ZETA-FUNCTION OF SUBELLIPTIC DIFFERENTIAL OPERATORS

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On a compact contact manifold of dimension $2n+1$
the complex powers of non-negative self-adjoint second
order differential operators doubly characteristic on the
contact line bundle are considered.

Via the symbolic calculus on the group $\mathbb{R} \times \mathbb{H}^n$ ($\mathbb{H}^n$ is
a Heisenberg group), the asymptotic expansion for the trace
of the heat Kernel has been obtained. This allows us to get
the analytic continuation for the zeta-function to the
whole complex plane excluding the finite number of points
$Z_j = -(n+1)+j, j=0,\ldots,n$, at which the zeta-function has simple
poles.
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INTRODUCTION

Suppose $\mathcal{M}$ is a compact manifold of dimension $2n+1$ with a contact structure. It is defined by a line bundle $\Lambda \subset T^*\mathcal{M}$ of codimension $2n$ which is symplectic in $T^*\mathcal{M}\setminus \mathcal{O}$ or the symplectic form on $T^*\mathcal{M}\setminus \mathcal{O}$ is nondegenerate acting on tangent vectors to $\Lambda$. We study the complex powers of self-adjoint non-negative differential operators of second order on $\mathcal{M}$ doubly characteristic on $\Lambda$.

Another characterization of contact structure is as follows: if $\alpha$ is a local section of $\Lambda$, then $\alpha \wedge \alpha \wedge \ldots \wedge d\alpha \neq 0$ where there are $n$ factors of $d\alpha$. A choice of such $\alpha$ provides $\mathcal{M}$ with a local volume form $\alpha \wedge d\alpha^n$. Two $1$-forms associated with the same contact structure differ by a smooth nonvanishing multiple. It follows from the Darboux's theorem that any two contact manifolds of the same dimension are locally diffeomorphic via a map preserving the contact structure.

A $\text{CR}$-manifold with non-degenerate Levi form is an example of a contact manifold. A $\text{CR}$-structure is given by a complex $n$-dimensional subbundle $T_{i,0} \subset \text{CTM}$ satisfying $T_{i,0} \cap \overline{T_{i,0}} = \{0\}$ and assumed to be integrable (i.e., the Lie bracket $[T_{i,0}, T_{i,0}] \subset T_{i,0}$).
The Levi form is given by

$$\langle v, u \rangle_{\nu} = -id\nu(v \wedge \bar{u}), \quad v, u \in T_{i,0}.$$ 

The nondegeneracy of the Levi form is equivalent to the condition $\nu \wedge d\nu \neq 0$. Dual to the Levi form is the norm $|\omega|_{\nu}$ on real 1-forms $\omega$ given by

$$|\omega|_{\nu}^{2} = \langle \omega, \omega \rangle_{\nu} = \sum_{j=1}^{n} |\omega(Z_{j})|^{2}$$

where $(Z_{1},...,Z_{n})$ is an orthonormal basis for $T_{i,o}$ with respect to the Levi form. Since $|\lambda|_{\nu} = 0$ the norm $|\omega|_{\nu}$ is degenerate. It follows that the sublaplacian operator $\Delta_b$ defined on functions by

$$\int_{M} (\Delta_{b} u) v \nu \wedge d\nu^{n} = \int_{M} \langle du, dv \rangle_{\nu} \nu \wedge d\nu^{n},$$

$\nu \in C_{o}^{\infty}(M),$ is subelliptic.

Folland and Stein [1] introduced function spaces $S_{r}^{k}$ on $M$ analogous to the Sobolev spaces. For example, $f \in S_{1}^{2}(M)$ if

$$\|f\|_{2}^{2} = \int_{M} (|df|_{\nu}^{2} + f^{2}) \nu \wedge d\nu^{n} < \infty.$$
In the survey [2] it was noted that the embedding theorems also follow from [1]: \( S^2_1(M) < L^r(M) \) if \( r > \frac{1}{2} - \frac{1}{2n+2} \) and the inclusion is compact if \( r > \frac{1}{2} - \frac{1}{2n+2} \).

The Heisenberg group \( \mathbb{H}^n \) can be used as a standard model for a contact manifold as an Euclidean space for a Riemannian manifold (see [1] and [3]). If a point in \( \mathbb{H}^n \) is denoted by \( (t, q, p) \) the contact structure on \( \mathbb{H}^n \) is the line bundle invariant by right translations, whose fiber over the identity on \( \mathbb{H}^n \) is spanned by \( dt \).

In Taylor's book [3] a symbolic calculus has been developed to study the classes of pseudodifferential subelliptic operators. The symbols of convolution operators on \( \mathbb{H}^n \) are their images under the basic representation of the Heisenberg group which are operators in the Weyl functional calculus. Methods of [3] are extensively used in this work.

In Section 1.1 of Chapter I a symbolic calculus is introduced for the convolution operators on the group \( \mathbb{R} \times \mathbb{H}^n \). Based on that a parametrix for the heat equation on \( \mathbb{R} \times \mathbb{H}^n \) is obtained in Section 1.2. In Section 1.3 complex powers of the right invariant differential operators on the group \( \mathbb{H}^n \) are studied. Note that the complex powers of right invariant operators on Lie groups were considered by Polland [4].

In Chapter II subelliptic differential operators on compact contact manifolds are investigated. In Section 2.1 a class of operators with variable coefficients is obtained from the class
of convolution operators on the group $\mathbb{R} \times \mathbb{H}^n$ using methods of [3]. This allows further in Section 2.2 to get an asymptotic expansion for the theta-function in which the coefficients of the non-integer powers of the time parameter cancel out. Such expansion was obtained by Beals, Greener, and Stanton [5] using a different approach. Based on results of Section 2.2 the behavior of the zeta-function is studied in Section 2.3.

In the case of subelliptic differential operators of second order, the poles of the zeta-function occur only at integer points. This implies that the zeta-function has a finite number of poles on the complex $\mathbb{Z}$-plane, and there are no poles for $\text{Re} z > 0$, which would not be the case if the order of operators was other than two. The analogous behavior of the zeta-function of the special class of elliptic self-adjoint positive definite differential operators of second order on the compact Riemannian manifold of an even dimension and without boundary follows from [6]. The zeta-function of the harmonic oscillator Hamiltonian is considered in the Appendix.
CHAPTER I. RIGHT INVARIANT OPERATORS

Section 1.1. Convolution Operators on the Group \( \mathbb{R} \times \mathbb{H}^n \).

We will consider convolution operators on the group \( \mathbb{G} = \mathbb{R} \times \mathbb{H}^n \), where \( \mathbb{H}^n \) is the Heisenberg group.

As a \( C^\infty \) manifold, \( \mathbb{G} \) is \( \mathbb{R}^{2n+2} \). A point of \( \mathbb{R}^{2n+2} \) and its dual will be denoted by

\[
(t, z) = (t, s, q, p), \quad t \in \mathbb{R}, \quad s \in \mathbb{R}, \quad q \in \mathbb{R}^n, \quad p \in \mathbb{R}^n
\]

and

\[
(\sigma, \xi) = (\sigma, \tau, y, \eta), \quad \sigma \in \mathbb{R}, \quad \tau \in \mathbb{R}, \quad y \in \mathbb{R}^n, \quad \eta \in \mathbb{R}^n
\]

respectively. The group law is

\[
(t_1, s_1, q_1, p_1) \cdot (t_2, s_2, q_2, p_2) = (t_1 + t_2, s_1 + s_2 + \frac{1}{2} p_2 q_1 - \frac{1}{2} p_1 q_2, q_1 + q_2, p_1 + p_2)
\]

The dilation is defined for \( r \in \mathbb{R} \setminus \{0\} \) by

\[
r \cdot (t, s, q, p) = (r^2 t, \ r^2 s, \ r q, \ r p)
\]

\[
r \cdot (\sigma, \tau, y, \eta) = (r^2 \sigma, \ r^2 \tau, \ r y, \ r \eta)
\]  \hspace{1cm} (1)
Let \( \| \| \) be a Euclidean norm on \( \mathbb{R}^{2n} \), a "homogeneous norm" is defined on \( G \) by

\[
\|(t, Z)\| = \left[ |t| + |s| + \| (q, p) \|^2 \right]^{1/2}
\]

For \( \lambda \in (0, \infty) \) irreducible unitary representations of \( H^n \) on \( L^2 \mathbb{R}^n \) are

\[
\pi_{\pm \lambda} (s, q, p) = e^{i(\pm \lambda s I_+ \lambda^{1/2} q \cdot x + \lambda^{1/2} p \cdot D)}
\]

The infinite dimensional irreducible unitary representations of the group \( G \) are given by

\[
\pi_{\delta, \pm \lambda} (t, Z) = e^{i \delta t} \pi_{\pm \lambda} (Z), \quad \delta \in \mathbb{R}, \quad \lambda \in (0, \infty).
\]

For a representation \( \pi_{\delta, \pm \lambda} \) to a function \( u \) on \( G \) we associate

\[
\pi_{\delta, \pm \lambda} (u) = \int_G u(g) \pi_{\delta, \pm \lambda} (g) dg.
\]

For a compactly supported function (or distribution) \( \kappa \) on \( G \) let \( \hat{\kappa}(\epsilon, \tau, y, \eta) \) denote the Euclidean space Fourier transform.
We have

\[ \mathcal{K}_{\delta', \pm \lambda} (k) = \hat{\mathcal{K}}(\delta', \pm \lambda, \pm \chi^\frac{\alpha}{2} x, \chi^\frac{\alpha}{2} D) = \delta_k'(\delta', \pm \lambda)(X, D), \]

where

\[ \delta_k'(\delta', \pm \lambda)(x, \xi) = \hat{\mathcal{K}}(\delta', \pm \lambda, \pm \chi^\frac{\alpha}{2} x, \chi^\frac{\alpha}{2} \xi) \]  \hspace{1cm} (2)

and the operator \( \alpha(X, D) \) is defined by the Weyl functional calculus:

\[ \alpha(X, D) = \int \tilde{\alpha}(q, \rho) e^{i(q \cdot X + \rho \cdot D)} \, dq \, d\rho \]

(\( \tilde{\alpha}(q, \rho) \) is the inverse Fourier transform of \( \alpha \)).

Formula (2) implies that

\[ \hat{\mathcal{K}}(\delta', \pm \tau, y, \eta) = \delta_k'(\delta', \pm \tau)(\pm \tau^{-\frac{\alpha}{2}} y, \tau^{-\frac{\alpha}{2}} \eta), \tau > 0. \]

Definition 1. The class \( \Psi^m_0(g) \) \( \Psi^m_+ (\mathfrak{g}) \) consists of functions \( \hat{\mathcal{K}}(\delta', \tau, y, \eta) \), smooth except at \( 0 \), and homogeneous of degree \( m \) with respect to the dilation (1), i.e.,

\[ \hat{\mathcal{K}}(r \cdot (\delta', \xi)) = r^m \hat{\mathcal{K}}(\delta', \xi) \]  \hspace{1cm} (3)

for \( r \in \mathbb{R} \setminus \{0\} (r > 0) \).
If \( K \in \psi^m_o(\Gamma) \psi^m_+(\Gamma) \), we say that the convolution operator \( K(Ku = Ku) \) belongs to the class \( \text{OP}\psi^m_o(\Gamma) \psi^m_+(\Gamma) \).

Let \( S^m_{\psi, \#} \) be the Frechet space with the seminorm:

\[
[p]_{\psi, m, \rho}^\alpha = \sup_x \left\{ \left[ 1 + |x|^2 \right]^{\alpha} \right\}^{-m + \rho \delta} \left| \partial_x^\alpha \rho(x) \right|.
\]

Neglecting the singularity at the origin, the elements of \( \psi^m_o(\Gamma) \) belong to \( S^m_{\psi, \#} \) if \( m > 0 \) and to \( S^{m/2}_{\psi, \#} \) if \( m < 0 \).

Note that (3) is equivalent to

\[
\delta^\alpha_K (r\delta, \pm r\zeta)(x, \xi) = r^{-m} \delta^\alpha_K (\delta, \pm \zeta)(x, \xi).
\]

In order to characterize the class \( \psi^m_o(\Gamma) \), we will consider an auxiliary classes of functions on \( \mathbb{R}^{2n+1} \).

Let \( \Sigma \) be a union of rays through the origin in the complex plane.

**Definition 2.** We say \( \alpha(\delta, x, \xi) \in S^m_{\psi, \Sigma} \) if \( \text{real}(x, \xi) \in \mathbb{R}^{2n} \), \( \delta \in \Sigma \), if \( \alpha(\delta, x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^{2n}) \) for each fixed \( \delta \) and for each multi-index \( \alpha \) there is a constant \( C_{m, \alpha} \) such that

\[
\left| \partial_{x, \xi}^\alpha a(\delta, x, \xi) \right| \leq C_{m, \alpha} \left( 1 + |x| + |\xi| + |\delta| \right)^{m - |\alpha|}. \tag{4}
\]
As usual $S_{m,\Sigma} = \cap S_{m,\Sigma}^{m}$. For $\alpha(\xi, x, \xi) \in S_{m,\Sigma}^{m}$, the family of operators $\alpha(\xi, x, \mathcal{D})$ is defined by the Weyl functional calculus:

$$\alpha(\xi, x, \mathcal{D}) u(x) = (2\pi)^{-n} \iiint e^{i(x-y) \xi} \alpha(\xi, \frac{1}{2}(x+y), \xi) u(y) dy dy' \xi.$$

If $\alpha(\xi, x, \xi) \in S_{m,\Sigma}^{m}$, we say that the operator $\alpha(\xi, x, \mathcal{D})$ belongs to the class $\mathcal{OPS}_{m,\Sigma}^{m}$.

The classes $\mathcal{OPS}_{m,0}$ were considered by A. Voros [7], Gromman, Loupias and Stein [8], and, in a much more general case by Hormander [9]. The symbolic calculus can be extended from classes $\mathcal{OPS}_{m,0}$ to $\mathcal{OPS}_{m,\Sigma}^{m}$. For example, the multiplication law is written as follows. If $\alpha(\xi, x, \mathcal{D}) \in \mathcal{OPS}_{m,\Sigma}^{m}$, $b(\xi, x, \mathcal{D}) \in \mathcal{OPS}_{m,\Sigma}^{m}$, then

$$c(\xi, x, \mathcal{D}) = \alpha(\xi, x, \mathcal{D}) b(\xi, x, \mathcal{D}) \in \mathcal{OPS}_{m,\Sigma}^{m+m}$$

and

$$c(\xi, x, \xi) = \sum_{j \geq 0} \left( \frac{i}{j} \right) \{ a, b \}_{j} (\xi, x, \xi),$$

where

$$\{ a, b \}_{0} (\xi, x, \xi) = a(\xi, x, \xi) b(\xi, x, \xi),$$

$$j \geq 1, \{ a, b \}_{j} (\xi, x, \xi) =$$

$$= \left( \frac{i}{2} \right)^{j} \sum_{\kappa=1}^{n} \left( \frac{\partial^{2}}{\partial y_{\kappa} \partial y_{\kappa}} - \frac{\partial^{2}}{\partial x_{\kappa} \partial x_{\kappa}} \right)^{j} a(\xi, x, \xi) b(\xi, y, \xi)|_{y = x}. \xi \mathcal{D}$$
The meaning of (6) is that the difference between \( c(\delta, x, \xi) \)
and the sum of the right side of (6) over \( 0 \leq j \leq N \) belongs to \( S^{m+\mu-2N}_{1, \Sigma} \).

Following Voros [7], we introduce a class of comparison operators: powers of the harmonic oscillator. Let \( h(x, \xi) = |x|^2 + |\xi|^2 \), \( H = h(x, D) \); \( H^K \) be the operator \( (I + H)^K \in OPS_{i, o}^{2K} \) for each integer \( K \). For each integer \( K \), let \( W^K \) be the Hilbert space obtained by completion of the domain of \( H^K \) in \( L^2(\mathbb{R}^n) \) for the inner product
\[
(u, v)_K = (H^K u, H^K v).
\]

We obtain the sequence of the spaces
\[ \ldots < W^K < \ldots < W_i < W_0 < W_{-i} < \ldots < W_{-K} < \ldots \]
for \( K > 0 \), \( W^K \) and \( W_{-K} \) are dual of each other for the inner product of \( W_0 \). Also, \( S(\mathbb{R}^n) = \bigwedge W^K \) and its topology is given by the directed family of seminorms \( \| \cdot \|_K \), and \( S'(\mathbb{R}^n) = \bigvee W^K \) [10].

It was shown by Voros [7] that if \( a(x, D) \in OPS_{i, o}^m \), then for any integers \( K, L \) with \( L > m/2 \), \( a(x, D) \) is a continuous operator \( W^K \rightarrow W^{K-L} \). Let \( \| a(\delta, x, D) \|_{K, K-L} \) be the norm of the operator
\[
a(\delta, x, D) : W^K \rightarrow W^{K-L}.
\]

To determine how \( \| a(\delta, x, D) \|_{K, K-L} \) depends on \( \delta \) for \( |\delta| \) suf-
ficiently large, we use the Calderon-Vaillancourt theorem, which states the following:

**Calderon-Vaillancourt theorem.**
If \( a(x, \xi) \) satisfies the estimate \( |D_{x, \xi}^\infty a(x, \xi)| \leq A \)
for \( |\xi| \leq K(n) \), then
\[
\|a(x, D)\| \leq C(n)A .
\]

The Calderon-Vaillancourt theorem and Definition 2 imply that if \( m \geq 0 \), then any operator \( a(\sigma, X, D) \in OPS_{1, \sigma}^{-m} \) is bounded as an operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \), and
\[
\|a(\sigma, X, D)\| \leq C(1 + |\sigma|^{1/2})^{-m} .
\]

Estimate (8) yields the more general

**Proposition 1.** An operator \( a(\sigma, X, D) \in OPS_{1, \Sigma}^m \) is bounded as an operator from \( \mathcal{W}_k \) to \( \mathcal{W}_{k-l} \) for \( k, l \) integers, \( l \geq m/2 \), and
\[
\|a(\sigma, X, D)\|_{k, k-l} \leq \begin{cases} 
C_{k, \ell}(1 + |\sigma|^{1/2})^{m/2-l}, & \ell \leq 0, \\
C_{k, \ell}(1 + |\sigma|^{1/2})^{m/2}, & \ell > 0,
\end{cases}
\]
\(|\sigma| - \) is sufficiently large.

**Proof.**
Denote as \( H_{\ell}(\sigma) \) the operator-function \( (I + H + |\sigma|^{1/2})^\ell, \sigma \in \Sigma \).
It suffices to show that $a(\sigma, X, D)$ is bounded on the domain of $H_\kappa$ in $W_\sigma$. Denote as $b(\sigma, X, D)$ the operator $H_{\kappa - \epsilon} a(\sigma, X, D) H_{-\kappa}$. For any vector $u$ in the domain of $H_\kappa$:

$$a(\sigma, X, D) u = H_{\kappa - \epsilon} b(\sigma, X, D) H_\kappa u.$$ 

The operator $H_\kappa$ is an isometry of $W_\kappa$ into $W_\sigma$, operator $H_{\kappa - \epsilon}$ is an isometry of $W_\sigma$ into $W_{\kappa - \epsilon}$. The operator $b(\sigma, X, D)$ belongs to the class $O_{PS}^{m - \epsilon}$, hence it is a bounded operator on $W_\sigma$.

In order to proof (9), we check it at first for the operator $H_{m_\sigma}(\sigma)$. We have

$$\| H_{m_\sigma}(\sigma) \|_{k, \kappa - \epsilon} = \| H_{\kappa - \epsilon} H_{m_\sigma}(\sigma) H_{-\kappa} \|,$$

and

$$H_{\kappa - \epsilon} H_{m_\sigma}(\sigma) H_{-\kappa} = H_{m_\sigma}(\sigma) H_{-\epsilon}.$$

The estimate

$$\| H_{m_\sigma}(\sigma) H_{-\epsilon} \| \leq C \begin{cases} (1 + |\sigma|) \frac{m_\sigma}{\kappa}, & \kappa > 0, \\ (1 + |\sigma|)^{m_\sigma - \epsilon}, & \kappa \leq 0, \end{cases}$$

can be deduced from the formula (6), Calderon-Vaillancourt theorem, and the formula

$$\sup_{y > 0} (1 + y + t)^m (1 + y)^{-\ell} = \begin{cases} (1 + t)^m, & \ell > 0, \\ C_{m, \ell} t^{m - \ell}, & \ell \leq 0, t \geq R_{m, \ell}. \end{cases}$$
Now, if \( \alpha(\sigma, X, D) \in \text{OPS}_{\nu, \Sigma}^m \), then

\[ \| \alpha(\sigma, X, D) \|_{k, k \pm \epsilon} = \| H_{m^2}(\sigma)(H_{-m^2}(\sigma) \alpha(\sigma, X, D)) \|_{k, k \pm \epsilon} \leq \]

\[ \leq \| H_{m^2}(\sigma) \|_{k, k \pm \epsilon} \| H_{-m^2}(\sigma) \alpha(\sigma, X, D) \|_{k, k} . \]

We have to show that

\[ \| H_{-m^2}(\sigma) \alpha(\sigma, X, D) \|_{k, k} \leq C , \quad (10) \]

where \( C \) does not depend on \( \sigma \).

Again,

\[ \| H_{-m^2}(\sigma) \alpha(\sigma, X, D) \|_{k, k} = \| H_k H_{-m^2}(\sigma) \alpha(\sigma, X, D) H_{-k} \| . \]

Denote as \( h(\sigma, x, \xi) \) the symbol of the operator \( H_{-m^2}(\sigma) \alpha(\sigma, X, D) \). The function \( h(\sigma, x, \xi) \in S^0_{\nu, \Sigma} \), so \( h(\sigma, x, \xi) \in S^0_{\nu, \Sigma} \), uniformly for \( \sigma \) or

\[
\sup_{x, \xi, \sigma, \xi} \left[ \frac{d^\infty}{d x, \xi} h(\sigma, x, \xi) (1 + |x| + |\xi|)^{1/\epsilon'} \right] \leq C .
\]
and the same is true for the symbol of the operator

\[ H_\kappa H^{-\eta_2} (\sigma) a(\sigma, x, D) H_\kappa , \]

which proves the estimate (10).

**Definition 3.** The class \( H^m \) consists of functions \( a(\sigma, x, \xi), \sigma \in \mathcal{S}, (x, \xi) \in \mathbb{R}^{2n} \), which are smooth on \( \mathbb{R}^{2n+1} \) and satisfy the following condition:

\[ a(\sigma, x, \xi) \sim \sum_{j \geq 0} q_j(\sigma, x, \xi), \quad 16^j + 1x^2 + 1\xi^2 \to \infty, \quad (11) \]

where \( q_j(\sigma, x, \xi) \) is smooth off \((0, 0, 0)\) and satisfies the homogeneity condition:

\[ q_j(r^2 \sigma, r x, r \xi) = r^{m-2j} q_j(\sigma, x, \xi), \quad r > 0. \]

**Definition 4.** We say that the function \( a(\sigma, x, \xi) \) belongs to the class \( H^m \) if \( a(\sigma, x, \xi) \in H^m \) and

\[ a(\sigma, x, \xi) = (-i)^m a(\sigma, x, \xi), \quad (12) \]
Definition 5. We say that the pair \( \alpha_{\pm}(\sigma, x, \xi) \) belongs to the class \( H_{\pm}^{\mu}(H_{\pm}^{\mu}, \sigma) \) if both \( \alpha_{+}(\sigma, x, \xi) \) and \( \alpha_{-}(\sigma, x, \xi) \) belong to \( H_{\pm}^{\mu}(H_{\pm}^{\mu}, \sigma) \), and if their expansions are compatible in the following sense:

\[
\alpha_{\pm}(\sigma, x, \xi) \sim \sum_{j \geq 0} (\pm 1)^{j} \phi_{j}(\sigma, \pm x, \xi). \tag{13}
\]

Proposition 2. The function

\[
\hat{K}(\sigma, \pm \tau, y, \eta) = \tau^{m/2} \alpha_{\mp}(\sigma^{1/2}, \pm \tau^{-1/2} y, \tau^{-1/2} \eta), \quad \tau > 0, \tag{14}
\]

belongs to the class \( \psi_{+}^{m}(G) \) (or \( \psi_{-}^{m}(G) \)) with

\[
\psi_{\pm}^{m}(\sigma_{+}^{1/2})(x, \xi) = \alpha_{\pm}(\sigma, x, \xi), \text{ if and only if } \alpha_{\pm}(\sigma, x, \xi) \text{ belong to } H_{\pm}^{m}(H_{\pm}^{m}, \sigma).
\]

Proof. It is needed to show that the function \( \hat{K}(\sigma, \pm \tau, y, \eta) \) defined by (14) is smooth at \( \tau = 0 \), \( \sigma^{1/2}(y, \eta) \neq 0 \). From the formula (13) we have

\[
\alpha_{\mp}(\sigma^{1/2}, \pm \tau^{-1/2} y, \tau^{1/2} \eta) \sim \sum_{j \geq 0} \tau^{-m/2} (\pm 1)^{j} \phi_{j}(\sigma, y, \eta), \tag{15}
\]

as \( \tau \to 0 \), \( 161 + 1y^{2} + 1\eta^{2} = 1 \).

It follows from (15) that if \( \tau \to 0 \), \( 161 + 1y^{2} + 1\eta^{2} = 1 \),

\[
\hat{K}(\sigma, \pm \tau, y, \eta) \sim \sum_{j \geq 0} (\pm \eta)^{j} \phi_{j}(\sigma, y, \eta).
\]
Assume that $\hat{k}(\xi, \tau, y, \eta)$ belongs to the class $\Psi_0^m(G)$. By the homogeneity relation (3) we have

$$\hat{k}(\xi, \tau, y, \eta) = \tau^{m/2} \hat{k}(\xi^{1/2}, \pm 1, \xi^{1/2} y, \xi^{1/2} \eta), \ \tau > 0. \quad (16)$$

Denote the function $\hat{k}(\xi, \pm 1, y, \eta)$ as $a_\pm(\xi, y, \eta)$. It is known that the function $\hat{k}(\xi, \tau, y, \eta)$ is smooth at $\tau \to 0$, $|\xi| + |y|^2 + |\eta|^2 = 1$ or

$$\hat{k}(\xi, \pm 1, y, \eta) \sim \sum_{j \geq 0} (-\xi)^j \phi_j(\xi, y, \eta).$$

It follows that

$$a_\pm(\xi^{1/2}, \pm \xi^{1/2} y, \xi^{1/2} \eta) \sim \sum_{j \geq 0} \xi^{-m/2} (-\xi)^j \phi_j(\xi, y, \eta)$$

or

$$a_\pm(r^{1/2} \xi, ry, r\eta) \sim \sum_{j \geq 0} r^{-m/2 - 2j} (-i)^j \phi_j(\xi, \pm y, \eta), \quad (17)$$

$$r \to +\infty.$$

Note that

$$D^i_{\xi} D^j_{\tau} D^\infty_{y, \eta} \hat{k}(\xi, \tau, y, \eta) \in \Psi_0^{m-2i-2j-\infty}(G), \quad (18)$$
which follows from differentiation of the relation (3). In particular,

\[ \mathcal{D}^\alpha_{y,\eta} \hat{k}(\theta, \tau, y, \eta) \in \psi^{m-1\alpha'}(\Gamma). \]

Denote the function \( \mathcal{D}^\alpha_{y,\eta} \alpha_{\pm}(\theta, y, \eta) \) as \( b^\pm_{\alpha}(\theta, y, \eta) \).

From the formula (16) we have

\[ \mathcal{D}^\alpha_{y,\eta} \hat{k}(\theta, \tau, y, \eta) = \mathcal{C} \frac{m-1\alpha'}{2} b^\pm_{\alpha}(\theta \tau, \tau^\frac{1}{2} y, \tau^\frac{3}{2} \eta). \]

It can be shown similarly to (17) that

\[ b^\pm_{\alpha}(r^2 \theta, r \tau, r \eta) \sim \sum_{j \geq 0} r^{m-2j-1\alpha}(\pm) \phi_{\alpha, j}(\theta, y, \eta) \]

as \( r \to +\infty \).

It is clear that the function \( \hat{k}(\theta, \pm \tau, y, \eta) \) defined by (14) is homogeneous with respect to dilation (3) for \( r > 0 \). Assume that (13) is satisfied. If \( r < 0 \) we have by (14) and (12)

\[ \hat{k}(r^2 \theta, \tau r^2 \tau, r \tau, r \eta) = \]

\[ = |r|^m \mathcal{C}^{m/2} a_{\pm}(\tau^{-1} \theta, \pm (-i) \tau^{-1/2} y, -\tau^{-1/2} \eta) = \]

\[ = |r|^m (-i)^m \mathcal{C}^{m/2} a_{\pm}(\tau^{-1} \theta, \pm \tau^{-1/2} y, \tau^{-1/2} \eta) = \]

\[ = r^m \hat{k}(\theta, \pm \tau, y, \eta). \]
Note that if the functions \( \alpha_{\pm}(\sigma, x, \xi) \) belong to the space \( S(\mathbb{R}^{2n+1}) \) (Schwartz space of rapidly decreasing functions), then (14) defines an element of \( OP \Psi^m_{\pm}(G) \).

If the functions \( \alpha_{\pm}(\sigma, x, \xi) \in H_{\pm}^m (H_{\pm, o}^m) \), we say that \( \alpha_{\pm}(\sigma, x, \xi) \in OP H_{\pm}^m (OP H_{\pm, o}^m) \).

Proposition 2 is similar to Proposition 2.2 (Chapter I) in [3]. This allows to consider products, adjoints, and hypoellipticity of convolution operators from the class \( OP \Psi^m_0(G) \) in the manner it was performed in [3] for the similar class \( OP \Psi^m_0(H^m) \).

Proposition 3. If \( \alpha_{\pm}(\sigma, x, \xi) \in OP H_{\pm}^m (OP H_{\pm, o}^m) \),
\( b_{\pm}(\sigma, x, \xi) \in OP H_{\pm}^\nu (OP H_{\pm, o}^\nu) \), then

\[
\alpha_{\pm}(\sigma, x, \xi) b_{\pm}(\sigma, x, \xi) = c_{\pm}(\sigma, x, \xi) \in OP H_{\pm}^{m+\nu} (OP H_{\pm, o}^{m+\nu}).
\]

Proof. Assume that \( \alpha(\sigma, x, \xi) \in OP H^m, b(\sigma, x, \xi) \in OP H^\nu \).

The class \( H^m \) is a subset of the class \( S_{l, \Sigma}^m \), so the product \( c(\sigma, x, \xi) \) belongs to the class \( OP S_{l, \Sigma}^{m+\nu} \) and \( c(\sigma, x, \xi) \) has the asymptotic expansion by the formulas (6) and (7).

It follows from formula (11) that if \( \alpha(\sigma, x, \xi) \in H^m(H^m_0), \)
\( b(\sigma, x, \xi) \in H^\nu (H^\nu_0) \), then \( \{\alpha, b\}_l(\sigma, x, \xi) \in H^{m+\nu-2j}(H^{m+\nu-2j}_0) \)
and \( \alpha(\sigma, x, \xi) b(\sigma, x, \xi) \in OP H^{m+\nu}(OP H^{m+\nu}_0) \).
Now we have that $c_+(\sigma, X, \xi)$ and $c_-(\sigma, X, \xi)$ belong to $\text{OPH}_{m+\mu}^m$ (OPH$_0^{m+\mu}$) and

$$c_{\pm}(\sigma, X, \xi) \sim \sum_{j \geq 0} \left( \frac{1}{j!} \right) \{ \alpha_{\pm}, b_{\pm} \}_j (\sigma, X, \xi).$$

(19)

The $j$ term of (19) belongs to the class $H_{m+\mu-2j}^{m+\mu-2j}$, so the series of (19) asymptotically sum to the element of $H_{m+\mu}^{m+\mu}$ (H$_{m+\mu}$). As a consequence of Proposition 3, we have

**Proposition 4.** If $K_1 \in \text{OPH}_{m}^{m}(\sigma, \psi^m_0)$, $K_2 \in \text{OPH}_{m}^{m}(\sigma, \psi^m_+)$, then $K_1K_2 \in \text{OPH}_{m}^{m}(\sigma, \psi^m_0)$, and

$$\delta_{K_1K_2}(\sigma, \pm \lambda)(X, \xi) = \delta_{K_1}(\sigma, \pm \lambda)(X, \xi)\delta_{K_2}(\sigma, \pm \lambda)(X, \xi).$$

Assume that the operator $K$ belongs to the class $\text{OPH}_{m}^{m}(\sigma, \psi^m_0)$ (OPH$_m^{m}(\sigma, \psi^m_0)$). It follows from Proposition 2 that the operators

$$\delta_K(\sigma, \pm)(X, \xi) = \alpha_{\pm}(\sigma, X, \xi)$$

belong to the class $\text{OPH}_{m}^{m}, 0$ (OPH$_m^{m}$). It is known from the Weyl calculus that if $\alpha(\sigma, X, \xi) \in \text{OPS}_{m}^{m}(\sigma, \psi^m_0)$, then $\alpha(\sigma, X, \xi)^* = \alpha(\sigma, X, \xi)^*$, and

$$\alpha(\sigma, X, \xi)^* = \overline{\alpha(\sigma, X, \xi)}.$$

It follows from the last formula that $\alpha_{\pm}(\sigma, X, \xi)^* \in \text{OPH}_{m}^{m}, 0$ (OPH$_m^{m}$).
This implies

**Proposition 5.** If \( k \in O \psi^m_\sigma(G)(O \psi^m_+(G)) \), then \( K^* \in O \psi^m_\sigma(G)(O \psi^m_+(G)) \), and

\[ \mathcal{G}_K^*(\mathcal{G}_s \mp \lambda)(X, D) = \mathcal{G}_K^*(\mathcal{G}_s \pm \lambda)(X, D) \]*

Consider the case when the operators \( \mathcal{G}_K^*(\mathcal{G}_s \pm 1)(X, D) \) are elliptic.

**Definition 6.** We say, operator \( \alpha(\mathcal{G}, X, D) \in O \phi^m \) is elliptic with parameter \( \mathcal{G} \), if \( q_\phi(\mathcal{G}, X, \xi) \neq 0 \) for \( |s| + |x_1^2| + |\xi_1^2| \neq 0 \).

Let \( R_M \) be the set \( \{ \mathcal{G} \in R^*; |s| \geq M \} \).

**Proposition 6.** If the operator \( \alpha(\mathcal{G}, X, D) \in O \phi^m \) and it is elliptic with parameter \( \mathcal{G} \), then there exists an operator \( \alpha(\mathcal{G}, X, D)^{-1} \in O \phi^m \) for some \( M > 0 \).

**Proof.** At first we show that \( \alpha(\mathcal{G}, X, D) \) has a parametrix \( b(\mathcal{G}, X, D) \) \( \in O \phi^m \). Let \( \beta(\mathcal{G}, X, D) \) be the operator with Weyl symbol \( \beta(\mathcal{G}, X, \xi) = q_\phi(\mathcal{G}, X, \xi) \) for \( |s| + |x_1^2| + |\xi_1^2| \) large. By the formulas (6) and (7):

\[ \beta(\mathcal{G}, X, D) \alpha(\mathcal{G}, X, D) = \mathbb{I} + r(\mathcal{G}, X, D), \]
where $r(\sigma, x, \xi) \sim \sum r^i(\sigma, x, \xi)$, $r^i(\sigma, x, \xi)$ are homogeneous of degree $m-2i$ in $(\sigma, x, \xi)$. Therefore, the operator $r(\sigma, x, \mathcal{D}) \in \mathcal{OPH}^{m-i}$.

By (6) and (7) $r^i(\sigma, x, \mathcal{D})$, $i \geq 1$, belongs to the class $\mathcal{OPH}^{m-i}$, so the operator

$$b(\sigma, x, \mathcal{D}) = (I - r(\sigma, x, \mathcal{D}) + r^2(\sigma, x, \mathcal{D}) - \ldots) \beta(\sigma, x, \mathcal{D})$$

is a parametrix for the operator $\alpha(\sigma, x, \mathcal{D})$ and belongs to the class $\mathcal{OPH}^{-m}$.

Now it has to be shown that $\alpha(\sigma, x, \mathcal{D})$ is invertible as an operator on $\mathcal{W}_k$ for all integer $k$ for $|\sigma|$ sufficiently large. The product

$$b(\sigma, x, \mathcal{D}) \alpha(\sigma, x, \mathcal{D}) = I + \mathcal{R}_g$$

where the operator $\mathcal{R}_g \in \mathcal{OPS}_{-\infty}^{\infty}$. It follows from Proposition 1, that there is $M > 0$, such that $\|\mathcal{R}_g\|_{k, k - \mathcal{E}} < \frac{1}{2}$ for $|\sigma| \geq M$, so the operator $I + \mathcal{R}_g$ is invertible as an operator on $\mathcal{W}_k$ for any integer $k$. Denote as $\mathcal{Q}_g$ the operator $(I + \mathcal{R}_g)^{-i} - I$. We have $\mathcal{Q}_g : S(R^n) \rightarrow S(R^n)$ is continuous, hence by the relation $\mathcal{Q}_g = -\mathcal{R}_g - Q_g \mathcal{R}_g$, $\mathcal{Q}_g : S'(R^n) \rightarrow S(R^n)$ is continuous. It follows that

$$\alpha(\sigma, x, \mathcal{D})^{-i} - b(\sigma, x, \mathcal{D}) \in \mathcal{OPS}_{i, k}^{\infty}.$$

Assume that the operator $k \in \mathcal{OPH}^{-m}(G)$ and the operators $\mathcal{G}_{\sigma}(\sigma, t)(x, \mathcal{D})$ are elliptic with parameter $\sigma$, $\sigma \in \mathcal{R}$. 
Assume also that the operators $\sigma'_K(\sigma, \pm \iota)(X, \bar{D})$ have the left inverses $[\sigma'_K(\sigma, \pm \iota)(X, \bar{D})]^{-1} \in OPH_{\pm}^{-m} (OPH_{\pm, o}^{-m})$. Then the operator $L$, such that

$$\sigma'_L(\sigma, \pm \iota)(X, \bar{D}) = \lambda^{-m/2} [\sigma'_K(\sigma, \pm \iota)(X, \bar{D})]^{-1},$$

is a left inverse for the operator $K$ and $L \in O\!P\!Y_{+}^{-m}(G)$ ($O\!P\!Y_{o}^{-m}(G)$).

The next proposition is an extension for the case of the group $R \times \mathbb{H}^n$ of the Proposition 2.10[3].

**Proposition 7.** If $K \in O\!P\!Y_{+}^{-m}(G)$ and $\sigma'_K(\sigma, \pm \iota)(X, \bar{D})$ are elliptic with parameter $\sigma$, then $K$ has a left inverse $L \in O\!P\!Y_{+}^{-m}(G)$ if and only if $\sigma'_K(\sigma, \pm \iota)(X, \bar{D})$ are injective on $S(R^n)$, and such a right inverse if and only if $\sigma'_K(\sigma, \pm \iota)(X, \bar{D})^*$ are injective on $S(R^n)$.

**Corollary 1.** If $K \in O\!P\!Y_{+}^{-m}(G)$ and $\sigma'_K(\sigma, \pm \iota)(X, \bar{D})$ are elliptic with parameter $\sigma$ and injective on $S(R^n)$, then $K$ is hypoelliptic.

Denote as $A_{\sigma}$ the operator $a(X, \bar{D}) - i \sigma$ where $a(X, \bar{D})$ is a differential operator of second order with the symbol
\[
\alpha(x, \xi) = \sum_{i,j=1}^{2n} a_{ij} x_i x_j , \quad x_i = x_i, \quad x_{i+n} = \xi_i, \quad 1 \leq i \leq n .
\]

The matrix \( \{a_{ij}\} \) is strictly positive definite and symmetric.

The operator \( A_\sigma \) satisfies the condition of Proposition 6.

Note that in this case the Weyl symbol for parametrix of \( A_\sigma \):

\[
B(\sigma, x, \xi) \sim \sum_{j \geq 0} \phi_j(\sigma, x, \xi), \quad 161 + 1 |x|^2 + |\xi|^2 \to \infty ,
\]

where the functions \( \phi_j(\sigma, x, \xi) \) are solutions of the following equations:

\[
(\alpha(x, \xi) - \iota \sigma) \phi_j(\sigma, x, \xi) = 1
\]

\[
\{ a - \iota \sigma, \phi_{2j-2} \}_2 + (a - \iota \sigma) \phi_{2j} = 0 , \quad j = 1, 2, ...
\]

The function \( \phi_{2j}(\sigma, x, \xi) \) is homogeneous of degree \(-2-4j\)
in \( (161^{1/2}, x, \xi) \) so the operator \( B(\sigma, x, \xi) \in \text{OPH}^{-2} \).

As a consequence of the Propositions 6 and 7, we have

**Corollary 2.** Assume that \( \kappa \in \text{OPH}^m (G) \) and

\[
\sigma_\kappa(\sigma, \pm i)(x, \xi) = a_\pm (x, \xi) - \iota \sigma = A_\sigma .
\]

If there exist \( a_\pm(x, \xi)^{-1} \), then the operator \( \kappa \) is hypoelliptic.
Definition 7. We say a convolution operator $K$ belongs to $OP\psi^m(G)$ if

$$K \sim \sum_{j \geq 0} K_j, \quad K_j \in OP\psi^{m-j}(G),$$

in the sense that the difference $K - \sum_{j=0}^{N} K_j$ is arbitrary smoothing for any sufficiently large $N$. 
Section 1.2. Parametrix for the Heat Equation on the Group $\mathbb{R} \times \mathbb{H}^n$.

Let $C_-$ be the half-plane $\{\text{Im} \xi < 0\}$ with the closure $\overline{C}_-$.

Definition 8. The class $\Psi^m_r(G)$ is the subclass of $\Psi^m_o(G)$, consisting of functions which extend to $(\mathbb{C}_- \times \mathbb{R}^{2n+1}) \setminus \mathcal{O}$ in such a way to be $C^\infty$ in all variables and holomorphic with respect to $g, \xi \in \mathbb{C}_-$.

We will consider the operator $\partial_{\partial t} - L_\alpha$ on $G$ where

$$L_\alpha = \sum_{j=1}^n (L_j^2 + M_j^2) + i\alpha \tau$$

on $\mathbb{H}^n$. Note that

$$\tilde{\mathcal{G}}_{\sigma, \pm \lambda} (\partial_{\partial t} - L_\alpha) = i\sigma + \lambda \left\{ \sum_{j=1}^n \left( -\partial_{\partial \xi_j}^2 + \xi_j^2 \right) \mp \alpha \right\}.$$ 

Obviously, $(\partial_{\partial t} - L_\alpha) \in Op \Psi^2_o$ and

$$\mathcal{G}'_{(\partial_{\partial t} - L_\alpha)} (\sigma, \pm \lambda) (x, \xi) = i\sigma + \lambda |x|^2 + |\xi|^2 \mp \alpha.$$ 

The operator $\mathcal{G}'_{(\partial_{\partial t} - L_\alpha)} (\sigma, \pm \lambda) (X, D)$ is elliptic with parameter $i\sigma \mp \alpha$ for all $\sigma \in \mathbb{R}$ and invertible on $L^2(\mathbb{R}^n)$ if and only if $-n \mp \alpha \notin \{0, 2, 4, \ldots\}$. 
Denoting as \( b_{\alpha}(\xi, \eta, \xi) \) the Weyl symbol of the inverse operator, we have the following equation

\[
(1x^2 + 1y^2 + i(\xi + \eta)\alpha) b_{\alpha}(\xi, \eta, \xi) - \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial^2}{\partial y_k \partial \xi_k} - \frac{\partial^2}{\partial \eta_k \partial \xi_k} \right)^2 \]

\[
\cdot \left( 1x^2 + 1y^2 + i(\xi + \eta)\alpha \right) b_{\alpha}(\xi, \eta, \xi) \bigg|_{y=x, \eta=\xi} \equiv 1.
\]

After differentiation it becomes

\[
-\frac{1}{4} \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2} \right) b_{\alpha} + (1x^2 + 1y^2 + i(\xi + \eta)\alpha) b_{\alpha} \equiv 1. \quad (20)
\]

Denote by \( H \) operator \(-\Delta + 1x^2\), the resolvent of \( H \) by

\[
R_{\eta}(X, D) = (H + \eta I)^{-1}.
\]

A solution of the equation (20) could be obtained as a Weyl symbol for the operator \( R_{\eta}(X, D) \) for \( \eta = i(\xi + \eta)\alpha \). In turn,

\[
R_{\eta}(x, \xi) = \int_{0}^{\infty} e^{-\eta t} h_t(x, \xi) dt,
\]
where \( h_t(x, \xi) \) is the well known Weyl symbol of the operator \( e^{-tH} \) (see [3]):

\[
h_t(x, \xi) = (\cosh t)^{-n} \exp \left[ - \left( 1 + \frac{1}{15} t^2 \right) \tanh t \right]. \tag{21}
\]

Using formula (21) and changing the variables, we have

\[
R_{\gamma'}(x, \xi) = C_n \int_0^1 (1-t)^{-1+(\sigma+n)/2} (1+t)^{-1-(\sigma-n)/2} e^{-t(1+15/2)} \, dt. \tag{22}
\]

The integral (22) converges for \( Re \gamma' > -n \). Using the Taylor series expansions for the functions \( (1+t)^{-1-(\sigma-n)/2} \) and \( e^{-t(1+15/2)} \), the integral (22) can be rewritten as the convergent series:

\[
R_{\gamma'}(x, \xi) = C_n \sum_{j \geq 0} \int_0^1 (1-t)^{-1+(\sigma+n)/2} C_j t^j dt,
\]

where \( C_j \) depends on \( x, \xi, \gamma', n, j \). For each \( j \) the integral

\[
\int_0^1 (1-t)^{-1+(\sigma+n)/2} t^j \, dt = \frac{(j-n)!}{(\sigma+n)/2 \cdots (\sigma+n+j)/2}.
\]
It follows that the integral (22) can be continued analytically to all complex \( \gamma \) excluding \(-n-2j, \ j=0,1, \ldots \). Therefore, the solution of the equation (20) can be written as follows:

\[
b_{\alpha}(\sigma, x, \xi) = \int_{0}^{\infty} e^{-i\sigma t + \alpha t} h_{t}(x, \xi) \, dt
\]

for \(|Re\alpha| < n\).

Consider now an operator \(\partial_{t}^{2} + \partial_{\alpha} \) on \(G\), where \(\partial_{\alpha} \) is a more general second order differential operator on \(H^{n}\):

\[
\partial_{\alpha} = \sum_{j,k=1}^{n} a_{j,k} X_{j} X_{k} + i\alpha T,
\]

(23)

where \(X_{j} = L_{j}, X_{j+n} = M_{j}, \ l \leq j \leq n\), and \(\{a_{j,k}\}\) is a symmetric, positive definite matrix of real numbers. We have

\[
\Pi_{G, \pm \lambda} (\partial_{t}^{2} + \partial_{\alpha}) = i\delta \mp \lambda \alpha - \lambda Q(x, z),
\]

where

\[
Q(x, \xi) = \sum_{j,k=1}^{n} a_{j,k} x_{j} x_{k}, \ x_{j} = x_{j}, \ x_{j+n} = \xi_{j}, \ (24)
\]

\[
1 \leq j \leq n.
\]

The operator \(Q(\pm x, z)\) is a positive self-adjoint differential
operator of the second order. Let \( S \) be the symplectic form on \( R^{2n} \):

\[
S((x, \xi), (x', \xi')) = x \cdot \xi' - x' \cdot \xi.
\]

If \( Q(u, v) \) is the symmetric bilinear form on \( R^{2n} \) polarizing the quadratic form \( Q(\omega)Q(\omega) = Q(\omega, \omega) \), the Hamilton map of \( Q(x, \xi) \) is defined to be the linear map \( \tilde{F} \) on \( R^{2n} \):

\[
\tilde{S}(u, \tilde{F}v) = Q(u, v), \quad u, v \in R^{2n}.
\] (25)

\( \tilde{F} \) is positive definite (if \( Q \) is positive definite) so its eigenvalues are of the form \( \pm i \mu_j \), \( 1 \leq j \leq n \), \( \mu_j > 0 \).

It was shown in [3] that \( Q(x, \xi) \) is unitarily equivalent to the operator

\[
\sum_{j=1}^{n} \mu_j \left( -\frac{\partial^2}{\partial x_j^2} + x_j^2 \right),
\]

so the spectrum of \( Q(x, \xi) \) is of the form:

\[
\left\{ \sum_{j=1}^{n} (a k_j + b) \mu_j, \ k_j \in Z^+ \cup \{0\} \right\}.
\]
If $Q(x, \xi)$ takes the form

$$Q(x, \xi) = \sum_{j=1}^{n} \mu_j (x_j^2 + \xi_j^2), \quad \mu_j > 0,$$

then the equation (20) changes to

$$-\frac{1}{4} \sum_{j=1}^{n} \mu_j \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2} \right) b + \sum_{j=1}^{n} \mu_j (x_j^2 + \xi_j^2) b$$

$$+ \left( i6 \pi \alpha \right) b \equiv 1.$$  

(26)

The formula for the solution of the equation (26) can be written as

$$\int_{-\infty}^{\infty} e^{i\theta t + \alpha t} h_t^Q(x, \xi) dt,$$

(27)

where

$$h_t^Q(x, \xi) = \prod_{j=1}^{n} \left( \cosh t \mu_j \right)^{-1}$$

$$\cdot \exp \left\{ -\sum_{j=1}^{n} (x_j^2 + \xi_j^2) \tanh t \mu_j \right\}. $$
and it is necessary to take \( \text{Re} \alpha \) small enough in order to avoid the spectrum of the operator \( \mathcal{Q}(\mathcal{X}, \mathcal{D}) \) i.e.,

\[
|\text{Re} \alpha| < \sum_{j=1}^{n} \mu_j
\]

(28)

The formula (27) will be expressed in an invariant form as it was done in [3]. Denote by \( F_{\alpha} \) the map \( \frac{1}{t} F_{\alpha} \). We have

\[
\prod_{j=1}^{n} (\cosh t \mu_j)^2 = \det \cosh t F_{\alpha}
\]

so

\[
\prod_{j=1}^{n} (\cosh t \mu_j)^{-1} = (\det \cosh t F_{\alpha})^{-\frac{1}{2}}
\]

Now let

\[
A_{\alpha} = (F_{\alpha}^2)^{\frac{1}{2}}
\]

be the unique square root of the matrix \( F_{\alpha}^2 \) with positive spectrum, and \( \gamma \) is a quadratic form on \( \mathbb{R}^{2n} \) defined by

\[
\gamma(A_{\alpha}z, z) = \mathcal{Q}(z, z).
\]

In the symplectic coordinate system on \( \mathbb{R}^{2n} \) such that
\[ Q(z, z) = \sum_j \mu_j (x_j^2 + s_j^2), \quad z = (x, s), \]

we have

\[ \mathcal{V}(z, z) = \sum_{j=1}^{n} (x_j^2 + s_j^2), \]

so

\[ \mathcal{V}(f(A_\alpha) z, z) = \sum_{j=1}^{n} f(\mu_j) (x_j^2 + s_j^2) \]

and

\[ \sum_{j=1}^{n} (x_j^2 + s_j^2) \tanh(t \mu_j) = \mathcal{V}(\tanh t A_\alpha z, z) = \]

\[ = Q(A^{-1}_\alpha \tanh t A_\alpha z, z). \]

Thus the formula (27) can be written invariantly as

\[ b_{\alpha}(\sigma, x, s) = \int_0^\infty e^{-i \sigma t + \alpha t} \Phi_{\alpha}(t, x, s) dt, \quad (29) \]

where

\[ \Phi_{\alpha}(t, z) = (\det \cosh t F_\alpha)^{-1/2}. \]

\[ \cdot \exp\left\{ -Q(A^{-1}_\alpha \tanh t A_\alpha z, z) \right\}, \quad z = (x, s), \quad (30) \]
and

$$|\Re \alpha| < \sum_{j=1}^{n} \mu_j.$$ 

The function $b_{\alpha}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{2n+1})$.

So, we can define an element of $\Psi_{\hbar}^{-2}$ by formula

$$b_{\alpha} (\xi, \pm \xi) (x, \xi) = \tau^{-1} b_{\alpha} (\tau^{-1} \xi, \pm \tau^{-\frac{1}{2}} x, \tau^{-\frac{1}{2}} \xi).$$

Using the formula (29) we obtain

$$\hat{R}(\xi, \pm \xi, y, \eta) = \int_{0}^{\infty} e^{-i \xi t + \alpha t} (\det \cosh \tau F_\alpha)^{-\frac{1}{2}}$$

$$\exp \left\{ - \frac{\omega}{Q} (A_{\alpha}^{-1} \tanh (t \tau) A_{\alpha} \tau^{-\frac{1}{2}} z, \tau^{-\frac{1}{2}} z) \right\} dt,$$

$$z = (y, \eta).$$

The function $\hat{R}(\xi, \pm \xi, y, \eta) = b_{\alpha} (\xi, \pm \xi)(\pm \tau^{-\frac{1}{2}} y, \tau^{-\frac{1}{2}} \eta)$ belongs to the class $\Psi_{\hbar}^{-2}$. It follows from the Proposition 1.17 of Beals, Grieener, and Stanton [5] that the function $k$ (inverse Fourier transform of $\hat{R}$) vanishes for $t \leq 0$.

Furthermore,

$$k_{\alpha}(t, s, q, p) = t^{-\frac{n+1}{2}} k_{\alpha}^{\varphi} (s/t, q/\sqrt{t}, p/\sqrt{t}), \quad t > 0,$$  

(31)
where the function $k_{i,\alpha}^Q$ belongs to the Schwartz space $S(R^{2n+1})$.

If $\alpha = 0$, the invariant form for the function $k_{i}^Q(s, q, p)$ is

$$k_{i}^Q(s, q, p) = C_n \int_{-\infty}^{+\infty} e^{is\tau} \psi_{i}^Q(\tau, q, p) d\tau$$

(32)

with

$$\psi_{i}^Q(\tau, q, p) = (-\tau^{-2n} \text{det} \sinh \tau F_{iQ})^{1/2}$$

$$\cdot \exp \left\{-\tau Q(A_{Q}^{-1} \coth \tau A_{Q} z, z) \right\}, \quad z = (q, p).$$

If $\alpha \neq 0$, $|\text{Re}\alpha| < \sum_{j=1}^{n} \eta_j$, then

$$k_{i,\alpha}^Q(s, q, p) = k_{i}^Q(s/\tau + i\alpha, q/\tau, p/\tau)$$

(33)

where $k_{i}^Q(s/\tau + i\alpha, q/\tau, p/\tau)$ is defined from (32) by an analytic continuation.
Section 1.3. Complex Powers on the Heisenberg Group

We consider complex powers of the right invariant differential operator\((-P)\) defined by (23). If \(\beta \in \mathbb{R}\), then

\[
\mathcal{G}^{\beta}(\pm \lambda)(X,D) = \lambda^{\beta} Q^{\beta}(X,D),
\]

where the Weyl symbol of the operator \(Q(X,D)\) defined by (24). In accordance with the last formula we will analyze the complex powers of the operator \((-P)\) as an operator \((-P)^{z}\), such that

\[
\mathcal{G}^{z}(-P)^{z}(X,D) = \lambda^{z} Q^{z}(X,D).
\]

Let \(q_{-z}(\xi)\) be the Weyl symbol of the complex power \(-z, \Re z > 0\) of the operator \(Q(X,D)\). It connects with the Weyl symbol of the operator \(e^{-tQ}\) (function \(\Phi_{Q}(t,\xi)\) from (30)) by the formula:

\[
q_{-z}(\xi) = \frac{1}{\rho(z)} \int_{0}^{\infty} t^{z-1} \Phi_{Q}(t,\xi) dt, \quad \Re z > 0. \tag{34}
\]
The integral (34) converges for \( \text{Re} \, z > 0 \) and the function
\[ q_z(x, \xi) \] belongs to the class \( \mathcal{H}^{-2z}(\mathbb{R}^n) \). If \( Q = H(X, D) \), then by (21)

\[
h_{-z}(x, \xi) = \frac{1}{\rho(z)} \int_0^\infty t^{2z-1} \cos \left( t \right) \exp \left[ -t^2 \tan \xi t \right] dt,
\]

(35)

\[ V^2 = \|x\|^2 + 15l^2. \]

Since \( q_z(x, \xi) \) belongs to the class \( \mathcal{H}^{-2z}(\mathbb{R}^n) \), we can define an element \( \hat{K}_{-z} \) of \( \mathcal{H}^{-2z}(\mathbb{H}^n) \) by the formula

\[
6_{K_{-z}} (\pm \xi) (x, \xi) = \mathcal{C}_Z q_z (\pm \mathcal{C}_Z^{-1} x, \mathcal{C}_Z^{-1} \xi).
\]

(36)

By (34)

\[
6_{K_{-z}} (\pm \xi)(x, \xi) = \frac{1}{\rho(z)} \int_0^\infty t^{2z-1} \mathcal{C}_Q (\pm \xi t, \mathcal{C}_Z^{-1} x, \mathcal{C}_Z^{-1} \xi) d\xi.
\]

(37)

If \( Q = H(X, D) \), then the formula (37) obtains a simpler form:
\[
\delta_{k_Z}(\pm \varepsilon)(x, \xi) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} (\cosh \varepsilon t)^{-n} \exp \left\{-\frac{r^2}{\varepsilon} \tanh \varepsilon t\right\} dt.
\]

It follows from the above considerations that the operator \((-P)_{k_Z}\) belongs to the class \(O \psi^{-2z}_{\alpha}(L^1)\), \(\text{Re} z > 0\).

Note that

\[
\Phi Q(\varepsilon t, \varepsilon^{-1/2} x, \varepsilon^{-1/2} \xi) = \delta_{e^{t\rho}}(\pm \varepsilon)(x, \xi)
\]

So the formula (37) can be rewritten as follows:

\[
\delta_{(\rho)_{k_Z}}(\pm \varepsilon)(x, \xi) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} \delta_{e^{t\rho}}(\pm \varepsilon)(x, \xi) dt.
\] (38)

If \(0 < \text{Re} z < n+1\) then an explicit formula for the convolution kernel \(k_{k_Z}\) of the operator \((-P)_{k_Z}\) can be found from the formula (37):

\[
k_{k_Z}(s, q, p) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} k_0(t, s, q, p) dt,
\] (39)

where the function \(k_0(t, s, q, p)\) defined by (31) with \(\zeta = \mathcal{O}\). Similarly, we will analyze complex powers of the operator
\((-p)\) \(\alpha \neq 0\), on \(\mathcal{H}^n\) as convolution operators \(K_{-z,\alpha} u = k_{-z,\alpha} \ast u\) where

\[
k_{-z,\alpha}(s, q, p) = \frac{1}{r(z)} \int_0^\infty t^{z-1} k_\alpha(t, s, q, p) \, dt.
\]  

(40)

The function \(k_\alpha(t, s, q, p)\) was defined by (31).

For each \((s, q, p)\), the function \(k_\alpha(t, s, q, p) = O(t^{-(n+1)})\) as \(t \to \infty\), if \(t \to 0\), \((s, q, p) \neq 0\), \(k_\alpha(t, s, q, p) = O(t^N)\) for any \(N > 0\).

It follows that the integral (40) converges for \(0 < \text{Re}z < n+1\) and \((s, q, p) \neq 0\). It can be shown similarly that the function \(K_{-z,\alpha}(s, q, p)\) is \(C^\infty\) on \(\mathcal{H}^n \setminus 0\). Note that

\[
k_{-z,\alpha}(r(s, q, p)) = \frac{1}{r(z)} \int_0^\infty t^{z-1} k_\alpha(t, r^2 s, r q, r p) \, dt = \]

\[
r^{2(z-(n+1))} k_{-z,\alpha}(s, q, p),
\]

which shows that the operator \(K_{-z,\alpha}\) belongs to the class \(O \psi^{-2z}_\alpha(\mathcal{H}^n)\).

In case of the operator \(L_\alpha\) the formula for the function \(L_{-z,\alpha}\) can be written explicitly.

The heat Kernel in this case is given by the formula
\[ l_{\alpha}(t,s,q,p) = \int_{-\infty}^{\infty} e^{i\tau [s/t + i\alpha]} t^{-(n+i)} \left( \tau / \sinh \tau \right)^n \cdot \exp \left\{ -\tau \coth \tau \frac{(1q_1^2 + 1p_1^2)}{t} \right\} d\tau. \]

Substituting \( l_{\alpha}(t,s,q,p) \) in (40), we have

\[ l_{-z,\alpha}(s,q,p) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{z-1} \left\{ \int_{-\infty}^{\infty} e^{i\tau [s/t + i\alpha]} t^{-(n+i)} \right\} \left( \tau / \sinh \tau \right)^n \exp \left\{ -\tau \coth \tau \frac{(1q_1^2 + 1p_1^2)}{t} \right\} d\tau dt. \]

We will integrate at first with respect to \( t \). Apparently,

\[ \frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{z-(n+2)} e^{-t} (i\tau s - \tau \coth \tau) (1q_1^2 + 1p_1^2) dt = \]

\[ = (i\tau s - (\tau \coth \tau) (1q_1^2 + 1p_1^2))^{z-(n+i)}. \]

We have
\[ \ell_{z, \alpha}(s, q, p) = \int_{-\infty}^{\infty} e^{-\alpha \tau} \left( \frac{\tau}{\sinh \tau} \right)^{z-1} \cdot \left\{ \im \sin \tau - \cosh \tau (1q^2 + 1p^2) \right\}^{z-(n+1)} \]  

(41)

This formula is valid for \( 0 < \Re z < n+1 \) and \( 1 \Re \alpha < n \). In the case of \( Z = k \in \mathbb{Z}^+ \), formula (41) can be continued analytically (by integration by parts) to \( \alpha \in \mathbb{C} \) such that \( \neq \alpha \) avoids the set \( \left\{ n+2j, \ j = 0, 1, \ldots \right\} \).
CHAPTER II. OPERATORS WITH VARIABLE COEFFICIENTS

Section 2.1. Operator class

Let $M$ be a compact contact manifold of dimension $2n+1$.

We aim to obtain the class of operators with variable coefficients on $M$ from the operator class $\mathcal{OP}\Psi^m(R \times \mathbb{H}^n)$ in the way it was developed by Taylor [3]. In order to have symbolic operator calculi for this class we need to verify certain hypotheses that were stated in Chapter I of [3]. These hypotheses for the class $\mathcal{OP}\Psi^m(\mathbb{R})$ can be written as follows:

$$\Psi^m_0(G) \subset \begin{cases} S^m_{\frac{1}{2} + \varepsilon}, & m > 0, \\ S^m_{\frac{1}{2} - \varepsilon}, & m \leq 0. \end{cases}$$  \hspace{1cm} (42)

$$K_1 \in \mathcal{OP}\Psi^m_0(G), \quad K_2 \in \mathcal{OP}\Psi^m_0(G) \rightarrow K_1K_2 \in \mathcal{OP}\Psi^{m+\mu}(G).$$  \hspace{1cm} (43)

$$\hat{K}(\xi, x, y, \eta) \in \Psi^m_0(G) \rightarrow D^i_\xi D^j_x D^k_y \hat{K} \in \Psi^{m-2\varepsilon-2j-1\varepsilon}(G).$$  \hspace{1cm} (44)

$$K \in \mathcal{OP}\Psi^m_0(G) \rightarrow K^* \in \mathcal{OP}\Psi^m_0(G).$$  \hspace{1cm} (45)
If \( \mathcal{K}_j \in \mathcal{O}\mathcal{P}\psi^m_j(G) \), \( j = 0, 1, \ldots \), then there exists \( \mathcal{K} \in \mathcal{O}\mathcal{P}\psi^m(G) \) such that
\[
\mathcal{K} \sim \mathcal{K}_0 + \mathcal{K}_1 + \cdots
\] (46)

The property (42) follows from Definition 1; properties (43) and (45) were stated as propositions 4 and 5, respectively; property (44) was stated by formula (18), and property (46) follows from Definition 7.

By Darboux's theorem an open set \( \mathcal{U} \subset M \) can be mapped diffeomorphically to an open set \( \Omega \subset \mathbb{R}^n \), preserving the contact form. For each \( \mathbf{v} \in \Omega \), if \( \tilde{K}(\mathbf{v}, \mathbf{\sigma}, \mathbf{\tau}, \mathbf{y}, \mathbf{\rho}) \) is a smooth function of \( \mathbf{v} \) with values in \( \psi^m(G) \) then \( K(v) \) defined by
\[
K(v)(\mathbf{u}) = K(v, \cdot) \ast \mathbf{u}
\]
is a smooth function of \( \mathbf{v} \) taking values in \( \mathcal{O}\mathcal{P}\psi^m(G) \). Then we say that the operator \( \mathcal{K} \) defined by
\[
(K\omega)(v) = K(v)\omega(v)
\]
belongs to the class \( \mathcal{O}\mathcal{P}\tilde{\psi}^m(R \times M) \). The symbol of the operator \( \mathcal{K} \) we denote by
\[
\sigma'_{\mathbf{K}}(v, \mathbf{\sigma}, \pm \lambda)(\mathbf{X}, \mathbf{D}) = \pi_{\mathbf{\sigma}, \pm \lambda}(K(v)).
\]
As a consequence of Proposition 1.2 [3] we have the following:

**Proposition 8.** If $A \in \mathcal{O}\mathcal{P}\mathcal{W}^{m}(R \times M)$, $B \in \mathcal{O}\mathcal{P}\mathcal{W}^{m}(R \times M)$, then $AB \in \mathcal{O}\mathcal{P}\mathcal{W}^{m+\mu}(R \times M)$. If $C \in \mathcal{O}\mathcal{P}\mathcal{W}^{m+\mu}(R \times M)$ is defined by

$$g_{c}^{\ast}(\nu', \sigma', \pm \lambda)(\chi, \mathcal{I}) = g_{A}^{\ast}(\nu, \sigma', \pm \lambda)(\chi, \mathcal{I})g_{B}^{\ast}(\nu, \sigma', \pm \lambda)(\chi, \mathcal{I}),$$

then

$$AB-C \in \mathcal{O}\mathcal{P}\mathcal{W}^{m+\mu}(R \times M).$$
Section 2.2. Parametrix for the Heat Equation

Suppose $P$ is a negative self adjoint second order differential operator, its principal symbol $p_2 \geq 0$ and vanishes to exactly second order on $\Lambda \subset T^*M \setminus \mathcal{O}$, the span of the contact form on $M$. Denote by $F$ the Hamilton map of $p_2$ and by $tr^+ F$ the sum of the positive eigenvalues of $\frac{1}{t} F$.

It was shown in [3] that if the condition

$$|s_{ob} 6'(P)| < tr^+ F \quad \text{on} \quad \Lambda \quad (47)$$

is satisfied, then $P$ is hypoelliptic. The operator $P$ also has a discrete spectrum, since the embedding $S^2_1(M) \subset L^2(M)$ is compact.

For $\nu \in \Omega$, we assume that

$$(Pu)(\nu) = P_\infty(\nu)u(\nu),$$

where

$$P_\infty(\nu) = \sum_{j,k=1}^{2n} \alpha_{jk}(\nu)X_jX_k + i\alpha(\nu)T,$$

the matrix $\{\alpha_{jk}(\nu)\}$ is symmetric and positive definite for each $\nu$, the functions $\alpha_{jk}(\nu), \alpha(\nu)$ are smooth functions of $\nu$. 
The symbol of the operator $\hat{p}_{\omega}(u)$ is 

$$\hat{\sigma}_{\hat{p}_{\omega}}(u, \pm i)(x, D),$$

where

$$\hat{\sigma}_{\hat{p}_{\omega}}(u, \pm i)(x, \xi) = \sum \alpha_{j\omega} X_j X_k \mp \alpha(u),$$

$$X_j = x_j, \quad X_{j+n} = \xi_j, \quad 1 \leq j \leq n.$$

The operators $\hat{\sigma}_{\hat{p}_{\omega}}(u, \pm i)(x, D)$ are elliptic and invertible on $L^2(\mathbb{R}^n)$ if the following condition is satisfied:

$$\mp \alpha(u) \neq \left\{ \sum_j (2\kappa_j + i) \mu_j(u), \quad \kappa_j \in \mathbb{Z}^+ \oplus \{0\}, \right\},$$

$$u \in \Omega,$$

$\mu_j(u)$ is the eigenvalue of the Hamilton map $\frac{1}{i} F_{\omega}(u)$.

For $u \in \Omega$, we consider the operator $\frac{\partial}{\partial t} + \hat{p}_{\omega}(u)$ with the symbol $\hat{\sigma}_{\frac{\partial}{\partial t} + \hat{p}_{\omega}}(u, \delta, \pm \lambda)(x, D)$, where

$$\hat{\sigma}_{\frac{\partial}{\partial t} + \hat{p}_{\omega}}(u, \delta, \pm i)(x, \xi) = \sum_{j, k} \alpha_{j\omega} X_j X_k \mp \alpha(u) + i \delta.$$
The operators $\mathcal{O}_{\mathcal{S}^{\mathcal{G} + \mathcal{P}_\alpha}}(\mathcal{U}, \mathcal{S}, \pm) (\mathcal{X}, \mathcal{D})$ are elliptic with parameter $i \mathcal{S}$ and invertible on $L^2(\mathbb{R}^n)$ if condition (48) is satisfied.

**Proposition 9.** If condition (48) is satisfied, then the operator $\mathcal{O}_{\mathcal{S}^{\mathcal{G} + \mathcal{P}_\alpha}}$ is hypoelliptic on $\Omega$ with parametrix $K_\alpha$ in the class $\text{OP} \tilde{\varphi}_h^{-2}$.

**Proof.** If the function $K_{1, \alpha}^{\mathcal{Q}}$ is defined by (29), we consider the function

$$
K_{\alpha}(v)(t, s, q, p) = t^{-(n+i)} K_{1, \alpha}(v)(s/t, q/\sqrt{t}, p/\sqrt{t}),
$$

for each $v$, and the corresponding operator

$$(K_\alpha u)(v) = (K_{\alpha}(v) u)(v), \quad K_{\alpha}(v) u = K_{\alpha}(v) (\cdot) * u.
$$

The operator $K_\alpha$ belongs to the class $\text{OP} \tilde{\varphi}_h^{-2}(\mathbb{R} \times \mathcal{M})$. It follows from Proposition 8 that

$$(\mathcal{O}_{\mathcal{S}^{\mathcal{G} + \mathcal{P}_\alpha}}) K_\alpha = I + R, \quad R \in \text{OP} \tilde{\varphi}_h^{-1}(\mathbb{R} \times \mathcal{M}).$$

So, the operator $\mathcal{O}_{\mathcal{S}^{\mathcal{G} + \mathcal{P}_\alpha}}$ has a left parametrix

$$
K \sim K_\alpha - K_\alpha R + K_\alpha R^2 - \ldots \equiv K_0 + K_1 + K_2 + \ldots
$$
It follows from Proposition 8 that $K_j \in \text{OP} \tilde{\psi}^{-2-j}(R \times M)$, $j=0, 1, \ldots$

Similarly it can be shown that $K$ is a right parametrix. After the rearrangement, we can write

$$K \sim \sum_{j \geq 0} K'_j, \quad K'_j \in \text{OP} \tilde{\psi}^{-2-j}(R \times M).$$

The function $\hat{K}'_j(u, s, \tau, y, \gamma)$ belongs to the class $\tilde{\psi}^{-2-j}$

It is homogeneous with respect to dilation, i.e.,

$$\hat{K}'_j(u, r^2 s, r^2 \tau, ry, r\gamma) = r^{-2j} \hat{K}'_j(u, s, \tau, y, \gamma).$$

It follows from propositions 1.9 and 1.17 [5] that $K'_j(u, t, s, q, \rho)$ is homogeneous of degree $2+j-2n-4$ and it vanishes for $t \leq 0$.

Substitution of $r=1$ shows that $\hat{K}'_j(u, s, \tau, y, \gamma)$ is an odd function of $(y, \gamma)$ if $j$ is odd. So $K'_j(u, t, 0) = 0$ if $j$ is odd, and it is homogeneous of degree $\frac{1}{2}(2+j-2n-4)$ in $t$ when $j$ is even.

Therefore,

$$K(u, t, 0) \sim t^{-(n+1)} \sum_{i \geq 0} t^i K_i(u), \quad t \to 0.$$  \hspace{1cm} (50)
If

\[ K(u(t, v) = k(t, v, \cdot) \ast u(t, v), \quad (t, v) \in \mathbb{R} \times \mathbb{H}^n, \]

then the kernel of the operator \( K \) is the function \( k(t, v, t'; v') \) independent on \( t \) and

\[ \text{tr} e^{t \mathcal{P}} = \int_M K(0, v, t, 0) d\text{vol}(v) + A(t), \quad (51) \]

where \( A(t) \in C^\infty(\mathbb{R}^+). \)

From formula (50) it follows

**Proposition 10.** If \( \mathcal{P} \) is a negative self-adjoint differential operator of a second order on a compact contact manifold, and its principal symbol vanishes to exactly second order on \( \Lambda \subset T^*M \setminus O \), and the hypothesis (47) is satisfied, then

\[ \text{tr} e^{t \mathcal{P}} \sim t^{-(n+\delta)} (c_0 + c_1 t + \cdots), \quad t \to 0. \quad (52) \]
Section 2.3. Analytical Continuation for Zeta-Function

Let \( \lambda_j \), \( j = 0, 1, \ldots \) be the eigenvalues of the operator \((-p)\), \( \lambda_j \geq 0 \),

\[
N(\lambda) = \sum_{\lambda_j < \lambda} 1.
\]

The result on the eigenvalue asymptotics for \((-p)\) is known [3]; it follows from the asymptotic expansion (52) and Karamata's Tauberian theorem:

\[
\lim_{\lambda \to \infty} \lambda^{-(n+1)} N(\lambda) = C. \tag{53}
\]

Denote by \( \zeta_{(-p)}(z) \) the zeta-function of the operator \((-p)\):

\[
\zeta_{(-p)}(z) = \sum_{j \geq 0} \lambda_j^{-z}, \quad z \in \mathbb{C}.
\]

The formula (53) implies that \( \zeta_{(-p)}(z) \) is a holomorphic function of \( z \) for \( \Re z > n+1 \). We aim to continue analytically \( \zeta_{(-p)}(z) \) for \( \Re z \leq n+1 \).
Proposition 11. If $\rho$ is a differential operator of second order on a compact contact manifold satisfying the above hypotheses, then the function $\zeta_{(-\rho)}(-Z)$ has a finite number of simple poles at the points $Z = 1, 2, \ldots, n+1$.

Proof. The zeta-function of the operator $(-\rho)$ and $t \exp^{\rho}$ are connected by the formula:

$$
\zeta_{(-\rho)}(-Z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} t \exp^{\rho} dt, \quad \text{Re} Z > n+1. \tag{54}
$$

Note that the integral (54) converges and defines a function holomorphic for $\text{Re} Z > n+1$ ($t \exp^{\rho} \sim_{t \to 0} c_0 t^{-(n+1)}$ and $t \exp^{\rho} = O(t^{-N})$ as $t \to \infty$ for any $N > 0$). The function

$$
\frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} t \exp^{\rho} dt
$$

is a holomorphic function of $Z$. Consider separately for $\text{Re} Z > (n+1)-\delta$, $0 < \delta < n+1$, the integral

$$
\frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left\{ \int_M K_i(x, t, \omega) d\nu(\omega) \right\} dt \tag{55}
$$
The function $K_i(x, t, o)$ for each $x$ is homogeneous in $t$ of degree $-(n+1) + i$ or $K_i(x, t, o) = t^{-(n+1) + i} K_i(x, 1, o)$.

Let $\gamma$ be the contour consisting of the real axis from 1 to $\rho$, $0 < \rho < 1$, the circle $|s| = \rho$, and the real axis from $\rho$ to 1. Denote by $I_i(z)$ the function

$$\int_{\gamma} s^{z-1-(n+1)+i} \left[ \int_{M} K_i(x, t, o) \, d\text{vol}(x) \right] ds.$$

If $\text{Re} z > (n+1) - i$ then the integral over the circular part of $\gamma$ tends to zero with $\rho$. It follows that

$$I_i(z) = -\int_{0}^{1} t^{z-1} \left[ \int_{M} K_i(x, t, o) \, d\text{vol}(x) \right] dt +$$

$$+ \int_{0}^{1} (te^{2\pi i})^{z-1} \left[ \int_{M} K_i(x, te^{2\pi i}, o) \, d\text{vol}(x) \right] dt.$$

So we have
\[
\int_0^1 \frac{1}{\Gamma(z)} \left\{ \int_M K_i(x,t,0) \, d\omega(x) \right\} dt =
\]

\[
\frac{\Gamma(1-z)}{2\pi i} e^{-i\pi z} \frac{I_i(z)}{\Gamma(z)}.
\]

The integral \( I_i(z) \) converges uniformly in any finite region of the \( \mathbb{C} \) plane and so defines an entire function of \( z \).

Hence, the formula (56) gives the analytic continuation of (55) over the complex \( \mathbb{C} \) plane. The possible singularities are the poles of the function \( \Gamma(1-z) \): points \( z = 1, 2, \ldots \).

The function \( I_i(z) \) at the points \( z = (n+j) - i, \ j = 2, 3, \ldots \), vanishes by Cauchy's theorem so the integral (55) does not have poles at these points. As a consequence, (55) has a finite number of simple poles at the points \( z = 1, 2, \ldots, (n+1) - i, \ 0 \leq i < n+1 \).

In accordance with (51) and (50) now we have to continue analytically for \( \Re z \leq n+1 \) the expression

\[
\frac{1}{\Gamma(z)} \left\{ \int_0^1 t^{z-1} \left\{ \sum_{i \geq n} \int_M K_i(x,t,0) \, d\omega(x) + B(t) \right\} dt \right\},
\]

where \( B \in C^\infty(\mathbb{R}^+) \). By integration by parts it continues analytically to a holomorphic function for \( z \in \mathbb{C} \).
APPENDIX

ZETA-FUNCTION OF THE HARMONIC OSCILLATOR HAMILTONIAN

Consider the case when $\alpha(x,D) = -\Delta + ilx^2 \equiv H$. The Weyl symbol of operator $e^{-tH}$ ([3]) is equal to

$$h_t(x,\xi) = c_n (\cosh t)^{-n} \exp \left\{ - (lx^2 + l\xi^2) \tanh t \right\}.$$

For $\Re z > 0$, using the formula

$$h^{-z} = \left\{ \Gamma(z) \right\} \int_0^\infty t^{z-1} e^{-t} dt,$$

we define the operator $H^{-z}(x,D)$ as the operator with Weyl symbol

$$h_{-z}(x,\xi) = \frac{1}{\Gamma(z)} \left\{ \int_0^\infty t^{z-1} (\cosh t)^{-n} \exp \left\{ - (lx^2 + l\xi^2) \tanh t \right\} dt \right\}.$$

Denote $lx^2 + l\xi^2$ as $r^2$. Using polar coordinates in $(x,\xi)$-space, we obtain

$$\text{tr } H^{-z}(x,D) = \int \int h_{-z}(x,\xi) dx d\xi =$$
\[
\int_0^\infty \left\{ \int_0^\infty z^{-1} (\cos \theta)^{-n} \exp (-r^2 \tan \theta) r^{2n-1} \, dt \right\} \frac{dr |S_{2n}|}{r(z)},
\]

where \( |S_{2n}| \) is the measure of the unit sphere in \( \mathbb{R}^{2n} \).

Inverting the order of integration by \( r \) and by \( t \), we get

\[
tr H^\mathbb{Z}(x, \mathcal{D}) = \frac{|S_{2n}| \Gamma(n)}{2 r(z)} \int_0^\infty t^{n-1} (\sin t)^{-n} \, dt. \tag{a.1}
\]

**Proposition A1.** The \( tr H^\mathbb{Z}(x, \mathcal{D}) \) extends from \( \text{Re} z > n \) to a meromorphic function on the complex plane with finite number of simple poles at the points \( z_j = n - 2j, \; 0 \leq j < n/2 \).

**Proof.** Consider the integral

\[
I(z) = \int \frac{S^{z-1}}{\rho (e^s - e^{-s})^n}, \quad z = \sigma + i \tau, \tag{a.2}
\]

with contour \( \mathcal{C} \) consisting of the real axis from \( \infty \) to \( \rho \),

\( 0 < \rho < \pi \), the circle \( |z| = \rho \), and the real axis from \( \rho \) to \( \infty \).
Assume that $\delta' > n$. On the circle $|s| = \rho$ we have

$$|s^{z-1}| \leq |s^{\delta'-1}| e^{2\pi i \rho}$$

and

$$|(e^s - e^{-s})|^n \geq |s|^n$$

so the integral over the circular part of $\mathcal{P}$ tends to zero with $\rho$ if $\delta' > n$. We have

$$\mathcal{I}(z) = -\int_0^\infty \frac{t^{z-1} dt}{(e^t - e^{-t})^n} + \int_0^\infty \frac{(te^{2\pi i})^{z-1}}{(e^t - e^{-t})^n} dt$$

so

$$\int_0^\infty t^{z-1} (\sinh t)^{-n} dt = 2^n \left[ (e^{2\pi i})^{z-1} \right] \mathcal{I}(z)$$

and

$$\text{tr} H^{-z}(x, \delta) = \frac{2^n |s_{2n}| r(n) \rho(1-z)}{2\pi i} e^{i\pi z} \mathcal{I}(z). \quad (a.3)$$
The integral \( \mathcal{I}(z) \) converges uniformly in any finite region of the \( z \)-plane and so defines an entire function of \( z \). Hence, the formula (a.3) gives the analytic continuation of \( \text{tr} \mathcal{H}^z(\mathcal{X},\mathcal{D}) \) over the complex \( z \)-plane. The possible singularities are the poles of function \( \Gamma(-z) \), points \( z = 1, 2, \ldots \). The aim now is to show that the function \( \mathcal{I}(z) \) vanishes at the points \( z = n+1, n+2, \ldots \) and \( z_j = n - (2j+1), 0 \leq j < (n-1)/2 \). The integral (a.2) after the change of variable \( u = e^{2z-1} \) can be written as

\[
\mathcal{I}(z) = \frac{1}{2z} \int \frac{(u+i)^{n/2-1}}{u^n} \ln^{z-1}(u+i) \, du.
\]

By Cauchy's theorem

\[
\mathcal{I}(z) = \frac{d^{(n-1)}}{du^{(n-1)}} \left\{ (u+i)^{n/2-1} \ln^{z-1}(u+i) \right\}_{u=0} \quad \text{(a.4)}
\]

It follows from (a.4) that \( \mathcal{I}(z) \) for \( z = n+1, n+2 \ldots \). Assume that \( n \) is an odd number: \( n = 2m+1, s-1 = k, k \) integer, \( 0 \leq k \leq 2m \).

To find \( \mathcal{I}(k+i) \) we use the Taylor series for the function

\[
f(u) = (u+i)^{m-1/2} \ln^k (u+i).
\]
Note that

\[
f(\omega) = \frac{d^K}{dt^K} \left[ (1+\omega)^t \right]_{t=m^{-\frac{1}{2}}}.\]

If \( \binom{t}{j} \) is the coefficient of \( \omega^j \) in the Taylor series for the function \((1+\omega)^t\), then

\[
(1+\omega)^t = \sum_{j \geq 0} \binom{t}{j} \omega^j
\]

and

\[
f(\omega) = \sum_{j \geq 0} \frac{d^K}{dt^K} \binom{t}{j} \omega^j.
\]

So \( I(k+1) \) is equal to \( j! \frac{d^K}{dt^K} \binom{t}{j} / d t^K \) for \( j = 2m, \ t = m^{-\frac{1}{2}} \).

Let \( \sigma = t - m + \frac{1}{2} \),

\[
\frac{d^K}{dt^K} \binom{t}{2m} \bigg|_{t=m^{-\frac{1}{2}}} = \frac{1}{(2m)!} \frac{d^K}{d\sigma^K} Q(\sigma) \bigg|_{\sigma=0},
\]

where

\[
Q(\sigma) = \left[ \sigma^2 - \frac{(2m-1)^2}{4} \right] \left[ \sigma^2 - \frac{(2m-3)^2}{4} \right] \ldots \left[ \sigma^2 - \frac{1}{4} \right].
\]
It follows that \( I(k+1) \) coincides with the coefficient of \( \sigma^k \) in the polynomial \( Q(\sigma) \), and

\[
I(2r+1) > 0, \quad I(2r) = 0, \quad r = 0, 1, \ldots, m.
\]

The case of even \( n \) can be considered similarly. Note that, if \( n = 1 \), \( \text{tr} \ H(z) \) has one simple pole at the point \( z = 1 \) with residue \( \frac{1}{2} \); in fact, \( \text{tr} \ H(z) = (\frac{1}{2})^2 \zeta(z; \frac{1}{2}) \), where

\[
\zeta(z; \frac{1}{2}) = \sum_{n \geq 0} \frac{1}{(n + \frac{1}{2})^z}.
\]

If \( n = 2 \), \( \text{tr} \ H(z) \) has one simple pole at the point \( z = 2 \).
REFERENCES


