ABSOLUTE HODGE CYCLES IN KUGA FIBER VARIETIES

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Salman Abdulali

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Salman Abdulali

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Michio Kuga

Michio Kuga, Professor of Mathematics
Dissertation Director

Marie-Louise Michelsohn, Associate Professor of Mathematics
Chairman of Defense

Chih-Han Sah, Professor of Mathematics

Sei Sujishi, Dean of Physical Sciences & Mathematics
Outside member

This dissertation is accepted by the Graduate School.

Barbara Bentley
Dean of the Graduate School

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Abstract of the Dissertation

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We show that if $A \to V$ is a Kuga fiber variety satisfying the H_2-condition, then the space $H^{\langle 0,2p \rangle}(A,\mathbb{Q})(p)$, of cohomology classes coming from a fiber, consists of absolute Hodge cycles. We use this to find relations between the zeta functions of two Kuga fiber varieties over the same base.

To my parents,
who created a family
of non-abelian varieties.

Table of Contents

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CHAPTER 0. Introduction

A group theoretical abelian scheme, or Kuga fiber variety, is a family of abelian varieties, $f : A \to V$, parametrized by an arithmetic variety $V = {}_{\Gamma}\backslash X$, and constructed from a symplectic representation, $\rho : G \to Sp(F, \beta)$, of an algebraic group G (Kuga [8]). In the rigid case, a group theoretical abelian scheme has a model defined over an algebraic number field; the study of such fiber varieties has led to deep results in the arithmetic theory of automorphic functions: for example, the proof of the Ramanujan-Petersson conjecture (Deligne [3]).

The Hasse-Weil zeta functions of rigid group theoretical abelian schemes over a Shimura curve are now known (Kuga-Shimura [11], Ohta [17]). Langlands [13] has investigated the case of higher dimensional base spaces. However, this remains a vast subject where little is known.

The aim of this investigation is to find relations between the zeta functions of two rigid Kuga fiber varieties, $A \to V$ and $B \to V$, over the same base V. In particular, suppose that $V = V_1 \times V_2$ is a product of two Shimura curves. Then, there are abelian schemes, $A_1 \to V_1$ and $A_2 \to V_2$, whose zeta functions are known (Ohta [17]). If V_1 and V_2 are suitably chosen, then there exists a rigid abelian scheme $B \to V_1 \times V_2$, which is not a product (Kuga [10]). By comparing the zeta functions of $A = A_1 \times A_2$ and B we may then expect to obtain nontrivial information about abelian schemes over products of Shimura curves.

Our technique is to use absolute Hodge cycles (Deligne [4]) in the product $A \times B$ to describe homomorphisms between the étale cohomology groups of A and B. These homomorphisms commute with the action of a Frobenius element (up to a Tate twist), for a sufficiently large field of definition, and thus give rise to relations between zeta functions. Loosely speaking, we think of zeta functions as the objects of a category, whose morphisms are absolute Hodge cycles.

We shall now describe our main results.

Let $A \to V$ be a Kuga fiber variety satisfying the H_2-condition. Then, the space $H^{<0,2p>}(A,\emptyset)(p)$, of cohomology classes coming from a fiber, consists of absolute Hodge cycles (Theorem (3.3.5)).

Let $A \to V$ and $B \to V$ be Kuga fiber varieties satisfying the H_2-condition, and let $\rho_A : G \to Sp(F_A, \beta_A)$ and $\rho_B : G \to Sp(F_B, \beta_B)$ be the representations defining them. For $b_o \equiv b_1 \pmod 2$, consider the representations $\wedge^{b_o} \rho_A$

and $\overset{b_1}{\wedge} \rho_B$ of G on $\overset{b_o}{\wedge} F_A$ and $\overset{b_1}{\wedge} F_B$, respectively, and write them as direct sums of irreducible representations: $\overset{b_o}{\wedge} \rho_A = \underset{\alpha}{\oplus} R^{b_o}_{A,\alpha}$, $\overset{b_1}{\wedge} \rho_B = \underset{\beta}{\oplus} R^{b_1}_{B,\beta}$. For each pair (α, β) with $R^{b_o}_{A,\alpha}$ equivalent to $R^{b_1}_{B,\beta}$ as representations of G over $\emptyset$, we find an absolute Hodge cycle in A $\times$ B, and thus obtain a relation between the zeta functions of A and B (Theorems (3.4.4), (3.4.6)).

Let us point out that though our main results are valid over a sufficiently large field of definition, we do not know how to find such a field.

Finally, we look at examples (Section 3.5). We compute the dimension of the space $H^{<0,2p>}(A \underset{V}{\times} B, \emptyset)(p)$ of absolute Hodge cycles, and we compare the representations $\overset{b_o}{\wedge} \rho_A$ and $\overset{b_1}{\wedge} \rho_B$. In order to do so we first describe (Section 1.3) the representations defining families of abelian varieties over quaternion Hilbert modular varieties. This is a generalization of Addington's classification [1]. We show that families of abelian varieties defined by "rigid polymers" satisfy the H_2-condition and are rigid.

<u>Notations</u> <u>and</u> <u>Conventions</u>. All algebraic varieties are assumed to be connected and smooth. We usually use the same symbol to represent an algebro-geometric object considered as a scheme, an algebraic variety in the sense of

Weil, or as a complex manifold. When it is necessary to make the distinction, we write X^{an} for the complex manifold associated to a variety defined over a subfield of $\mathbb{C}$. If a variety X is defined over a field k, and K is an extension of k, we write X_K for the set of K-rational points of X. $X_{\mathbb{R}}$ and $X_{\mathbb{C}}$ are often considered as real and complex manifolds, respectively.

A vector space or algebra W over a field k determines an algebraic variety, again called W, such that $W_K = W \otimes_k K$ for any field K containing k. If $K \supset k$ then $\mathrm{Res}_{K/k}$ denotes the restriction of scalar functor from the category of K-varieties to the category of k-varieties, if $[K:k] < \infty$.

Notations for cohomology groups are explained in detail in Section 2.1.

CHAPTER 1. Group Theoretical Abelian Schemes

This chapter deals with the construction of group theoretical families of abelian varieties, with emphasis on families of abelian varieties parametrized by quaternion Hilbert modular varieties.

1.1. Kuga's Construction of Fiber Varieties

A group theoretical family of abelian varieties, $f : A \to V$, is constructed out of data $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$, as described below. For details see Kuga [8], or Satake [18].

Let G be a semisimple, linear algebraic group defined over $\mathbb{Q}$. We assume that $G_{\mathbb{R}}$ is a connected, semisimple, real Lie group with finite center, and that G has no connected, normal $\mathbb{Q}$-subgroup $N \neq \{1\}$ such that $N_{\mathbb{R}}$ is compact. Let Γ be a torsion-free arithmetic subgroup of $G_{\mathbb{Q}}$, with $\Gamma \backslash G_{\mathbb{R}}$ compact.

Proposition (1.1.1) (Borel [2], Theorem 1). Γ is Zariski-dense in G.

We assume that $X = G_{\mathbb{R}}/K$ has a $G_{\mathbb{R}}$-invariant complex structure, where K is a maximal compact subgroup of $G_{\mathbb{R}}$. It is known that X is diffeomorphic to a Euclidean space. Let $\underline{g}$ be the Lie algebra of $G_{\mathbb{R}}$, $\mathfrak{k}$ the Lie subalgebra of $\underline{g}$

corresponding to K, and $\mathfrak{P}$ the orthogonal complement of $\mathfrak{K}$ in $\underline{g}$ with respect to the Killing form. If $\nu : G_{\mathbb{R}} \to X$ is the natural map, then $d\nu|\mathfrak{P}$ is an isomorphism of $\mathfrak{P}$ onto $T_o(X)$, the tangent space of X at $0 = \nu(1)$. Since X is a complex manifold, $T_o(X)$ has a complex structure J_o, and it is known that there exists a unique $H_o \in Z(\mathfrak{K})$, the center of $\mathfrak{K}$, such that $(\mathrm{ad}\, H_o)|\mathfrak{P} = J_o$, where we have considered J_o as a complex structure on $\mathfrak{P}$ via the isomorphism $d\nu$.

Γ acts properly discontinuously and without fixed points on X, so $V = {}_{\Gamma}\backslash^{X}$ is a compact, connected, complex manifold. We shall construct a family of abelian varieties parametrized by V.

Let F be a finite dimensional vector space over $\mathbb{Q}$, L a lattice in F, and β a nondegenerate alternating bilinear form on F with $\beta(L,L) \subset \mathbb{Z}$. Then, the symplectic group,

$$G' = Sp(F,\beta) = \{x \in GL(F) \,|\, \beta(xu,xv) = \beta(u,v)\ \forall\, u,v \in F\},$$

is an algebraic group defined over $\mathbb{Q}$. Let $\rho : G \to G'$ be an algebraic representation of G, defined over $\mathbb{Q}$, such that $\rho(\gamma)L = L$ for all $\gamma \in \Gamma$. Since Γ acts on L, we can form the semidirect product $\Gamma \ltimes L$ which acts on $X \times F_{\mathbb{R}}$ by $(\gamma,\ell)\cdot(x,u) = (\gamma x, \rho(\gamma)u+\ell)$. Then the quotient $A = {}_{\Gamma\ltimes L}\backslash(X \times F_{\mathbb{R}})$

is a smooth manifold, and we denote by f the natural projection: $A = {}_{\Gamma\ltimes L}\backslash(X \times F_{\mathbb{R}}) \to V = {}_{\Gamma}\backslash X$.

$f : A \to V$ is a smooth fiber bundle with fiber $T = F_{\mathbb{R}}/L$, and structure group $\rho(\Gamma)$.

The Siegel space is defined by

$$X' = \mathfrak{S}(F_{\mathbb{R}}, \beta) = \{J \in GL(F_{\mathbb{R}}) \mid J^2 = -I,\ \beta(x, Jy) \text{ is a positive definite symmetric form on } F_{\mathbb{R}}\}.$$

$G'_{\mathbb{R}}$ acts transitively on X' by $g(J) = gJg^{-1}$. Choose a base point $J_o \in X'$ and let K' be the isotropy subgroup of $G'_{\mathbb{R}}$ at J_o, $\underline{g}'$ the Lie algebra of $G'_{\mathbb{R}}$, $\mathfrak{k}'$ the Lie subalgebra of $\underline{g}'$ corresponding to K', and $\mathfrak{p}'$ the orthogonal complement of $\mathfrak{k}'$ in $\underline{g}'$ with respect to the Killing form. Then, it is known (Satake [18], Chapter 2, Section 7) that $H'_o = \frac{1}{2} J_o$ belongs to the center of $\mathfrak{k}'$, and $\operatorname{ad} H'_o$ is a complex structure on $\mathfrak{p}'$. The map $\nu' : G'_{\mathbb{R}} \to X'$ given by $\nu'(g) = gJ_o g^{-1}$ induces an isomorphism $d\nu' : \mathfrak{p}' \to T_{J_o}(X')$. With this identification, $\operatorname{ad} H'_o$ is a complex structure on $T_{J_o}(X')$. There exists a unique $G'_{\mathbb{R}}$-invariant complex structure on X' which induces the complex structure $\operatorname{ad} H'_o$ on $T_{J_o}(X')$. We shall always consider X' as a complex manifold with this complex structure; it is independent of the choice of the base point J_o.

Let $\tau : X \to X'$ be equivariant with respect to ρ, i.e., $\tau(g(x)) = \rho(g)(\tau(x))$ $\forall x \in X$ $\forall g \in G_{\mathbb{R}}$, and let $J_o = \tau(0)$ where $0 = \nu(1)$ is the base point in X. We say that the H_1-condition is satisfied if

$$[d\rho(H_o) - \frac{1}{2} J_o, d\rho(a)] = 0 \quad \forall a \in \underline{g}. \tag{1.1.2}$$

We say that the H_2-condition is satisfied if

$$d\rho(H_o) = \frac{1}{2} J_o. \tag{1.1.3}$$

We note that (1.1.3) implies (1.1.2), and (1.1.2) implies that τ is holomorphic. We assume that τ satisfies the H_1-condition, but not necessarily the H_2-condition.

For each $x \in X$, $\tau(x)$ is a complex structure on $F_{\mathbb{R}}$ such that the torus $T_{\tau(x)} = F_{\mathbb{R}}/L$ with the complex structure $\tau(x)$, is an abelian variety with polarization β. Since $T_{\tau(x)}$ and $T_{\tau(\gamma x)}$ are isomorphic abelian varieties for $\gamma \in \Gamma$, $A_P = T_{\tau(x)}$ is well-defined for $P = q(x) \in V$, where $q : X \to V = {}_{\Gamma}\backslash X$ is the natural map. Identifying A_P with the fiber $f^{-1}(\{P\})$ of the bundle $f : A \to V$, we see that A is a family of polarized abelian varieties parametrized by V.

<u>Theorem (1.1.4)</u> (Kuga [8], Theorem II-6-3). Let $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ be as above. Then, there exists a unique complex

structure on A such that

(i) $f : A \to V$ is holomorphic;

(ii) the underlying real analytic structure on A coincides with the one it already has;

(iii) the restriction of the complex structure to the fiber over $P \in V$ is $\tau(x)$, where $P = q(x)$ as in the previous paragraph;

(iv) lifting the complex structure to the universal covering $\tilde{A} = X \times F_{\mathbb{R}}$ of A defines a holomorphic vector bundle $\tilde{A} \to X$, where the fiber over $x \in X$ is $F_{\mathbb{R}}$ with the complex structure $\tau(x)$.

Now suppose $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ are as above and $f : A \to V$ is the fiber bundle constructed above with the unique complex structure given by the theorem. Then, it is known that A and V can be biholomorphically embedded in a complex projective space, i.e., they can be identified with projective algebraic varieties, with f a regular map of algebraic varieties. Such a fiber variety, considered as a projective algebraic variety, a scheme, or a complex manifold, will be called a <u>Kuga fiber variety</u>, a <u>group theoretical family of abelian varieties</u>, or a <u>group theoretical abelian scheme</u>.

We shall need the following facts.

Lemma (1.1.5) (Satake [18], p. 173). Let ρ be a symplectic representation of G, as above. Then ρ is self-dual, i.e., the representation ρ of G on F is equivalent over $\mathbb{Q}$ to the representation $g \mapsto {}^t\rho(g)^{-1}$ of G on F*.

Lemma (1.1.6). (a) If $A_1 \to V_1$ and $A_2 \to V_2$ are Kuga fiber varieties, then so is $A_1 \times A_2 \to V_1 \times V_2$. Furthermore, if $A_1 \to V_1$ and $A_2 \to V_2$ satisfy the H_2-condition, then so does $A_1 \times A_2 \to V_1 \times V_2$.

(b) If $A \to V$ and $B \to V$ are Kuga fiber varieties, then so is $A \underset{V}{\times} B \to V$. Furthermore, if $A \to V$ and $B \to V$ satisfy the H_2-condition, then so does $A \underset{V}{\times} B$.

Proof: (a) Let $A_i \to V_i$, (i=1,2), be defined by data (G_i, K_i, X_i, Γ_i, F_i, L_i, β_i, ρ_i, τ_i). Then $A_1 \times A_2 \to V_1 \times V_2$ is defined by data ($G_1 \times G_2$, $K_1 \times K_2$, $X_1 \times X_2$, $\Gamma_1 \times \Gamma_2$, $F_1 \oplus F_2$, $L_1 \oplus L_2$, $\beta_1 \oplus \beta_2$, $\rho_1 \circ pr_1 \oplus \rho_2 \circ pr_2$, $\tau_1 \circ pr_1 \oplus \tau_2 \circ pr_2$), where $pr_i : G_1 \times G_2 \to G_i$ and $pr_i : X_1 \times X_2 \to X_i$ are the projections. If H_{oi} is the element of $\underline{g}_i$, the Lie algebra of $G_{i,\mathbb{R}}$, defining the complex structure on X_i, then $(H_{o1}, H_{o2}) \in \underline{g}_1 \times \underline{g}_2$ defines the complex structure on $X_1 \times X_2$. Both statements of (a) are easy consequences of these facts.

(b) If $A \to V$ and $B \to V$ are defined by $(G,K,X,\Gamma,F_A,L_A,\beta_A,\rho_A,\tau_A)$ and $(G,K,X,\Gamma,F_B,L_B,\beta_B,\rho_B,\tau_B)$ respectively, then it is easy to see that $A \underset{V}{\times} B$ is defined by $(G,K,X,\Gamma,F_A \oplus F_B,\ L_A \oplus L_B,\ \beta_A \oplus \beta_B,\ \rho_A \oplus \rho_B,\ \tau_A \oplus \tau_B)$. The second statement is an immediate consequence of this fact, since

$$d(\rho_A \oplus \rho_B(H_o)) = \frac{1}{2}\ \tau_A(0) \oplus \frac{1}{2}\ \tau_B(0) = \frac{1}{2}(\tau_A \oplus \tau_B)(0).$$

Q.E.D.

1.2. Rigidity of Abelian Schemes

Let $f : A \to V$ be a group theoretical family of abelian varieties defined by data $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$. We say that $A \to V$ is rigid if τ is uniquely determined by the rest of the data: G,K,X,Γ,F,L,β, and ρ.

Let $\nu : G_{\mathbb{R}} \to X$ be the natural map, and $0 = \nu(1)$. Since τ is equivariant with respect to ρ, we have $\tau(g(0)) = \rho(g)\tau(0)\rho(g)^{-1}$ for all $g \in G_{\mathbb{R}}$. Thus, τ is uniquely determined by specifying $\tau(0)$. Let $X\rho$ be the set of all $J \in X' = \mathfrak{S}(F_{\mathbb{R}},\beta)$ such that the map $g(0) \mapsto \rho(g)J\rho(g)^{-1}$ is well-defined, satisfies the H_1-condition (1.1.2), and thus defines a Kuga fiber variety. We view $X\rho$ as the space of all deformations of the fiber variety $A \to V$. Thus, A is rigid if and only if $X\rho$ reduces to a point.

<u>Theorem (1.2.1)</u> (Satake [18], Chapter 4, Proposition 4.1). $X\rho$ is a complex submanifold of X'. Furthermore, $G\rho$, the Zariski-connected component of the centralizer of $\rho(G)$ in G', is a reductive subgroup of G' which acts transitively on $X\rho$.

<u>Proposition (1.2.2)</u>. If the H_2-condition is satisfied, then the fiber variety $A \to V$ is rigid.

<u>Proof</u>: Let $J \in X\rho$ and $J_o = \tau(0)$. Then, by Theorem (1.2.1), there exists $g \in G\rho$ such that $J = gJ_og^{-1}$.

Now, since J_o is a complex structure on $F_{\mathbb{R}}$, we can write $J_o = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ with respect to some basis of $F_{\mathbb{R}}$, where $2n = \dim F$. Then, an easy calculation shows that

$$\exp(t\,J_o) = \cos t \cdot I_{2n} + \sin t \cdot J_o \quad \forall t \in \mathbb{R}. \tag{1.2.3}$$

We have the following commutative diagram:

$$\begin{array}{ccc} \underline{g} & \xrightarrow{\;d\rho\;} & \underline{g}' \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G_{\mathbb{R}} & \xrightarrow{\;\rho\;} & G'_{\mathbb{R}} \end{array} \tag{1.2.4}$$

(cf. Helgason [7], Chapter 2, Lemma 1.12). Then,

$$\begin{aligned} J_o &= \exp(\tfrac{\pi}{2} J_o) && \text{(from (1.2.3))} \\ &= \exp(d\rho(\pi H_o)) && \text{(by the } H_2\text{-condition)} \\ &= \rho(\exp(\pi H_o)) && \text{(from (1.2.4))}, \end{aligned}$$

and we conclude that $J_o \in \rho(G_{\mathbb{R}})$. Since g belongs to the centralizer of $\rho(G_{\mathbb{R}})$ we have $J = gJ_og^{-1} = J_o$. Thus, $X\rho$ reduces to a point and $A \to V$ is rigid.

Q.E.D.

1.3. Abelian Schemes arising from Quaternion Algebras

Addington [1] constructed families of abelian varieties over an arithmetic variety obtained from a quaternion algebra, in terms of a combinatorial device called "chemistry." Kuga [10] constructed families of abelian varieties over a product of two such arithmetic varieties; these were obtained as families of deformations of the fiber varieties constructed by Addington, and were described in terms of a "sharing" between two chemistries. Here, we describe a generalization of Addington's method which includes the families obtained by "sharing." For simplicity, we deal mainly with the rigid case.

For $i = 1,\ldots,n$, let B^i be a division quaternion algebra over a totally real number field k_i of finite degree over $\mathbb{Q}$,

S^i the set of all embeddings of k_i into $\mathbb{R}$, $S_o^i = \{\alpha \in S^i | B^i \otimes_\alpha \mathbb{R} \cong M_2(\mathbb{R})\}$, $S_1^i = S^i - S_o^i$, $S = \coprod_{i=1}^n S^i$ (disjoint union), $S_o = \coprod_{i=1}^n S_o^i$, and, $S_1 = S - S_o$. We assume that each S_o^i is nonempty.

For any quaternion algebra B, let ν be the reduced norm on B, and put $SL_1(B) = \{x \in B | \nu(x) = 1\}$. Then, $SL_1(B^i)$ is an algebraic group over k_i, $G^i = Res_{k_i/\mathbb{Q}} SL_1(B^i)$ is an algebraic group over $\mathbb{Q}$, and $G = \prod_{i=1}^n G^i$ is an algebraic group over $\mathbb{Q}$ satisfying the assumptions of Section 1.1. The Lie group $G_{\mathbb{R}}$ is isomorphic to $SL_2(\mathbb{R})^{S_o} \times SU_2^{S_1}$; we fix an isomorphism and identify them. A maximal compact subgroup of $G_{\mathbb{R}}$ is $K = SO_2^{S_o} \times SU_2^{S_1}$; the quotient is $X = G_{\mathbb{R}/K} = (SL_2(\mathbb{R})/SU_2)^{S_o}$. The element $H_o = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}^{S_o} \times 0^{S_1}$ of the Lie algebra $\underline{g} = \mathfrak{sl}_2(\mathbb{R})^{S_o} \times \mathfrak{su}_2^{S_1}$ of $G_{\mathbb{R}}$ determines a complex structure on X. $G_{\mathbb{R}}$ acts holomorphically on $\mathfrak{H}^{S_o}$, the product of $|S_o|$ copies of the upper half plane $\mathfrak{H}$ by linear fractional transformations, and the isotropy group of $(\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathfrak{H}^{S_o}$ is K; hence we can identify $G_{\mathbb{R}/K} = X$ with $\mathfrak{H}^{S_o}$. If Γ is an arithmetic subgroup of $G_{\mathbb{Q}}$, then $V = {}_{\Gamma}\backslash X$ is called a quaternion Hilbert modular variety.

We shall partially classify representations ρ of $G_{\mathbb{C}}$ on a complex vector space W which define families of abelian varieties, i.e., for which there exist Γ, F, L, β, and τ such that $F_{\mathbb{C}} = W$, and $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ satisfy the assumptions of Section 1.1.

Let $k \subset \mathbb{R}$ be a finite Galois extension of $\mathbb{Q}$ such that $\alpha(k_i) \subset k$ for all $i = 1,\ldots,n$, and all $\alpha \in S^i$. Put $\mathfrak{G} = \mathrm{Gal}(k/\mathbb{Q})$. Then, $\mathfrak{G}$ acts on S by $g(\alpha) = g\circ\alpha$, and the orbits of this action are the sets S^i. The triple $(\mathfrak{G},S,S_o)$ is called a <u>chemistry</u>. Elements of S are called <u>atoms</u>, subsets of S are called <u>molecules</u>, and formal sums $\sum_{j=1}^{t} M_j$ of molecules are called <u>polymers</u>. Since $\mathfrak{G}$ acts on S, $\mathfrak{G}$ acts on the set of all molecules, and on the set of all polymers.

A molecule M is called <u>stable</u> if $|M \cap S_o| \le 1$, and <u>rigid</u> if $|M \cap S_o| = 1$. A polymer $P = \sum_{j=1}^{t} M_j$ is called <u>stable</u> (resp. <u>rigid</u>) if P is $\mathfrak{G}$-invariant and each M_j $(j=1,\ldots,t)$ is stable (resp. rigid).

We extend the isomorphism $G_{\mathbb{R}} \cong SL_2(\mathbb{R})^{S_o} \times SU_2^{S_1}$ to an isomorphism of $G_{\mathbb{C}}$ with $SL_2(\mathbb{C})^S$, and identify them. An atom $\alpha \in S$ determines a representation ρ_α of $G_{\mathbb{C}}$ by projection to the α-th factor, $\rho_\alpha : G_{\mathbb{C}} \to SL_2(\mathbb{C}) \subset GL_2(\mathbb{C})$. A molecule $M \subset S$ determines a representation $\rho_M = \bigotimes_{\alpha\in M} \rho_\alpha$ of $G_{\mathbb{C}}$, with the understanding that $\rho_\emptyset$ is a trivial one-dimensional representation. Finally, a polymer $P = \sum_{j=1}^{t} M_j$ determines a representation

$\overset{t}{\underset{j=1}{\oplus}} \rho_{M_j}$ of $G_{\mathbb{C}}$.

Theorem (1.3.1). Let G, K, and X be as above, and suppose that ρ is a representation of $G_{\mathbb{C}}$ that defines a family of abelian varieties. Then there exists a stable polymer P such that ρ is equivalent to ρ_P over $\mathbb{C}$.

Proof: This is proved by Addington ([1], Section 8) under the assumption that n = 1. However, the same proof applies for general n, mutatis mutandis, if we observe that even though the action of $\mathfrak{g}$ on S is not transitive, each $\mathfrak{g}$-orbit of S contains an element of S_o.

Q.E.D.

Conversely, let M be a molecule such that $P = \sum_{\sigma \in \mathfrak{g}} \sigma M$ is rigid. We will show that either ρ_P or ρ_{2P} defines a family of abelian varieties.

For an atom $\alpha \in S$, let $i(\alpha)$ be the unique index such that $\alpha \in S^{i(\alpha)}$. Put $B_\alpha = B^{i(\alpha)} \otimes_\alpha k$, and let $\tilde{\alpha} : B^{i(\alpha)} \to B_\alpha$ be the extension to $B^{i(\alpha)}$ of the embedding $\alpha : k_{i(\alpha)} \to k$. Restricting scalars to $\emptyset$, $\tilde{\alpha}$ induces a map of algebraic varieties over $\emptyset$, $\alpha_o : \mathrm{Res}_{k_{\alpha(i)}/\emptyset} B^{i(\alpha)} \to \mathrm{Res}_{k/\emptyset} B_\alpha$. Form the k-algebra $\tilde{\mathfrak{a}} = \otimes_{\alpha \in M} B_\alpha$, and put $\mathfrak{a} = \mathrm{Res}_{k/\emptyset} \tilde{\mathfrak{a}}$. Let $\tilde{F}$ be a minimal left ideal in $\tilde{\mathfrak{a}}$; then $F = \mathrm{Res}_{k/\mathbb{Q}} \tilde{F}$ is a minimal left ideal in $\mathfrak{a}$. Let $\tilde{\lambda}$ be the action of $\tilde{\mathfrak{a}}$ on $\tilde{F}$ by left multiplication,

$$\tilde{\lambda} : \tilde{\mathfrak{a}} \to \mathrm{End}_k(\tilde{F});$$

then $\tilde{\lambda}$ is defined over k, and we put $\lambda = \mathrm{Res}_{k/\mathbb{Q}} \tilde{\lambda}$, $\lambda : \mathfrak{a} \to \mathrm{End}_{\mathbb{Q}}(F)$.

Next, define $\varphi : G_{\mathbb{Q}} = \prod_{i=1}^{n} G^i \to \mathfrak{a}_{\mathbb{Q}} = \bigotimes_{\alpha\in M} B_\alpha$ by

$$\varphi(g_i) = \bigotimes_{\alpha\in M} \tilde{\alpha}(g_{i(\alpha)}).$$

Then φ extends to a map $\varphi : G \to \mathfrak{a}$ of algebraic varieties over $\mathbb{Q}$, and $\rho = \lambda\circ\varphi$ is an algebraic representation of G on F, defined over $\mathbb{Q}$.

We shall now describe the above constructions in greater detail. For $\sigma \in \mathfrak{g}$, let $\tilde{\mathfrak{a}}_\sigma = \bigotimes_{\alpha\in\sigma M} B_\alpha$. Then, $\mathfrak{a} = \mathrm{Res}_{k/\mathbb{Q}} \tilde{\mathfrak{a}} = \bigoplus_{\sigma\in\mathfrak{g}} \tilde{\mathfrak{a}}_\sigma$. Also, $G = \prod_{i=1}^{n} G^i = \prod_{\alpha\in S} G_\alpha$, with $G_\alpha = SL_1(B_\alpha)$. In these terms, we may describe φ as follows: if $g = (g_\alpha) \in \prod_{\alpha\in S} G_\alpha$, then $\varphi(g) = \sum_{\sigma\in\mathfrak{g}} \varphi_\sigma(g)$, where $\varphi_\sigma(g) = \bigotimes_{\alpha\in\sigma M} g_\alpha \in \tilde{\mathfrak{a}}_\sigma$. We note that each φ_σ is defined over k, while φ is defined over $\mathbb{Q}$.

Next, we take a closer look at ρ. Recall that a central, simple k-algebra is equivalent to a division quaternion algebra if and only if it determines an element of order 2 in $B(k)$, the Brauer group of k. Since $B(k)$ is commutative, this implies that a tensor product of division quaternion algebras is either trivial, or equivalent to a division quaternion algebra. Hence, we can write $\tilde{\mathfrak{a}} = M_N(C)$ where either $C = k$, or C is a division quaternion algebra over k. Then, for some $e = 1,\ldots,N$, the minimal left ideal $\tilde{F}$ of $\tilde{\mathfrak{a}}$ consists of all $x = (x_{ij}) \in M_N(C)$ such that $x_{ij} = 0$ when $j \neq e$. In particular, $\tilde{F} \cong C^N$, and the action of $\tilde{\mathfrak{a}}$ on $\tilde{F}$ is just the left multiplication of matrices on column vectors.

__Lemma (1.3.2)__. If $\tilde{\mathfrak{a}}$ is trivial, then $\rho \sim \rho_P$ over $\mathbb{C}$; otherwise $\rho \sim \rho_{2P}$ over $\mathbb{C}$.

__Proof__: ρ_P maps $G = \prod_{\alpha\in S} G_\alpha$ into $\mathfrak{a} = \bigoplus_{\sigma\in\mathfrak{g}} \bigotimes_{\alpha\in\sigma M} B_\alpha$, via the map $\sum_{\sigma\in\mathfrak{g}} \bigotimes_{\alpha\in\sigma M} \rho_\alpha$, where $\rho_\alpha : \prod_{\alpha\in S} G_\alpha \to G_\alpha \subset B_\alpha$ is the projection map. Since $\mathfrak{a} \otimes_{\mathbb{Q}} \mathbb{C}$ is isomorphic to $\bigoplus_{\sigma\in\mathfrak{g}} \bigotimes_{\alpha\in\sigma M} M_2(\mathbb{C}) \subset M_{2^m g}(\mathbb{C})$, $g = |\mathfrak{g}|$, $m = |M|$, ρ_P is a representation of G on a minimal left ideal of $M_{2^m g}(\mathbb{C})$, which is $2^m g$-dimensional.

Suppose $\tilde{\mathfrak{a}} = M_N(k)$. Then $N = 2^m$, $\dim_k \tilde{F} = 2^m$, and $\dim_{\mathbb{Q}} F = 2^m g$. Since $F_{\mathbb{C}}$ is an ideal of $\mathfrak{a} \otimes_{\mathbb{Q}} \mathbb{C}$ of dimension $2^m g$, it must be a minimal left ideal. Therefore, $\rho \sim \rho_P$ over $\mathbb{C}$.

Next, suppose $\widetilde{\mathfrak{A}} = M_N(C)$, with C a division quaternion algebra over k. Then, $N = 2^{m-1}$, $\dim_k \widetilde{F} = 2^{m+1}$, and $\dim_{\mathbb{Q}} F = 2^{m+1} g = \dim_{\mathbb{C}} F_{\mathbb{C}}$. Since $F_{\mathbb{C}}$ is a left ideal in $\mathfrak{A} \otimes_{\mathbb{Q}} \mathbb{C}$, it must be a direct sum of two minimal left ideals. Therefore, $\rho \sim \rho_P \oplus \rho_P$ over $\mathbb{C}$.

Q.E.D.

Next, we define an alternating form β on F. For each i (i=1,...,n) there exists $\eta_i \in B^i$ such that $\eta_i' = -\eta_i$ and $\nu(\eta_i)$ is totally positive, where $(\ )'$ denotes the canonical involution of B^i and ν is the reduced norm (Addington [1], Lemma 11.1). Such an η_i is not unique; for the time being we choose one arbitrarily, later we will need to adjust it.

For an atom $\alpha \in S$, define k-bilinear forms e_α, f_α on B_α by

$$e_\alpha(u,v) = \tau(u\, v'),$$
$$f_\alpha(u,v) = \tau(u\, \widetilde{\alpha}(\eta_{i(\alpha)})v'),$$

where τ denotes the reduced trace on B_α. Then, e_α is symmetric and f_α is alternating ([1], Lemma 11.7). For an atom $\beta \in M$ define an alternating k-bilinear form E_β on $\widetilde{\mathfrak{A}} = \bigotimes_{\alpha \in M} B_\alpha$ by

$$E_\beta(\otimes u_\alpha, \otimes v_\alpha) = f_\beta(u_\beta, v_\beta) \cdot \prod_{\substack{\alpha \in M \\ \alpha \neq \beta}} e_\alpha(u_\alpha, v_\alpha).$$

Then, $E_M = \sum_{\beta \in M} E_\beta$ is again an alternating k-bilinear form on $\tilde{\mathfrak{a}}$. Since $\mathfrak{a}_\emptyset = \tilde{\mathfrak{a}}_K$, we can define a $\emptyset$-bilinear form on $\mathfrak{a}_\emptyset$ by

$$\beta(u,v) = tr_{k/\emptyset}(E_M(u,v)) = \sum_{\sigma \in \mathfrak{a}} \sigma(E_M(u,v)).$$

Since E_M is alternating, β is again an alternating form on $\mathfrak{a}$ or on $F \subset \mathfrak{a}$.

Lemma (1.3.3). ρ maps G into $Sp(F,\beta)$.

Proof: Since $G_\emptyset$ is Zariski-dense in $G_{\mathbb{C}}$, it is enough to show that ρ maps $G_\emptyset$ into $Sp(F_\emptyset,\beta)$. Let $g = (g_i) \in G_\emptyset = \prod_{i=1}^{n} G^i$, and $u,v \in F_\emptyset = \tilde{F}_K \subset \tilde{\mathfrak{a}}_K = \bigotimes_{\alpha \in M} B_\alpha$. Then, it is sufficient to show that $\beta(\rho(g)(u),\rho(g)(v)) = \beta(u,v)$ for all u,v of the form $u = \otimes u_\alpha$, $v = \otimes v_\alpha$ with $u_\alpha, v_\alpha \in B_\alpha$. Let $\beta \in M$. Then,

$$E_\beta(\otimes \tilde{\alpha}(g_{i(\alpha)})u_\alpha, \otimes \tilde{\alpha}(g_{i(\alpha)})v_\alpha)$$

$$= f_\beta(\tilde{\beta}(g_{i(\beta)})u_\beta, \tilde{\beta}(g_{i(\beta)})v_\beta) \cdot \prod_{\substack{\alpha \in M \\ \alpha \neq \beta}} e_\alpha(\tilde{\alpha}(g_{i(\alpha)})u_\alpha, \tilde{\alpha}(g_{i(\alpha)})v_\alpha)$$

$$= \tau(\tilde{\beta}(g_{i(\beta)})u_\beta\, \tilde{\beta}(\eta_{i(\beta)})v'_\beta\, \tilde{\beta}(g_{i(\beta)})') \prod_{\substack{\alpha \in M \\ \alpha \neq \beta}} \tau(\tilde{\alpha}(g_{i(\alpha)})u_\alpha v'_\alpha (g_{i(\alpha)})')$$

$$= \tau(\tilde{\beta}(g_{i(\beta)})u_\beta\, \tilde{\beta}(\eta_{i(\beta)})v'_\beta \tilde{\beta}(g_{i(\beta)})^{-1}) \prod_{\substack{\alpha \in M \\ \alpha \neq \beta}} \tau(\tilde{\alpha}(g_{i(\alpha)})u_\alpha v'_\alpha \tilde{\alpha}(g_{i(\alpha)})^{-1})$$

(because $g'_i = g_i^{-1}$ for $g_i \in SL_1(B^i)$)

$$= \tau(u_\beta\, \tilde{\beta}(\eta_{i(\beta)})v'_\beta) \prod_{\substack{\alpha \in M \\ \alpha \neq \beta}} \tau(u_\alpha\, v'_\alpha)$$

(because τ is invariant under conjugation)

$= E_\beta(\otimes u_\alpha, \otimes v_\alpha)$.

Thus,

$$\begin{aligned}
&\beta(\rho(g)(u), \rho(g)(v)) \\
&= \beta(\varphi(g)u, \varphi(g)v) \\
&= \mathrm{tr}_{k/\mathbb{Q}} \sum_{\beta \in M} E_\beta(\otimes \tilde{\alpha}(g_{i(\alpha)})u_\alpha, \otimes \tilde{\alpha}(g_{i(\alpha)})v_\alpha) \\
&= \mathrm{tr}_{k/\mathbb{Q}} \sum_{\beta \in M} E_\beta(\otimes u_\alpha, \otimes v_\alpha) \\
&= \beta(u,v).
\end{aligned}$$

Q.E.D.

Next, let $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{S_0} \times I^{S_1} \in G_{\mathbb{R}} = SL_2(\mathbb{R})^{S_0} \times SU_2^{S_1}$, and set $J = \rho(j)$.

<u>Lemma (1.3.4)</u>. J is a complex structure on $F_{\mathbb{R}}$.

<u>Proof</u>: We know that $\rho \sim \rho_{\mu P}$ over $\mathbb{C}$ for $\mu = 1$ or 2. Hence, $\rho(j)$ is conjugate to $\bigoplus_i \rho_{M_i}(j)$, where each M_i is a rigid molecule. $\rho_{M_i}(j)$ is the tensor product of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $|M_i| - 1$ copies of the identity matrix; hence $\rho_{M_i}(j)$ can be written as $\begin{pmatrix} 0 & I_{m_i} \\ -I_{m_i} & 0 \end{pmatrix}$ with respect to a suitable basis. Then $\bigoplus_i \rho_{M_i}(j) = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ with respect to a suitable basis of

$F_{\mathbb{C}}$. This shows that $J^2 = \rho(j)^2 = -I$, and J is a complex structure on $F_{\mathbb{R}}$.

Q.E.D.

Lemma (1.3.5). $\beta(u,v)$ is nondegenerate, and $\beta(u,Jv)$ is a positive definite symmetric bilinear form on $F_{\mathbb{R}}$ for a suitable choice of the η_i's.

Proof: It is sufficient to show that $\beta(u,Jv)$ is symmetric and positive definite, since that implies the nondegeneracy of $\beta(u,v)$. We begin by proving that $\beta(u,Jv)$ is symmetric.

Since $F_{\mathbb{R}} \subset C_{\mathbb{R}} = \bigoplus_{\sigma\in\mathcal{G}} \bigotimes_{\alpha\in\sigma M} (B_\alpha \otimes_k \mathbb{R})$, it is enough to show $\beta(u,Jv) = \beta(v,Ju)$ for $u = \sum_{\sigma\in\mathcal{G}} \bigotimes_{\alpha\in\sigma M} u_\alpha^\sigma$, $v = \sum_{\sigma\in\mathcal{G}} \bigotimes_{\alpha\in\sigma M} v_\alpha^\sigma$ with $u_\alpha^\sigma, v_\alpha^\sigma \in B_\alpha \otimes_k \mathbb{R}$. Then, $Jv = \rho(j)(v) = \varphi(j)v$

$= \sum_{\sigma\in\mathcal{G}} (\varphi_\sigma(j) \bigotimes_{\alpha\in\sigma M} v_\alpha^\sigma) = \sum_{\sigma\in\mathcal{G}} \bigotimes_{\alpha\in\sigma M} j_\alpha v_\alpha^\sigma$, where $j_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if $\alpha \in S_0$, and $j_\alpha = I$ if $\alpha \in S_1$. Note that $\nu(j_\alpha) = 1$ and $j'_\alpha = j_\alpha^{-1}$ for all $\alpha \in S$, $j_\alpha^{-1} = j_\alpha$ for $\alpha \in S_1$, and $j_\alpha^{-1} = -j_\alpha$ for $\alpha \in S_0$. Using the fact that the reduced trace is invariant under conjugation, and under the canonical involution, we get,

$\beta(u,Jv)$

$$= \sum_{\sigma\in\mathcal{G}} \sum_{\beta\in\sigma M} f_\beta(u_\beta^\sigma, j_\beta v_\beta^\sigma) \prod_{\substack{\alpha\in\sigma M \\ \alpha\neq\beta}} e_\alpha(u_\alpha^\sigma, j_\alpha v_\alpha^\sigma)$$

$$= \sum_{\sigma\in G} \sum_{\beta\in\sigma M} \tau(u_\beta^\sigma \, \tilde{\beta}(\eta_{i(\beta)}) v_\beta^{\sigma'} \, j_\beta') \prod_{\substack{\alpha\in\sigma M\\ \alpha\neq\beta}} \tau(u_\alpha^\sigma \, v_\alpha^{\sigma'} j_\alpha')$$

$$= \sum_{\sigma\in G} \sum_{\beta\in\sigma M} \tau(j_\beta \, v_\beta^\sigma \, \tilde{\beta}(\eta_{i(\beta)})' u_\beta^{\sigma'}) \prod_{\substack{\alpha\in\sigma M\\ \alpha\neq\beta}} \tau(j_\alpha \, v_\alpha^\sigma \, u_\alpha^{\sigma'})$$

$$= \sum_{\sigma\in G} \sum_{\beta\in\sigma M} \tau(v_\beta^\sigma \, \tilde{\beta}(\eta_{i(\beta)})' u_\beta^{\sigma'} \, j_\beta) \prod_{\substack{\alpha\in\sigma M\\ \alpha\neq\beta}} \tau(v_\alpha^\sigma \, u_\alpha^{\sigma'} j_\alpha)$$

$$= -\sum_{\sigma\in G} \sum_{\beta\in\sigma M} \tau(v_\beta^\sigma \, \tilde{\beta}(\eta_{i(\beta)}) u_\beta^{\sigma'} \, j_\beta) \prod_{\substack{\alpha\in\sigma M\\ \alpha\neq\beta}} \tau(v_\alpha^\sigma \, u_\alpha^{\sigma'} \, j_\alpha)$$

(because $\eta_i' = -\eta_i$ for all i)

$$= \sum_{\sigma\in G} \sum_{\beta\in\sigma M} \tau(v_\beta^\sigma \, \tilde{\beta}(\eta_{i(\beta)}) u_\beta^{\sigma'} \, j_\beta') \prod_{\substack{\alpha\in\sigma M\\ \alpha\neq\beta}} \tau(v_\alpha^\sigma \, u_\alpha^{\sigma'} \, j_\alpha')$$

(because $j_\alpha' = -j_\alpha$ for the unique $\alpha \in S_o \cap \sigma M$)

$$= \sum_{\sigma\in G} \sum_{\beta\in\sigma M} f_\beta(v_\beta^\sigma, j_\beta \, u_\beta^\sigma) \prod_{\substack{\alpha\in\sigma M\\ \alpha\neq\beta}} e_\alpha(v_\alpha^\sigma, j_\alpha \, u_\alpha^\sigma)$$

$$= \beta(v, Ju).$$

It remains to show that $\beta(u,Jv)$ is positive definite. We note that we can replace $\eta_i \in B^i$ by $\alpha_i \, \eta_i$ for any nonzero $\alpha_i \in k_i$, and still have $(\alpha_i \, \eta_i)' = -(\alpha_i \, \eta_i)$ and $\nu(\alpha_i \, \eta_i)$ totally positive. We will show that after adjusting η_i by a suitable choice of α_i in this way, we get a positive definite form $\beta(u,Jv)$. Replacing η_i by $\alpha_i \, \eta_i$, we have with u,v as before,

$$\beta(u,Jv) = \sum_{\sigma\in\mathfrak{g}} \sum_{\beta\in\sigma M} \tau(u_\beta^\sigma \tilde{\beta}(\alpha_{i(\beta)} \eta_{i(\beta)}) v_\beta^{\sigma'} j'_\beta) \prod_{\substack{\alpha\in\sigma M \\ \alpha\neq\beta}} \tau(u_\alpha^\sigma v_\alpha^{\sigma'} j'_\alpha)$$

$$= \sum_{\sigma\in\mathfrak{g}} \sum_{\beta\in\sigma M} \beta(\alpha_{i(\beta)}) \tau(u_\beta^\sigma \tilde{\beta}(\eta_{i(\beta)}) v_\beta^{\sigma'} j'_\beta) \prod_{\substack{\alpha\in\sigma M \\ \alpha\neq\beta}} \tau(u_\alpha^\sigma v_\alpha^{\sigma'} j'_\alpha).$$

Let $Q_\beta^\sigma(v) = \tau(v_\beta^\sigma \tilde{\beta}(\eta_{i(\beta)}) v_\beta^{\sigma'} j'_\beta) \cdot \prod_{\substack{\alpha\in\sigma M \\ \alpha\neq\beta}} \tau(v_\alpha^\sigma v_\alpha^{\sigma'} j'_\alpha)$.

Then, $\beta(v,Jv) = \sum_{\sigma\in\mathfrak{g}} \sum_{\beta\in\sigma M} \beta(\alpha_{i(\beta)}) Q_\beta^\sigma(v)$, and our proof of the symmetry of $\beta(u,Jv)$ shows that $Q_\beta^\sigma(v)$ is a quadratic form. It follows from [1] (Lemmas 13.7, 13.12, 13.13) that $Q_\beta^\sigma(v)$ is positive definite if $\beta \in S_o$. For $\sigma \in \mathfrak{g}$, let β_σ be the unique element of $\sigma M \cap S_o$. Then, Corollary 13.8 of [1] implies the existence of a positive integer N_σ such that $N_\sigma Q_{\beta_\sigma}^\sigma(v) + Q_\alpha^\sigma(v)$ is positive definite for all $\alpha \in \sigma M$. Let $N \geq N_\sigma$ for all $\sigma \in \mathfrak{g}$. Then, using the approximation theorem of Artin-Whaples (cf. Lang [12], Chapter 2, Section 1, Theorem 1) we can find $\alpha_i \in k_i$ such that $\psi(\alpha_i) > N\cdot|M|$ for $\psi \in S_o^i$, and,

$$0 < \psi(\alpha_i) < 1 \text{ for } \psi \in S_1^i.$$

Fixing such α_i, we can see that

$$\beta(v,Jv) = \sum_{\sigma\in g} \sum_{\beta\in\sigma M} \beta(\alpha_{i(\beta)}) Q_\beta^\sigma(v) \text{ is positive definite.}$$

Q.E.D.

Define a map $\tau : X \to X' = \mathfrak{S}(F_{\mathbb{R}}, \beta)$ by $\tau(g(0)) = \rho(gjg^{-1})$, where $0 = \nu(1) = (\sqrt{-1}, \ldots, \sqrt{-1})$ is the base point in $X = \mathfrak{H}^{S_o}$. It is easy to check that τ is well-defined: if $g(0) = g'(0)$, then $g^{-1}g' \in K$, and an easy calculation shows that $g^{-1}g'j(g^{-1}g')^{-1} = j$, so that $gjg^{-1} = g'jg'^{-1}$.

Lemma (1.3.6). The map τ defined above is equivariant with respect to ρ, and satisfies the H_2-condition.

Proof: The equivariance of τ is obvious from the definition. We shall check the H_2-condition.

$$\text{Let } j(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{S_o} \times I^{S_1} \in G_{\mathbb{R}}, \text{ for } \theta \in \mathbb{R}.$$

Then, we know that $J_o = \tau(0) = \rho(j(\frac{\pi}{2}))$, and $H_o = \frac{1}{2} j(\frac{\pi}{2})$. Since $\rho \sim \rho_{\mu P}$ for $\mu = 1$ or 2, and P is a rigid polymer, we see that, as in the proof of Lemma (1.3.4), $\rho(j(\theta))$ is conjugate to $\cos\theta \cdot I + \sin\theta \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, i.e., there exists a complex invertible matrix T such that

$$\rho(j(\theta)) = T\left(\cos\theta \cdot I + \sin\theta \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right)T^{-1}$$

$$= \cos\theta \cdot I + \sin\theta \, T\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}T^{-1}$$

Setting $\theta = \frac{\pi}{2}$, gives $\tau(0) = \rho(j(\frac{\pi}{2})) = T\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}T^{-1}$, and we conclude that $\rho(j(\theta)) = \cos\theta\cdot I + \sin\theta\cdot J_o$ for all $\theta \in \mathbb{R}$.

An easy calculation shows that $\exp(2\theta H_o) = j(\theta)$. Then, (1.2.4) shows that

$$\begin{aligned}\exp(d\rho(2\theta H_o)) &= \rho(\exp(2\theta H_o))\\ &= \rho(j(\theta))\\ &= \cos\theta\cdot I + \sin\theta\cdot J_o\end{aligned}$$

But, as in (1.2.3),

$$\exp(\theta J_o) = \cos\theta\cdot I + \sin\theta\cdot J_o.$$

Since $d\rho(2H_o)$ and J_o are elements of $\underline{g}'$ which generate the same 1-parameter subgroup of $G'_{\mathbb{R}} = Sp(F_{\mathbb{R}},\beta)$, they must be equal: $d\rho(2H_o) = J_o$, or, $d\rho(H_o) = \frac{1}{2} J_o$, which is the H_2-condition.

Q.E.D.

<u>Theorem (1.3.7)</u>. Let P be a rigid polymer. Then, for some positive integer μ, $\rho_{\mu P}$ defines a family of abelian varieties which satisfies the H_2-condition and is therefore rigid.

<u>Proof</u>: Because of Lemma (1.1.6(a)), it is enough to prove the theorem for polymers of the type we have been considering so far: $P = \sum_{\sigma\in\mathcal{G}} \sigma M$, with each σM rigid.

For each $i = 1,\ldots,n$, there exists a cocompact, torsion-free, arithmetic subgroup Γ_i of $G^i_{\mathbb{Q}}$ (Vignéras [20], Chapter IV, Theorem 1.1 and Proposition 1.6). Then, $\Gamma = \prod_{i=1}^{n} \Gamma_i$ is a cocompact, torsion-free arithmetic subgroup of $G_{\mathbb{Q}}$. Let L be any Γ-lattice in $F_{\mathbb{Q}}$ on which β takes integer values (such an L always exists, see Satake [18], p. 198, Remark 2). Then, $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ define a Kuga fiber variety which satisfies the H_2-condition by Lemma (1.3.6), is rigid by Proposition (1.2.2), and is defined by either ρ_P or ρ_{2P} by Lemma (1.3.2).

Q.E.D.

CHAPTER 2. Absolute Hodge Cycles

In this chapter, we review Deligne's theory of absolute Hodge cycles, as described in [4] and [5]. We include only those definitions and results which are needed for our purpose. In this chapter all algebraic varieties are assumed to be connected, smooth, and projective.

2.1. Cohomology Theories

We now review the three cohomology theories we need, namely topological (singular) cohomology, algebraic de Rham cohomology, and étale cohomology. Our aim is to fix notations and state some results which will be used later; we follow Deligne ([4], Section 1).

Topological Theory. Let X be an algebraic variety over $\mathbb{C}$. We write $H^n_B(X)$ for $H^n(X^{an},\mathbb{Q})$, the singular cohomology of the complex manifold associated to X. If $K = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, we also write $H^n_B(X,K)$ for $H^n(X^{an},K) = H^n_B(X) \otimes_{\mathbb{Q}} K$.

Now let X be an algebraic variety over a field k, and $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Then, $\sigma X = X \otimes_\sigma \mathbb{C}$ is a variety over $\mathbb{C}$ and we write $H^n_\sigma(X)$ for $H^n_B(\sigma X)$, and $H^n_\sigma(X,K)$ for $H^n_B(\sigma X,K)$ if $K = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. There is a canonical Hodge decomposition:

$$H^n_\sigma(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(\sigma X^{an}), \overline{H^{p,q}_\sigma(\sigma X^{an})} = H^{q,p}(\sigma X^{an}).$$

Setting $F^pH^n_\sigma(X) = \bigoplus_{p'\geq p} H^{p',q'}(\sigma X^{an})$ defines a canonical filtration on $H^n_\sigma(X,\mathbb{C})$, called the Hodge filtration.

<u>Algebraic de Rham Theory</u> (See Hartshorne [6] for details.) Let X be an algebraic variety over a field k of characteristic zero. We denote by $\Omega^\bullet_{X/k}$ the complex in which $\Omega^n_{X/k}$ is the sheaf of algebraic differential n-forms on X. Define the algebraic de Rham cohomology $H^n_{DR}(X/k)$ to be $\mathbb{H}^n(X_{Zar},\Omega^\bullet_{X/k})$, the hypercohomology of $\Omega^\bullet_{X/k}$ relative to the Zariski topology on X. For any embedding $\sigma : k \hookrightarrow k'$ there is a canonical isomorphism $H^n_{DR}(X/k) \otimes_{k,\sigma} k' \cong H^n_{DR}(X \otimes_\sigma k'/k')$. There is a canonically defined filtration $F^pH^n_{DR}(X/k)$ on $H^n_{DR}(X/k)$, which is stable under base change.

<u>Étale Theory</u> (See Milne [14] for details.) Let X be an algebraic variety over an algebraically closed field k of characteristic zero. Let $\mathbb{A}^f = \left(\varprojlim_{m>0} \mathbb{Z}/m\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q}$. We write $H^n_{et}(X)$ for $\left(\varprojlim_{m>0} H^n(X_{et},\mathbb{Z}/m\mathbb{Z})\right) \otimes_{\mathbb{Z}} \mathbb{Q}$. $H^n_{et}(X)$ is a free $\mathbb{A}^f$-module of finite rank.

For a prime number ℓ, let $H^n(X_{et},\mathbb{Z}_\ell) = \varprojlim_{m>0} H^n(X_{et},\mathbb{Z}/\ell^m\mathbb{Z})$ and $H^n(X_{et},\mathbb{Q}_\ell) = H^n(X_{et},\mathbb{Z}_\ell) \otimes_{\mathbb{Z}} \mathbb{Q}$. An inclusion $k \hookrightarrow k'$ of algebraically closed fields induces canonical isomorphisms

$H^n_{et}(X) \cong H^n_{et}(X \otimes_k k')$ and $H^n(X_{et}, \mathbb{Q}_\ell) \cong H^n((X \otimes_k k')_{et}, \mathbb{Q}_\ell)$. $H^n_{et}(X)$ may also be described as the restricted direct product of the spaces $H^n(X_{et}, \mathbb{Q}_\ell)$, ℓ prime, with respect to the subspaces $H^n(X_{et}, \mathbb{Z}_\ell)$.

Tate Twist. In each of our cohomology theories, a Tate twist is introduced as described below.

Let k be a field of characteristic zero, and $\mu_n = \{\zeta \in \bar{k} \mid \zeta^n = 1\}$. Then, define:

$$\mathbb{Q}_B(1) = 2\pi i\, \mathbb{Q};$$

$$\mathbb{Q}_{DR}(1) = k;$$

$$\mathbb{Q}_{et}(1) = \left(\varprojlim_{n>0} \mu_n\right) \otimes_{\mathbb{Z}} \mathbb{Q};$$

$$\mathbb{Q}_\ell(1) = \left(\varprojlim_{n>0} \mu_{\ell^n}\right) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad (\ell \text{ a prime number}).$$

For a nonnegative integer m, $\mathbb{Q}(m)$ denotes the tensor product of m copies of $\mathbb{Q}(1)$. Thus, $\mathbb{Q}_B(m) = (2\pi i)^m \mathbb{Q}$ is a one-dimensional $\mathbb{Q}$-subspace of $\mathbb{C}$, $\mathbb{Q}_{DR}(m) = k$, $\mathbb{Q}_{et}(m)$ is a free $\mathbb{A}^f$-module of rank one on which $\mathrm{Aut}(\bar{k})$ acts, and $\mathbb{Q}_\ell(m)$ is a one-dimensional $\mathbb{Q}_\ell$-vector space on which $\mathrm{Aut}(\bar{k})$ acts.

Now let X be an algebraic variety over k.

If $\sigma : k \hookrightarrow \mathbb{C}$ is an embedding, define $H^n_\sigma(X)(m) = H^n_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{Q}_B(m)$. If $k = \mathbb{C}$, define $H^n_B(X)(m) = H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_B(m)$.

The Tate twist is trivial on algebraic de Rham cohomology: $H^n_{DR}(X)(m) = H^n_{DR}(X) \otimes_k \mathbb{Q}_{DR}(m) = H^n_{DR}(X)$.

Next suppose k is algebraically closed, and define $H^n_{et}(X)(m) = H^n_{et}(X) \otimes_{\mathbb{A}^f} \mathbb{Q}_{et}(m)$, and, for a prime number ℓ, $H^n(X_{et},\mathbb{Q}_\ell)(m) = H^n(X_{et},\mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(m)$. If X is defined over a subfield k_o of k with $\overline{k_o} = k$, then $H^n_{et}(X)(m)$ and $H^n(X_{et},\mathbb{Q}_\ell)(m)$ are $\mathrm{Gal}(k/k_o)$-modules.

If F is a subspace of $H^n(X)$, then $F(m)$ is defined as a subspace of $H^n(X)(m)$ in the obvious way, in any cohomology theory.

<u>Comparison Isomorphisms</u>. Let X be an algebraic variety over a field k of characteristic zero, and $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Then, there is a canonical isomorphism

$$H^n_{DR}(X/k)(m) \underset{k,\sigma}{\otimes} \mathbb{C} \cong H^n_\sigma(X)(m) \otimes_{\mathbb{Q}} \mathbb{C}, \tag{2.1.1}$$

inducing an isomorphism

$$F^p H^n_{DR}(X/k) \underset{k,\sigma}{\otimes} \mathbb{C} \cong F^p H^n_\sigma(X). \tag{2.1.2}$$

Assume, further, that k is algebraically closed, and ℓ is a prime number. Then, there are canonical isomorphisms

$$H^n_\sigma(X)(m) \otimes_{\mathbb{Q}} \mathbb{A}^f \cong H^n_{et}(X)(m), \qquad (2.1.3)$$

$$H^n_\sigma(X)(m) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^n(X_{et}, \mathbb{Q}_\ell)(m). \qquad (2.1.4)$$

2.2. Absolute Hodge Cycles

Let X be a variety over an algebraically closed field k, which we assume to be embeddable in $\mathbb{C}$. For $p \geq 0$, write $H^{2p}_{\mathbb{A}}(x)(p)$ for the free $k \times \mathbb{A}^f$-module $H^{2p}_{DR}(X/k)(p) \times H^{2p}_{et}(x)(p)$.

An embedding $\sigma : k \hookrightarrow \mathbb{C}$ gives rise to canonical isomorphisms:

$$H^{2p}_{DR}(X/k)(p) \underset{k,\sigma}{\otimes} \mathbb{C} \xrightarrow{\cong} H^{2p}_{DR}(\sigma X/\mathbb{C})(p),$$

and

$$H^{2p}_{et}(X)(p) \xrightarrow{\cong} H^{2p}_{et}(\sigma X)(p).$$

Taking the product, we get a canonical embedding:

$$\sigma_* : H^{2p}_{\mathbb{A}}(x)(p) \hookrightarrow H^{2p}_{\mathbb{A}}(\sigma X)(p). \qquad (2.2.1)$$

Also, the comparison isomorphisms (2.1.1) and (2.1.3) induce a diagonal embedding:

$$\Delta : H^{2p}_\sigma(X)(p) \hookrightarrow H^{2p}_{\mathbb{A}}(\sigma X)(p). \qquad (2.2.2)$$

An element $t = (t_{DR}, t_{et}) \in H^{2p}_{\mathbb{A}}(X)(p)$ is called <u>rational</u> relative to σ if $\sigma_*(t)$ belongs to the image of Δ. t is called <u>Hodge</u> relative to σ if t is rational relative to σ and $t_{DR} \in F^p H^{2p}_{DR}(X/k)$. t is called an <u>absolute Hodge cycle</u>

if t is Hodge relative to every embedding of k into $\mathbb{C}$. The space of all absolute Hodge cycles in $H^{2p}_{\mathbb{A}}(X)(p)$ is denoted by $C^p_{AH}(X/k)$; it is a finite dimensional vector space over $\mathbb{Q}$.

By abuse of language, an element $t \in H^{2p}_{\sigma}(X)(p)$ is sometimes called an absolute Hodge cycle when $\Delta(t)$ is an absolute Hodge cycle.

If $k = \mathbb{C}$, then $t = (t_{DR}, t_{et}) \in H^{2p}_{\mathbb{A}}(X)(p)$ is an absolute Hodge cycle if and only if t is rational relative to every $\sigma \in \mathrm{Aut}(\mathbb{C})$, and $t_{DR} \in F^p H^{2p}_{DR}(X/\mathbb{C})$.

Let $k \hookrightarrow k'$ be an extension of algebraically closed fields, both embeddable in $\mathbb{C}$. The natural map

$$H^{2p}_{\mathbb{A}}(X)(p) \rightarrow H^{2p}_{\mathbb{A}}(X \otimes_k k')(p),$$

induces an isomorphism:

$C^p_{AH}(X/k) \xrightarrow{\cong} C^p_{AH}(X \otimes_k k'/k')$, (Deligne [4], Proposition 2.9(a)).

For this reason, we shall usually just write $C^p_{AH}(X)$ for $C^p_{AH}(X/k)$, when k is algebraically closed.

<u>Theorem (2.2.3)</u> (Deligne [4], Main Theorem 2.11). If X is an abelian variety over an algebraically closed field k, and t is a Hodge cycle on X relative to one embedding $\sigma : k \hookrightarrow \mathbb{C}$, then t is an absolute Hodge cycle.

Now suppose X is defined over a subfield k_o of k with $\overline{k_o} = k$, i.e., there exists a variety X_o over k_o with $X = X_o \otimes_{k_o} k$. Then $\mathrm{Gal}(k/k_o)$ acts on $C^p_{AH}(X/k)$, and the subspace of $C^p_{AH}(X/k)$ fixed by $\mathrm{Gal}(k/k_o)$ is denoted by $C^p_{AH}(X/k_o)$.

Proposition (2.2.4) (Deligne [4], Proposition 2.9(b)). If X is defined over a subfield k_o of k with $\overline{k_o} = k$, then there exists a finite extension k' of k_o such that $\mathrm{Gal}(k/k')$ acts trivially on $C^p_{AH}(X/k)$.

2.3. Maps Induced on Cohomology

Let X and Y be varieties over an algebraically closed field k which is embeddable in $\mathbb{C}$. Let k_o be a subfield of k with $\overline{k_o} = k$, and assume that X and Y are defined over k, i.e., there exist varieties X_o and Y_o over k_o such that $X = X_o \otimes_{k_o} k$ and $Y = Y_o \otimes_{k_o} k$. Let n = dim X, and let p be a positive integer, $0 \le p \le 2n$. Then, in any cohomology theory (singular, étale, or algebraic de Rham), we have the following canonical isomorphisms:

$$H^{2n+2p}(X\times Y)(p+n) \cong \bigoplus_{r+s=2n+2p} H^r(X) \otimes H^s(Y)(p+n)$$

$$\cong \bigoplus_{s=r+2p} H^r(X)^* \otimes H^s(Y)(p)$$

$$\cong \bigoplus_{r=0}^{2n} \mathrm{hom}(H^r(X), H^{r+2p}(Y)(p)).$$

<u>Theorem (2.3.1)</u> (Deligne and Milne [5], Proposition 6.1).
An element $Z \in C_{AH}^{n+p}(X \times Y/k_o)$ gives rise to

(a) for each prime ℓ, homomorphisms

$$Z_\ell^r : H^r(X_{et}, \mathbb{Q}_\ell) \to H^{r+2p}(Y_{et}, \mathbb{Q}_\ell)(p);$$

(b) homomorphisms

$$Z_{DR}^r : H_{DR}^r(X_o/k_o) \to H_{DR}^{r+2p}(Y_o/k_o)(p);$$

(c) for each embedding $\sigma : k \hookrightarrow \mathbb{C}$, homomorphisms

$$Z_\sigma^r : H_\sigma^r(X) \to H_\sigma^{r+2p}(Y)(p).$$

These maps satisfy the following conditions:

(d) for all $\gamma \in \mathrm{Gal}(k/k_o)$ and all primes ℓ,

$$\gamma(Z_\ell^r) = Z_\ell^r;$$

(e) Z_{DR}^r is compatible with the Hodge filtrations;

(f) for each $\sigma : k \hookrightarrow \mathbb{C}$, the maps Z_ℓ^r, Z_{DR}^r, and Z_σ^r correspond under the comparison isomorphisms (2.1.1), (2.1.4).

Conversely, any family of maps Z_ℓ^r, Z_{DR}^r as in (a) and (b) arises from a unique $Z \in C_{AH}^{n+p}(X \times Y/k_o)$ provided that Z_ℓ^r and Z_{DR}^r satisfy (d) and (e), and, for every $\sigma : k \hookrightarrow \mathbb{C}$ there exist Z_σ^r as in (c) such that Z_ℓ^r, Z_{DR}^r, and Z_σ^r satisfy (f).

2.4. Functorial Properties

In this section we investigate the behavior of absolute Hodge cycles under maps of algebraic varieties.

Let $f : X \to Y$ be a map of algebraic varieties defined over $\mathbb{C}$, $n = \dim X$, $m = \dim Y$. Since cohomology is a contravariant functor, f induces maps $f^* : H^r(Y) \to H^r(X)$ in any cohomology theory (singular, algebraic de Rham, or étale). Then, by Poincaré duality, we get maps

$$f_! = {}^t f^* : H^r(X) \to H^{r+2m-2n}(Y)(m-n).$$

In the analytic de Rham theory we may describe $f_!$ as follows: If $\alpha \in H^r(X^{an},\mathbb{C})$, then $f_!\alpha$ is the unique element of $H^{r+2m-2n}(Y^{an},\mathbb{C})$ such that

$$\int_Y f_!\alpha \wedge \zeta = (2\pi i)^{m-n} \int_X \alpha \wedge f^*\zeta, \quad \forall \zeta \in H^{2n-r}(Y^{an},\mathbb{C}). \tag{2.4.1}$$

<u>Lemma (2.4.2)</u>. Let $f : X \to Y$ be a map of varieties over $\mathbb{C}$, $n = \dim X$, $m = \dim Y$. If $\omega \in C^p_{AH}(Y)$, then $f^*\omega \in C^p_{AH}(X)$. If $\alpha \in C^p_{AH}(X)$, then $f_!\alpha \in C^{p+m-n}_{AH}(Y)$.

<u>Proof</u>: First of all we observe that $f^*\omega$ is a Hodge cycle since f^* takes rational cycles to rational cycles, and is compatible with the Hodge filtration.

Next, since ω is absolute Hodge, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$ there exists $\omega^{(\sigma)} \in H^{2p}_\sigma(Y)(p)$ whose image in $H^{2p}_{\mathbb{A}}(\sigma Y)(p)$ is

$\sigma_*(\omega)$ (see (2.2.1), (2.2.2)). Then, the commutative diagram

$$\begin{array}{ccccc} H^{2p}_{\sigma}(Y)(p) & \xrightarrow{\Delta} & H^{2p}_{\mathbb{A}}(\sigma Y)(p) & \xleftarrow{\sigma_*} & H^{2p}_{\mathbb{A}}(Y)(p) \\ \downarrow{\scriptstyle (\sigma f)^*} & & \downarrow{\scriptstyle (\sigma f)^*} & & \downarrow{\scriptstyle f^*} \\ H^{2p}_{\sigma}(X)(p) & \xrightarrow{\Delta} & H^{2p}_{\mathbb{A}}(\sigma X)(p) & \xleftarrow{\sigma_*} & H^{2p}_{\mathbb{A}}(X)(p) \end{array}$$

shows that $\sigma_*(f^*\omega) = (\sigma f)^*(\sigma_*\omega) = (\sigma f)^*(\Delta(\omega^{(\sigma)}))$ $= \Delta((\sigma f)^*(\omega^{(\sigma)}))$. Hence, $f^*\omega$ is rational relative to σ, and this completes the proof of the fact that $f^*\omega$ is an absolute Hodge cycle.

The proof that α is rational relative to each $\sigma \in \mathrm{Aut}(\mathbb{C})$ is similar. Since α is absolute Hodge, there exists $\alpha^{(\sigma)} \in H^{2p}_{\sigma}(X)(p)$ such that $\Delta(\alpha^{(\sigma)}) = \sigma_*(\alpha)$. Then, $\sigma_*(f_!\alpha)$ $= (\sigma f)_!(\sigma_*(\alpha)) = (\sigma f)_!(\Delta(\alpha^{(\sigma)})) = \Delta((\sigma f)_!(\alpha^{(\sigma)}))$, shows that $f_!\alpha$ is rational relative to σ.

It remains to show that $f_!\alpha$ is Hodge. Because of the comparison isomorphism (2.1.2), it is enough to show that $f_!\alpha \in H^{q,q}(Y^{an},\mathbb{C})$, where $q = p+m-n$. Write $f_!\alpha = \sum_{s+t=2q} z^{s,t}$ with $z^{s,t} \in H^{s,t}(Y^{an},\mathbb{C})$, and let $\zeta \in H^{a,b}(Y^{an},\mathbb{C})$ with $a+b = 2n-2p$. Then $f^*\zeta \in H^{a,b}(X^{an},\mathbb{C})$, and we know from (2.4.1) that

$\int_Y f_! \alpha \wedge \zeta = (2\pi i)^{m-n} \int_X \alpha \wedge f^*\zeta$. If $a = b$, then,

$f_! \alpha \wedge \zeta = Z^{q,q} \wedge \zeta$, so $\int_Y Z^{q,q} \wedge \zeta = (2\pi i)^{m-n} \int_X \alpha \wedge f^*\zeta$. If

$a \neq b$, then we see that both $\int_Y Z^{q,q} \wedge \zeta$ and

$\int_X \alpha \wedge f^*\zeta$ are zero. Thus $\int_Y Z^{q,q} \wedge \zeta = (2\pi i)^{m-n} \int_X \alpha \wedge f^*\zeta$

for all $\zeta \in H^{2n-2p}(Y^{an}, \mathbb{C})$. Therefore, $f_! \alpha = Z^{q,q}$; this completes the proof.

Q.E.D.

Remark: The first statement of the Lemma is property (b) in the definition of an accessible cycle (Deligne [4], p. 10).

CHAPTER 3. Zeta Functions

This chapter contains our main results.

3.1. Cohomology of Kuga Fiber Varieties

Let $f : A \to V$ be a group theoretical family of abelian varieties, constructed from data $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$. We shall now summarize some known results on the topology of A (Kuga [8], [9], Tjiok [19]).

Since X is diffeomorphic to a Euclidean space, we have a global coordinate system $(x^1,\ldots,x^{2d})$ on X. Let $(u^1,\ldots,u^{2m})$ be a basis of the dual space of F. Then, $(x^1,\ldots,x^{2d},u^1,\ldots,u^{2m})$ is a global coordinate system on $X \times F_{\mathbb{R}} = \tilde{A}$. Such a coordinate system is called an admissible coordinate system. Since $A = {}_{\Gamma KL}\backslash(X \times F_{\mathbb{R}})$, the cohomology of A (real or complex coefficients) may be computed as the cohomology of the complex $\Omega^{\cdot}(X \times F_{\mathbb{R}})^{\Gamma KL}$, of ΓKL-invariant differential forms on $X \times F_{\mathbb{R}}$. Let $p : X \times F_{\mathbb{R}} \to A$ be the universal covering map. We shall identify a differential form ω on A with its pullback $p^*\omega$ on $\tilde{A}$.

Let $q : X \to V$ be the universal covering map, $x \in X$, and $P = q(x)$. Then, there is a canonical isomorphism (depending on x), $\Gamma \cong \pi_1(V,P)$, under which $\gamma \in \Gamma$ corresponds to the image in V of any path connecting $\gamma(x)$ to x in X. $\pi_1(V,P)$ acts on the cohomology of the fiber, $H^b(A_P,\mathbb{Q})$. Since A_P is a torus $F_{\mathbb{R}}/L$, identifying L with $H_1(A_P,\mathbb{Z})$ gives a canonical isomorphism $H^b(A_P,\mathbb{Q}) \cong \overset{b}{\wedge} F_{\mathbb{Q}}^*$. Since Γ acts on $F_{\mathbb{Q}}$ via ρ, Γ acts on $\overset{b}{\wedge} F_{\mathbb{Q}}^*$. The isomorphisms $\pi_1(V,P) \cong \Gamma$ and $H^b(A_P,\mathbb{Q}) \cong \overset{b}{\wedge} F_{\mathbb{Q}}^*$ are compatible with the actions of $\pi_1(V,P)$ on $H^b(A_P,\mathbb{Q})$ and of Γ on $\overset{b}{\wedge} F_{\mathbb{Q}}^*$. Since ρ is self-dual (Lemma (1.1.5)), we have an isomorphism of Γ-modules:

$$H^b(A_P,\mathbb{Q}) \cong \overset{b}{\wedge} F_{\mathbb{Q}}. \qquad (3.1.1)$$

Now let $f : A \to V$ be any abelian scheme, (not necessarily group theoretical), defined over a subfield k of $\mathbb{C}$. We assume that V is a smooth, projective variety. For any integer N, let $\theta(N) : A \to A$ be the map which is multiplication by N on each fiber. In each cohomology theory (singular, algebraic de Rham, and étale), let

$$H^{\langle a,b\rangle}(A) = \{\omega \in H^{a+b}(A) \mid \theta(N)^*\omega = N^b\omega \ \forall\ N \in \mathbb{Z}\}. \qquad (3.1.2)$$

Then, we have the following fiber-base decompositions:

$$H_B^r(A,K) = \bigoplus_{a+b=r} H_B^{\langle a,b\rangle}(A,K),\ K = \mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C}; \qquad (3.1.3)$$

$$H_{DR}^r(A/\mathbb{C}) = \bigoplus_{a+b=r} H_{DR}^{\langle a,b\rangle}(A/\mathbb{C}),\ (k=\mathbb{C}); \qquad (3.1.4)$$

$$H_{et}^r(A) = \bigoplus_{a+b=r} H_{et}^{\langle a,b\rangle}(A),\ (k \text{ algebraically closed}). \qquad (3.1.5)$$

For a proof of (3.1.5), see the proof of Lemma 5.3 of Deligne [3]. The same argument (due to Lieberman) works for (3.1.3), and then (3.1.4) follows from the comparison isomorphism (2.1.1).

Returning to the group theoretical case, we have canonical isomorphisms:

$$H^{\langle a,b\rangle}(A,\mathbb{Q}) \cong H^a(V,R^b f_*(\mathbb{Q})) \cong H^a(\Gamma, \overset{b}{\wedge} F_{\mathbb{Q}}). \qquad (3.1.6)$$

With respect to an admissible coordinate system $(x^1,\ldots,x^{2d}, u^1,\ldots,u^{2m})$ on $\tilde{A}$, a cohomology class in $H^{\langle a,b\rangle}(A,\mathbb{C})$ can be represented by a differential form

$$\omega = \sum_{C,D} \varphi_{C,D}(x)\ dx^C \wedge du^D, \qquad (3.1.7)$$

where, C runs over all a-tuples $(i_1,\ldots,i_a)$ with $1 \le i_1 < i_2 <\ldots< i_a \le 2d$, D runs over all b-tuples $(j_1,\ldots,j_b)$ with $1 \le j_1 < j_2 <\ldots< j_b \le 2m$, $dx^C = dx^{i_1} \wedge\ldots\wedge dx^{i_a}$, $dt^D = dt^{j_1} \wedge\ldots\wedge dt^{j_b}$, and, $\varphi_{C,D}(x)$ are C^∞-functions on X. Note that $\varphi_{C,D}(x)$ depends only on $x \in X$, and not on $u \in F_{\mathbb{R}}$ (see Kuga [8], Lemma II-3-5).

Lemma (3.1.8). If D_Γ is a fundamental domain for the action of Γ on X, and D_L is a fundamental domain for the lattice $L \subset F_{\mathbb{R}}$, then $D_\Gamma \times D_L$ is a fundamental domain for the action of $\Gamma \ltimes L$ on $X \times F_{\mathbb{R}}$.

Proof: Let (x,u), $(x',u') \in D_\Gamma \times D_L$ and suppose there exists $(\gamma,\ell) \in \Gamma \ltimes L$ such that $(\gamma,\ell)\cdot(x,u) = (x',u')$. Then, $(\gamma x, \rho(\gamma)u+\ell) = (x',u')$. Since D_Γ is a fundamental domain for Γ, and Γ acts without fixed points on X, $\gamma x = x'$ implies $x = x'$ and $\gamma = 1$. Then $u + \ell = u'$ implies $u = u'$ and $\ell = \ell'$, since $u,u' \in D_L$. Thus $(x,u) = (x',u')$.

Conversely, let $(x,u) \in X \times F_{\mathbb{R}}$. Then there exist $x' \in D_\Gamma$ and $\gamma \in \Gamma$ such that $\gamma x = x'$. Let $\ell \in L$ be such that $u' = \rho(\gamma)u + \ell \in D_L$. Then, $(\gamma,\ell)(x,u) = (x',u') \in D_\Gamma \times D_L$.

Q.E.D.

3.2. Hodge Cycles in a Fiber

Hodge cycles in a fiber A_p of a group theoretical family of abelian varieties have been studied by many people: Addington (unpublished), Kuga [9], [10], Mumford [15], [16], and Weil [21]. The following proposition may be well known, but we include it for the sake of completeness.

Proposition (3.2.1). If $A \to V$ is a Kuga fiber variety satisfying the H_2-condition (1.1.3) then, for any $P \in V$, the space $H^{2p}(A_p,\mathbb{Q})^{\Gamma(p)}$, of Γ-invariant rational cycles, consists of Hodge cycles.

Proof: We have,

$$\rho(\exp(2\theta H_o)) = \exp(d\rho(2\theta H_o)) \qquad \text{(by 1.2.4)}$$
$$= \exp(\theta\cdot\tau(0)) \qquad \text{(by 1.1.3)}$$
$$= \cos\theta\cdot I + \sin\theta\cdot\tau(0) \qquad \text{(by 1.2.3).}$$

Let $x \in X$ be such that $q(x) = P$, where $q : X \to V$ is the natural map. If $x = g(0)$ for $g \in G_{\mathbb{R}}$, then

$$\cos\theta\cdot I + \sin\theta\cdot\tau(x) = \cos\theta\cdot I + \sin\theta\cdot\rho(g)\tau(0)\rho(g)^{-1}$$
$$= \rho(g)(\cos\theta\cdot I + \sin\theta\cdot\tau(0))\rho(g)^{-1}$$
$$= \rho(g\exp(2\theta H_o)g^{-1}),$$

and we conclude that $\cos\theta\cdot I + \sin\theta\cdot\tau(x) \in \rho(G_{\mathbb{R}})$.

Since Γ is Zariski-dense in G (Proposition (1.1.1)), any $\omega \in H^{2p}(A_P,\mathbb{Q})^{\Gamma}(p)$ is actually $G_{\mathbb{R}}$-invariant, and therefore invariant under the action of $\cos\theta\cdot I + \sin\theta\cdot\tau(x)$ for all $\theta \in \mathbb{R}$. This implies that ω is a Hodge cycle.

Q.E.D.

Remark: In the course of the above proof we have shown that $A \to V$ satisfies the Condition Inner of Kuga ([9], 1.4.10). Hence, for a generic point P, $H^{2p}(A_P,\mathbb{Q})^{\Gamma}(p)$ equals the space of Hodge cycles of degree 2p on A_P. However, the above proposition is sufficient for our purpose.

3.3. Absolute Hodge Cycles in the Total Space

In this section we show that for a Kuga fiber variety $f : A \to V$ satisfying the H_2-condition (1.1.3), the subspace $H^{\langle 0,2p \rangle}(A,\mathbb{Q})(p)$ of $H^{2p}(A,\mathbb{Q})(p)$ consists of absolute Hodge cycles.

Let $P \in V$, and let $\iota : A_P \to V$ be the inclusion map. Let $(x^1,\ldots,x^{2d},u^1,\ldots,u^{2m})$ be an admissible coordinate system on $\tilde{A}$, where $d = \dim V$ and $m = \dim A_P$. Then (3.1.7) shows that any element of $H^{\langle 0,2p \rangle}(A,\mathbb{C})$ can be represented by a differential form $\omega = \sum_D \varphi_D(x)du^D$. Since ω is closed, each $\varphi_D(x)$ is a constant $a_D \in \mathbb{C}$. Hence $\omega = \sum_D a_D\, du^D$ and is Γ-invariant. Therefore, the map

$$\iota^* : H^{\langle 0,2p \rangle}(A,\mathbb{C}) \to H^{2p}(A_P,\mathbb{C})^{\Gamma} \qquad (3.3.1)$$

is an isomorphism.

We denote by dV the cohomology class of the point P in any cohomology theory: $dV \in H^{2d}(V)(d)$, (Deligne [4], Section 1). In the analytic de Rham theory dV is represented by a volume form which we again call dV, and is characterized by $\int_V dV = (2\pi i)^d$. Since dV is an algebraic cycle, it is absolute Hodge (Deligne [4], Example 2.1 (a)).

Lemma (3.3.2). If $\omega_p \in H^{2p}(A_p,\mathbb{C})^{\Gamma}$, then $\iota_!(\omega_p) = f^*dV \wedge \omega$, where ω is the unique element of $H^{<0,2p>}(A,\mathbb{C})$ such that $\iota^*\omega = \omega_p$.

Proof: By (2.4.1) we have to show that

$$\int_A f^*dV \wedge \omega \wedge \zeta = (2\pi i)^d \int_{A_p} \omega_p \wedge \iota^*\zeta \tag{3.3.3}$$

for all $\zeta \in H^{2m-2p}(A,\mathbb{C})$. Because of the fiber-base decomposition (3.1.3), it is sufficient to prove (3.3.3) for all $\zeta \in H^{<a,b>}(A,\mathbb{C})$ with $a+b = 2m-2p$. Then, both sides of (3.3.3) are zero unless $a=0$ and $b=2m-2p$, which we assume. Let D_Γ be a fundamental domain for the action of Γ on X, and D_L a fundamental domain for $L \subset F_{\mathbb{R}}$. Then $D_\Gamma \times D_L$ is a fundamental domain for the action of $\Gamma \ltimes L$ on $X \times F_{\mathbb{R}}$ (Lemma (3.1.8)). Since $f^*dV \in H^{<2d,0>}(A,\mathbb{C})$ and $\omega \wedge \zeta \in H^{<0,2m>}(A,\mathbb{C})$, we have,

$$\int_A f^*dV \wedge \omega \wedge \zeta = \int_{D_\Gamma \times D_L} f^*dV \wedge \omega \wedge \zeta = \Big(\int_{D_\Gamma} dV\Big)\Big(\int_{D_L} \iota^*(\omega\wedge\zeta)\Big)$$

$$= (2\pi i)^d \int_{A_p} \omega_p \wedge \iota^*\zeta.$$

Q.E.D.

Lemma (3.3.4). The subspace $H^{<2d,0>}(A,\mathbb{Q})(d) \otimes H^{<0,2p>}(A,\mathbb{Q})(p)$ of $H^{<2d,2p>}(A,\mathbb{Q})(d+p)$ consists of absolute Hodge cycles.

Proof: The previous lemma shows that $\iota_!$ maps $H^{2p}(A_p,\mathbb{Q})^{\Gamma}(p)$ onto $H^{<2d,0>}(A,\mathbb{Q})(d) \otimes H^{<0,2p>}(A,\mathbb{Q})(p)$, Proposition (3.2.1) and Theorem (2.2.3) show that $H^{2p}(A_p,\mathbb{Q})^{\Gamma}(p)$ consists of absolute Hodge cycles, and Lemma (2.4.2) shows that $\iota_!$ maps absolute Hodge cycles to absolute Hodge cycles.

Q.E.D.

Theorem (3.3.5). Let $f : A \to V$ be a Kuga fiber variety satisfying the H_2-condition. Then, for $p \geq 0$, the subspace $H^{<0,2p>}(A,\mathbb{Q})(p)$ of $H^{2p}(A,\mathbb{Q})(p)$ consists of absolute Hodge cycles.

Proof: Proposition (3.2.1) and the isomorphism (3.3.1) imply that $H^{<0,2p>}(A,\mathbb{Q})(p)$ consists of Hodge cycles. It will therefore be sufficient to show that if $\sigma \in \mathrm{Aut}(\mathbb{C})$, and ω is rational in $H_{\mathbb{A}}^{<0,2p>}(A)(p) = H_{DR}^{<0,2p>}(A/\mathbb{C})(p) \times H_{et}^{<0,2p>}(A)(p)$, then $\sigma(\omega)$ is rational in $H_{\mathbb{A}}^{<0,2p>}(\sigma A)(p)$. We will chase around the following commutative diagram, where $\chi(\zeta) = f^* dV \wedge \zeta$.

$$
\begin{array}{ccc}
H_B^{<0,2p>}(A)(p) & \xrightarrow[\chi]{\cong} & H_B^{<2d,0>}(A)(d) \otimes H_B^{<0,2p>}(A)(p) \\
\Big\downarrow \Delta & & \Big\downarrow \Delta \\
H_{\mathbb{A}}^{<0,2p>}(A)(p) & \xrightarrow[\chi]{\cong} & H_{\mathbb{A}}^{<2d,0>}(A)(d) \otimes H_{\mathbb{A}}^{<0,2p>}(A)(p) \\
\sigma_* \Big\downarrow \cong & & \cong \Big\downarrow \sigma_* \\
H_{\mathbb{A}}^{<0,2p>}(\sigma A)(p) & \xrightarrow[\sigma\chi]{\cong} & H_{\mathbb{A}}^{<2d,0>}(\sigma A)(d) \otimes H_{\mathbb{A}}^{<0,2p>}(\sigma A)(p) \\
\sigma\Delta \Big\uparrow & & \Big\uparrow \sigma\Delta \\
H_\sigma^{<0,2p>}(A)(p) & \xrightarrow[\sigma\chi]{\cong} & H_\sigma^{<2d,0>}(A)(d) \otimes H_\sigma^{<0,2p>}(A)(p)
\end{array}
$$

Now let $\omega_B \in H_B^{<0,2p>}(A)(p)$ be such that $\Delta(\omega_B) = \omega$. Then Lemma (3.3.4) shows that $\chi(\omega) = \Delta\chi(\omega_B)$ is an absolute Hodge cycle. In particular, $\chi(\omega)$ is rational relative to σ, so there exists $\chi(\omega)_\sigma \in H_\sigma^{2d+2p}(A)(d+p)$ such that $(\sigma\Delta)(\chi(\omega)\sigma) = \sigma_*\chi(\omega)$. Since $\sigma_*\chi(\omega) \in H_{\mathbb{A}}^{<2d,0>}(\sigma A)(d) \otimes H_{\mathbb{A}}^{<0,2p>}(\sigma A)(p)$, we have $\chi(\omega)_\sigma \in H_\sigma^{<2d,0>}(A)(d) \otimes H_\sigma^{<0,2p>}(A)(p)$. But then, $(\sigma\Delta)((\sigma\chi)^{-1}(\chi(\omega)_\sigma)) = \sigma_*(\omega)$, showing that ω is rational relative to σ.

Q.E.D.

3.4. Application to Zeta Functions

Let $A \to V$ and $B \to V$ be two Kuga fiber varieties with the same base V. In this section we make the following two assumptions:

Assumption (3.4.1). A and B are defined over algebraic number fields;

Assumption (3.4.2). $A \to V$ and $B \to V$ satisfy the H_2-condition (1.1.3).

We shall now find relations between the zeta functions of A and B.

We denote the data defining A by $(G,K,X,\Gamma,F_A,L_A,\beta_A,\rho_A,\tau_A)$, and the data defining B by $(G,K,X,\Gamma,F_B,L_B,\beta_B,\rho_B,\tau_B)$. Choose a base point $P \in V$, and let $d = \dim V$, $m = \dim A_P$, and $n = \dim B_P$. We know (3.1.1) that $H^b(A_P,\mathbb{Q}) \cong \overset{b}{\wedge} F_{A,\mathbb{Q}}$ and $H^b(B_P,\mathbb{Q}) \cong \overset{b}{\wedge} F_{B,\mathbb{Q}}$ as Γ-modules. Since the representations $\overset{b}{\wedge} \rho_A$ and $\overset{b}{\wedge} \rho_B$ of Γ on $\overset{b}{\wedge} F_{A,\mathbb{Q}}$ and $\overset{b}{\wedge} F_{B,\mathbb{Q}}$, respectively, extend to representations of the semisimple algebraic group G defined over $\mathbb{Q}$, and Γ is Zariski-dense in G (Proposition (1.1.1)), these representations are completely reducible. Write $\overset{b}{\wedge} F_{A,\mathbb{Q}} = \bigoplus_{\alpha} F^b_{A,\alpha}$ and $\overset{b}{\wedge} F_{B,\mathbb{Q}} = \bigoplus_{\beta} F^b_{B,\beta}$ with $F^b_{A,\alpha}$ and $F^b_{B,\beta}$ irreducible Γ-modules. Then, from (3.1.3) and (3.1.6) we have:

$$H^r(A,\mathbb{Q}) \cong \bigoplus_{a+b=r} H^a(\Gamma, \overset{b}{\wedge} F_{A,\mathbb{Q}}) = \bigoplus_{a+b=r} \bigoplus_{\alpha} H^a(\Gamma, F^b_{A,\alpha});$$

$$H^r(B,\mathbb{Q}) \cong \bigoplus_{a+b=r} H^a(\Gamma, \overset{b}{\wedge} F_{B,\mathbb{Q}}) = \bigoplus_{a+b=r} \bigoplus_{\beta} H^b(\Gamma, F^b_{B,\beta}).$$

Therefore, for a prime number ℓ, we have:

$$H^r(A_{et},\mathbb{Q}_\ell) \cong \bigoplus_{a+b=r} \bigoplus_{\alpha} H^a(\Gamma, F^b_{A,\alpha}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell;$$

$$H^r(B_{et},\mathbb{Q}_\ell) \cong \bigoplus_{a+b=r} \bigoplus_{\beta} H^a(\Gamma, F^b_{B,\beta}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \qquad (3.4.3)$$

Now suppose that $F^{b_o}_{A,\alpha_o} \cong F^{b_1}_{B,\beta_1}$ as Γ-modules, with $b_o \equiv b_1$ (mod 2). Without loss of generality we may assume that $b_o \le b_1$. Let $p = \frac{1}{2}(b_1 - b_o) + m$, $a \ge 0$, $r = a + b_o$, and $s = a + b_1 = r + 2p - 2m$.

<u>Theorem (3.4.4)</u>. There exists an absolute Hodge cycle $z \in C^{p+d}_{AH}(A \times B)$ such that the induced map

$$z^r_\ell : H^r(A_{et},\mathbb{Q}_\ell) \to H^s(B_{et},\mathbb{Q}_\ell)(p-m),$$

is an isomorphism of the subspace $H^a(\Gamma, F^{b_o}_{A,\alpha_o}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ onto $H^a(\Gamma, F^{b_1}_{B,\beta_1}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(p-m)$, and, z^r_ℓ is the zero map on $H^a(\Gamma, F^{b_o}_{A,\alpha}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ if $\alpha \ne \alpha_o$.

<u>Proof</u>: We have the following canonical isomorphisms:

$$H^{2p}(A_p \times B_p, \mathbb{Q})^\Gamma(p) \cong \bigoplus_{i+j=2p} [H^i(A_p,\mathbb{Q}) \otimes H^j(B_p,\mathbb{Q})]^\Gamma(p)$$

$$\cong \bigoplus_{i+j=2p} \hom_\Gamma(H^{2m-i}(A_p,\mathbb{Q}), H^j(B_p,\mathbb{Q})(p-m))$$

$$\cong \bigoplus_{b=0}^{2m} \hom_{\Gamma}(H^{b}(A_{p},\emptyset), H^{b+2p-2m}(B_{p},\emptyset)(p-m))$$

$$\cong \bigoplus_{b=0}^{2m} \bigoplus_{\alpha,\beta} \hom_{\Gamma}(F^{b}_{A,\alpha}, F^{b+2p-2m}_{B,\beta}(p-m)).$$

Let $\eta_* \in \bigoplus_{b=0}^{2m} \bigoplus_{\alpha,\beta} \hom_{\Gamma}(F^{b}_{A,\alpha}, F^{b+2p-2m}_{B,\beta}(p-m))$ be an isomorphism of $F^{b_o}_{A,\alpha_o}$ onto $F^{b_1}_{B,\beta_1}(p-m)$, and the zero map on all other components. Let η be the corresponding element of $H^{2p}(A_p \times B_p, \emptyset)^{\Gamma}(p)$ under the above isomorphisms. Then η is a Hodge cycle by Lemma (1.1.6(b)) and Proposition (3.2.1), and therefore an absolute Hodge cycle by Theorem (2.2.3).

Let $\iota : A_p \times B_p \to A \underset{V}{\times} B$ be the inclusion map, and $\tilde{\eta}$ the unique element of $H^{<0,2p>}(A \underset{V}{\times} B, \emptyset)(p)$ such that $\iota^*(\tilde{\eta}) = \eta$ (3.3.1). Then, $\tilde{\eta}$ is an absolute Hodge cycle by Theorem (3.3.5). Let $j : A \underset{V}{\times} B \to A \times B$ be the natural embedding, and put $z = j_!(\tilde{\eta})$. z is an absolute Hodge cycle by Lemma (2.4.2), and we claim that z satisfies the properties stated in the theorem.

Let z_* be the map induced by z on singular cohomology, $z_* : H^r(A,\emptyset) \to H^s(B,\emptyset)(p-m)$. Then, because of Theorem (2.3.1(f)) it is sufficient to show that z_* is an isomorphism of $H^a(\Gamma, F^{b_o}_{A,\alpha_o})$ onto $H^a(\Gamma, F^{b_1}_{B,\beta_1})(p-m)$, and is zero on $H^a(\Gamma, F^{b_o}_{A,\alpha})$ for $\alpha \neq \alpha_o$. In other words, it is sufficient to show that $z_* = H^a(\eta_*)$. To do this, we may work with complex coefficients instead of rational coefficients (using Theorem (2.3.1(f)) again).

Let $(x^1,\ldots,x^{2d})$ be global coordinates on X and $(u^1,\ldots,u^{2m})$ a basis of F_A^*, so that $(x^1,\ldots,x^{2d},u^1,\ldots,u^{2m})$ is an admissible coordinate system on $\tilde{A}$. Let $(y^1,\ldots,y^{2d})$ be coordinates on a second copy of X, with $x^k = y^k (k=1,\ldots,2d)$, and let $(v^1,\ldots,v^{2n})$ be a basis of F_B^*. Then, $(y^1,\ldots,y^{2d}, v^1,\ldots,v^{2n})$ is an admissible coordinate system on $\tilde{B}$. Also, $(x^1,\ldots,x^{2d},u^1,\ldots,u^{2m},v^1,\ldots,v^{2n})$ is an admissible coordinate system on $\widetilde{A \underset{V}{\times} B}$, and, $(x^1,\ldots,x^{2d},y^1,\ldots,y^{2d},u^1,\ldots,u^{2m}, v^1,\ldots,v^{2n})$ is an admissible coordinate system on $\widetilde{A \times B}$. In what follows, all differential forms are expressed in terms of these coordinates.

The map $\eta_* : H^{b_o}(A_p,\mathbb{C}) \to H^{b_1}(A_p,\mathbb{C})$ is given by $\eta_*(\mu) = (2\pi i)^{-m} \int_{A_p} \eta \wedge \mu$. The induced map $H^a(\eta_*)$ may be described as follows: If $\omega = \sum_{C,D} \varphi_{C,D}(x)dx^C \wedge du^D \in H^{<a,b_o>}(A,\mathbb{C})$, then,

$$H^a(\eta_*)(\omega) = (2\pi i)^{-m} \sum_{C,D} \varphi_{C,D}(y)dy^C \wedge \int_{A_p} \eta \wedge du^D \in H^{<a,b_1>}(B,\mathbb{C}).$$

Use the Künneth formula to write $z = \sum_{e+f=2p+2d} z^{e,f}$ with $z^{e,f} \in H^e(A,\mathbb{C}) \otimes H^f(B,\mathbb{C})$. Then,

$$z_*(\omega) = (2\pi i)^{-d-m} \int_A \sum_{C,D} z \wedge \varphi_{C,D}(x)dx^C \wedge du^D$$

$$= (2\pi i)^{-d-m} \int_A \sum_{C,D} z^{e_o,s} \wedge \varphi_{C,D}(x)dx^C \wedge du^D,$$

where $e_o = 2d + 2m - r$, and $s = r + 2p - 2m$.

We have an isomorphism

$$\psi : H^{2m+2d}(A,\mathbb{C}) \otimes H^s(B,\mathbb{C}) \to H^s(B,\mathbb{C})$$

given by $\psi(\theta) = (2\pi i)^{-d-m} \int_A \theta$. In order to show that $H^a(\eta_*)(\omega) = z_*(\omega)$, we will show that $\psi^{-1}(H^a(\eta_*)(\omega)) = \psi^{-1}(z_*(\omega))$. We have

$$\psi^{-1}(z_*(\omega)) = \sum_{C,D} z^{e_o,s} \wedge \varphi_{C,D}(x)\,dx^C \wedge du^D;$$

$$\psi^{-1}(H^a(\eta_*)(\omega)) = \sum_{C,D} \varphi_{C,D}(y)\,dy^C \wedge \eta \wedge du^D \wedge dV_A,$$

where dV_A is a volume form on V, expressed in the coordinates $(x^1,\ldots,x^{2d})$, and normalized so that $\int_V dV_A = (2\pi i)^d$.

By Poincaré duality it is enough to show that

$$\int_{A\times B} \sum_{C,D} \varphi_{C,D}(y)\,dy^C \wedge \eta \wedge du^D \wedge dV_A \wedge \zeta$$

$$= \int_{A\times B} \sum_{C,D} z^{e_o,s} \wedge \varphi_{C,D}(x)\,dx^C \wedge du^D \wedge \zeta$$

for all $\zeta \in H^{2m+2n+2d-r-2p}(A\times B,\mathbb{C})$. In fact, we may assume that $\zeta \in H^{2m+2n+2d-r-2p}(B,\mathbb{C})$, since $\psi^{-1}(H^a(\eta_*)(\omega))$ and $\psi^{-1}(z_*(\omega))$ belong to $H^{2m+2d}(A,\mathbb{C}) \otimes H^s(B,\mathbb{C})$.

Let D_Γ, D_{L_A}, and D_{L_B} be fundamental domains for the actions of Γ, L_A, and L_B on X, $F_{A,\mathbb{R}}$, and $F_{B,\mathbb{R}}$ respectively. Then, using Lemma (3.1.8), we have

$$\int_{A\times B} \sum_{C,D} \varphi_{C,D}(y)\,dy^C \wedge \eta \wedge du^D \wedge dV_A \wedge \zeta$$

$$= \int_{D_\Gamma \times D_\Gamma \times D_{L_A} \times D_{L_B}} \sum_{C,D} \varphi_{C,D}(y)\,dy^C \wedge \eta \wedge du^D \wedge dV_A \wedge \zeta$$

$$= (2\pi i)^{-d} \int_{D_\Gamma \times D_{L_A} \times D_{L_B}} \sum_{C,D} \varphi_{C,D}(y)\,dy^C \wedge \eta \wedge du^D \wedge \zeta$$

$$= (2\pi i)^{-d} \int_{D_\Gamma \times D_{L_A} \times D_{L_B}} \sum_{C,D} \varphi_{C,D}(x)\,dx^C \wedge \eta \wedge du^D \wedge j^* \zeta$$

$$= (2\pi i)^{-d} \int_{\substack{A\times B \\ V}} \sum_{C,D} \varphi_{C,D}(x)\,dx^C \wedge \eta \wedge du^D \wedge j^* \zeta$$

$$= (2\pi i)^{-d} (2\pi i)^{d} \int_{A\times B} \sum_{C,D} z \wedge \varphi_{C,D}(x)\,dx^C \wedge du^D \wedge \zeta$$

(using $z = j_! \tilde{\eta}$ and (2.4.1))

$$= \int_{A\times B} \sum_{C,D} z^{e_o,s} \wedge \varphi_{C,D}(x)\,dx^C \wedge du^D \wedge \zeta .$$

Q.E.D.

Assumption (3.4.1) and Proposition (2.2.4) imply the existence of an algebraic number field k of finite degree over $\mathbb{Q}$, such that A and B are defined over k, and $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ acts trivially on $C^k_{AH}(A\times B)$, $C^k_{AH}(A\times A)$ and $C^k_{AH}(B\times B)$ for all k. We shall now examine the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ on the étale cohomology groups of A and B.

Corollary (3.4.5). In the direct sum decompositions (3.4.3), each $H^a(\Gamma, F^b_{A,\alpha}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is a $Gal(\overline{\mathbb{Q}}/k)$-submodule of $H^r(A_{et}, \mathbb{Q}_\ell)$, and each $H^a(\Gamma, F^b_{B,\beta}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is a $Gal(\overline{\mathbb{Q}}/k)$-submodule of $H^r(B_{et}, \mathbb{Q}_\ell)$.

Proof: It is enough to prove the first statement. Apply the theorem with $B = A$. For any α_o, b_o, $F^{b_o}_{A,\alpha_o}$ and $F^{b_o}_{A,\alpha_o}$ are isomorphic as Γ-modules, hence there exists $z \in C^{m+d}_{AH}(A \times A)$ such that

$$z^r_\ell : H^a(\Gamma, F^{b_o}_{A,\alpha_o}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^a(\Gamma, F^{b_o}_{A,\alpha_o}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

is an isomorphism, and z^r_ℓ is the zero map on all other summands. By assumption, $Gal(\overline{\mathbb{Q}}/k)$ acts trivially on $C^{m+d}_{AH}(A \times A)$; Theorem (2.3.1(d)) then shows that z^r_ℓ commutes with the action of $Gal(\overline{\mathbb{Q}}/k)$. This implies that $H^a(\Gamma, F^{b_o}_{A,\alpha_o}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, the image of z^r_ℓ, is a $Gal(\overline{\mathbb{Q}}/k)$-submodule of $H^r(A_{et}, \mathbb{Q}_\ell)$.

Q.E.D.

Let $\mathfrak{P}$ be a finite prime of k, $D_{\mathfrak{P}}$ a decomposition subgroup of $Gal(\overline{\mathbb{Q}}/k)$ for $\mathfrak{P}$, $I_{\mathfrak{P}}$ the inertia subgroup of $D_{\mathfrak{P}}$, and $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}$ a Frobenius element. For a smooth, projective variety X over k, and a prime ℓ which is prime to $\mathfrak{P}$, the local zeta function of X is defined by

$$Z^{(i)}_{\mathfrak{P}}(u; X/k) = \det(1 - u \cdot \sigma^{-1}_{\mathfrak{P}} | H^i((X \otimes_k \overline{\mathbb{Q}})_{et}, \mathbb{Q}_\ell)^{I_{\mathfrak{P}}})^{-1},$$

$$Z_{\mathfrak{P}}(u; X/k) = \prod_{i=1}^{2 \dim X} Z^{(i)}_{\mathfrak{P}}(u; X/k)^{(-1)^i}.$$

It is known that the local zeta function is independent of the choice of ℓ when X has good reduction at $\mathfrak{p}$. The global zeta function is defined by

$$Z(s;X/k) = \prod_{\mathfrak{p}} Z_{\mathfrak{p}}((N_{\mathfrak{p}})^{-s};X/k),$$

the product being taken over all finite primes of k. For more details see Ohta ([17], Part 1, Section 4.5) and the references cited there.

Corollary (3.4.5) gives us factorizations of the zeta functions of A and of B:

$$Z_{\mathfrak{p}}^{(r)}(u;A/k) = \prod_{a+b=r} \prod_{\alpha} Z_{\mathfrak{p}}^{a,b,\alpha}(u;A/k);$$

$$Z_{\mathfrak{p}}^{(r)}(u;B/k) = \prod_{a+b=r} \prod_{\beta} Z_{\mathfrak{p}}^{a,b,\beta}(u;B/k),$$

where,

$$Z_{\mathfrak{p}}^{a,b,\alpha}(u;A/k) = \det(1-u\cdot\sigma_{\mathfrak{p}}^{-1} | (H^{a}(\Gamma,F^{b}_{A,\alpha}) \otimes_{\emptyset} \emptyset_{\ell})^{I_{\mathfrak{p}}})^{-1},$$

$$Z_{\mathfrak{p}}^{a,b,\beta}(u;B/k) = \det(1-u\cdot\sigma_{\mathfrak{p}}^{-1} | (H^{a}(\Gamma,F^{b}_{B,\beta}) \otimes_{\emptyset} \emptyset_{\ell})^{I_{\mathfrak{p}}})^{-1}.$$

We know from Theorem (2.3.1(d)) that the map $z_{\ell}^{r}(A_{et},\emptyset_{\ell}) \to H^{s}(B_{et},\emptyset_{\ell})(p-m)$, defined in Theorem (3.4.4) commutes with the action of $\sigma_{\mathfrak{p}}$. We have an isomorphism:

$$\psi : H^{s}(B_{et},\emptyset_{\ell})(p-m) \to H^{s}(B_{et},\emptyset_{\ell}).$$

Since $\sigma_{\mathfrak{p}}$ acts on $\emptyset_{\ell}(p-m)$ as multiplication by $(N_{\mathfrak{p}})^{p-m}$, ψ satisfies the relation

$$\psi(\sigma_{\mathfrak{P}}\cdot\omega) = (N_{\mathfrak{P}})^{p-m}\sigma_{\mathfrak{P}}(\psi(\omega))$$

for all $\omega \in H^s(B_{et}, \mathbb{Q}_\ell)(p-m)$.

We have proved our main theorem:

<u>Theorem (3.4.6)</u>. Let the notations and assumptions be as above, with $F^{b_o}_{A,\alpha_o} \cong F^{b_1}_{B,\beta_1}$ as Γ-modules. Then,

$$Z^{a,b_o,\alpha_o}_{\mathfrak{P}}((N_{\mathfrak{P}})^{p-m}u;\ A/k) = Z^{a,b_1,\beta_1}_{\mathfrak{P}}(u;\ B/k).$$

3.5. <u>Examples</u>

<u>Example 1</u>.

Let $k = \mathbb{Q}(\sqrt{d})$ where $d > 1$ is a square-free integer. Let k_1 and k_2 be fields isomorphic to k, $S^1 = \{\sigma_1, \sigma_{-1}\}$ the set of embeddings of k_1 into $\mathbb{R}$, and $S^2 = \{\varphi_1, \varphi_{-1}\}$ the set of embeddings of k_2 into $\mathbb{R}$. Let B^i be quaternion algebras over k_i $(i=1,2)$, such that B^1 splits at σ_1 and ramifies at σ_2, and B^2 splits at φ_1 and ramifies at φ_2. Then, we have a chemistry $(\mathfrak{G}, S, S_o)$ with $\mathfrak{G} = \mathrm{Gal}(k/\mathbb{Q})$, $S = \{\sigma_1, \sigma_{-1}, \varphi_1, \varphi_{-1}\}$, and $S_o = \{\sigma_1, \varphi_1\}$.

Ohta ([17]), Part 2) constructed group theoretical abelian schemes $A_1 \to V_1$ and $A_2 \to V_2$ with $V_i = \Gamma_i\backslash\mathfrak{H}$, Γ_i an arithmetic group contained in B^i, belonging to the polymers $P_1 = 2S^1$ and $P_2 = 2S^2$ respectively. The zeta functions of A_1 and A_2

have also been determined by Ohta. Taking the product of these two abelian schemes we get an abelian scheme $A = A_1 \times A_2 \to V = V_1 \times V_2$ defined by the polymer $P_A = 2S^1 + 2S^2$.

We can also define an abelian scheme $B \to V$ by the rigid polymer $P_B = 2\{\sigma_1, \varphi_{-1}\} + 2\{\sigma_{-1}, \varphi_1\}$. B is not a product of two abelian schemes.

Since A and B are defined by rigid polymers, they satisfy the H_2-condition (Theorem (1.3.7)). By Lemma (1.1.6(b)), $A \underset{V}{\times} B \to V$ is defined by the polymer $P = P_A + P_B$ and again satisfies the H_2-condition. Therefore $H^{\langle 0,2p\rangle}(A \underset{V}{\times} B, \mathbb{Q})(p)$ consists of absolute Hodge cycles (Theorem (3.3.5)). Let us calculate its dimension:

$$\dim_{\mathbb{Q}} H^{\langle 0,2p\rangle}(A \underset{V}{\times} B, \mathbb{Q})(p) = \dim_{\mathbb{Q}} H^{2p}(A_P \times B_P, \mathbb{Q})^{\Gamma_1 \times \Gamma_2}$$

$$= \dim_{\mathbb{Q}}\left(\overset{2p}{\wedge}(F_{A,\mathbb{Q}} \times F_{B,\mathbb{Q}})\right)^{\Gamma_1 \times \Gamma_2} = \dim_{\mathbb{C}}\left(\overset{2p}{\wedge}(F_{A,\mathbb{C}} \times F_{B,\mathbb{C}})\right)^{G_{\mathbb{C}}}.$$

Now, $G_{\mathbb{C}} = SL_2(\mathbb{C})^4$ and the representation of $G_{\mathbb{C}}$ on $F_{A,\mathbb{C}} \times F_{B,\mathbb{C}}$ is equivalent to the representation of $SL_2(\mathbb{C})^4$ in an example of Kuga ([10], p.280). By Kuga's calculations the dimensions $d_{2p} = \dim_{\mathbb{Q}} H^{\langle 0,2p\rangle}(A \underset{V}{\times} B, \mathbb{Q})(p)$ are:

$d_0 = d_{32} = 1$, $d_2 = d_{30} = 4$, $d_4 = d_{28} = 82$, $d_6 = d_{26} = 452$,

$d_8 = d_{24} = 2600$, $d_{10} = d_{22} = 8208$, $d_{12} = d_{20} = 20574$,

$d_{14} = d_{18} = 33224$, $d_{16} = 40790$.

We shall now compare the representations $\overset{b_o}{\wedge}\rho_A$ and $\overset{b_1}{\wedge}\rho_B$, where ρ_A and ρ_B are the representations defining A and B respectively. We know from [17], Part 2, that A is defined over a number field; if B has a model defined over a number field then Theorem (3.4.6) combined with our calculations gives relations between the zeta functions of A and B.

We have $G = \prod_{\alpha\in S} G_\alpha$ with $G_\alpha = SL_1(B_\alpha)$, (see 1.3). Each projection $\rho_\alpha : G \to G_\alpha$ is a representation of G defined over k. We abbreviate ρ_α by α, denote by F_α the representation space of α, and write $S^k(\alpha)$ for the representation of G on $S^k(F_\alpha)$, the space of symmetric tensors of degree k on F_α. With these notations, $\overset{2}{\wedge}\rho_A$ is equivalent over $\mathbb{C}$ to

$$3\ S^2(\sigma_1) \oplus 3\ S^2(\sigma_{-1}) \oplus 3\ S^2(\varphi_1) \oplus 3\ S^2(\varphi_{-1})$$

$$\oplus\ S^2(\sigma_1) \otimes S^2(\varphi_{-1}) \oplus S^2(\sigma_{-1}) \otimes S^2(\varphi_1)$$

$$\oplus\ 4\ \sigma_1 \otimes \sigma_{-1} \otimes \varphi_{-1} \oplus 2\ \phi,$$

where Φ denotes a trivial one-dimensional representation.

The subrepresentations

$$\rho^2_{B,1} = 3\ S^2(\sigma_1) \oplus 3\ S^2(\sigma_{-1}),$$

$$\rho^2_{B,2} = 3\ S^2(\varphi_1) \oplus 3\ S^2(\varphi_{-1}),$$

$$\rho^2_{B,3} = 4\ \sigma_1 \otimes \sigma_{-1} \otimes \varphi_1 \otimes \varphi_{-1},$$

$$\rho^2_{B,4} = 2\phi,$$

$$\rho^2_{B,5} = S^2(\sigma_1) \otimes S^2(\varphi_{-1}) \oplus S^2(\sigma_{-1}) \otimes S^2(\varphi_1),$$

are defined over $\mathbb{Q}$, and are multiples of $\mathbb{Q}$-irreducible representations. Furthermore, $\rho^2_{B,\beta}$ is also a subrepresentation of $\overset{2}{\wedge}\rho_A$ for $\beta = 1,2,3,4$, while $\rho^2_{B,5}$ is a subrepresentation of $\overset{4}{\wedge}\rho_A$. We conclude that each $\mathbb{Q}$-irreducible subrepresenatation of $\overset{2}{\wedge}\rho_B$ is also a subrepresentation of $\overset{b_o}{\wedge}\rho_A$ for $b_o = 2(\mathrm{mod}\ 2)$.

Example 2.

Let $k \subset \mathbb{R}$ be a Galois extension of $\mathbb{Q}$, of degree 3; for example, take $k = \mathbb{Q}(\cos\frac{2\pi}{9})$. Let k_1 and k_2 be fields isomorphic to k, $S^1 = \{\sigma_o,\sigma_1,\sigma_{-1}\}$ the set of embeddings of k_1 into $\mathbb{R}$, and $S^2 = \{\varphi_o,\varphi_1,\varphi_{-1}\}$ the set of embeddings of k_2 into $\mathbb{R}$. Let B^1 be a quaternion algebra over k_1 with discriminant $\sigma_1\cdot\sigma_{-1}$ and B^2 a quaternion algebra over k_2 with discriminant $\varphi_1\cdot\varphi_{-1}$. Then, we have a chemistry $(\mathfrak{G},S,S_o)$ with $\mathfrak{G} = \mathrm{Gal}(k/\mathbb{Q})$, $S = S^1 \cup S^2$, and $S_o = \{\sigma_o,\varphi_o\}$.

The rigid polymers $P_1 = S^1$ and $P_2 = S^2$ define abelian schemes $A_1 \to V_1 = {}_{\Gamma_1}\backslash\mathfrak{H}$ and $A_2 \to V_2 = {}_{\Gamma_2}\backslash\mathfrak{H}$, where Γ_i is an arithmetic subgroup of $SL_1(B^i)$ (see Mumford [16], Section 4, or the remark at the end of Kuga [9]). Ohta ([17], Part 1) has found models of A_1 and A_2 defined over algebraic number fields, and determined their zeta functions. Taking the product we get an abelian scheme $A = A_1 \times A_2 \to V = V_1 \times V_2 = \Gamma\backslash \mathfrak{H}^2$, where $\Gamma = \Gamma_1 \times \Gamma_2$. $A_1 \times A_2$ is defined by the polymer $P_A = S^1 + S^2$.

We define an abelian scheme $B \to V$ by the rigid polymer

$P_B = \{\sigma_o, \varphi_1, \varphi_{-1}\} + \{\sigma_1, \varphi_o, \varphi_{-1}\} + \{\sigma_{-1}, \varphi_o, \varphi_1\}$.

The fiber varieties A, B, and $A \underset{V}{\times} B$ over V, all satisfy the H_2-condition. As in Example 1, we write α for the projection of $G = \prod_{\alpha \in S} G_\alpha$ to the α-th component.

We shall now compare the representations $\overset{3}{\wedge} \rho_A$ and $\overset{3}{\wedge} \rho_B$. Using Lemma 2.2.1 of Kuga [9], we can write $\overset{3}{\wedge} \rho_A$ and $\overset{3}{\wedge} \rho_B$ as direct sums of irreducible representations over $\mathbb{C}$. $\overset{3}{\wedge} \rho_A$ has 14 irreducible subrepresentations, while $\overset{3}{\wedge} \rho_B$ has 52 irreducible subrepresentations. Four subrepresentations are common to $\overset{3}{\wedge} \rho_A$ and $\overset{3}{\wedge} \rho_B$, namely:

$$\sigma_o \otimes \sigma_1 \otimes \sigma_{-1},$$

$$\sigma_o \otimes \sigma_1 \otimes \sigma_{-1} \otimes S^2(\varphi_o) \otimes S^2(\varphi_1),$$

$$\sigma_o \otimes \sigma_1 \otimes \sigma_{-1} \otimes S^2(\varphi_o) \otimes S^2(\varphi_{-1}),$$

$$\sigma_o \otimes \sigma_1 \otimes \sigma_{-1} \otimes S^2(\varphi_1) \otimes S^2(\varphi_{-1}).$$

Of these, the first one, $\sigma_o \otimes \sigma_1 \otimes \sigma_{-1}$ is defined over $\mathbb{Q}$; in fact $\sigma_o \otimes \sigma_1 \otimes \sigma_{-1} = \rho_{A_1}$. Also, the sum of the last three is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-irreducible: it is equal to $\rho_{A_1} \otimes (\overset{3}{\wedge} \rho_{A_2})$.

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