where $P_{c(1-u)} \left(\frac{dc}{du}_{| \ u=l}\right)$ denotes the parallel translation of $\frac{dc}{du}_{| \ u=l}$, backward along $c(u)_{| \ u\in[0,l]}$ to a vector in $T_{c(0)}M$. The definition does not depend on the choice of the orthonormal frame which is used to identify $T_{c(0)}M$ with R^n. We, also, note from [KN] that a path $\underline{c}(s)$ in T_pM can be undeveloped into a curve $c(s)$ in M with $c(0) = p$, when M is complete.

The development of a horizontal path $\sigma(s)$ into the q dimensional horizontal distribution $D_{\sigma(0)}$ is more complicated. The difficulty with the previous definition is that parallel translation preserves D if and only if D is integrable, so $P_{\sigma(1-u)}\ (\frac{d\sigma}{du}|_{u=l})$ may not be in $D_{\sigma(0)}$, in general. For σ lying in a submersion chart $\{\ U, \varphi\ \}$ of a codimension q foliation $\mathcal{F}$, the development, $\underline{\sigma}$, of $\sigma(s)$ into $D_{\sigma(0)}$ is defined, locally, by letting $\overline{\sigma}$ be the development of the path $\varphi \circ \sigma(s)$ into the tangent space

$$d\varphi\ D_{\sigma(0)} \ = \ T_{\sigma(0)}\ \varphi(U),$$

then letting $\underline{\sigma}(s) \ = \ d\varphi^{-1}\ \overline{\sigma}(s)$ be the lift of $\overline{\sigma}$ to $D_{\sigma(0)}$. This definition is independent of the choice of local submersion chart. When σ is covered by intersecting submersion charts $\{\ U_i, \varphi_i\ \}$ and $\{\ U_j, \varphi_j\ \}$, parallel translation along $\varphi_i \circ \sigma(s)$ can be continued to parallel translation along $\varphi_j \circ \sigma(s)$ by a change of coordinates $\gamma_{ji} \ = \ \varphi_j \circ \varphi_i^{-1}$, so development can be defined along σ, globally. We, also, mention that paths in $D_{\sigma(0)}$ can be undeveloped when M is horizontally complete, that is, when horizontal geodesics can be extended indefinitely.

Returning to the example in [BH1], the horizontal paths of a rectangle δ with initial vertical edge τ and initial horizontal edge σ are determined as follows. The path σ can be developed into the distribution D_p to obtain a path C_0. The riemannian foliation $\mathcal{F}$ on M, with a bundlelike metric, is determined by local submersion charts $\{U, \varphi\}$, where φ is a riemannian submersion, ie. $d\varphi : D_p \to T_{\varphi(p)}\varphi(U)$ is a linear isometry. It follows that *the linear germinal holonomy along τ is an isometry from D_p to $D_{\tau(t')}$, for each t'. Let $C_{t'}$ be the image of C_0 under the holonomy along τ. Since M is complete, there exists a horizontal path which develops into $D_{\tau(t')}$ as the path $C_{t'}$. We let $\delta(s,t')$ be this horizontal path.*

Example 1.5 *A distribution $D = (T\mathcal{F})^{\perp}$, for a totally geodesic foliation $\mathcal{F}$ on a compact riemannian manifold M is an Ehresmann connection. From [BH1], the vertical paths of a rectangle δ with initial vertical edge τ and initial horizontal edge σ are determined as follows. Using the riemannian connection on the leaves, τ can developed into $T_p\mathcal{F}$ as a curve C_0. The differential of the element of holonomy along $\sigma(s)$, $0 \leq s \leq s'$ is a map which sends C_0 to a curve $C_{s'}$ in $T_{\sigma(s')}\mathcal{F}$. Since the leaves are complete in the induced riemannian metrics, there exists a vertical curve whose development into $T_{\sigma(s')}\mathcal{F}$ is $C_{s'}$. We let $\delta(s',t)$ be this vertical curve.*

We mention some basic properties of rectangles, for foliations with an Ehresmann connection.

Remark 1.6 In a leaf L, let τ and τ' be fixed endpoint homotopic paths (with homotopy through paths in L) which begin at a point p. Let σ be a horizontal path beginning at p. Then from [BH1], the rectangles δ and δ' with initial vertical edges τ and τ', respectively, and with the same initial horizontal edge σ, have a common terminal horizontal edge and terminal vertical edges which are homotopic through vertical paths. (See Fig. 2)

We mention, without proof, a result which appears in [BH1].

Proposition 1.7 *If M is connected and D is an Ehresmann connection for $\mathcal{F}$ then any two leaves of $\mathcal{F}$ may be joined by a horizontal curve.*

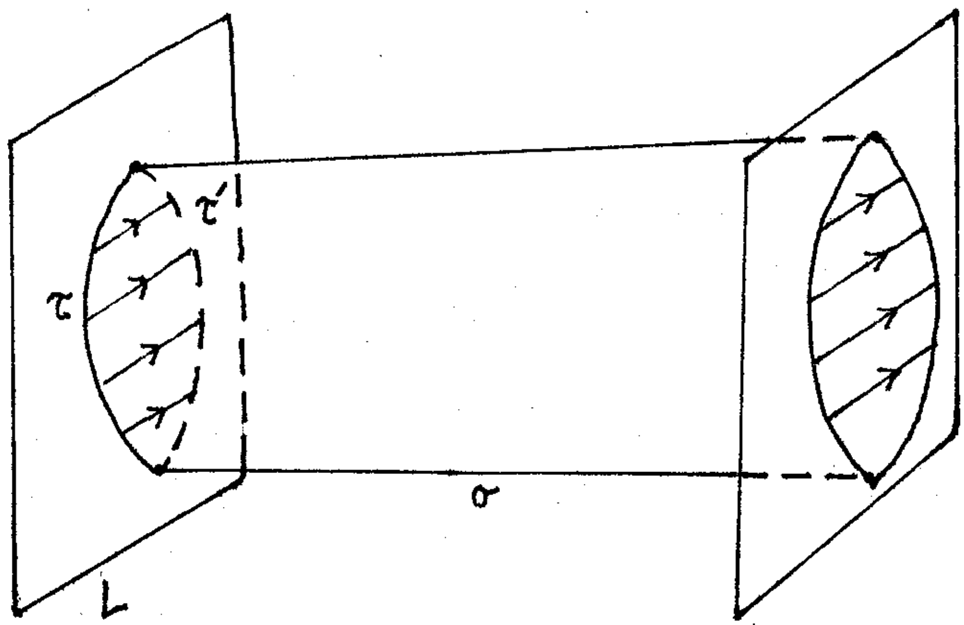

Figure 2

Example 1.8 *The Reeb foliation $\mathcal{F}$ of an annulus (see Fig. 3) cannot have an Ehresmann connection since any path from L_1 to L_2 must be tangent to $\mathcal{F}$ at some point. Also, the Reeb foliation of a solid torus, M, does not have an Ehresmann connection, otherwise, from Proposition 3.1 of [BH1],* $\tilde{M} = R^2 \times R$.

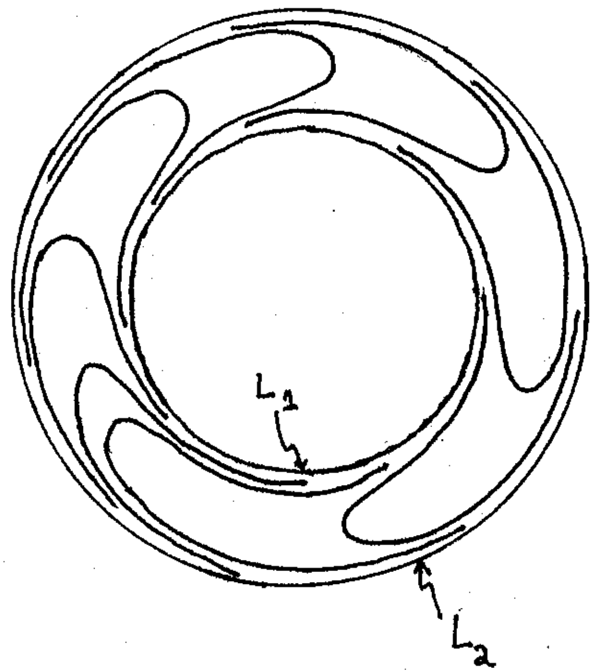

Figure 3

CHAPTER 2

In this section, estimates are derived to compare the lengths of vertical paths in rectangles and applied to the case of foliated flows. This comparison is used in a proof of a sufficient condition for the orthogonal distribution to a flow to be an Ehresmann connection. We show that riemannian flows satisfy this condition and, thus, have an Ehresmann connection.

For most of this chapter, let $\mathcal{F}$ be a smooth m - dimensional foliation on a compact riemannian manifold M. For each leaf $L \in \mathcal{F}$, consider the second fundamental form $S_L = \nabla - \nabla^L$, for riemannian connection ∇ on M and the connection ∇^L of the induced metric on L. On the intersection of a leaf L with a neighborhood U_l with local coordinates, $\{x_1^l, ..., x_m^l\}$, we can express

$$S_L(\mathbf{v},\mathbf{w}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{h}_{ij}^l dx_i^l \otimes dx_j^l(\mathbf{v},\mathbf{w})$$

where $\mathbf{v},\mathbf{w} \in TL$ and the $\mathbf{h}_{ij}^l$ in $(T\mathcal{F})^\perp$ are locally defined vector fields. By choosing a partition of unity $\{f_l\}$ subordinate to a finite cover $\{U_l\}$ of M, we can express

$$S_L(\mathbf{v},\mathbf{w}) = \sum_l \sum_{i=1}^{m}\sum_{j=1}^{m} f_l \mathbf{h}_{ij}^l dx_i^l \otimes dx_j^l(\mathbf{v},\mathbf{w}).$$

The vector fields $f_l \mathbf{h}_{ij}^l$ are continuous on M so for some K, the finite sum $\sum_l \|f_l \mathbf{h}_{ij}^l\| < K$. For vectors in TL expressed, locally, as

$$\mathbf{v} = \sum_{i=1}^{m} v_i^l \frac{\partial}{\partial x_i^l}$$

and

$$\mathbf{w} = \sum_{j=1}^{m} w_j^l \frac{\partial}{\partial x_j^l},$$

an estimate

$$|dx_i^l \otimes dx_j^l(\mathbf{v},\mathbf{w})| = |v_i^l||w_j^l| \leq \|\mathbf{v}\|\|\mathbf{w}\|$$

holds. The preceeding inequalities imply that

$$\|S_L(\mathbf{v},\mathbf{w})\| \leq \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l}\|f_l\mathbf{h}_{ij}^l\|\|\mathbf{v}\|\|\mathbf{w}\| \leq \sum_{i=1}^{m}\sum_{j=1}^{m} K\|\mathbf{v}\|\|\mathbf{w}\|,$$

that is,

$$\|S_L(\mathbf{v},\mathbf{w})\| \leq m^2 K\|\mathbf{v}\|\|\mathbf{w}\| \tag{2.1}$$

Proposition 2.2 *Let the distribution $D = (T\mathcal{F})^{\perp}$, with respect to some metric on M. Suppose the following holds:*

For each horizontal $\sigma(s)$, $\exists$ an integrable function $f : R \mapsto R$ so that any rectangle $\delta(s,t), (s,t) \in [0,1] \times [0,T]$, with initial horizontal edge $\delta(s,0) = \sigma(s)$ satisfies

$$\|\frac{\partial\delta(s,t)}{\partial s}\| \leq f(\|\frac{\partial\delta(s,0)}{\partial s}\|) \tag{2.3}$$

Then $\exists$ a constant $C > 0$, which is determined by the second fundamental form on M, such that

$$E_T(a)e^{-C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds} \leq E_T(b) \leq E_T(a)e^{C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds},$$

where $E_T(s)$ denotes the energy of vertical path $\delta(s,t)_{|\ t\in[0,T]}$. That is, the energies of vertical paths in δ have the same growth type, as functions of T.

proof

Let $0 \leq a < b \leq 1$. We wish to compare the energies $E_T(a)$ and $E_T(b)$ of vertical paths $\delta(a,t)$ and $\delta(b,t)$, $t \in [0,T]$, respectively. Observe that

$$\begin{aligned}\frac{dE_T(s)}{ds} &= \frac{d}{ds}\int_0^T < \frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t} > dt \\ &= \int_0^T \frac{d}{ds} < \frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t} > dt \\ &= 2\int_0^T < \nabla_{\frac{\partial\delta}{\partial s}} \frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t} > dt \\ &= 2\int_0^T < \nabla_{\frac{\partial\delta}{\partial t}} \frac{\partial\delta}{\partial s}, \frac{\partial\delta}{\partial t} > dt\end{aligned}$$

since $[\frac{\partial\delta}{\partial s}, \frac{\partial\delta}{\partial t}] = \mathbf{O}$.

Because $\frac{\partial\delta}{\partial t}$ is vertical and $\frac{\partial\delta}{\partial s}$ is horizontal, we have

$$\frac{dE_T(s)}{ds} = 2\int_0^T (\frac{d}{dt} < \frac{\partial\delta}{\partial s}, \frac{\partial\delta}{\partial t} > - < \frac{\partial\delta}{\partial s}, \nabla_{\frac{\partial\delta}{\partial t}} \frac{\partial\delta}{\partial t} >)dt = -2\int_0^T < \frac{\partial\delta}{\partial s}, \nabla_{\frac{\partial\delta}{\partial t}} \frac{\partial\delta}{\partial t} > dt$$

Letting L_s denote the leaf which contains path $t \mapsto \delta(s,t)$, we can write

$$\begin{aligned}< \frac{\partial\delta}{\partial s}, \nabla_{\frac{\partial\delta}{\partial t}} \frac{\partial\delta}{\partial t} > &= < \frac{\partial\delta}{\partial s}, S_{L_s}(\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t}) + \nabla^{L_s}_{\frac{\partial\delta}{\partial t}} \frac{\partial\delta}{\partial t} > \\ &= < \frac{\partial\delta}{\partial s}, S_{L_s}(\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t}) >,\end{aligned}$$

the last inequality follows since $\frac{\partial\delta}{\partial s} \perp \nabla^{L_s}_{\frac{\partial\delta}{\partial t}} \frac{\partial\delta}{\partial t}$ because D is orthogonal to $T\mathcal{F}$. Using the Schwarz inequality, hypothesis [2.3], and inequality [2.1], we get

$$| < \frac{\partial\delta}{\partial s}, S_{L_s}(\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t}) > | \leq \|\frac{\partial\delta}{\partial s}\|\|S_{L_s}(\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t})\| \leq f(\|\frac{\partial\delta}{\partial s}(s,0)\|)m^2K\|\frac{\partial\delta}{\partial t}\|^2.$$

therefore,

$$\begin{aligned}|\frac{dE_T(s)}{ds}| &\leq 2\int_0^T | < \frac{\partial\delta}{\partial s}, S_{L_s}(\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t}) > |dt \leq 2\int_0^T f(\|\frac{\partial\delta}{\partial s}(s,0)\|)m^2K\|\frac{\partial\delta}{\partial t}\|^2 dt \\ &= 2f(\|\frac{\partial\delta}{\partial s}(s,0)\|)2m^2KE_T(s). \qquad (2.4)\end{aligned}$$

Let

$$C = 2m^2K, \tag{2.5}$$

then

$$|\frac{dlnE_T(s)}{ds}| = |\frac{1}{E_T(s)}\frac{dE_T(s)}{ds}| \leq f(\|\frac{\partial\delta}{\partial s}(s,0)\|)\ C\ .$$

By integrating on $[a, b]$, we get

$$|\int_a^b \frac{dlnE_T(s)}{ds}ds| \leq \int_a^b |\frac{dlnE_T(s)}{ds}|ds \leq C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds$$

or

$$|ln\frac{E_T(b)}{E_T(a)}| \leq C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds,$$

therefore,

$$-C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds \leq ln\frac{E_T(b)}{E_T(a)} \leq C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds$$

and

$$E_T(a)e^{-C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds} \leq E_T(b) \leq E_T(a)e^{C\int_a^b f(\|\frac{\partial\delta}{\partial s}(s,0)\|)ds}. \tag{2.6}$$

Thus, $E_T(a)$ and $E_T(b)$ have the same growth type, as functions of T. □

Let $\mathcal{F}$ be a riemannian foliation and let $D = (T\mathcal{F})^\perp$, with respect to a bundlelike metric on M. Recall, from [1.4], p.9, that $\mathcal{F}$ is locally determined by riemannian submersions and that the linear germinal holonomy along vertical paths $t \mapsto \delta(s,t)$ is an isometry, so

$$\|\frac{\partial\delta}{\partial s}(s,t)\| = \|\frac{\partial\delta}{\partial s}(s,0)\|, for\ all\ t.$$

Since the hypothesis [2.3] is satisfied, with the function $f(x) = x$, we have

Corollary 2.7 *Let $\mathcal{F}$ be a riemannian foliation and let $D = (T\mathcal{F})^{\perp}$, with respect to a bundlelike metric on M. Then*

$$E_T(a)e^{-C\int_a^b \|\frac{\partial\delta}{\partial s}(s,0)\|ds} \leq E_T(b) \leq E_T(a)e^{C\int_a^b \|\frac{\partial\delta}{\partial s}(s,0)\|ds}$$

where C is as in the proof of Proposition 2.2.

Corollary 2.8 *Let $\mathcal{F}$ be a totally geodesic foliation and let $D = (T\mathcal{F})^{\perp}$. Then the vertical paths of any rectangle δ have the same energy.*

proof

On every leaf L, the second fundamental form $S_L \equiv 0$ so it follows from the proof of Proposition 2.2 that $|\frac{dE_T(s)}{ds}| \equiv 0$. □

Let $L_T(a)$ denote the length of the vertical path $\delta(a,t)$ in a rectangle $\delta(s,t), (s,t) \in [0,1] \times [0,T]$. We can say

$$L_T(a) = \int_0^T \|\frac{\partial\delta}{\partial t}(a,t)\|dt \leq (\int_0^T \|\frac{\partial\delta}{\partial t}\|^2 dt)^{\frac{1}{2}}(\int_0^T 1^2 dt)^{\frac{1}{2}} = (E_T(a))^{\frac{1}{2}}(T)^{\frac{1}{2}}. \tag{2.9}$$

If $\|\frac{\partial\delta}{\partial t}(a,t)\| \equiv constant$, then

$$L_T(a) = (E_T(a))^{\frac{1}{2}}(T)^{\frac{1}{2}} \tag{2.10}$$

Lemma 2.11 *If $\delta(a,t)$ is a piecewise geodesic curve in leaf L_a, then $L_T(a) = (E_T(a))^{\frac{1}{2}}(T)^{\frac{1}{2}}$*

proof

$$\frac{d}{dt}\|\frac{\partial\delta}{\partial t}(a,t)\|^2 = 2 < \nabla_{\frac{\partial\delta}{\partial t}}\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t} > = 2 < \nabla^{L_a}_{\frac{\partial\delta}{\partial t}}\frac{\partial\delta}{\partial t} + S_{L_a}(\frac{\partial\delta}{\partial t}, \frac{\partial\delta}{\partial t}), \frac{\partial\delta}{\partial t} > = 0$$

since $\delta(a,t)$ is a geodesic with respect to ∇^{L_a} and $S_{L_a} \perp TL_a$. The conclusion follows from the preceeding remarks.□

We wish to compare the lengths $L_T(0)$ and $L_T(1)$.

Proposition 2.12 *Suppose, for some C and rectangle δ, that the following condition holds:*

$$E_T(0)e^{-C} \leq E_T(1) \leq E_T(0)e^{C}. \tag{2.13}$$

Suppose, further, that $\delta(0,t)$ is a piecewise geodesic curve in L_0. Then $L_T(1) \leq L_T(0)e^{\frac{C}{2}}$

proof

From remarks preceeding the last lemma,

$$L_T(1) \leq (E_T(1))^{\frac{1}{2}}(T)^{\frac{1}{2}} \leq (E_T(0)e^{C})^{\frac{1}{2}}(T)^{\frac{1}{2}} = L_T(0)e^{\frac{C}{2}}.□$$

For the remainder of Chapter 2, suppose that $\mathcal{F}$ is a 1 dimensional foliation.

Proposition 2.14 *Let a distribution $D = (T\mathcal{F})^{\perp}$, for a 1 dimensional foliation $\mathcal{F}$. Suppose that $\mathcal{F}$ is totally geodesic, or that hypothesis [2.3] is satisfied. Then for any rectangle $\delta(s,t), (s,t) \in [0,1]\times[0,T]$, $\exists$ a constant C such that*

$$L_T(0)e^{\frac{-C}{2}} \leq L_T(1) \leq L_T(0)e^{\frac{C}{2}}.$$

proof

From the hypothesis [2.3] and Proposition [2.2],

$$E_T(0)e^{-C} \leq E_T(1) \leq E_T(0)e^{C}$$

where

$$C = 2m^2K\int_0^1 f(\|\frac{\partial\delta(s,0)}{\partial s}\|)ds$$

By changing the t parameter of δ so that $\delta(0,t)$ has the arclength parameter, the path $\delta(0,t)$ is a geodesic in the leaf L_0. Length remains unchanged under reparameterization, so we can express the conclusion of Proposition 2.12, in terms of the original t parameter of δ, as $L_T(1) \leq L_T(0)e^{\frac{C}{2}}$. By changing t to make $\delta(1,t)$ a geodesic in leaf L_1, Proposition 2.12, applied to the rectangle $\delta(1-s,t)$, implies that $L_T(0) \leq L_T(1)e^{\frac{C}{2}}$, with C as above, therefore,

$$L_T(0)e^{\frac{-C}{2}} \leq L_T(1) \leq L_T(0)e^{\frac{C}{2}}.$$

When $\mathcal{F}$ is totally geodesic, $C = 0$ in [2.13], so $L_T(0) = L_T(1)$.□

The next proposition gives a sufficient condition for 1 dimensional foliations to have an Ehresmann connection. We note that in [2.6], the term $\int_a^b f(\|\frac{\partial\delta}{\partial s}\|)ds$ can be replaced with $sup_{s\in[a,b]}f(\|\frac{\partial\delta}{\partial s}(s,0)\|)(b-a)$ to give the inequality,

$$E_T(a)e^{C(a-b)} \leq E_T(b) \leq E_T(b)e^{C(b-a)}, \qquad (2.15)$$

where $C = 2m^2Ksup_{s\in[0,1]}f(\|\frac{\partial\delta}{\partial s}(s,0)\|)$.

Proposition 2.16 *Let $\mathcal{F}$ be a 1 dimensional foliation on M and let $D = (T\mathcal{F})^{\perp}$. Suppose that the hypothesis [2.3] is satisfied. Then D is an Ehresmann connection.*

proof

Let $\tau(t)_{t\in[0,T]}$ and $\sigma(s)_{s\in[0,1]}$ be vertical and horizontal paths, respectively, beginning at point p. For sufficiently small $\epsilon > 0$, τ and $\sigma(s)_{s\in[0,\epsilon]}$ determine a rectangle. Let ϵ_M be the supremum of $\epsilon \in [0,1]$ such that there is a rectangle $\delta(s,t)$ with $\delta(s,0) = \sigma(s)$ and $\delta(0,t) = \tau(t)$ for $(s,t) \in [0,\epsilon] \times [0,T]$. (see Fig. 4)

We, first, show that $\epsilon_M = 1$, so that τ and σ determine a rectangle for $(s,t) \in [0,1) \times [0,T]$.

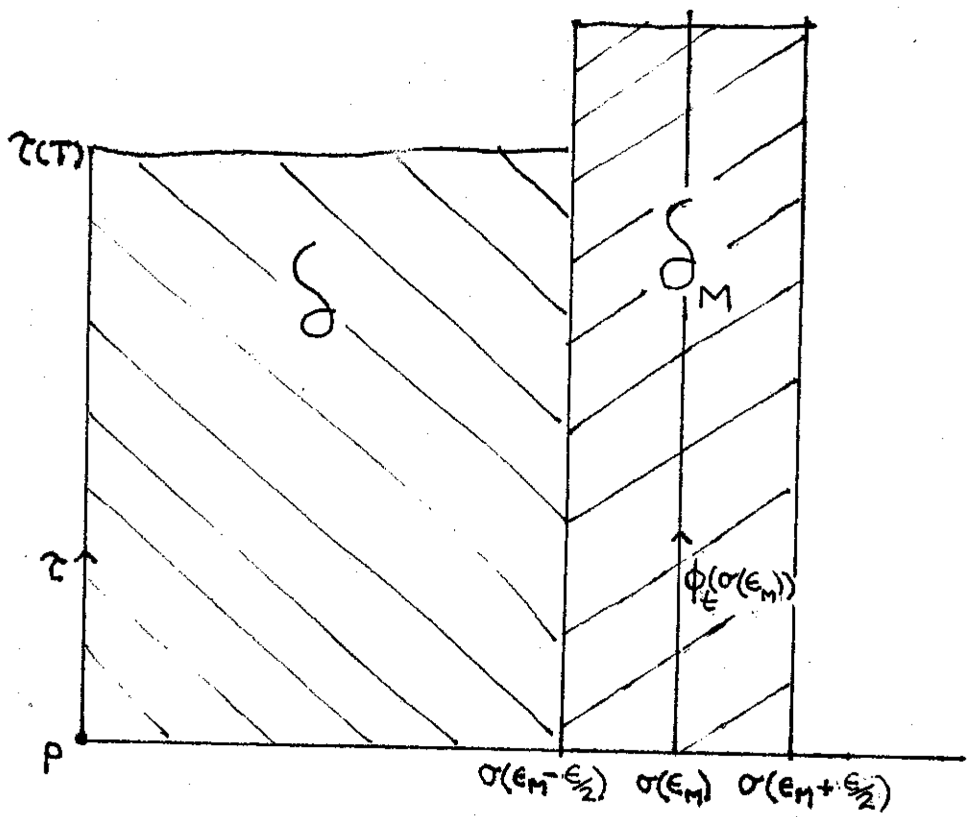

Figure 4

Suppose $\epsilon_M < 1$. Let $\phi_t(\sigma(\epsilon_M))$ be a vertical path through $\sigma(\epsilon_M)$ when $t = 0$ and choose T_M so that the length of path $\phi_t(\sigma(\epsilon_M))_{t\in[0,T_M]}$ is equal to $(length\ \tau(t)_{t\in[0,T]})e^{\frac{C}{2}\epsilon_M}$, where $C = 2m^2 K sup_{s\in[0,1]} f(\|\frac{d\sigma(s)}{ds}\|)$ with K as in [2.1].

For sufficiently small $\epsilon > 0$, there is a rectangle $\delta_M(s,t), (s,t) \in [\frac{-\epsilon}{2}, \frac{\epsilon}{2}] \times [0, T_M]$ such that $\delta_M(0,t) = \phi(\sigma(\epsilon_M))$ and $\delta_M(s,0) = \sigma(\epsilon_M - s)$. Let $E_{T_M}(s)$ and $L_{T_M}(s)$ denote the energy and length, respectively, of the vertical path $\delta(s,t)_{t\in[0,T_M]}$. From [2.3] and [2.15], we have $E_{T_M}(0)e^{-c\frac{\epsilon}{2}} \leq E_{T_M}(\frac{\epsilon}{2}) \leq E_{T_M}(0)e^{c\frac{\epsilon}{2}}$, with C as above. From proposition [2.14], it follows that

$$L_{T_M}(0)e^{-C\frac{\epsilon}{4}} \leq L_{T_M}(\frac{\epsilon}{2}) \leq L_{T_M}(0)e^{C\frac{\epsilon}{4}}.$$

From the inequality on the left, the length of the leftmost vertical edge in δ_M is bounded from below by length$(\phi_t(\sigma(\epsilon_M)_{t\in[0,T_M]})e^{-C\frac{\epsilon}{4}}$, which equals length$(\tau(t)_{t\in[0,T]})e^{\frac{C}{2}\epsilon_M}e^{\frac{-C\epsilon}{4}}$.

Choose a rectangle $\delta(s,t)$ with $(s,t) \in [0, \epsilon_M - \frac{\epsilon}{2}] \times [0,T]$ such that $\delta(s,0) = \sigma(s)$ and $\delta(0,t) = \tau(t)$. The rightmost vertical edge $\delta(\epsilon_M - \frac{\epsilon}{2}, t)$ lies on the same leaf as the path $\delta_M(\frac{\epsilon}{2}, t)$. We will show that δ_M can be patched to δ to form a rectangle determined by τ and $\sigma(s)_{s\in[0,\epsilon_M + \frac{\epsilon}{2}]}$.

Let $L_T(s)$ denote the length of $\delta(s,t)_{t\in[0,T]}$, then we have the following inequality,

$$L_T(0)e^{\frac{-C}{2}(\epsilon_M - \frac{\epsilon}{2})} \leq L_T(\epsilon_M - \frac{\epsilon}{2}) \leq L_T(0)e^{\frac{C}{2}(\epsilon_M - \frac{\epsilon}{2})}.$$

From the inequality on the right, the length of the rightmost vertical edge of δ is less than length$(\tau(t)_{t\in[0,T]})e^{\frac{C}{2}(\epsilon_M - \frac{\epsilon}{2})}$, which is less than the length of the leftmost vertical edge of δ_M. The t parameter of δ_M can be changed so that $\delta_M(\frac{\epsilon}{2}, t) = \delta(\epsilon_M - \frac{\epsilon}{2}, t)$ on $t \in [0,T]$.

The s parameter of δ_M can be reversed to define a rectangle $\delta'_M(s,t) =$

$\delta_M(\frac{\epsilon}{2} - s, t)$. The union

$$\{\delta(s,t),(s,t) \in [0,\epsilon_M - \frac{\epsilon}{2}] \times [0,T]\} \cup \{\delta'_M(s,t),(s,t) \in [0,\epsilon] \times [0,T]\}$$

is a rectangle determined by τ and σ for $(s,t) \in [0,\epsilon_M + \frac{\epsilon}{2}] \times [0,T]$, contrary to the definition of ϵ_M.

It follows that $\epsilon_M = 1$. By patching a rectangle δ_M to δ when $\epsilon_M = 1$, as in the preceeding arguments, we obtain a rectangle determined by τ and all of the horizontal path $\sigma(s)_{s\in[0,1]}\square$.

Corollary 2.17 *If $\mathcal{F}$ is a 1 dimensional riemannian foliation, then $D = (T\mathcal{F})^{\perp}$ is an Ehresmann connection.*

CHAPTER 3

In this chapter, we describe a foliation, which is not riemannian, totally geodesic, nor a product of these types, and we show, directly, that it has an Ehresmann connection.

Consider $S^3 \subset C^2$ as the unit circle

$$\{(z_1, z_2) \in C^2 : \ \|z_1\|^2 + \|z_2\|^2 = 1\}$$

in the complex plane. Using the circular orbits $\{z_1 = 0\}$ and $\{z_2 = 0\}$ as references, we assign to a point $p \in S^3$ coordinates $(r(p), \theta(p), \psi(p))$, as in (Fig. 5).

These are related to the coordinates on the complex plane by

$$(z_1, z_2) = (re^{i\theta}, \sqrt{1 - r^2}e^{i\psi}).$$

Notice that $\{z_1 = 0\} = \{p \in S^3 \mid r(p) = 0\}$ and $\{z_2 = 0\} = \{p \in S^3 \mid r(p) = 1\}$. We choose the metric on S^3 so that the vector fields $\partial_\theta, \partial_\psi$ and ∂_r are mutually perpendicular and

$$\|\partial_r\| = 1, \|\partial_\theta\| = r \ and \ \|\partial_\psi\| = \sqrt{1 - r^2}$$

so that ∂_θ and ∂_ψ vanish on $\{z_1 = 0\}$ and $\{z_2 = 0\}$, respectively. The tangent space T_pS^3 is spanned by ∂_θ, ∂_ψ, and ∂_r at points p such that $0 < r(p) < 1$. On $\{z_1 = 0\}$, T_pS^3 is spanned by ∂_ψ and a perpendicular plane $\{\psi \equiv constant\}$ and on $\{z_2 = 0\}$, by ∂_θ and the perpendicular plane $\{\theta \equiv constant\}$.

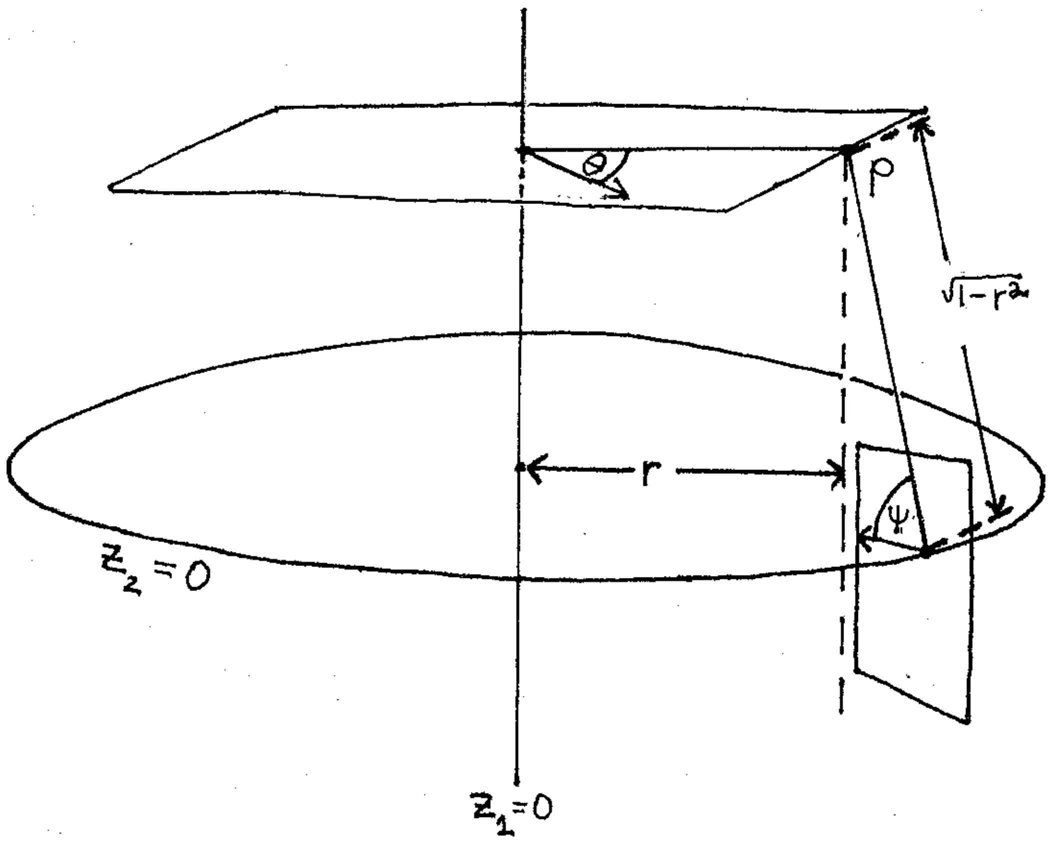

Figure 5

The perturbed Hopf flow on S^3 (see Fig. 6) is defined as the integral flow of the vector field

$$\frac{\partial \phi_t}{\partial t}_{|(r(p),\theta(p),\psi(p))} = \lambda_p(r)X_2 + \partial_\theta + \partial_\psi, \tag{3.1}$$

where $X_2 = \partial_r$ and λ_p is the bump function (see Fig. 7), with $\frac{d^{(k)}}{dr^{(k)}}\lambda_p(r) = 0$, $\forall k$ at $r = 0$, $r = \frac{1}{2}$ and $r = 1$. Note that this integral flow includes the closed orbits $\{z_1 = 0\}$ and $\{z_2 = 0\}$.

The perturbed Hopf flow on S^3 is not riemannian since, the distance between leaves is not constant. In fact, from [Ca1], the lens space $L_{p,q}$ and $S^2 \times S^1$ are the only 3 dimensional manifolds which can have a riemannian flow with exactly 2 closed orbits.

Further, the perturbed Hopf flow is not geodesible for any choice of metric

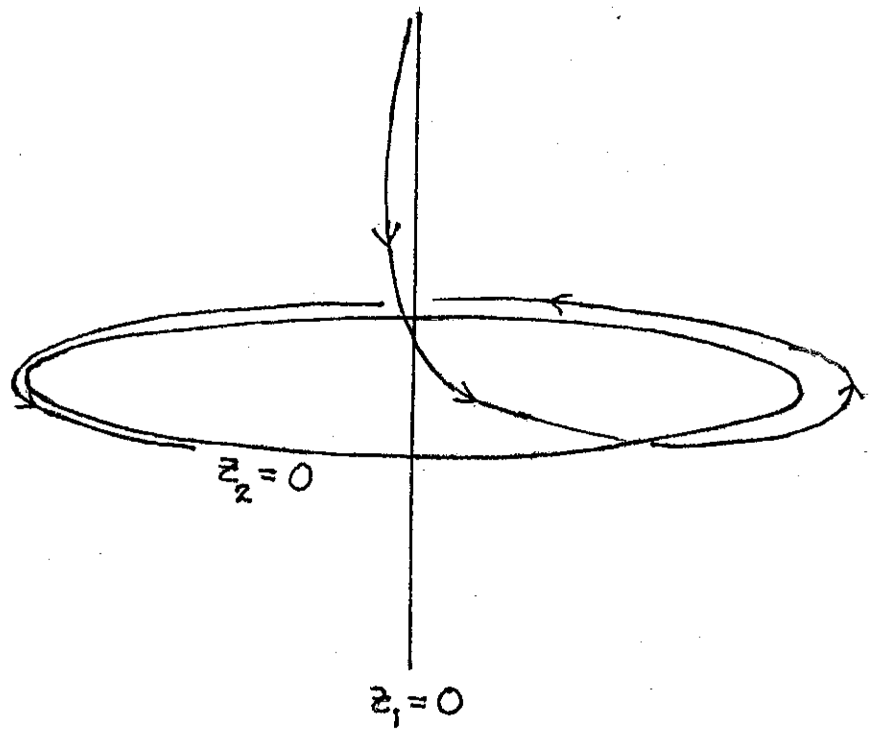

Figure 6: orbit of perturbed Hopf flow

on S^3, since [Gl] shows that a closed orbit can be approximated, arbitrarily closely, by the boundary of a 2 chain which is tangent to the flow, and this, together with a theorem of Sullivan, implies the result.

In the remainder of this chapter, we show that the perturbed Hopf flow has an Ehresmann connection.

We describe a distribution D on S^3. Let

$$X_1 = (1 - \lambda(r))\partial_\theta - \lambda(r)\partial_\psi, \tag{3.2}$$

where the function λ (see Fig.8) satisfies $\frac{d^{(k)}}{dr^{(k)}}\lambda(r) = 0,\ \forall k$, when $r \leq \frac{1}{4}$ and $r \geq \frac{3}{4}$. Both X_1 and X_2 vanish at p on $\{z_1 = 0\}$ and $\{z_2 = 0\}$ and as $r(p) \to 0$ *and* 1, the plane spanned by X_1 and X_2 becomes tangent to the planes $\{\psi \equiv constant\}$ and $\{\theta \equiv constant\}$, respectively. Let D be spanned by X_1 and X_2 when $0 < r(p) < 1$, and let D be tangent to $\{\psi \equiv constant\}$ and $\{\theta \equiv constant\}$ when $r = 0$ and $r = 1$, respectively.

We will show, eventually, that D is an Ehresmann connection for the

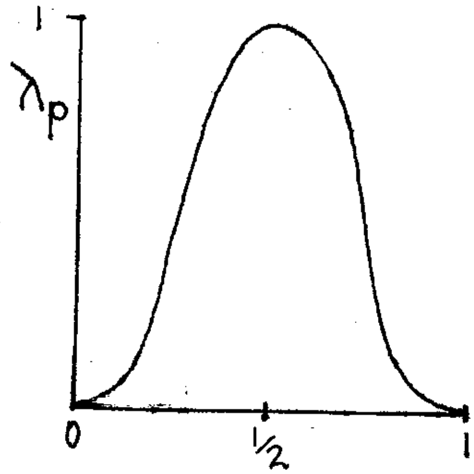

Figure 7

perturbed Hopf flow foliation by constructing a rectangle δ such that $\delta(0,t) = \tau(t)$ and $\delta(s,0) = \sigma(s)$, for any vertical path τ and horizontal path σ beginning at a common point.

Paths σ and τ , as above, lie in a flow surface

$$\{\phi_t(\sigma(s)) \mid 0 \le s \le 1,\ t \in R\}$$

with $\phi_0(\sigma(s)) = \sigma(s)$ and $\phi_t(\sigma(0)) = \tau(t)$. At any point $\phi_t(\sigma(s))$, there is a horizontal vector $\mathbf{V}_{\phi_t(\sigma(s))}$ which is parallel to the line at which the tangent plane to the flow surface (spanned by $\frac{\partial \phi_t}{\partial t}$ and $\frac{\partial \phi_t}{\partial s}$) and the plane spanned by X_1 and X_2 intersect. A rectangle δ corresponding to edges τ and σ is determined by letting $s \mapsto \delta(s,t)$ be the integral curve of $\mathbf{V}$ such that $\delta(0,t) = \tau(t)$. We must show that these integral curves exist over $0 \le s \le 1$. For simplicity, we will assume that $0 < r(\sigma(s)) < 1$, $\forall s \in [0,1]$. The case where the endpoints of σ lie on $\{z_1 = 0\}$ and $\{z_2 = 0\}$ will be considered, later.

We define $\mathbf{V}$, more precisely, as follows. At any point (r, θ, ψ), with

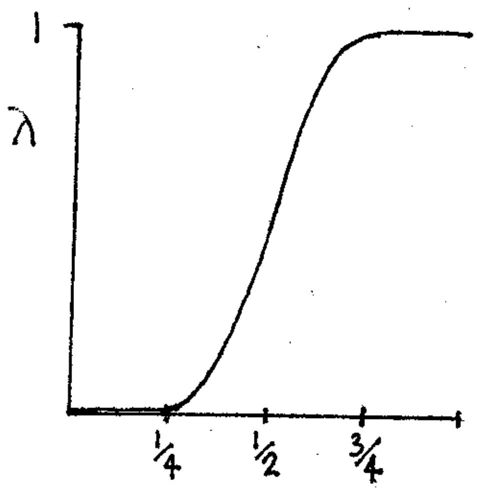

Figure 8

$0 < r < 1$, the vectors X_1, X_2 and $\frac{\partial \phi_t}{\partial t} = \lambda_p(r)X_2 + \partial_\theta + \partial_\psi$ are linearly independent so

$$\frac{\partial \phi_t}{\partial s}(\sigma(s)) = a(s,t)\frac{\partial \phi_t}{\partial t}(\sigma(s)) + b(s,t)X_{1_{\phi_t(\sigma(s))}} + c(s,t)X_{2_{\phi_t(\sigma(s))}} \qquad (3.3)$$

for some coefficient functions $a, b,$ *and* c. Let

$$\mathbf{V}_{\phi_t(\sigma(s))} = \frac{\partial \phi_t}{\partial s}(\sigma(s)) - a(s,t)\frac{\partial \phi_t}{\partial t}(\sigma(s)). \qquad (3.4)$$

Clearly, $\mathbf{V}$ is tangent to the flow surface and $\mathbf{V}$ is horizontal since

$$\mathbf{V}_{\phi_t(\sigma(s))} = b(s,t)X_{1_{\phi_t(\sigma(s))}} + c(s,t)X_{2_{\phi_t(\sigma(s))}} \qquad (3.5)$$

The integral flow of $\mathbf{V}$ preserves orbits of the perturbed Hopf flow, since $[\mathbf{V}_{\phi_t(\sigma(s))}, \frac{\partial \phi_t}{\partial t}(\sigma(s))]$ is proportional to $\frac{\partial \phi_t}{\partial t}(\sigma(s))$, so $\delta(s,t)$ lies on the leaf through $\sigma(s)$.

In the following, we solve for coefficient functions a, b, and c in [3.3]. Since $\frac{d\sigma}{ds} = \frac{\partial \phi_0}{\partial s}(\sigma(s))$ is horizontal, the expression in [3.3] with $t = 0$ has $a(s,0) \equiv 0$, so $\frac{d\sigma}{ds} = b(s,0)X_1 + c(s,0)X_2$. To compute $\frac{\partial \phi_t}{\partial s}(\sigma(s))$, we need expressions for $d\phi_t X_{i_{\sigma(s)}}$, $i = 1, 2$. Since $\frac{\partial \phi_t}{\partial t}$ commutes with ∂_θ, ∂_ψ, and $\lambda_p(r)X_2$, these vector fields are invariant under $d\phi_t$ so

$$d\phi_t X_{1_{\sigma(s)}} = (1 - \lambda(r(\sigma(s)))\partial_{\theta\,|\phi_t(\sigma(s))} - \lambda(r(\sigma(s))\partial_{\psi\,|\phi_t(\sigma(s))} \tag{3.6}$$

and

$$d\phi_t X_{2_{\sigma(s)}} = \frac{\lambda_p(r(\phi_t(\sigma(s)))}{\lambda_p(r(\sigma(s))} X_{2_{\phi_t(\sigma(s))}}. \tag{3.7}$$

From the preceeding, we can express

$$\begin{aligned} \tfrac{\partial \phi_t}{\partial s}(\sigma(s)) \;&=\; b(s,0)(1 - \lambda(r(\sigma(s)))\partial_{\theta\,\phi_t(\sigma(s))} \\ &-\; b(s,0)\lambda(r(\sigma))\partial_{\psi\,\phi_t(\sigma(s))} \\ &+\; c(s,0)\tfrac{\lambda_p(r(\phi_t(\sigma(s))))}{\lambda_p(r(\sigma(s))}X_{2_{\phi_t(\sigma(s))}} \end{aligned} \tag{3.8}$$

Substituting [3.1] and [3.8] into [3.4], we can write

$$\begin{aligned} \mathbf{V}_{\phi_t(\sigma(s))} \;&=\; [A - a(s,t)]\partial_{\theta\,\phi_t(\sigma(s))} + [B - a(s,t)]\partial_{\psi\,\phi_t(\sigma(s))} \\ &+\; [(\tfrac{c(s,0)}{\lambda_p(r(\sigma(s))} - a(s,t))\lambda_p(r(\phi_t(\sigma(s))))]X_{2_{\phi_t(\sigma(s))}}, \end{aligned} \tag{3.9}$$

where

$$A \;=\; b(s,0)(1 \;-\; \lambda(r(\sigma(s))) \tag{3.10}$$

and

$$B \;=\; -b(s,0)\lambda(r(\sigma(s))). \tag{3.11}$$

Comparing with [3.5], observe that

$$c(s,t) = (\frac{c(s,0)}{\lambda_p(r(\sigma(s)))} - a(s,t))\lambda_p(r(\phi_t(\sigma(s)))) \tag{3.12}$$

and $[A - a(s,t)]\partial_{\theta\,\phi_t(\sigma(s))} + [B - a(s,t)]\partial_{\psi\,\phi_t(\sigma(s))} = b(s,t)X_{1_{\phi_t(\sigma(s))}}$.

Let

$$E = 1 - \lambda(r(\phi_t(\sigma(s)))) \tag{3.13}$$

$$F = -\lambda(r(\phi_t(\sigma(s)))) \tag{3.14}$$

so that $X_{1_{\phi_t(\sigma(s))}} = E\partial_{\theta\phi_t(\sigma(s))} + F\partial_{\psi\phi_t(\sigma(s))}$. Solving for $a(s,t)$ and $b(s,t)$, we have

$$a = \frac{FA - EB}{E - F} \; and \; b = \frac{A-a}{E} = \frac{B-A}{F-E}$$

so that

$$a(s,t) = b(s,0)(\lambda(r(\phi_t(\sigma(s)))) - \lambda(r(\sigma(s))) \tag{3.15}$$

and

$$b(s,t) = b(s,0) \tag{3.16}$$

We remark that $a(s,t)$ and $b(s,0)$ have the same sign, since $r(\phi_t(\sigma(s))) > r(\sigma(s))$ for $t > 0$.

To show that rectangles exist, in general, it is sufficient to show that they exist when the initial horizontal edge σ is a piecewise smooth curve which joins $\{z_1 = 0\}$ to $\{z_2 = 0\}$. Assume that $0 < r(\sigma(s)) < 1$ for $0 < s < 1$. As spelled out for me by my advisor, Professor Dusa McDuff, the problem can be reduced to showing that integral curves exist for a vector field

$$\mathbf{W}_{(s,t)} = \frac{\partial}{\partial s} - a(s,t)\frac{\partial}{\partial t},$$

defined on $(s,t) \in [0,1] \times R^+$. Let

$$\Phi : [0,1] \times R^+ \to S^3$$

by

$$\Phi(s,t) = \phi_t(\sigma(s)),$$

then

$$d\Phi : T_{(s,t)}([0,1] \times R^+) \to T_{\phi_t(\sigma(s))}(S^3)$$

by

$$d\Phi \; \mathbf{W}_{(s,t)} = d\Phi\frac{\partial}{\partial s} - a(s,t)d\Phi \; \frac{\partial}{\partial t}$$

$$= \frac{\partial \phi_t}{\partial s}(\sigma(s)) - a(s,t)\, \frac{\partial \phi_t}{\partial t}(\sigma(s)) = \mathbf{V}_{\phi_t(\sigma(s))}.$$

Observe that integral curves of $\mathbf{V}$ join $\{z_1 = 0\}$ to $\{z_2 = 0\}$ if and only if integral curves of $\mathbf{W}$ join $\{0 \times R\}$ to $\{1 \times R\}$, and the latter happens when $a(s,t)$ is bounded, for $0 \leq s \leq 1$. From [3.15], it suffices to show that $b(s,0)$ is bounded.

Observe from

$$\begin{aligned} \tfrac{d\sigma}{ds} &= b(s,0)(1 - \lambda(r(\sigma(s)))\partial_{\theta\,\sigma(s)} - b(s,0)\lambda(r(\sigma(s)))\partial_{\psi\,\sigma(s)} + c(s,0)X_{2_{\sigma(s)}} \\ &= \tfrac{d\theta}{ds}\,\partial_{\theta\sigma(s)} + \tfrac{dr}{ds}\,\partial_{r\sigma(s)} + \tfrac{d\psi}{ds}\,\partial_{\psi\sigma(s)} \end{aligned}$$

that

$$\frac{d\theta}{ds} = b(s,0)(\,1 - \lambda(r(\sigma(s)))\,) \tag{3.17}$$

and

$$\frac{d\psi}{ds} = -b(s,0)\;\lambda(r(\sigma(s))) \tag{3.18}$$

so that b is bounded for $\sigma(s)$ near $\{z_1 = 0\}$ when $\frac{d\theta}{ds}$ is and for $\sigma(s)$ near $\{z_2 = 0\}$ when $\frac{d\psi}{ds}$ is.

The following argument that $\frac{d\theta}{ds}$ is bounded has been spelled out for me by my advisor, Professor Dusa McDuff. Since motion in the θ direction lies in a plane $\{\psi \equiv\ \mathit{constant}\ \}$, assume, for simplicity, that path $\sigma(s)$ lies in such a plane for $r(\sigma(s))$ near 0. Let rectangular coordinates (x,y) on the plane satisfy

$$x^2 + y^2 = r^2$$

and

$$\theta = \mathit{arctan}\,\frac{y}{x}.$$

We are free to parameterize $\sigma(s)$ so that

$$x(\sigma(s)) = s$$

and

$$y(\sigma(s)) = f(s)$$

where f is a real valued, smooth function which vanishes at $s = 0$. We can express

$$\frac{d\theta(\sigma)}{ds} = \frac{d\,arctan}{ds}\left(\frac{f(s)}{s}\right) = \frac{1}{1 + \frac{f^2}{s^2}} \cdot \frac{f's - f}{s^2}$$

so that

$$\left|\frac{d\theta}{ds}\right| \leq \frac{f's^2 - f}{s^2}.$$

The expression on the right side is bounded since the MacClaurin expansion is

$$f's - f = 0 + 0 \cdot s + f''(0)\, s^2 + \ldots,$$

therefore $b(s,0)$ is bounded when σ is appropriately parameterized for $r(\sigma(s))$ near 0.

The argument for boundedness of $\frac{d\psi}{ds}(\sigma(s))$ when $r(\sigma(s))$ is near 1 is similar, and involves the use of rectangular coordinates on a plane $\{ \theta \equiv constant \}$, so $b(s,0)$ is bounded when σ is appropriately parameterized for $r(\sigma(s))$ near 1.

CHAPTER 4

In this chapter, estimates from Chapter 2 are used to compare the lengths of the initial and terminal vertical edges of rectangles. According to [BH1], when a foliation has an Ehresmann connection, there is a diffeomorphism between the universal covers of any two leaves which are joined by a horizontal path. Using the comparison above, we show, under certain conditions, that this diffeomorphism is a quasi isometry with respect to the pullback metrics on the universal covers.

For the first two propositions of this chapter, assume only that $D = (T\mathcal{F})^{\perp}$, where $\mathcal{F}$ is totally geodesic or satisfies hypothesis [2.3] of Proposition 2.2. We do not necessarily assume that D is an Ehresmann connection, although D is, in some of preceeding cases.

The first proposition is a somewhat local version of the second proposition and both propositions are proved in the same way.

Proposition 4.1 *Let $\delta(s,t)$, $(s,t) \in [0,1] \times [0,T]$ be the rectangle determined by a sufficiently short minimal geodesic τ_0 in leaf L_0 and a horizontal path $\sigma(s)$ from L_0 to leaf L_1. Let $\tau_1(t)$ be the shortest geodesic in L_1 such that $\tau_1(0) = \delta(1,0)$ and $\tau_1(T) = \delta(1,T)$, then there is a constant C, which depends only on σ, such that*

$$(length\ \tau_0)e^{-\frac{C}{2}} \leq (length\ \tau_1) \leq (length\ \tau_0)e^{\frac{C}{2}}$$

(see Fig. 9)

proof

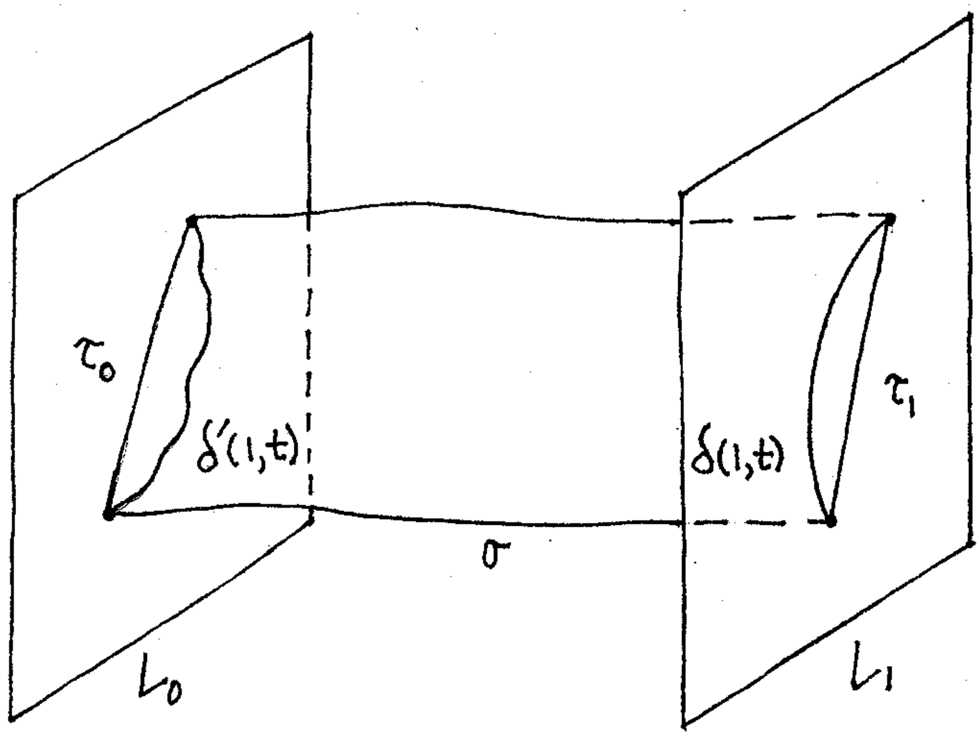

Figure 9.

By hypothesis, *length* $\tau_1 \leq$ *length* $\delta(1,t)$.

Since the hypothesis of Proposition 2.12 in Chapter 2 is satisfied,

$$\textit{length } \delta(1,t) \;\leq\; (\textit{length } \tau_0(t))e^{\frac{C}{2}},$$

where C depends only on σ. When τ_0 is sufficiently short, $\delta(1,t)$ and τ_1 lie in the range of an element of holonomy along σ. From Remark [1.6], τ_1 and $\sigma(1-s)$ determine a rectangle δ' such that the terminal horizontal edge satisfies $\delta'(s,T) \equiv \delta(1-s,T)$. Since the paths $\tau_0(t) = \delta(0,t)$ and $\delta'(1,t)$ in L_0 share the same endpoints, *length* $\tau_0 \leq$ *length* $\delta'(1,t)$. From Proposition 2.12, *length* $\delta'(1,t) \leq$ (*length* $\tau_1)e^{\frac{C}{2}}$, with the same C as above, therefore

$$\textit{length } \tau_0 \leq \; (\textit{length } \tau_1)e^{\frac{C}{2}}$$

and

$$\text{length } \tau_1 \leq (\text{length } \tau_0)e^{\frac{C}{2}}$$

so

$$(\text{length } \tau_0)e^{-\frac{C}{2}} \leq \text{length } \tau_1 \leq (\text{length } \tau_0)e^{\frac{C}{2}} \square$$

Proposition 4.2 *For D and $\mathcal{F}$ as above, any rectangle $\delta(s,t)$, $(s,t) \in [0,1] \times [0,T]$ satisfies*

$$(\text{length } \delta(0,t)|_{t\in[0,T]})e^{-\frac{C}{2}} \leq \text{length } \delta(1,t)|_{t\in[0,T]} \leq (\text{length } \delta(0,t)|_{t\in[0,T]})e^{\frac{C}{2}} \tag{4.3}$$

for some constant C which depends only on the path $\delta(s,0)$.

proof

Let L_s denote the leaf containing the path $s \mapsto \delta(s,t)$. Fix a partition

$$0 = t_0 < t_1 < ... < t_n = T$$

Let $\tau_0(t)_{|t\in[t_{i-1},t_i]}$ be the shortest geodesic in L_0 which shares endpoints with the segment $\delta(0,t)_{|t\in[t_{i-1},t_i]}$. Clearly,

$$\text{length } \tau_0(t)_{|t\in[t_{i-1},t_i]} \leq \text{length } \delta(0,t)_{|t\in[t_{i-1},t_i]}.$$

When *length* $\delta(0,t)_{|t\in[t_{i-1},t_i]}$ is sufficiently small (by making the partition of [0,T] fine), $\tau_0(t)_{|t\in[t_{i-1},t_i]}$ is in the domain of an element of holonomy along $\delta(s,t_{i-1})$, so there is a rectangle $\underline{\delta}(s,t), (s,t) \in [0,1] \times [t_{i-1},t_i]$ with $\underline{\delta}(0,t)_{|t\in[t_{i-1},t_i]} = \tau_0(t)_{|t\in[t_{i-1},t_i]}$ and $\underline{\delta}(s,t_{i-1}) = \delta(s,t_{i-1})$ as the initial vertical and horizontal edges, respectively, such that the terminal horizontal edge satisfies $\underline{\delta}(s,t_i) \equiv \delta(s,t_i)$.

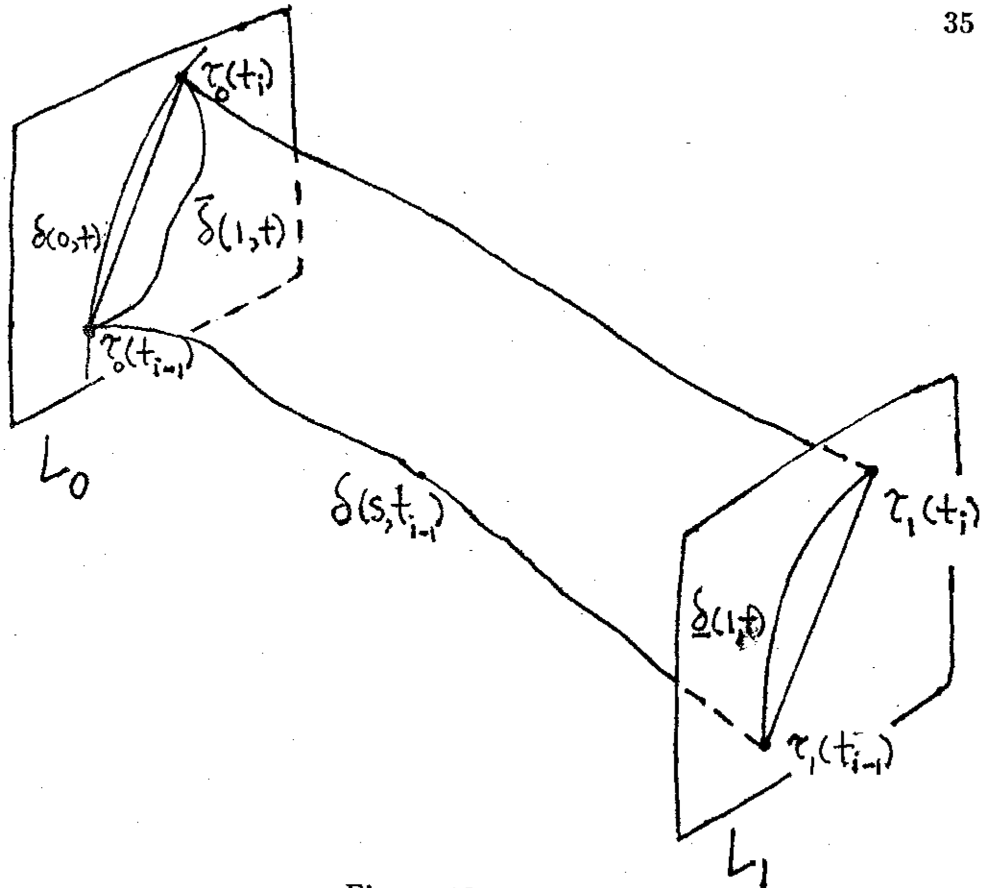

Figure 10

Let $\tau_1(t)_{|t\in[t_{i-1},t_i]}$ be the shortest geodesic path in L_1 which shares endpoints with $\underline{\delta}(1,t)_{|t\in[t_{i-1},t_i]}$. We have

$$\textit{length}\ \tau_1(t)_{|t\in[t_{i-1},t_i]} \leq \textit{length}\ \underline{\delta}(1,t)_{|t\in[t_{i-1},t_i]}. \tag{4.4}$$

The length $\underline{\delta}(1,t)_{|t\in[t_{i-1},t_i]}$ can be made sufficiently short (from Proposition 2.12, by making $\tau_0(t)_{|t\in[t_{i-1},t_i]}$ short) so that $\tau_1(t)_{|t\in[t_{i-1},t_i]}$ lies in the range of an element of holonomy along $\underline{\delta}(s,t_{i-1}) = \delta(s,t_{i-1})$. There is a rectangle $\overline{\delta}(s,t)$, $(s,t) \in [0,1]\times[t_{i-1},t_i]$ with $\overline{\delta}(0,t)_{|t\in[t_{i-1},t_i]} = \tau_1(t)_{|t\in[t_{i-1},t_i]}$ and $\overline{\delta}(s,t_{i-1}) = \delta(1-s,t_{i-1})$ as the initial vertical and horizontal edges, respectively, such that the terminal horizontal edge satisfies $\overline{\delta}(s,t_i) \equiv \delta(1-s,t_i)$.

Since paths $\overline{\delta}(1,t)_{|t\in[t_{i-1},t_i]}$ and $\delta(0,t)_{|t\in[t_{i-1},t_i]}$ in L_0 share endpoints,

$$\textit{length}\ \tau_0(t)_{|t\in[t_{i-1},t_i]} \ \leq\ \textit{length}\ \overline{\delta}(1,t)_{|t\in[t_{i-1},t_i]} \tag{4.5}$$

Taking unions, we see that the piecewise geodesic

$$\tau_0(t)_{t\in[0,T]} = \cup_{i=1}^{n}\tau_0(t)_{t\in[t_{i-1},t_i]}$$

in L_0 is the initial vertical edge of the rectangle

$$\underline{\delta}(s,t),\ (s,t)\in[0,1]\times[0,T]\ =\cup_{i=1}^{n}\underline{\delta}(s,t),\ (s,t)\in[0,1]\times[t_{i-1},t_i].$$

From Proposition 2.12, we have

$$\textit{length}\ \underline{\delta}(1,t)_{|t\in[0,T]}\ \leq\ (\textit{length}\ \tau_0(t)_{|t\in[0,T]})e^{\frac{C}{2}} \tag{4.6}$$

for some constant C which depends only on the initial horizontal edge, $\delta(s,0)$, of the rectangle δ.

Again taking unions, we see that the piecewise geodesic

$$\tau_1(t)_{|t\in[0,T]} = \cup_{i=1}^{n}\tau_1(t)_{t\in[t_{i-1},t_i]}$$

in L_1 is the initial vertical edge of the rectangle

$$\overline{\delta}(s,t),\ (s,t)\in[0,1]\times[0,T]\ =\ \cup_{i=1}^{n}\overline{\delta}(s,t),\ (s,t)\in[0,1]\times[t_{i-1},t_i]$$

From Proposition 2.12, we have

$$\textit{length}\ \overline{\delta}(1,t)_{|t\in[0,T]}\ \leq(\textit{length}\ \tau_1(t)_{|t\in[0,T]})e^{\frac{C}{2}}, \tag{4.7}$$

with the same C as before. Using [4.6] and summing up lengths in [4.4], we have

$$\textit{length}\ \tau_1\ \leq\ (\textit{length}\ \tau_0)e^{\frac{C}{2}}.$$

Doing the same with [4.7] and [4.5], we get

$$\textit{length}\ \tau_0 \ \leq\ (\textit{length}\ \tau_1)e^{\frac{C}{2}}.$$

By making the partition of $[0,T]$ fine, the lengths of paths τ_0 and τ_1 closely approximate the lengths of paths $\delta(0,t)$ and $\delta(1,t)$, respectively, so in the limit, the inequality

$$(\textit{length}\ \tau_0)e^{-\frac{C}{2}} \leq\ \textit{length}\ \tau_1 \leq\ (\textit{length}\ \tau_0)e^{\frac{C}{2}}$$

becomes

$$(\textit{length}\ \delta(0,t)_{t\in[0,T]})e^{-\frac{C}{2}} \leq\ \textit{length}\ \delta(1,t)_{t\in[0,T]}\ \leq\ (\textit{length}\ \delta(0,t)_{t\in[0,T]})e^{\frac{C}{2}}. \tag{4.8}$$

Note that when $\mathcal{F}$ is totally geodesic, $C = 0$, so

$$\textit{length}\ \tau_0(t)_{|t\in[0,T]} =\ \textit{length}\ \tau_1(t)_{|t\in[0,T]}.\Box$$

For the remainder of this chapter, assume, unless otherwise stated, that D is an Ehresmann connection.

Theorem 4.9 *Let $D = (T\mathcal{F})^{\perp}$ be an Ehresmann connection and suppose that condition [2.3] is satisfied. Then the universal covers of the leaves have the same growth type.*

proof

Let $\sigma(s)$ be a horizontal path starting at $p \in L_0$ and ending at $q \in L_1$. Choose $\tilde{p} \in \tilde{L}_0$ and $\tilde{q} \in \tilde{L}_1$ as lifts of p and q to the universal covers. A diffeomorphism $\phi : \tilde{L}_0 \to \tilde{L}_1$ will be constructed. For any $\tilde{x} \in \tilde{L}_0$, let $\tilde{\tau}$ be a

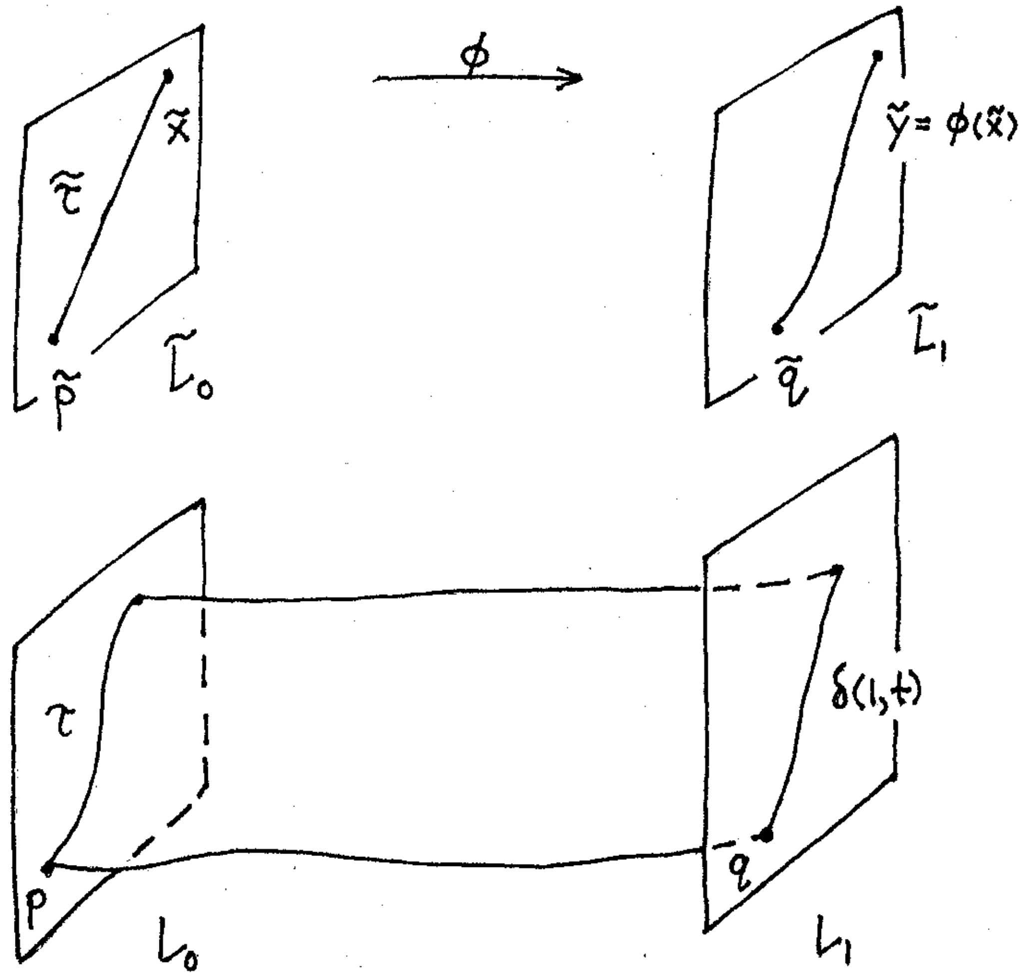

Figure 11

path from $\tilde{p}$ to $\tilde{x}$. Its projection to L_0 is a path, τ, which begins at p and, with σ, determines a rectangle $\delta(s,t)$. The path $\delta(1,t)$ in L_1 begins at q and lifts uniquely to a path in $\tilde{L_1}$ starting at $\tilde{q}$ and ending at some point $\tilde{y}$. Define $\phi : \tilde{L_0} \to \tilde{L_1}$ by $\phi(\tilde{x}) = \tilde{y}$. Note that $\phi(\tilde{p}) = \tilde{q}$. To see that $\tilde{y}$ is independent of the choice of path from $\tilde{p}$ to $\tilde{x}$, observe that all such paths project to fixed endpoint homotopic paths in L_0 which, from Remark [1.6], determine, along with σ, a family of rectangles which have fixed endpoint homotopic terminal vertical edges in L_1. The map ϕ has an inverse map determined by the path $\sigma(1-s)$, and the maps are smooth since the terminal vertical edge of a rectangle depends smoothly on the initial vertical edge. Therefore, $\phi : \tilde{L_0} \to \tilde{L_1}$ is a diffeomorphism. We will show that ϕ is a quasi isometry.

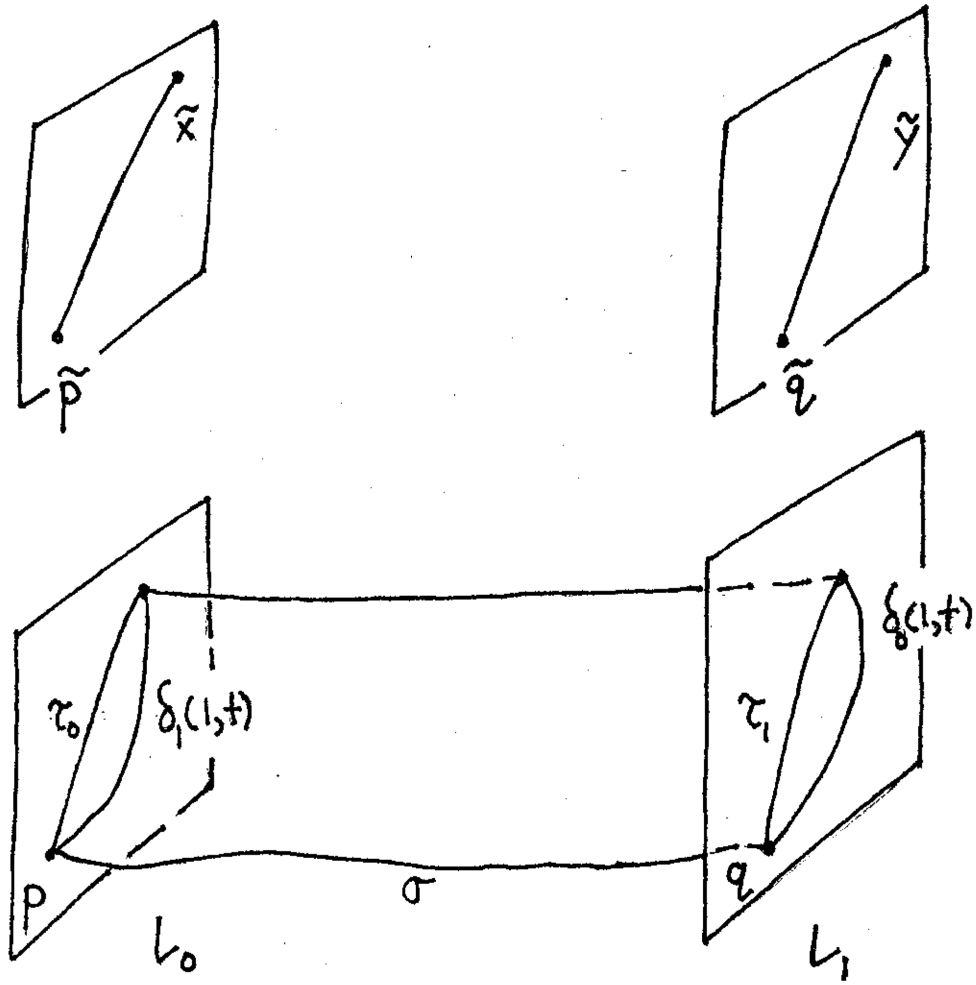

Figure 12

In L_0, let τ_0 be the shortest path starting at p which lifts to a path from $\tilde{p}$ to $\tilde{x}$. In L_1, let τ_1 be the shortest path starting at q which lifts to a path from $\tilde{q} = \phi(\tilde{p})$ to $\tilde{y} = \phi(\tilde{x})$. The rectangle $\delta_0(s,t)$ determined by τ_0 and σ satisfies, for some C which depends on σ,

$$\mathit{length}\ \delta_0(1,t) \ \leq\ (\mathit{length}\ \tau_0)e^{\frac{C}{2}}.$$

Since $\delta_0(1,t)$ is fixed endpoint homotopic to τ_1, *length* $\tau_1 \leq$ *length* $\delta_0(1,t)$. The rectangle $\delta_1(s,t)$, determined by τ_1 and $\sigma(1-s)$, satisfies

$$length\ \delta_1(1,t) \leq (length\ \tau_1)e^{\frac{G}{2}}.$$

Since τ_1 and $\delta_0(1,t)$ are fixed endpoint homotopic, so are $\delta_1(1,t)$ and τ_0, therefore *length* $\tau_0 \leq$ *length* $\delta_1(1,t)$. From the above, $(length\ \tau_0)e^{-\frac{G}{2}} \leq length\ \tau_1 \leq (length\ \tau_0)e^{\frac{G}{2}}$.

Let $\tilde{L_0}$ and $\tilde{L_1}$ have for riemannian metrics the pullbacks of the metrics on L_0 and L_1, respectively. The lengths of the shortest paths from $\tilde{p}$ to $\tilde{x}$ and from $\tilde{q}$ to $\phi(\tilde{x})$ are *length* τ_0 and *length* τ_1, respectively, so

$$dist(\tilde{p},\tilde{x}) = length\ \tau_1\ and\ dist(\tilde{q},\tilde{y}) = length\ \tau_1,$$

where *dist* is the distance in each covering space. Thus,

$$dist(\tilde{p},\tilde{x})e^{-\frac{G}{2}} \leq dist(\phi(\tilde{p}),\phi(\tilde{x})) \leq dist(\tilde{p},\tilde{x})e^{\frac{G}{2}}, \tag{4.10}$$

for any $\tilde{x} \in \tilde{L_0}$. From the following lemma, ϕ is a quasi isometry.

Lemma 4.11 *Let a diffeomorphism* $f : X \to Y$ *and suppose* $\forall x \in X$ *that* $k \cdot dist(p,x) \leq dist(f(p),f(x)) \leq K \cdot dist(p,x)$, *for some constants* $K, k > 0$. *Then* f *is a quasi isometry.*

proof

We'll show for any $\mathbf{v} \in T_pX$ that $k\|\mathbf{v}\| \leq \|f_*\mathbf{v}\| \leq K\|\mathbf{v}\|$.

Let $c(t)$ be the unique geodesic in X with $c(0) = p$ and $\frac{dc(s)}{ds}|_{s=0} = \mathbf{v}$.

By the hypothesis, $k \cdot dist(p,c(t)) \leq dist(f(p),f(c(t)))$, with equality when $t = 0$, so

$$k \cdot \frac{d}{dt}_{|t=0} dist(p, c(t)) \leq \frac{d}{dt}_{|t=0} dist(f(p), f(c(t))) \tag{4.12}$$

Near $t = 0$, $c(t)$ minimizes arclength, so $dist(p, c(t)) = \int_0^t \|c'(s)\| ds = t\|\mathbf{v}\|$ and

$$\frac{d}{dt}_{|t=0} dist(p, c(t)) = \|\mathbf{v}\|. \tag{4.13}$$

Since $f(c(t))$ need not be a geodesic in Y, its arclength

$$\int_0^t \|f_* \frac{d}{ds}(c(s))\| ds \geq \ dist(f(p), f(c(t))$$

with equality when $t = 0$, therefore

$$\frac{d}{dt}_{|t=0} dist(f(p), f(c(t))) \leq \|f_* \mathbf{v}\|. \tag{4.14}$$

From [4.12], [4,13], and [4.14], $k\|\mathbf{v}\| \leq \|f_* \mathbf{v}\|$.

Similar reasoning is used for obtaining the other inequality. From the hypothesis,

$$dist(f(f^{-1}(q)), f(f^{-1}(y))) \leq K \cdot dist(f^{-1}(q), f^{-1}(y))$$

for $f^{-1}(q), f^{-1}(y)$ in X, therefore

$$\frac{1}{K} dist(q, y) \leq dist(f^{-1}(q), f^{-1}(y))$$

for $q, y \in Y$. By working with a geodesic $b(t)$ such that $b(0) = q$ and $\frac{db(s)}{ds}_{|s=0} = f_* \mathbf{v}$, one shows

$$\frac{1}{K}\|f_* \mathbf{v}\| \leq \|f_*^{-1} f_* \mathbf{v}\| = \|\mathbf{v}\|,$$

therefore $k\|\mathbf{v}\| \leq \|f_* \mathbf{v}\| \leq K\|\mathbf{v}\|$.□

Suppose the hypothesis of Theorem 4.9 is satisfied and suppose some leaf L is compact. From [M], the growth type of $\tilde{L}$ agrees with the growth type of $\pi_1(L)$. For any other $L' \in \mathcal{F}$, the growth type of L' is no greater than that of $\tilde{L}'$, so Theorem 4.9 implies the following.

Corollary 4.15 *Let $D = (T\mathcal{F})^{\perp}$ and suppose that condition [2.3] is satisfied. For any compact leaf L, the growth type of $\pi_1(L)$ bounds the growth type of any other leaf L' from above.*

From [BH2], the universal covers of leaves are isometric when $\mathcal{F}$ is totally geodesic. This result, also, follows from observing that proof of Theorem 4.9 works with $C = 0$ in [4.10] when $\mathcal{F}$ is totally geodesic. We, also, remark that Corollary 4.15 holds when $\mathcal{F}$ is a riemannian foliation, since condition [2.3] is satisfied.

Let distribution D be transverse to $T\mathcal{F}$. Fix a base point p in leaf L. The vertical loop $\alpha(t)$ based at p and the horizontal path $\sigma(s)$ starting at p determine a rectangle $\delta(s,t)$, $(s,t) \in [0,1] \times [0,1]$. Denote by $\alpha \cdot \sigma$ the terminal horizontal edge $\delta(s,1)$.

Following [BH1], define

Definition 4.16

$$K_D(L,p) = \{\alpha \in \pi_1(L,p) \mid \alpha \cdot \sigma = \sigma\}$$

that is, $K_D(L,p)$ consists of homotopy classes of loops based at p for which the rectangles always close up.

It is easily shown that $K_D(L,p) \triangleleft \pi_1(L,p)$ and for $p, p' \in L$ that $K_D(L,p) \cong K_D(L,p')$. We remark that a horizontal path, σ, from $p \in L$

to $q \in L'$ determines an isomorphism from $K_D(L,p)$ onto $K_D(L',q)$ by the map

$$\alpha \mapsto \delta(1,t)$$

where $\delta(s,t)$, $(s,t) \in [0,1] \times [0,1]$ is the rectangle determined by $\alpha \in K_D(L,p)$ and σ.

For convenience, the base point p will sometimes be omitted from the notation.

We give the following definition,

Definition 4.17 $H_D(L,p) = \pi_1(L,p)/K_D(L,p)$.

$H_D(L,p)$ does not depend on the base point p and is independent of the choice of the Ehresmann connection D, from [BH2].

The following result is proved in [BH2].

Proposition 4.18 *Let the transverse distribution D be an Ehresmann connection for $\mathcal{F}$. Then $\tilde{L_0}/K_D(L_0) \cong \tilde{L_1}/K_D(L_1)$ for any L_0, $L_1 \in \mathcal{F}$.*

proof

The proof is almost identical to the first portion of the proof of Theorem 4.9.

Let $\sigma(s)$ be a horizontal path beginning at $p \in L_0$ and ending at $q \in L_1$. Let $\tilde{p} \in \tilde{L_0}/K_D(L_0,p)$ and $\tilde{q} \in \tilde{L_1}/K_D(L_1,q)$ be the lifts of p and q, respectively. On any $\tilde{x} \in \tilde{L_0}/K_D(L_0,p)$, define a map

$$\phi : \tilde{L_0}/K_D(L_0,p) \to \tilde{L_1}/K_D(L_1,q)$$

as follows. Let τ be the projection to L_0 of a path $\tilde{\tau}$ in $\tilde{L_0}/K_D(L_0,p)$ from $\tilde{p}$ to $\tilde{x}$, and let $\delta(s,t)$ be the rectangle determined by τ and σ. The path $\delta(1,t)$ in L_1 lifts to a unique path in $\tilde{L_1}/K_D(L_1,q)$ from $\tilde{q}$ to some point $\tilde{y}$. Define $\phi(\tilde{x}) = \tilde{y}$. Observe that $\phi(\tilde{p}) = \tilde{q}$.

To show that $\tilde{y}$ is independent of the choice of path $\tilde{\tau}$ from $\tilde{p}$ to $\tilde{x}$, let $\tilde{x}'$ be another such path and let τ' be its projection to L_0. The loop $\tilde{\tau} * \tilde{\tau'}^{-1} \in \pi_1(\tilde{L_0}/K_D(L_0,p))$ projects to a loop $\tau * \tau'^{-1} \in K_D(L_0,p)$. Let $\delta'(s,t)$ be the rectangle determined by τ' and σ. It follows that $\delta(1,t) * \delta'(1,1-t)$ is a loop in $K_D(L_1,q)$ which lifts to a loop in $\pi_1(\tilde{L_1}/K_D(L_1,q))$, therefore the lifts of $\delta(1,t)$ and $\delta'(1,t)$ both end at $\tilde{y}$.

An inverse to ϕ is determined by the path $\sigma(1-s)$, and the maps are smooth, therefore ϕ is a diffeomorphism.

Theorem 4.19 *Let $D = (T\mathcal{F})^{\perp}$ be an Ehresmann connection and suppose that $\mathcal{F}$ satisfies condition [2.3]. Then the map*

$$\phi : \tilde{L_0}/K_D(L_0,p) \to \tilde{L_1}/K_D(L_1,q)$$

from the previous proposition is a quasi isometry.

proof

Let $\tilde{L_0}/K_D(L_0,p)$ and $\tilde{L_1}/K_D(L_1,q)$ have as metrics the pullbacks of the metrics on L_0 and L_1, respectively. Let

$$\phi : \tilde{L_0}/K_D(L_0,p) \to \tilde{L_1}/K_D(L_1,q)$$

be determined by the horizontal path σ from p to q, as in the previous proposition.

In L_0, let τ_0 be the shortest path starting at p which lifts to a path in $\tilde{L}_0/K_D(L_0,p)$ from a base point $\tilde{p}$ to some point $\tilde{x}$. Let $\delta_0(s,t)$, $(s,t) \in [0,1] \times [0,1]$ be the rectangle determined by τ_0 and σ_0. By the way ϕ is defined, $\delta_0(1,t)$ lifts to a path in $\tilde{L}_1/K_D(L_1,q)$ from the base point $\tilde{q} = \phi(p)$ to the point $\tilde{y}$.

In L_1, let τ_1 be the shortest path, starting at q, which lifts to a path in $\tilde{L}_1/K_D(L_1,q)$ from $\tilde{q}$ to $\tilde{y}$. We can say

$$\textit{length } \tau_1 \leq \textit{ length } \delta_0(1,t) \tag{4.20}$$

Let $\delta_1(s,t)$ be the rectangle determined by τ_1 and $\sigma(1-s)$. By the way ϕ^{-1} is defined, $\delta_1(1,t)$ in L_0 lifts to a path in $\tilde{L}_0/K_D(L_0,p)$ from $\tilde{p} = \phi^{-1}(\tilde{q})$ to $\tilde{x} = \phi^{-1}(\tilde{y})$, so

$$\textit{length } \tau_1 \leq \textit{ length } \delta_1(1,t) \tag{4.21}$$

From Proposition 4.2,

$$\textit{length } \delta_0(1,t) \leq (\textit{length } \tau_0)e^{\frac{C}{2}} \tag{4.22}$$

and

$$\textit{length } \delta_1(1,t) \leq (\textit{length } \tau_1)e^{\frac{C}{2}} \tag{4.23}$$

for some C which depends on σ.

From [4.20 - 4.23], it follows that

$$(\textit{length } \tau_0)e^{\frac{-C}{2}} \leq \textit{ length } \tau_1 \leq (\textit{length } \tau_0)e^{\frac{C}{2}}.$$

In the metrics on $\tilde{L}_0/K_D(L_0,p)$ and $\tilde{L}_1/K_D(L_1,q)$, $\textit{length } \tau_0 = \textit{dist}(\tilde{p},\tilde{x})$ and $\textit{length } \tau_1 = \textit{dist}(\tilde{q},\tilde{y})$, where $\textit{dist}$ denotes the distance in each space, therefore

$$\textit{dist}(\tilde{p},\tilde{x})e^{\frac{-C}{2}} \leq \textit{dist}(\tilde{q},\tilde{y}) \leq \textit{dist}(\tilde{p},\tilde{x})e^{\frac{C}{2}}.$$

From Lemma 4.11, it follows that ϕ is a quasi isometry. □

When $\mathcal{F}$ is totally geodesic, $\tilde{L_0}/K_D(L_0,p)$ and $\tilde{L_1}/K_D(L_1,q)$ are isometric, from [BH2]. This result can be seen from the previous proof since $C = 0$ and ϕ is an isometry when $\mathcal{F}$ is totally geodesic.

Corollary 4.24 *Let $D = (T\mathcal{F})^{\perp}$ be an Ehresmann connection and suppose that $\mathcal{F}$ is totally geodesic or that [2.3] holds. For any compact $L \in \mathcal{F}$, the growth type of $H_D(L)$ bounds the growth type of any other leaf from above.*

proof

The growth type of $\tilde{L}/K_D(L,p)$ agrees with the growth type of the group of deck transformations,

$$\pi_1(L,p)/K_D(L,p) = H_D(L,p).$$

From the preceeding theorem, the growth type of any other leaf cannot exceed the growth type of $H_D(L,p)$.

CHAPTER 5

In this chapter, we discuss the growth of leaves in a transversally parallelisable foliation and prove a theorem of Molino that the transverse orthonormal frame bundle of a riemannian foliation has a transversally parallelisable foliation. Notation is developed for use in the following chapter.

The following definitions and observations are due to Molino [Mo].

Let $\mathcal{F}$ be a codimension q foliation of a manifold M. Given a vector field $\mathbf{X} \in \mathbf{TM}$, the following conditions are equivalent:

$$[\mathbf{X}, \mathbf{Y}] \in T\mathcal{F} \ \forall \mathbf{Y} \in T\mathcal{F} \tag{5.1}$$

$$\textit{The integral flow of } \mathbf{X} \textit{ preserves } \mathcal{F} \tag{5.2}$$

Using local coordinates $(x^1, .., x^{n-q}, y^1, ..., y^q)$ where the $x^1, .., x^{n-q}$ are coordinates along the leaf, we can express

$$\mathbf{X} = \sum_{i=1}^{n-q} \xi^i(x^1, .., x^{n-q}, y^1, .., y^q) \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \eta^j(y^1, .., y^q) \frac{\partial}{\partial y^j} \tag{5.3}$$

that is, the transverse part of $\mathbf{X}$ does not change along the leaves.

Definition 5.4 $\mathbf{X}$ *is said to be a foliated vector field with respect to $\mathcal{F}$ if one of the conditions above is satisfied.*

The collection of foliated vector fields is a Lie algebra which we denote by $\mathcal{L}(M, \mathcal{F})$.

We can assume that codimension q foliation $\mathcal{F}$ is determined by local submersion charts $\{U_i, \phi_i\}$ of rank q which satisfy the following conditions from [Ca1].

•

$$M \subseteq \bigcup_i U_i$$

- Whenever $U_i \cap U_j \neq \emptyset, \exists$ a local "change of transverse coordinates" diffeomorphism γ_{ij} such that

$$\varphi_j(x) = \gamma_{ji} \circ \varphi_i(x) \ for \ x \in \ U_i \cap U_j. \tag{5.5}$$

- The γ_{ij} satisfy a cocycle condition:

$$\gamma_{ki}(x) \ = \ \gamma_{kj} \circ \gamma_{ji}(x) \ for \ x \in \varphi_i(U_i \cap U_j \cap U_k) \tag{5.6}$$

We can think of the disjoint union $T = \bigcup_i^{\triangle} \varphi_i(U_i)$ as a transverse manifold of dimension q. The γ_{ij} generate a pseudogroup, G, of local diffeomorphisms on T.

Locally, a path with initial point $\varphi_1(p)$ can be lifted, via submersion chart $\{U_1, \varphi_1\}$, to a unique path in M, starting at p, which is tangent to $(T\mathcal{F})^{\perp}$, for some choice of metric on M. It is clear how a locally defined vector field at $\varphi_1(p)$ lifts to one at p. The lift of a globally defined, G- invariant vector field, $\mathbf{X}$, on T is a globally defined, foliated vector field, $\overline{\mathbf{X}}$. To see that $\overline{\mathbf{X}}$ does not depend on the local submersion chart, suppose that p is in another local submersion chart, $\{U_2, \varphi_2\}$, then $\varphi_2 = \gamma_{21} \circ \varphi_1$ for some $\gamma_{21} \in G$, and $\mathbf{X}_{\varphi_2(p)}$ lifts to $\overline{\mathbf{X}}_p$ since

$$\mathbf{X}_{\varphi_2(p)} = \mathbf{X}_{\gamma_{21} \circ \varphi_1(p)} = d\gamma_{21}\mathbf{X}_{\varphi_1(p)} = d\gamma_{21} \circ d\varphi_1(\overline{\mathbf{X}}_p) = d\varphi_2(\overline{\mathbf{X}}_p).$$

Definition 5.7 *A codimension q foliation $\mathcal{F}$ is transversally parallelisable if there exists a parallelism of vector fields $\mathbf{X_1},..,\mathbf{X_q}$ on T which are invariant under elements of G.*

It follows, when $\mathcal{F}$ is transversally parallelisable, that a parallelism $\overline{\mathbf{X}}_1,..,\overline{\mathbf{X}}_q$ of foliated vector fields in $(T\mathcal{F})^\perp$ is obtained as the lift of a G-invariant parallelism $\mathbf{X_1},..,\mathbf{X_q}$ on T. If T is given a riemannian metric g_T such that $g_T(\mathbf{X}_i,\mathbf{X}_j) = \delta_{ij}$, then elements of G become local isometries and $\mathcal{F}$ becomes a riemannian foliation when the riemannian metric g on M is redefined on $(T\mathcal{F})^\perp \times (T\mathcal{F})^\perp$ so that

$$g(\overline{\mathbf{X}}_i,\overline{\mathbf{Y}}_j) = \delta_{ij}.$$

From now on, assume that g on M is chosen as above and, also, assume that M is compact.

Since a transversally parallelisable foliation $\mathcal{F}$ with metric g, as above, is a riemannian foliation, we know from [BH1] that $D = (T\mathcal{F})^\perp$ is an Ehresmann connection. We can see this more directly, as follows. A rectangle is determined by a vertical curve $\tau(t),\ t \in [0,T]$, and a horizontal curve $\gamma(s),\ s \in [0,1]$ as follows. For some smooth functions $a_i(s)$, we can express

$$\frac{d\gamma(s)}{ds} = \sum_{i=1}^{q} a_i(s)\overline{\mathbf{X}}_{i_{\gamma(s)}}.$$

Observe that γ is the integral curve of the foliated vector field $\sum_{i=1}^q a_i(s)\overline{\mathbf{X}}_i$ with initial condition $\gamma(0) = \tau(0)$. For each t, let the horizontal path $\delta(s,t)$ be the integral curve of $\sum_{i=1}^q a_i(s)\overline{\mathbf{X}}_i$ with initial condition $\delta(0,t) = \tau(t)$. Since $\sum_{i=1}^q a_i\overline{\mathbf{X}}_i \in \mathcal{L}(M,\mathcal{F})$, the integral flow of the vector field preserves $\mathcal{F}$, so the paths $t \mapsto \delta(s,t)$ are vertical and it follows that δ is a rectangle.

The preceeding ideas lead to the following proposition.

Proposition 5.8 *Let $\mathcal{F}$ be a transversally parallelisable foliation of a compact manifold M, then all leaves have the same growth type.*

proof

From [BH1], we know that any two leaves, L_0 and L_1, are joined by some horizontal path $\gamma(s)_{s\in[0,1]}$. Let

$$\frac{d\gamma(s)}{ds} = \sum_{i=1}^{q} a_i \overline{\mathbf{X}}_{i_{\gamma(s)}},$$

for some functions $a_i(s)$, then the integral flow of $\sum_{i=1}^{q} a_i \overline{\mathbf{X}}_i$ gives a self diffeomorphism on M which preserves $\mathcal{F}$ and maps L_0 onto L_1. Since M is compact, the flow map is a quasi isometry on M and its restriction is a quasi isometry from L_0 onto L_1, so L_0 and L_1 have the same growth type.□

Remark 5.9 For a transversally parallelisable foliation $\mathcal{F}$ with Ehresmann connection $D = (T\mathcal{F})^{\perp}$, we have

$$K_D(L) = \pi_1(L),$$

and, therefore, $H_D(L) = \{e\}$, for any $L \in \mathcal{F}$. This follows, since vertical loops remain as loops under the flow of a foliated vector field, so that rectangles are always determined in which the initial and terminal horizontal edges coincide.

Definition 5.10 *Let $\mathcal{F}$ be a codimension q foliation of M, with a bundlelike metric. The transverse orthonormal frame bundle*

$$\begin{aligned} e(M) \;=\; & \{(p, \mathbf{f}_p) \mid p \in M \text{ and} \\ & \mathbf{f}_p \text{ is an orthonormal frame of } q \text{ vectors in } T_pM \cap (T\mathcal{F})^{\perp}\}. \end{aligned}$$

Note that $e(M)$ is a compact manifold of dimension $n + \frac{q(q-1)}{2}$, when M is compact.

The foliation $\mathcal{F}$ on M can be lifted to a foliation $\mathcal{F}_T$ on $e(M)$ with leaves of the same dimension as leaves on $\mathcal{F}$. A picture of the lifted leaf through $(p, \mathbf{f}_p)$ may be obtained by seeing how vertical paths through p lift to $e(M)$. A vertical path $\tau(t)$, beginning at p, lifts to a path $\tilde{\tau}(t) = (\tau(t), \mathbf{f}_{\tau(t)})$, which begins at $(p, \mathbf{f}_p)$, where the orthnormal frame $\mathbf{f}_{\tau(t)}$ satisfies $d\varphi(\mathbf{f}_{\tau(t)}) \equiv constant$, for any local riemannian submersion chart $\{U, \varphi\}$ at p. The lifted leaf through $(p, \mathbf{f}_p)$ can be viewed as the union of the lifts to $e(M)$ of all paths in L through p.

Molino has shown that:

Theorem 5.11 *$\mathcal{F}_T$ is transversally parallelisable when $\mathcal{F}$ is a riemannian foliation of M.*

Most of the remainder of Chapter 5 will be devoted to proving Molino's result. We will also show how $\mathcal{F}_T$ is induced by local submersions and describe an invariant parallelism in the submersion space. The notation introduced will be needed in Chapter 6.

Definition 5.12 *Let T be a q dimensional manifold. The orthonormal frame bundle*

$$B(T) = \{ (b, \mathbf{e}_b) \mid b \in T \text{ and } \mathbf{e}_b \text{ is an orthonormal frame of } q \text{ vectors in } T_b(T)\}$$

Note that $B(T)$ is a manifold of dimension $\frac{q(q-1)}{2} + q$.

Assume $T = \bigcup_i^{\triangle} \varphi_i(U_i)$, as before. We can regard

$$B(T) = \bigcup_i^{\triangle} B(\varphi_i(U_i)).$$

Define the projection map

$$\pi : B(T) \rightarrow T$$

by $\pi(b, \mathbf{e}) = b$.

Fix a neighborhood $\varphi(U)$ in T. A curve $c(s)$ in $\varphi(U)$ can be lifted in a canonical way to a path in $B(\varphi(U))$ which begins at $(b, \mathbf{e}_b)$. The lifted path is

$$\tilde{c}(s) = (\ c(s), \mathbf{P}_{c(s)}(\mathbf{e}_b)\ )$$

where $\mathbf{P}_{c(s)}(\mathbf{e}_b)$ denotes the parallel translation of the frame $\mathbf{e}_b$ along $c(s)$. The canonical horizontal distribution at $(b, \mathbf{e}_b)$ is the span of tangent vectors of canonical lifts to paths beginning at $(b, \mathbf{e}_b)$ of all paths in T beginning at b.

For a vector $\mathbf{X} = (x_1, .., x_q) \in R^q$ and a point $a = (b, \mathbf{e}_b)$ in $B(T)$, denote

$$a \bullet \mathbf{X} = \sum_{i=1}^{q} x_i \mathbf{e}_i, \tag{5.13}$$

where $\mathbf{e}_b$ is the orthonormal frame of q vectors

$$\{\ \mathbf{e}_1, .., \mathbf{e}_q\ \}$$

in $T_b(T)$.

Definition 5.14 *At a point $a = (b, \mathbf{e}_b)$ in $B(T)$, the solder form is a map*

$$\omega : T_a B(T) \rightarrow R^q$$

defined by

$$\omega(\mathbf{X}_a) = \mathbf{X},$$

where $d\pi(\mathbf{X}_a) = a \bullet \mathbf{X}$.

Note that ω vanishes on $T_a\ \pi^{-1}(a)$ since this subspace is the kernel of $d\pi$.

Denote by $\mathbf{E}(\mathbf{X})$ the vector field tangent to the canonical horizontal distribution on $TB(T)$ which satisfies, at each $a = (b, \mathbf{e}_b)$, the equation

$$\omega(\ \mathbf{E}(\mathbf{X})(a)\ ) \ = \ \mathbf{X}.$$

Equivalently, we can say

$$d\pi\ \mathbf{E}(\mathbf{X})(a) \ = \ a\ \bullet\ \mathbf{X}.$$

It is easy to show that the integral curve of $\mathbf{E}(\mathbf{X})$ which begins at $a = (b, \mathbf{e}_b)$ is the lift $\tilde{c}(s) \ = \ (\ c(s), \mathbf{P}_{c(s)}(\mathbf{e}_b)\ )$ of the unique geodesic $c(s)$ in T which satisfies $c(0) = b$ and $\frac{dc(s)}{ds}_{|s=0} \ = \ a\ \bullet\ \mathbf{X}$.

$\mathbf{E}(\mathbf{X})$ is called a **basic vector field**. Every basic vector field can be expressed as

$$\mathbf{E}(\ (x_1, .., x_q)\ ) \ = \ \sum_{i=1}^{q} x_i \mathbf{E}_i,$$

where the $\mathbf{E}_1, .., \mathbf{E}q$ are basic vector fields which satisfy

$$\mathbf{E}_i \ = \ \mathbf{E}(\ (0, .., 1, .., 0)\ ).$$

Now, consider the fiber $\pi^{-1}(b)$ over $b \in T$. $SO(q)$ acts freely and transitively on the fibers. For $a = (b, \mathbf{e}_b)$ in $B(T)$ and $M \ = \ (\ m_{ij}\ ) \ \in SO(q)$, let

$$aM \ = \ (b, M\mathbf{e}_b),$$

where the vectors of the frame $M\mathbf{e}_b$ are

$$\{\ \sum_{i=1}^{q} m_{i1}\mathbf{e}_i, ..., \sum_{i=1}^{q} m_{iq}\mathbf{e}_i\ \}.$$

Observe for the identity matrix $I \ \in \ SO(q)$ that $aI \ = \ a$. The point $a \ = \ (b, \mathbf{e}_b)$ determines a map from $SO(q)$ to $B(T)$ by

$$a :\ M \ \rightarrow \ aM,$$

with derivative map

$$da : S\wp(q) \rightarrow T_a\{\pi^{-1}(\pi(a))\}.$$

A vector field **A** is called **fundamental** if, at every $a \in B(T)$,

$$\mathbf{A}(a) = da\, A_I, \; for\ a\ fixed\ A_I \in S\wp(q),$$

where $S\wp(q)$ denotes the Lie algebra of $SO(q)$. Let $\mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}$ be fundamental vector fields which correspond to the usual basis of skew symmetric matrices in $S\wp(q)$. It is clear that the space of fundamental vector fields is spanned by this collection. We observe that the collection $\{ \mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}, \mathbf{E}_1, .., \mathbf{E}_q \}$ is a parallelism of nonvanishing vector fields on $B(T)$.

Definition 5.15 *The connection one form $\theta : T_aB(T) \rightarrow S\wp(q)$ is defined by $\theta(\mathbf{Y}_a) = A_I$, where $da\ A_I$ is the projection of $\mathbf{Y}_a$ onto to the fiber $\{ \pi^{-1}(\pi(a))\}$ in $B(T)$.*

Remark 5.16 For a fundamental vector field $\mathbf{A}(a) = da\ A_I$, we always have $\theta(\mathbf{A}(a)) = A_I$. For a basic vector field $\mathbf{E}(\mathbf{X})$, we have $\theta(\mathbf{E}(\mathbf{X})) = O$, the zero matrix. Also, $\omega(\mathbf{A}(a)) = (0, .., 0)$, the zero element of R^q.

Let φ_i and φ_j be riemannian submersions which locally determine $\mathcal{F}$ and suppose that their domains, U_i and U_j, respectively, intersect. The composition $\gamma_{ji} = \varphi_j \circ \varphi_i^{-1}$ is a local isometry from T into itself. A map

$$(\gamma_{ji}, d\gamma_{ji}) : B(\varphi_i(U_i)) \rightarrow B(\varphi_j(U_j))$$

is defined by

$$[(b, \mathbf{e}_b)](\gamma_{ji}, d\gamma_{ji}) = (\gamma_{ji}(b), d\gamma_{ji}(\mathbf{e}_b)).$$

From the following two lemmas, we conclude that the parallelism

$$\mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}, \mathbf{E}_1, .., \mathbf{E}_q$$

on $B(T)$ is invariant under the maps

$$(\gamma_{ji}, d\gamma_{ji})_* : TB(T) \rightarrow TB(T).$$

Lemma 5.17 *Let* $\gamma_{ji} = \varphi_j \circ \varphi_i^{-1}$, *then any basic vector field* $\mathbf{E}(\mathbf{X})$ *is invariant under* $(\gamma_{ji}, d\gamma_{ji})_*$.

proof

Let $p = (\ b, \mathbf{e}_b\ )$ in $B(\varphi_i(U_i))$. We will show that

$$(\gamma_{ji}, d\gamma_{ji})_*\ \mathbf{E}(\mathbf{X})(p) = \mathbf{E}(\mathbf{X})\ (\ p(\gamma_{ji}, d\gamma_{ji})\ ).$$

Note that on the left, $\mathbf{E}(\mathbf{X})$ is defined on $B(\varphi_i(U_i))$ and on the right, $\mathbf{E}(\mathbf{X})$ is defined on $B(\varphi_j(U_j))$.

Let $p(s)$ be the unique integral curve of $\mathbf{E}(\mathbf{X})$ with $p(0) = p$. We know that $p(s)$ is the canonical horizontal lift of some geodesic, $\varphi_i \circ c(s)$ in $\varphi_i(U_i)$ which satisfies

$$\varphi_i \circ c(0) = b$$

and

$$\frac{d\varphi_i \circ c(s)}{ds}_{|s=0} = p \bullet \mathbf{X}.$$

We can express

$$p(s) = (\ \varphi_i \circ c(s), \mathbf{P}_{\varphi_i \circ c(s)}\ (\mathbf{e}_b)\ ).$$

Consider the path

$$[p(s)]\ (\ \gamma_{ji}, d\gamma_{ji}\ ) = (\ \gamma_{ji} \circ \varphi_i \circ c(s), d\gamma_{ji}\ \mathbf{P}_{\varphi_i \circ c(s)}(\mathbf{e}_b)\ )$$

in $B(\varphi_j(U_j))$. Since γ_{ji} is a local isometry,

$$d\gamma_{ji}\,\mathbf{P}_{\varphi_i\circ c(s)}\,(\mathbf{e}_b) \;=\; \mathbf{P}_{\gamma_{ji}\circ\varphi_i\circ c(s)}\,(\,d\gamma_{ji}(\mathbf{e}_b)\,)$$

so we can express

$$[p(s)]\;(\,\gamma_{ji}, d\gamma_{ji}\,) \;=\; (\,\varphi_j\circ c(s), \mathbf{P}_{\varphi_j\circ c(s)}\,(d\gamma_{ji}(\mathbf{e}_b)).$$

Observe that the right hand side is the canonical horizontal lift of the geodesic $\varphi_j\circ c(s)$ to a path which starts at $[p]\;(\,\gamma_{ji}, d\gamma_{ji}\,)$. Since the geodesic $\varphi_j\circ c(s)$ satisfies initial conditions

$$\varphi_j\circ c(0) \;=\; \gamma_{ji}(b)$$

and

$$\frac{d\varphi_j\circ c(s)}{ds}_{|s=0} \;=\; d\gamma_{ji}\circ\frac{d\varphi_i\circ c(s)}{ds}_{|s=0} \;=\; d\gamma_{ji}(\,p\;\bullet\;\mathbf{X})$$
$$=[p]\;(\,\gamma_{ji}, d\gamma_{ji}\,)\;\bullet\;\mathbf{X},$$

it follows that $[p(s)]\;(\gamma_{ji}, d\gamma_{ji})$ is the integral curve of $\mathbf{E}(\mathbf{X})$ which begins at $[p]\;(\,\gamma_{ji}, d\gamma_{ji}\,)$.

This result follows from letting $s=0$ in the following:

$$(\gamma_{ji}, d\gamma_{ji})_*\;\mathbf{E}(\mathbf{X})\;(p(s)) \;=\; (\gamma_{ji}, d\gamma_{ji})_*\;\frac{dp(s)}{ds}$$

$$=\;\frac{d}{ds}(\;[p(s)]\;(\gamma_{ji}, d\gamma_{ji})\;) \;=\; \mathbf{E}(\mathbf{X})\;(\;[p(s)](\gamma_{ji}, d\gamma_{ji})\;).\Box$$

Lemma 5.18 *Let* $\gamma_{ji} \;=\; \varphi_j\circ\;\varphi_i^{-1}$, *then any fundamental vector field* $\mathbf{A}$ *is invariant under* $(\gamma_{ji}, d\gamma_{ji})_*$.

proof

At $a = (b, \mathbf{e}_b)$ in $B(\varphi_i(U_i))$, $\mathbf{A}(a) = da\ A_I$, for some $A_I \in S\wp(q)$. Let curve $A(t)$ in $SO(q)$ satisfy $A(0) = I$ and $\frac{dA(t)}{dt}_{|t=0} = A_I$, then

$$\mathbf{A}(a) = da\ \frac{dA(t)}{dt}_{|t=0} = \frac{d\ aA(t)}{dt}_{|t=0} = \frac{d}{dt}_{|t=0}(b, A(t)\mathbf{e}_b),$$

where the frame $A(t)\mathbf{e}_b$ has vectors as described near the end of p. 53.

We must show that

$$(\gamma_{ji}, d\gamma_{ji})_*\ \mathbf{A}(a) = \mathbf{A}(\ [a](\gamma_{ji}, d\gamma_{ji})\ ).$$

Certainly,

$$\begin{aligned}(\gamma_{ji}, d\gamma_{ji})_*\ \mathbf{A}(a) &= \tfrac{d}{dt}_{|t=0}(\ [(b, A(t)\mathbf{e}_b)]\ (\gamma_{ji}, d\gamma_{ji})\ ) \\ &= \tfrac{d}{dt}_{|t=0}(\ \gamma_{ji}(b), d\gamma_{ji}(A(t)\mathbf{e}_b)\ ).\end{aligned}$$

The frame $d\gamma_{ji}(A(t)\mathbf{e}_b)$ is the same as the frame $A(t)d\gamma_{ji}(\mathbf{e}_b)$, so

$$(\gamma_{ji}, d\gamma_{ji})_*\ \mathbf{A}(a) = \frac{d}{dt}_{|t=0}(\ \gamma_{ji}(b), A(t)d\gamma_{ji}(\mathbf{e}_b)\ ) =$$

$$\frac{d}{dt}_{|t=0}(\ \gamma_{ji}(b), d\gamma_{ji}(\mathbf{e}_b)\ )A(t) = \frac{d}{dt}_{|t=0}(\ [a](\gamma_{ji}, d\gamma_{ji})\ )A(t) =$$

$$d(\ [a](\gamma_{ji}, d\gamma_{ji})\ )\ \frac{dA(t)}{dt}_{|t=0} = d(\ [a](\gamma_{ji}, d\gamma_{ji})\ )\ A_I.$$

We have the desired result since the last expression is the same as

$$\mathbf{A}(\ [a](\gamma_{ji}, d\gamma_{ji})\ )\Box.$$

Thus, $B(T)$ has a parallelism which is invariant under elements of G, which is defined as the pseudogroup generated by the local diffeomorphisms $(\gamma_{ji}, d\gamma_{ji})$ of $B(T)$.

For future use, we must develop a metric on $B(T)$ in which elements of G are local isometries. Define a metric

$$< \mathbf{X}, \mathbf{Y} >_{B(T)} = < \omega(\mathbf{X}), \omega(\mathbf{Y}) >_{R^q} + < \theta(\mathbf{X}), \theta(\mathbf{Y}) >_{SO(q)}$$

where $< , >_{R^q}$ is the standard euclidean metric and $< A, B >_{SO(q)} = \frac{1}{2}\, trace(A^t B)$ on $SO(q)$. The fundamental vector fields $\mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}$ satisfy

$$< \theta(\mathbf{A}_i), \theta(\mathbf{A}_j) >_{SO(q)} = \delta_{ij}$$

when they are chosen, as on p.54, so that they correspond to the usual basis of skew symmetric matrices. The basic vector fields satisfy

$$< \omega(\mathbf{E}_i), \omega(\mathbf{E}_j) >_{R^q} = < (0, .., 0, 1, 0, .., 0), (0, .., 0, 1, 0, .., 0) >_{R^q} = \delta_{ij}.$$

The fundamental and basic vector fields are mutually orthogonal so the parallelism

$$\mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}, \mathbf{E}_1, .., \mathbf{E}_q$$

is orthonormal. It follows that elements of G are local isometries in the metric $< , >_{B(T)}$.

After the preceeding discussion of the transverse space $B(T)$, we return to our study of the lifted foliation $\mathcal{F}_T$. We can conclude that $\mathcal{F}_T$ is transversally parallelisable once we discuss how elements of G relate to the local submersions which determine $\mathcal{F}_T$.

Definition 5.19 *Let φ_i be a riemannian submersion which determines $\mathcal{F}$ on a neighborhood U_i. Define the map $(\varphi_i, d\varphi_i)$ from $e(U_i) \subset e(M)$ to $B(\varphi_i(U_i))$ by*

$$[(p, \mathbf{f}_p)](\varphi_i, d\varphi_i) = (\varphi_i(p), d\varphi_i(\mathbf{f}_p)).$$

The domain of $(\varphi_i, d\varphi_i)$ is diffeomorphic to $U_i \times SO(q)$, and we can say, for $U_i \cap U_j \neq \emptyset$, that

$$[(p, \mathbf{f}_p)](\varphi_j, d\varphi_j) = [\ [p, \mathbf{f}_p](\varphi_i, d\varphi_i)\ ](\gamma_{ji}, d\gamma_{ji}).$$

It is easily seen that the $(\varphi_i, d\varphi_i)$ are locally constant on the leaves of $\mathcal{F}_T$. To show, rigorously, that leaves are regular submanifolds of dimension $n-q$, the maps $(\varphi_i, d\varphi_i)$ must be shown to have the maximal rank, $\frac{q(q-1)}{2} + q$. It will follow that $\mathcal{F}_T$ is a transversally parallelisable foliation of $e(M)$.

To show that the $(\varphi_i, d\varphi_i)$ are local submersions, we will study a parallelism on $e(M)$ which is transverse to $\mathcal{F}_T$ and projects to the invariant parallelism on $B(T)$.

To describe the portion of the parallelism which projects to the basic vector fields in the parallelism on $B(T)$, we must first define some notion of lifting horizontal paths in M to paths in $e(M)$.

Let $c(s)$ be a horizontal path in M with $c(0) = p$, that is, let $c(s)$ be tangent to $(T\mathcal{F})^{\perp}$ for a riemannian foliation $\mathcal{F}$ with a bundlelike metric. (See [1.4], p.9 for the definition of a bundlelike metric.) A description will follow of a canonical way of lifting $c(s)$ to a path $\tilde{c}(s)$ in $e(M)$ which begins at point $q = (p, \mathbf{f}_p)$. Suppose that $c(s)$ lies in a neighborhood U_i, on which a riemannian submersion φ_i determines $\mathcal{F}$. The canonical horizontal lift of $\varphi_i \circ c(s)$ to $B(\varphi_i(U_i))$ is a path

$$\varphi \tilde{\circ} c(s) = (\ \varphi_i \circ c(s), \mathbf{P}_{\varphi_i \circ c(s)}\ (d\varphi_i(\mathbf{f}_p))$$

which begins at

$$(\ \varphi_i(p), d\varphi_i(\mathbf{f}_p)\ ).$$

We pause to introduce some new notation. A vector at $\varphi_i \circ c(s)$ can be lifted to a unique vector at $c(s)$ which lies in $(T\mathcal{F})^\perp$. Similarly, for an orthonormal frame $\mathbf{e}_{\varphi_i \circ c(s)}$ at $\varphi_i \circ c(s)$, there corresponds a unique orthonormal frame of horizontal vectors, denoted by

$$d\varphi_i^{-1}\ \mathbf{e}_{\varphi_i \circ c(s)},$$

at $c(s)$ such that $d\varphi_i(\ d\varphi_i^{-1}\ \mathbf{e}_{\varphi_i \circ c(s)}\ ) = \mathbf{e}_{\varphi_i \circ c(s)}$.

Define the lift of $c(s)$, locally, to be

$$\tilde{c}(s) = (\ c(s), d\varphi_i^{-1}\ \mathbf{P}_{\varphi_i \circ c(t)}(d\varphi_i(\mathbf{f}_p))\ ),$$

where $\mathbf{P}_{\varphi_i \circ c(t)}(d\varphi_i(\mathbf{f}_p))$ is a frame at $\varphi_i \circ c(s)$. We will show that $\tilde{c}(s)$ does not depend on the choice of riemannian submersion φ_i at p. Suppose p and $c(s)$ lie in $U_i \cap U_j$, where riemannian submersion φ_j determines $\mathcal{F}$ on U_j. Using φ_j, the lift of $c(s)$ is a path

$$(\ c(s), d\varphi_j^{-1}\ \mathbf{P}_{\varphi_j \circ c(t)}(d\varphi_j(\mathbf{f}_p))\ ).$$

To show that this lift agrees with the lift via φ_i, let $\gamma_{ji} \circ \varphi_i = \varphi_j$ for local isometry γ_{ji} and observe that

$$d\varphi_j(\ d\varphi_i^{-1}\ \mathbf{P}_{\varphi_i \circ c(t)}(d\varphi_i(\mathbf{f}_p))\ ) = d\gamma_{ji} \circ d\varphi_i(\ d\varphi_i^{-1}\ \mathbf{P}_{\varphi_i \circ c(t)}(d\varphi_i(\mathbf{f}_p))\ )$$

$$= d\gamma_{ji}\ \mathbf{P}_{\varphi_i \circ c(t)}(d\varphi_i(\mathbf{f}_p)) = \mathbf{P}_{\gamma_{ji} \circ \varphi_i \circ c(s)}(d\gamma_{ji} \circ d\varphi(\mathbf{f}_p)) = \mathbf{P}_{\varphi_j \circ c(t)}(d\varphi_j(\mathbf{f}_p)).$$

Since frames $d\varphi_j^{-1}\ \mathbf{P}_{\varphi_i \circ c(t)}(d\varphi_i(\mathbf{f}_p))$ and $d\varphi_j^{-1}\ \mathbf{P}_{\varphi_j \circ c(t)}(d\varphi_j(\mathbf{f}_p))$ at $c(s)$ have the same $d\varphi_j$ projection, they are equal. This local argument extends to give a unique lifting of all of $c(s)$ to $\tilde{c}(s)$, beginning at $(p, \mathbf{f}_p)$.

Define the projection map

$$p_T : e(M) \rightarrow M$$

by

$$p_T\ (p, \mathbf{f}_p) \ = \ p.$$

Consider a distribution which, at any point $q = (p, \mathbf{f}_p)$ in $e(M)$, is the span of tangent vectors of canonical lifts of all horizontal paths starting at p. For vector $\mathbf{X} \ = \ (x_1, .., x_q) \in R^q$, a **"basic"** vector field $\tilde{\mathbf{E}}(\mathbf{X})$ is a vector field which is tangent to the above distribution and satisfies

$$dp_T\ \tilde{\mathbf{E}}(\mathbf{X})(q) \ = \ q \bullet \mathbf{X},$$

at every $q \in e(M)$.

Existence and uniqueness of $\tilde{\mathbf{E}}(\mathbf{X})$ at any point $q = (p, \mathbf{f}_p)$ can be established by showing that the lift $\tilde{c}(s)$ of a geodesic $c(s)$ in M which satisfies

$$c(0) = p$$

and

$$\frac{dc(s)}{ds}_{|s=0} \ = \ q \bullet \mathbf{X}$$

is the integral curve of $\tilde{\mathbf{E}}(\mathbf{X})$ which begins at q. Observe that this geodesic is horizontal everywhere since it is horizontal at the point $c(0)$, and since this is sufficient when M has a bundlelike metric, from [H]. Clearly, $\tilde{\mathbf{E}}(\mathbf{X})$ is globally defined on $e(M)$ and we can express

$$\tilde{\mathbf{E}}(\ (x_1, .., x_q)\ ) \ = \ \sum_{i=1}^{q} x_i \tilde{\mathbf{E}}_i$$

where $\tilde{\mathbf{E}}_i \ = \ \tilde{\mathbf{E}}(\ (0, .., 0, 1, 0, .., 0)\ )$.

Lemma 5.20 *Let $\mathcal{F}$ be determined by a riemannian submersion φ on a neighborhood U of p. Then at $q = (p, \mathbf{f}_p)$,*

$$(\varphi, d\varphi)_*\ \tilde{\mathbf{E}}(\mathbf{X})(q) \ = \ \mathbf{E}(\mathbf{X})(\ [q](\varphi, d\varphi)\ ).$$

proof

Let $q(s)$ be the integral curve of $\tilde{\mathbf{E}}(\mathbf{X})$ with $q(0) = q$. We will show that $[q(s)](\varphi, d\varphi)$ is an integral curve of $\mathbf{E}(\mathbf{X})$. The path $\gamma(s) = p_T(q(s))$ in U is horizontal, since

$$\frac{d\gamma}{ds} = dp_T \frac{dq(s)}{ds} = dp_T \tilde{\mathbf{E}}(\mathbf{X})(q(s)) = q(s) \bullet \mathbf{X},$$

which is a linear combination of horizontal vectors in some orthonormal frame, so we can express

$$q(s) = (\ \gamma(s), d\varphi^{-1} \mathbf{P}_{\varphi\circ\gamma(s)}(d\varphi(\mathbf{f}_p))\ )$$

as the canonical lift of $\gamma(s)$ to $e(M)$. Clearly,

$$[q(s)](\varphi, d\varphi) = (\ \varphi \circ \gamma(s), \mathbf{P}_{\varphi\circ\gamma(s)}(d\varphi(\mathbf{f}_p))\ ).$$

Also,

$$d\pi \frac{d}{ds}(\ [q(s)](\varphi, d\varphi)\ ) = \frac{d}{ds}\varphi \circ \gamma(s) = d\varphi(\ q(s) \bullet \mathbf{X})$$

$$= [q(s)](\varphi, d\varphi) \bullet \mathbf{X}.$$

Recall that $\mathbf{E}(\mathbf{X})$ was defined by $d\pi\ \mathbf{E}(\mathbf{X})(a) = a \bullet \mathbf{X}$, so that

$$\frac{d}{ds}[q(s)](\varphi, d\varphi) = \mathbf{E}(\mathbf{X})(\ [q(s)](\varphi, d\varphi)\ ).$$

The lemma follows from setting $s = 0$ in the following:

$$(\varphi, d\varphi)_* \tilde{\mathbf{E}}(\mathbf{X})(q(s)) = \frac{d}{ds}[q(s)](\varphi, d\varphi) = \mathbf{E}(\mathbf{X})(\ [q(s)](\varphi, d\varphi)\ ).\square$$

From the lemma, the "basic" vector fields $\tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q$ project under $(\varphi_i, d\varphi_i)_*$ to the basic vector fields $\mathbf{E}_1, .., \mathbf{E}_q$ on $B(T)$.

To describe the portion of the parallelism on $e(M)$ which projects to the fundamental vector fields in the parallelism on $B(T)$, we must define something analogous to a fundamental vector field on $e(M)$. Using the same notation as on p.54, a vector field $\tilde{\mathbf{A}}$ on $e(M)$ is called **"fundamental"** if, at every $q = (p, \mathbf{f}_p) \in e(M)$,

$$\tilde{\mathbf{A}}(q) = dq\ A_I$$

for a fixed $A_I \in S\wp(q)$. We note that $\tilde{\mathbf{A}}$ is globally defined on $e(M)$ and $\tilde{\mathbf{A}} \in span\{\ \tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}\ \}$, where the $\tilde{\mathbf{A}}_i$ are images of the standard basis on $S\wp(q)$ under dq.

Lemma 5.21 *Let φ be a riemannian submersion which locally determines $\mathcal{F}$. Let $\tilde{\mathbf{A}} = dq\ A_I$ be a "fundamental" vector field at $q = (p, \mathbf{f}_p)$ in $e(M)$ which corresponds to $A_I \in S\wp(q)$. Then*

$$(\varphi, d\varphi)_*\ \tilde{\mathbf{A}} = d(\ [q](\varphi, d\varphi)\ )A_I$$

is a fundamental vector field on $B(T)$.

proof

Let $A(t)$ be a path in $S\wp(q)$ with $A(0) = I$ and $\frac{dA}{dt}_{|t=0} = A_I$. Let the frame have vectors

$$\mathbf{f}_p = \{\mathbf{f}_1, .., \mathbf{f}_q\}.$$

We can write

$$(\varphi, d\varphi)_*\ \tilde{\mathbf{A}} = (\varphi, d\varphi)_*\ dq\ \frac{dA}{dt}_{|t=0} =$$

$$(\varphi, d\varphi)_*\ \frac{d}{dt}_{|t=0}(p, A\mathbf{f}_p) = \frac{d}{dt}_{|t=0}(\ [p, A\mathbf{f}_p](\varphi, d\varphi)\ ) = \frac{d}{dt}_{|t=0}(\ \varphi, d\varphi A\mathbf{f}_p\ ).$$

The frame

$$d\varphi A\mathbf{f}_p \;=\; \{\; \sum_{i=1}^{q} d\varphi a_{i1}\mathbf{f}_i, .. \sum_{i=1}^{q} d\varphi a_{iq}\mathbf{f}_i \;\}$$

is orthonormal, since $A\mathbf{f}_p$ is an orthonormal frame in $(T\mathcal{F})^{\perp}$ and since $d\varphi$ is an isometry on $(T\mathcal{F})^{\perp}$. Clearly, $d\varphi\ A\ \mathbf{f}_p \;=\; A\ d\varphi\ \mathbf{f}_p$ since for $j = 1, ..q$,

$$\sum_{i=1}^{q} d\varphi a_{ij}\mathbf{f}_i \;=\; \sum_{i=1}^{q} a_{ij} d\varphi \mathbf{f}_i,$$

so that

$$(\varphi, d\varphi)_* \;\tilde{\mathbf{A}} \;=\; \frac{d}{dt}_{|t=0}(\varphi(p), Ad\varphi\ \mathbf{f}_p) \;=\; \frac{d}{dt}_{|t=0}(\ (\varphi(p), d\varphi\ \mathbf{f}_p)\ )A$$

$$= d(\ [q](\varphi, d\varphi)\ )\ \frac{dA}{dt}_{\ |t=0} \;=\; d(\ [q](\varphi, d\varphi)\ )\ A_I,$$

which is the fundamental vector field in $B(T)$ corresponding to A_I. □

From the lemma, the "fundamental" vector fields $\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}$ on $e(M)$ project to the fundamental vector fields $\mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}$ in $B(T)$ under $(\varphi, d\varphi)_*$. It follows that

$$\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}, \tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q$$

is a parallelism on $e(M)$, so $(\varphi, d\varphi)_*$ has the maximal rank $\frac{q(q-1)}{2} + q$.

CHAPTER 6

We saw in Chapter 5 that $e(M)$ has a transversally parallelisable foliation $\mathcal{F}_T$ when $\mathcal{F}$ is a riemannian foliation on M and we discussed properties of transversally parallelisable foliations. In this chapter, we study $\mathcal{F}_T$ to deduce properties of $\mathcal{F}$. Specifically, we give a proof different from the one mentioned in Chapter 1, that a riemannian foliation has an Ehresmann connection, and we use Proposition 5.8 on the growth of leaves in transversally parallelisable foliations to give an alternate proof of results in Chapter 4 on the growth of leaves for riemannian foliations.

We will suppose, throughout this chapter, that $\mathcal{F}$ is a riemannian foliation of codimension q and that a corresponding bundlelike metric has been chosen for M.

We begin by describing a metric on $e(M)$. Let $\mathcal{F}$ be determined locally at p by a riemannian submersion φ_i. At $q = (p, \mathbf{f}_p)$ in $e(M)$, define a metric by

Definition 6.1

$$< , >_{e(M)} = (\varphi_i, d\varphi_i)^*(< , >_{B(T)}) + p_T^*(< , >_{|\,T\mathcal{F}}),$$

where $< , >_{B(T)}$ *is described on p.58 of Chapter 5 and* $< , >_{|\,T\mathcal{F}}$ *is the restriction of the bundlelike metric on* M *to* $T\mathcal{F}$.

The definition does not depend on the choice of φ_i, for let φ_j be another local riemannian submersion at p, then

$$(\varphi_j, d\varphi_j)^*(< , >_{B(T)}) =$$

$$(\gamma_{ji} \circ \varphi_i, d\gamma_{ji} \circ d\varphi_i)^*(< , >_{B(T)}) = (\varphi_i, d\varphi_i)^* \ (\gamma_{ji}, d\gamma_{ji})^* \ (< , >_{B(T)})$$
$$= (\varphi_i, d\varphi_i)^* \ (< , >_{B(T)}),$$

since $(\gamma_{ji}, d\gamma_{ji})$ is an isometry in the $< , >_{B(T)}$ metric.

To show that the projection p_T induces a local isometry from leaves of $\mathcal{F}_T$ to leaves of $\mathcal{F}$, we show, first, that $p_{T*|\,T\mathcal{F}_T}$ has *rank* $n-q$. Observe that

$$\begin{array}{rcccccc} Te(M) & = & T\mathcal{F}_T & \oplus & span\ \{\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}\ \} & \oplus & span\ \{\tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q\ \} \\ TM & = & T\mathcal{F} & \oplus & O & \oplus & (T\mathcal{F})^\perp \end{array}$$

where $rank\ p_{T*} = n$ and $rank\ p_{T*|\,span\ \{\tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q\ \}} = dim(T\mathcal{F})^\perp = q$, therefore $rank\ p_{T*|\,T\mathcal{F}_T} = n - q$. It follows that $p_T^*(< , >_{|\,T\mathcal{F}})$ is a non singular metric on $T\mathcal{F}_T$. Since $(\varphi_i, d\varphi_i)_*$ annhilates $T\mathcal{F}_T$, we have

$$< \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} >_{e(M)} = p_T^*(< \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} >_{|\,T\mathcal{F}})$$

for $\tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in T\mathcal{F}_T$, so p_T is a local isometry.

To show that $T\mathcal{F}_T$ is perpendicular to "fundamental" and "basic" vector fields, observe that

$$(\varphi_i, d\varphi_i)^*(< \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} >_{|\,T\mathcal{F}}) = 0$$

when $\tilde{\mathbf{X}} \in T(\mathcal{F}_T)$ and that $p_{T*}(< \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} >_{T\mathcal{F}}) = 0$ when $\tilde{\mathbf{Y}}$ is "fundamental" or "basic", since the images of *span* $\{\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}\ \}$ and *span* $\{\tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q\ \}$ under p_{T*} are O and $(T\mathcal{F})^\perp$, respectively. It follows that

$$(T\mathcal{F}_T)^\perp = span\{\ \tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}, \tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q\ \},$$

since

$$dim(T\mathcal{F}_T)^\perp = codim\ \mathcal{F}_T = \frac{q(q-1)}{2} + q.$$

Since $(\varphi_i, d\varphi_i)_*\ \tilde{\mathbf{A}}_j = \mathbf{A}_j$ and $(\varphi_i, d\varphi_i)_*\ \tilde{\mathbf{E}}_k = \mathbf{E}_k$, we can regard the parallelism

$$\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}, \tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q$$

as the lift to $(T\mathcal{F}_T)^\perp$ of the parallelism

$$\mathbf{A}_1, .., \mathbf{A}_{\frac{q(q-1)}{2}}, \mathbf{E}_1, .., \mathbf{E}_q$$

on $B(T)$, therefore the parallelism on $e(M)$ consists of foliated vector fields which are orthonormal in the $<\ ,\ >_{e(M)}$ metric. From remarks on p.49 of Chapter 5, it follows that $\mathcal{F}_T$ is a riemannian foliation on $e(M)$ with the metric $<\ ,\ >_{e(M)}$.

Also, from these remarks, it follows that $(T\mathcal{F}_T)^\perp$ is an Ehresmann connection for the transversally parallelisable foliation $\mathcal{F}_T$. That is, horizontal paths of rectangles in $e(M)$ are determined by a time dependent flow along a vector field in

$$span\{\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}, \tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q\ \}.$$

For future reference, we will sometimes denote $\tilde{D} = (T\mathcal{F}_T)^\perp$.

We, now, discuss how rectangles in M lift to rectangles in $e(M)$. Note that we are not assuming that $D = T\mathcal{F}$ is an Ehresmann connection for $\mathcal{F}$, that is, we are not assuming that rectangles necessarily exist for any horizontal and vertical paths beginning at a common point in M. The relationship between rectangles on M and on $e(M)$ in the following lemma, together with the property that $\tilde{D}$ is an Ehresmann connection for $\mathcal{F}_T$, will, later, be used to show that D is an Ehresmann connection for $\mathcal{F}$.

Lemma 6.2 *$\tilde{D}$ is the lift of D, that is, rectangles in M lift to rectangles in $e(M)$.*

proof

Suppose $\delta(s,t)$, $(s,t) \in [0,1] \times [0,1]$, is a rectangle in M with initial horizontal and vertical edges $\delta(s,0) = c(s)$ and $\delta(0,t) = \tau(t)$, respectively.

For sufficiently small $s_1 \in [0,1]$, there is a partition

$$0 = t_0 < t_1 < .. < t_n = 1$$

and a family of neighborhoods $\{ U_i \}_{i=1}^n$, with $U_l \cap U_{l-1} \neq \emptyset$ for each l, such that $\delta(s,t)$, $(s,t) \in [0,s_1] \times [t_{i-1}, t_i]$ lies in U_i. On each U_i, let $\mathcal{F}$ be determined by a riemannian submersion φ_i. We will show that $\delta(s,t)$, $(s,t) \in [0,1] \times [0,s_1]$ lifts to a rectangle in $e(M)$.

For future use, we remark, for any $0 \leq s' \leq s_1$, that the portion of the vertical path $\delta(s',t)$ which lies in U_i is related to $c(s') = \delta(s',0)$ by

$$\varphi_i \circ \delta(s',t) = \gamma_{i\ i-1} \circ ... \gamma_{2\ 1} \circ \varphi_1 \circ c(s'), \tag{6.3}$$

where the local isometries $\gamma_{l\ l-1}$ satisfy

$$\varphi_l = \gamma_{l\ l-1} \circ \varphi_{l-1}$$

on $U_l \cap U_{l-1}$.

Let $\tilde{c}(s)$ and $\tilde{\tau}(t)$ be lifts of horizontal and vertical paths $c(s)$ and $\tau(t)$, respectively, to paths starting at $(p, \mathbf{f}_p)$ in $e(M)$. For $0 \leq s \leq s_1$,

$$\tilde{c}(s) = (c(s), d\varphi_1^{-1}\ \mathbf{P}_{\varphi_1 \circ c(s)}\ (d\varphi_1(\mathbf{f}_p))\ )$$

and, certainly $\tilde{c}(s)$ is tangent to $\tilde{D}$. (For notation, see Chapter 5, top of p.60.) We can express

$$\tilde{\tau}(t) = (\tau(t), \mathbf{f}_{\tau(t)}),$$

where the frame $\mathbf{f}_{\tau(t)}$, for $\tau(t)$ in U_i, satisfies

$$d\varphi_i\ \mathbf{f}_{\tau(t)} = d\gamma_{i\ i-1} \circ .. \circ d\gamma_{2\ 1} \circ d\varphi_1\ \mathbf{f}_p. \tag{6.4}$$

Certainly, $\tilde{\tau}(t)$ lies in a leaf of $\mathcal{F}_T$ since, locally, $[\tilde{\tau}(t)](\varphi_i, d\varphi_i)$ is constant in t. (See Chapter 5, p.51.)

For the portion of rectangle δ in U_i, a horizontal path $\delta(s,t')$, lifts to a path

$$\tilde{\delta}(s,t') \;=\; (\;\delta(s,t'), d\varphi_i^{-1}\mathbf{P}_{\varphi_i\circ\delta(s,t')}(d\varphi_i(\mathbf{f}_{\tau(t')}))\;)$$

in $e(M)$ with starting point $\tilde{\tau}(t') \;=\; (\tau(t'), \mathbf{f}_{\tau(t')})$. A vertical path $\delta(s',t)$ lifts to a path

$$\tilde{\delta}(s',t) \;=\; (\delta(s',t), \mathbf{f}_{\delta(s',t)}),$$

which starts at $\tilde{c}(s') \;=\; (c(s'), d\varphi_1^{-1}\mathbf{P}_{\varphi_1\circ c(s')}(d\varphi_1(\mathbf{f}_p))\;)$. Note that the frame $\mathbf{f}_{\delta(s',t)}$ satisfies

$$d\varphi_i\;\mathbf{f}_{\delta(s',t)} \;=\; d\gamma_{i\;i-1}\circ..\circ d\gamma_{2\;1}\circ d\varphi_1\;(\;d\varphi_1^{-1}\mathbf{P}_{\varphi_1\circ c(s)}(d\varphi_1(\mathbf{f}_p))\;),$$

when $\delta(s',t)\;\in U_i$.

We observe that frames $\mathbf{f}_{\delta(s',t)}$ and $d\varphi_i^{-1}\mathbf{P}_{\varphi_i\circ\delta(s',t)}(d\varphi_i(\mathbf{f}_{\tau(t)}))$ are the same since, from [6.3] and [6.4], the projection of the second frame is

$$\mathbf{P}_{\varphi_i\circ\delta(s',t)}(d\varphi_i(\mathbf{f}_{\tau(t)})) \;=\; \mathbf{P}_{\gamma_{i\;i-1}\circ\ldots\gamma_{2\;1}\circ\varphi_1\circ c(s')}(d\gamma_{i\;i-1}\circ\ldots\circ d\gamma_{2\;1}\circ d\varphi_1\;(\mathbf{f}_p),$$

which agrees with the projection $d\varphi_i(\mathbf{f}_{\delta(s',t)})$.

Thus, the lift, $\tilde{\delta}(s',t)$, of vertical path $\delta(s',t)$ is

$$(\delta(s',t), \mathbf{f}_{\delta(s',t)}) \;=\; (\delta(s',t), d\varphi_i^{-1}\mathbf{P}_{\varphi_i\circ\delta(s',t)}(d\varphi_i(\mathbf{f}_{\tau(t)}))\;),$$

which meets the lift, $\tilde{\delta}(s,t')$, of horizontal path $\delta(s,t')$ when $(s',t) \;=\; (s,t')$. It follows that $\tilde{\delta}(s,t)$, $(s,t)\in[0,s_1]\times[0,1]$, is a rectangle in $e(M)$ with horizontal and vertical paths lifted from those in $\delta(s,t)$, $(s,t)\in[0,s_1]\times[0,1]$. From repeating this argument, we see that all of $\delta(s,t)$, $(s,t)\in[0,1]\times[0,1]$ lifts to a rectangle in $e(M)$.□

Since $\mathcal{F}_T$ is a transversally parallelisable foliation of $e(M)$, we know from Chapter 5 that the horizontal paths of any rectangle in $e(M)$ are solutions to a differential equation, which can be expressed in terms of the initial horizontal edge.

For thoroughness, we will describe that equation, for a lifted rectangle in $e(M)$, and show, directly, that the horizontal paths of the lifted rectangle are solutions.

We need the following definition.

Definition 6.5 *At a point* $q = (p, \mathbf{f}_p)$ *in* $e(M)$, *the solder form on* $e(M)$ *is a map*

$$\tilde{\omega} : T_q e(M) \rightarrow R_q$$

defined on span $\{\tilde{\mathbf{E}}_1, .., \tilde{\mathbf{E}}_q\}$ *by*

$$\tilde{\omega}(\tilde{\mathbf{X}}_q) = \mathbf{X},$$

where $dp_T(\tilde{\mathbf{X}}_q) = q \bullet \mathbf{X}$. *On* $T\mathcal{F}_T$, $\tilde{\omega}$ *is defined to be* 0.

Also, note that $\tilde{\omega}$ vanishes on *span* $\{\tilde{\mathbf{A}}_1, .., \tilde{\mathbf{A}}_{\frac{q(q-1)}{2}}\}$ since $T_q\ p_T^{-1}(p)$ is the kernel of dp_T.

Observe that $\tilde{\omega}(\tilde{\mathbf{E}}(\mathbf{X})(q)) = \mathbf{X}$, for any q, since

$$dp_T\ \tilde{\mathbf{E}}(\mathbf{X})(q) = q \bullet \mathbf{X}.$$

For horizontal path $c(s)$ starting at p with lift $\tilde{c}(s)$ starting at $(p, \mathbf{f}_p)$,

$$\frac{dc(s)}{ds} = dp_T \frac{d\tilde{c}}{ds}(s) = \tilde{c}(s) \bullet \tilde{\omega}(\frac{d\tilde{c}}{ds}),$$

so

$$\frac{d\tilde{c}}{ds}(s) = \tilde{\mathbf{E}}(\tilde{\omega}(\frac{d\tilde{c}}{ds}))(\tilde{c}(s)),$$

that is, $\tilde{c}(s)$ is a solution for $q(s)$ in the equation

$$\frac{dq(s)}{ds} = \tilde{\mathbf{E}}(\tilde{\omega}(\frac{d\tilde{c}}{ds}))\ (q(s)).$$

We remark that since

$$\tilde{\mathbf{E}}(\ (a_1, .., a_q)\ ) = \sum_{i=1}^{q} a_i \tilde{\mathbf{E}}_i,$$

the preceeding differential equation can be written, in terms of the parallelism on $e(M)$, as

$$\frac{dq(s)}{ds} = \sum_{i=1}^{q} \tilde{\omega}_i\ (\frac{d\tilde{c}}{ds})\ \tilde{\mathbf{E}}_i(q(s)),$$

where

$$\tilde{\omega}(\frac{d\tilde{c}}{ds}) = (\tilde{\omega}_1(\frac{d\tilde{c}}{ds}), .., \tilde{\omega}_q(\frac{d\tilde{c}}{ds})).$$

Lemma 6.6 *Let $\delta(s,t)$, $(s,t) \in [0,1] \times [0,1]$ be a rectangle with initial horizontal and terminal edges $\delta(s,0) = c(s)$ and $\delta(0,t) = \tau(t)$, respectively. Let the lifted path*

$$\tilde{\tau}(t) = (\tau(t), \mathbf{f}_{\tau(t)})$$

begin at $(p, \mathbf{f}_p)$. Then the lift of any horizontal path $\delta(s,t')$ to a path $\tilde{\delta}(s,t')$ beginning at $\tilde{\tau}(t')$ is a solution for $q(s)$ in the equation

$$\frac{dq(s)}{ds} = \tilde{\mathbf{E}}(\tilde{\omega}(\frac{d\tilde{c}}{ds}))\ (q(s)).$$

proof

Once the lemma is proved on a portion $\delta(s,t)$ $(s,t) \in [0,s_1] \times [0,1]$ of the rectangle, then it will hold for all $(s,t) \in [0,1] \times [0,1]$ by repeating the argument on other portions. Assume s_1 is sufficiently small so that

$\delta(s,t)$, $(s,t) \in [0,s_1] \times [0,1]$ is covered by a family $\{ U_i \}_{i=1}^{n}$ of neighborhoods as described early in the proof of Lemma 6.2.

First, consider $\delta(s,t)$, $(s,t) \in [0,s_1] \times [0,t_1]$, which is contained in U_1, the domain of a riemannian submersion φ_1. For a fixed $t' \in [0,t_1]$, the lifted path

$$\tilde{\delta}(s,t') = (\delta(s,t'), d\varphi_1^{-1} \mathbf{P}_{\varphi_1 \circ \delta(s,t')}(d\varphi_1(\mathbf{f}_{\tau(t')})))$$

is a solution for $q(s)$ in

$$\frac{dq(s)}{ds} = \tilde{\mathbf{E}}(\tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t'))) (q(s)).$$

It suffices to show that $\tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t')) = \tilde{\omega}(\frac{d\tilde{c}}{ds}(s))$. From the definition of $\tilde{\omega}$,

$$\frac{d\delta}{ds}(s,t') = dp_T \frac{d\tilde{\delta}}{ds}(s,t')$$

$$= \tilde{\delta}(s,t') \bullet \tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t')).$$

Since $\delta(s,t')$ is in U_1, $\varphi_1 \circ \delta(s,t') = \varphi_1 \circ c(s)$, so

$$d\varphi_1 \frac{d\delta}{ds}(s,t') = d\varphi_1(\tilde{\delta}(s,t') \bullet \tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t'))$$

agrees with

$$d\varphi_1 \frac{dc(s)}{ds} = d\varphi_1(\tilde{c}(s) \bullet \tilde{\omega}(\frac{d\tilde{c}}{ds}(s))).$$

Recalling the notation in the proof of [5.20] , an equation, using the right hand sides of the preceeding two equations, can be rewritten as

$$([\tilde{\delta}(s,t')] (\varphi_1, d\varphi_1)) \bullet \tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t')) = ([\tilde{c}(s)] (\varphi_1, d\varphi_1)) \bullet \tilde{\omega}(\frac{d\tilde{c}}{ds}(s)).$$

Observe that

$$[\tilde{\delta}(s,t')](\varphi_1, d\varphi_1) = (\varphi_1 \circ \delta(s,t'), \mathbf{P}_{\varphi_1 \circ \delta(s,t')}(d\varphi_1(\mathbf{f}_{\tau(t')}))$$

agrees with

$$[\ \tilde{c}(s)\ ](\varphi_1, d\varphi_1) = (\ \varphi_1 \circ c(s), \mathbf{P}_{\varphi_1 \circ c(s)}(d\varphi_1(\mathbf{f}_p))\ )$$

since $\varphi_1 \circ \delta(s,t') = \varphi_1 \circ c(s)$ and $d\varphi_1\ \mathbf{f}_{\tau(t')} = d\varphi_1\ \mathbf{f}_p$, for $\delta(s,t')$ in U_1. It follows that

$$\tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t')) = \tilde{\omega}(\frac{d\tilde{c}}{ds}(s)),$$

for $0 \leq t' \leq t_1$.

Now, consider $\delta(s,t)$, $(s,t) \in [0,s_1] \times [t_1,t_2]$, which is contained in U_2. The preceeding argument implies that

$$\tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t')) = \tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t_1))$$

for $t_1 \leq t' \leq t_2$, therefore $\tilde{\omega}(\frac{d\tilde{\delta}}{ds}(s,t')) = \tilde{\omega}(\frac{d\tilde{c}}{ds}(s))$. By repeating this argument over the partition $0 = t_0 < \ldots < t_n = 1$, we conclude that

$$\tilde{\omega}(\frac{d\delta}{ds}(s,t')) = \tilde{\omega}(\frac{d\tilde{c}}{ds}(s))$$

for $(s,t') \in [0,s_1] \times [0,1]$.□

We wish to prove the result of Blumenthal and Hebda that riemannian foliations have Ehresmann connections.

Proposition 6.7 *Let $\mathcal{F}$ be a riemannian foliation on M with a bundlelike metric. Then the horizontal distribution $D = (T\mathcal{F})^{\perp}$ is an Ehresmann connection.*

proof

Let $\tau(t)$ and $c(s)$ be vertical and horizontal paths, respectively, with $\tau(0) = p\ c(0)$. Let $\tilde{\tau}(t)$ and $\tilde{c}(s)$ be their lifts to paths in $e(M)$ starting at

$(p, \mathbf{f}_p)$. From Chapter 5, p.49 and the remarks above, the rectangle $\tilde{\delta}(s,t)$ with initial horizontal edge $\tilde{c}(s)$ and initial vertical edge $\tilde{\tau}(t)$ is determined by the flow of the differential equation

$$\frac{dq(s)}{ds} = \tilde{\mathbf{E}}(\frac{d\tilde{c}(s)}{ds})\ (q(s))$$

by letting, for each t', $\tilde{\delta}(s,t')$ be the solution which satisfies $\tilde{\delta}(0,t') = \tilde{\delta}(t')$. Clearly, horizontal paths $\tilde{\delta}(s,t')$ in $\mathcal{F}_T$ project under p_T to horizontal paths in M starting at $\tau(t')$ and vertical paths $\tilde{\delta}(s',t)$ project to vertical paths starting at $c(s')$. It follows that $\delta(s,t) = p_T\tilde{\delta}(s,t)$ is a rectangle with initial vertical edge $\tau(t)$ and initial horizontal edge $c(s)$. $\square$

When M is compact, so is $e(M)$ and the leaves of $\mathcal{F}_T$ are complete in the $<\ ,\ >_{e(M)}$ metric. Let $L_T \in \mathcal{F}_T$ project to leaf $L = p_T(L_T) \in \mathcal{F}$. From [Ch], we know that $\mathrm{p}_T : L_T \rightarrow L$ is a covering map since p_T is a local isometry. Denote the induced map on the fundamental groups by $p_{T*} : \pi_1(L_T) \rightarrow \pi_1(L)$.

Recall, from Chapter 4, that $K_D(L)$ consists of homotopy classes of loops, based at some point p, which determine, together with any horizontal path starting at p, rectangles in which initial and terminal horizontal edges coincide.

Proposition 6.8 $p_{T*}\ \pi_1(L_T) = K_D(L)$

proof

Choose $p \in L$ and $(p, \mathbf{f}_p) \in L_T$ as base points.

We, first, show that $p_{T*}\ \pi_1(L_T) \supseteq K_D(L)$ by showing that loops in $K_D(L,p)$ lift to loops in $\pi_1(L_T, (p, \mathbf{f}_p))$. Let loop τ in L satisfy $\tau(0) = p =$

$\tau(1)$ and $[\tau] \in K_D(L,p)$. Let the lifted path $\tilde{\tau}(t) = (\tau(t), \mathbf{f}_\tau(t))$ satisfy $\tilde{\tau}(0) = (p, \mathbf{f}_p)$. We must show that $\tilde{\tau}$ is a loop, that is, $\mathbf{f}_p = \mathbf{f}_\tau(1)$. Let $\{ U_i \}_{i=1}^n$ be a family of neighborhoods which covers τ , with $p \in U_1 \cap U_n$, and suppose that $\mathcal{F}$ is determined on the U_l's by riemannian submersions φ_l, which satisfy

$$\varphi_l = \gamma_{l\ l-1} \circ \varphi_{l-1},$$

for a local isometry $\gamma_{l\ l-1}$. For any horizontal path, $c(s)$, with $c(0) = p$, the rectangle $\delta(s,t)$, with initial horizontal edge $\delta(s,0) = 0$, satisfies

$$\varphi_n \circ \delta(s,1) = \gamma_{n\ n-1} \circ .. \circ \gamma_{\ 1} \circ \varphi_1 \circ c(s).$$

Since $[\tau] \in K_D(L,p)$, we have

$$\delta(s,1) \equiv c(s),$$

so

$$\varphi_n \circ c(s) = \gamma_{n\ n-1} \circ .. \circ \gamma_{2\ 1} \circ \varphi_1 \circ c(s).$$

For any $\mathbf{v} \in (T\mathcal{F}_p)^\perp$, there exists a horizontal curve $c(s)$ which satisfies

$$\frac{dc(s)}{ds}_{s=0} = \mathbf{v}$$

(infact, the geodesic with those initial conditions is horizontal everywhere, according to [H], so we can let $c(s)$ be this geodesic). Thus,

$$d\varphi_n \mathbf{v} = d\gamma_{n\ n-1} \circ .. \circ d\gamma_{2\ 1} \circ d\varphi_1\ \mathbf{v}$$

for any $\mathbf{v} \in (T\mathcal{F}_p)^\perp$, in particular,

$$d\varphi_n\ \mathbf{f}_p = d\gamma_{n\ n-1} \circ .. \circ d\gamma_{2\ 1} \circ d\varphi_1\ \mathbf{f}_p.$$

From [6.4], the frame $\mathbf{f}_\tau(1)$ at $p = \tau(t)$ satisfies

$$d\varphi_n\ \mathbf{f}_\tau(1) = d\gamma_{n\ n-1} \circ .. \circ d\gamma_{2\ 1} \circ d\varphi_1\ \mathbf{f}_p,$$

so

$$d\varphi_n\ \mathbf{f}_p = d\varphi_n\ \mathbf{f}_\tau(1),$$

therefore $\mathbf{f}_p = \mathbf{f}_\tau(1)$.

We, next, show that $p_{T*}\ \pi_1(L_T) \subseteq K_D(L)$. A loop $\tilde{\tau}$ in $\pi_1(L_T,\ (p, \mathbf{f}_p)\ )$ projects to a loop $\tau = p_T \circ \tilde{\tau}$ in $\pi_1(L, p)$. To see that $[p_T \circ \tilde{\tau}] \in K_D(L, p)$, first choose a rectangle $\delta(s,t)$ with initial vertical edge $\delta(0,t) = \tau(t)$ and initial horizontal edge $\delta(s,0) = c(s)$, where $c(s)$ is any horizontal path starting at p. The lifted rectangle $\tilde{\delta}(s,t)$ in $e(M)$ has initial vertical edge $\tilde{\delta}(0,t) = \tilde{\delta}(t)$ and $\tilde{\delta}(s,0) = \tilde{c}(s)$, the lift of $c(s)$ which begins at $(p, \mathbf{f}_p)$. Observe that $[\tilde{\tau}] \in K_{\tilde{D}}(L_T,\ (p, \mathbf{f}_p))$, where $\tilde{D} = (T\mathcal{F}_T)^\perp$, since $K_{\tilde{D}}(L_T) = \pi_1(L_T)$, from remark [5.9]. Thus, $\tilde{\delta}(s,0) \equiv \tilde{\delta}(s,1)$ and it follows that $\delta(s,0) \equiv \delta(s,1)$. Since $c(s)$ was an arbitrary horizontal path, we have $[\tau] \in K_D(L,p)$.□

Corollary 6.9 *L_T and $\tilde{L}\ /\ K_D(L)$ are diffeomorphic and*

$$H_D(L) = \pi_1(L)\ /\ K_D(L)$$

is the group of deck transformations on L_T.

proof

The group of deck transformations on $L_T \cong \tilde{L}\ /\ p_{T*}(\ \pi_1(L_T)\ )$ is $\pi_1(L)\ /\ p_{T*}\ \pi_1(L_T)$, so the results follow from the previous proposition.□

Remark 6.10 The metric $< \, , \, >_{e(M)}$, restricted to L_T, is

$$< \, , \, >_{e(M)} = p_T^*(< \, , \, >_{T\mathcal{F}}),$$

the pullback of the leaf metric. From Chapter 4, the metric on $\tilde{L} \, / \, K_D(L)$ is the pullback, via the covering map, of the leaf metric on L, so the homeomorphism between L_T and $\tilde{L} \, / \, K_D(L)$ is an isometry.

The last remark allows us to reprove Corollary 4.24, for a riemannian foliation $\mathcal{F}$.

Corollary 4.24

Let $\mathcal{F}$ be a riemannian foliation with Ehresmann connection $D = (T\mathcal{F})^{\perp}$. For any compact leaf L, the growth type of $H_D(L)$ bounds the growth type of any other leaf L' from above.

proof

The growth type of L_T with respect to the metric $< \, , \, >_{e(M)}$ restricted to $T\mathcal{F}_T$, is the same as growth type of $H_D(L)$, since L_T covers L and $< \, , \, >_{e(M)}$ is the pullback metric, via covering map p_T. From Propostion 5.8, all leaves in $\mathcal{F}_T$ have the same growth type, so the conclusion follows.□

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