ARITHMETIC CLASSIFICATION OF FAMILIES OF
ABELIAN VARIETIES OF QUATERNION TYPE.

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Abstract of the Dissertation

Arithmetic Classification of Families of Abelian Varieties of quaternion type.

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A family of Abelian varieties of quaternion type, \( A \overset{f}{\to} V \) is a fiber space whose fibers are Abelian varieties and which is parametrized by a Hilbert modular variety \( \Gamma \backslash \mathbb{H}^t \).

We present a classification of such families with a given endomorphism ring and Hodge structure. The main result is that the bottom field of \( A \) is an abelian extension of the bottom field of \( V \). An example of family of "Satake" type is constructed to show that the bottom field of \( A \) can be different from the bottom field of \( V \).
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Introduction

A family of Abelian varieties of quaternion type \( A \overset{f}{\to} V \) is a fiber space whose fibers are Abelian varieties and which is parameterized by a Hilbert modular variety \( V = \Gamma \backslash \mathbb{H}^t \).

The aim of this dissertation is to study the arithmetic of the family. The main result is that, assuming the family is characterized by its Hodge structure, the bottom field of \( A \) is an abelian Galois extension of the bottom field of \( V \). Definitions will be given later, but the best known example is the bottom field of a polarized Abelian variety (field of moduli), the central notion in the theory of complex multiplication.

The construction of \( \Gamma \backslash \mathbb{H}^t \) is classical. Let \( k \) be a totally real algebraic number field \( B \) a quaternion algebra over \( k \). Then we have an isomorphism

\[
B \otimes \mathbb{R} \cong M_2(\mathbb{R}) \times \ldots \times M_2(\mathbb{R}) \times \mathbb{K} \times \ldots \times \mathbb{K}
\]

where \( M_2(\mathbb{R}) \) is the total matrix algebra of degree two and \( \mathbb{K} \) the algebra of real quaternions. Let \( t > 0 \) be the number of copies of \( M_2(\mathbb{R}) \) and \( G = \text{Res}_{k/\mathbb{Q}} \{ a \in B | a a^i = 1 \} \) where \( i \) is the canonical involution.

Any arithmetic subgroup \( \Gamma \) of \( G \) is a discontinuous group of transformations of the product of upper half planes \( \mathbb{H}^t \). It defines a Hilbert modular variety \( \Gamma \backslash \mathbb{H}^t \).
A family over $\Gamma \backslash H^c$ is constructed from a symplectic representation. In particular we consider only rigid families, i.e., $G$ is isomorphic to the Hodge group of the generic fiber of $A$.

The first chapter is dedicated to the classification of the families induced by a given rigid symplectic representation $\rho$. We will prove

**Theorem**

$$\# \text{ (isomorphism classes of families defined by } \rho \text{ over } \Gamma \backslash H^c) = h(k) \cdot \nu$$

where $k$ is the smallest Galois extension containing $k$ and $\nu$ is a constant.

In the second chapter we prove our main result via the study of the analytic structure of $A^g \overset{f^g}{\rightarrow} V^g$, where $g$ is an automorphism of $G$. This proof will proceed in several steps.

In Section 2.3, assuming $V^g = V$ we prove that $A^g \overset{f^g}{\rightarrow} V^g$ can be defined by the same analytic data as $A \overset{f}{\rightarrow} V$ with the exception of the lattice defining the fibers. We will have, at that point, reduced the problem to the classification in Chapter 1 that establishes a correspondence between possible lattices and ideal classes of $k$. It is via the lattices that we construct the homomorphism

$$1 \rightarrow \text{Gal}(K_A/K_V) \rightarrow \text{Ideal class group } (k).$$

In the third chapter we construct an example to show that $[K_A:K_V]$ can be different from 1. We apply the
Shimura-Taniyama [Sh-T] theory of complex multiplication to the fibers of CM type.

Notation and Conventions. All algebraic varieties are assumed to be connected and smooth. If a variety is defined over a field \( k \), and \( K \) is an extension of \( k \), we write \( X_K \) for the set of \( K \)-rational points of \( X \). A vector space or algebra \( W \) over a field \( k \) determines an algebraic variety, again called \( W \), such that \( W_K = W \otimes_k K \) for any field \( K \) containing \( k \). If \( K > k \) and \( [K:k] < \infty \) then \( \text{Res}_{K/k} \) denotes the restriction to the category of \( k \)-varieties.
1.1 Group Theoretic Families of Abelian Varieties

Kuga developed the general theory of families of Abelian varieties parameterized by a projective variety $V$, i.e. holomorphic fiber spaces, $A \xrightarrow{f} V$, whose fibers are Abelian varieties, and where $A$ and $V$ are projective varieties. Kuga's construction is purely analytic and makes use of Kodaira's Embedding Theorem to show the existence of an algebraic structure for $A$ and $V$. We shall summarize this construction following Kuga [K] and Satake [S].

The starting point is an arithmetic variety $V$ that we construct from a connected semisimple linear algebraic group $G$, defined over $\mathbb{Q}$ and with finite center. Let $K$ be a maximal compact subgroup of $G$ and $\Gamma$ an arithmetic torsion free subgroup of $G$ such that $\Gamma \backslash G$ is compact. If we also assume that $X = G_{\mathbb{R}}/K$ has a $G_{\mathbb{R}}^\ast$-invariant complex structure, then $V = \Gamma \backslash X$ with the induced complex structure is a compact complex manifold. $V$ is known to be holomorphically equivalent to an algebraic submanifold of a complex projective space and is called an arithmetic variety.

It is over $V$ that Kuga constructs families of Abelian varieties and uses a long list of ingredients to do so:
Let $G$, $K$, $X$, $\Gamma$ and $V$ be given as above. Let $F$ be an even dimensional vector space over $Q$ and $\beta: F \times F \to Q$ a non degenerate bilinear form on $F$ so that $\text{Sp}(F, \beta) = \{g \in \text{GL}(F_R) \mid \beta(gx, gy) = \beta(x, y)\}$, the symplectic group is an algebraic group defined over $Q$.

Let $\rho: G \to \text{Sp}(F, \beta)$ be an algebraic representation of $G$ defined over $Q$, which we will call a "symplectic representation" of $G$.

Let $L$ be a lattice in $F$ satisfying $\rho(\Gamma)L \subseteq L$ and $\beta(L, L) \subseteq L$.

A lattice satisfying the first condition exists since $\Gamma$ is an arithmetic subgroup of $G$ and the second condition is always satisfied replacing $L$ by $nL$ if necessary.

Via $\rho$ we can define the semidirect product $G \times F_R$ with the following multiplication law: $(g, w)(g', w') = (gg', \rho(g)w' + w)$.

Furthermore consider the product space $X \times F_R$ on which the group $G \times F_R$ acts:

$$
(g, w) \in G \times F_R \\
(x, u) \in X \times F_R
$$

$$
\rightarrow (g, w)(x, u) = (g(x), \rho(g)u + w)
$$

The condition $\rho(\Gamma)L \subseteq L$ makes $\Gamma \times L$ into a discrete subgroup of $G \times L$ acting of $X \times F_R$ freely and discontinuously. So finally we can define the fiber space $A \xrightarrow{f} V$, where $A = \Gamma \times L \setminus X \times F_R$ and $f$ is defined as follows:

$$
A \xleftarrow{\nu} X \times (F_R/L) \\
\downarrow \nu \quad \downarrow \text{proj}_X
$$

$$
V = \Gamma \setminus X \xrightarrow{\nu} X
$$
A $f \to V$ is a smooth fiber bundle with fiber $F_R/L$ and structure group $\rho(G)$. To make it into a holomorphic fiber space we need one more object.

We define the Fichler map $\tau$ associated to a symplectic representation $\rho$ as a holomorphic map from $X$ into the Siegel space,

$$\tau : X \to X' = \mathfrak{S}(F_R, \beta) = \{ J \in GL(F_R) \mid J^2 = -1, \beta(x, Jy) > 0 \text{ and symmetric} \}$$

which is weakly equivalent with respect to $\rho$, i.e. $\tau(g(x)) = \rho(g)(\tau(x))$. Note that neither the existence nor the uniqueness of $\tau$ is assured.

We identify $X' = \mathfrak{S}(F_R, \beta)$ with $SO(F, \beta) \backslash Sp(F, \beta)$ via the transitive action $Sp(F, \beta) \times X' \to X'$, $(g, J) \to gJg^{-1}$.

If $J_0 \in X'$ and $H'_0 = 1/2J_0$ it is known (Satake [S], Chapter 2. §7), that $\text{ad}H'_0$ is a complex structure on $T_{J_0}(X')$ and there is a unique Hermitian complex structure on $X'$ which induces the complex structure $J' = \text{ad}H'_0$ on $T_{J_0}(X')$. We shall always consider $X'$ as a complex manifold with this complex structure, and this structure is independent of $J_0$. Returning now to $\tau : X \to X'$ we recall that $\tau$ is $\rho$-equivariant. Therefore if we choose $0 \in X = G/K$ for every $x \in X$ we have $g(0) = x$ for some $g \in G$ and $\tau(x) = \tau(g(0)) = \rho(g)(\tau(0)) = \rho(g)\tau(0)\rho(g^{-1})$. Thus $\tau$ is completely determined by $\rho$ and $\tau(0)$. We set $\tau(0) = J_0 \in X'$. 

Let now \( J \) be the complex structure on \( \tau^{-1}(J_0)(X) \equiv \mathfrak{g} \) where \( \mathfrak{g} \) is the Lie algebra of \( G_\mathbb{R} \). Then \( J = \text{ad} H_0 \), \( H_0 \) uniquely determined. We define the following conditions:

Condition (H1) \( [\text{d} \rho H_0 - H_0^\prime, \text{d} \rho(Y)] = 0 \), for all \( Y \in \mathfrak{g} \), \( H_0^\prime = 1/2J_0 \).

Condition (H2) \( \text{d} \rho H_0 = H_0^\prime \).

Condition (H2) implies (H1) and (H1) is equivalent to \( \tau \) being holomorphic (Satake [S], chapter 11.§8). \( \tau \) makes \( X \times (F_\mathbb{R}/L) \to X \) into a fiber space whose fibers are Abelian varieties with polarization \( \beta \); in fact:

**Lemma 1.1.1.** \( \tau(x) \) is a complex structure on \( F_\mathbb{R} \) such that \( (F_\mathbb{R}/L, \tau(x)) \) is an Abelian variety with polarization \( \beta \).

Moreover if \( \gamma \in \Gamma \) we have:

**Lemma 1.1.2.** \( (F_\mathbb{R}/L, \tau(x), \beta) \) is isomorphic to \( (F_\mathbb{R}/L, \tau(\gamma x), \beta) \).

So also \( A \xrightarrow{f} V \) is a fiber space whose fibers are Abelian varieties with a fixed polarization. We have:

**Theorem 1.1.3.** (Kuga [K] Theorem 11.6.3) Let \( A \xrightarrow{f} V \) be the fiber space constructed above from the data: \( (G, K, X, \Gamma, F, \beta, \rho, \tau) \). Then, if \( \tau \) is holomorphic, or equivalently if the (H1) condition is satisfied, \( A \) has a unique structure \( J_\mathbb{A} \) such that:
1. \( J_A \) restricted to the zero section coincides with the given structure on \( V \) (recall that \( V = T\backslash X \) where \( X \) is a Hermitian symmetric space).

2. \( f: A \xrightarrow{f} V \) is a holomorphic map.

3. \( J_A \) restricted to each fiber \( A_x \) coincides with \( \tau(x) \).

We will call the holomorphic fiber space \( A \xrightarrow{f} V \) a group theoretic family of Abelian varieties or a Kuga fiber variety, since using Kodaira's theorem we have the following.

**Theorem 1.1.4.** (Kuga [K] Theorem 11.6.8) Let \( A \xrightarrow{f} V \) be a group theoretic family of Abelian varieties. If \( V \) is compact then \( A \) has a projective embedding.

Throughout this work we shall always consider a fixed projective realization of \( A \xrightarrow{f} V \).

**Remark:** A natural problem arises from Theorem 1.1.3 and was posed by Kuga in the 1960's: Classify all of the representations of \( G \), \( G \) a \( \mathbb{Q} \)-simple algebraic group of hermitian type, into a symplectic group and investigate the existence of corresponding Eichler maps. Let \( G_{\mathbb{R}} \) be the set of real points of \( G \) and \( K \) a maximal compact subgroup. \( G \) of Hermitian type means that \( G_{\mathbb{R}}/K \) is a Hermitian symmetric space. So, given such a symplectic representation the existence of a corresponding Eichler map is the only obstruction
to the construction of Kuga fiber varieties over $\Gamma \backslash G/K$.

In the case when $G_R$ has no compact factors the problem was solved by Satake. In the case when $G_R$ has compact factors and the corresponding symmetric domains are of type II or III the solution is due to Addington ([Ad2],[Ad1]). We shall describe the solution of the quaternion case in the next Section.

1.2 Families arising from quaternion algebras.

In this paper we will investigate the arithmetic structure of families of Abelian varieties parameterized by a quaternion Hilbert modular variety.

**Definition 1.2.1.** A quaternion Hilbert modular variety $V$ is the quotient space $\Gamma \backslash X$, where $X$ is the symmetric space associated to $G = \text{Res}_{k/Q}(\text{SL}_1(B))$, $B$ a quaternion algebra over $k$, and $\Gamma$ is an arithmetic subgroup of $G$.

Addington [Ad1] classified the representations of $G$ into $\text{Sp}(F,\beta)$ which define families of Abelian varieties over $V$, equivalently for which there exist $\Gamma, L, \tau$ such that $(G, K, X, \Gamma, F, L, \beta, \rho, \tau)$ satisfy the assumptions of Section 1.1. These symplectic representations of $G = \text{Res}_{k/Q}(\text{SL}_1(B))$ are describable by a combinatorial scheme called chemistry. Since they are our main tools we will give an explicit description, following Addington.
Let \( k \) be a totally real number field and \( B \) a division quaternion algebra over \( k \), i.e., a central simple algebra of dimension 4 over \( k \). We will denote by \( \nu: B \to k \), the reduced norm of \( B \).

Assuming \( k \) of degree \( n \) over \( \mathbb{Q} \), let \( S = \{ \phi_1, \ldots, \phi_n \} \) be the distinct embeddings of \( k \) into \( \mathbb{R} \). We define \( S_0 = \{ \alpha \in S | B \cong \mathbb{R} \} \) and \( S_1 = S - S_0 \).

If we now consider \( \text{SL}_1(B) = \{ x \in B | \nu(x) = 1 \} \), it is an algebraic group defined over \( k \) and so \( G = \text{Res}_{k/\mathbb{Q}}(\text{SL}_1(B)) \) is a \( \mathbb{Q} \)-simple algebraic group defined over \( \mathbb{Q} \) (Weil [W]).

**Proposition 1.2.2.** (a) \( G_\mathbb{R} \cong \Pi_{\phi \in S} \text{SL}_1(B^\phi \mathbb{R}) \cong \text{SL}_2(\mathbb{R}) \cdot |S_0| \times \mathbb{R}^1|S_1| \)

\[ \cong \text{SL}_2(\mathbb{R}) \cdot |S_0| \times \text{SU}(2)|S_1| \]

\[ G_\mathbb{C} \cong \text{SL}_2(\mathbb{C})|S| \]

(b) Identifying \( G_\mathbb{R} \) and \( \text{SL}_2(\mathbb{R}) \cdot |S_0| \times \text{SU}(2)|S_1| \),

a maximal compact subgroup is

\[ K = \text{SO}(2) \cdot |S_0| \times \text{SU}(2)|S_1| \]

So \( G_\mathbb{R} \) is a semisimple Lie group and

\[ G/K \cong (\text{SL}_2(\mathbb{R})/\text{SO}(2)) \cdot |S_0| \times \{ p \}|S_1| \cong \mathbb{H}|S_0| \]

is a Hermitian symmetric space of non compact type as it is a product of Hermitian symmetric spaces.

Henceforth \( G \) will always mean \( \text{Res}_{k/\mathbb{Q}}(\text{SL}_1(B)) \).

Put \( \mathbb{F} = \phi_1(k) \cdots \phi_n(k) \), then \( \mathbb{F} \) is a totally real Galois extension of \( \mathbb{Q} \). \( G = \text{Gal}(\mathbb{F}/\mathbb{Q}) \) acts transitively on \( S \) via \( g(\alpha) = \ldots \)
The triple $(\Omega, S, S_0)$ is called *chemistry*. Elements of $S$ are called *atoms*, subsets of $S$ are called *molecules* and finite sums $M_j$ of molecules are called *polymers*.

Since $G$ acts on $S$, $G$ acts on the set of all molecules and polymers.

**Definition 1.2.3.** We say that a molecules is *stable* if $|M \cap S_0| \leq 1$ and *rigid* if $|M \cap S_0| = 1$. Analogously a polymer $P = \Sigma M_j$ is stable (resp. rigid) if $P$ is $G$-invariant and each $M_j$ is stable (resp. rigid).

For any polymer $P$ we will construct a representation $\rho_P$ of the algebraic group $G$.

We have seen in proposition 1.2.1 that $$G = G_C = \prod_{\phi \in S} SL_1(B \otimes C) = SL_2(C) |S|$$

Let $proj_\phi$ be the projection map of $G$ into its simple factor $$G_\phi = SL_1(B \otimes C) \cong SL_2(C).$$

For every atom $\alpha$ let $\rho_\alpha$ be the map $proj_\alpha$ considered as a representation of $G$ on the vector space $C^2$.

For a molecule $M = \{\alpha_1, \ldots, \alpha_r\}$, set $\rho_M = \rho_{\alpha_1} \ldots \ast \rho_{\alpha_r}$ For a polymer $P = \Sigma M_i$, set $\rho_P = \rho_{M_1} \ast \ldots \ast \rho_{M_t}$.

We are finally able to state:

**Theorem 1.2.4.** (Addington [Ad1]). Let $G$ be the algebraic group $Res_k/Q(SL_1(B))$. 

1. Suppose that $\rho$ is a symplectic representation of $G$ defined over $Q$ that defines a group theoretic family of Abelian varieties. Then there exists a stable polymer $P$ such that $\rho$ is equivalent to $\rho_P$ over $C$.

2. Let $P$ be a stable polymer. Then some multiple of $\rho_P$ is an algebraic group representation of $G$ defined over $Q$ and defines a group theoretic family of Abelian varieties.

**Proof.** We will only describe in detail the representation and the construction for part 2. In fact it is these families that we will use throughout this paper.

Let $P$ be an invariant polymer and $M$ any molecule of $P$. The smallest orbit of $M$ under the action of the Galois group is a subpolymer of $P$ and we can write $P$ as $P = \sum_i P_i$, $P_i$ generated by any single molecule.

**Definition 1.2.5.** We call a minimal $G$-invariant polymer prime when it is generated by a single molecule.

For a prime polymer $P$ there exists an integer $\mu \geq 1$ such that

$$\mu P = \sum_{g \in \Omega} g M$$

In general $\rho_P$ is an algebraic group homomorphism. We shall prove that there exists and integer $\mu$ such that either $\rho_{\mu P}$ or $\rho_{\mu P} \circ \rho_{\mu P}$ is an algebraic group representation defined over $Q$. Since $\rho_P = \sum P_i$, it is indeed enough to consider $\rho_P$ when $P$ is prime.
For any $\alpha \in \Omega = \text{Gal}(\bar{k}/Q)$ we define $B^\alpha = B \circ \bar{k}$; Similarly, for any molecule $M = \{\alpha_1, \ldots, \alpha_r\}$, $B^M = B^{\alpha_1} \cdots B^{\alpha_r}$, and for any polymer $P = \sum M_i$, $B^P = B^{M_1} \cdots B^{M_t}$. $B^\alpha$ is a central simple algebra over $\bar{k}$ for any $\alpha$, as is $B^M$ for any $M$.

Let $F_M$ be a minimal left ideal in $B^M$. $B^M$ acts on $F_M$ by left multiplication so that we have a representation $\rho: B^M \to \text{End}_\bar{k}(F_M)$ and $\rho$ is defined over $\bar{k}$.

**Lemma 1.2.6.** Let $P$ be a prime polymer, $P = \sum gM$. Then $B^P$ is a central simple algebra over $\bar{k}$ and $B^P = \text{Res}_{\bar{k}/Q}(B^P)$.

$B^P$ and $B^M$ are isomorphic as algebras and $F_P = \text{Res}_{\bar{k}/Q}(F_M)$ is a minimal left ideal in $B^P$.

Recalling that $\text{Res}_{\bar{k}/Q}(B) = \Pi_{\alpha \in B^G} B^\alpha$ we call $\text{proj}_{\alpha} : \text{Res}_{\bar{k}/Q}(B) \to B^\alpha$ the natural projections.

For any molecule $M$ we define $i_M : \text{Res}_{\bar{k}/Q}(B) \to B^M$ as $i_M = \oplus_{\alpha \in M} \text{proj}_{\alpha}$ and for any polymer $P = \sum M_i$, $i_P = \oplus i_{M_i}$.

**Lemma 1.2.7.** Assuming that $P$ is as in Lemma 1.2.3, the map $i_P : \text{Res}_{\bar{k}/Q}(B) \to B^P$ is defined over $Q$.

We are now prepared to define a representation of $G$. From the previous results we have that

$$\text{Res}_{\bar{k}/Q}(B) \xrightarrow{i_P} B^P \xrightarrow{\rho} \text{End}_Q(F_P)$$

is an algebra homomorphism defined over $Q$, so restructuring to $G_Q$:
\[ G_\mathbb{Q} = \text{Res}_{\mathbb{k}/\mathbb{Q}}(\text{SL}_1(\mathbb{B})) \overset{i_P}{\longrightarrow} B^P \overset{\rho}{\longrightarrow} \text{Aut}_{\mathbb{Q}}(F_P) \]
is an algebraic group representation defined over \( \mathbb{Q} \) that we will denote again by \( \rho \). Extending \( \rho \) to \( G_\mathbb{C} \) we get

\[ \rho_\mathbb{C} : G_\mathbb{C} \to \text{Aut}_{\mathbb{C}}(F_\mathbb{P} \otimes \mathbb{C}) \]

Returning to \( \rho_\mathbb{P} \) we have:

**Lemma 1.2.8.** If \( B^M \) is a trivial simple algebra then \( \rho = \rho_\mathbb{P} \) over \( \mathbb{C} \). Otherwise \( \rho = \rho_2 \mathbb{P} \) over \( \mathbb{C} \).

**Proof.** By definition if \( M = \{ \alpha_1, \ldots, \alpha_r \} \), \( B^M = B^{\alpha_1} \otimes \ldots \otimes B^{\alpha_r} \).

Recalling that a tensor product of division quaternion algebras is either trivial or equivalent to a division quaternion algebra, we can write:

\[ B^M \cong M_N(B_0) \quad B_0 = \begin{cases} \kappa & \text{and } N = 2^r \\ \text{division quaternion algebra over } \kappa & \text{and } N = 2^{r-1} \end{cases} \]

It follows that \( F_M = B_0^N \) viewed as column vectors. Now

\[ \rho_\mathbb{C} : G_\mathbb{C} \to \bigotimes_{\alpha \in \Omega} \text{M}_2(C) \subset M_{2^r m}(C) \]

if \( m = |\Omega| \), \( r = |M| \) and the representation space is \( C^{2^r m} \).

On the other hand \( \rho : G_\mathbb{C} \to (\text{Res}_{\mathbb{k}/\mathbb{Q}}(B^M)) \otimes \mathbb{C} \) is the same map but the representation space may differ. In fact

\[ \dim_{\mathbb{C}}(F_N \otimes \mathbb{C}) = \dim_{\mathbb{Q}}(F_M) = \begin{cases} 2^{r m} & \text{if } N = 2^r \\ 2^{r+1} m & \text{if } N = 2^{r-1} \end{cases} \]

Then \( \rho_\mathbb{C} = \rho_\mathbb{P} \) if \( B^M \cong M_N(\kappa) \) and \( \rho_\mathbb{C} = 2 \rho_\mathbb{P} \) if \( B^M \cong M_N(B_0) \).
Now to complete the proof of part 2 (of 1.2.2) Addington considers $\Gamma \subset G_{Q}$ an arithmetic subgroup of $G$ and constructs a family of Abelian varieties over $V = \Gamma \backslash H|S_{0}|$ using $\rho_{R} : G_{R} \to \text{Aut}(F_{P} \circ R)$. For this purpose she defines $\beta^{P}$ a non degenerate bilinear form on $F_{P}$ such that $\text{Sp}(F_{P}, \beta^{P})$ and $\tau^{P}$ a holomorphic Eichler map associated to $\rho$. Moreover she chooses $L^{P}$ a $\rho(G)$-invariant lattice in $F_{P}$. The data $(G, K, X, \Gamma, F_{P}, L^{P}, \beta^{P}, \rho, \tau^{P})$ satisfy the hypotheses of theorem 1.1.3 and give a Kuga fiber variety $A_{P} \xrightarrow{f} V$.

To extend the construction to any stable polymer $P = \Sigma P_{i}$ is immediate. In fact we can define the following data:

$(G, K, X, \Gamma, F, L, \beta, \rho, \tau)$ where $\rho = \rho_{P}$ or $\rho_{P} \circ \rho_{P}$ whichever is indicated by lemma 1.2.5

$F = \oplus F_{P_{i}}$ or $F = \oplus (F_{P_{i}} \circ F_{P_{i}})$

$L = \oplus L_{P_{i}}$ or $L = \oplus (L_{P_{i}} \circ L_{P_{i}})$

$\beta = \oplus \beta_{P_{i}}$ or $\beta = \oplus (\beta_{P_{i}} \circ \beta_{P_{i}})$

$\tau = \oplus \tau_{P_{i}}$ or $\tau = \oplus (\tau_{P_{i}} \circ \tau_{P_{i}})$

The family defined by $\rho$ will then be the fiber product over $V$ of families:

$$A = A_{P_{1}} \times_{\cdots} \times A_{P_{d}} \downarrow$$

$$V = \Gamma \backslash H|S_{0}|$$

This completes the sketch of the proof of 1.2.4, part 2. ■
Remark: A family of Abelian varieties, defined by a polymer representation $\rho_p$ will not necessarily be the fiber product of families defined by the irreducible components of $\rho_p$. In general it is only isogeneous to such a product.

Definition 1.2.9 Let $G$ be an algebraic group defined over a field $F$. A representation $(V, \rho)$ of $G$ defined over $F$ is $F$-primary if for any $F$-irreducible invariant subspaces $W$ and $W'$ of $V$, $\rho|W' = (\rho|W)^{\sigma}$ for some $\sigma \in \text{Gal}(F/F)$.

In this paper we will restrict ourselves to Kuga fiber varieties defined by $Q$-primary symplectic representations of $G$, or in other words by prime polymers $P = \Sigma_g M$.

Proposition 1.2.10 (Satake [S]) Let $A \xrightarrow{\varphi} V$ be a Kuga fiber variety defined by a symplectic representation $\rho: G \to \text{Sp}(F, \beta)$. Then if $F = \prod F[i]$ is the primary decomposition we have:

$$
\rho = \prod \rho[i], \quad \beta = \prod \beta[i], \quad \tau = \prod \tau[i]
$$

It follows that we can find an integer $n$ such that:

$$
\begin{array}{ccc}
A & \xrightarrow{\text{id} \times n\mathbb{I}} & A[i] \times \ldots \times A[i] \\
V & \xrightarrow{\text{id}} & V
\end{array}
$$

where the map $n\mathbb{I}: A Xu A_x[i] \times \ldots \times A_x[i]$ is an isogeny.
1.3 Rigid Kuga varieties and Hodge Kuga varieties.

Let \( A \overset{f}{\rightarrow} V \) be a Kuga fiber variety defined by \((G,K,X,T,F,L,\beta,\rho,\tau)\).

**Definition.** We say that \( A \overset{f}{\rightarrow} V \) is rigid if \( \tau \) is uniquely determined by the rest of the data: \( G,K,X,T,F,L,\beta, \) and \( \rho \).

Rigid families were introduced by Abdulali[Ab1] and we follow his beautiful exposition.

We recall that \( \tau \) is completely defined by \( \rho \) and \( \tau(0) \), where \( 0 \in X \) is an arbitrary point. In fact \( \tau: X \rightarrow \mathfrak{C}(F_{\mathbb{R}},\beta) \) is equivariant with respect to \( \rho \), so that we have \( \tau(g(0)) = \rho(g)\tau(0)\rho(g)^{-1} \) for all \( g \in G_{\mathbb{R}} \).

**Definition 1.3.1.** Let \( X_\rho \) be the set of possible \( \tau(0) \)'s i.e. the set of \( J_0 \in \mathfrak{C}(F_{\mathbb{R}},\beta) \) such that the map \( g(0) \rightarrow \rho(g)J_0\rho(g)^{-1} \) is well defined and satisfies the (H1)-condition (see 1.1).

**Lemma 1.3.2.** A \( A \overset{f}{\rightarrow} V \) is rigid if and only if \( X_\rho \) reduces to a point.

**Theorem 1.3.3.** (Satake [S], chapter 4, prop. 4.1). \( X_\rho \) is a complex submanifold of \( \mathfrak{C}(F_{\mathbb{R}},\beta) \). Furthermore, \( G_\rho \) the Zariski connected component of the centralizer of \( \rho(G) \) in \( Sp(F,\beta) \) is a reductive subgroup which acts transitively on \( X_\rho \).
Theorem 1.3.4. (Abdulali [Ab1], proposition 1.2.2). If the (H2)-condition is satisfied then the fiber variety $A \rightarrow V$ is rigid.

We return to families arising from quaternion algebras. Recall that for the chemistry $(S, S_0, \Omega)$ a polymer $P = \mathbb{E} M_i$ is rigid if $|M_i \cap S_0| = 1$ for every $i$.

Theorem 1.3.5. (Abdulali [Ab1], theorem 1.3.7) Let $P$ be a rigid stable polymer and suppose that $\mu$ is an integer such that $\rho_{\mu P}$ is defined over $Q$. Then any family of Abelian varieties induced by $\rho_{\mu P}$ is rigid and in particular satisfies the (H2)-condition.

Proof. The idea is to construct an Eichler map $\tau$ associated to $\rho_{\mu P}$ and show that $\tau$ satisfies the (H2)-condition and is therefore unique by theorem 1.3.3. We shall define $\tau$ after recalling some notation from section 1.2.

Let $B$ be a division algebra over a totally real number field $k$. Define $G = \text{Res}_{k/Q}(\text{SL}_1(B))$ and identify $G_R$ with $\text{SL}_1(R)|S_0| \times \text{SU}(2)|S_1|$, $K_R$ maximal compact subgroup with $\text{SO}(2)|S_0| \times \text{SU}(2)|S_1|$, and $X = G_R/K_R$ with $H|S_0|$. The element $H_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} |S_0| \times 0 |S_1|$ of the Lie algebra of $G_R$ determines a complex structure on $X$.

Set $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} |S_0| \times 1 |S_1|$ and $J_0 = \rho_{\mu P}(j)$, $\rho_{\mu P} : G_R \rightarrow \text{Sp}(F_R, \beta)$
By definition \( \rho_M = \rho_{a_1} \ast \ldots \ast \rho_{a_r}, \ M = \{a_1, \ldots, a_r\} \). Therefore

if \( M \in \mathcal{P}, \ |M \cap S_0| = 1 \) and \( \rho_M(j) = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \)

with respect to a suitable basis, so \( J_0 = \rho_{\mu_F(j)} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \)

with respect to a suitable basis of \( F_G \) and it is a complex structure on \( F_R \).

We are ready to define a map \( \tau: X \rightarrow \mathfrak{F}(F_R; \beta) \) equivariant with respect to \( \rho \), as \( \tau(g(0)) = \rho(g)J_0 \rho(g)^{-1} \) where

\( 0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in X = \mathfrak{H} |S_0| \). Abdulali shows that \( \tau \) satisfies the (H2)-condition, \( d\rho(H_0) = 1/2J_0 \).

Let \( X \) be an Abelian variety given by \((F, J, L, \beta)\):

- \( F \) an even dimensional vector space,
- \( J \) complex structure on \( F \),
- \( L \subset F \) a \( \mathbb{Z} \)-lattice,
- \( \beta \) a non degenerate bilinear form on \( F \), such that
  i. \( \beta(L, L) < \mathbb{Z} \)
  ii. \( \beta(u, Jv) \) is symmetric and positive definite.

Consider \( \phi: T = \{ z \in \mathbb{C} \mid |z| = 1 \} \rightarrow \text{GL}(F) \)

with \( \phi(\theta) = \cos \theta I + \sin \theta J \).

**Definition 1.3.7.** The Hodge group of \( X \), \( Hg(X) \), is the smallest algebraic subgroup of \( \text{GL}(V) \) defined over \( \mathbb{Q} \) and containing \( \phi(T) \).
Definition 1.3.8. A group theoretic family of Abelian varieties \( A \rightarrow V \) defined by a symplectic representation \( \rho \) is of Hodge type if \( \rho(G) = \mathbb{Hg}(A_x) \) where \( A_x \) is a generic fiber.

Theorem 1.3.8. (Addington [Ad2]). Let \( P \) be a polymer. Then the families defined by \( \rho_\mu P \) are of Hodge type if and only if \( P \) is rigid.

Definition 1.3.9. We say that an Abelian variety \( X \) has complex multiplication (is of CM type) if \( \text{Hom}_\mathbb{Q}(X) = \text{Hom}(X,X) \) contains a commutative semisimple \( \mathbb{Q} \)-algebra \( R \) such that \( [R: \mathbb{Q}] = 2 \dim X \).

Theorem 1.3.10. (Mumford [M2]). a) Every family of Abelian varieties of Hodge type contains Abelian varieties of CM type. b) If a family contains an Abelian variety of CM type then it is isomorphic to a family of Hodge type.

Corollary 1.3.11. (Addington [Ad1]). A family defined by a polymer representation contains a fiber with complex multiplication if and only if the polymer is rigid.

We will give an explicit description of the endomorphism ring of a CM fiber in the next section.
1.4. Endomorphism Ring of the fibers.

Let \( A \xrightarrow{f} V \) be a rigid Kuga fiber variety given by the data: 
\((G, k, \Gamma, F, L, \beta, \rho, \tau)\), where as in 1.2.: \( k \) is an algebraic number field, \( \bar{k} \) the smallest Galois extension containing \( k \), \( \Omega = \text{Gal} (\bar{k}/Q) \), \( B \) a quaternion algebra over \( k \), \( B \ncong M_2(k) \), 
\( G = \text{Res}_{k/Q}(SL_1(B)) \)
\( \rho = \rho_{\mu \mu} \) symplectic representation defined over \( Q \) associated to a rigid polymer \( P \).

In this section we compute the endomorphism ring of the generic fiber of \( A \xrightarrow{f} V \) and the endomorphism ring of the fibers of CM type.

Let \( \rho = \bigoplus \rho[i] \) be the primary decomposition. Any \( \rho[i] \) determines naturally a family of Abelian varieties \( A[i] \to V \) and we know that \( A \xrightarrow{f} V \) is isogeneous to \( A[1] \times \cdots \times A[n] \to V \) (see proposition 1.2.6).

So since we are concerned with the endomorphism ring we may consider \( \rho \) to be \( Q \)-primary.

Let \( \rho \) be a \( Q \)-primary representation; \( \rho = \rho_P \), with \( P = \sum gM \) and we will also assume \( gM \neq \gamma M \) if \( g \neq \gamma \).

Remembering the definition: \( \rho_P = \text{Res}_{k/Q} \rho_{\mu} \)
with \( \rho_{\mu} : \text{Res}_{k/Q}(SL_1(B)) \to B^M \subset \text{End}_k(F^M) \) the regular representation.

Let \( A_x \) be the generic fiber of \( A \), \( x \) in \( V \), isomorphic to \( (F, L, J_x = \tau(x)) \). Any endomorphism of \( A_x \) has a natural rational representation:
Hom \( A_x \) \( \cong \{ g \in \text{End}_Q(F) \mid gL = L, \ gJ_x = J_xg \} \)

Hom\(_Q\) \( A_x \) \( \cong \{ g \in \text{End}_Q(F) \mid gJ_x = J_xg \} \)

Since \( H_g(A_x) \), the Hodge group of \( A_x \), is generated by 
\( \cos \theta I + \sin \theta J_x \) and their conjugates over \( Q \), we have:

\( \text{Hom}_Q(A_x) \cong \{ g \in \text{End}_Q(F) \mid gg' = g'g \text{ for any } g' \in H_g(A_x) \} \).

But \( A \stackrel{f}{\longrightarrow} V \) is a rigid family so that \( H_g(A_x) = \rho(G) \) and

\( \text{Hom}_Q(A_x) \cong \{ g \in \text{End}_Q(F) \mid gg' = g'g \text{ for any } g' \in \rho(G) \} \).

**Proposition 1.4.1.** Let \( A \stackrel{f}{\longrightarrow} V \) be a Kuga fiber variety of quaternion type defined by a symplectic representation \( \rho \). Let \( A_x \) be a generic fiber, \( \mathbb{k} \) the smallest Galois extension of \( Q \) containing \( k \). If \( \rho \) is a polymer \( Q \)-primary representation, \( \rho = \rho_p, P = \Sigma gM, \) and \( gM \neq \gamma M \) for \( g \neq \gamma \), then

\( \text{End}_Q(A_x) = \text{Res}_{\mathbb{k}/Q}(B_0) \)

where \( B_0 = \mathbb{k} \) if \( \rho \) is \( G \)-irreducible and \( B_0 \) is a division quaternion algebra over \( \mathbb{k} \) otherwise.

**Proof.** We assumed \( P = \Sigma gM, \Omega = \text{Gal}(\mathbb{k}/Q), M = \{a_1, \ldots, a_r\} \subset \Omega \)

Therefore

\[ \text{Res}_{\mathbb{k}/Q}(B) \quad \longrightarrow \quad B^M \subset \text{End}_{\mathbb{k}}(F_M) \]

\[ G = \text{Res}_{\mathbb{k}/Q}SL_1(B) \quad \stackrel{\rho_P}{\longrightarrow} \quad \text{Res}_{\mathbb{k}/Q}^B \quad \longrightarrow \quad \text{Res}_{\mathbb{k}/Q}^{B^M} \subset \text{End}_Q(F) \]

\( F_M \) is a minimal left ideal and \( F = \text{Res}_{\mathbb{k}/Q}F_M \) is a minimal left ideal in \( \text{Res}_{\mathbb{k}/Q}^{B^M} \).

As we have seen in the proof of Lemma 1.3, we have: \( B^M \cong M_N(B_0) \),
\( \mathbb{F}_M \cong \mathbb{B}_0^N \), \( \mathbb{B}_0 \) a simple division algebra over \( \overline{\mathbb{k}} \) (\( \mathbb{B}_0 \) is either \( \mathbb{k} \) or a division quaternion algebra). Under these identifications:

\[
\{ g \in \text{End}_k \mathbb{F}_M \mid gg' = g'g, \ g' \in \rho(G) \}
\]

\( = \text{Res}_{k/\mathbb{Q}} \{ g \in \text{End}_k \mathbb{F}_M \mid gg' = g'g, g' \in \rho_M(\text{SL}_1(\mathbb{B})) \} \).

In fact for \( M = \{ \alpha_1, \ldots, \alpha_r \} \),

\[
\rho_p = \rho \rho_{gM} = \rho_{8\alpha_1} \ast \ldots \ast \rho_{8\alpha_r} \text{ and } F = \bigotimes \mathbb{F}_E \]

by construction. Since \( gM \neq \gamma M \) if \( g \neq \gamma \), we have that \( id_{gM} \in \rho(G) \) for any \( g \).

Therefore \( \{ g \in \text{End}_k \mathbb{F}_M \mid gg' = g'g, g' \in \rho(G) \} \)

\( \subseteq \{ g \mid g(\mathbb{F}_\alpha M) = \mathbb{F}_\alpha M \text{ for any } \alpha \} = \text{Res}_{k/\mathbb{Q}} \text{End}_k \mathbb{F}_M \).

We must look closely at \( \text{End}_k \mathbb{F}_M \) and \( \rho_M(\text{SL}_1(\mathbb{B})) \).

We distinguish two cases.

(i) \( \mathbb{B}^N \cong \mathbb{M}_N(\overline{\mathbb{k}}) \), \( \mathbb{F}_M \cong \overline{\mathbb{k}}^N \); then \( \text{End}_k \mathbb{F}_M = \mathbb{B}^M \).

(ii) \( \mathbb{B}^M \cong \mathbb{M}_N(\mathbb{B}_0) \), \( \mathbb{F}_M \cong \mathbb{B}_0^N \), \( \mathbb{B}_0 \) division quaternion algebra

over \( \overline{k} \); then \( \text{End}_k \mathbb{F}_M = \mathbb{B}^M \ast \mathbb{B}_0 \).

In fact \( \text{End}_k(\mathbb{F}_M) \) contains \( \mathbb{B}^M \ast \mathbb{B}_0 \) where \( \mathbb{B}_0 \) is identified with \( \mathbb{B}_0 I \) in the dual \( (\mathbb{B}^M)^* \) and acts on \( \mathbb{F}_M \) by right multiplication.

Moreover \( \dim_k (\mathbb{B}^M \ast \mathbb{B}_0) = N^2 4^2 = \dim_k \mathbb{F}_M \).

So we can write \( \text{End}_k(\mathbb{F}_M) = \mathbb{B}^M \ast \mathbb{B}_0 \) with the convention that \( \mathbb{B}_0 \) is either \( \overline{k} \) or a division quaternion algebra, depending on the class of \( \mathbb{B}^M \) in the Brauer group.

Finally we can compute \( \text{End}(A_{\overline{k}}) \ast \mathbb{Q} \).

1) \( \mathbb{B}^M \cong \mathbb{M}_N(\overline{\mathbb{k}}) \), \( \mathbb{F}_M \cong \overline{\mathbb{k}}^N \).
Then $\rho_M : \text{Res}_{k/Q}(\text{SL}_1(B)) \to B^M = \text{End}_k(F_M)$ is an irreducible representation over $\bar{k}$ as well as over $C$, since $B_M \cdot C = M_N(C) = \text{End}_C(F_M \cdot C)$.

So by Schur's lemma, $\bar{k} = \{ g \in \text{End}_k(F_M) \mid gg' = g'g \text{ g' e } \rho(\text{SL}_1(B)) \}$.

ii) $B^M \cong M_N(B_0)$, $F_M \cong B_0^N$.

Then $\rho_M : \text{Res}_{k/Q}(\text{SL}_1(B)) \to B^M \subseteq \text{End}_{\bar{k}}(F_M)$ is irreducible over $\bar{k}$ but reducible over $C$. In fact $B^M \cdot C = M_{2N}(C), F_M \cdot C \cong C^{4N}$. So by Schur's lemma we can only conclude that

$\{ g \in \text{End}_k(F_M) \mid gg' = g'g \text{ g' e } \rho(\text{SL}_1(B)) \}$ is a division algebra.

We need to define a new representation

$$\tilde{\rho} = \rho_M \otimes \text{id} : \text{Res}_{k/Q}(\text{SL}_1(B)) \cdot \text{SL}_1(B_0) \to B^M \otimes B_0 = \text{End}_{\bar{k}}(F_M).$$

$\tilde{\rho}$ is an irreducible representation over $\bar{k}$ as a tensor product of irreducible representations and is moreover irreducible over $C$, since $(B_M \otimes B_0) \cdot C = \text{End}_C(F_M \cdot C)$. We can now conclude as in (i), that $\bar{k} = \{ g \in \text{End}_k(F_M) \mid gg' = g'g \text{ g' e } \rho(\text{SL}_1(B)) \otimes \text{SL}_1(B_0) \}$.

Now let $H = \{ g \in \text{End}_{\bar{k}}(F_M) = B^M \otimes B_0 \mid gg' = g'g \text{ g' e } \rho_{\tilde{\rho}}(C) \}$.

$H \supset 1 \cdot B_0$ and any $h$ in $H$ can be written as $h = \Sigma h_i \cdot b_i$, where $B_0 = \langle b_1, \ldots, b_n \rangle_k$, $h_i \in B^M$.

Let $g \circ 1$ be an element of $\rho(\text{SL}_1(B)) \subseteq B_M \subseteq \text{End}_{\bar{k}}(F)$.

We have: $h \cdot g \circ 1 = g \circ 1 \cdot h$

and $\Sigma h_i g \cdot b_i = g Nh_i \cdot b_i$

if and only if $h_i g \cdot b_i = gh_i \cdot b_i$ for any $i$, i.e.

$$(h_i \circ 1)(g \circ b_i) = (g \circ b_i)(h_i \circ 1)$$

for any $i$ and therefore
\[ h_1 \circ 1 \in \text{centralizer}_{\text{End}_{K}^{\text{EM}}}(\rho(SL_1(B)) \circ SL_2(B_0)) = \bar{k}. \]

We have shown that \( H = 1 \circ B_0. \)

**Remark 1.** \( \text{End}(A_{\chi}) = 0, \) an order in \( \text{Res}_{E/\mathbb{Q}} B_0. \)

If \( O \) is a maximal order, then

\[ \text{End}(A_{\chi}) = \text{Res}_{E/\mathbb{Q}} O^\prime, \] an \( \mathcal{O}_K^\prime \) order in \( B_0. \)

**Remark 2.** If \( A \overset{f}{\rightarrow} V \) is a Kuga fiber variety of quaternion type induced by a \( \mathbb{Q} \)-primary symplectic representation then the generic fiber is a simple Abelian variety.

The study of the fibers of CM type is far more complicated than the study of the generic fiber.

Let \( A \overset{f}{\rightarrow} V = \Gamma \setminus X \) be a rigid Kuga variety of quaternion type induced by a polymer representation irreducible over \( \mathbb{Q}. \)

We have seen in 1.2 that \( X = \mathbb{H}^t. \) We shall construct an elliptic point \( \lambda \) in \( V, \) i.e. \( \lambda = (\lambda_1, \ldots, \lambda_t) \) with \( \lambda_i \) elliptic for every \( i, \) such that \( A_{\lambda} = f^{-1}(\lambda) \) is of CM type.

We need to consider the quaternion algebra \( B. \) We defined in Section 1.2:

\[ S = \{\phi_1, \ldots, \phi_n\} \] the set of real embeddings of \( k \) and

\[ S_0 = \{\phi \in S| B \circ \mathbb{R} \cong M_2(\mathbb{R})\}, \] \( |S_0| = t.\)

For every \( \phi \in S_0 \) we consider the natural immersion \( i_{\phi}: \)

\[ B \circ B \overset{i_{\phi}}{\rightarrow} B \circ \mathbb{R} \cong M_2(\mathbb{R}). \]
Recalling that \( \det i_\phi = v \), where \( v \) is the reduced norm, we have:

\[
\{ \alpha \in B \mid v(\alpha)^\phi > 0, \phi \in S_0 \} \overset{i_\phi}{\rightarrow} S_\phi \mathbb{GL}_2^+ (\mathbb{R})^t
\]

**Proposition 1.4.2.** [Sh4, Proposition 9.4] Let \( L \) be a totally imaginary quadratic extension of \( k \) and \( q \) a \( \mathbb{R} \)-linear isomorphism of \( L \) into \( B \). Then \( q(L^\times) \) is contained in \( \{ \alpha \in B \mid v(\alpha)^\phi > 0, \phi \in S_0 \} \) and every element of \( q(L^\times) \) not contained in \( k \) has a unique fixed point \( \lambda \) on \( \mathbb{H}^t \), which is common to all such elements of \( q(L^\times) \). Moreover, \( q(L^\times) = \{ \gamma \in B \mid v(\gamma)^\phi > 0, \phi \in S_0, \gamma(\lambda) = \lambda \} \).

Conversely, if an element \( \alpha \in B \), \( v(\alpha)^\phi > 0 \) for any element in \( S_0 \), has a fixed point in \( \mathbb{H}^t \) then \( k(\alpha) \) is isomorphic to a totally imaginary number field.

Returning to the base space of our family:

\[
\mathcal{V} = \Gamma \backslash X, \quad X = \mathbb{C}_R / \mathbb{K}_R
\]

We recall the identification of \( X \) with \( \mathbb{H}^t \):

\[
G = \text{Res}_{k/\mathbb{Q}}(\mathbb{SL}_2(B)) = \prod_S \mathbb{SL}_1(B^\phi)
\]

\[
G_\mathbb{R} = \prod_S \mathbb{SL}_1(B^\phi) \rtimes \mathbb{R} = \mathbb{SL}_2(\mathbb{R}) \times \mathbb{SU}(2)
\]

\[
K_\mathbb{R} = \mathbb{SO}(2) \times \mathbb{SU}(2)
\]

\[
X = \mathbb{G}_\mathbb{R} / K_\mathbb{R} \cong \mathbb{SL}_2(\mathbb{R}) / \mathbb{SO}(2) \cong \mathbb{H}^t\]
So we can consider \( \lambda \) the fixed point of \( q(L^x) \) as a point in \( X \) and therefore in \( V \). We shall show that under suitable hypotheses \( A_\lambda \) is of CM type.

We need to construct a field \( R \) contained in \( \text{End}_Q(A_\lambda) \) and \([R:Q] = 2 \dim A_\lambda\). As in the case of the generic fiber \( A_\lambda \) is isomorphic to \((\mathcal{V}, L, J_\lambda)\) and the rational representation for the endomorphism ring gives:

\[
\text{End}_Q(A_\lambda) = \{ g' \in \text{End}_Q^F, g'J_\lambda = J_\lambda g' \},
\]

where \( g(\sqrt{-1}x_1 \cdots \sqrt{-1}x_d) = \lambda, J_\lambda = \tau(\lambda) = \rho(g) J_\sigma \rho(g)^{-1} \).

From the previous considerations \( \lambda \) is a fixed point in \( \mathbb{H}^c \) for the group \( \Pi(L^x)^S \), which group we will denote by \((L^x)^S \). Let \( \alpha \) be in \((L^x)^S, \alpha = (\alpha^{\phi_1}, \ldots, \alpha^{\phi_n}) \). Since \( v(\alpha)^{\phi} > 0 \) for any \( \phi \) we can define \( \overline{\alpha} = \left( \frac{\alpha^{\phi_1}}{\sqrt{v(\alpha)^{\phi_1}}}, \ldots, \frac{\alpha^{\phi_n}}{\sqrt{v(\alpha)^{\phi_n}}} \right) \) an element of \( G \), so that \( \overline{\alpha} \) belongs to the isotropy group of \( \lambda, K_\lambda \). \( \rho \) and \( \tau \) form an equivariant pair so it follows that \( \rho(\overline{\alpha}) \in \text{End}_Q(A_\lambda) \). Moreover, \( \rho = \ast \rho_{\overline{\alpha}} \) and \( J_\lambda = \ast J_{\overline{\alpha}} \) by construction and so also \( \rho(\alpha) \) belongs to \( \text{End}_Q(A_\lambda) \) and \( \rho((L^x)^S) \subseteq \text{End}_Q(A_\lambda) \). If we put \( L^\phi = L \ast \overline{\alpha} \) and \( L^M = L^{\phi_1} \ast \cdots \ast L^{\phi_r} \) then we can state the previous result as

\[
\text{Res}_{K/Q} L^M \subseteq \text{End}_Q(A_\lambda).
\]

On the other hand \( J_\sigma \) and \( J_\lambda = \rho(g) J_\sigma \rho(g)^{-1} \) are in \( \rho(G_R) \) so that \( J_\lambda \) commutes with \( \text{Res}_{K/Q} (1 \ast B_\sigma) \).

Finally: \( \text{End}_Q(A_\lambda) \supset \text{Res}_{K/Q} (L^M \ast B_\sigma) \).
We shall look closely at $L^M$ and show that:

**Proposition 1.4.2.** There exist $L$ in $B$ and $K$ in $B_0$ such that:

$L^{\phi_1} \cdots L^{\phi_d} \cdot K \cong L^M \cdot B_0$

so that: $A_{L}^{\lambda}$ is of CM type

We will break the proof into several steps:

**Lemma 1.** Let $B$ a quaternion algebra over $k$, $k$ totally real algebraic number field. There exists $L \subset B$, $L$ totally imaginary quadratic extension of $k$ such that: $L^{\phi_1}, \ldots, L^{\phi_n}$ are linearly disjoint where $L^{\phi_i} = L \cdot e_i$ and $S = \{\phi_1, \ldots, \phi_n\}$ is the set of real embeddings of $k$.

**Lemma 1a.** Let $k$ be an algebraic number field, and $a_1, \ldots, a_n, \gamma$ elements of $k$. If $\sqrt{\gamma} \in k(\sqrt{a_1}, \ldots, \sqrt{a_n})$ then

$\gamma \cdot a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \in k^2$ where $\epsilon_i = \{0, 1\}$

**Proof.** $k(\sqrt{a_1}, \ldots, \sqrt{a_n})$ is a Galois extension of $k$ of degree $2^n$, $\text{Gal}(k(\sqrt{a_1}, \ldots, \sqrt{a_n})/k) = Z_2^n$

We define $\alpha_{I} = a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$, $\epsilon_i$ either 0 or 1, so that we can write $k(\sqrt{a_1}, \ldots, \sqrt{a_n}) = \bigoplus I k \sqrt{\alpha_{I}}$.

$\{\alpha_{I}\}_I$ are the $2^n$ eigenvectors of the Galois group.

If $\sqrt{\gamma} \in k(\sqrt{a_1}, \ldots, \sqrt{a_n})$ then $\sqrt{\gamma}$ is an eigenvector of the Galois
group and so
\[ \sqrt{\gamma} = a \sqrt{\alpha I} \] for some \( a \in k \) and \( I \). Therefore \( \gamma a I \in k^2 \). ●

**Lemma 1.b.** ([v], Ch.3, lemma 3.8)

Suppose \( B \) and \( k \) are as in lemma 1. \( L \) a quadratic extension of \( k \). \( L \) is contained in \( B \) if and only if \( L_v \) is a quadratic extension of \( k_v \) for every \( v \) place of \( k \) such that \( B_v = B \otimes k_v \) is a division algebra.

**Lemma 1.c.** ([v]), Ch.3, lemma 3.6) 1

There exists a quadratic extension \( L \) of \( k \) such that \( L_v = L \otimes k_v \) is a given quadratic extension of \( k_v \) for a finite set of places of \( k \).

We are finally able to prove lemma 1.

**Proof.** Let \( d(B) = \prod_{I} p_i^I \) be the discriminant of \( B \). Let \( p \) be a prime ideal of \( k \), \( p \neq p_i^I \phi \) for any \( i \) in \( I \) and \( \phi \) in \( S \) and \( \text{char} \left( O_k/p \right) \neq 2 \). Set \( L \) an unramified quadratic extension of \( k \) for any \( i \in I \).

- \( L_p \) a ramified quadratic extension of \( k \)
- \( L_{p_i}^\phi = k_{p_i^i}^\phi \) if \( \phi \neq \text{id} \)
- \( L_q = C \) for any \( q \) infinite prime of \( k \)

By lemma 1.c. there exists an \( L \), imaginary quadratic extension of \( k \), \( L = k(\sqrt{-d}) \) that coincides locally with the given extensions.

By lemma 1.b. \( L \) is contained in \( B \). From our construction
d(L) = p·b, b ideal of k, (b,pφ) = 1 for any φ in S, so (d) = d(L)·a², a an ideal of k.

Now \( L^φ = k(\sqrt{-d^φ}) \) so \( \prod_{S} (d)^φ \prod_{pφ} \prod_{(a^2)^φ} \) with \( (p^φ, b^ψ) = 1 \) for any \( φ, ψ \) in S.

The lemma is then proved applying lemma 1.a. ■

Moreover we can show:

Lemma 2. Let \( B, B_0 \) be two quaternion algebras over k. Let \( L ⊂ B \) be as in lemma 1. There exists \( K ⊂ B_0 \) such that \( L^φ_1, ..., L^φ_n, K \) are linearly disjoint.

**Proof.** In fact if we assume \( L = k(\sqrt{-α}) \) we can choose, in the above notation, a prime ideal \( q \) of k such that \( ((α), q) = 1, q ≠ p^φ \) for any \( φ \) in S and char \( (0_k/q) = 2 \). Then we can construct as in lemma 1, \( K ⊂ B_0 \) an imaginary quadratic extension of k such that \( d(K) = q^b^1, (b^1, q^ψ) = 1 \) for any \( φ \) in S. \( K = k(\sqrt{-β}), (β) = d(K)(a^1)^2 \) and

\[ β^αφ_1...αφ_n = p^φ_1...p^φ(nq) \tilde{a}, (\tilde{a}, p^φ) = 1, (\tilde{a}, q) = 1 \]

So \( β^αφ_1...αφ_n \) is not in \( k^2 \) and again by lemma 1.a. the lemma follows ■

**Proof of Proposition 1.4.2.**

It is now easy to extend \( L \) and \( K \) to \( B_0^φ \tilde{k} \) and \( B_0^φ \tilde{k} \) respectively, so that they will satisfy proposition 1.4.2.
For $L$ we just need to choose $p$, see proof of Lemma 1 such that $p \neq p_i^\phi$ for any $p_i \mid d(B)$, $p_i \mid d(B \otimes \kappa)$ and $\phi$ in $S$. Proceed analogously for $K$. We obtain then $L_1^\phi, \ldots, L_n^\phi, K$ linearly disjoint where $L_1^\phi = L \otimes \kappa$ so that $L_1^\phi \ldots L_n^\phi K = L \otimes K$.

1.5 Lattices in $F$

Let $\kappa = M_n(k)$ be a trivial central simple algebra over $k$. $k$ an algebraic number field. Let $F$ be a minimal left ideal in $\kappa$, and $0$ an order in $\kappa$. In this section we clarify the $0$-invariant lattices of $F$. The main result will be:

**Theorem.** The number of isomorphism classes of $0$-invariant lattices in $F$ is up to a constant, equal to the class number of $k$.

All the lattices are assumed to be $O_k$-lattices, $O_k$ the ring of integers of $k$.

The main reference for this chapter is I. Reiner [R]

$F$ is a minimal ideal of and since is trivial we have $\kappa = \text{End}_k F$. Let $L$ be a lattice in $\kappa$, we define:

$O_r(L) = \{ x \in \kappa \mid Lx \subseteq L \}$

$O_\ell(L) = \{ x \in \kappa \mid xL \subseteq L \}$

$O_r(L)$ and $O_\ell(L)$ are orders in $\kappa$ and called right order of $L$ and left order of $L$ respectively.
Proposition 1.5.1. There is a natural correspondence between lattices in F and lattices in \( \mathcal{M} \) with right order \( M_n(O_k) \).

Proof. Let \( L \) be a lattice in F. \( \mathcal{M} \) is isomorphic to the direct sum of \( n \) copies of F so we can define the lattice \( M = L \oplus \ldots \oplus L \), \( n \) copies of \( L \).

We have \( O_r(M) = M_n(O_k) \).

Vice versa given \( M \) a lattice in \( \mathcal{M} \) such that \( O_r(M) = M_n(O_k) \)
then \( M = L \oplus \ldots \oplus L \).

Lemma 1.5.2. Let \( L \) be lattice in F.

Then \( O_L(L) = \{ x \in \mathcal{M} \mid xL \subset L \} \) is a maximal order of \( \mathcal{M} \).

Proof. Let \( M = L \oplus \ldots \oplus L \) be the associated lattice to \( L \) in \( \mathcal{M} \).

We have \( O_r(M) = M_n(O_k) \) a maximal order and therefore \( O_L(M) \) is a maximal order. But \( O_L(L) = O_L(M) \) so \( O_L(L) \) is maximal.

Let \( L_1 \) and \( L_2 \) be \( O \)-invariant lattices in F, \( O \) any order of \( \mathcal{M} \). Put \( a = \{ x \in \mathcal{M} \text{ such that } xL_1 \subset L_2 \} \). \( a \) is a two sided \( O \)-ideal.

Proposition 1.5.3. \( L_2 = aL_1 \)

Proof. \( \mathcal{M} \) coincide with \( \text{End}_k F \), therefore there is \( \psi \) in \( \mathcal{M} \) such that \( L_2 = \psi L_1 \).
Corollary 1.5.4. The action of the set of two sided 0-ideals on the 0-invariant lattices of $F$ is transitive.

Proposition 1.5.5. Let $L_1$, $L_2$ be as above. Then $L_1$ is isomorphic to $L_2$ as $O$-modules if and only if $L_1 = \alpha L_2$ where $\alpha$ is in $k^\times$.

Proof. Suppose $L_1 = \phi L_2$, $\phi$ isomorphism of $O$-modules.
Then $\phi x = x\phi$ for any $x$ in $O$, but $0$ is an order so $\phi$ belongs to center of $\mathfrak{M}$, i.e. $k^\times$. 

Corollary 1.5.6. Let $L_1$, $L_2$ be as above. If $L_1$ is isomorphic to $L_2$ as $O$-module, then: $O_\phi(L_1) = O_\phi(L_2)$.

By lemma 1.5.2 we know that if $L$ is a lattice in $F$ then $O_\phi(L)$ is a maximal order, this allows us to classify the $O$-invariant lattices. Put $C(F,0) = \{L$ lattice in $F\}/\approx_0$,
where $L_1 \equiv L_2$ means that $L_1$ and $L_2$ are isomorphic as $O$-modules.

Theorem 1.5.7. $C(F,0) = \bigcup_i C(F, O_i)$,
where $O_i$ are the maximal orders of $\mathfrak{M}$ containing $0$.

Proof. Let $L_1$, $L_2$ be two $O$-invariant lattices in the same isomorphism class. Then $L_1 = \alpha L_2$, $\alpha \in k^\times$, and $O_\phi(L_1) = O_\phi(L_2)$ is a maximal order by lemma 1.5.2., containing $0$. The theorem is
proved. ■

To estimate the number of isomorphism classes of $O$-invariant lattices in $F$ we need:

Lemma 1.5.6. The number of maximal orders of $\mathcal{U}$ containing a given order $O$ is finite.

Proof. $\mathcal{U}$ is a trivial simple algebra then:

$\mathcal{U}^1 = \{ a \in A^1 \mid \nu(a) = 1 \}$ is a $k$ simple group. $O^1$ is an arithmetic subgroup of $A^1$. Let $\tilde{O}$ be a maximal order in $\mathcal{U}$ containing $O$. Then $\tilde{O}^1$ is a subgroup of $\mathcal{U}^1$ containing $O^1$ as a subgroup of finite index. By Borel's result [B] there are only finitely many subgroups of $\mathcal{U}^1$ containing $O^1$ as a subgroup of finite index. But since a maximal order in a trivial algebra is generated by its units elements the lemma is proved. ■

Theorem 1.5.9. Let $\mathcal{U}$ be a trivial central simple algebra over $k$.

$$|C(F,O)| = ch(k)$$

Where $c$ is the number of maximal orders containing $O$ and $h(k)$ is the class number of $k$.

Proof. Let $\tilde{O}$ be a maximal order in $\mathcal{U}$. Since $\mathcal{U}$ is a trivial central simple algebra we have that the non zero prime ideals $p$ of $k$ and the prime ideals $\beta$ of $\tilde{O}$ are in one to one correspondence:
\[ p = O_k \cap \beta, \quad p\bar{O} = \beta. \]

Therefore by proposition 1.5.3 we have \( |C(F, \bar{O})| = h(k) \) and applying lemma 1.5.8 the theorem is proved. ■

Let \( L_1 \) and \( L_2 \) be two \( O \)-invariant lattices in \( F \) as above.

**Definition 1.5.10.** We say that \( L_1 \) and \( L_2 \) have the same genus if \( L_1, p \) is isomorphic as \( O_p \)-module to \( L_2, p \) for any prime \( p \).

**Theorem 1.5.10.** Let \( \mathcal{A} \) be a trivial central simple algebra over \( k \). The number of genus classes of \( O \)-invariant lattices in \( F \) is equal to the class number of \( k \).

**Proof.** Let \( L_1 \) and \( L_2 \) be lattices in \( F \), \( O \)-invariant. If \( L_1 \) and \( L_2 \) have the same genus then:

\[ O_k(L_1, p) = O_k(L_2, p) \]

for every prime \( p \).

But \( O_k(L_p) = (O_k(L))^p \) and two orders are equal if they coincide at every localization. So \( O_k(L_1) = O_k(L_2) \) and it is a maximal order. The theorem follows. ■

1.6 Classification of Kuga fiber varieties arising from quaternion algebras.

Let \( A \xrightarrow{f} \Gamma \backslash \mathbf{H}^t \) be a Kuga fiber variety defined, as in 1.2, by a symplectic representation \( \rho \) of \( G = \text{Res}_{k/Q}	ext{SL}_1(B) \). By Addington's
Theorem we can assume \( \rho \) to be a polymer representation. In this section we will classify, up to isomorphism, the rigid Kuga fiber varieties defined by a given polymer representation.

Let \( A \xrightarrow{f} V \) and \( B \xrightarrow{g} V \) be Kuga fiber varieties given by \((G,K,X,\Gamma,F,L,\beta,\rho,\tau)\) and \((G,K,X,\Gamma,F',L',\beta',\rho',\tau')\) respectively.

**Definition 1.6.1.** We say that \( A \xrightarrow{f} V \), \( B \xrightarrow{g} V \) are isomorphic if there exists a \( Q \)-linear isomorphism \( \psi: F \rightarrow F' \) such that:

\[
\begin{align*}
\rho &= \psi^{-1}\rho'\psi \\
\beta(x,y) &= \beta'(\psi(x,y)), \quad x, y \in F \\
\tau &= \psi^{-1}\tau'\psi \\
L &= \psi^{-1}L'.
\end{align*}
\]

The fiber of \( A \xrightarrow{f} V \) as a real torus is isomorphic to \( F/L \) where \( F \) is a minimal left ideal in a \( Q \)-algebra and \( L \) is a lattice in \( F \). We will show that the classification of the families induced by a given representation can be reduced to the classification of lattices in a given central simple algebra. Therefore this classification is easily obtained from the results of section 1.5.

**Lemma 1.6.2.** Let \( \psi: F \rightarrow F' \) be a \( Q \)-linear isomorphism. Then \( \rho(g) = \psi^{-1}\rho'(g)\psi \) and \( L = \psi^{-1}L' \) if and only if \( L \) is isomorphic via \( \psi \) to \( L' \).
as $\Gamma$-modules.

**Proof.** Sufficiency is obvious so we want to prove necessity. Suppose $\psi: L \to L'$ is an isomorphism of $\Gamma$-modules; then

$\psi(\rho(\gamma)z) = \rho'(\gamma)\psi(z)$ for any $z$ in $L$ and $\gamma$ in $\Gamma$. Since $\Gamma$ is an arithmetic subgroup of $G$, it is Zariski dense in $G$ (Borel [B]) so that $\psi(\rho(g)z) = \rho'(g)\psi(z)$ for any $g$ in $G$ and $z$ in $L$.

Therefore $\rho = \psi^{-1}\rho'\psi$ and the lemma is proved. ■

In most cases the isomorphism of $L$ and $L'$ as $\rho(\Gamma)$ modules determines the isomorphism of the two families. We have the following:

**Proposition 1.6.3.** (Satake [S]). Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be Kuga fiber varieties defined by $(G, X, X, \Gamma, F, L, \beta, \rho, \tau)$ and $(G, X, X, \Gamma, F', L', \beta', \rho', \tau')$. If $\rho$ satisfies the $H2$ - condition and $\rho'$ is equivalent to $\rho$ over $\mathbb{Q}$, then $B \xrightarrow{g} V$ can be defined by $(G, X, X, \Gamma, F, L', \beta, \tau)$.

Rigid families of quaternion type are defined by polymer representation satisfying the $H2$ - condition, so we have:

**Corollary 1.6.4.** Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be rigid Kuga fiber varieties defined by the same polymer representation. Then $A \xrightarrow{f} V$ is isomorphic to $B \xrightarrow{g} V$ in the sense of definition
1.5.1 if and only if $L$ and $L'$ are isomorphic as $\Gamma$-modules.

**Proof.** Lemma 1.5.2 and proposition 1.5.3 give this result. 

We can now find a classification.
Suppose the data $(C, K, X, F, \beta, \rho)$ is given.

**Notation:** We will denote by $C(\rho, \Gamma)$ the set of isomorphism classes of families of Abelian varieties defined by $\rho$ over $V = \Gamma \backslash X$.

**Definition 1.6.5.** Let $R$ be an algebra over $Q$ and $O_R$ a $Z$-order in $R$. An Abelian variety of $O_R$-type is a pair $(X, \iota)$ formed by an Abelian variety $X$ and an isomorphism $\iota$ of $O_R$ into $\text{End}(X)$ such that $\iota_X(1) = 1_X$. We say that a Kuga fiber variety is of $O_R$-type if the fibers are of $O_R$ type.

**Definition 1.6.6.** We say that two Kuga fiber varieties $A \xrightarrow{\varphi} V$ and $B \xrightarrow{\varphi'} V$ of $O_R$ type are isomorphic as $O_R$ type if:

(i) They are isomorphic (in the sense of 1.6.1), and
(ii) If $\psi : F \to F'$ is the $Q$-linear map inducing the isomorphism then $\varphi_A = \iota_B \psi$. 

**Notation:** Let $C(\rho, \Gamma, O_R)$ be the set of classes of isomorphism of families of Abelian varieties of $O_R$-type defined by $\rho$.

**Main Theorem 1.6.7.** Let $\rho$ be a rigid polymer representation of $G = \text{Res}_{\mathbb{K}/\mathbb{Q}}(\text{SL}_1(B))$, irreducible over $\mathbb{Q}$. $B$ a quaternion algebra over $k$, $B \neq M_2(k)$. $k$ a totally real number field, $\mathbb{K}$ the smallest Galois extension of $\mathbb{Q}$ containing $k$. $\Omega = \text{Gal}(\mathbb{K}/\mathbb{Q})$.

Let $R$ be a $\mathbb{Q}$-algebra of dimension $n$ if $\rho = \rho_P$ and $4n$ otherwise.

Let $O_R$ be a maximal order in $R$.

Then we have:

$$|C(\rho, \Gamma, O_R)| = c \cdot h(\mathbb{K}).$$

Where $c$ is a constant, $h(\mathbb{K})$ is the class number of $\mathbb{K}$ and $\Gamma$ is an arithmetic subgroup of $G$.

**Proof.** Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be two Kuga fiber varieties defined by $\rho$. By proposition 1.6.3 we may assume:

A $\xrightarrow{f} V$ defined by $(G, K, X, \Gamma, F, L_1, \beta, \rho, \tau)$

and $B \xrightarrow{g} V$ defined by $(G, K, X, \Gamma, F, L_2, \beta, \rho, \tau)$

Let $P = \Sigma gM$, $gM \neq \gamma M$ if $\gamma \neq g$, the polymer such that $\rho = \rho_P$. In the notation of 1.2 we have:

(i) $\rho = \text{Res}_{\mathbb{K}/\mathbb{Q}^M}(\rho_M^1 \cdot \text{Res}_{\mathbb{K}/\mathbb{Q}}(\text{SL}_1(B)) \rightarrow \text{End}_{\mathbb{K}^F}^M$.

(ii) $F_M$ minimal left ideal in $B^M$, $F = \text{Res}_{\mathbb{K}/\mathbb{Q}^F}^M$.

(iii) The endomorphism ring of the generic fiber of any family defined by $\rho$ depends on the class of $B^M$ in the Brauer group."
fact by proposition 1.4.1:

Let $A_X$ be the generic fiber

$$\begin{align*}
\text{if } B^M &= M_N(B_0) \\
\text{then } \text{End}_{Q^F} &= \text{Res}_{E/Q}(B^M \cdot B_0) \\
\text{End}_{A^A_X} &= 1 \ast \text{Res}_{E/Q}B_0
\end{align*}$$

$$\begin{align*}
\text{if } B^M &= M_N(\overline{k}) \\
\text{then } \text{End}_{Q^F} &= \text{Res}_{E/Q}B^M \\
\text{End}_{A^A_X} &= \text{Res}_{E/Q}\overline{k}
\end{align*}$$

If $A_X$ is any fiber we have only inclusions.

We can identify $O_R$ with a maximal order in the endomorphism ring of the generic fiber, since $\text{dim}_Q R = 2 \text{dim} A_X$.

Therefore $O_R$ contains $O_{\overline{k}}$ (prop. 1.4.1) and we can consider $O_R$ as an $O_{\overline{k}}$-order, $L_1$ and $L_2$ as $O_{\overline{k}}$-lattices.

A $F \rightarrow V$ and $B \rightarrow V$ are isomorphic as families of $O_R$ type if and only if $L_1$ and $L_2$ are isomorphic as $\mathbb{Z}[\rho(\Gamma)] \ast O_R$-modules.

Therefore if and only if $L_1$ and $L_2$ are isomorphic as

$$O_{\overline{k}}[\rho_M(\Gamma)] \ast O_R$$

If $\rho_M$ is a $C$-irreducible representation of $G$, then, since $\Gamma$ is Zariski dense in $G$, we have that $O_{\overline{k}}[\rho_M(\Gamma)]$ is an order.

If $\rho_M$ is a $C$-reducible representation of $G$, then we have to consider $\rho_M \ast \text{id}: G \times SL_1(B_0) \rightarrow \text{End}_R(F_M)$. $\rho_M \ast \text{id}$ is defined over $\mathbb{R}$ and $C$-irreducible.

$\Gamma \times O_R^1$ is an arithmetic subgroup of $G \times SL_1(B_0)$ so as above $O_{\overline{k}}[\rho(\Gamma) \ast O_R^1]$ is an order in $\text{End}_R(F_M)$.

We can conclude: $\text{C}(\rho, \Gamma, O_R) = \text{C}(F_M, O)$,
where $O = \begin{cases} \mathcal{O}_K[\rho_M(\Gamma)] \\ \text{or} \\ \mathcal{O}_K[\rho_M(\Gamma)] = \mathcal{O}_R \end{cases}$ is an order in $\text{End}_K^\mathbb{F}_M$.

By theorem 1.5.9. $|C(\rho, \Gamma, \mathcal{O}_R)| = c \cdot h(\mathcal{E})$ where $c$ is the number of maximal orders containing $O$.

**Corollary 1.6.8.** Suppose that $O$ is a maximal order. Then we have a bijection:

$$C(\rho, \Gamma, \mathcal{O}_R) \rightarrow \text{Ideal class group } (\mathcal{E}).$$
Chapter 2.

2.0 Definitions
Let $V$ be a projective variety defined over a subfield of $\mathbb{C}$. Let $\sigma \in \text{Aut}(\mathbb{C})$ be an automorphism of the complex numbers. If $V$ is the variety determined by $I(V)$ we define $V^\sigma$ as the variety determined by $I(V)^\sigma$.

Definition. A subfield $h$ of $\mathbb{C}$ is called the bottom field of $V$ if an automorphism $\sigma$ of $\mathbb{C}$ is the identity mapping on $h$ if and only if $V^\sigma$ is birationally equivalent to $V$.

The definition of bottom field is due to Shimura, as are all the results quoted in this section [Sh1,2]. The best known example is the bottom field of a polarized Abelian variety (moduli field), the central notion in the theory of complex multiplication [Sh-T]. The moduli field is interpreted in terms of class field theory Shimura described also the bottom field of the Hilbert modular variety $\Gamma \backslash \mathcal{H}$ in terms of class field theory. In particular he constructed a Hilbert modular variety with bottom field different from $\mathbb{Q}$.

Shimura's original definition is weaker: it requires $V$ and $V^\sigma$ to be only birationally equivalent. But in the case of Hilbert modular varieties the two definitions coincide. Let $D$ be a
bounded symmetric domain and $\Gamma$ a properly discontinuous group of transformations of $D$ with compact quotient and without fixed points. Then $\Gamma \backslash D$ is known to be isomorphic (analytically) to a non singular projective variety $V$. By Igusa $V$ is a minimal model, i.e. every rational mapping of a variety $W$ into $V$ is defined at every simple point of $W$.

Therefore for $V$ the biregular equivalence coincides with the birational equivalence.

Following Shimura we have:

2.0.0 If $W$ is biregularly equivalent to $V$ then the bottom field of $W$ and $V$ coincide.

2.0.1 Every field of definition of $V$ contains the bottom field of $V$, if the latter exists.

2.0.2 Assume $V$ is defined over an algebraic number field, then the bottom field of $V$ exists and is an algebraic number field of finite degree.

Let $V = \Gamma \backslash \mathbb{H}^k$ be as above then the existence of the bottom field of $V$ is assured:

$\Gamma \backslash D$ is rigid if the dimension is bigger then 1. (Calabi-Vesentini). $\Gamma \backslash D$ is therefore defined over an algebraic number field [Sh3] so that its bottom field exists by 2.0.2.

If $\Gamma \backslash D = V$ has dimension 1 then the argument is different.

By Torelli's theorem the bottom field of $V$ coincides with the
field of moduli of \((J, \ )\), \(J\) the Jacobian variety of the curve \(V\) and the canonical polarization.

Let \(A \xrightarrow{f} V\) be a family of Abelian varieties parametrized by a Hilbert modular variety. As we mentioned \(A\) and \(V\) are biregularly equivalent to a projective variety \([K]\) \((V = \Gamma\Backslash \mathbb{H}^t\) is always compact if the quaternion algebra is a division algebra).

If we assume that \(V\) and \(A\) have models defined over an algebraic number field then by 2.0.0 and 2.0.1 the bottom fields of \(A\) and \(V\) are well defined, exist and are algebraic number fields.

The main result of this chapter relates the two fields:

**Theorem.** The bottom field of \(A\) is a Galois extension of the bottom field of \(V\).

From now on we will assume that \(A \xrightarrow{f} V\) has a model defined over an algebraic number field.

2.1 \(A^\sigma \xrightarrow{f^\sigma} V^\sigma\)

Let \(\sigma \in \text{Aut}(\mathbb{C})\) be an automorphism of the complex numbers. Let \(A \xrightarrow{f} V\) be a family of Abelian varieties parametrized by a compact arithmetic variety. We shall identify \(A\) and \(V\) with their embedded images in the complex projective space so that \(f\) is a morphism of algebraic varieties and \(f^\sigma\) is defined as in 2.0.
Theorem 2.1.1. (M.H. Lee [L]) Let \( f: A \rightarrow V \) be a Kuga fiber variety over a compact arithmetic variety \( V \). Then \( f^\sigma: A^\sigma \rightarrow V^\sigma \) is a Kuga fiber variety over \( V^\sigma \).

\[ A^\sigma \xrightarrow{f^\sigma} V^\sigma \] is therefore defined analytically by the data:

\[ (\mathcal{G}(\sigma), K(\sigma), X(\sigma), \Gamma(\sigma), \mathcal{L}(\sigma), \beta(\sigma), \rho(\sigma), \tau(\sigma)) \]

and

\[ A^\sigma \cong (\Gamma(\sigma) \times \mathcal{L}(\sigma)) \setminus (X(\sigma) \times \mathcal{L}(\sigma)) \]

\[ \rho(\sigma) \]

\[ f^\sigma(\sigma) \]

\[ V(\sigma) = \Gamma(\sigma) \backslash X(\sigma) \]

Corollary 2.1.1. If \( A \overset{f}{\rightarrow} V \) is of Hodge type then \( A^\sigma \overset{f^\sigma}{\rightarrow} V^\sigma \) is of Hodge type.

Proof. A family of Abelian varieties \( A \overset{f}{\rightarrow} V \) is of Hodge type if and only if it contains \( A_X \), a fiber of CM type. But then \( A_X^\sigma \) is contained in \( A^\sigma \overset{f^\sigma}{\rightarrow} V^\sigma \) and \( A_X^\sigma \) is of CM type. \( \Box \)

Corollary 2.1.3. Let \( L_p \) and \( L_p^{(\sigma)} \) be the \( p \)-adic completions of \( L \) and \( L^{(\sigma)} \) respectively. Let \( \hat{\Gamma} \) and \( \hat{\Gamma}^\sigma \) be the completions of \( \Gamma \) and \( \Gamma^{(\sigma)} \) respectively in the topology of the subgroups of finite index. Then \( L_p \) and \( L_p^{(\sigma)} \) are isomorphic as \( \hat{\Gamma} \) modules for any prime \( p \). In other words

\[ \hat{\Gamma} \times L_p^{\sigma} \cong \hat{\Gamma}(\sigma) \times L_p^{\sigma} \]

\[ \hat{\rho}_p \cong \hat{\rho}_p^{(\sigma)} \]

for any prime \( p \).
Proof. Let $\{\Gamma_i\}$ be a cofinal system of subgroups of $\Gamma$, $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$

Then for any $i, X_i = \Gamma_i \setminus X$ is a finite unramified covering manifold of $V$ and $\{X_i\}$ is a cofinal system of covering manifolds. For each $i$ we consider $\Gamma_i \times p^i L$, so that $A_i = (\Gamma_i \times p^i L) \setminus (X \times F)$ is a finite unramified covering of $A = (\Gamma \times L) \setminus (X \times F)$. Moreover $A_i \xrightarrow{f_i} V_i$ is a family of Abelian varieties.

Applying $\sigma$ to every $A_i$, we obtain a cofinal system $\{A_i^\sigma\}$ of unramified covering manifolds of $A^\sigma$. We have the following diagrams:

```
  .   .   .   .   .   .   .   .   .
  .   .   .   .   .   .   .   .   .
  |   |   |   |   |   |   |   |   |
A_2 \rightarrow V_2   A_2^\sigma \rightarrow V_2^\sigma
  |   |   |   |   |   |   |   |   |
A_1 \rightarrow V_1   A_1^\sigma \rightarrow V^\sigma
  |   |   |   |   |   |   |   |   |
A \rightarrow V   A^\sigma \rightarrow V^\sigma
```

By theorem 2.1.1 $A_i^\sigma \rightarrow V_i^\sigma$ is a family of Abelian varieties for any $i$. Let $J(\lambda)$ be the $\lambda$-adic points of any fiber of $A$; it doesn't matter which fiber as the $\lambda$-adic points of any two fibers are isomorphic. $\sigma$ induces an isomorphism of $J(\lambda)$ onto $J(\lambda^\sigma)$ and therefore:

$$A_i^\sigma = (\Gamma_i^\sigma \times p^i L^\sigma) \setminus (\lambda^\sigma \times F^\sigma)$$

We proved that $\sigma: \hat{\Gamma} \times \Gamma \rightarrow \hat{\Gamma}^\sigma \times L^\sigma$ is a bijection. To show that $\sigma$ is a homomorphism of groups we need the definition of $\rho^\sigma$ as...
given in the proof of theorem 2.1.1. If \( \hat{\gamma} \in \hat{\Gamma}^\sigma \), \( \hat{\gamma} = \lim \gamma_1^\sigma \)
then \( \rho(\hat{\gamma}) = \lim (\rho(\gamma_1))^\sigma \).
So the conclusion follows at once:

\[
\hat{\Gamma} \times L_p \cong \hat{\Gamma}^\sigma \times L_\sigma^p \quad \text{as groups.}
\]

\[
\hat{\rho}_p \cong \hat{\rho}_\sigma^p
\]

2.2 Bottom fields.
Let \( A \stackrel{f}{\rightarrow} V \) be a Kuga fiber variety parametrized by an arithmetic
variety \( V = \Gamma \backslash X \). We assume that \( A \stackrel{f}{\rightarrow} V \) has a projective model
defined over a number field. We define \( K_A \) as the bottom field of
\( A \) and \( K_V \) as the bottom field of \( V \).

Theorem 2.2.1. \( K_A \) and \( K_V \) exist and \( K_A \supseteq K_V \).

Proof. The existence follows from 2.0.2 and the hypothesis. Let
\( \sigma \) be an element of \( \text{Aut}(G) \). By theorem 2.1.1 we know that
\( A^\sigma \stackrel{f^\sigma}{\rightarrow} V^\sigma \) is a Kuga fiber variety over an arithmetic variety
\( V^\sigma = \Gamma(\sigma) \backslash X(\sigma) \).

Assume that \( \sigma|_{K_A} = \text{id} \), i.e., there exists a biholomorphic map
\( b(\sigma) : A^\sigma \rightarrow A \). If we show that \( b(\sigma) \) induces a biholomorphic map
between \( V \) and \( V^\sigma \) the theorem is proved. In fact then \( \sigma|_{K_A} = \text{id} \)
implies \( \sigma|_{K_V} = \text{id} \) so that \( K_A \supseteq K_V \).
Consider the fiber \( A^\sigma_\lambda = (f^\sigma)^{-1}(\lambda) \), \( \lambda \in V^\sigma \), and the holomorphic
map \( fb(\sigma) : A^\sigma_{\lambda} \to V \).

\( A^\sigma_{\lambda} \) is an Abelian variety and \( V \) an arithmetic variety, \( V = \Gamma \backslash X \).

So there exists a lifting \( \tilde{f}_{\sigma, \lambda} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
A^\sigma_{\lambda} & \xrightarrow{fb(\sigma)} & V \\
\downarrow{\pi} & & \downarrow{\tilde{n}} \\
C^n & \xrightarrow{\tilde{f}_{\lambda, \sigma}} & X \\
\end{array}
\]

Then \( \tilde{f}_{\lambda, \sigma} \) is a bounded holomorphic function and therefore a constant, i.e. \( fb(\sigma)(A^\sigma_{\lambda}) = x \) for some \( x \in V \). This implies that \( b(\sigma) \) sends fibers of \( A^\sigma \) to fibers of \( A \) and induces in the natural way, identifying \( V^\sigma \) with the zero section, \( \eta(\sigma) : V^\sigma \to V \), \( \eta(\sigma) \) holomorphic.

Replacing \( b(\sigma) \) with \( b(\sigma)^{-1} \) we conclude that \( \eta(\sigma) \) is biholomorphic and \( \sigma|_{K_V} = id. \)

2.3 \( V^\sigma \neq V \).

Let \( \sigma \) be an element of \( \text{Ant}(C) \)
We examined in 2.2 the structure of \( A^\sigma \xrightarrow{f^\sigma} V^\sigma \) as a family of Abelian varieties.

In this section we show that in the rigid quaternion case, if \( V \cong V^\sigma \) then \( A^\sigma \xrightarrow{f^\sigma} V^\sigma \) can be defined by the data \((G, K, X, \Gamma, F, L(\sigma), \beta, \rho, \tau)\). So it is the lattice \( L(\sigma) \) that characterizes the new family. We need some preliminary results:
Let $Y_1$ and $Y_2$ be Abelian varieties and $\sigma: Y_1 \to Y_2$ a morphism. We will denote by $\alpha_Q$ the rational representation of $\alpha$ and by $\alpha_\ell$ its $\ell$-adic representation. The following is well known:

**Lemma 2.3.1.** The representations $\alpha_Q$ and $\alpha_\ell$ are equivalent.

**Lemma 2.3.2.** Given the morphism $\alpha$, apply $\sigma$ to get $\alpha^\sigma: Y_1^\sigma \to Y_2^\sigma$. Then $\alpha_Q \sim (\alpha^\sigma)_Q$.

**Proof.** Let $J_\ell(Y_1)$, $J_\ell(Y_2)$ be the groups of $\ell$-adic points. Then $\sigma$ induces the isomorphisms:

$$J_\ell(Y_1) \cong J_\ell(Y_1^\sigma)$$

and the commutative diagram

$$
\begin{array}{ccc}
J_\ell(Y_1) & \xrightarrow{\alpha_\ell} & J_\ell(Y_2) \\
\downarrow{\cong} & & \downarrow{\cong} \\
J_\ell(Y_1^\sigma) & \xrightarrow{\alpha_\ell^\sigma} & J_\ell(Y_2^\sigma)
\end{array}
$$

Therefore $\alpha_\ell \sim \alpha_\ell^\sigma$ and $\alpha_Q \sim (\alpha^\sigma)_Q$. $\blacksquare$

We assume that $A \xrightarrow{\Phi} V$ is defined by $(G, \mathcal{K}, X, \Gamma, F, L, \beta, \rho, \tau)$ where $G = \text{Res}_{K/Q}(\mathcal{L}_1(B))$, and $A^\sigma \xrightarrow{\Phi^\sigma} V^\sigma$ is defined by $(G(\sigma), \mathcal{K}(\sigma), X(\sigma), \Gamma(\sigma), F(\sigma), L(\sigma), \beta(\sigma), \rho(\sigma), \tau(\sigma))$. Following Lee [L] we analyze the representation $\rho(\sigma): G(\sigma) \to \text{Sp}(F(\sigma), \beta(\sigma))$ that defines $A^\sigma \xrightarrow{\Phi^\sigma} V^\sigma$. Let $X(\sigma)$ be
the universal covering of $V^\sigma$, then:

$\Gamma(\sigma)$ is the fundamental group of $V^\sigma$,

$G(\sigma)$ is the connected component of the identity in $\text{Aut}(X(\sigma))$.

We define $C(\Gamma(\sigma))$ the commensurability subgroup of $\Gamma(\sigma)$ in $G(\sigma)$ as:

$$C(\Gamma(\sigma)) = \{ g^\sigma \in \text{Aut}(X(\sigma)) \mid [\Gamma(\sigma) : g^\sigma \Gamma(\sigma)(g^\sigma)^{-1} \cap \Gamma(\sigma)] < \infty \}$$

We have:

$\Gamma(\sigma) \subset G(\sigma)$ and Zariski dense,

$$C(\Gamma(\sigma)) \rightarrow \hat{G}(\sigma) = \text{Aut}(\hat{X}(\sigma))$$

The representation $\rho(\sigma)$ is actually defined on

$$C(\Gamma(\sigma)) \rightarrow \hat{G}(\sigma)$$. Let $\Gamma(\sigma) \supset \Gamma_1(\sigma) \supset \ldots$ be a cofinal sequence of groups. Then the corresponding system of coverings of $V(\sigma)$ is a cofinal system $\ldots \rightarrow V_2(\sigma) \rightarrow V_1(\sigma) \rightarrow V(\sigma)$. In the same way we can consider the cofinal system $\{\Gamma_\sigma^i \times \rho(\sigma) \cdot L^{\sigma}\}$. We then have

\[
\begin{array}{c}
A_1(\sigma) = (\Gamma_1(\sigma) \times L(\sigma)) \backslash X(\sigma) \times \rho(\sigma) \\
\downarrow \\
A_{i-1}(\sigma) \rightarrow \downarrow \\
\vdots \\
\uparrow \\
A(\sigma) \rightarrow V(\sigma)
\end{array}
\]

Applying $\sigma^{-1}$ to these constructions yields

\[
\begin{array}{c}
\downarrow \rightarrow V_i \\
\downarrow \\
\downarrow \rightarrow V_{i-1} \\
\vdots \\
\uparrow \\
\rightarrow V
\end{array}
\]
where \( \{ V_i \} \) corresponds to a cofinal sequence \( \{ \Gamma_i \} \) and for each \( i \) \( A_i \rightarrow V_i \) is a family of Abelian varieties. Therefore
\[ A_i = (\Gamma_i \times L) \backslash (X \times F). \]

Let \( \gamma \) be an element of \( \Gamma(\sigma) \). Since \( \Gamma(\sigma) \rightarrow \hat{\Gamma}(\sigma) \) we have
\[ \gamma = \lim \gamma_i \text{ and for each } i, (\gamma_i, \rho(\gamma)(\gamma)) : A_i \rightarrow A_i \text{ is a covering transformation over } A. \]

Applying \( \sigma^{-1} \) yields the set
\[ \{ \gamma_i \sigma^{-1}, [\rho(\gamma)] \sigma^{-1} \} \]

of covering transformations for the cofinal system \( \rightarrow A_i \rightarrow A_{i-1} \rightarrow \ldots \rightarrow A. \)

Hence \( \gamma_i \sigma^{-1} = \gamma_i' \) for some \( \gamma_i' \in \Gamma_i\) and the rational representation \( ([\rho(\gamma)] \sigma^{-1})_\mathbb{Q} = \rho(\gamma_i') \) is independent of the fiber.

This implies

**Lemma 2.3.3.**

(i) \( \sigma|\Gamma : \Gamma \rightarrow \hat{\Gamma}(\sigma) \) and

(ii) \( \sigma|\Gamma \) is an isomorphism.

By lemma 2.3.2 we have \( ([\rho(\gamma)] \sigma^{-1})_\mathbb{Q} = \rho(\sigma). \)

Thus there exists \( \psi : F \rightarrow \hat{F}(\sigma) \) such that \( \rho(\sigma)(\gamma') \subset \psi p(G)\psi^{-1} \) for any \( \gamma' \in \hat{\Gamma}(\sigma) \). Since \( \Gamma(\sigma) \) is dense in \( \hat{\Gamma}(\sigma) \),
\[ \rho(\sigma)(\hat{G}(\sigma)) \subset \psi p(G)\psi^{-1}. \]

By interchanging \( \rho(\sigma) \) and \( \rho \) we can proceed in the same way to conclude:
Lemma 2.3.4. \( \rho(G) = \psi^{-1}\rho(\sigma)(G(\sigma))\psi \)

This result is the crucial step in the proof of the following:

Theorem 2.3.5. Let \( A \xrightarrow{f} V \) be a rigid Kuga fiber variety of quaternion type defined by the data \((G,K,X,\Gamma,F,L,\beta,\rho,\tau)\). If \( \sigma \) is an element of \( \text{Aut}(G) \) and \( V \cong V^\sigma \), then the Kuga fiber variety \( \xrightarrow{f^\sigma} V^\sigma \) can be defined by the data \((G,K,X,\Gamma,F,L(\sigma),\beta,\rho,\tau)\).

**Proof.** We assume that \( A^\sigma \xrightarrow{f^\sigma} V^\sigma \) is defined by the data \((G(\sigma),K(\sigma),X(\sigma),\Gamma(\sigma),F(\sigma),L(\sigma),\beta(\sigma),\rho(\sigma),\tau(\sigma))\). Let \( \phi \) be the biregular isomorphism \( \phi: V \rightarrow V^\sigma \). \( \phi \) induces an isomorphism of the fundamental group \( \phi_\pi : \pi_1(V) \rightarrow \pi_1(V^\sigma) \).

\( \Gamma \) and \( \Gamma(\sigma) \) are arithmetic subgroups of \( G \) and \( G(\sigma) \) (respectively) so that \( \pi_0(V) = \{e\} \), \( \pi_0(V(\sigma)) = \{e\} \) and \( \pi_1(V) = \Gamma, \pi_1(V(\sigma)) = \Gamma(\sigma) \) (see Kajdan [Kj]).

Therefore \( \phi \) induces and isomorphism \( \phi_\pi : \Gamma \rightarrow \Gamma(\sigma) \). Since \( \Gamma \) is Zariski dense in \( G \) we can extend \( \phi_\pi \) to a \( G \)-morphism \( \phi_\pi : G \rightarrow G(\sigma) \).

By lemma 2.3.3 we have \( \rho(G) = \psi^{-1}\rho(\sigma)(G(\sigma))\psi \) so that:

\[
\begin{array}{ccc}
G & \xrightarrow{\rho} & \text{Sp}(F,\beta) \\
\phi_\pi \downarrow & & \downarrow \psi_\pi \\
G(\sigma) & \xrightarrow{\rho(\sigma)} & \text{Sp}(F(\sigma),\beta(\sigma))
\end{array}
\]
The map of $\rho^{-1}I_{\phi_p}(\sigma)\phi_\tau : G \to G$ is an automorphism defined over $Q.G = \text{Res}_{K/Q} \text{SL}_1(\mathbb{B})$ so that any $Q$-automorphism is inner and thus there exists $x \in G$ such that any $\rho^{-1}I_{\phi_p}(\sigma)\phi_\tau = I_x$.

Therefore $\rho(\sigma)\phi_\tau \sim \rho$.

Now the two representations $\rho$ and $\rho(\sigma)\phi_\tau$ satisfy the hypotheses of proposition 1.6.3. We conclude that $A^\sigma \overset{\phi_\tau}{\longrightarrow} V^\sigma$ can be defined by $(G,K,X,\Gamma,F,L(\sigma),\beta,\rho,\tau)$. ■

Corollary 2.3.6. Let $L_p = L \circ Z_p$ and $L_p(\sigma) = L(\sigma) \circ Z_p$ for any prime $p$. Then $L_p$ and $L_p(\sigma)$ are isomorphic as $\Gamma$-modules:

$$\Gamma \times L_p \cong \Gamma \times L_p(\sigma).$$

Proof. The map $\psi:F \to F(\sigma)$ introduced before Lemma 2.3.4 locally coincides with $\sigma$, and therefore:

by lemma 2.1.3

$$\Gamma \times L_p \overset{\sigma \times \psi_p}{\cong} \Gamma(\sigma) \times L_p(\sigma).$$

by lemma 2.3.3

$$\sigma|\Gamma:\Gamma \longrightarrow \Gamma(\sigma)$$

so that

$$\Gamma \times L_p \overset{\sigma \times \psi_p}{\cong} \Gamma(\sigma) \times L_p(\sigma).$$

Moreover in the proof above (2.3.5) we have shown that there exists $x \in G$ such that

$$\Gamma \times L_p(\sigma) \overset{\phi_\tau \times \psi_p}{\cong} \Gamma(\sigma) \times L_p(\sigma).$$
where, for simplicity, we write $L(\sigma)$ for $[\rho(x)]^{-1}L(\sigma)$. $\Phi_k^{-1} \times \sigma$ is an automorphism of $\Gamma$, so, as we have seen, it is given by conjugation by an element of $G$. The corollary is then proved.

2.4 Gal (C/Bottom field of V) $\rightarrow$ Ideal class group ($\mathcal{K}$).

Let $\sigma$ be an element of Gal(C/Bottom V), let $A \xrightarrow{f} V$ be a rigid family of quaternion type defined as usual by $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$. We proved that $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is defined by $(G,K,X,\Gamma,F,L(\sigma),\beta,\rho,\tau)$.

In this section we will show that $L(\sigma) = a(\sigma)L$, $a(\sigma)$ an ideal of $\tilde{k}$, and show that the map Gal(C/KV) $\rightarrow$ IC($\tilde{k}$) is a group homomorphism. We first need to make some assumptions, as in Section 1.6.

i) $\rho = \rho_F$ is rigid, Q-primary, Q-irreducible

i.e., $P = \sum gM$ ; $gM \neq \gamma M$ when $g \neq \gamma$ 

$\forall g \in \Omega$

ii) $A \xrightarrow{f} V$ is of type $O_R$ maximal order in the Q-algebra $R$,

$\dim Q_R = \dim Q[\text{End}_Q A_\chi]$ where $A_\chi$ is a generic fiber.

Lemma 2.4.1. If $A \xrightarrow{f} V$ is of type $O_R$ then so is $A^\sigma \xrightarrow{f^\sigma} V^\sigma$.

Proof. In the notation of proposition 1.4.1 we may identify $O_R$ with a $O_{\tilde{k}}$ maximal order in $\text{Res}_{\tilde{k}/Q} B_\chi = \text{End}_Q(A_\chi)$. As before $B_\chi$ is either $\tilde{k}$ or a division algebra over $\tilde{k}$ and $A_\chi$ is isomorphic to $(F,L,J_\chi)$ where $F = \text{Res}_{\tilde{k}/Q} B_\chi^N$. 
Let $b$ be an element of $B_0$ and $r_b: B_0 \to B_0$ the right multiplication by $b$. Put $R_b = \text{Res}_{k/Q}^\sigma r_b$. The rational representation of $B_0$ is $N$ copies of $R_b$:

\begin{enumerate}
\item $B_0 \to \text{GL}(F)$
\item $b \to \text{Res}_{k/Q}^\sigma \text{Nr}_b = N\text{R}_b$
\end{enumerate}

We can therefore write $F$ as a sum of invariant subspaces

$F = \left(\text{Res}_{k/Q}^\sigma B_0\right)^N$. If $S: O_R \to \text{End} A_\sigma$ is the analytic representation then the generic fiber $A^0_\sigma$ of $A^0 \xrightarrow{f^\sigma} V^0$ is of type $O_R$ via the representation $S^\sigma: O_R \to \text{End}(A_\sigma^\sigma)$. We show that $S^\sigma = S$. Let $\text{Res}_{k/Q}^\sigma B_0$ be an $i$-invariant subspace of $F$. Then $J_\sigma(\text{Res}_{k/Q}^\sigma B_0)$ is $i$-invariant and therefore coincides with a copy of $\text{Res}_{k/Q}^\sigma B_0$.

We can conclude that $S(b) = \frac{N}{2} R_b$ and

$S^\sigma(b) = \frac{N}{2} R^\sigma_b = \frac{N}{2} R_b$, as $R_b$ is defined over $Q$.

The choice of $A_\sigma$ is not restrictive and the lemma is proved. 

**Lemma 2.4.2.** (i) $L, L^\sigma$ can be considered as $O_k^\sigma$ lattices in $F_M$,

$F = \text{Res}_{k/Q}^\sigma F_M$.

(ii) $L, L^\sigma$ are $O_k^\sigma [\rho_M(\Gamma)] \ast O_R$ - lattices of the same genus, so that $O_k(L) = O_k(L^\sigma)$.

(iii) $L = a(\sigma)L^\sigma$, where $a(\sigma)$ is the $O_k^\sigma$-ideal in $\bar{k}$:

$a(\sigma) = [L^\sigma : L] = \{g \in \text{End}_{Q_F M}|gL^\sigma \subseteq L\}$.

**Proof.** (i) is an obvious consequence of lemma 2.4.1, recalling that since $O_R$ is a maximal order, $O_R \supset O_k^\sigma$. 

ii) follows from corollary 2.3.6 since all the local isomorphisms considered commute with the elements of $O_{\mathcal{R}}$.

(iii) is just Proposition 1.5.3. 

We will denote the order $O_{\mathcal{R}}[\rho_M(\Gamma)] \cdot O_{\mathcal{R}}$ by $0$.

**Lemma 2.4.3.** Let $L$ and $L^{(\sigma)}$ be $O_{\mathcal{R}}^{\sigma}$-lattices associated to $A \xrightarrow{\underline{f}} V$ and $A^{\sigma} \xrightarrow{\underline{f}^{\sigma}} V^{\sigma}$ respectively.

Then $L^{(\sigma)} = a(\sigma)L$,

the ideal class of $a(\sigma)$ is independent of $L$ and $L^{(\sigma)}$, and the map $\text{Gal}(C/\text{Bottom field (V)}) \rightarrow \text{Ideal class group (}\mathbb{K}\text{)}$ is well defined.

**Proof.** We must show that the ideal class of $a(\sigma)$ is independent of the choice of $L$ and $L^{(\sigma)}$. Suppose $A \xrightarrow{\underline{f}} V$ has two analytic models as a family of $O_{\mathcal{R}}$-type. Let $L$ and $L'$ be the lattices associated to the two models. Since the two models are biregularly isomorphic as $O_{\mathcal{R}}$-type, $L$ and $L'$ are isomorphic as $0$-modules, i.e., $L = \alpha L'$, $\alpha \in \bar{k}^\times$.

Suppose $L = [L^{(\sigma)}:L] L^{(\sigma)}$ and $L' = [L^{(\sigma)}:L'] L^{(\sigma)}$, then $\alpha[L^{(\sigma)}:L'] = [L^{(\sigma)}:L]$ and the ideal class of $a(\sigma)$ is independent on the choice of $L$. We can proceed analogously for $L^{(\sigma)}$. 

Lemma 2.4.4. Let \( A \xrightarrow{f} V \) and \( A' \xrightarrow{f'} V \) be two families of quaternion type defined by \( (G,K,X,\Gamma,F,\beta,\rho,\tau) \). Let \( L \) and \( L' \) be the corresponding lattices and let \( \sigma \in \text{Gal}(C/K_V) \). If we assume that \( L = [L':L]L' \) where \([L':L]\) is an \( \mathcal{O}_k \) ideal, then
\[
[L':L] \equiv [L'(\sigma);L(\sigma)]
\]
as ideal classes, here \( L'(\sigma) \) and \( L(\sigma) \) are the \( \mathcal{O}_k \)-lattices associated to \( A'^\sigma \rightarrow V \) and \( A^\sigma \rightarrow V \).

**Proof.** Since we are concerned with ideal classes, we can assume:
\( [L':L] \subset \mathcal{O}_k \) so that \( L' \subset L \). Let \( x \) be in \( V \) and \( \pi_x \) the natural projection \( \pi_x: A_x \rightarrow A'_x \). Let \( \Lambda \) be \( \text{Ker} \pi_x \cong (\text{Res}_{k/Q}^L)/\text{Res}_{k/Q}^L \).

Now for any \( \alpha \) in \( \mathcal{O}_k^\sigma \) we have the following diagrams:

\[
\begin{array}{ccc}
\text{Nra} & & \\
L' & \xrightarrow{\text{NR}_\alpha} & L' & \text{and} & 0 \rightarrow \Lambda \rightarrow A'_x \rightarrow A_x \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\text{Res}_{k/\mathcal{Q}}^{L'} & \rightarrow & \text{Res}_{k/\mathcal{Q}}^{L'} & 0 \rightarrow \Lambda \rightarrow A'_x \rightarrow A_x \rightarrow 0 \\
\end{array}
\]

Then \([L':L] = \{ \alpha \in \mathcal{O}_k \mid \frac{N}{2}R_\alpha \Lambda = 0 \}\)

Applying \( \sigma \):
\[
\begin{array}{ccc}
& & \\
0 & \rightarrow & \Lambda^\sigma \rightarrow A_x^\sigma \rightarrow A_x^\sigma \rightarrow 0 \\
& & \downarrow & & \\
0 & \rightarrow & \Lambda^\sigma \rightarrow A'_x^\sigma \rightarrow A_x^\sigma \rightarrow 0 \\
\end{array}
\]

where \( \Lambda^\sigma = \text{Res}_{k/\mathcal{Q}}^{L'(\sigma)}/\text{Res}_{k/\mathcal{Q}}^{L'(\sigma)} \) and \( \left( \frac{N}{2}R_\alpha \right)^\sigma = \frac{N}{2}R_\alpha \).
\[
[\mathcal{L}_1^\sigma;\mathcal{L}_1^\sigma] = \{ \alpha \mid \frac{N}{2} R_\alpha \Lambda^\sigma = 0 \} \\
= \{ \alpha \mid \left( \frac{N}{2} R_\alpha \right)^\sigma \Lambda^\sigma = 0 \} \\
= \{ \alpha \mid \frac{N}{2} R_\alpha \Lambda = 0 \} = [\mathcal{L}_1;\mathcal{L}]
\]

Lemma 2.4.5.

The map \( \text{Gal}(\mathcal{G}/\mathcal{K}_Y) \to \text{Ideal class group } (\mathcal{K}) \) is an homomorphism of groups.

Proof. Let \( \sigma, \delta \) be elements of \( \text{Gal}(\mathcal{G}/\mathcal{K}_Y) \).

Consider the following ideal classes:

\[
\begin{align*}
a(\sigma) &= [\mathcal{L}(\sigma);\mathcal{L}] \\
a(\delta) &= [\mathcal{L}(\delta);\mathcal{L}] \\
a(\delta \sigma) &= [\mathcal{L}(\delta \sigma);\mathcal{L}]
\end{align*}
\]

By definition

\[
\mathcal{L}(\sigma \delta) = [\mathcal{L}(\sigma \delta);\mathcal{L}(\delta)]^{-1} [\mathcal{L}(\sigma);\mathcal{L}]^{-1} \mathcal{L}
\]

and

\[
\mathcal{L}(\sigma \delta) = [\mathcal{L}(\sigma \delta);\mathcal{L}]^{-1} \mathcal{L}.
\]

By the uniqueness of the associated ideal:

\[
[\mathcal{L}(\sigma \delta);\mathcal{L}] = [\mathcal{L}(\sigma \delta);\mathcal{L}(\delta)][\mathcal{L}(\sigma);\mathcal{L}]
\]

and we just prove that \([\mathcal{L}(\sigma \delta);\mathcal{L}(\delta)] \equiv [\mathcal{L}(\delta);\mathcal{L}] \) as ideal classes. So \( a(\sigma \delta) = a(\delta) \cdot a(\sigma) \). \( \blacksquare \)
2.5 $1 \to \text{Gal}(K_A, 0/K_V) \to \text{Ideal class group } (\tilde{k})$

Let $A \xrightarrow{f} V$ be a rigid Kuga fiber variety. In Section 2.2, we proved that the bottom field of $A$ is an extension of the bottom field of $V$. Our discussion of $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ in the previous section suggests that the definition of bottom field of $A$ must be modified to give an interpretation of the extension.

In fact if $V \cong V^\sigma$ then $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is a Kuga fiber variety defined by the same data as $A \xrightarrow{f} V$ and with the same endomorphisms ring of the generic fiber [same type]. We will therefore define the bottom field of a family with a given ring of endomorphisms $K_{A, 0}$ and prove that $K_{A, 0}$ is abelian over $K_V$.

**Main Theorem 2.5.1.** Let $A \to V$ be a Kuga fiber variety defined by:

$(G, K, X, \Gamma, F, L, \beta, \rho, \tau)$.

We assume:

(i) $B$ is a quaternion algebra over $k$, $k$ a totally real number field, $B \cong M_2(k)$ and $G = \text{Res}_{\tilde{k}/Q}(\text{SL}_1(B))$.

$\tilde{k}$ is the smallest Galois extension of $Q$ containing $k$.

(ii) $\rho = \rho_p$ rigid, $Q$ irreducible, $Q$ primary polymer representation

(iii) $A \xrightarrow{f} V$ is of $0$-type, $0$ a maximal order in $\text{End}_Q(A_x)$. $A_x$ is a generic fiber.

Then $1 \to \text{Gal}(K_A, 0/K_V) \to \text{Ideal class group } (\tilde{k})$
We need the definition of $K_{A,0}$.
We showed that $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is of the same 0-type as $A \xrightarrow{f} V$, naturally we can modify the definition of bottom field of $A$.

**Definition 2.5.2.** We call the subfield $K_{A,0}$ of $C$ the bottom field of $A$ respect to $0$ if an automorphism $\sigma$ of $C$ is the identity mapping on $K_{A,0}$ if and only if $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is isomorphic to $A \xrightarrow{f} V$ as a family of 0-type.

**Proof.** Let $L \supset K_{A,0} \supset K_V$ be a Galois extension of $Q$. We already constructed the homomorphism:

$$\text{Gal}(L/K_V) \to \text{Ideal class group } (\widetilde{k})$$

What is yet to be determined is the kernel.
Let $\sigma \in \text{Gal } (L/K_V)$ such that $a(\sigma) = 1$, i.e. $L^\sigma = (q)L$, $q \in k^\times$.
But $L^{(\sigma)} = (q)L$ if and only if $A$ and $A^\sigma$ are isomorphic. So we have

$$1 \to \ker \to \text{Gal}(L/K_V) \to \text{Ideal class group } (\widetilde{k})$$

and $L^H = K_{A,0}$. 
Chapter 3.

In this chapter we will construct a Kuga fiber variety of quaternion type, \( A \xrightarrow{f} V \), with bottom field of \( A \) strictly bigger then the bottom field of \( V \).

In particular we consider a totally real number field \( k \) of class number 2. Any rigid Kuga fiber variety, \( A \xrightarrow{f} V \), constructed from a quaternion algebra over \( k \) has \( [K_A:K_V] \leq 2 \). We will show that with a suitable choice of the data we have \( [K_A:K_V] = 2 \). To do so we need to state several results of Complex Multiplication Theory. In fact a rigid Kuga fiber variety contains fibers of CM type (Section 1.3.) and the action of an automorphism of the complex numbers on such varieties is determined by the main theorem of Complex Multiplication.

The universal reference for this chapter is Shimura-Taniyama "Complex multiplication of Abelian varieties and its application to number theory".

3.1 The Endomorphism Ring of an Abelian variety

Let \( A \) be an Abelian variety defined over the complex numbers.

We denote \( \text{Hom}(A) \) the set of all homomorphism of \( A \) into \( A \). \( \text{Hom}(A) \) is a finitely generated free \( \mathbb{Z} \)-module, we put \( \text{Hom}_Q(A) = \text{Hom}(A) \otimes \mathbb{Q} \).
Theorem 3.1.1. Let $A$ be an Abelian variety of dimension $n$ and $F$ be a subfield of $\text{Hom}_Q(A)$. Then we have:

(i) $[F:Q]$ divides $2n$

Suppose $[F:Q] = 2n$

(ii) $F$ is totally imaginary

(iii) The commutant of $F$ in $\text{Hom}_Q(A)$ is equal to $F$.

(iv) $A$ is isogeneous to a product $B \times \cdots \times B$ where $B$ is a simple Abelian variety.

(v) For every $\alpha \in F$, we have:

$$\nu(\alpha) = N_{F/Q}(\alpha), \quad \text{tr}(\alpha) = \text{Tr}_{F/Q}(\alpha)$$

Theorem 3.1.2. Let $B$ a simple Abelian variety defined over $\mathbb{C}$.

Let $K$ be the center of $\text{Hom}_Q(B)$

(i) $K = \text{Hom}_Q(B)$

(ii) $[K:Q] = 2 \dim(B)$

(iii) $K$ is totally imaginary

Let $R$ be an algebra over $Q$, with identity element $1$.

Definition 3.1.3. An Abelian variety of type (R) is a pair $(A, i)$ formed by an Abelian variety $A$ and an isomorphism $i$ of $R$ into $\text{Hom}_Q(A)$
3.2 CM type

Let $F$ be an algebraic number field of degree $2n$. Let $\phi_1, \cdots, \phi_n$ be $n$ distinct isomorphisms of $F$ into $\mathbb{C}$, we say that $(F, \{\phi_i\})$ is of CM type if: $F$ contains to subfields $K$ and $K_0$ satisfying the following conditions:

(CM1) $K_0$ is totally real and $K$ is a totally imaginary quadratic extension of $K_0$.

(CM2) There are no two isomorphisms among the $\phi_i$ which are complex conjugate to each other on $K$.

**Theorem 3.2.1.** In order that $(F, \{\phi_i\})$ be a CM type, it is necessary and sufficient that there exists an Abelian variety of dimension $n$ of type $(F, \{\phi_i\})$.

**Theorem 3.2.2.** Let $A$ be an Abelian variety of type $(F, \{\phi_i\})$, $A = B \times \cdots \times B$. Let $K$ be the subfield of $F$ defined above, then:

$\text{Hom}_Q(B) = K$.

3.3 The Reflex of a CM type.

**Proposition 3.3.1.** Let $F$ be an extension of $Q$ of degree $2n$ and $\{\phi_1, \cdots, \phi_n\}$ be a set of $n$ distinct isomorphisms of $F$ into $\mathbb{C}$.

$(F, \{\phi_i\})$ of CM type.

Let $E$ be a Galois extension of $Q$ containing $F$ and $\Omega$ be the Galois group of $E$ over $Q$. Denote by $\rho$ the element of $\Omega$ such that $E^\rho$ is
the complex conjugate of $\xi$ for every $\xi \in E$, and by $S$ the set of all elements of $\Omega$ inducing some $\phi_i$ on $F$.

Put: $S^* = \{ \sigma | \sigma \in S \}$, $\mathcal{H}^* = \{ \gamma | \gamma \in \Omega, \gamma S^* = S^* \}$.

Let $K^*$ be the subfield of $E$ corresponding to $\mathcal{H}^*$ and $\{ \psi_i \}$ the set of all the isomorphisms of $K^*$ into $C$ obtained from the elements of $S^*$. Then $(K^*, \{ \psi_i \})$ is a CM type and we have

$$K^* = \mathbb{Q}(\prod_{\xi \in S^*} \xi^{\psi_i} | \xi \in E)$$

$(K^*, \{ \psi_i \})$ is determined only by $(F, \{ \phi_i \})$ and independent on the choice of $E$.

**Corollary 3.3.2.** The Abelian varieties of $(K^*, \{ \psi_i \})$ type are simple.

We call the CM type $(K^*, \{ \psi_i \})$ of the above proposition the reflex of $(F, \{ \phi_i \})$.

For every CM type $(F, \{ \phi_i \})$ we can find two subfields $K$ and $K_0$ of $F$ satisfying the conditions CM1 and CM2. Let $\{ \phi_i \}$ be the set of distinct isomorphisms of $K$ into $C$ induced by the $\phi_i$. Then it is easy to see that $(F, \{ \psi_i \})$ and $(K, \{ \phi_i \})$ have the same reflex.

**Proposition 3.3.3.** Let $(F, \{ \phi_i \})$ be a CM type and $(K^*, \{ \psi_i \})$ the reflex of $(F, \{ \phi_i \})$. Let $a$ be an element of $K^*$; put $\beta = \prod a^{\phi_i}$.
Then $\beta$ is an element of $F$ and we have $\beta \beta^\rho = N_{K^\times/Q}(\alpha)$. Set $a$ be an ideal of $K^\times$.
Put $b = \prod a_i^{\psi_i}$, then $b$ is an ideal of $F$; and we have

$$b \beta^\rho = N_{K^\times/Q}(a).$$

3.4 The Main Theorem of Complex Multiplication.

Let $(K, \{\phi_i\})$ be a field of CM type, $[K:Q] = 2n$, and $(K^\times, \{\psi_i\})$ be the reflex of $(K, \{\phi_i\})$.
Let $(A, i)$ be an Abelian variety of type $(K, \{\phi_i\})$. Let $k_0$ be the field of moduli of $(A, i)$ where $i$ is a polarization of $A$.
Put $k_0^\times = K^\times \cdot k_0$, $k_0^\times$ is an abelian extension of $K^\times$.

We assume $\text{Hom}(A) = 0_K$, $0_K$ ring of integers in $K$. Then there exists $a$ an ideal of $K$ such that $A$ is isomorphic to the complex torus $C^n/D(a)$. We say that $(A, i)$ is of type $(K, \{\phi_i\}, a)$.

Let $\sigma$ be an automorphism of $C$ such that $\sigma \equiv \text{id}$ on $K^\times$. Then $(A^\sigma, i^\sigma)$ is again of $(K, \{\phi_i\})$ type and isomorphic to the complex torus $C^n/D(b)$ for some $b$ ideal of $K$.

**Theorem 3.4.1.** Let $(A, i)$ be an Abelian variety of type $(K, \{\phi_i\}, a)$ as above. Let $\sigma \in \text{Aut}(C/K^\times)$ and let $a$ an ideal of $K^\times$ such that $\sigma = (s, K^\times)$ on $K^{ab}$.

Then $(A^\sigma, i^\sigma)$ is of type $(K, \{\phi_i\}, N_{\psi}(s^{-1}) \ a)$, where $N_{\psi}(s^{-1}) = \prod_{i} (s^{-1})_{\psi_i}$. 


Theorem 3.4.2. Let \((A, i)\) be an Abelian variety of type \((K, \{\phi_i\})\) as above. Let \(H_0\) be the group of all ideals \(a\) of \(K^\times\) such that there exists an element \(\mu \in K\) for which we have
\[
\prod_i a_i \psi_i = (\mu), \quad N(a) = \mu \overline{\mu}.
\]
Let \(k_0\) be the field of moduli of \((A, i)\). Then \(H_0\) is an ideal group of \(K^\times\) and \(k_0^\times\) is the unramified class field over \(K^\times\) corresponding to the ideal group \(H_0\).

3.5 The Example

Let \(A \xrightarrow{f} V\) be a Kuga fiber variety defined by \(B\) a division quaternion algebra over \(k\) a totally real Galois number field.

If we assume the defining representation rigid and irreducible over \(\mathbb{Q}\), we proved the following:

(i) The bottom field of \(A, K_A\), is an abelian Galois extension of the bottom field of \(V, K_V\).

(ii) If \(\sigma \in \text{Gal}(K_A/K_V)\), then \(A^\sigma \xrightarrow{f^\sigma} V^\sigma\) is defined by the same data as \(A \xrightarrow{f} V\) except the lattice \(L(\sigma)\). In fact \(L(\sigma) = a(\sigma)L\), \(a(\sigma)\) an ideal in \(k\) and we have:

\[
1 \to \text{Gal}(K_A/K_V) \to \text{Ideal class group (k)}
\]

If \(k\) has class number 1, then \(K_A = K_V\). If \(k\) has class number bigger than 1 then be can apply the theory of complex multiplication to determine \(a(\sigma)\).
Let \( A \xrightarrow{f} V \) be defined by the data:
\[(G = \text{Res}_{\mathbb{Q}/\mathbb{Q}} \text{SL}_1(B), K, X, \Gamma, F, L, \beta, \rho, \tau)\]

We make the following assumptions:
1. \( k \) is Galois over \( \mathbb{Q} \), \( \Omega = \text{Gal}(k/\mathbb{Q}) \).
2. \( \rho = \rho_p \); \( P = E_g M \), \( P \) rigid, \( g M \neq \gamma M \) if \( g \neq \gamma \).
3. \( B^M = M \Omega(k) \), \( F = \text{Res}_{\mathbb{Q}/\mathbb{Q}}^F \).
4. \( A \xrightarrow{f} V \) is of type \( O_k \), \( O_K \) ring of integers in \( k \).
5. Let \( A_{\lambda} \) be the fiber with complex multiplication constructed in Section 1.4 of type \( (K, \{\phi_k\}) \). Then we assume \( \text{Hom}(A_{\lambda}) = O_K \), \( O_K \) the ring of integers of \( K \).

\( A_{\lambda} \) as a complex torus is isomorphic to \( (F/L, J_{\lambda} = \tau(\lambda)) \),
and if \( \sigma \in \text{Aut}(G_K \mathbb{V}) \), then \( (A_{\lambda})^\sigma = A_{\lambda}^{\sigma} \) is isomorphic to
\((F/L(\sigma), J_{\lambda}^{\sigma} = \tau(\lambda^\sigma))\).

From assumption 4 it follows that \( A_{\lambda} \) is of type \( (K, \{\phi_k\}, a) \) where \( a \) is an \( O_K \) ideal. Let \( (K^*, \{\psi_i\}) \) be the reflex of \( K \) and \( \sigma \) an element of \( \text{Aut}(G/K^*) \). If \( \sigma = (\mathbb{S}, K^*) \) on \( K^{ab} \) then \( (A_{\lambda})^\sigma \) is of type \( (K, \{\phi_k\}, N_{\mathbb{S}}(\mathbb{S}^{-1}) \mathbb{S} \) A formula. Moreover if we assume \( \sigma \in \text{Aut}(G/K^* \mathbb{V}) \),
we have:
\[N_{\mathbb{S}}(\mathbb{S}^{-1}) = \{x \in K \mid xL \subset L(\sigma)\}.\]

On the other hand:
\[[L:L(\sigma)] = \{x \in \text{End}_K^F M \mid xL \subset L(\sigma)\}.\]

So we can conclude:
\[[L:L(\sigma)] \cap k = N_{\mathbb{S}}(\mathbb{S}^{-1}) \cap k = a(\sigma)\]
and we proved the following lemma:

**Lemma 3.5.1** If $\sigma$ is an element of $\text{Aut}(C/K^*K_V)$ and $\sigma = (g, K^*)$ on $K_{ab}$, then:

$$[N_{\tilde{g}}(g^{-1})] \cap k = a(\sigma).$$

We want to use this result to show that $a(\sigma)$ may not be principal.

As a first step we need to analyze the CM type $(K, \{\phi_i\})$. $K$ was constructed in Section 1.4 as:

$$K = L^{\alpha_1} \cdots L^{\alpha_r} = L^{\alpha_1} \ast \cdots \ast L^{\alpha_r}.$$  

$L = k(\sqrt{-\zeta})$ is a totally imaginary extension of $k$, contained in $B$ and $\zeta$ is totally positive.

$K$ is of CM type together with a set of embeddings into $C$: $\phi_1, \ldots, \phi_n$. The analytic representation of $K$ is equivalent to the direct sum of the $\phi_i$. So to determine $\phi_1, \ldots, \phi_n$ we need to discuss the embeddings of $K$ into $C$ and the analytic representation.

Let $\Omega = \text{Gal}(k/Q) = \{\alpha_1, \ldots, \alpha_g\}$, then $K^\Omega = L^{\alpha_1} \cdots L^{\alpha_g}$ is the smallest extension of $Q$ containing $K$. We can assume, by Proposition 1.4.2, that $[K^\Omega : Q] = 2^g \cdot g$.

**Lemma 3.5.2** The Galois group of $K^\Omega$ over $Q$ is isomorphic to $\Omega \times Z^\Omega_2$, where the product is defined in the following way:
\[ \psi, \phi \in \Omega \times \mathbb{Z}_2^\Omega; \ \phi = (\alpha, (\sigma_i)), \ \psi = (\gamma, (\delta_i)) \]

\[ \phi \cdot \psi = (\alpha \gamma, (\sigma_i \cdot \delta_i)) \]

**Proof.** Let \( \phi \) be an element of \( \text{Gal}(K^\Omega/Q) \)

Then \( \phi \) restricted to \( k \) coincides with an element of \( \Omega \) that we call \( \alpha \). For any \( i \in \Omega \), \( \phi : \mathbb{L}^i \rightarrow \mathbb{L}^i \alpha \) is an extension of \( \alpha \) to \( \mathbb{L}^i \). So either \( \phi(\sqrt{-\zeta^i}) = \sqrt{-\zeta^i} \alpha \) or \( \phi(\sqrt{-\zeta^i}) = -\sqrt{-\zeta^i} \alpha \).

Put \( \iota : \text{Gal}(K^\Omega/Q) \rightarrow \Omega \times \mathbb{Z}_2^\Omega \), \( \iota(\phi) = (\alpha, (\sigma_i)) \), where \( \sigma_i = 0 \)

if \( \phi(\sqrt{-\zeta^i}) = \sqrt{-\zeta^i} \alpha \) and \( \sigma_i = 1 \) if \( \phi(\sqrt{-\zeta^i}) = -\sqrt{-\zeta^i} \alpha \).

\( \iota \) is a bijection and it is easy to see that it is also an homomorphism of groups. \( \blacksquare \)

**Corollary 3.5.2** The embeddings of \( K \) into \( \mathbb{C} \) are the restrictions of the elements of \( \Omega \times \mathbb{Z}_2^\Omega \) to \( K \).

**Lemma 3.5.3** In the above notations. \( A_\lambda \) is of type \( (K, \{\phi, S_0\}) \).

Where \( \phi, S_0 : K \rightarrow \mathbb{C} \) is defined in the following way:

\[ \phi_{\alpha, S_0} = (\alpha, (\sigma_i)), \ \sigma_M \cap \alpha^{-1}S_0 = 0 \]

**Proof.** \( A_\lambda \) as complex torus is isomorphic to \( (F/L, J_\lambda = \tau(\lambda)) \).

\( J_\lambda \) is defined in the following way. Let \( g \) be an element of \( G_{\mathbb{R}} \)

such that \( \lambda = \rho(g)(\sqrt{-1}, ..., \sqrt{-1}) \) and \( J \) the element of \( G_{\mathbb{R}} \).

\[ J = (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})|S_0| \times |S_1| \]
Then $J_\lambda = \rho(g)\rho(J)\rho(g)^{-1}$. In other words:

$F = \ast F_{\lambda_1}$ and $J_\lambda = \ast J_{\lambda,1}$, where

$J_{\lambda,1} = \rho_{M_1}(g) I \times \ldots \times (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \times \ldots \times I \rho_{M_1}(g)^{-1}$ and $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ occurs at the place $gM \cap S_\alpha$.

The rational representation of $K$ is just $\rho : K \to \text{End}_Q(F)$ and

$\rho = \ast \rho_{M_1}$. Now $\rho_{M_1} : K \to \text{End}_C(F_{M_1})$ is equivalent to $\ast \phi_i$ where the $\phi_i$ are all the extension of $i \in \Omega$ to $K$. So by the construction of $J_{\lambda,M_1}$ we obtain that the analytic representation of $K$ is equivalent to $\ast \phi_i, S_\alpha$.

A field of CM type

Let $k$ be $Q(\sqrt{10})$ and $\Omega = \text{Gal} (k/Q) = \{\text{id}, \gamma\}$.

$k$ has class number 2 and its ideal classes are $(1), (2,\sqrt{10})$.

Let $\zeta$ be in $k$, $\zeta$ totally positive, for instance we will consider

$\zeta = \frac{4-\sqrt{10}}{2}$.

Put $K = k(\sqrt{-\zeta})$ and $K^\gamma = k(\sqrt{-\zeta^\gamma})$. $K$ and $K^\gamma$ are linearly disjoint.

The smallest Galois extension of $Q$ containing $K$ is

$E = K \cdot K^\gamma = k(\sqrt{-\zeta}, \sqrt{-\zeta^\gamma})$, $[E:Q] = 8$. As we have seen in Lemma

3.5.2. $\text{Gal}(L/Q) = \Omega \times Z_2^\Omega$.

$L$ is of CM type with respect to $\{\phi_1, \phi_2\}$ where $\phi_1 = \text{id}$ and $\phi_2 : k \to k$, $\phi_2 = \gamma$ on $k$ and $\phi_2(\sqrt{-\zeta}) = (\sqrt{-\zeta^\gamma})$. $L$ is also of CM type respect to $\{\phi_i\}$ where:

$\phi_1 = (\text{id}, 0, 0)$, $\phi_2 = (\text{id}, 0, 1)$, $\phi_3 = (\gamma, 0, 0)$, $\phi_4 = (\gamma, 0, 1)$. 
We want to compute the reflex of $E$. Note that $K$ and $k$ are the fields satisfying condition CM1 and CM2. Therefore the reflex of $E$ coincide with the reflex of $K$.

For $E$, in the notation of proposition 3.3.1, we have:

$$S = \{\phi_1\} \quad S^* = \{\sigma^{-1} \mid \sigma \in S\} = S$$
$$H^* = \{\gamma \mid \gamma \in \text{Gal}(E/Q) \quad \gamma S^* = S^*\} = \Omega \times \{0\}$$
$$K^* = \mathbb{E}^{H^*} = Q(\sqrt{\zeta} + \sqrt{\zeta}^\gamma) \quad [K^* : Q] = 4$$

$K^*$ is a CM type respect to $\{\psi_1, \psi_2\}$:

$$\psi_1(\sqrt{\zeta} + \sqrt{\zeta}^\gamma) = \sqrt{\zeta} + \sqrt{\zeta}^\gamma$$
$$\psi_2(\sqrt{\zeta} + \sqrt{\zeta}^\gamma) = \sqrt{\zeta} - \sqrt{\zeta}^\gamma$$

The diagram is as follows:

```
E = K K^*

K

/ \  /
/   \ /
/     /
K^*

\_\_\_
| k |
\_\_\_
|   |
\_\_\_

Q
```

Let $a$ be an ideal of $K^*$ then $N_{\psi}(a) = a^{\psi_1} \cdot a^{\psi_2}$ is an ideal of $K$.

In particular any ideal class of $k$ contains an ideal of the form $N_{\psi}(a) \cap k$ for some $a$ in $K^*$.

Since there are only two ideal classes in $k$: (1) and $(2, \sqrt{10})$ we need to verify that there exists $a$ such that $N_{\psi}(a) = (2, \sqrt{10})$.

Put $a = (2, \sqrt{\zeta} + \sqrt{\zeta}^\gamma)$, then:
\[ N_\Psi(a) = a_{\Psi_1} a_{\Psi_2} = (2, \sqrt{10}, 4\sqrt{\zeta}) = (2, \sqrt{10}). \]

\textbf{A Kuga fiber variety}

Let \( k \) be \( \mathbb{Q}(\sqrt{10}) \) and \( \Omega = \text{Gal}(k/\mathbb{Q}) = \{ \text{id}, \gamma \} \).

Let \( B \) be a quaternion algebra containing \( K = k(\sqrt{-\zeta}), \zeta = \frac{4-\sqrt{10}}{2} \).

For instance we consider. \( B = \begin{pmatrix} -\zeta & 1 \\ k & \end{pmatrix} \cong \begin{pmatrix} 1 & \zeta \\ k & \end{pmatrix} \)

so that we have: \( B \supset K, \ B \supset KY \) and

\[ B \otimes \mathbb{R} = M_2(\mathbb{R}) \times M_2(\mathbb{R}) \]

\[ \mathbb{Q} \]

In the notation of Section 1.2.: \( S_0 = S = \Omega \).

Put \( G = \text{Res}_{k/\mathbb{Q}} \text{SL}_1(B) \) and \( \rho = \rho_p, \ P = \{ \text{id} \} \times \{ \gamma \} \).

Note that \( \rho \) is the only rigid polymer representation for the chemistry \( (\Omega, S, S_0) \). On the other hand to have a representation defined over \( \mathbb{Q} \) we need to increase the multiplicity and consider \( \rho_{2P} = \rho_p \otimes P \).

We want to construct a family of Abelian varieties, \( A \rightarrow V \),

associated to \( \rho_{2P} \). We need many ingredients.

1. The space of parameters \( V \). Let \( O \) be a maximal order of \( B \).

Put \( \Gamma = \text{Res}_{k/\mathbb{Q}}^{O_1} \). Then \( \Gamma \) is an arithmetic subgroup of \( G \).

\( \Gamma \) defines a Hilbert modular variety: \( V = \Gamma \backslash \mathbb{H}^2 \).
2. The vector space. In our case $P = \{id\} + \{\gamma\}$ is the simple
polymer generated by $\{id\}$. So $F = \text{Res}_{k/Q}F_1$.
$F_1$ minimal left ideal in $B$, therefore $B$ itself.

3. The lattice. Let $\tilde{O}$ be a maximal order of $B \ast B$ containing $O$
and $O_E$ the ring of integers of $E$. Let $L$ be a lattice in $F_1$, $\tilde{O}$
variant.

So we can define: 

\[
A = \Gamma \times \text{L} \times F_R \\
\downarrow f \\
V = \Gamma \backslash \mathbb{H}^2
\]

$A \xrightarrow{f} V$ is a family of Abelian varieties, Sec. 1.2.

Since $\Gamma = O^1$, $O$ a maximal order, $V = \Gamma \backslash \mathbb{H}^2$ has a projective
model defined over $Q$, Shimura [Sh3]. Therefore the bottom field
of $V$ is $Q$ and $K_A$ is an Abelian extension of $Q$.

We want to show that $[K_A:K_V = Q] = 2$

Let $A_\lambda$ be the fiber of $A \xrightarrow{f} V$, of type $(E,\{\phi_i\})$ constructed in
Section 1.4. $A_\lambda$ is isomorphic to $F/L$ as real torus.

In our case $B^M = B$ is not a trivial algebra, but it is easy to
generalize Lemma 3.5.3 and show that $\{\phi_i\}$ is the set of embeddings
of $E$ we discussed previously.

Let $K^*$ be the reflex of $E$. Let $\mathfrak{a}$ be the ideal of $K^*$,
$\mathfrak{a} = (2, \sqrt{\zeta} + \sqrt{\zeta}^{-1})^{-1}$, and $\sigma \in \text{Aut}(C/K^*)$ such that $\sigma = (\mathfrak{a}, K^*)$ on
$\mathfrak{g}ab$. 
We note then that \( \sigma \) is the identity map on \( K_V \) so \( (A_{\lambda})^\sigma = A_{\lambda^\sigma} \) is isomorphic to \( F/L(\sigma) \).

By Lemma 3.5.1. \( L^{(\sigma)} = A(\sigma)L = (N_{\overline{F}/F}(a^{-1}) \cap k)L \) and as computed previously:

\[
N_{\overline{F}/F}(a^{-1}) \cap k = (2, \sqrt{10})
\]

So we can conclude:

1 \( \rightarrow \) \( \text{Gal}(K_A/K_V) \rightarrow \text{Ideal class group}(k) \rightarrow 1 \)

and \([K_A:K_V] = 2\).
References


