P-CONVEXITY OF MANIFOLDS WITH BOUNDARY

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Ji-ping Sha

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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We say that the boundary of a connected compact n-dimensional Riemannian manifold M is p-convex (where p is an integer with $1 \leq p \leq n - 1$) if at each point of the boundary the sum of any p principal curvatures, defined with respect to the outward normal, is positive. By generalizing Bochner's formula to differential forms with proper boundary conditions, we show that if the curvature operator of M is non-negative and the boundary is p-convex then $H_1(M,\partial M;\mathbb{R}) = \cdots = H_{n-p}(M,\partial M;\mathbb{R}) = 0$. By applying Morse theory to a modified distance function, a stronger theorem is proved: If M carries a Riemannian metric with non-negative sectional curvature and

p-convex boundary, then M has the homotopy type of a CW-complex of dimension $\leq p - 1$. We also prove the converse of this result, i.e. if a compact n-dimensional manifold with boundary is a handlebody only with handles of dimension $\leq p - 1$, then it supports a Riemannian metric of positive sectional curvature and p-convex boundary. Taken together, these theorems give a complete characterization of such manifolds.

To My Parents

To Jinping

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0. Introduction

The purpose of this dissertation, roughly speaking, is to study the global topological structure of a Riemannian manifold with boundary which has positive sectional curvature in the interior and proper geometric conditions on the boundary. Our main interest is in the effect of the geometric behavior of the boundary.

In [G], M. Gromov proved a theorem which states that on an open n-manifold $M(n>1)$ there exist Riemannian metrics both with positive and with negative curvature. An immediate consequence then is that an n-manifold $M(n>1)$ with $\partial M \neq \phi$ (i.e., each connected component of M has non-empty boundary) always carries a Riemannian metric with positive sectional curvature. However, in this theorem there are no assumptions on the behavior of the metric at the boundary. If we impose some geometric conditions on ∂M, then the situation is certainly different. For instances:

B. Lawson showed in [L] that if M carries a Riemannian metric of positive Ricci curvature with positive mean curvature at ∂M, then $\pi_1(M,\partial M) = 0$ and in particular, ∂M is connected.

It follows from the work of J. Cheeger, D. Gromoll and W. Meyer in [CG] and [GM] that if M carries Riemannian metric of positive sectional curvature with convex ∂M, then M is diffeomorphic to the standard n-disc.

We shall generalize these results. For each integer p, $1 \leq p \leq (n-1)$, we shall formulate a notion of "p-convexity" at the boundary, and show that this condition together with non-negative sectional curvature in the interior implies that the manifold is at most (p-1)-dimensional, i.e., is homotopy equivalent to a complex of dimension $\leq (p-1)$. We shall also show that any manifold with this property can carry such a metric.

Let us now be more specific. Let X be an (n-1)-dimensional (normally oriented) hypersurface in a Riemannian manifold Ω and let

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$$

be its principal curvature functions.

Definition (0.1). X is called p-convex if

$$\lambda_1 + \lambda_2 + \ldots + \lambda_p > 0$$

at each point of X. (X is called weakly p-convex if $\lambda_1 + \lambda_2 + \cdots + \lambda_p \geq 0$).

Note in particular that "1-convexity" is the usual notion of local convexity; "(n-1)-convexity" means that X has positive mean curvature. It is also obvious that p-convexity implies (p+1)-convexity.

Let M be an n-dimensional compact connected Riemannian manifold with non-empty boundary ∂M. ∂M can be considered as a normally oriented hypersurface in M.

Definition (0.2). M is said to be with p-convex (or weakly p-convex) boundary if ∂M is p-convex (or weakly p-convex respectively) with respect to the outward normal vector.

In Section 2, we shall prove the following.

Theorem (2.0.1). If M is with non-negative sectional curvature in the interior and with p-convex boundary, then M has the homotopy type of a CW-complex of dimension $\leq (p-1)$.

It should be mentioned that there has been a concept of p-convexity in both real and complex geometry for a long time. It is generally defined by the requirement that at least (n-p) eigenvalues of the second fundamental form (or the Levi form in the complex case) be positive. However, with this definition of p-convexity, Theorem (2.0.1) is false. A counter-example was pointed out by M. Gromov as follows.

Suppose S_1 and S_2 are two non-intersecting great circles in the standard 3-sphere S^3. Let T_1 and T_2 be the tubular δ-neighborhoods of S_1 and S_2 respectively. If $\delta > 0$ is small enough, $M \equiv S^3 \setminus (T_1 \cup T_2)$ is a compact connected 3-manifold with boundary and, at each point of the boundary, one of the two principal curvatures is positive. However, ∂M here is certainly not connected. (Note that in this example, the sum of the two principal curvature is negative.)

Our stronger notion of p-convexity was first suggested by a Bochner-type vanishing theorem that we proved for harmonic forms on Riemannian manifolds with boundary. This is discussed in Section 1. The main result there is the following.

Theorem (1.3.2). *Suppose the curvature operator of* M *is nonnegative and positive somewhere. Suppose furthermore* ∂M *is weakly p-convex. Then* $H^k(M,\partial M;\mathbb{R}) = 0$ *for* $k = 0,1,\cdots,(n-p)$.

In [LM_1], B. Lawson and M.-L. Michelsohn showed the following. *Suppose* X *has positive mean curvature and let* X' *be a hypersurface obtained from* X *by attaching an ambient k-handle to the positive side of* X. *If the codimension* (n-k) *of the handle is* ≥2, *then* X' *can be constructed also to have positive mean curvature*. Their method is generalized in Section 3. We shall prove the following.

Theorem (3.0.1). *Suppose* X *is a (normally oriented) p-convex hypersurface in a Riemannian manifold* Ω, *and let* X' *be a hypersurface obtained from* X *by attaching a k-handle to the positive side of* X. *If* $k \leq p - 1$, *then* X' *can be constructed also to be p-convex*.

Applying this together with the theorem of Gromov in [G], a striking consequence then is the following.

Theorem (3.0.3). *Let* M *be a compact connected manifold with* non-empty boundary and suppose M is a handlebody with handles only of dimension $\leq(p-1)$. Then M supports a Riemannian metric with positive sectional curvature and p-convex boundary.

Hence the Theorem (2.0.1) is "sharp" and it leads to a complete characterization of such manifolds.

Remark. J. D. Moore and T. Schulte recently proved a special case of the Theorem (2.0.1) for $p = (n-2)$ ([MS]). Very recently, we learned that the Theorem (2.0.1) has also been proved independently by H. Wu ([Wu]).

1. Harmonic forms on Riemannian manifolds with boundary and Bochner-type vanishing theorem.

Among the great achievements in geometry and topology are the Hodge Decomposition Theorem and Bochner Formula. We summarize very briefly those facts we need (we refer to [F], [W], [Gi], [LM_2] for example, for general references) in this section and then use Bochner's method for harmonic forms on Riemannian manifolds with boundary to give some vanishing theorems.

1.1 Hodge's decomposition theorem on Riemannian manifolds with boundary.

Let M be a n-dimensional compact Riemannian manifold and Δ be the Hodge Laplacian on

$$\Gamma(\Lambda^*(M)) \equiv \Gamma(\Lambda^0(M)) \oplus \Gamma(\Lambda^1(M)) \oplus \cdots \oplus \Gamma(\Lambda^n(M))$$

where $\Gamma(\Lambda^k(M))$ is the space of real k-forms on M.

If $\partial M = \phi$, Hodge Theorem tells us that the space of harmonic k-forms

$$\mathbb{H}^k \equiv \{\omega \in \Gamma(\Lambda^k(M)) : \Delta\omega = 0\}$$

is isomorphic to the real cohomology space $H^k(M;\mathbb{R})$.

In the case $\partial M \neq \phi$, some elliptic boundary conditions are needed. Let $i : \partial M \to M$ be the inclusion. As usual, we denote by $*$ the Hodge "star-operator" and $\delta \equiv \pm * d *$ the adjoint of the exterior derivative d. There are two sets of standard elliptic boundary conditions for Δ, i.e., the absolute and relative boundary conditions. They are defined by

$$B_a(\omega) \equiv (i^* * \omega, i^* * d\omega) \text{ and } B_r(\omega) \equiv (i^*\omega, i^*\delta\omega).$$

Let

$$E_a^k(M) \equiv \{\omega \varepsilon \Gamma(\Lambda^k(M)) : B_a(\omega) = 0\} \text{ and } E_r^k(M) = \{\omega \varepsilon \Gamma(\Lambda^k(M)) : B_r(\omega) = 0\}.$$

We denote by the following the two specific sets of harmonic forms on M.

$$\mathbb{H}_a^k(M) \equiv \{\omega \varepsilon E_a^k(M) : \Delta\omega = 0\} \text{ and } \mathbb{H}_r^k(M) \equiv \{\omega \varepsilon E_r^k(M) : \Delta\omega = 0\}.$$

Then by the generalized Hodge Theorem

$$\mathbb{H}_a^k(M) \cong H^k(M;\partial M;\mathbb{R}) \text{ and } \mathbb{H}_r^k(M) \cong H^k(M,\partial M;\mathbb{R})$$

1.2. Bochner's formula.

One of the major results in Riemannian geometry is the following formula discovered by S. Bochner.

$$\Delta = \nabla^* \nabla + R.$$

For the convenience of calculation, we use the canonical isomorphism $\Lambda^*(M) \cong C\ell(M)$. Then $\nabla = D^2$, where $D = d+\delta$ is the "Dirac operator" on $\Gamma(C\ell(M))$. $\nabla^*\nabla$ is the connection Laplacian defined by

$$\nabla^*\nabla \equiv -\sum_{i=1}^{n} (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})$$

using local orthonormal vector fields $e_1,\cdots,e_n$.

$R \in \Gamma(\mathrm{Hom}(C\ell(M),C\ell(M)))$ is a symmetric bundle endomorphism. When acting on 1-forms, it is equal to the Ricci tensor. While acting on higher degree forms, it has the property that $R \geq 0$ (or > 0) at any point of M where the curvature operator is ≥ 0 (or > 0 respectively).

1.3. Vanishing theorems.

Suppose M is a n-dimensional Riemannian manifold with boundary $\partial M \neq \phi$. Let

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$$

be the principal curvature functions of ∂M with respect to the outward normal. We first prove the following.

Lemma (1.3.1). *Suppose there is a fixed* $p > 0$ $(1 \leq p \leq n-1)$, *such that*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_p \geq 0$$

everywhere on ∂M. *Then for any* $\varphi \in \mathbb{H}_a^k(M)$ *where* $0 \leq k \leq n-p$,

$$\int_M \langle \nabla^* \nabla\varphi, \varphi\rangle \geq 0 .$$

Furthermore, $\int_M \langle \nabla^* \nabla\varphi, \varphi\rangle = 0$ only if φ is parallel.

Proof. By a direct calculation using Stoke's theorem,

$$\int_M \langle \nabla^* \nabla\varphi, \varphi\rangle = \int_M \langle \nabla\varphi, \nabla\varphi\rangle + \int_{\partial M} \langle \nabla_N \varphi, \varphi\rangle$$

where N is the inner unit normal of ∂M. Hence, it will suffice to show that $\langle \nabla_N \varphi, \varphi\rangle \geq 0$ on ∂M.

Observe that

$$0 = \int_M \langle \Delta\varphi, \varphi\rangle = \int_M d\delta\varphi \wedge *\varphi + \int_M \delta d\varphi \wedge *\varphi =$$

$$= \int_M d(\delta\varphi \wedge *\varphi) - (-1)^{k-1} \int_M \delta\varphi \wedge d*\varphi + \int_M \varphi \wedge *\delta d\varphi =$$

$$= \int_M d(\delta\varphi \wedge *\varphi) + \int_M \delta\varphi \wedge *\delta\varphi + (-1)^{k-1} \int_M \varphi \wedge d*d\varphi =$$

$$= \int_M d(\delta\varphi \wedge *\varphi) + \int_M \delta\varphi \wedge *\delta\varphi + \int_M d\varphi \wedge *d\varphi - \int_M d(\varphi \wedge *d\varphi) =$$

$$= (-1)^{n-1} \int_{\partial M} i^*\delta\varphi \wedge i^* *\varphi + \int_M \|\delta\varphi\|^2 + \int_M \|d\varphi\|^2 - (-1)^{n-1} \int_{\partial M} i^*\varphi \wedge i^* *d\varphi =$$

$$= \int_M \|\delta\varphi\|^2 + \int_M \|d\varphi\|^2 .$$

Here we have used the condition that $i^*\varphi = i^*\delta\varphi = 0$. We conclude that $d\varphi = \delta\varphi = 0$, and hence $D\varphi = 0$.

Fix $x \in \partial M$, and choose a smooth orthonormal tangent frame field $e_1, \cdots, e_n$ in a neighborhood of x such that $e_n = N$ on the boundary, and such that at the point x

(i) $\nabla_{e_j} N = -\lambda_j e_j$ for $j = 1, \cdots, (n-1)$;

(ii) $(\nabla_{e_j} e_k)^T = 0$ (and therefore $\nabla_{e_j} e_k = \delta_{j_k} \lambda_k N$)

for $j,k = 1,\ldots,(n-1)$;

(iii) and $\nabla_N e_i = 0$ and $i = 1,\cdots,n$.

Suppose that φ has the form

$$\varphi = \sum_{i_1<\cdots<i_k<n} f_I e_{i_1} \cdots e_{i_k} + \sum_{j_1<\cdots<j_{k-1}<n} g_J e_{j_1} \cdots e_{j_{k-1}} e_n .$$

Then the condition $i^*\varphi = 0$ implies that $f_I\big|_{\partial M} = 0$ and hence that $e_i f_I\big|_{\partial M} = 0$ for $i = 1,\cdots,(n-1)$ and all $i_1<\cdots<i_k<n$.

Therefore, at x we find that

$$0 = D\varphi = \sum_{i=1}^{n} e_i \nabla_{e_i} \Big(\sum_{i_1<\cdots<i_k<n} f_I e_{i_1} \cdots e_k + \sum_{j_1<\cdots<j_{k-1}<n} g_J e_{j_1} \cdots e_{j_{k-1}} e_n \Big)$$

$$= \sum_{i=1}^{n} e_i \{ \sum_I [(e_i f_I) e_{i_1} \cdots e_{i_k} + f_I \nabla_{e_i} (e_{i_1} \cdots e_{i_k})]$$

$$+ \nabla_{e_i} (\sum_J g_J e_{j_1} \cdots e_{j_{k-1}}) e_n + (\sum_J g_J e_{j_1} \cdots e_{j_{k-1}}) \nabla_{e_i} e_n \}$$

$$= (-1)^k \sum_I (e_n f_I) e_{i_1} \cdots e_{i_k} e_n + \sum_J \sum_{i=1}^{n-1} (e_i g_J) e_i e_{j_1} \cdots e_{j_{k-1}} e_n$$

$$+ (-1)^k \sum_J (e_n g_J) e_{j_1} \cdots e_{j_{k-1}} + \sum_J g_J [\sum_{i=1}^{n} e_i \nabla_{e_i} (e_{j_1} \cdots e_{j_{k-1}}) e_n]$$

$$- \sum_J g_J (\sum_{i=1}^{n-1} \lambda_i e_i e_{j_1} \cdots e_{j_{k-1}} e_i). \qquad (*)$$

Note that

$$\sum_{i=1}^{n} e_i \nabla_{e_i}(e_{j_1}\cdots e_{j_{k-1}})e_n = (-1)^{k-1}(\lambda_{j_1}+\cdots+\lambda_{j_{k-1}})e_{j_1}\cdots e_{j_{k-1}};$$

$$\sum_{i=1}^{n-1} \lambda_i e_i e_{j_1}\cdots e_{j_{k-1}} e_i = (-1)^{k-1}(\lambda_{j_1}+\cdots+\lambda_{j_{k-1}})e_{j_1}\cdots e_{j_{k-1}}$$

$$+ (-1)^k(\sum_{\substack{i\notin J\\ i\neq n}} \lambda_i)e_{j_1}\cdots e_{j_{k-1}}.$$

Then we see that

$$0 = \text{the coefficient of } e_{j_1}\cdots e_{j_{k-1}} \text{ in } (*)$$

$$= (-1)^k[Ng_J-(\sum_{\substack{i\notin J\\ i\neq n}} \lambda_i)g_J].$$

Consequently we find that

$$\langle\nabla_N\varphi,\varphi\rangle = \langle\sum_I(Nf_I)e_{i_1}\cdots e_{i_k} + \sum_J(Ng_J)e_{j_1}\cdots e_{j_{k-1}}e_n,$$

$$\sum_I f_I e_{i_1}\cdots e_{i_k} + \sum_J g_J e_{j_1}\cdots e_{j_{k-1}}e_n\rangle = \sum_J(Ng_J)g_J$$

$$= \sum_J(\sum_{\substack{i\notin J\\ i\neq n}} \lambda_i)g_J^2 \geq 0$$

because, by the assumption on k, each sum $\sum_{\substack{i\notin J\\ i\neq n}} \lambda_i$ takes over at least p λ_i's.

Furthermore, if $\mathring{\int}_M \langle\nabla^*\nabla\varphi,\varphi\rangle = 0$, then $\int_M \langle\nabla\varphi,\nabla\varphi\rangle = \|\nabla\varphi\|^2 = 0$ and therefore, φ is parallel. #

Taking this lemma together with the material in 1.1 and 1.2, the following theorems follows immediately from Bochner's vanishing argument.

Theorem (1.3.2). Suppose M is an n-dimensional compact Riemannian manifold with weakly p-convex boundary ∂M and non-negative curvature operator which is positive somewhere. Then $\mathbb{H}_r^k(M) = 0$ and hence $H^k(M,\partial M;\mathbb{R}) = 0$ for $0 \le k \le (n-p)$. #

We also have a similar version of the theorem in [L].

Theorem (1.3.3). Suppose the compact manifold M supports a Riemannian metric with non-negative Ricci curvature which is positive somewhere and with non-negative mean curvature on the boundary ∂M. Then $H^1(M,\partial M;\mathbb{R}) = 0$. Consequently, ∂M is connected if M is connected. #

The corresponding vanishing theorems for $H^*(M;\mathbb{R})$ follows from the Poincaré duality. They can also be derived from a similar lemma by using the absolute boundary conditions.

Theorem (1.3.2)'. Let M be as in(1.3.2). Then $\mathbb{H}_a^k(M) = 0$ and hence $\mathbb{H}^k(M;\mathbb{R}) = 0$ for $p \le k \le n$. #

2. Homotopy type of p-convex Riemannian manifolds

Let M be an n-dimensional compact connected manifold with $\partial M \neq \phi$. We shall prove the following in this section.

Theorem (2.0.1). If M carries a Riemannian metric with non-negative sectional curvature and p-convex boundary, then M has the homotopy type of a CW-complex of dimension $\leq(p-1)$.

We use Morse theory to prove this theorem. The principal idea is to show the "p-convexity" of the distance function to the boundary and then to construct a delicate smoothing.

2.1. An algebraic lemma.

Let V be a real vector space of dimension d with inner product $<,>$, and let A be a symmetric linear transformation from V into itself. We call the quadratic form $<A\cdot,\cdot>$ p-positive if for all orthonormal sets of p vectors $\{X_1,\cdots,X_p\} \subset V$, we have

$$<AX_1,X_1> + \cdots + <AX_p,X_p> > 0.$$

Lemma (2.1.1). The following are equivalent.

i) $<A\cdot,\cdot>$ is p-positive;

ii) The sum of any p eigenvalues of A is positive.

Proof. That i) $\Rightarrow$ ii) is obvious.

To prove that ii) $\Rightarrow$ i), let $\lambda_1 \geq \cdots \geq \lambda_d$ be the eigenvalues of A, with corresponding orthonormal eigenvectors $Y_1,\cdots,Y_d$.

Define q by the condition that $\lambda_q > 0 \geq \lambda_{q+1}$, and note that $q > d-p$. Now suppose that $X_i = \sum_{j=1}^{d} a_{ij}Y_j$, $i = 1,\cdots,p$ are p arbitrary orthogonal unit vectors. Then we have that

$$\sum_{i=1}^{p} \langle AX_i, X_i\rangle = \sum_{i=1}^{p} \sum_{j=1}^{d} \lambda_j a_{ij}^2 = \sum_{j=1}^{d} \lambda_j \left(\sum_{i=1}^{p} a_{ij}^2\right)$$

$$\geq \lambda_{d-p+1}\left(\sum_{j=1}^{d-p+1} \sum_{i=1}^{p} a_{ij}^2\right) + \lambda_{d-p+2} \sum_{i=1}^{p} a_{i,d-p+2}^2 + \cdots + \lambda_q \sum_{i=1}^{p} a_{iq}^2$$

$$+ \lambda_{q+1} + \cdots + \lambda_d \geq \lambda_{d-p+1}\Big[(d-p+1)-(d-p) + \sum_{i=p+1}^{d} a_{i,d-p+2}$$

$$+ \cdots + \sum_{i=p+1}^{d} a_{iq}^2\Big] + \lambda_{d-p+2} \sum_{i=1}^{p} a_{i,d-p+2}^2 + \cdots + \lambda_q \sum_{i=1}^{p} a_{iq}^2 + \cdots + \lambda_{q+1}$$

$$+ \cdots + \lambda_d \geq \lambda_{d-p+1} + \lambda_{d-p+2} + \cdots + \lambda_q + \lambda_{q+1} + \cdots + \lambda_d > 0.$$

Note that we have extended (a_{ij}) to a d×d orthogonal matrix. #

From this lemma, it is easy to see the following.

<u>Corollary</u> (2.1.2). <u>There exist</u> $\delta_0 > 0$ <u>and</u> $\lambda > 0$ <u>such that for all</u> $y \in \partial M$ <u>and for all sets of</u> p <u>unit tangent vectors</u> $\{Y_1,\cdots,Y_p\} \subset T_y\partial M$ <u>with</u> $|\langle Y_i, Y_j\rangle| < \delta_0$ <u>for</u> $i \neq j$, <u>one has that</u>

$$a_1 S(Y_1,Y_1) + \cdots + a_p S(Y_p,Y_p) > \lambda$$

<u>where</u> S <u>is the second fundamental form of</u> ∂M <u>in</u> M, <u>and where</u> $a_1,\cdots,a_p$ <u>are arbitrary real numbers with</u> $|a_i - 1| < \delta_0$ <u>for all</u> i.

2.2. The modified boundary distance function.

Let $c_1 > 0$ and $c_2 > 0$ be two generic constants whose value will be specified later on. Set

$$\tau(t) = c_1 e^{c_2 t}$$

and consider the function

$$\tilde{F} = \tau \circ F : M \to \mathbb{R}$$

where $F(x) \equiv -\mathrm{dist}(x, \partial M)$ is the negative of the distance to the boundary in M.

We call a function f on M uniformly p-convex if for any given $C \subset\subset \overset{\circ}{M}$, there exist $\delta > 0$ and $\eta > 0$ such that for all $x \in C$ and for all orthonormal sets $\{X_1, \cdots, X_p\} \subset T_x M$, we have that

$$\sum_{i=1}^{p} [f(\exp_x(-tX_i)) + f(\exp_x(tX_i)) - 2f(x)] \geq \eta t^2$$

for all $t \in \mathbb{R}$ with $|t| < \delta$.

Our main purpose in this section is to prove the following.

Theorem (2.2.1). $\tilde{F}$ is uniformly p-convex.

Proof. Suppose $x \in \overset{\circ}{M}$. Choose a minimal geodesic $c_0(s) = \exp_x sX_0$, $0 \leq s \leq s_0 = \mathrm{dist}(x, \partial M)$, from x to ∂M. Let $X_1, \cdots, X_p \in T_x M$ be orthogonal unit vectors. Suppose the angle between X_0 and the linear subspace spanned by $\{X_1, \cdots, X_p\}$ is α. We have

$0 \leq \alpha \leq \frac{\pi}{2}$, and there are three cases to consider.

Case I. Suppose $\alpha = \frac{\pi}{2}$. Set

$$c_t^i(s) = \exp_{c_o(s)} t\overline{X}_i$$

and $\overline{X}_i$ is the vector field generated by parallel translation of X_i along c_o.

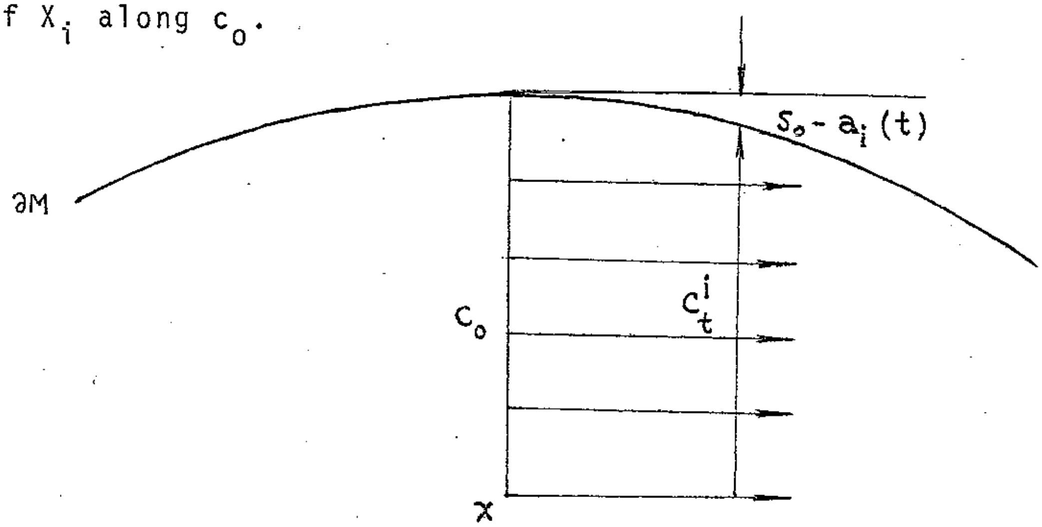

Fig. 1

It is not hard to see that for some $\delta_1 > 0$ there exist smooth functions $a_i(t)$, $1 \leq i \leq p$, with the property that the curves c_t^i are well defined for $0 \leq s \leq a_i(t)$ and have $c_t^i(a_i(t)) \in \partial M$ for each i. A calculation in Fermi coordinates shows that $a_i'(0) = 0$ and $a_i''(0) = -S(X_i, X_i)$. We then have that,

$$L(c_t^i) = \int_0^{a_i(t)} \|\dot{c}_t^i\| ds$$

and $$\frac{d}{dt}L(c_t^i) = \|\dot{c}_t^i(a_i(t)\| a_i'(t) + \int_0^{a_i(t)} \frac{\langle \nabla_{\dot{c}_t} \overline{X}_i, \dot{c}_t^i \rangle}{\|\dot{c}_t^i\|} ds,$$

from which we conclude that

$$\frac{d}{dt}L(c_t^i)\Big|_{t=o} = 0$$

and $$\frac{d^2}{dt^2}L(c_t^i)\Big|_{t=0} = a_i''(0) - \int_0^{s_0} \langle R(\dot{c}_0,\overline{X}_i)\overline{X}_i, \dot{c}_0 \rangle ds \le -S(\overline{X}_i,\overline{X}_i).$$

Corollary (2.1.2) implies that

$$\sum_{i=1}^{p} \left(-\frac{d^2}{dt^2}L(c_t^i)\Big|_{t=0}\right) > \lambda.$$

Choose $c_1 > \frac{1}{c_2} e^{c_2 d}$, where d is the diameter of M.

Then we have $\tau'(-s_0) > 1$, and therefore,

$$\frac{d^2}{dt^2}\left(\sum_{i=1}^{p} \tau(-L(c_t^i))\right)\Big|_{t=0} = \tau'(-s_0) \sum_{i=1}^{p} \left(-\frac{d^2}{dt^2}L(c_t^i)\Big|_{t=0}\right) > \lambda.$$

It follows that there exists $\delta_2 > 0$ such that for $|t| < \delta_2$,

$$\sum_{i=1}^{p} [\tilde{F}(\exp_x(-tX_i) + \tilde{F}(\exp_x tX_i) - 2\tilde{F}(x)]$$

$$\ge \sum_{i=1}^{p} [\tau(-L(c_{-t}^i)) + \tau(-L(c_t^i)) - 2\tau(-s_0)] > \frac{\lambda}{2}t^2. \qquad (1)$$

<u>Case II</u>. Suppose $\frac{\pi}{2} > \alpha \ge \alpha_0$ where α_0 is a constant such that $|\sin^2\alpha_0 - 1| < \delta_0$ and $|\langle X_1', X_j' \rangle| < \delta_0$ for $i \ne j$, where X_i' denotes the normalized projection of X_i onto the orthogonal complement $X_0^{\perp}$ of X_0.

For simplicity, we momentarily omit the subscripts i ($1 \leq i \leq p$).

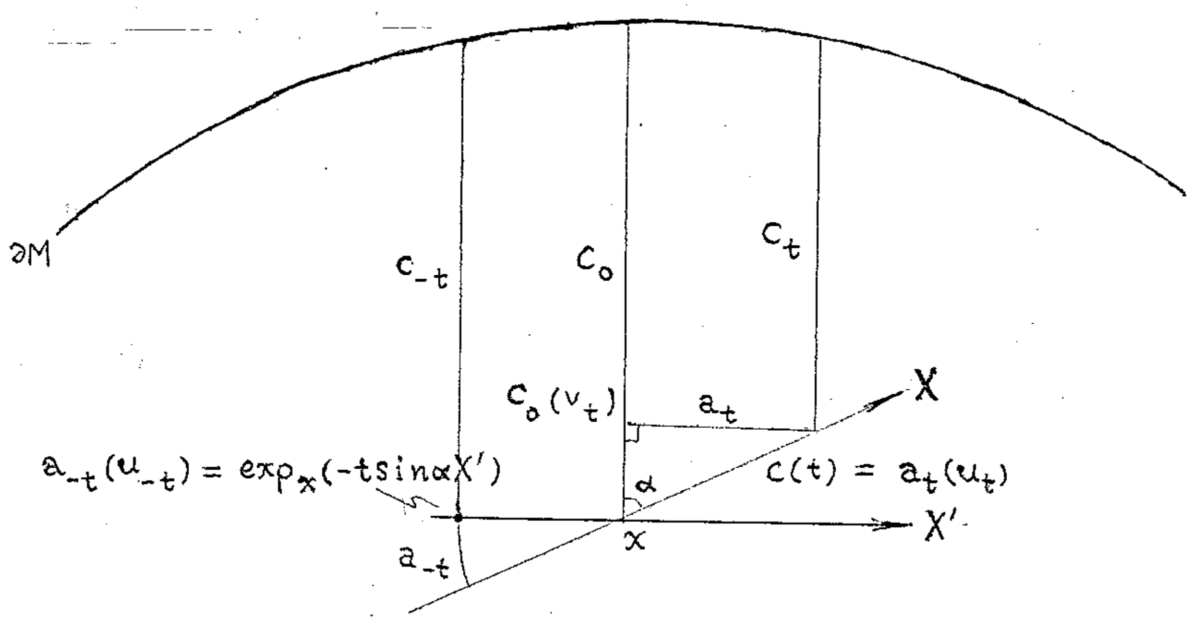

Fig. 2

For each sufficiently small $t > 0$, let a_t be the arc-length parametrized minimal geodesic segment from the point $c(t) = \exp_X tX$ to the curve c_o. (See Fig. 2.) Suppose $a_t(0) = c_o(v_t)$ and $c(t) = a_t(u_t)$. Let

$$c_t(s) = \exp_{c_o(s)} u_t \bar{\dot{a}}_t(0), \quad s \geq v_t$$

be a curve from $c(t)$ to ∂M, where $\bar{\dot{a}}_t(0)$ is the parallel translation of $\dot{a}_t(0)$ along c_o. By the argument given in Case I, there exists $\delta_1 > 0$ so that for $t < \delta_1$,

$$-L(c_t) \geq -s_o + v_t + \frac{1}{2}s(\bar{\dot{a}}_t(0), \bar{\dot{a}}_t(0))u_t^2.$$

By Rauch's comparison theorem, we have that

$$u_t^2 \leq v_t^2 + t^2 - 2tv_t \cos$$

$$t^2 \leq u_t^2 + v_t^2.$$

Consequently, we find that

$$2v_t t \cos\alpha \leq 2v_t^2 \quad \text{i.e.,} \quad v_t \geq t \cos\alpha.$$

Furthermore, $\frac{u_t}{t \sin\alpha} \to 1$ as $t \to 0^+$, and $\sin\alpha_o \leq \sin\alpha \leq 1$. Note also that $\bar{\dot{a}}_t(0) \to \bar{X}'$ as $t \to 0^+$. Combining these facts, we get a $\delta_2 > 0$, such that for $t < \delta_2$, we have

$$-L(c_t) \geq -s_o + t\cos\alpha + \frac{1}{2}aS(\bar{X}', \bar{X}')t^2$$

where a is some real number with $|a-1| < \delta_o$.

From this we conclude, after noting that $\frac{d^+}{dt}(-L(c_t))\Big|_{t=0} = \cos\alpha$, that

$$\frac{d^{2+}}{dt^2}(-L(c_t))\Big|_{t=0} \geq a\cdot S(\overline{X}',\overline{X}').$$

On the other hand, let a_{-t} be a minimal geodesic segment from the point $c(-t) = \exp_s(-tX)$ to the point $a_{-t}(u_{-t}) = \exp_x(-t\sin\alpha X')$. Let

$$c_{-t}(s) = \begin{cases} a_{-t}(s), & 0 \leq s \leq u_{-t} \\ \exp_{c_0}(s-u_{-t})(-t\sin\alpha\overline{X}'). & s \geq u_{-t} . \end{cases}$$

By the same use of Case I arguments and Rauch's comparison theorem, we have that

$$-L(c_{-t}) \geq -s_0 - t\cos\alpha + \tfrac{1}{2}S(\overline{X}',\overline{X}')t^2\sin^2\alpha.$$

We see that $L(c_{-t})$ is right differentiable and that $\frac{d^+}{dt}(-L(c_{-t}))\Big|_{t=0} = -\cos\alpha$. This implies that

$$\frac{d^{2-}}{dt^2}(-L(c_t))\Big|_{t=0} \geq aS(\overline{X}_i',\overline{X}_i')$$

and therefore that

$$\frac{d^2}{dt^2}\left(\sum_{i=1}^{p}\tau(-L(c_t^i))\right)\Big|_{t=0} \geq \sum_{i=1}^{p} a_i S(\overline{X}_i',\overline{X}_i') > \lambda.$$

The same final arguments as those given in Case I complete the proof of Case II.

<u>Case III</u>. Suppose $0 \leq \alpha \leq \alpha_0$. Let α_i denote the angle between c_0 and X_i. If $\alpha_i \geq \alpha_0$ for all i, then the arguments of Case II apply, so we shall assume that $\alpha_1 < \alpha_0$. Set

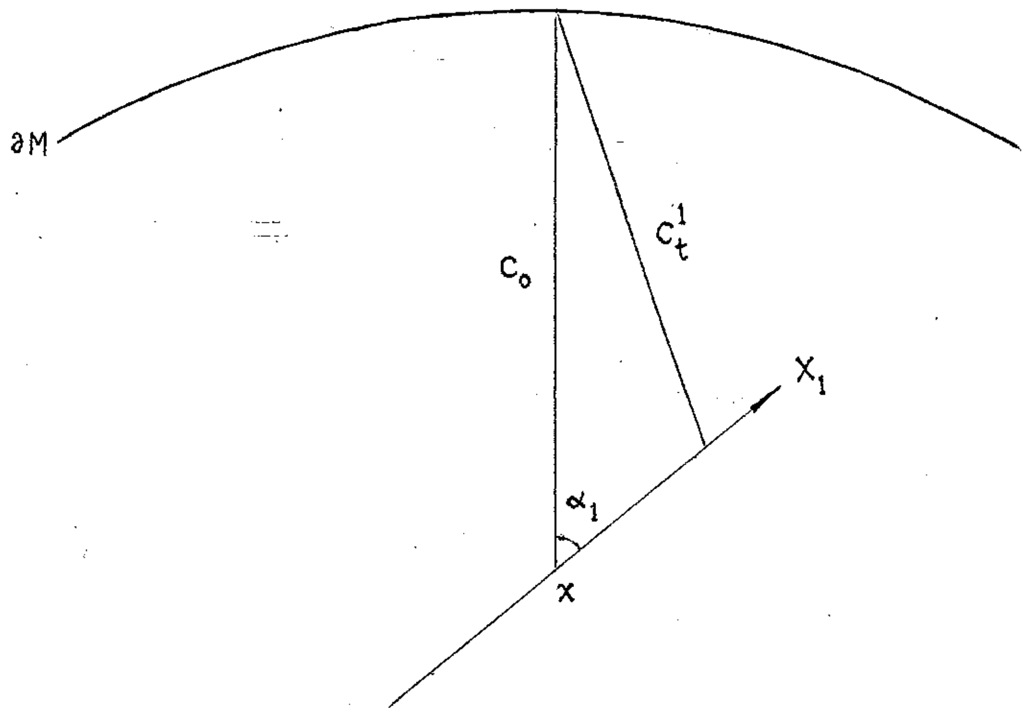

Fig. 3

$$c_t^1(s) = \exp_{c_0(s)} a(s) t\overline{X}_1, \quad 0 \leq s \leq s_0$$

where a(s) is a non-negative smooth function on $0 \leq s \leq s_0$ such that $a(0) = 1$, $a(s_0) = 0$. Then we have

$$\frac{d}{dt} L(c_t^1)\Big|_{t=0} = -\cos\alpha \leq -\cos\alpha_0,$$

and

$$\frac{d^2}{dt^2}\tau(-L(c_t^1))\Big|_{t=0} = c_1c_2^2e^{-c_2s_o}\left[\frac{d(-L(c_t^1))}{dt}\Big|_{t=0}\right]^2$$

$$+ c_1c_2e^{-c_2s_o}\frac{d^2(-L(c_t^1))}{dt^2}\Big|_{t=0}$$

$$\geq c_1c_2^2e^{-c_2s_o}\cos^2\alpha_o - c_1c_2e^{-c_2s_o}K$$

where K is a positive constant depending only on the manifold M.

For the other X_i''s, we clearly have

$$\frac{d^2}{dt^2}\tau(-L(c_t^i))\Big|_{t=0} \geq -c_1c_2e^{-c_2s_o}K$$

where

$$c_t^i = \exp_{c_o(s)}a(s)t\overline{X}_i,\ 0 \leq s \leq s_o.$$

By choosing $c_2 > \frac{(n-1)K}{\cos^2\alpha_o}$, $c_1 > \max\{\frac{1}{c_2}e^{c_2d}, \frac{\lambda}{c_2}e^{c_2d}(c_2\cos^2\alpha_o-(n-1)K)^{-1}\}$

we get

$$\frac{d^2}{dt^2}\left[\sum_{i=1}^{p}\tau(-L(c_t^i))\right]\Big|_{t=0} > \lambda .$$

The same argument as before now gives (1).

To complete the proof of Theorem (2.2.1) we first observe by careful checking that the δ_1,δ_2 in the proof can always be chosen uniformly in small open sets. This implies that δ can be chosen uniformly on compact subsets of $\mathring{M}$; and the argument is complete.

2.3. Smooth approximation of $\tilde{F}$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-negative C^∞ function which has support contained in $[-1,1]$, is constant in $[-\frac{1}{2},\frac{1}{2}]$, and has the property that $\int_{v\in\mathbb{R}^n} \omega(\|v\|)d\mu = 1$, where $\|v\| =$ the duclidean length of v and $d\mu$ is the usual Lebesgue measure on $\mathbb{R}^n$.

There is a collar neighborhood $U \approx \partial M \times [0,t_o]$ of ∂M in M. Choose a C^∞ function $\xi : M \to [0,1]$ such that $\xi \equiv 0$ on $\partial M \times [0,\frac{t_o}{4}]$ and $\xi \equiv 1$ on $\tilde{M} \equiv M - M \times [0,\frac{t_o}{2})$. Define f_ε, $0 < \varepsilon < \frac{t_o}{4}$, by

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{v\varepsilon T_xM} \varphi(\frac{\|v\|}{\varepsilon}) \cdot \tilde{F}(\exp_x \xi(x)v)d\Omega_x$$

where $\| \ \|$ and $d\Omega_x$ are the euclidean metric and volume on T_xM obtained from the Riemannian metric of M.

Lemma (2.3.1). For all sufficiently small $\varepsilon > 0$, the function f_ε is C^∞ on all of M and has the property that $f_\varepsilon(x) = \tilde{F}(x)$ for $x \in \partial M \times [0,\frac{t_o}{4}]$.

Proof. The second statement is obvious for all ε, so we only need to check the differentiability of f_ε at $x \in \tilde{M}$. For sufficiently small $\varepsilon > 0$,

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int \varphi(\frac{\|\theta x^{-1} y\|}{\varepsilon}) \cdot \tilde{F}(y) d_x y$$

where

$$\theta_x(v) = \exp_x v$$

$$d_x y = \theta_x^{-1^*}(d\Omega_x).$$

Standard results about differentiation under the integral sign complete the argument. #

A little modification of the arguments as in [GW] then gives the following.

Proposition (2.3.2). There exists $\eta > 0$ such that for all $x \in \tilde{M}$ and for arbitrary orthonormal vectors $X_1, \cdots, X_p \in T_x M$, one has

$$\sum_{i=1}^{p} \frac{d^2}{dt^2} f_\varepsilon(\exp_x tX_i)\Big|_{t=0} > \eta$$

for all sufficiently small $\varepsilon > 0$.

2.4. Completion of the proof.

It is a well-known fact that the Morse functions form an open dense subset of $C^\infty(M)$ in the C^2 topology. By 2.3, we can get a Morse function $\psi : M \to [0,1]$ with only non-degenerate critical points in $\tilde{M}$, such that $\psi^{-1}(1) = \partial M$ and such that the Hessian H at any critical point is p-positive. This implies that the index of H is $\leq (p-1)$. The Theorem (2.0.1) now follows from standard Morse theory. In fact we conclude that M is a handlebody with handles only of dimension $\leq (p-1)$.

3. Handlebodies and p-convexity

Let X be a hypersurface in a n-dimensional Riemannian manifold Ω. In [LM_1], by a handle-attaching process, Lawson and Michelsohn showed the following. Suppose X has positive mean curvature and let X' be a hypersurface obtained from X by attaching an ambient k-handle to the positive side of X. If the codimension (n-k) of the handle is ≥ 2, then X' can be constructed also to have positive mean curvature. (That is to say that X' is ambiently isotopic to a hypersurface of positive mean curvature.)

Our central result of this section is a generalization of this theorem to the p-convex case. Specifically we shall prove the following.

Theorem (3.0.1). Let X be a (normally oriented) p-convex hypersurface in a Riemannian manifold Ω, and let X' be a hypersurface obtained from X by attaching a k-handle D^k to the positive side of X. If $k \leq p - 1$, then X' can be constructed also to be p-convex.

Arguing as in [LM_1] we get the following.

Corollary (3.0.2). Let X be a compact manifold embedded as the boundary of a domain D in a Riemannian manifold Ω. Orient X with respect to the inward pointing normal vector. If D is diffeomorphic to a handlebody of dimension $\leq p - 1$, then X is ambiently isotopic through mutually disjoint embeddings to a p-convex hypersurface X' in Ω. The new hypersurface X' bounds a domain D' which is diffeomorphic to D.

Applying this together with the fundamental results of Gromov in [G] we then obtain the following result which is a converse to the theorem in Section 2.

Theorem (3.0.3). Let M be a compact connected manifold with non-empty boundary. If M is a handlebody with handles only of dimension $\leq p-1$, then M supports a Riemannian metric with positive sectional curvature and p-convex boundary.

In fact, by the theorem of Gromov the sectional curvature of M can be ε-pinched for any $\varepsilon > 0$. If M is parallelizable, then by immersion-submersion theory (cf. [Hi]) there exists an immersion $M \hookrightarrow S^n(1)$ where $n = \dim M$. By pulling back the constant curvature metric from $S^n(1)$ and proceeding as in Theorem (3.0.3), we have the following.

Theorem (3.0.4). Let M be as in Theorem (3.0.3). If M is parallelizable and is a handlebody with handles only of dimension $\leq p-1$, then M supports a Riemannian metric with constant sectional curvature 1 and p-convex boundary.

The remainder of the section is devoted to proving Theorem (3.0.1). Since our basic set-up here closely follows Lawson and Michelsohn [LM_1], our presentation will be brief. The basic picture is shown in Fig. 4.

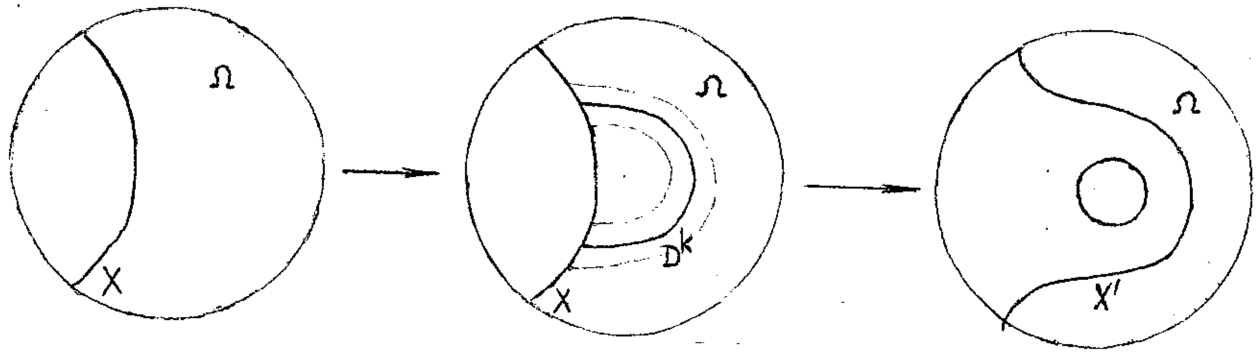

Fig. 4

3.1. The basic set-up.

Assume Ω is connected. Let X be as in Theorem 1. Positive mean curvature (implied by p-convexity) implies a well defined normal direction to X; i.e., we have an embedding of $X \times (-1,1)$ in Ω with the image of $X \times 0$ identified to X. Let X^+ be the union of components of $\Omega \setminus X$ containing $X \times (0,1)$, and X^- be the union of components of $\Omega \setminus X$ containing $X \times (-1,0)$.

Let D^k be a k-dimensional disk orthogonally attached to X in X^+. Set, for $x \in \Omega$,

$$s(x) \equiv \mathrm{dist}(x,X)$$

$$r(x) \equiv \mathrm{dist}(x,D^k).$$

Then there exists a neighborhood Ω_1 of X in Ω such that s is smooth in $\Omega_1' \equiv \Omega_1 \setminus X^-$ and $\|\nabla s\| \equiv 1$. Similarly, there exists a neighborhood Ω_2 of D^k such that r is smooth in

$\Omega_2' \equiv \Omega_2 \setminus (X^- \cup D^k)$ and $\|\nabla r\| \equiv 1$, then $r^{-1}(r_o) \cap \Omega_2'$ is a hypersurface in Ω_2' for any sufficiently small $r_o > 0$.

Hence, the map

$$(r,s) : \Omega_1' \cap \Omega_2' \to \mathbb{R}^2$$

is a smooth submersion. Our idea is to construct a regular curve γ which is essentially the graph of some function $s = f(r)$ in $\mathbb{R}^2$, so that the hypersurface $S_\gamma \equiv (r,s)^{-1}(\gamma)$ joins $r^{-1}(\varepsilon_o)$ to X smoothly for some $\varepsilon_o > 0$, and the whole new hypersurface obtained will still be p-convex.

Recall that the second fundamental form of the level hypersurface of a function is closely related to its Hessian form. We summarize this fact in the following.

Lemma (3.1.1). *Let u be a smooth function on a domain of Ω. Then at every point the 2-form $\nabla^2 u$ defined by*

$$\nabla^2 u(\cdot,\cdot) = \mathrm{Hess}_u(\cdot,\cdot) = \langle\nabla\cdot(\nabla u),\cdot\rangle$$

is symmetric. Furthermore, if $\|\nabla u\| \equiv 1$, then ∇u lies in the null space of $\nabla^2 u$, and when restricted to $\nabla u^\perp$, $\nabla^2 u$ is the second fundamental form of the level hypersurface of u with respect to ∇u.

Proof. See [LM_1]. #

Suppose u is a function as in Lemma (3.1.1). Let

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

be the eigenvalues of $\nabla^2 u$. We denote by $\sigma_u(m)$ the sum

$$\lambda_1 + \cdots + \lambda_m$$

for $m = 1, \cdots, n$.

Remark. Note that by Lemma (3.1.1), ∇u is an eigenvector of $\nabla^2 u$, the corresponding eigenvalue is 0. The other $(n-1)$ eigenvalues are the principal curvatures of the level hypersurface of u. We then clearly have that the level hypersurface is p-convex if and only if $\sigma_u(p+1)$ is positive.

Lemma (3.1.2). (i) We can choose Ω_1 such that there exists a constant $\delta > 0$ for which

$$\sigma_s(p+1) > \delta$$

in Ω_1. (Here δ could be replaced by a smooth positive function.)

(ii) We can choose Ω_2 such that

$$\sigma_r(p+1) > \frac{c}{r}$$

in $\Omega_2 \setminus (X^- \cup D^k)$, where $c > 0$ is a constant.

Proof. (i) is from the p-convexity of X.

(ii) is by a calculation in Fermi coordinates and the fact that $k \leq p - 1$ as follows.

Choose locally smooth orthonormal vector fields $e_1, \cdots, e_n$ along D^k such that $e_1, \cdots, e_k$ are tangent to D^k and that

$e_{k+1},\cdots,e_n$ are normal to D^k. Then for $\xi \in D^k$, $(x_1,\cdots,x_{n-k}) \in \mathbb{R}^{n-k}$ with $x_1^2+\cdots+x_{n-k}^2$ small, the map

$$(\xi,(x_1,\cdots,x_{n-k})) \mapsto \exp_\xi(x_1 e_{k+1}+\cdots+x_{n+k}e_n)$$

gives a local coordinate in some open set $W \subset \Omega_2$. Extend $e_1,\cdots,e_n$ to smooth vector fields $\tilde{e}_1,\cdots,\tilde{e}_n$ on W, where each $\tilde{e}_i$ is obtained by parallel translation of e_i along the geodesic

$$\alpha(t) = \exp_\xi[t(x_1 e_{k+1}+\cdots+x_{n-k}e_n)], \; 0 \le t \le 1.$$

On W, it is clear that

$$r(\xi,(x_1,\cdots,x_{n-k})) = \sqrt{x_2^2+\cdots+x_{n-k}^2}$$

and that

$$\nabla r = \frac{1}{r}(x_1\tilde{e}_{k+1}+\cdots+x_{n-k}\tilde{e}_n).$$

If the metric were euclidean, i.e., if all the $\tilde{e}_i$'s wer parallel, we would obviously have

$$\sigma_r(p+1) = \frac{p-k}{r}.$$

In general, let $V_1,\cdots,V_{p+1}$ be arbitrary (p+1) orthonormal tangent vectors at some point in W, we have that

$$\sum_{i=1}^{p+1}\nabla^2 r(V_i,V_i) = \sum_{i=1}^{p+1}\tilde{\nabla}^2 r(V_i,V_i) + \sum_{i=1}^{p+1}\left(\frac{x_1}{r}\langle\nabla_{V_i}\tilde{e}_{k+1},V_i\rangle+\cdots+\frac{x_{n-k}}{r}\langle\nabla_{V_i}\tilde{e}_n,V_i\rangle\right) \qquad (*)$$

Where $\tilde{\nabla}^2 r$ denotes the Hessian of r under the euclidean metric, then the first sum in (*) is $\geq \frac{p-k}{r}$. The second sum in (*) can clearly be bounded by some fixed constant which is independent of r. Therefore, by choosing Ω_2 properly and noting that $p - k \geq 1$, there exists a constant $c > 0$ such that

$$\sigma_r(p+1) > \frac{c}{r}$$

in $\Omega_2 \backslash (X^- \cup D^k)$. #

3.2. The bending function.

Let δ, ε_1, ε_2 and c_0 be fixed positive constants. Our aim in this section is to construct a smooth function f which is defined on $r > \varepsilon_0$, for some $0 < \varepsilon_0 < \varepsilon_1$, such that

$$f(r) = 0 \quad \text{for} \quad r \geq \varepsilon_1;$$

$$f'(r) \leq 0 \quad \text{for} \quad r > \varepsilon_0;$$

$$f(r) \to \varepsilon_3 < \varepsilon_2 \quad \text{as } r \to \varepsilon_0^+.$$

All the derivatives of $f \to \infty$ in absolute value as $r \to \varepsilon_0^+$. (See Fig. 5.)

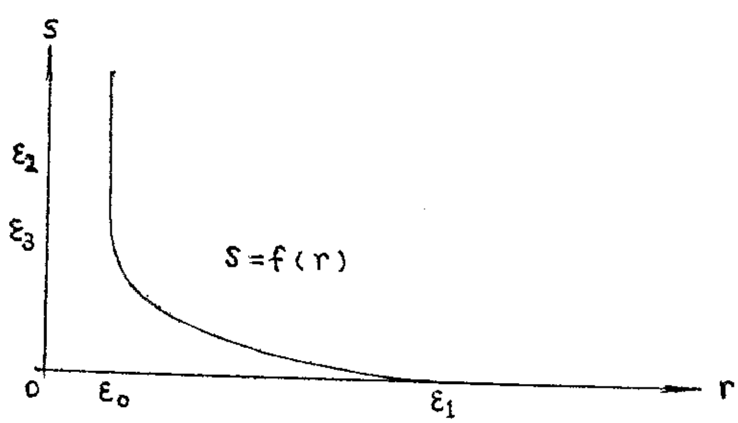

Fig. 5

Furthermore, f satisfies either of the following conditions for $r > \varepsilon_0$:

$$\delta - f''(r) - \frac{c_0 f'(r)}{r} > 0$$

or

$$\delta - \frac{f''(r)}{f'(r)^2} - \frac{c_0 f'(r)}{r} > 0.$$

We begin by choosing f_1'' properly to get a smooth function f_1 such that

$$f_1(r) = 0 \qquad \text{for } r \geq \varepsilon_1;$$

$$f_1'(r) \leq 0 \qquad \text{for all } r;$$

$$0 < f_1''(r) = \text{constant} < \delta \text{ for } r < \frac{\varepsilon_1}{2};$$

$$\exp\left[\frac{1}{2c_0 f_1'\left(\frac{\varepsilon_1}{2}\right)^2}\right] > 1;$$

$$\frac{c_0[-f_1'(\frac{\varepsilon_1}{2})]^3}{f_1''(0)} < \frac{\varepsilon_1}{2},$$

$$f_1(0) + \frac{c_0[-f_1'(\frac{\varepsilon_1}{2})]^3}{f_1''(0)} \cdot \frac{1}{\ell} \int_1^{\ell} \frac{dt}{\sqrt{2c_0 \ln t}} < \varepsilon_2 \text{ for all } \ell > 1.$$

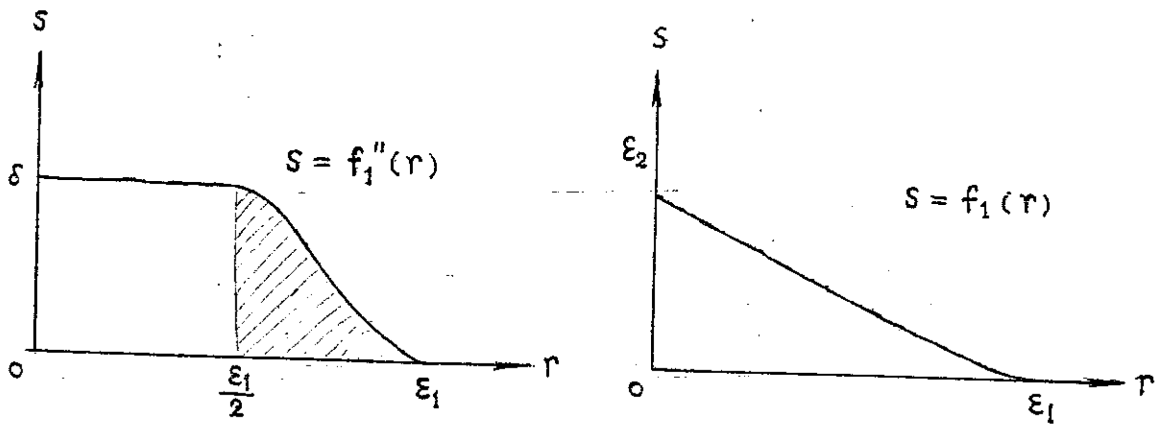

Fig. 6

All the requirements can be satisfied by choosing $f_1''(0)$ small and then by choosing the area of the shaded part in Fig. 6 small and also by noting that

$$\frac{1}{\ell}\int_1^{\ell} \frac{dt}{\sqrt{2c_0 \ln t}} \to 0 \qquad \text{as } \ell \to \infty$$

therefore in particular it is bounded for $\ell > 1$.

Now set

$$a = \exp\left[\frac{1}{2c_0 f_1'\left(\frac{\varepsilon_1}{2}\right)^2}\right]$$

$$\varepsilon_0 = \frac{c_0[-f_1'(\frac{\varepsilon_1}{2})]^3}{a f_1''(0)}$$

Then by the construction of f_1

$$a > 1 \quad \text{and} \quad a\varepsilon_0 < \frac{\varepsilon_1}{2} .$$

Define for $r > \varepsilon_0$

$$f_2(r) = \int_r^{a\varepsilon_0} \frac{dt}{\sqrt{2c_0 \ln\frac{t}{\varepsilon_0}}} .$$

We have

$$f_2'(r) = -\frac{1}{\sqrt{2c_0 \ln\frac{r}{{}_0}}}$$

$$f_2''(r) = \frac{1}{2\sqrt{2c_0}(\ln\frac{r}{\varepsilon_0})^{3/2} r} .$$

Hence

$$\frac{f_2''(r)}{f_2'(r)^2} + \frac{c_0 f_2'(r)}{r} = 0 .$$

Finally, let

$$f_3(r) = \begin{cases} f_1(\frac{\varepsilon_1}{2}) + f_2(r) & \text{for } \varepsilon_0 < r \le a\varepsilon_0 \\ f_1(r - a\varepsilon_0 + \frac{\varepsilon_1}{2}) & \text{for } r \ge a\varepsilon_0 \end{cases}$$

Then it is easy to verify that f_3 is C^2 and satisfies all the conditions required for f. In fact when $r \ge a\varepsilon_0$

$$\delta - f_3''(r) - \frac{c_0 f_3'(r)}{r} > 0$$

by the construction of f_1 and when $\varepsilon_0 < r \le a\varepsilon_0$

$$\delta - \frac{f_3''(r)}{f_3'(r)^2} - \frac{c_0 f_3'(r)}{r} = \delta - \frac{f_2''(r)}{f_2''(r)^2} - \frac{c_0 f_2'(r)}{r} = \delta > 0.$$

The required f is then gotten by a smoothing of f_3.

3.3. The construction of X'.

Let

$$D_\varepsilon = \{x\varepsilon\Omega : r(x)<\varepsilon\}$$

$$X_\varepsilon = \{x\varepsilon\Omega : s(x)<\varepsilon\}$$

be tubular neighborhoods of D^k and X respectively.

There exists ε_1, $\varepsilon_2 > 0$, such that

$$D_{2\varepsilon_1} \subset \Omega_2, \; X_{2\varepsilon_2} \subset \Omega_1$$

and such that

$$|\langle\nabla r, \nabla \;\rangle| < 1$$

in $U = \{x\varepsilon D_{2\varepsilon_1} \cap X_{2\varepsilon_1} \cap X^+ : r(x) > 0\}$.

Let γ be the curve $s = f(r)$ as in Fig. 5 . The hypersurface $S_\gamma = (r,s)^{-1}(\gamma)$ smoothly joining $X \setminus (X \cap U)$ to $\partial D_{\varepsilon_0} \setminus (\partial D_{\varepsilon_0} \cap U)$ produces a new hypersurface which will be our hypersurface X' obtained from X by attaching the handle D^k (see Fig. 7).

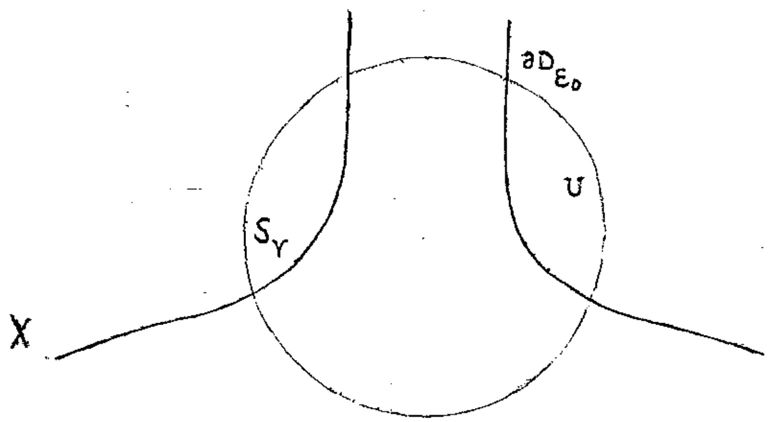

Fig. 7

We claim that X' is p-convex. It only needs to verify at the part of S_γ where $r > \varepsilon_0$. For this part, S_γ is the level set of the smooth function

$$F(x) = s(x) - f(r(x)).$$

We have

$$\nabla F = \nabla s - f'(r)\nabla r$$

$$\nabla^2 F = \nabla^2 s - f'(r)\nabla^2 r - f''(r)(\nabla r)^2.$$

Let $e_n = \frac{\nabla F}{\|\nabla F\|}$. The second fundamental form of S_γ is given by

$$B_F(\cdot,\cdot) = \langle\nabla_{\cdot} e_n,\cdot\rangle = \frac{\nabla^2 F}{\|\nabla F\|} - \frac{\nabla(\|\nabla F\|^2)\nabla F}{2\|\nabla F\|^3} .$$

Clearly

$$B_F(e_n,e_n) = 0$$

$$\nabla_{e_n}(\|\nabla F\|^2)\nabla_{e_n} F = \nabla_{e_n}(\|\nabla F\|^2) \langle e_n,\nabla F\rangle$$

$$= \nabla_{\frac{\nabla F}{\|\nabla F\|}}\langle\nabla F,\nabla F\rangle\cdot\langle\frac{\nabla F}{\|\nabla F\|},\nabla F\rangle$$

$$= 2\langle\nabla_{\nabla F}\nabla F,\nabla F\rangle = 2\nabla^2 F(\nabla F,\nabla F)$$

$$= 2[\nabla^2 s(\nabla F,\nabla F) - f'(r)\nabla^2 r(\nabla F,\nabla F)] - 2\|\nabla F\|^2 f''(r)(\nabla_{e_n} r)^2$$

$$= 2[f'(r)^2\nabla^2 s(\nabla r,\nabla r) - f'(r)\nabla^2 r(\nabla s,\nabla s)] - 2\|\nabla F\|^2 f''(r)(\nabla_{e_n} r)^2$$

where the last equality is obtained by recalling that ∇s is in the null space of $\nabla^2 s$ and that ∇r is in the null space of $\nabla^2 r$.

Then

$$\frac{\nabla_{e_n}(\|\nabla F\|^2)\nabla_{e_n} F}{2\|\nabla F\|^3} = -\frac{f''(r)}{\|\nabla F\|}(\nabla_{e_n} r)^2$$

$$-\frac{1}{\|\nabla F\|^3}[f'(r)\nabla^2 r(\nabla s,\nabla s) - f'(r)^2\nabla^2 s(\nabla r,\nabla r)]$$

Now suppose that $e_1,\cdots,e_p$ are orthonormal vectors tangent to s_γ. Then $\nabla_{e_i} F = 0$ for $i = 1,\cdots,p$.

Therefore,

$$\sum_{i=1}^{p} B_F(e_i,e_i) = \sum_{i=1}^{p} B_F(e_i,e_i) + B_F(e_n,e_n)$$

$$= \frac{1}{\|\nabla F\|}\sum_{i=1}^{p}[\nabla^2 s(e_i,e_i) - f'(r)\nabla^2 r(e_i,e_i) - f''(r)(\nabla_{e_i} r)^2]$$

$$+ \frac{1}{\|\nabla F\|}[\nabla^2 s(e_n,e_n) - f'(r)\nabla^2 r(e_n,e_n) - f''(r)(\nabla_{e_n} r)^2]$$

$$+ \frac{1}{\|\nabla F\|} f''(r)(\nabla_{e_n} r)^2 + \frac{1}{\|\nabla F\|^3}[f'(r)\nabla^2 r(\nabla s,\nabla s) - f'(r)^2\nabla^2 s(\nabla r,\nabla r)]$$

$$\geq \frac{1}{\|\nabla F\|}[\sigma_s(p+1) - f'(r)\sigma_r(p+1) - f''(r)\sum_{i=1}^{p}(\nabla_{e_i} r)^2]$$

$$+ \frac{1}{\|\nabla F\|^3}[f'(r)\nabla^2 r(\nabla s,\nabla s) - f'(r)^2\nabla^2 s(\nabla r,\nabla r)]$$

$$\geq \frac{1}{\|\nabla F\|}[\delta - f'(r)(\frac{c}{r} - \frac{1}{\|\nabla F\|^2}\left|\frac{r\nabla^2 e(\nabla s,\nabla s)}{r}\right|$$

$$- \frac{1}{\|\nabla F\|^2}|f'(r)\nabla^2 s(\nabla r,\nabla r)|) - f''(r)\sum_{i=1}^{p}(\nabla_{e_i} r)^2].$$

Note that

$$\lim_{r\to 0} r\nabla^2 r(\nabla s,\nabla s) = 0$$

in U, and that

$$\nabla^2 s(\nabla r,\nabla r),\ \frac{f'(r)}{\|\nabla F\|^2} = \frac{f'(r)}{1 + f'(r)^2 - 2f'(r)\langle\nabla r,\nabla s\rangle}$$

are bounded in U. It is then easy to see that we can choose ε_1, ε_2, c_0 so that

$$\sum_{i=1}^{p} B_F(e_i, e_i) \geq \frac{1}{\|\nabla F\|}\left[\delta - \frac{c_0 f'(r)}{r} - f''(r)\right]$$

or (note that $\nabla_{e_i} r = \dfrac{\nabla_{e_i} s}{f'(r)}$)

$$\sum_{i=1}^{p} B_F(e_i, e_i) \geq \frac{1}{\|\nabla F\|}\left[\delta - \frac{c_0 f'(r)}{r} - \frac{f''(r)}{f'(r)^2}\right].$$

Therefore by the construction of f, s_γ is p-convex.

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