# Equivariant Reidemeister Torsion

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Abstract of the Dissertation

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For a finite simplical complex X with flat orthogonal bundle $E$ over X and G = <g> (the group generated by g) a finite group acting simplically on X such that g extends to a bundle automorphism we define the "torsion" invariants $\tau_\rho$ and $\tau_g$ where $\rho$ is an orthogonal irreducible representation of G. These invariants depend on "preferred volume forms" being chosen for a certain chain complex associated to the pair $(X,E)$ and its cohomology groups. A basic property of $\tau_\rho$ and $\tau_g$ is they are combinatorial invariants. Their invariance is established by developing a formula for $\tau_\rho$ and $\tau_g$ that involves expressing $\tau_\rho$ and $\tau_g$ in terms of the torsions of the isotropy spaces $X(H) = \{x \in X | G_x = H\}$

where H is a subgroup of G and $G_x$ is the stabilizer of x. In addition we show that $\tau_\rho$ and $\tau_g$ have many of the same properties of the classical Reidemeister torsion $\tau$.

To Grace, for her love and support during the preparation of this thesis.

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## 0. Introduction

One of the basic aims of Riemannian geometry is to understand the relationship between the geometry and the topology of a manifold. One fruitful line of investigation has been the connection between the topology and the analysis (e.g., Hodge theory, Atiyah-Singer index theory) on compact manifolds. In particular the study of the Laplace and heat operators has given rise to many interesting results.

In 1935 the concept of "torsion" was introduced to study certain finite simplicial complexes by Franz, Reidemeister and de Rham (see [F], [R], [DR]). A particular type of torsion known as the Reidemeister torsion, $\tau$, is a real number obtained from a certain chain complex associated to a CW-complex. This complex is defined in terms of an orthogonal representation of the fundamental group of the CW-complex and the torsion depends on "preferred volume forms" being chosen for the chain groups and their cohomology groups (see §1). A basic property of the torsion is that it is a combinatorial invariant [M p. 378]. It was conjectured but not proved until 1974 by T.A. Chapman [Ch] that the torsion is a topological invariant for compact, connected CW-complexes.

Arnold Shapiro suggested that an expression involving the spectrum for the Laplacian on forms should exist for the Reidemeister torsion. In [RS], [R] Ray and Singer suggested the following formula. Let ${}_i\zeta(s) = \sum_{\lambda_j>0} {}_i\lambda_j^{-s}$ where $\{{}_i\lambda_j\}$ is the spectrum of $\Delta_i$ the Laplacian on i-forms with coefficients in a flat bundle $E$. Then define the analytic torsion

$$\ln T(M,E) = \Sigma(-1)^i i\, {}_i\zeta'(0).$$

Ray and Singer proved that T was a manifold invariant and presented some evidence that $\tau = T$. Finally, in [C] and [Mü] Cheeger and Müller independently succeeded in proving the equality between $\tau$ and T.

In [C] and [RS] heat equation methods play a major role. That heat equation methods are useful can be understood from the following identity. Letting ${}_iE(x,y,t)$ denote the fundamental solution to the heat equation on i-forms we have

$${}_i\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{tr}\, {}_iE(t)\,dt.$$

Thus in studying the behavior of the analytic torsion T one is lead to the study of the trace of the heat kernel about which much is known.

In [C] Cheeger suggested a generalization of the equality $\tau = T$ to the case where an isometry of finite order acts on a manifold. One can define an analog of the Reidemeister torsion, $\tau_g(M,E)$, which is a combinatorial invariant (see §2). In addition one defines $T_g(M,E)$ to be

$$\sum_i (-1)^i i \, {}_i\zeta_g'(0),$$

where

$${}_i\zeta_g(s) = \sum_{{}_i\lambda_j} {}_i\lambda_j^{-s} \langle g\phi_j, \phi_j \rangle$$

and $\phi_j$ is an eigenform of ${}_i\lambda_j$. With these definitions we expect the following to be ture.

<u>Conjecture 0.1</u>. Let M be a closed compact Riemannian manifold, g an isometry of finite order which extends to an automorphism of the flat orthogonal bundle $E^M$. Then

$$\tau_g(M,E) = T_g(M,E).$$

It is the purpose of this thesis to begin such a generalization. In Section 1 we begin with a review of Reidemeister torsion and give a sketch of the proof of the combinatorial combinatorial invariance of $\tau$. In Section 2 we introduce the torsions $\tau_\rho$ and $\tau_g$ (where $\rho$ is an orthogonal representation of G) and show they have many of the same properties as $\tau$. We also show $\tau_\rho(\tau_g)$ can be determined from the $\{\tau_g\}(\{\tau_\rho\})$. In Section 3 we prove $\tau_\rho$ and $\tau_g$ are combinatorial invariants for G a finite group acting freely and in Section 4 extend the proof of combinatorial invariance to the case where G has fixed points. In Section 5 we compute the torsions $\tau_\rho$ and $\tau_g$ in some simple cases.

## 1. Review of Reidemeister Torsion

In this section the definition and basic properties of Reidemeister Torsion are stated. The basic references for this material are Milnor [M] and Cheeger [C].

1.1

Let C* be a real cochain complex

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} 0$$

i.e., each $C^i$ is a real vector space and $d_{i+1}d_i = 0$. Let $\dim C^i = \ell_i$ and $\dim H^i = b_i$ where $H^i$ denotes the $i^{th}$ cohomology group of C*. Suppose volume element $\omega_i \in \Lambda^{\ell_i}(C^i)^*$ and $\mu_i \in \Lambda^{b_i}(H^i)^*$ ($\omega_i$ and $\mu_i \neq 0$) have been chosen. Let $B^i = d_{i-1}(C^{i-1})$, $Z^i = \ker d_i$ and let $\dim B^i = t_i$. Let $\rho_i$ be an element of $\Lambda^{t_i}(C^i)^*$ such that $\rho_i|_{B^i} \neq 0$, then using the exact sequence

$$0 \to B^i \to Z^i \xrightarrow{\pi} H^i \to 0$$

we have

$$\rho_i \wedge d_i^*(\rho_{i+1}) \wedge \pi^*(\mu_i) = m_i\omega_i$$

for some $m_i \neq 0$ (this follows as $\dim \Lambda^{\ell_i}(C^i)^* = 1$).

<u>Definition 1.1.1</u>. The <u>Reidemeister</u> <u>Torsion</u> is defined as

$$(1.1.2) \qquad \tau(C,\omega,\mu) = \Pi \frac{m_{2i}}{m_{2i+1}} .$$

It is straightforward to check that $\tau(C,\omega,\mu)$ is independent of the choices of the $\rho_i$.

We state several useful properties of the Reidemeister torsion. Proposition 1.1.5 and its analog in Section 2 play a key role in the proof of combinatorial invariance of $\tau$ and $\tau_\rho, \tau_g$ respectively.

Proposition 1.1.3. If $\gamma_i$ represents another choice of volume elements of $\Lambda^{b_i}(H^i)^*$ with $\gamma_i = k_i\mu_i$ then

(1.1.4) $$\tau(C,\omega,\gamma) = \Pi \frac{k_{2i}}{k_{2i+1}} \tau(C,\omega,\mu)$$

Proof. Let

$$\rho_i \wedge d_i^*(\rho_{i+1}) \wedge \pi^*(\mu_i) = m_i\omega_i$$

then

$$m_i' w_i = \rho_i \wedge d_i^*(\rho_{i+1}) \wedge \pi^*(\gamma_i) = k_i\rho_i \wedge d_i^*(\rho_{i+1}) \wedge \pi^*(\mu_i^*) = k_i m_i \omega_i .$$

Thus

$$\tau(C,\omega,\gamma) = \Pi \frac{m_{2i}'}{m_{2i+1}'} = \Pi \frac{k_{2i}m_{2i}}{k_{2i+1}m_{2i+1}} = \Pi \frac{k_{2k}}{k_{2i+1}} \tau(C,\omega,\mu).$$

Q.E.D.

Proposition 1.1.5. Let $0 \to C_1^* \to C_2^* \to C_3^* \to 0$ be an exact sequence of chain complexes with volumes ${}_j\omega_i$ (j=1,2,3) and ${}_j\mu_i$ for $H^i(C_j)$. Suppose that for all i the torsion of the complex $0 \to C_1^i \to C_2^i \to C_3^i \to 0$ equals one. Then

(1.1.6) $$\tau(C_2,\omega_2,\mu_2) = \tau(C_1,\omega_1,\mu_1)\tau(C_3,\omega_3,\mu_2)\tau(H)$$

where $H$ is the long exact sequence in cohomology associated to the short exact sequence of chain complexes $0 \to C_1^* \to C_2^* \to C_3^* \to 0$.

Proof. First we use the fact that $(0 \to C_1^i \to C_2^i \to C_3^i \to 0) = 1$ to obtain a relationship between ${}_2\omega_i$ and ${}_1\omega_i$, ${}_3\omega_i$. Since $0 \to C_1^i \to C_2^i \to C_3^i \to 0$ is acyclic we have

$$\tau(0 \to C_1^i \to C_2^i \to C_3^i \to 0) = 1 = \frac{n_0 n_2}{n_1}$$

where

$$
\begin{aligned}
& {}_i\eta_0 \wedge \alpha^*({}_i\eta_1) = n_0 \; {}_i\omega_1 \\
(1.1.7) \qquad & {}_i\eta_1 \wedge \psi^*({}_i\eta_2) = n_1 \; {}_i\omega_2 \\
& {}_i\eta_2 \wedge \beta^*({}_i\eta_2) = n_2 \; {}_i\omega_3 .
\end{aligned}
$$

$\beta^*({}_i\eta_3) = 1$ since $\beta^*$ pulls back the zero volume form. Thus we choose ${}_i\eta_2 = \omega_3$ which implies $\eta_2 = 1$. Since ${}_i\eta_0 = 1$ we have $\phi^*({}_i\eta_1) = n_0 \; {}_i\omega_2$, wo choose ${}_i\eta_1$ such that $\phi^*({}_i\eta_1) = {}_i\omega_2$ which implies $n_0 = 1$. This implies $n_2 = 1$ and

$$(1.1.8) \qquad {}_2\omega_i = {}_i\eta_1 \wedge \psi^*({}_3\omega_i) = \overline{{}_i\omega_1} \wedge \psi^*({}_3\omega_i).$$

For the long exact sequence in cohomology $H$ let ${}_1k_i$, ${}_2k_i$, ${}_3k_i$ denote the kernels of the homomorphisms $H_1^i \xrightarrow{\partial} H_3^{i+1}$, $H_2^i \xrightarrow{\partial} H_1^i$, and $H_3^i \xrightarrow{\partial} H_2^i$ respectively. Then $\tau(H) = \Pi^{0_{2i}}/0_{2i+1}$ where

$$0_{3i}\ {}_3\mu_i = {}_3\phi_i \wedge \partial^*({}_2\phi_i)$$

(1.1.9) $$0_{3i+1}\ {}_2\mu_i = {}_2\phi_i \wedge \partial^*({}_1\phi_i)$$

$$0_{3i+2}\ {}_1\mu_i = {}_1\phi_i \wedge \partial^*({}_3\phi_{i+1})$$

and ${}_j\phi_i$ denotes a volume form for ${}_jk_i$. Set

(1.1.10) $$\Omega = \frac{\tau(C_3)\tau(C_1)\tau(H)}{\tau(C_2)} .$$

We want to show $\Omega = 1$. We have $\Omega$ equals

(1.1.11) $$\Pi \frac{{}_1m_{2i}}{{}_1m_{2i+1}} \Pi \frac{0_{6i+2}}{0_{6i+5}} \Pi \frac{{}_2m_{2i+1}}{{}_2m_{2i}} \Pi \frac{0_{6i+4}}{0_{6i+1}} \Pi \frac{{}_3m_{2i}}{{}_3m_{2i+1}} \Pi \frac{0_{6i}}{0_{6i+3}}$$

with

(1.1.12) $${}_jm_i\, {}_j\omega_i = {}_j\rho_i \wedge {}_jd_i^*({}_j\rho_{i+1}) \wedge \pi^*({}_j\mu_i).$$

Letting ${}_j\gamma_i = {}_jp_i\ {}_j\mu_i$ be another choice of volume forms and using Proposition 1.1.3 it follows after a little calculation that $\Omega$ is independent of the choice ${}_j\mu_i$. Thus we may as well assume that ${}_3\mu_i = {}_3\phi_i \wedge \partial^*({}_2\phi_i)$, ${}_2\mu_i = {}_2\phi_i \wedge \partial^*({}_1\phi_i)$, and ${}_1\mu_i = {}_1\phi_i \wedge \partial^*({}_3\phi_{i+1})$ yielding $\tau(H) = 1$.

If ${}_1\omega_i = p\ {}_1\omega_i'$ and ${}_3\omega_i' = q\ {}_3\omega_i$ then via (1.1.12) we have ${}_1m_i = {}_1m_i'p$ and ${}_3m_i = {}_3m_i'q$. Use of (1.1.8) then gives ${}_2m_i = {}_2m_i'pq$. With these relations another simple computation shows $\Omega$ is independent of the choices of ${}_1\omega_i$ and ${}_2\omega_i$. So we may as well assume ${}_1\omega_i = {}_1\rho_i \wedge {}_1d_i^*({}_1\rho_{i+1}) \wedge \pi^*({}_1\mu_i)$ and ${}_3\omega_i = {}_3\rho_i \wedge {}_3d_i^*({}_3\rho_{i+1}) \wedge \pi^*({}_3\mu_i)$ yielding $\tau(C^1) = \tau(C^3) = 1$.

With these assumptions it remains to show $\tau(C^2) = 1$. But using the fact that $\tau(0 \to C_1^i \to C_2^i \to C_3^i \to 0) = 1$ we have as before that

$$
\begin{aligned}
{}_2m_i \; {}_2\omega_i &= {}_2\rho_i \wedge {}_2d_i^*({}_2\rho_{i+1}) \wedge \pi^*({}_2\mu_i) \\
&= \overline{{}_1\rho_i \wedge {}_1d_i^*({}_1\rho_{i+1}) \wedge \pi^*({}_1\mu_i)} \wedge \psi^*({}_3\rho_i \wedge {}_3d_i^*({}_3\rho_{i+1}) \wedge \pi^*({}_3\mu_i)) \\
&= \overline{{}_1\omega_i} \wedge \psi^*({}_3\omega_i) \\
&= {}_2\omega_i
\end{aligned}
$$

implying that ${}_2m_i = 1$.

Q.E.D.

Observe that if we take $C_2^* = C_1^* \oplus C_3^*$ that the above proposition implies that $\tau(C_1^* \oplus C_3^*) = \tau(C_1^*)\tau(C_2^*)\tau(H)$.

Now let $C_1^*$ and $C_2^*$ be two complexes equipped with inner product ${}_jh_k$ $(j=1,2)$ inducing volume forms $\omega_j^*$ and inner products ${}_jf_k$ on $H^k(C_j^*)$ inducing volume forms ${}_j\mu_k$. Let $C_1^* \otimes C_2^*$ denote the tensor product complex with its standard differential. By the Künneth formula

$$(1.1.13) \qquad H^i(C_1^* \otimes C_2^*) = \bigoplus_{k=1}^{i} H^k(C_1^*) \otimes H^{i-k}(C_2^*).$$

Let w denote the volume form on $C_1^* \otimes C_2^*$ induced from the inner product $\Sigma_1 h_k \otimes {}_2h_{i-k}$ and suppose the volume form $\mu_i$ on $H^i(C_1^* \otimes C_2^*)$ is induced by the inner product $\eta_i = \Sigma_1 f_k \otimes {}_2f_{i-k}$. Then we have the following Proposition.

Proposition 1.1.14.

$$\ln\tau(C_1^* \otimes C_2^*) = \chi(C_2^*)\ln\tau(C_1^*) + \chi(C_1^*)\ln\tau(C_2^*)$$

where $\chi(C_j^*)$ denotes the Euler characteristic of the complex.

Proof. See Cheeger [C].

The following Proposition comes into play when we establish the combinatorial invariance of $\tau$ for a geometrically defined complex.

Let

$$F^0 \xrightarrow{d_0} F^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} F^n \xrightarrow{d_n} 0$$

be a complex of free abelian groups and set $C^i = F^i \otimes \mathbb{R}$. Each $C^i$ has a preferred equivalence class of bases coming from bases of $F^i$ and any two such bases differ by a matrix with integral entries. Thus each $C^i$ has a canonical volume element. With respect to these bases $d_i$ is represented by a matrix with integral entries, hence $H^i(C)$ has canonical volume element. With these choices of volumes we have

Proposition 1.1.15.

$$\tau(C,\omega,\mu) = \Pi \frac{0_{2k+1}}{0_{2k}}$$

where $0_i$ represents the order of the torsion subgroup of $H^i$.

Proof. See Cheeger [C].

## 1.2

Let X be a finite simplical (cell) complex with flat orthogonal bundle $E$ over X. $E$ can be thought of as a vector bundle over X with a distinguished family of local trivializations having transition functions $g\alpha\beta : U_\alpha \cap U_\beta \to O(n)$ which are locally constant. In this situation the horizontal lift of a cell in any dimension makes sense. The collection of horizontal lifts of an i-cell ${}_i\sigma_j$, denoted as $L({}_i\sigma_j)$ forms a vector space equal to the dimension of the fiber of $E$. Setting $C^i(X,E) = \bigoplus_j L({}_i\sigma_j)$ and taking the set of $C^i(X,E)$ forms a complex in a natural way. The cohomology groups $H^i(X,E)$ defined from the dual complex are topological invariants of the pair $(X,E)$ [BT]. When $E$ is the trivial flat $\mathbb{R}^n$-bundle $H^*(X,E) = H^*(X,\mathbb{R}) \otimes \mathbb{R}^n$. Therefore if we are given a choice of volume forms $\mu$ for $H^i(X,E)$, the torsion $\tau(X,E,\mu)$ is defined. For the case of real coefficients, as explained before Proposition 1.1.15, there is a natural choice of volume elements corresponding to a basis of integral classes. The key property enjoyed by $\tau(X,E,\mu)$ is that we get the same value of $\tau$ for all subdivisions of X, i.e., $\tau(X,E,\mu)$ is a combinatorial invariant.

## 1.3

Here we outline two methods of proving the combinatorial invariance of $\tau$. The key point in each case is to reduce computing the torsion $\tau(X, E, \mu)$ to computing torsions of complexes where the bundle $E$ is trivial and then using Proposition 1.1.15 to get these torsions in terms of purely topological data. In order to use Proposition 1.1.15 we have made a particular choice of volume forms; however, we can use Proposition 1.1.3 to get the general case.

We begin with a lemma which relates the torsion of a complex C to that of the relative torsions obtained from a filtration of C.

<u>Lemma 1.3.1</u>. Let $C^*$ be a cochain complex and $C_n^*, \cdots, C_0^*$ a filtration of $C^*$ by a subcomplexes such that $C^* = C_n^* \supset C_{n-1}^* \supset \cdots \supset C_1^* \supset C_0^* \supset C_{-1}^* = \emptyset$. Suppose $0 \to (C_i^*, C_{i-1}^*) \to C_i^* \to C_{i-1}^* \to 0$ satisfies the conditions of Proposition 1.1.5, then

$$(1.3.2) \qquad \tau(C^*, \omega, \mu) = [\prod_{j=0}^{n} \tau(C_j^*, C_{j-1}^*)\tau(H_{j-1})]$$

where $H_j$ denotes the long exact sequence in cohomology associated to $0 \to (C_{j+1}^*, C_j^*) \to C_{j+1}^* \to C_j^* \to 0$.

<u>Proof</u>. The proof proceeds via induction over the number of elements in the filtration. So suppose we have $C^* = C_1^* \supset C_0^* \supset C_{-1}^* = \emptyset$. Then $0 \to (C_1^*, C_0^*) \to C_1^* \to C_0^* \to 0$ satisfies the conditions of Proposition 1.1.5 and hence by Proposition 1.1.5 we have

$$(1.3.3) \qquad \tau(C^*) = \tau(C_1^*, C_0^*)\tau(C_0^*, C_{-1}^*)\tau(H_0).$$

Thus the lemma is true for n=1. So assume true for n-1.

$$(1.3.4) \qquad \tau(C_n^*) = [\prod_{j=0}^{n-1} \tau(C_j^*,C_{j-1}^*)\tau(H_{j-1})]$$

So consider the exact sequence

$$0 \to (C_n^*,C_{n-1}^*) \to C_n^* \to C_{n-1}^* \to 0.$$

Using Proposition 1.1.5 gives

$$(1.3.5) \qquad \tau(C_n^*) = \frac{\tau(C_{n+1}^*)}{\tau(C_{n+1}^*,C_n^*)\tau(H_{n-1})}.$$

Plugging (1.3.5) into (1.3.4) yields the lemma.

Q.E.D.

Suppose X is a finite n-dimensional symplical complex. Then X is naturally filtered by letting $X_j$ = j skeleton of X. So if $C(X_j,E)$ denotes the complex of horizontal lifts of cells in $X_j$ and $C^*(X_j,E)$ the dual complex we have that $C^*(X_j,E)$ is filtration of $C^*(X,E)$. In order to apply the above lemma we need one additional fact contained in the following lemma.

Lemma 1.3.6. Let X be a simplical complex with flat orthogonal bundle $E$ over X and Y a subcomplex of X. Let $\mu_i$ be a volume form for $H^i(X,E)$ and $\underline{\mu}_i$, $\underline{\underline{\mu}}_i$ volume forms for $H^i(Y,E)$ and $H^i(X,Y,E)$ respectively. Then.

$$(1.3.7) \qquad \tau(X,E,\mu) = \tau(X,Y,E,\underline{\underline{\mu}})\,\tau(Y,E,\underline{\mu})\tau(H)$$

where $H$ denotes the long exact sequence in cohomology of $0 \to C(Y,E) \to C(X,E) \to C(X,Y,E) \to 0$.

Proof. This follows immediately from Proposition (1.1.5) after observing that the right hand side of (1.3.7) is independent of the choices of $\underline{\underline{\mu}}$, and $\underline{\mu}$ by an argument similar to the one in the proof of Proposition 1.1.5 using Proposition 1.1.3.

Q.E.D.

Remark. For $Y = X_j$ we have $H^i(X,E) = H^i(Y,E)$ hence we may as well assume $\underline{\mu} = \mu$.

Theorem 1.3.8. $\tau(X,E,\mu)$ is a combinatorial invariant.

Sketch of Proof. Suppressing explicit mention of the volume forms we have from Lemma 1.3.1

$$\tau(X,E) = [\prod_{j=0}^{n} \tau(X_j,X_{j-1},E)\tau(H_j)]. \tag{1.3.9}$$

By excision we have $C^*(X_i,X_{i-1},E) \cong \bigoplus_{\sigma^i} (\sigma^i,\partial\sigma^i)^*$ where $\sigma_i$ denotes the i-cells in X. The bundle $E$ is trivial over $\sigma^i$ (and $\partial\sigma^i$) since $\sigma^i$ is simply connected and hence Proposition 1.1.15 applies giving $\tau(X_i,X_{i-1},E) = 1$. Since the cohomology groups are combinatorially invariant we have $\tau(H_i)$ is a combinatorial invariant thus proving the theorem.

Q.E.D.

Our second method involves repeated application of a Mayer-Vietoris type lemma, Lemma 1.3.10.

Lemma 1.3.10. Let X be a finite simplical complex with flat orthogonal bundle $E$ over X. Let $X_1$ and $X_2$ be two subcomplexes of X. Then

$$\tau(X_1,E)\tau(X_2,E) = \tau(X_1 \cap X_2,E)\tau(X_1 \cup X_2,E)\tau(H_m) \tag{1.3.11}$$

where $H_m$ is the long exact sequence in cohomology obtained from

$$0 \to C(X_1 \cap X_2,E) \to C(X_1,E) \oplus C(X_2 E) \to C(X_1 \cup X_2,E) \to 0$$

Proof. This is an immediate consequence of Proposition 1.1.5.

Q.E.D.

We now generalize Lemma 1.3.10 to the case where we have n subcomplexes $X_1,\cdots,X_n$. In fact we will restrict ourselves to considering the case where $X = \bigcup_i X_i$.

Lemma 1.3.12. Let X be a finite n-dimensional simplical complex with flat orthogonal bundle $E$ over X. Let $X_1,\cdots,X_\ell$ be complexes of X such that $X = \cup X_i$. Then

$$\tau(X,E) = \frac{\displaystyle\prod_{\substack{n=1\\ m\ \text{odd}}}^{\ell} \ \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)}{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell} \ \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)} \ \tau(\overline{H}) \tag{1.3.13}$$

where

$$\tau(\overline{H}) = \frac{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell} \ \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell-m}} \tau(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}i_{m+1},\cdots,i_1\cdots i_{m-1}\ell})}{\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell} \ \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell-m}} \tau(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}i_{m+1},\cdots,i_1\cdots i_{m-1}\ell})} \tag{1.3.14}$$

where $\tau(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1},i_{m+1},\ldots,i_1\cdots i_{m-1}\ell})$ denotes the torsion of the long exact sequence in cohomology coming from the short exact sequence

$$(1.3.15)\quad 0 \to C((X_{i_1}\cap\cdots\cap X_{i_{m-1}})\cap((X_{i_1}\cap\cdots\cap X_{i_{m-1}})\cup\cdots\cup(X_{i_1}\cap X_{i_{m-1}}\cap X_\ell),E)$$

$$\to C(X_{i_1}\cap\cdots\cap X_{i_{m-1}}\cap X_{i_m},E)\oplus C((X_{i_1}\cap\cdots\cap X_{i_{m+1}})\cup\cdots\cup(X_{i_1}\cap\cdots\cap X_{i_{m-1}}$$

$$\cap X_\ell),E) \to C((X_{i_1}\cap\cdots\cap X_{i_m})\cup\cdots\cup(X_{i_1}\cap\cdots\cap X_{i_{m-1}}\ X_\ell),E) \to 0.$$

Proof. This proof is by induction over the number of $X_i$. We verify the lemma for the case $\ell=2$ first. So suppose $X = X_1 \cup X_2$, and consider the exact sequence

$$0 \to C(X_1,X_2,E) \to C(X_1,E) \oplus C(X_2,E) \to C(X,E) \to 0.$$

Using Lemma 1.3.10 we have

$$\tau(X,E) = \frac{\tau(X_1,E)\tau(X_2,E)}{\tau(X_1\cap X_2,E)\tau(E_1^2)}$$

and thus the lemma is verified for $\ell=2$. So assume true for $\ell-1$. Then

$$(1.3.16)\quad \tau(X,E) = \tau(X_1 \cup \bigcup_{k=2}^{\ell} X_k,E) = \frac{\tau(X_1,E)\tau(\bigcup_{k=2}^{\ell} X_k,E)}{\tau((X_1\cap X_2)\cup\cdots\cup(X_1\cap X_\ell),E)\tau(H_1^{2\cdots\ell})}$$

Applying the lemma to the terms $\tau(\bigcup_{k=2}^{\ell} X_k,E)$ and $\tau(\bigcup_{k=2}^{\ell}(X_1\cap X_k),E)$ yields

$$(1.3.17)\qquad \tau\Big(\bigcup_{k=2}^{\ell} X_k,\ E\Big) = \frac{\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)}{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)}\ \tau(\bar{\bar{H}})$$

and

$$(1.3.18)\qquad \tau\Big(\bigcup_{k=2}^{\ell}(X_1\cap X_k),E\Big) = \frac{\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_1\ X_{i_1}\cap\cdots\cap X_{i_m},E)}{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_1\ X_{i_1}\cap\cdots\cap X_{i_m},E)}\ \tau(\bar{\bar{H}})$$

Plugging (1.3.17) and (1.3.18) into (1.3.16) yields

$$\frac{\tau(X_1,E)\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_1\cap X_{i_1}\cap\cdots\cap X_{i_m},E)\,\tau(\bar{\bar{H}})}{\tau\Big(\bigcup_{k=2}(X_1\cap X_k),E\Big)\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 2\le i_1\le \ell}} \tau(X_1\cap X_{i_1}\cap\cdots\cap X_{i_m},E)\,\tau(\bar{\bar{\bar{H}}})}$$

which equals

$$\frac{\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell-1}\ \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)\,\tau(\bar{\bar{H}})}{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell}\ \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau(X_{i_1}\cap\cdots\cap X_{i_m},E)\,\tau(H_1^{2\cdots\ell})\,\tau(\bar{\bar{\bar{H}}})}.$$

Thus it remains to examine the product

$$\frac{\tau(\bar{\bar{H}})}{\tau(H^{2\cdots\ell})\tau(\bar{\bar{\bar{H}}})} \tag{1.3.19}$$

where

$$\tau(\bar{\bar{H}}) = \frac{\prod\limits_{\substack{m=1 \\ m \text{ even}}}^{\ell-1} \prod\limits_{\substack{i_1<\cdots<i_m \\ 2\le i_1\le \ell-m}} \tau(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}i_{m+1},\cdots,i_1\cdots i_{m-1}\ell})}{\prod\limits_{\substack{m=1 \\ m \text{ odd}}}^{\ell-1} \prod\limits_{\substack{i_1<\cdots<i_m \\ 2\le i_1\le \ell-m}} \tau(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}i_{m+1},\cdots,i_1\cdots i_{m-1}\ell})} \tag{1.3.20}$$

and

$$\tau(\bar{\bar{\bar{H}}}) = \frac{\prod\limits_{\substack{m=1 \\ m \text{ even}}}^{\ell-1} \prod\limits_{\substack{i_1<\cdots<i_m \\ 2\le i_1\le \ell-m}} \tau(H_{1i_1\cdots i_m}^{1i_1\cdots i_{m-1}i_{m+1},\cdots,1i_1\cdots i_{m-1}\ell})}{\prod\limits_{\substack{m=1 \\ m \text{ odd}}}^{\ell-1} \prod\limits_{\substack{i_1<\cdots<i_m \\ 2\le i_1\le \ell-m}} \tau(H_{1i_1\cdots i_m}^{1i_1\cdots i_{m-1}i_{m+1},\cdots,1i_1\cdots i_{m-1}\ell})}. \tag{1.3.21}$$

Plugging (1.3.20) and (1.3.21) into (1.3.19) yields $\tau(\bar{H})$ and completes the proof.

Q.E.D.

By taking $X_1,\cdots,X_\ell$ such that the $X_i$ are simply connected and all their intersections are simply connected (this can be achieved for instance by letting the $X_i$ be the n-cells in X) we can repeat the arguement of Theorem (1.3.8) to show that $\tau(X,E,\mu)$ is a combinatorial invariant.

1.4

We end this section with a slight generalization of Proposition 1.1.5 and obtain from this one last formula for the torsion $\tau(X,E,\mu)$.

Lemma 1.4.1. Let $0 \to C_1^* \to C_2^* \to \cdots \to C_n^*$ denote a long exact sequence of chain complexes. Denote by $K_i^*$ the kernel of the map between $C_i^*$ and $C_{i+1}^*$ and $H_i$ the long exact sequence in cohomology obtained from the short exact sequence

$$(1.4.2) \qquad 0 \to K_i^* \to C_i^* \to K_{i+1}^* \to 0.$$

Suppose that all the short exact sequences in (1.4.2) satisfy the conditions of Proposition 1.1.5 then

$$(1.4.3) \qquad \frac{\prod_{i \text{ even}} \tau(C_i^*)}{\prod_{i \text{ odd}} \tau(C_i^*)} = \frac{\prod_{\substack{i \text{ even} \\ 2 \le i \le n-1}} \tau(H_i)}{\prod_{\substack{i \text{ odd} \\ 3 \le i \le n-1}} \tau(H_i)} .$$

Proof. Since we have $K_2^* = C_1^*$ and $K_n^* = C_n^*$ consider the n-2 short exact sequences

$$0 \to C_1^* \to C_2^* \to K_3^* \to 0$$

$$0 \to K_3^* \to C_3^* \to K_4^* \to 0$$

$$\vdots$$

$$0 \to K_{n-1}^* \to C_{n-1}^* \to C_n^* \to 0.$$

Using Proposition 1.1.5 we have

$$\tau(C_i^*) = \tau(K_i^*)\tau(K_{i+1}^*)\tau(H_i).$$

Thus

$$\frac{\prod\limits_{\substack{i \text{ even} \\ 2\le i\le n-1}} \tau(C_i^*)}{\prod\limits_{\substack{i \text{ odd} \\ 2\le i\le n-1}} \tau(C_i^*)} = \frac{\prod\limits_{\substack{i \text{ even} \\ 2\le i\le n-1}} \tau(K_i^*)\tau(K_{i+1}^*)\tau(H_i)}{\prod\limits_{\substack{i \text{ odd} \\ 2\le i\le n-1}} \tau(K_i^*)\tau(K_{i+1}^*)\tau(H_i)}$$

Taking $\tau(K_2^*) = \tau(C_1^*)$ and $\tau(K_n^*) = \tau(C_n^*)$ to the left hand side and cancelling the $\tau(K_i^*)$ yields the lemma.

Q.E.D.

Using the generalized Mayer-Vietoris sequence we can obtain a formula for $\tau(X,E,\mu)$ using the above lemma. As before let $C^i(X,E)$ denote the cochain complex associated to complex of horizontal lifts and let $\bar{\partial}$ be its boundary operator. So let $X_1,\cdots,X_\ell$ be subcomplexes of X such that $X = \bigcup_{i=1}^{\ell} X_i$. Then one has the exact cochain complex given by

$$0 \to C^*(X, \ ) \xrightarrow{D} \prod_\alpha C^*(X_{\alpha_1},E) \xrightarrow{D} \prod_{\alpha<\beta} C^*(X_\alpha \cap X_\beta, E) \to \cdots$$
$$\xrightarrow{D} \prod_{\alpha_1<\cdots<\alpha_\ell} C^*(X_{\alpha_1} \cap \cdots \cap X_{\alpha_\ell}) \to 0$$

where $D = \delta + (-1)^p\,\bar{\partial}$ and $\delta$ is the difference operator. See [BT]. Thus applying Lemma 1.4.1 to the above yields a formula for $\tau(X,E,\mu)$ in terms of torsions of all the various intersections of the $X_i$.

The formula obtained in the above manner avoids the asymmetry that appears in Lemma 1.3.12. By this we mean that in Lemma 1.3.12 the terms

$\tau(H)$ occur for some of the Mayer-Vietoris sequences but not all possible such sequences. However, in the above formulation all the Mayer-Vietoris sequences are used.

## 2. Equivariant Torsion

We begin by defining the torsions $\tau_\rho$ and $\tau_g$. A simple example shows that $\tau_\rho$ and $\tau_g$ are finer invariants than the classical torsion $\tau$. In this section we also show that $\tau_\rho$ and $\tau_g$ enjoy many of the same properties of the classical torsion.

### 2.1

Let $C^*$ be a finite cochain complex on which $G = \langle g \rangle$ (the group generated by g) acts. Suppose that g is of finite order. Let $C^*_\rho$ be the largest subcomplex on which the action G splits as a direct sum of complexes on which G acts by a given irreducible $\rho(G)$. Let $\omega_{i,\rho}$ be a g invariant volume element for $C^i_\rho$ and $\mu_{i,\rho}$ a g-invariant volume element for $H^i(C^i_\rho)$.

Definition 2.1.1. The g-torsion is defined as

$$\tau_g(C^*,\omega,\mu) = \prod_\rho \tau(C^*_\rho,\omega_\rho,\mu_\rho)^{tr(\rho(g))} \tag{2.1.2}$$

and

$$\tau_\rho(C^*) = \tau(C^*_\rho,\omega_\rho,\mu_\rho). \tag{2.1.3}$$

The above definition for $\tau_\rho$ works equally well where G is any finite group. In the following discussion we will give the correspondence between $\tau_\rho$ and $\tau_g$ for an arbitrary finite group G. In order to get this correspondence between $\tau_\rho$ and $\tau_g$ we first derive some formulas for $\tau_\rho$ and $\tau_g$ in terms of zeta functions.

Now suppose the $C^i$ come equipped with G invariant inner-products and these inner-products induce volume forms. Then taking $\omega_{i,\rho}$ and $\mu_{i,\rho}$ to be the induced volume forms where we have identified $H^i_\rho$ with $(B^i_\rho)^\perp \cap Z^i_\rho$ gives G-invariant volume forms for $C^i_\rho$ and $H^i_\rho$ respectively.

Let ${}_ie^\rho_1, \cdots, {}_ie^\rho_{t^\rho_{i+1}}$ be an orthonormal basis of eigen-vectors for the operator

$$(2.1.4) \qquad (d^\rho_i)^* d^\rho_i \big|_{(Z^i_\rho)^\perp} \subset C^i_\rho$$

where $d^\rho_i$ denotes $d_i \big|_{C^i_\rho}$ and the adjoint $(d^\rho_i)^*$ is computed with respect to the given inner-products. Let ${}_i\lambda_j$ denote the eigen-values of the above operators. Then we have

$$(2.1.5) \qquad \left\langle \frac{d^\rho_i({}_ie^\rho_j)}{({}_i\lambda^\rho_j)^{\frac{1}{2}}}, \frac{d^\rho_i({}_ie^\rho_k)}{({}_i\lambda^\rho_k)^{\frac{1}{2}}} \right\rangle = \frac{1}{({}_i\lambda^\rho_j \, {}_i\lambda^\rho_k)^{\frac{1}{2}}} \left\langle (d_i)^* d_i({}_ie_j), {}_ie_k \right\rangle$$

$$= \delta_{jk}$$

Thus using the orthogonal decompositions

$$B^i_\rho \oplus ((B^i_\rho)^\perp \cap Z^i_\rho) \oplus (Z^i)^\perp \to B^i_\rho \oplus ((B^{i+1}_\rho)^\perp \cap Z^{i+1}_\rho) \oplus (Z^{i+1}_\rho)^\perp$$

one has $M^i_\rho = \Pi({}_i\lambda_j)^{\frac{1}{2}}$ where $M^i_\rho$ denotes the numbers in (1.1) for the complex $C^*_\rho$. Thus

$$(2.1.6) \qquad \ln \tau_\rho = \frac{1}{2} \sum_{i=0} (-1)^i \Big[ \sum_{j=0}^{t^\rho_i} \ln {}_i\lambda^\rho_j \Big].$$

Letting ${}_i\zeta_\rho(s)$ denote the zeta function for the operator $\Delta_i^\rho = d_{i-1}^\rho d_{i-1}^{\rho^*} + d_i^{\rho^*} d_i^\rho$ where the zeta function for any non-negative symmetric operator A is defined as $\sum_{\lambda_k > 0} \lambda_k^{-s}$ where $\lambda_k$ are the nonzero eigenvalues of A. Then as in Ray and Singer [RS] (2.1.6) can be rewritten as

$$\text{(2.1.7)} \qquad \ln \tau_\rho = \frac{1}{2} \Sigma(-1)^i i\, {}_i\zeta_\rho'(0).$$

Letting ${}_if_1, \cdots, {}_if_{t_i+1}$ be an orthonormal basis of eigenvectors for the operator $\Delta_i = d_{i-1}d_{i-1}^* + d_i^* d_i$ such that each ${}_if_j^\rho = {}_if_j$, where the $\{{}_if_j^\rho\}$ are an orthonormal basis of eigenvector $\Delta_i^\rho$. Then for $g \in G$ let

$$\text{(2.1.8)} \qquad {}_i\zeta_g(s) = \Sigma\, {}_i\lambda_j^{-s} \langle g\, {}_if_j, {}_if_j \rangle$$

where ${}_i\lambda_j$ are the eigenvalues of ${}_if_j$ and define

$$\text{(2.1.9)} \qquad \ln \tau_g(C^*) = \frac{1}{2} \Sigma(-1)^i i\, {}_i\zeta_g'(0).$$

That this definition is justified is a consequence of our next proposition which gives the relationship between $\tau_g$ and the $\tau_\rho$ for a finite group G. That is by taking $G = \langle g \rangle$ in the next proposition gives the same expression as (2.1.2).

<u>Proposition 2.1.10</u>. With the setup as above

$$\text{(2.2.11)} \qquad \ln \tau_g(C^*) = \sum_\rho \chi_\rho(g) \ln \tau_\rho(C^*)$$

where $\chi_\rho(g)$ is the character of the representation $\rho$.

Proof. Let $F_\lambda$ denote the eigenspace of $\Delta_i$ on $C^i$ with eigenvalue $\lambda$. For $f \in F_\lambda$ we have $gf \in F_\lambda \ \forall_g \in G$ since $\Delta_i(gf) = g\Delta_i(f) = g(\lambda f) = \lambda(gf)$. Thus G acts on $F_\lambda$ and we can write

$$F_\lambda = \sum_\rho \dim(F_{\lambda,\rho}) \otimes [\rho]$$

where $[\rho]$ denotes an irreducible subspace of $\rho$. Consequently for all $g \in G$ we have

$$(2.1.12) \qquad tr(g|_{F_\lambda}) = \sum_\rho \dim(F_{\lambda,\rho})\chi_\rho(g).$$

From (2.1.8) we have

$$\begin{aligned} {}_i\zeta_g(s) &= \sum_{{}_i\lambda_j \neq 0} {}_i\lambda_j^{-s} \langle g\,{}_if_j, {}_if_j \rangle \\ &= \sum_{{}_i\lambda_j \neq 0} {}_i\lambda_j^{-s} Tr(g|_{F_\lambda}) \\ &= \sum_{{}_i\,{}_j\neq 0} (\sum_\rho \dim(F_{\lambda,\rho})\chi_\rho(g))\,{}_i\lambda_j^{-s} \\ &= \sum_\rho (\sum_{{}_i\lambda_j \neq 0} \dim(F_{\lambda,\rho})\,{}_i\lambda_j^{-s})\chi_\rho(g) \\ & \sum_\rho {}_i\zeta_\rho(s)\chi_\rho(g) \end{aligned}$$

Thus using the above in (2.1.7) and (2.1.9) the proposition follows.

Q.E.D.

In order to get $\tau_\rho$ in terms of the $\tau_g$ it will be convenient to complexify the $C^i$. Let $K^i = C^i \otimes_{\mathbb{R}} \mathbb{C}$ and take hermitian inner product on $K^i$ given by the product of the inner product on $C^i$ and the standard hermitian inner product on $\mathbb{C}$. Then $K^i$ is complex vector space of complex dimension $\ell_i$.

Let $K^*$ denote the complex

$$K^0 \xrightarrow{d_0 \otimes 1} K^1 \xrightarrow{d_1 \otimes 1} K^2 \longrightarrow \cdots \xrightarrow{d_{i-1} \otimes 1} 0.$$

Denoting $d_i \otimes 1$ as $\overline{d}_i$ we can as before define the operator $\overline{\Delta}_i = \overline{d}_{i-1}\overline{d}^*_{i-1} + \overline{d}^*_i\overline{d}_i$. This operator has the same eigenvalues as $\Delta_i$ and hence has basis of eigenvectors consisting of ${}_if_1 \otimes 1, \ldots, {}_if_{t_{i+1}} \otimes 1$. Thus the zeta function $\overline{{}_i\zeta_g(s)}$ associated to $\overline{\Delta}_i$ equals the zeta functions ${}_i\zeta_g(s)$ of $\Delta_i$ and the torsions $\overline{\tau}_g$ and $\tau_g$ are equal.

Let $[\rho]$ be a subspace of $C^i$ on which G acts by a given irreducible $\rho(G)$. Then

$$[\rho] \otimes \mathbb{C} = \begin{cases} [\rho'] & \chi_\rho = \chi_{\rho'} \\ [\rho'] \otimes [\overline{\rho}'] & \text{if } \chi_\rho = \chi_{\rho'} + \chi_{\overline{\rho'}} \end{cases}$$

where $[\rho']$ is a subspace of $K^i$ on which G acts by a given irreducible $\rho'(G)$ and the bar denotes conjugation. Thus we have that

$$C^i \otimes \mathbb{C} = \begin{cases} K^i_{\rho'} & \chi_\rho = \chi_{\rho'} \\ K^i_{\rho'} \oplus K^i_{\overline{\rho'}} & \chi_\rho = \chi_{\rho'} + \chi_{\overline{\rho'}} \end{cases} .$$

Letting $\zeta^i_\rho(s)$ denote the zeta function for $\Delta^i_\rho$ and $\overline{{}_i\zeta_\rho(s)}$ the zeta function corresponding to the operator $\overline{\Delta^i_\rho}$

we have as before that ${}_i\zeta_\rho(s) = \overline{{}_i\zeta_\rho(s)}$. Thus by the above decomposition

$$ {}_i\zeta_\rho(s) = \begin{cases} {}_i\zeta_{\rho'}(s) & \chi_\rho = \chi_{\rho'} \\ {}_i\zeta_{\rho'}(s) + {}_i\zeta_{\overline{\rho'}}(s) & \chi_\rho = (\chi_{\rho'} + \chi_{\overline{\rho'}} \end{cases} $$

where ${}_i\zeta_{\rho'}(s)$ is the zeta function of the operator $\Delta^i_{\rho'}$. Since the eigenvalues of $\Delta^i_{\rho'}$ are real we have that ${}_i\zeta_{\rho'}(s) = \overline{{}_i\zeta_{\rho'}(s)}$. From the above we have

$$ \tau_\rho = \begin{cases} \tau_{\rho'} & \chi_\rho = \chi_{\rho'} \\ \tau_{\rho'} + \tau_{\overline{\rho'}} & \chi_\rho = 0(\chi_{\rho'}, \chi_{\overline{\rho'}}). \end{cases} $$

The same proof as in Proposition 2.1.10 shows that

(2.1.14) $$\overline{\tau}_g(K^*) = \sum_{\rho'} \chi_{\rho'}(g)\tau_{\rho'}(K^*).$$

Using the above relations for $\tau_g$, $\overline{\tau}_g$, $\tau_\rho$, $\tau_{\rho'}$, and $\chi_\rho$ and $\chi_{\rho'}$ we can always recover the real case from the complex case. Before giving the relation between $\tau_\rho$ and the $\tau_g$ we first recall the following theorem.

<u>Theorem 2.1.15</u>. Let G be a finite group of order $|G|$. If $\chi_\alpha$, $\chi_\beta$ are the characters of two irreducible complex unitary representation $\alpha, \beta$ of G, then

(2.1.16) $$\frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g)\overline{\chi}_\beta(g) = \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha$ is equivalent to $\beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha$ is not equivalent to $\beta$.

Proof. See [S].

Proposition 2.1.17.

(2.1.18) $$\ln \tau_{\rho'} = \frac{1}{|G|} \sum_{g\varepsilon G} \chi_{\rho'}(g) \ln \bar{\tau}_g$$

Remark. It follows from Proposition 2.1.17 that

$$\ln \tau_{\rho} = \frac{1}{|G|} \sum_{g\varepsilon G} \chi_{\rho}(g) \ln \tau_g$$

by using the fact that $\bar{\tau}_g = \tau_g$ and the relations in (11).

Proof. From Proposition 2.1.10 we have

$$\ln \bar{\tau}_g = \sum_{\rho'} \bar{\chi}_{\rho'}(g) \ln \tau_{\rho'} .$$

Thus

$$\frac{1}{|G|} \sum_{g\varepsilon G} \bar{\chi}_{\rho''}(g) \ln \tau_g = \frac{1}{|G|} \sum_{g\varepsilon G} \bar{\chi}_{\rho''}(g) \sum_{\rho'} \chi_{\rho'}(g) \ln \bar{\tau}_{\rho}$$

$$= \sum_{\rho'} \frac{1}{|G|} \sum_{g\varepsilon G} \bar{\chi}_{\rho''}(g) \chi_{\rho'}(g) \ln \tau_{\rho'}$$

$$= \sum_{\rho'} \delta_{\rho'\rho''} \ln \tau_{\rho'}$$

$$= \ln \tau_{\rho'}$$

where the last step follows from (2.1.16).

Q.E.D.

## 2.2

We now show that $\tau_\rho$ and $\tau_g$ have many of the same properties as that of classical Reidemeister torsion $\tau$. By Propositions 2.1.10 and 2.1.17 it will be enough to establish these properties for $\tau_\rho$ with G a finite cyclic group. So although the following propositions will be stated for G finite cyclic they will be understood to hold more generally.

<u>Proposition 2.2.1</u>. Suppose $0 \to C_1^* \xrightarrow{\alpha} C_2^* \xrightarrow{\psi} C_3^* \xrightarrow{\beta} 0$ is an exact sequence of chain complexes with volume ${}_j\omega_i^\rho$ of $C_j^{i\rho}$ (j=1,2,3) and ${}_j\mu_i^\rho$ for $H_\rho^i(C_j^\rho)$. In addition assume that for all i and $\rho$ the torsion of the complex $(0 \to C_1^{i\rho} \to C_2^{i\rho} \to C_3^{i\rho} \to 0)$ is 1, then

$$\tau_\rho(C_2,\omega_2^\rho,\mu_2^\rho) = \tau_\rho(C_1,\omega_1^\rho,\mu_1^\rho)\tau_\rho(C_3,\omega_3^\rho,\mu_3^\rho)\tau_\rho(H)$$

where $H$ is the long exact sequence in cohomology associated to the short exact sequence $0 \to C_1^* \to C_2^* \to C_3^* \to 0$.

<u>Proof</u>. The Proposition will follow from Proposition 1.1.5 if we can show that (i) G acts on ${}_jH^i$ and (ii) ${}_jH_\rho^i = {}_j\overline{H}_\rho^i$ where ${}_jH_\rho^i$ the largest subcomplex on which the action of G splits as a direct sum of complexes on which G acts by a given irreducible $\rho(G)$ and ${}_j\overline{H}_\rho^i$ is the cohomology of the complex $C_j^{*\rho}$.

<u>Proof of (i)</u>. Let $h \in {}_jH^i$, then if z and z' are two cycles representing h we have z - z' = b. Since $g \circ d = d \circ g$ for all

$g \varepsilon G$ we have $g(z-z') = g(b) = g(da) = d(ga)$. Thus G acts on ${}_jH^i$.

Q.E.D.

Proof of (ii). We have

$$ {}_jC^i = \bigoplus_\rho m_\rho[\rho] $$
$$ {}_jZ^i = \bigoplus_\rho n_\rho[\rho] $$
$$ {}_jB^i = \bigoplus_\rho s_\rho[\rho] $$

where $s_\rho < n_\rho < m_\rho$. Thus ${}_jH^i_\rho = (n_\rho - s_\rho)[\rho]$. Therefore we need to show that ${}_jZ^i_\rho = n_\rho[\rho]$ and ${}_jB^i_\rho = s_\rho[\rho]$ where ${}_jZ^i_\rho = \ker {}_jd^\rho_i \equiv \ker {}_jd_i\big|_{{}_jC^i_\rho}$ and ${}_jB^i_\rho = {}_jd^\rho_{i-1}(C^i_\rho)$. $Z^i_\rho = (n_\rho - s_\rho)[\rho]$ follows immediately from the definitions. Now suppose $b \varepsilon s_\rho[\rho]$ then $d_{i-1}(a) = b$ for $a \varepsilon C^{i-1}$. We can assume $a \varepsilon C^{i-1}_\rho$ since by Schur's lemma the mapping $d_{i-1} : C^{i-1}_\rho \to C^i_{\rho'}$ is the zero map unless $\rho = \rho'$.

Q.E.D.

Let $K_1^*$ and $K_2^*$ be the complexified complexes of $C_1^*$ and $C_2^*$ respectively with hermitian inner products ${}_jh_k (j=1,2)$ obtained as before inducing volume forms ${}_j\mu_k$ and hermitian inner products ${}_jf_k$ on $H^k(K_j^*)$ inducing volume forms ${}_j\mu_k$. Suppose $K_1^*$ and $K_2^*$ are acted on in such a way that $G = \langle g \rangle$ preserves the given inner products. Let ${}_jf^{\rho'}_k$ and ${}_jh^{\rho'}_k$ denote the induced inner products on $H^k(K_j^*)$ and $K_j^*$ respectively with $\omega^\rho_j$ and ${}_j\mu^\rho_k$ their induced volume forms.

Let $K_1^* \otimes K_2^*$ denote the tensor product complex with its standard differential and $(K_1^* \otimes K_2^*)^{\rho'}$ the largest subcomplex on which the action of G splits as the direct sum of complexes on which G acts by a given irreducible $\rho'(G)$. Since G is finite cyclic we have the irreducible representations of G have degree 1. Using a simple dimension counting arguement we have

$$(K_1^* \otimes K_2^*)^{\rho'_k} = \bigoplus_{\ell+m=k} K_1^{*^{\rho'_\ell}} \otimes K_2^{*^{\rho_m}}$$

where $\chi_{\rho'_k}(g) = e^{\frac{2\ ik}{|G|}}$. By the Künneth formula and Schur's Lemma.

$$(2.2.2) \qquad H^i((K_1^* \otimes K_2^*)^{\rho'_k}) = \bigoplus_{\ell+m=k} H^j(K_1^{*^{\rho'_\ell}}) \otimes H^{i-j}(K_2^{*^{\rho'_m}}).$$

Let $\omega^{\rho_k}$ denote the volume form of $(K_1^* \otimes K_2^*)^{\rho'_k}$ induced form $\Sigma_i h_j^{\rho'_\ell} \otimes {}_2h_{i-j}^{\rho'_m}$ and suppose that the volume form $\mu_i^\rho$ on $H^i((K_1^* \otimes K_2^*)^{\rho'_k})$ is induced from the inner product $\Sigma_i f_j^{\rho_\ell} \otimes {}_2f_{i-j}^{\rho_m}$. Then we have the following proposition.

<u>Proposition 2.2.3</u>.

$$\ln \tau_{\rho'_k}(K_1^* \otimes K_i^*) = \sum_{\ell+m=k} (\chi_{\rho_\ell}(K_1^*)\ln \tau_{\rho_m}(K_2^*) + \chi_{\rho_m}(K_2^*)\ln \tau_{\rho_\ell}(K_1^*))$$

where $\chi_{\rho_\ell}(K_j^*)$ denotes the Euler characteristic of the complex $K_j^{*^{\rho_\ell}}$.

<u>Proof</u>. This Proposition follows from Propositions 1.1.4, 2.2.1 and remarks (i) and (ii) of the previous proof.

Q.E.D.

Proposition 2.2.4. Let $C^*$ be a cochain complex on which $G = \langle g \rangle$ acts. Let $C_\rho^*$ be the largest subcomplex on which the action of G splits as a direct sum of the complexes on which G acts by a given irreducible $\rho(G)$. Let $C_n^*, \cdots, C_1^*$ be a filtration of $C^*$ such that $C^* = C_n^* \supset C_{n-1}^* \supset \cdots \supset C_1^* \supset C_0^* \supset C_{-1}^* = \emptyset$ and such that G acts on each $C_i^*$. Let $C_{i,\rho}^*$ be the largest subcomplex on which the action of G splits as a direct sum of the complexes on which G acts by a given irreducible $\rho(G)$. If $0 \to (C_i^*, C_{i-1}^*) \to C_i^* \to C_{i-1}^* \to 0$ satisfies the conditions of Proposition 2.2.1, then

$$(2.2.5) \qquad \tau_\rho(C^*) = \prod_{j=0}^{n} \tau_\rho(C_j^*, C_{j-1}^*)\tau_\rho(H_{j-1})$$

where $H_j$ denotes the long exact sequence in cohomology associated to $0 \to (C_{i+1}^*, C_i^*) \to C_{i+1}^* \to C_i^* \to 0$.

Proof. The proof is the same as Lemma 1.3.1 except one uses Proposition 2.2.1 instead of Proposition 1.1.5.

Q.E.D.

2.3

Let X be a finite simplical complex, $E$ a flat orthogonal bundle over X and g a simplical automorphism of X that extends to a bundle automorphism , $\tilde{g}$. Suppose further that g is of finite order and let $G = \langle g \rangle$ the group generated by g. Let F(H) denote the set of points fixed by every element of H. We say G satisfies condition S if given $x \in F(H) \cap \sigma$ then $\sigma \subset F(H)$.

Condition S which guarantees that F(H) is a subcomplex of X can always be achieved by taking barycentric subdivision. Thus without essential loss of generality we will always assume condition S is satisfied.

Let $C_\rho^*(X,E)$ be the largest subcomplex on which the action of G splits as the direct sum of complexes on which G acts by a given irreducible $\rho(G)$. Given g-invariant volume forms for $C^i(X,E)$ and $H_\rho^i(X,E)$ (the cohomology of the complex $C_\rho^i(\,,E)$) the torsion $\tau_\rho(X,E)$ is defined.

The next lemma plays a role in the proof of combinatorial invariance (Section 4).

<u>Lemma 2.3.1</u>. Let X be a simplical complex with orthogonal bundle $E$ over X. Let g be a simplical automorphism of finite which extends to a bundle automorphism and G = <g> the group generated by G. Let $X_1, X_2$ be invariant subcomplexes of X such that $X_2 \subset X_1$, then

$$(2.3.2) \qquad \tau_\rho(X,E) = \tau_\rho(X_2,E)\tau_\rho(X_1,X_2,E)\tau_g(H_r)$$

where $H_r$ is the long exact sequence in cohomology obtained from the relative sequence.

<u>Proof</u>. This follows immediately from Proposition 2.2.1.

Q.E.D.

Now let $X = M^n$ be a smooth compact manifold, then X has a unique g equivariant combinatorial structure [I]. By this we mean given two simplical decompositions of M for which G acts simplically there is a common refinement such that G continues to act simplically. Let g be an isometry of M which extends to an automorphisms of $E^m$. As above assume g is of finite order and G = <g> the group generated by G. Since $E^m$ is a G-bundle, the induced inner product on $C^i(M,E)$ is g-invariant. By identifying $H^i(M,E)$ with the space of harmonic forms with coefficients in $E$ and using the global inner product on these forms, we can define volume element $\mu_i^\rho$ for $H_\rho^i(M,E)$ and the torsion $\tau_\rho(M,E)$ is defined.

Let $M^n$ be a closed compact oriented n-manifold with flat orthogonal bundle E over M. Let g be as above and assume further that g is orientation preserving. Then by the standard proof of Poincaré duality there is an isomorphism $\lambda_i^\rho : (H_\rho^i(M,E))^* \to H_\rho^{n-i}(M,E)$.

<u>Proposition 2.3.3</u>. Let $M^n$ be a closed, compact orientable even dimensional manifold. Let $\{\mu_i^\rho\}$ denote volume elements for $H^i(M,E)$ and $\{\mu_i^{\rho^*}\}$ the induced volume elements for $H^i(M,E)^*$. Let g act as above, g orientation preserving, then if $\lambda_i^\rho : (H^i(M,E))^* \to H^{n-i}(M,E)$ is the isomorphism defined by Poincaré duality and $\lambda_i^{\rho^*}(\mu_{n-i}^\rho) = \mu_i^{\rho^*}$, then $\ln \lambda_\rho(M,E,\mu) = 0$.

<u>Proof</u>. Let K be a g-invariant triangulation of M and $\tilde{K}$ the resulting g-invariant dual cell complex. Let $\underline{*} : C^i(K,E) \to C^{n-i}(\tilde{K},E)$ be induced by mapping an orientable simplex of K

to the dual cell of $\tilde{K}$ with suitable orientation. Let $\underline{*}\rho : C^i(K,E) \to C^{n-i}(\tilde{K},E)$ denote the restriction of $\underline{*}$ to $C^i_\rho(K,E)$. Then $\underline{*}\rho$ induces a map which takes harmonic cocycles of $C^i(K,E)$ to harmonic cocycles of $C^{n-i}(\tilde{K},E)$, and we also denote by $\underline{*}\rho$ the induced on cohomology. The standard inner product and hence volume element $\gamma_\rho$ for which (2.1.9) holds is induced by $\langle\langle x,y\rangle\rangle = X \cup \underline{*}\rho y$. The arguement of [RS] implies $\ln \tau_\rho(M,E,\gamma_\rho)$ = the $\{\gamma^\rho\}$ satisfy the condition $\lambda^\rho_{n-i}(\gamma^\rho_{n-i}) = \gamma^{\rho^*}_i$. For if $\alpha : H^{n-i}_\rho(M,E) \to (H^{n-i}(M,E))^*$ is the isomorphism induced by the inner product then $\alpha\lambda^\rho$ is an isometry. Now let $\{\mu^\rho_i\}$ be any other set of volume forms satisfying $\lambda^{\rho^*}_{n-i}(\mu^\rho_{n-i}) = \mu^{\rho^*}_i$. Then if

$$\gamma^\rho_{n-i} = k\ \mu^\rho_{n-i}$$

it follows that

$$\lambda^\rho_{n-i}\gamma_{n-i}(\gamma^\rho_i) = k\ \lambda^{\rho^*}_{n-i}\mu^\rho_{n-i}(\gamma^\rho_i).$$

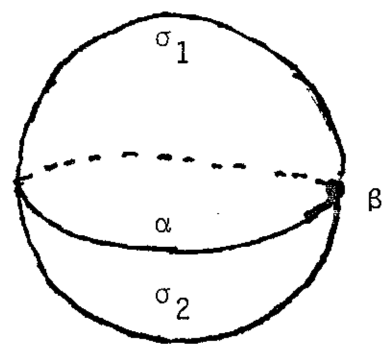
Since $\lambda^{\rho^*}_{n-i}(\gamma^\rho_{n-i}) = \gamma^{\rho^*}_i$, $\lambda^{\rho^*}_{n-i};\lambda_{n-i} = \mu^{\rho^*}_{n-i}$ and by definition we have $1 = \lambda^{\rho^*}_i(\lambda^\rho_i)$, yielding $1 = k\ \mu^{\rho^*}_i(\gamma^\rho_i)$. Since also $1 = \mu^{\rho^*}_i(\mu^\rho_i)$, $k\ \mu^{\rho^*}_i(\gamma^\rho_i) = \mu^{\rho^*}_i(\mu^\rho_i) = k(\mu^{\rho^*}_i(\frac{1}{k}\ \mu^\rho_i))$ which implies $\gamma^\rho_i = \frac{1}{k}\ \mu^\rho_i$. Using this and the fact n is even implies $\mathrm{lm}\ \tau_\rho(M,E,\mu_\rho) = \ln \tau_\rho(M,E,\gamma_\rho)$.

Q.E.D.

## 2.4

We conclude this section with some examples.

Example 2.4.1. Let $S^2$ denote the standard 2-sphere with cell structure consisting of 2 two-cells $\sigma_1, \sigma_2$, 1 one-cell $\alpha$, and 1 zero-cell $\beta$. Let $E$ be the trivial line bundle over




$S^2$. Assume $\alpha$ is the equator and let g be reflection through the equator. We consider two extensions $\tilde{g}_1$, $\tilde{g}_2$ of g to E defined by

$$\tilde{g}_1(x,v) = (rx,v)$$

(2.4.2)

$$\tilde{g}_1(x,v) = (rx,-v)$$

where r denotes reflection in $S^2$, $x \in S^2$, $v \in \mathbb{R}$. Since $G = \langle g \rangle = \mathbb{Z}_2$, there are two irreducible representations (1) and (-1). We will compute $\tau_{\rho_1}$ and $\tau_{\rho_{-1}}$ for each of the two extensions of g given above. We handle the $\tilde{g}_1$ case first.

Let $\{v_1,v_2\} = \{\sigma_1 \otimes 1 + \sigma_2 \otimes 1,\ \sigma_1 \otimes 1 - \sigma_2 \otimes 1\}$, $\{u_1\} = \{\alpha \otimes 1\}$, and $\{w_1\} = \{\beta \otimes 1\}$ be bases of horizontal lifts. Then $\tilde{g}_1(v_1) = -\sigma_2 \otimes 1 - \sigma_1 \otimes 1 = -v_1$, $\tilde{g}_1(v_2) = -\sigma_2 \otimes 1 + \sigma_1 \otimes 1 = v_2$,

$\tilde{g}_1(u_1) = u_1$ and $g_1(w_1) = w_1$. Then $\tilde{g}_1$ has matrix representation $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ on $C^2(S^2,E)$ and (1) on $C^1(S^2,E)$ and $C^2(S^2,E)$.

Computing the boundary maps with respect to the above basis gives

$$\partial_2(v_1) = \partial_i(\sigma_1\ 1+\sigma_2\otimes 1)$$

$$= \alpha \otimes 1 + (-\alpha\otimes 1)$$

$$= 0$$

$$\partial_i(v_2) = (\alpha\otimes 1) - (-\alpha\otimes 1)$$

$$= 2w_1$$

$$\partial_1(u_1) = \partial(w_1) = 0.$$

This gives rise to the two cochain complexes

(2.4.3) $$\{w_1^*\} \xrightarrow{d_0^1} \{u_1^*\} \xrightarrow{d_1^1} \{v_2^*\}$$

with $d_0^1 = 0$ and $d_1^1 = 2$ for the representation (1) and

(2.4.4) $$\{0\} \xrightarrow{d_0^{-1}} \{0\} \xrightarrow{d_1^{-1}} \{v_1^*\}$$

with $d_0^{-1} = 0$ and $d_1^{-1} = 0$ for the representation (-1).

To compute the torsion, $\tau_{\rho_1}(S^2,E)$, of (2.4.3) take volume forms $\pi^*(\mu_0^1) = w_1 = \omega_0^1$, $\rho_0 = 1$, $\rho_1 = \omega_1^1 = u_1, \omega_2^1 = v_2 = \pi^*(\mu_2^1)$ and $\rho_2 = 1$. Then from the relations

$$\rho_0 \wedge d_0^{1*}(\rho_1) \wedge \pi^*(\mu_0^1) = m_0\omega_0^1$$

$$\rho_1 \wedge d_1^*(\rho_2) = m_1\omega_1^1$$

$$\rho \wedge \pi^*(\mu_i^1) = m_2\omega_2^1$$

we get $m_0 = m_2 = 1$ and $m_1 = 2$. Thus $\tau_{\rho_1}(S^2,E) = \frac{1}{2}$. Clearly $\tau_{\rho_{-1}}(S^3,E) = 1$ and thus $\tau_g(S^2,E) = \tau_{\rho_1}(S^2,E) \cdot (\tau_{\rho-1}(S^2,E))^{-1} = \frac{1}{2}$. This shows that $\tau_g$ is a finer invariant than the classical Reidmeister torsion, $\tau$, since $\tau = 1$ for simply connected spaces.

For the case of $\tilde{g}_2$ we have the complexes

$$(2.4.5) \qquad \{0\} \xrightarrow{d_0^1} \{0\} \xrightarrow{d_p^1} \{v_1^*\}$$

with $d_0^1 = d_p^1 = 0$ and

$$(2.4.6) \qquad \{w_1^*\} \xrightarrow{d_0^{-1}} \{u_1^*\} \xrightarrow{d_1^{-1}} \{v_2^*\}$$

with $d_0^{-1} = 0$ and $d_1^{-1} = 2$. Then by a similar set of computations as above we have $\tau_{\rho_1}(S^2,E) = 1$ and $\tau_{\rho_{-1}}(S^2,E) = \frac{1}{2}$. It follows then that $\tau_g(S^2,E) = 2$. Thus the torsion $\tau_g$ depends on the extension of g to $E$.

<u>Example 2.4.7</u>. Our second example will be for $X = S^1$. Although there is a more efficient way of computing the torsion in this case (see Section 3.2) the technique used here will be typical of other torsion computations (see §5).

Let $E_\alpha$ denote the flat SO(2) bundle over $S^1$ having holonomy given by

$$\begin{bmatrix} \cos 2\pi\alpha & -\sin 2\pi\alpha \\ \sin 2\pi\alpha & \cos 2\pi\alpha \end{bmatrix} .$$

Suppose that $S^1$ has length 1 and has cell decomposition consisting of q one-cells, $\sigma_j$ $0 \le j \le q - 1$, and q zero-cells $\alpha_j$ $0 \le j \le q - 1$. Let g be rotation by $\frac{1}{q}$ and assume each one-cell has length $\frac{1}{q}$.

Let $E_1 = (1,0)$ and $E_2 = (0,1)$ be a basis for the fiber at each point and $e_1 = (\cos 2\pi\alpha\theta, \sin 2\pi\alpha\theta)$, $e_2 = (-\sin 2\pi\alpha\theta, \cos 2\pi\alpha\theta)$ be parallel sections in the bundle. We can pick the following bases for the horizontal lifts of the zero and one cells, namely, $\{\sigma_0 \otimes e_1, \sigma_0 \otimes e_2, \cdots, \sigma_{q-1} \otimes e_1, \sigma_{q-1} \otimes e_2\}$ for the one-cells and $\{\alpha_0 \otimes e_1, \alpha_0 \otimes e_2, \cdots, \alpha_{q-1} \otimes e_1, \alpha_{q-1} \otimes e_2\}$ for the zero-cells.

The boundary map is

(2.4.8)
$$\partial_1(\sigma_j \otimes e_i) = \begin{cases} \alpha_{j+1} \otimes e_i - \alpha_j \otimes e_i & 0 \le j \le q-2 \quad i = 1,2 \\ \alpha_0 \otimes R_\alpha e_i - \alpha_{q-1} \otimes e_i & j = q-1 \quad i = 1,2 \end{cases} .$$

We extend g to $E_\alpha$ as follows

(2.4.9)
$$\tilde{g}(\sigma_j \otimes e_i) = \begin{cases} \sigma_{j+1} \otimes R_{-\frac{\alpha}{q}} e_i & 0 \le j \le q-2 \quad i = 1,2 \\ \sigma_0 \otimes R_{\frac{q-1}{q}\alpha} e_i & j = q-1 \quad i = 1,2 \end{cases}$$

(replace $\sigma_j$ by $\alpha_j$ for the zero-cells) where

$$R_\theta = \begin{bmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

It is easy to verify with the above extension that $\tilde{g}$ is of finite order and $\partial \circ g = g \circ \partial$.

With respect to the bases given earlier the matrix representation of g on $C_i(S^1, E_\alpha)$ has the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & R_{-\frac{q-1}{q}\alpha} \\ R_{-\frac{\alpha}{q}} & 0 & & \vdots & \vdots & 0 \\ 0 & R_{-\frac{\alpha}{q}} & & \vdots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots \\ \vdots & 0 & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & R_{-\frac{\alpha}{q}} & 0 & \vdots \\ 0 & 0 & \cdots & 0 & R_{-\frac{\alpha}{q}} & 0 \end{bmatrix}_{2q \times 2q}.$$

We need to work out a basis which gives an orthogonal representation of $\tilde{g}$. Let $\{v_1, v_{-1}, \cdots v_{q-1}, v_{-q-1}\}$ and $\{u_1, u_{-1}, \ldots, u_{q-1}, u_{-q-1}\}$ be bases for $C_1(S^1, E\ )$ and $C_2(S^1, E\ )$ respectively where

$$v_\ell = \sum_{j=0}^{q-1} \sigma_j \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1$$

(2.4.10)

$$v_{-\ell} = \sum_{j=0}^{q-1} \sigma_j \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_2$$

and $u_\ell$, $u_{-\ell}$ are defined by replacing $\sigma_j$ by $\alpha_j$ in the above formulas.

$$\tilde{g}(v_\ell) = \sum_{j=0}^{q-1} \tilde{g}(\sigma_j \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1)$$

$$= \sum_{j=0}^{q-2} \sigma_{j+1} \otimes R_{-\frac{\alpha}{q}} R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1$$

$$+ \sigma_0 \otimes R_{\frac{q-1}{q}\alpha} R_{-\frac{\alpha(q-1)}{q}} R_{\frac{(q-q+1)\ell}{q}} e_1$$

$$= \sigma_0 \otimes R_{\frac{\ell}{q}} e_1 + \sum_{j=0}^{q-2} \sigma_{j+1} \otimes R_{\frac{\ell}{q}} R_{-\frac{\alpha(j+1)}{q}} R_{\frac{(q-(j+1))\ell}{q}} e$$

$$= \cos \frac{2\pi\ell}{q} v_\ell + \sin \frac{2\pi\ell}{q} v_{-\ell}.$$

A similar computation shows

$$\tilde{g}(v_{-\ell}) = -\sin \frac{2\pi\ell}{q} v_\ell + \cos \frac{2\pi\ell}{q} v_{-\ell}.$$

Thus the matrix representation of g with respect to these bases has the form

$$\begin{bmatrix} I & & & & \\ & R_{\frac{1}{q}} & & & \\ & & R_{\frac{1}{q}} & & \\ & & & \ddots & \\ & & & & R_{\frac{q-1}{q}} \end{bmatrix}.$$

We now compute the boundary map $\partial_1$ with respect to these bases.

$$\partial_1(v_\ell) = \sum_{j=0}^{q-1} \partial_1(\sigma_j \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1)$$

$$= \sum_{j=0}^{q-2} \alpha_{j+1} \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1 - \sum_{j=0}^{q-2} \sigma_j \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1$$

$$+ \alpha_0 \otimes R_\alpha R_{-\frac{\alpha(q-1)}{q}} R_{\frac{\ell}{q}} e_1 - \alpha_{q-1} \otimes R_{-\frac{\alpha(q-1)}{q}} R_{\frac{\ell}{q}} e_1$$

$$= \sum_{j=0}^{q-2} \alpha_{j+1} \otimes R_{-\frac{\alpha j}{q}} R_{\frac{(q-j)\ell}{q}} e_1 + \alpha_0 \otimes R_{\frac{\ell+\alpha}{q}} e_1 - u_\ell$$

$$= \sum_{j=0}^{q-2} \alpha_{j+1} \otimes R_{\frac{\ell}{q}} R_{\frac{\alpha}{q}} R_{-\frac{\alpha(j+1)}{q}} R_{\frac{(q-j-1)\ell}{q}} e_1 + \alpha_0 \otimes R_{\frac{(\ell+\alpha)}{q}} e_1$$

$$- u_\ell$$

$$= \sum_{j=0}^{q-2} \alpha_{j+1} \otimes R_{-\frac{\alpha(j+1)}{q}} R_{\frac{(q-j-1)\ell}{q}} \cos\frac{2\pi(\ell+\alpha)}{q} e_1 + \sin\frac{2\pi(\ell+\alpha)}{q} e_2)$$

$$+ \cos\frac{2\pi(\ell+\alpha)}{q} \alpha_0 \otimes e_1 + \sin\frac{2\pi(\ell+\alpha)}{q} \alpha_0 \otimes e_2 - u_\ell$$

$$= (\cos\frac{2\pi(\ell+\alpha)}{q} - 1)u_\ell + \sin\frac{2\pi(\ell+\alpha)}{q} u_{-\ell}.$$

In an analogous way one has

$$\partial_1(v_{-\ell}) = -\sin\frac{2\pi(\ell+\alpha)}{q} u_\ell + \cos\frac{2\pi(\ell+\alpha)}{q} u_{-\ell}.$$

Therefore we have the cochain complex $C_{\rho_\ell}(S^1, E_\alpha)$

$$\{u_\ell^*,\ u_{-\ell}^*\} \xrightarrow{d_0^\ell} \{v_\ell^*, v_{-\ell}^*\}$$

where

$$d_0^\ell = (R_{\frac{\ell+\alpha}{q}} - I)^t \qquad \text{(t denotes transpose)}$$

for each irreducible representation $\rho_\ell = R_{\frac{\ell}{q}}$.

Taking preferred volume forms $\omega_0^\ell = u_\ell \wedge u_{-\ell}$, $\omega_1^\ell = v_\ell \wedge v_{-\ell}$ and $\rho_0 = 1$ and $\rho_1 = v_\ell \wedge v_{-\ell}$ and using the fact that $C_{\rho_\ell}(S^1, E_\alpha)$ is acyclic we have

$$\rho_0 \wedge d_0^{\ell *}(\rho_1) = m_0 \omega_0^\ell$$

$$\rho_1 = m_1 \omega_1^\ell .$$

Thus $m_1 = 1$ and $m_0 = \det(R_{\frac{\ell+\alpha}{q}} - I)^t = \det(R_{\frac{\ell+\alpha}{q}} - I)$, and hence $\tau_{\rho_\ell}(S^1, E_\alpha) = 4 \sin^2 \frac{\pi(\ell+\alpha)}{q}$ . From this we also have

$$\tau_g(S^1, E_\alpha) = \prod_{\ell=0}^{q-1} \left(4 \sin^2 \frac{\pi(\ell+\alpha)}{q}\right)^{2 \cos \frac{2\pi\ell}{q}} .$$

3. Combinatorial Invariance for a Free G Action.

Suppose X is a finite simplical complex with flat orthogonal bundle $E$ over X. Let g be a simplical automorphism of finite order such that g extends to a bundle automorphism $\tilde{g}$ and let G = <g> the group generated by G. It is the purpose of this section to show that if G acts freely on X then $\tau_\rho(X,E)$ and $\tau_g(X,E)$ are combinatorial invariants. Recall it is enough to show $\tau_\rho(X,E)$ is a combinatorial invariant where G is a finite cyclic group by Propositions 2.1.10 and 2.1.16. The basic idea is to construct a suitable bundle $E_\rho$ over the quotient $^X/G$ such that $\tau_\rho(X,E) = \tau(^X/G,E_\rho)$. By the arguement in the first section $\tau(^X/G,E_\rho)$ is a combinatorial invariant and thus $\tau_\rho(X,E)$ will be as well.

3.1

Let X be a simplical complex with flat orthogonal bundle $E$ over X. Let g be a finite simplical automorphism of X such that g extends to a bundle automorphism which we denote as $\tilde{g}$. Let G = <g> the group generated by g and suppose G acts freely on X. Denote the quotient $^X/G$ by X' and let $\pi$ be the projection map from X to X'. We construct a bundle $\overleftarrow{\pi}(E)$ over X' as follows. The fiber $\overleftarrow{\pi}(E)_y$ over y will be the vector space $\bigoplus_{x\in\pi^{-1}(y)} E_x$. Thus $\overleftarrow{\pi}(E)$ is a bundle over X' such that G acts trivially on the base X' and acts via the regular representation tensored with the trivial representations on the fiber $\overleftarrow{\pi}(E)_y$.

Definition 3.1.1. With X and E as above we will say s is a equivariant section of the representation $\rho_\ell = R_\ell$ if there is a section s' such that

$$(3.1.2) \qquad \tilde{g}(s)(x) = \cos \frac{2\pi\ell}{|G|} s(x) + \sin \frac{2\pi\ell}{|G|} s'(x)$$

and

$$(3.1.3) \qquad \tilde{g}(s')(x) = -\sin \frac{2\pi\ell}{|G|} s(x) + \cos \frac{2\pi\ell}{|G|} s'(x).$$

The space of $\rho$ equivariant sections of $E$ do not coincide with the space of all sections of some subbundle of $E$ since G does not act strictly fiberwise on $E$; however, the $\rho$ equivariant sections of $\overleftarrow{\pi}(E)$ are the sections of a subbundle since G acts trivially on X'. Define the subbundle $E_\rho$ to be bundle whose fiber over y is the largest subspace of $\overleftarrow{\pi}(E)_y$ on which the action of G splits as the direct sum of subspaces on which G acts by a given irreducible $\rho(G)$.

Proposition 3.1.4. The $\rho$ equivariant sections of $E$ are in one-to-one correspondence with the sections of $E_\rho$.

Proof. We begin by establishing a correspondence between $\Gamma(E)$ (the sections of $E$) and $\Gamma(\overleftarrow{\pi}(E))$. Define

$$S : \Gamma(E) \to \Gamma(\overleftarrow{\pi}(E)) \text{ as}$$

$$(3.1.5) \qquad S(s)(y) = \bigoplus_{x \in \pi^{-1}(y)} s(x).$$

We can write $\bigoplus_{x\varepsilon\pi^{-1}(y)} s(x)$ as $\bigoplus_{i=1}^{G} s(g^i(x))$ where we pick some $x \varepsilon \pi^{-1}(y)$. $\mathcal{S}$ is immediately seen to be injective since only the zero section is sent to the zero section. To see that  is surjective suppose $s' \varepsilon \Gamma(\overleftarrow{\pi}(E))$, $s'(y) = (s_1(y),\cdots,s_{|G|}(y))$, and take $s(x) = s_1(y)$ and $s(g^k(x)) = s_k(y)$. Then $\mathcal{S}(s) = s'$. Therefore $\mathcal{S}$ gives a one-one correspondence between $\Gamma(E)$ and $\Gamma(\overleftarrow{\pi}(E))$.

Now suppose s is a $\rho$ equivariant section of $E$. The claim is that $\mathcal{S}(s) \varepsilon \overleftarrow{\pi}(E)_\rho$. To see this suppose s' is such that (3.1.2) and (3.1.3) hold. Then

$$(g\mathcal{S}(s))(y) = \mathcal{S}(gs)(y) = \bigoplus_{x\varepsilon\pi^{-1}(y)} (gs)(x)$$

$$= \bigoplus_{x\varepsilon\pi^{-1}(y)} \left(\cos\frac{2\pi\ell}{|G|}\, s(x) + \sin\frac{2\pi\ell}{|G|}\, s'(x)\right)$$

$$= \bigoplus_{x\varepsilon\pi^{-1}(y)} \cos\frac{2\pi\ell}{|G|}\, s(x) + \bigoplus_{x\varepsilon\pi^{-1}(y)} \sin\frac{2\pi\ell}{|G|}\, s'(x)$$

$$= \cos\frac{2\pi\ell}{|G|} \bigoplus_{x\varepsilon\pi^{-1}(y)} s(x) + \sin\frac{2\Pi\ell}{|G|} \bigoplus_{x\varepsilon\pi^{-1}(y)} s'(x)$$

$$= \cos\frac{2\pi\ell}{|G|}\, \mathcal{S}(s)(y) + \sin\frac{2\pi\ell}{|G|}\, \mathcal{S}(s')(y).$$

Similarly one has $g\mathcal{S}(s')(y) = -\sin\frac{2\pi\ell}{|G|}\, \mathcal{S}(s)(y) + \cos\frac{2\pi\ell}{|G|}\, \mathcal{S}(s')(y)$. Thus $\mathcal{S}(s)$ is a $\rho$ equivariant section.

Q.E.D.

Let $C^*(X,E)$ and $C^*(X',E_\rho)$ be the cochain complexes consisting of the collection of horizontal lifts and $C^*_\rho(X,E)$ the largest subcomplex on which the action of G splits as a direct sum of complexes on which G acts by a given irreducible $\rho(G)$. Let $\mu_{i,\rho}$ be a g-invariant volume form for $H^i(C_\rho(X,E))$ then we get a volume form $\mu_i$ on $C^i(X',E_\rho)$ defined by $\pi^*\mu_i(h_1,\ldots h_{b_i^\rho}) = \pi^*\mu_{i,\rho}(S^{-1}(h_1,\ldots,S^{-1}(h_{b_i^\rho}))$. With this choice of volume forms we have the following proposition.

<u>Proposition 3.1.6</u>.

$$\tau_\rho(X,E) = \tau(X',E_\rho),$$

<u>Proof</u>. If h denotes the horizontal lift of a cell in X we have $\partial h = S(\partial h)$ since the boundary map commutes with the projection $\pi$. Together with Proposition 3.1.4 this implies the complexes $C^*_\rho(X,E)$ and $C^*(X',E_\rho)$ are the same and hence their torsions are equal.

Q.E.D.

<u>Proposition 3.1.7</u>. $\tau_\rho(X,E)$ is a combinatorial invariant.

<u>Proof</u>. By Proposition 3.1.6 $\tau_\rho(X,E) = \tau(X',E_\rho)$. If we take an equivariant subdivision of X then it passes to a subdivision of X'. By Theorem 1.3.8 $\tau(X',E_\rho)$ is a combinatorial invariant hence $\tau_\rho(X,E)$ is invariant under equivariant subdivision.

Q.E.D.

3.2

In Example 2.4.7 we mentioned there was a more expeditious way to compute the torsion $\tau_{\rho_\ell}(S^1, E_\alpha)$. Recall $E_\alpha$ was the flat $\mathbb{R}^2$ bundle with holonomy $R_\alpha$ and $S^1$ had cell decomposition consisting of q one-cells and q zero-cells. Also we had a g action where g acted by rotation of $\frac{2\pi}{q}$. We now use the above procedure to construct the bundle $E_{\rho_\ell}$ associated the representation $R_{\frac{\ell}{q}}$ and compute $\tau_{\rho_\ell}(S^1, E_\alpha)$.

A section s of $E_\alpha$ is a vector valued function on $\mathbb{R}$ that satisfies the condition $s(x+1) = R_\alpha s(x)$. The extension $\tilde{g}$ of g sends $s(x)$ to $R_{-\frac{\alpha}{q}}\, s(x + \frac{1}{q})$. Suppose $s_\ell(x)$ is a section that transforms via $\rho_\ell$, i.e. $\tilde{g}(s_\ell(x)) = R_{\frac{\ell}{q}}\, s_\ell(x)$. Now $S^1/G = S^1$, and as a complex consists of 1 one-cell $\sigma$ (of length $\frac{1}{q}$) and 1 zero-cell $\alpha$. Let $\overleftarrow{\pi}(E_\alpha)$ be the bundle over $S^1/G$ constructed as before. $\mathcal{S}(s_\ell)$ is a section of $\overleftarrow{\pi}(E_\alpha)$ which transforms via the representation $\rho_\ell$. Now notice

$$\begin{aligned}
\mathcal{S}(s_\ell)(y + \tfrac{1}{q}) &= \bigoplus_{j=0}^{q-1} s_\ell(x + \tfrac{j+1}{q}) \\
&= \bigoplus_{j=0}^{q-1} s_\ell(x + \tfrac{j}{q} + \tfrac{1}{q}) \\
&= \bigoplus_{j=0}^{q-1} R_{\frac{\alpha}{q}}\, \tilde{g}(s_\ell(x + \tfrac{j}{q})) \\
&= \bigoplus_{j=0}^{q-1} R_{\frac{\alpha}{q}}\, R_{\frac{\ell}{q}}\, s_\ell(x + \tfrac{j}{q}) \\
&= R_{\frac{\alpha+\ell}{q}}\, \mathcal{S}(s_\ell)(y).
\end{aligned}$$

Thus $\overleftarrow{\pi}(E)_{\rho_\ell}$ is a flat $\mathbb{R}^2$ bundle with holonomy given by $R_{\frac{\alpha+\ell}{q}}$.

Let $\{v_1,v_2\} = \{\sigma \otimes e_1, \sigma \otimes e_2\}$ and $\{u_1,u_2\} = \{\alpha \otimes e_1, \alpha \otimes e_2\}$ denote bases for $C_1(S^1,\overleftarrow{\pi}(E)_{\rho_\ell})$ and $C_0(S^1,\overleftarrow{\pi}(E)_{\rho_\ell})$ respectively. The boundary map is

$$\begin{aligned}
\partial_1(v_1) &= \partial_1(\sigma \otimes e_1)\\
&= \alpha \otimes R_{\frac{\alpha+\ell}{q}}\, e_1 - \alpha \otimes e_1\\
&= \left(\cos \frac{2\pi(\alpha+\ell)}{q} - 1\right)u_1 + \sin \frac{2\pi(\alpha+\ell)}{q}u_2\\
\partial_1(v_2) &= -\sin \frac{2\pi(\alpha+\ell)}{q}u_1 + \cos \frac{2\pi(\alpha+\ell)}{q}u_2.
\end{aligned}$$

Thus we have the cochain complex

$$\{u_1^*,u_2^*\} \xrightarrow{d_0} \{v_1^*,v_2^*\}$$

where $d_0 = (R_{\frac{\ell+\alpha}{q}} - I)^t$. Then as in Example 2.4.7 we have $\tau(S^1,\overleftarrow{\pi}(E)_{\rho_\ell}) = 4\sin^2\left(\frac{\pi(\ell+\alpha)}{q}\right)$.

## 3.3

Again let X be a simplical complex with flat orthogonal bundle $E$ over X. Let g be a finite simplical automorphism of X and G the group generated by g. Let Y be an invariant subcomplex of X such that G acts freely on X-Y. Denote the quotient $^{X-Y}/G$ by X'. X' is to be viewed as a complex whose boundary operator is induced from the relative boundary operator of the pair (X,Y). Let $\overleftarrow{\pi}(E)$ denote the bundle over X' constructed as before, then we have the following proposition.

<u>Proposition 3.3.1</u>. Let $\tau_\rho(X,Y,E)$ denote the torsion of the complex $C_\rho(X,Y,E)$ where $C_\rho(X,Y,E)$ is the largest subcomplex on which the action of G splits as the direct sum of complexes on which G acts by a given irreducible $\rho(G)$. Consider the complex of horizontal lifts $C(X',E_\rho)$ where the boundary is understood to be the one induced from the relative boundary. Then

$$\tau_\rho(X,Y,E) = \tau(X',E_\rho). \tag{3.3.2}$$

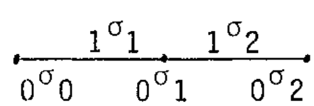
<u>Proof</u>. The proof is essentially the same as Proposition 3.1.6 except instead of using the boundary map we use the relative boundary map.

Q.E.D.

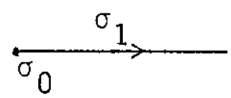
As before we have a consequence.

<u>Proposition 3.3.3</u>. $\tau_\rho(X,Y,E)$ is a combinatorial invariant.

We conclude this section with a proposition that is necessary in the proof of combinatorial invariance when G does not act freely. Let X, E be as before and suppose $X_1$ is an invariant subcomplex acted trivially on by some subgroup $H \subset G$. Let $X_2$ be an invariant subcomplex of X such that $X_2 \subset X_1$ and $^G/H$ acts freely on $X_1 - X_2$. We emphasize that $X' = \frac{X_1 \setminus X_2}{^G/_H}$ is an open complex. For example if X is the 1-simplex



with $\mathbb{Z}_2$ action given by reflection through ${}_0\sigma_1$ $X_1 = X$, $X_2 = F(\mathbb{Z}_2) = {}_0\sigma_1$ and $H = e$ we have $X' = \frac{X_1 \setminus X_2}{G/H}$ is the complex




with boundary $\partial\sigma_1 = \sigma_0$. Then we can construct a bundle $\overleftarrow{\pi}(E)$ over ${}^{X_1 \setminus X}/_{G/H} = {}^{X_1 \setminus X_2}/_G$ as before. Using the proof of Proposition 3.3.1 we have

<u>Proposition 3.3.4</u>. $\tau_\rho(X_1, X_2, E) = \tau(X', E_\rho)$.

## 4. Combinatorial Invariance.

In this section we prove that for X a simplical complex and G a finite cyclic group acting simplically that $\tau_\rho(X,E,\mu)$ and hence $\tau_g(X,E,\mu)$ and $\tau_\rho(X,E,\mu)$ for an arbitrary finite group G are combinatorial invariants. In fact we give two proofs. Both proofs work by breaking X up in such a way as to have a free action and then applying the results of the previous section.

### 4.1

We begin with some notation. Let $G_x = \{g \in G | gx=x\}$ and define for H a subgroup of G the space

(4.1.1) $\quad X(H) = \{x \in X | G_x = H\}$

which we call the isotropy space of H and let

(4.1.2) $\quad F(H) = \{x \in X | gx=x \quad \forall g \in H\}$

denote the fixed point set of H.

The next proposition gives various relationships between the isotropy spaces and the fixed point sets.

Proposition 4.1.3. Let G be a group acting on a space X, then

(i) $F(H) = \bigcup_{H' \supseteq H} X(H')$

(ii) $X(H) = F(H) - F^H$ where $F^H = \bigcup_{H' \supsetneq H} F(H')$

(iii) $F(H) \cap F(H') = F(H+H')$.

Proof.

(i) Suppose $x \in F(H)$. Then $x$ must have $G_x \supseteq H$ hence $x \in \bigcup_{H' \geq H} X(H)$. Conversely given $x \in \bigcup_{H' \supseteq H} X(H')$ we must have $G_x = \overline{H}$ for some $\overline{H} \supseteq H$, hence $x \in F(\overline{H}) \subseteq F(H)$.

(ii) Suppose $x \in X(H)$, then $G_x = H$ and $x \in F(H)$. Now $x \notin F^H$ otherwise $G_x = H'$ for some $H' \supset H$ contradicting the fact that $G_x = H$. Conversely given $x \in F(H) - F^H$ implies $G_x = H$ thus $x \in X(H)$.

(iii) Suppose $x \in F(H+H')$. Since $H + H' \supseteq H, H'$ we have $F(H), F(H') \supseteq F(H+H')$ and consequently $x \in F(H) \cap F(H')$. Now if $x \in F(H) \cap F(H')$ then $x$ is fixed by all products of elements in $H$ and $H'$ and thus $x \in F(H+H')$.

Q.E.D.

Definition 4.1.4. We say that $H$ is an isotropy subgroup of $G$ if $X(H) \neq \emptyset$ and we denote the set of all isotropy subgroups as $I$.

Remark. It should be remarked that the $X(H)$ are mutually disjoint, i.e. $X(H) \cap X(H') = \emptyset$ for $H' \neq H$. We also have that $X = \bigcup_{H \in I} X(H)$ with each $X(H)$ having a free $G/H$ action. It is this observation that is essential in what follows.

Definition. A subgroup $H \subset G$ will be said to be of maximal height if $F(H)$ can not be written as the union of fixed point sets of subgroups properly containing H. Denote the set of all such subgroups as M.

Proposition 4.1.5. $M = I$.

Proof. Let $\bar{J}$ denote the set of all subgroups $\bar{H} \supset H$. Since $F^H = \bigcup_{\bar{H}\varepsilon J} F(\bar{H}) = \bigcup_{\bar{H}\subset J} ( \bigcup_{H' \supsetneq \bar{H}} X(H'))$, $F(H) = \bigcup_{H' \supsetneq H} X(H')$ and $X(H) = F(H) - F^H$. The result follows.

Q.E.D.

## 4.2

Our goal now is to give a filtration of the complex $C(X,E)$ coming from the lattice of isotropy subgroups. We then develop a formula for $\tau_\rho(X,E)$ using Proposition 2.2.4 and this filtration. Upon examination of the relative complexes obtained from the filtration we see we can split these up in correspondence with the isotropy spaces $X(H)$. Then using the results of Section 3 and the fact that $X(H)$ has a free $^G/H$ action obtain the combinatorial invariance of $\tau_\rho$.

Define $I_1$ to be the set of all minimal isotropy groups, i.e. those groups in I which have no proper subgroups in I and let $I_j$ be all those groups in I that contain a group in $I_{j-1}$ as a maximal subgroup. For $H \varepsilon I_j$ we say that H is at level j. Let $X_j = \bigcup_{H\varepsilon I_j} F(H)$. It is immediate from Proposition 4.1.3 part (ii) and Proposition 4.1.5 that the $X_j$ filter X and thus we may filter the complex $C(X,E)$ by the $C(X_j,E)$. Now observe that G acts on $X_j$, for if $x \in X_j$ then $hx = x$ for all $h \in H \subset I_j$ and since G is abelian we have $ghx = gx = hgx$ implies $gx \in X_j$. As before let $C_\rho(X,E)$ and $C_\rho(X_j,E)$ denote the direct sum of the subcomplexes on which G acts irreducibly by a given $\rho(G)$, and by the above remark the $C_\rho(X_j,E)$ filter $C_\rho(X,E)$.

At this point it will be convenient to make one more observation before proceeding to our main lemma.

Proposition 4.2.1. $$C(X_{j-1},X_j,E) \cong \bigoplus_{H \in I_{j-1}} C(F(H),F^H,E).$$

Remark. Let J denote the subgroups in $I_j$ that contain H as a subgroup. Then $F^H = \bigcup_{H' \supsetneq H} F(H) = \bigcup_{H' \in J} F(H')$, hence $F^H \subset X_j$.

Proof. This immediate from Proposition 4.1.3 (ii) and the fact that the X(H) are mutually disjoint.

Q.E.D.

Remark. G acts on $C(F(H),F^H,E)$ by the same proof as given earlier and hence $C_\rho(X_{j-1},X_j,E) \cong \bigoplus_{H \in I_{j-1}} C_\rho(F(H),F^H,E)$.

Our next lemma which is the key ingredient in proving combinatorial invariance gives the formula for $\tau_\rho$ mentioned previously.

Lemma 4.2.2. Let X be a simplical complex with flat orthogonal bundle E over X. Let g be a simplical automorphism of X which extends to the bundle E and G = <g> the group generated by G.

Suppose further that G is of finite order. Setting $F^G = \emptyset$ and $\tau_\rho(H_G^r) = 1$ we have

$$(4.2.3) \qquad \tau_\rho(X,E) = \prod_{H\varepsilon I} \tau_\rho(F(H),F^H,E)\tau_\rho(H_H^r)$$

where $H_H^r$ denotes the long exact sequence in cohomology obtained from the short exact sequence $0 \to C(F^H,E) \to C(F(H),E) \to C(F(H),F^H,E) \to 0$.

Proof. Using Proposition 2.2.4 we have

$$(4.2.4) \qquad \tau_\rho(X,E) = \prod_{j=0}^{n} \tau(C_\rho(X_{j-1},X_j,E))\tau_\rho(H_j)$$

where n denotes the number of levels and $H_i$ denote the long exact sequence in cohomology obtained from the short exact sequence

$$0 \to C\ (X_{j-1},E) \to C_\rho(X_j,E) \to C\ (X_j,X_{j-1},E) \to 0.$$

Now using the decomposition $C_\rho(X_{j-1},X_j,E) = \bigoplus_{H\varepsilon I_j} C_\rho(F(H),F^H,E)$ and the fact that $\tau_\rho(\bigoplus_j C_j) = \prod_j \tau(C_j)$ we have that (4.2.4) can be expressed as

$$\tau_\rho(X,E) = \prod_{H\varepsilon I} \tau_\rho(F(H),F^H,E)\tau_\rho(H_H^r).$$

Q.E.D.

Theorem 4.2.5. $\tau_\rho(X,E)$ is a combinatorial invariant.

Proof. As ${}^G/H$ acts freely on $X(H) = F(H) - F^H$ we can use the construction of Section 3 to construct a bundle $E_\rho$ over

$X\{H\} = X(H)/G/H$ such that $\tau_\rho(F(H), F^H, E) = \tau(X\{H\}, E_\rho)$. Thus 4.2.3 can be rewritten as

$$\tau_\rho(X, \ ) = \prod_{H\varepsilon I} \tau(X\{H\}, E_\rho)\tau_\rho(H_H^r)$$

where each of the terms in the above is a combinatorial invariant by Proposition 3.3.3.

Q.E.D.

Before giving our second proof of combinatorial invariance we give an additional application of Lemma 4.2.2. Let $X_1$ and $X_2$ be two simplical complexes with flat orthogonal bundle $E_1$ and $E_2$ respectively. Suppose $g_1$, $g_2$ are simplical automorphisms of finite order of $X_1$ and $X_2$ respectively such that $g_1$ and $g_2$ extend to the bundles $E_1$ and $E_2$. Suppose further that the groups generated by $g_1$, $g_2$ are isomorphic. Then we have $X \times Y(H) = X(H) \times Y(H)$ and $F_{X\times Y}(H) = F_X(H) \times F_Y(H)$. Denote by $\pi_i : X_1 \times X_2 \to X_i$.

<u>Theorem 4.2.6</u>. With the conditions as above

$$\ln \tau_\rho(X_1 \times X_2, \pi^*(E_1) \otimes \pi_2^*(E_2)) = \sum_{H\varepsilon I} \chi((E_2)_\rho) \ln \tau(X_1\{H\}, (E_1)_\rho)$$
$$+ \chi((E_1)_\rho) \ln \tau(X_2\{H\}, (E_2)_\rho)$$

where $\chi((E_i)_\rho)$ denotes the Euler characteristic of the bundle $(E_i)_\rho$ and $X_i\{G\}$ is understood to mean $X_i(G) = F(G)$.

<u>Proof</u>. Using Lemma 4.2.2 we have

$$\tau_\rho(X\times Y,\pi_1^*(E_1)\otimes\pi_2^*(E_2)) = [\prod_{H\subseteq G}\tau_\rho(F_{X_1}(H)\times F_{X_2}(H),F_{X_1}^H\times F_{X_2}^H,\pi_1^*(E_1)\otimes\pi_2^*(E_2))\cdot\tau_\rho(H_H^r)].$$

Then as in the proof of Theorem 4.2.5 we can write this as

$$=[\prod_{\substack{H\subset G\\ H\neq G}}\tau(X_1\{H\}\times X_2\{H\},(\pi_1^*(E_1)\otimes\pi_2^*(E_2))_\rho)\tau_\rho(H_H^r(X_1)\otimes H_H^r(X_2))]$$

Then making use of the fact $\chi(H_H^r(X_i)) = 0$ and applying Proposition 1.1.4 yields the result.

Q.E.D.

## 4.3

Our second proof of combinatorial invariance uses a Mayer-Vietoris argument and is somewhat more complex than the previous proof. We give this proof to point out an interesting cancellation phenomenon that occurs but is not a priori apparent. We start with a straightforward generalization of Lemma 1.3.10.

Lemma 4.3.1. Let X be a simplical complex with orthogonal bundle $E$ over X. Let g be a simplical automorphism of finite order of X which extends to the bundle $E$. Suppose further the G = <g> the group generated by g satisfies condition S. Let $X_1, X_2$ be invariant subcomplexes of X, then

(4.3.2) $$\tau_\rho(X_1,E)\tau_\rho(X_2,E) = \tau_\rho(X_1\ X_2,E)\tau_\rho(X_1\cup X_2,E)\tau_\rho(H_M)$$

where $H_M$ denotes the long exact sequence in cohomology obtained from the short exact sequence.

$$0\to C_\rho(X_1,X_2,E)\to C_\rho(X_1,E)\oplus C_\rho(X_2,E)\to C_\rho(X_1\cup X_2,E)\to 0$$

Proof. This follows as an immediate Corollary of Proposition 2.2.1.

Q.E.D.

This next lemma is analogous to Lemma 1.3.12 and is the starting point to developing a formula like 4.2.4.

Lemma 4.3.3. Let X be a simplical complex with flat orthogonal bundle $E$ over X. Let g be a simplical automorphism of X of finite order which extends to the bundle $E$. Let G = <g> the group generated by G. Then if $H_i$ $1 \leq i \leq \ell$ are distinct subgroups of G we have

$$(4.3.4) \qquad \tau_\rho(\bigcup_i F(H_i),E) = \frac{\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m\\ 1\leq i_1\leq \ell}} \tau_\rho(F(H_{i_1}+\cdots+H_{i_m}),E)}{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m\\ 1\leq i_1\leq \ell}} \tau_\rho(F(H_{i_1}+\cdots+H_{i_m}),E)} \tau_\rho(\overline{H})$$

where

$$(4.3.5) \qquad \tau_\rho(\overline{H}) = \frac{\displaystyle\prod_{\substack{m=1\\ m\ \text{even}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m\\ 1\leq i_1\leq \ell-m}} \tau_\rho(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}\ i_{m+1},\ldots,i_1\cdots i_{m-1}\ell})}{\displaystyle\prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m\\ 1\leq i_1\leq \ell-m}} \tau_\rho(H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}\ i_{m+1},\ldots,i_1\cdots i_{m-1}\ell})}$$

and $H_{i_1\cdots i_m}^{i_1\cdots i_{m-1}\ i_m,\ldots,i_1\cdots i_{m-1}\ell}$ denotes the long exact sequence in cohomology obtained from the short exact sequence

$$0 \to C(F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_m}) \cap [F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_{m+1}}) \cup \cdots \cup F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_\ell})],E)$$

$$\to C(F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_m}),E) \oplus C(F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_{m+1}}) \cup \cdots \cup F(H_{i_1}+\cdots+H_{i_{m-1}}+H_\ell),E)$$

$$C(F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_m}) \cup \cdots \cup F(H_{i_1}+\cdots+H_{i_{m-1}}+H_{i_\ell}),E) \to 0.$$

Proof. Making the minor notational changes of $X_i$ replaced by $F(H_i)$ and using the fact that $F(H_1+H_2) = F(H_1) \cap F(H_2)$ the proof follows essentially verbatim from the proof of Lemma 1.3.12.

Q.E.D.

After giving a corollary to the previous lemma we will state our two main lemma's. The combinatorial invariance of $\tau_\rho$ and $\tau_g$ will then follow as before. We begin with some notation for $F^H$.

Let G be a cyclic group, then G can be written as $\mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$ where $p_1,\cdots,p_k$ are distinct primes. If H is a subgroup of G then $H = \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}$ where $0 \le r_j \le n_j$. Recall that $F^H = \bigcup_{H' \neq H} F(H') = \bigcup_{H' \in J} F(H')$ where J denotes those $H'$ such that H is a maximal proper subgroup. Thus if $H = \mathbb{Z}_{p_{i_1}^{t_{i_1}}} \oplus \cdots \oplus \mathbb{Z}_{p_{i_\ell}^{t_{i_\ell}}}$ and $\overline{p}$ denotes those $p_j$ for which $n_j \neq r_j$ then J consists of all groups of the form $p_{i_1}^{t_{i_1}} \cdots p_{i_\ell}^{t_{i_\ell}}$ $(\mathbb{Z}_{\overline{p}_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{\overline{p}_{j_s}})$ where the $\overline{p}_{j_n} \in \overline{p}$ are distinct and

$p_{i_1}^{t_{i_1}} \cdots p_{i_\ell}^{t_{i_\ell}} (\mathbb{Z}_{\overline{p}_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{\overline{p}_{j_s}})$ means the group with factors

$\mathbb{Z}_{p_{i_n}^{t_{i_n}+1}}$ if $p_{i_n} = \overline{p}_{j_q}$ for some $\overline{p}_{j_q}$ and factors $\mathbb{Z}_{p_{i_n}^{t_{i_n}}}$, $\mathbb{Z}_{\overline{p}_{i_q}}$

otherwise. We denote by $F_{p_{i_1}^{t_{i_1}} \cdots p_{i_\ell}^{t_{i_\ell}} (\mathbb{Z}_{\overline{p}_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{\overline{p}_{j_s}})}$ the fixed

point set of $p_{i_1}^{t_{i_1}} \cdots p_{i_\ell}^{t_{i_\ell}} (\mathbb{Z}_{\overline{p}_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{\overline{p}_{j_s}})$.

<u>Lemma 4.3.6</u>. Let X, g, and G be as in Lemma 4.3.3, then

$$(4.3.7) \qquad \tau_\rho(F^H, E) = \frac{\displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\ell} \prod_{\substack{j_1 < \cdots < j_m \\ 1 \le j_1 \le \ell}} \tau_\rho(F_{p_{i_1}^{t_{i_n}} \cdots p_{i_s}^{t_{i_s}} (\mathbb{Z}\overline{p}_{j_1} \oplus \cdots \oplus \mathbb{Z}\overline{p}_{j_m})}, E)}{\displaystyle\prod_{\substack{m=1 \\ m \text{ even}}}^{\ell} \prod_{\substack{j_1 < \cdots < j_m \\ 1 \le j_1 \le \ell}} \tau_\rho(F_{p_{i_1}^{t_{i_1}} \cdots p_{i_s}^{t_{i_s}} (\mathbb{Z}_{\overline{p}_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{\overline{p}_{j_m}})}, E)} \cdot \tau_\rho(H_H^M)$$

where $H = \mathbb{Z}_{p_{i_1}^{t_{i_1}}} \oplus \cdots \oplus \mathbb{Z}_{p_{i_s}^{t_{i_s}}}$, $\overline{p} = \{\overline{p}_1, \ldots, \overline{p}_\ell\}$ and $\tau_\rho(H_H^M)$ denotes

the torsion term of the previous lemma with the $H_i$ replaced by

$p_{i_1}^{t_{i_1}} \cdots p_{i_s}^{t_{i_s}} (\mathbb{Z}_{\overline{p}_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{\overline{p}_{j_m}})$.

<u>Proof</u>. This is an immediate Corollary to Leamma 4.3.3.

Q.E.D.

After stating our next lemma (whose proof we postpone until the end of this section) we prove a lemma which gives a formula similar to that of (4.2.3). The combinatorial invariance of $\tau_\rho, \tau_g$ will then follow as before.

Let $G = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$ where the $p_1, \cdots, p_k$ are distinct primes. Suppose also that $n_1, \ldots, n_s > t$, $n_{s+1}, \ldots, n_\ell < t$, and $n_{\ell+1}, \ldots, n_k < t$ and recall $r_H = \max_i\{r_i\}$ with $H = \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}$. In the succeeding lemmas we adopt the convention that $\tau_\rho(F(H), F^H, E) = \tau_\rho(H_H^r) = \tau_\rho(H_H^M) = 1$ for $H \notin I$.

<u>Lemma 4.3.8</u>. With notation as above and hypothesis as in Lemma 4.3.3 we have

$$(4.3.9) \quad \frac{\displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\ell} \prod_{\substack{i_1 < \cdots < i_m \\ 1 \le i_1 \le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}), E)}{\displaystyle\prod_{\substack{m=1 \\ m \text{ even}}}^{\ell} \prod_{\substack{i_1 < \cdots < i_m \\ 1 \le i_1 \le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}), E)} = \prod_{\substack{H \\ r_H = t}} \tau_\rho(F(H), F^H, E)\tau_\rho(H_H^r)\tau_\rho(H_H^M)$$

$$\frac{\displaystyle\prod_{\substack{q=1 \\ q \text{ odd}}}^{s} \prod_{\substack{j_1 < \cdots < j_q \\ 1 \le j_1 \le s}} \tau_\rho(F(\mathbb{Z}_{p_{j_1}^{t+1}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_q}^{t+1}}), E)}{\displaystyle\prod_{\substack{q=1 \\ q \text{ odd}}}^{s} \prod_{\substack{j_1 < \cdots < j_q \\ 1 \le j_1 \le s}} \tau_\rho(F(\mathbb{Z}_{p_{j_1}^{t+1}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_q}^{t+1}}), E)} .$$

<u>Lemma 4.3.10</u>. With the hypothesis as in Lemma 4.3.3 we have

$$\tau_\rho(X,E) = \prod_{H\in I} \tau_\rho(F(H),F^H,E)\tau_\rho(H_H^r)\tau_\rho(H_H^M) \tag{4.3.11}$$

<u>Proof</u>. The proof proceeds by induction over $r_H$. The above lemma follows by verifying the following formula and taking $r_H = n$ where $n = \max_i\{n_i\}$.

$$\tau_\rho(X,E) = \Big[ \prod_{\substack{H \\ r_H \le t-1}} \tau_\rho(F(H),F^H,E)\tau_\rho(H_H^r)\tau_\rho(H_H^M)\Big] \tag{4.3.12}$$

$$\frac{\displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m \\ 1\le i_1\le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}),E)}{\displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m \\ 1\le i_1\le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}),E)}$$

We check (4.3.12) for the case $t = 1$. By the relative lemma we have

$$\tau_\rho(X,E) = \tau_\rho(F(e),F^e,E)\tau_\rho(F^e,E)\tau_\rho(H_e^M)$$

where e denotes the identity element in G and $F^e = F(\mathbb{Z}_{p_1}) \cup\cdots\cup (\mathbb{Z}_{p_k})$. Applying Lemma 4.3.6 to $\tau_\rho(F^e,E)$ yields

$$\tau_\rho(X,E) = \tau_\rho(F(e),F^e,E)\tau_\rho(H_e^r)\tau_\rho(H_e^M).$$

$$\frac{\displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m \\ 1\le i_1\le k}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}}),E)}{\displaystyle\prod_{\substack{m=1 \\ m \text{ even}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m \\ 1\le i_1\le k}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}}),E)}$$

Thus verifying (4.3.12) for $t=1$. So if we assume (4.3.12) for the case $r_H = t-1$ then Lemma 4.3.8 completes the induction step.

Q.E.D.

We make two remarks. First one sees that the proof of combinatorial invariance proceeds exactly as before using (4.3.11). Secondly we remark that formulas (4.3.11) and (4.2.4) are identical except for terms of $\tau_\rho(H_H^M)$. This implies that $\prod_{H\varepsilon I} \tau_\rho(H_H^M) = 1$ and thus some very interesting cancellation must go on that is not a priori clear.

## 4.4

The remainder of this section is devoted to the proof of Lemma 4.3.8. We start with two counting problems which are necessory in the proof of Lemma 4.3.8.

As was the case earlier we need to partition the subgroups of G. If $H = \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}$ set $r_H = \max_i\{r_i\}$. Let $J_t = \{H \varepsilon G \mid r_H = t\}$. In the proof of Lemma 4.3.8 it will be necessary to know how many times a subgroup in $J_{t+1}$ or $J_t$ occurs in all the expressions $F^H$ for H in $J_t$. Actually a slightly more refined counting scheme is needed and it is this problem which is addressed in the following sequence of propositions.

The first counting problem may be formulated as follows: Given $A = \{p_{i_1}^t, \cdots, p_{i_g}^t, p_{i_{g+1}}^s, \cdots p_{i_{g+h}}^s, p_{i_{g+h+1}}, \cdots, p_{i_{g+h+q}}\}$

where the $p_{i_j}$ are all distinct, we wish to distribute them between two boxes (the left box and right box) subject to the conditions

(i) we use all the objects in A and we use them only once

(ii) at least one $p^t_{i_j}$ is in the left box

(iii) the right box is not empty

(iv) the $p^t_{i_j}$ and $p^s_{i_j}$ behave according to the rules

a) $p^t_{i_j}$, $p^s_{i_j}$ appears in the left box

or

b) you place a $p^{t-1}_{i_j}$, $p^{s-1}_{i_j}$ in the left box and a $p_{i_j}$ in the right box.

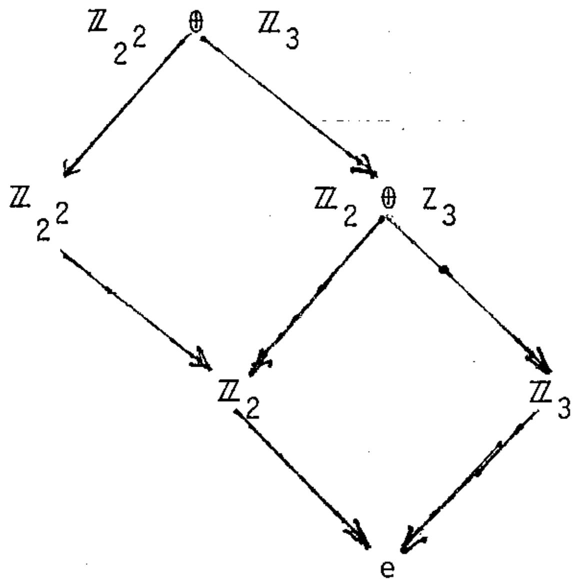
Proposition 4.4.1. Given the situation described above there are $2^{g+h+q} - 2^{h+q} - 1$ states.

Proof. Since each $p_{i_j}$ has two states there is a total of $2^{g+h+q}$ possible states. As we required that at least one $p^t_{i_j}$ be in the left box we must eliminate all those cases in which none appears. There are $2^{h+q}$ states. Finally, since we required that right box be nonempty we eliminate the one case where this occurs. This leaves $2^{g+h+q} - 2^{h+q} - 1$ states.

Q.E.D.

Remark. In relation to our previous discussion this proposition counts the number of times a subgroup occurs in $J_{t+1}$ or $J_t$

in all expressions $F^H$ for $H$ $J_t$. As a simple example, consider $G = \mathbb{Z}_{2^2} \oplus \mathbb{Z}_3$ which has the lattice of subgroups shown below.




Here $J_2 = \{\mathbb{Z}_{2^2} \oplus \mathbb{Z}_3, \mathbb{Z}_{2^2}\}$, $J_1 = \{\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_3\}$, and $J_0 = \{e\}$. If we consider the case $H = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ we have $g = 2$, $h = 0$ and $q = 0$ giving us $2^2 - 2^0 - 1 = 2$ possible states. This agrees with what one determines by direct examination of the lattice.

We mentioned earlier that we actually want a more refined counting scheme. This is necessary since in proving the next lemma we will make repeated use of Lemma 4.16. In the proof we will need to keep track of the number times the torsion of given fixed point set occurs in the numerator and denominator of an expression after use of Lemma 4.16.

Before stating the next proposition we make the following observation. Given n objects to distribute between two boxes there are $2^{n-1}$ ways of placing an (odd, even) number of objects in the (left, right) box. To see this note there are $\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}$ ways of placing even number of objects in the (left, right) box. Since $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$ and $0^n = (1-1)^n = \sum_{k=0}^{n} (-1)^n \binom{n}{k}$ and $2^n = 2^n + 0^n = 2 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}$ implies $2^{n-1} = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}$.

It turns out we want to partition the aforementioned counting problem into the following four cases.

(i) The number of $p_{i_j}^t$ in the left box is odd, the number of objects in the right box is even.

(ii) The number of $p_{i_j}^t$ in the left box is odd, the number of object in the right box is odd.

(iii) The number of $p_{i_j}^t$ in the right box is even, the number of objects in the right box is even.

(iv) The number of $p_{i_j}^t$ in the right box is even, the number of objects in the right box is odd.

<u>Proposition 4.4.2</u>. The number of states in case (i)-(iv) is given in the following table.

| Case | g odd | | g even | |
|---|---|---|---|---|
| | $g + h > 0$ | $q = h = 0$ | $g + h > 0$ | $q = h = 0$ |
| (i) | $(2^{g-1})(2^{q+h-1}) - 1$ | $2^{g-1} - 1$ | $(2^{q-1})(2^{q+h-1})$ | 0 |
| (ii) | $(2^{g-1})(2^{q+h-1})$ | 0 | $(2^{g-1})(2^{q+h-1})$ | $2^{g-1}$ |
| (iii) | $(2^{q-1}-1)(2^{q+h-1})$ | 0 | $(2^{g-1}-1)(2^{q+h-1}) - 1$ | $2^{g-1} - 2$ |
| (iv) | $(2^{g-1}-1)(2^{q+h-1})$ | $2^{g-1} - 1$ | $(2^{g-1}-1)(2^{q+h-1})$ | 0 |

Proof. We give the proofs of (i) and (ii) for g odd and (iii) and (iv) for g even and $g + h > 0$. All the other cases follow by similar reasoning.

Case (i) g odd

If g is odd then there are $2^{g-1}$ ways of putting an odd number of the $p_{i_j}^t$ in the left box leaving an even number of the $p_{i_j}^t$ in the right box. Since we require an even number of objects in the right box we must place an even number of objects from the remaining $q + h$ objects in the right box of which there are $2^{q+h-1}$ ways of doing this. Finally, we exclude the case where all the $p_{i_j}^t$ are in the left box and the right box is empty. Thus there are $(2^{g-1})(2^{q+h-1}) - 1$ states.

Case (ii) g odd

There are $2^{g-1}$ ways of placing an odd number of the $p_{i_j}^t$ in the left box leaving an even number of the $p_{i_j}^t$ in the right box. Thus we must place an odd number of the remaining q+h objects in the right box of which there are $2^{q+h-1}$ ways. Thus the total number of states is $(2^{g-1})(2^{q+h-1})$.

Case (iii) g even

There are $2^{g-1} - 1$ ways of placing an even number of the $p^t_{i_j}$ in the left box where the case of putting zero objects is excluded. This leaves an even number of $p^t_{i_j}$ in the right box and so we must put an even number of the remaining q+h objects in the right box. There are $2^{q+h-1}$ ways of doing this. Finally, we exclude the case where all the objects are in the left box yielding $(2^{g-1}-1)(2^{q+h-1}) - 1$ possible states.

Case (iv) g even

There are $2^{g-1} - 1$ ways of placing an even number the $p^t_{i_j}$ in the left box where we exclude the case of putting zero $p^t_{i_j}$ in the left box. Thus we have an even number $p^t_{i_j}$ in the right box and must add an odd number of the remaining q+t objects to the right box. Since there are $2^{q+h-1}$ ways of doing this there is a total of $(2^{g-1}-1)(2^{q+h-1})$ states.

Q.E.D.

The second problem is a slight variation of the first. Let $B = \{p^{t+1}_{i_1},\ldots,p^{t+1}_{i_f}, p^t_{i_{f+1}},\ldots,p^t_{i_{f+g}},\ldots,p^s_{i_{f+g+1}},\ldots,p^s_{i_{f+g+h}},$ $p_{i_{f+g+h+1}},\ldots,p_{i_{f+g+h+q}}\}$. Our goal once again is to place all the objects in B into two boxes subject to the following rules.

(a) (i) and (iv) of the previous problem hold

and

(b) Given a $p_{i_j}^{t+1}$ you place $p_{i_j}^{t}$ in the left box and a $p_{i_j}$ in the right box.

It is immediately clear that there are $2^{g+h+q}$ possible states. Again we want a slightly more refined counting result so we break the above states into four possible cases I,$\cdots$,IV. These cases are basically the same as before only we change the initial phrase to read "The number of $p_{i_j}^{t+1}$, $p_{i_j}^{t}$ in the (left, right) box is (odd, even),... ."

Proposition 4.4.3. The number of possible states in (I) + (III) and (II) + (IV) is $2^{g+h+q-1}$.

Proof. Since (I) + (IV) or (II) + (III) is exactly half the number of possible states the result follows.

Q.E.D.

Proof of 4.18. In order to make the notation less cumbersome we will denote $F^H$ by F[H] in this proof.

Applying Lemma 2.3.1 to $\tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}), E)$ with the subcomplex F(H) yields

$$(4.4.4) \quad \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}), E) = \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}), F[\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}], E) \cdot \tau_\rho(H^r_{\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t}})$$

Thus we can rewrite the left hand of (4.3.9) as $\Gamma \cdot \Omega$ where

$$\Gamma = \frac{\prod\limits_{\substack{m=1\\ m\ \text{odd}}}^{\ell} \prod\limits_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}), F[\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}], E)\tau_\rho(H^r_{\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}})}{\prod\limits_{\substack{m=1\\ m\ \text{even}}}^{\ell} \prod\limits_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}), E)}$$

$$\Omega = \prod_{\substack{m=1\\ m\ \text{odd}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m\\ 1\le i_1\le \ell}} \tau_\rho(F[\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}], E).$$

Consider the $\Omega$ term. Apply Lemma 4.3.6 to each term in $\Omega$ and call the resulting expression $\Omega_1$. For terms $\tau_\rho(F(H),E)$ appearing in the numerator of $\Omega_1$ with $r_H = t$ apply Lemma 2.3.1 using the subcomplex $F[H]$. Factor away the relative terms and call the resulting expression $\Omega_2$. Now for terms $\tau_\rho(F(H),E)$ appearing in the numerator of $\Omega_2$ with $r_H = t$ and such that Lemma 2.3.1 has not been previously applied to it, apply Lemma 2.3.1. Factor away the relative terms and call the resulting expression $\Omega_3$. Continue this process untill all terms with $r_H = t$ have had Lemma 2.3.1 applied to them. In order to carry out this process we need the following claim.

<u>Claim 1</u>. Every term of the form $\tau_\rho(F(H),E)$ with $r_H = t$ appears in the numerator of some $\Omega_i$.

Suppose for the moment we have proved Claim 1. A similar arguement shows that all terms of the form $\tau_\rho(F(H),E)$ with $r_H = t+1$ occurs in some $\Omega_i$. Let $\Omega'$ denote the final expression after all the relative terms have been factored out and $\Gamma'$ the factored out relative terms.

Claim 2

$$\Omega' = \prod_{\substack{m=1 \\ m \text{ even}}}^{\ell} \prod_{\substack{i_1<\cdots<i_m \\ 1\le i_1\le \ell}} \tau_\rho(F(\mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_m}^t}),E)\cdot \left[\frac{\displaystyle\prod_{\substack{q=1 \\ q \text{ odd}}}^{s} \prod_{\substack{j_1<\cdots<j_q \\ 1\le j_1\le s}} \tau_\rho(F(\mathbb{Z}_{p_{j_1}^{t+1}} \oplus\cdots\oplus \mathbb{Z}_{p_{j_q}^{t+1}}),E)}{\displaystyle\prod_{\substack{q=1 \\ q \text{ even}}}^{s} \prod_{\substack{j_1<\cdots<j_q \\ 1\le j_1\le s}} \tau_\rho(F(\mathbb{Z}_{p_{j_1}^{t+1}} \oplus\cdots\oplus \mathbb{Z}_{p_{j_q}^{t+1}}),E)}\right]$$

Given Claim 2 the lemma follows by taking the product of $\Gamma\cdot\Gamma'\cdot\Omega'$.

Proof of Claim 2. By the above claims we know that every term of the $\tau_\rho(F(H),E)$ $r_H=t$, $t+1$ occurs in $\Omega'$. Thus it remains to count how many factors of $\tau_\rho(F(H),E)$ occur in the numerator and denominator of $\Omega'$. A term $\tau_\rho(F(H),E)$ appears whenever $H = p_{i_1}^t\cdots p_{i_n}^t,\ p_{j_1}^{s_{i_j}}\cdots p_{j_m}^{s_{j_m}}(\mathbb{Z}_{p_{k_1}} \oplus\cdots\oplus \mathbb{Z}_{p_{k_m}})$ for some subgroup

$H = \mathbb{Z}_{p_{i_1}^t} \oplus\cdots\oplus \mathbb{Z}_{p_{i_n}^t} \oplus \mathbb{Z}_{p_{j_1}^{s_{j_1}}} \oplus\cdots\oplus \mathbb{Z}_{p_{j_m}^{s_{j_m}}}$. By examination of (4.3.7) we see that $\tau_\rho(F(H),E)$ is in the numerator when r is odd and in the numerator when r is even ($r_H=t$). Thus we need to count the number of possible ways H can be represented as above taking care to count separately the cases when r is even or r is odd. This is precisely the content of Proposition 4.4.2, cases (i) and (iii) count terms is the denominator and cases (ii) and (iv) count terms in the numerator. We must subtract from cases (ii) and (iv) since we removed a factor of $\tau_\rho(F(H),E)$ to apply Lemma 2.3.1 to when obtaining $\Omega'$. A similar discussion applies to subgroups with $r_H = (t+1)$. Table I complies the data for the various subgroups.

Q.E.D.

Proof of Claim I. After the first step all the terms of the form $\tau_\rho(F(H),E)$ with $H = \mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t} \oplus \mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}$ m and q odd appear in the numerator. After the second iteration one gets all subgroups of the form $H = \mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t} \oplus \mathbb{Z}_{p_{j_1}^2} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}^2} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}$ m odd, hold q even or h even q even since

$$H = \begin{cases} \mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t} \oplus \mathbb{Z}_{p_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}} (\mathbb{Z}_{p_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}} \oplus \mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}) & \text{h odd q even} \\ \mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t} \oplus \mathbb{Z}_{p_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_m}} \oplus \mathbb{Z}_{p_{k_1}} (\mathbb{Z}_{p_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}} \oplus \mathbb{Z}_{p_{k_q}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}) & \text{h even q even} \end{cases}$$

for instance. After the next iteration one has the remaining subgroups of the form $\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t} \oplus \mathbb{Z}_{p_{j_1}^2} \oplus \cdots \oplus \mathbb{Z}_{p_{j_m}^2} \oplus \mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}$ since

$$H = \mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_m}^t} \oplus \mathbb{Z}_{p_{i_1}^2} \oplus \cdots \oplus \mathbb{Z}_{p_{i_h}^2} (\mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}) \text{ for } q \text{ odd.}$$

Now using induction and the same process outlined above the result follows.

Q.E.D.

TABLE I

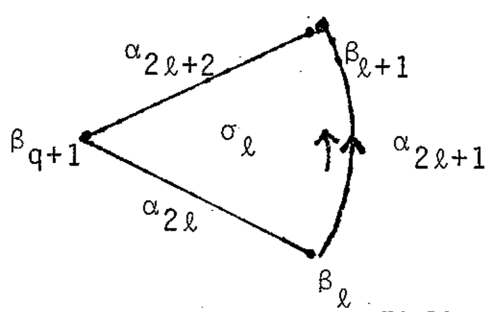
| TYPE OF SUBGROUP | Number of Factors in Numerator $\Omega'$ | Number of Factors in Denominator $\Omega'$ | Number of Factors in $\Omega'$ |
|---|---|---|---|
| $\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_g}^t} \oplus \mathbb{Z}_{p_{j_1}^{s_{j_1}}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}^{s_{j_n}}} \oplus \mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}$<br>g odd $1 \le i_1, \cdots, i_g \le \ell$ $q+h>0$ | $(2^{g-1})(2^{g+h-1})$<br>$+(2^{g-1}-1)(2^{g+h-1})$<br>$-1$ | $(2^{q-1})(2^{q+h-1})-1$<br>$+(2^{g-1}-1)(2^{q+h-1})$ | 0 |
| $\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_g}^t} \oplus \mathbb{Z}_{p_{j_1}^{s_{j_1}}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}^{s_{j_n}}} \oplus \mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k}$<br>g even $1 \le i_1, \cdots, i_g \le \ell$ $q+h>0$ | $(2^{g-1})(2^{q+h-1})$<br>$+(2^{g-1}-1)(2^{q+h-1})$<br>$-1$ | $(2^{g-1})(2^{q+h-1})$<br>$+(2^{g-1}-1)(2^{q+h-1})$<br>$-1$ | 0 |
| $\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_g}^t}$ g odd $1 \le i_1, \cdots, i_g \le \ell$ | $2^{g-1}$ | $2^{g-1}$ | 0 |
| $\mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_g}^t}$ g even $1 \le i_1, \cdots, i_g \le \ell$ | $2^{g-1}-1$ | $2^{g-1}-2$ | 1 |
| $\mathbb{Z}_{p_{\ell_1}^{t+1}} \oplus \cdots \oplus \mathbb{Z}_{p_{\ell_f}^{t+1}}$ f odd $1 \le \ell_1, \cdots, \ell_f \le s$ | 1 | 0 | 1 |
| $\mathbb{Z}_{p_{\ell_1}^{t+1}} \oplus \cdots \oplus \mathbb{Z}_{p_{\ell_f}^{t+1}}$ f even $1 \le \ell_1, \cdots, \ell_f \le s$ | 0 | 1 | -1 |
| $\mathbb{Z}_{p_{\ell_1}^{t+1}} \oplus \cdots \oplus \mathbb{Z}_{p_{\ell_f}^{t+1}} \oplus \mathbb{Z}_{p_{i_1}^t} \oplus \cdots \oplus \mathbb{Z}_{p_{i_g}^t} \oplus \mathbb{Z}_{p_{j_1}^{s_{j_1}}} \oplus \cdots \oplus \mathbb{Z}_{p_{j_n}^{s_{j_n}}} \oplus$<br>$\mathbb{Z}_{p_{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_{k_q}}$ $1 \le \ell_1, \cdots, \ell_f \le s_1$, $1 \le i_1, \cdots, i_g \le \ell$ | $2^{g+h+q-1}$ | $2^{g+h+q-1}$ | 0 |

## 5. Further Examples.

In the previous section we developed two formulas for the torsion $\tau_\rho(X,E)$ in terms of the torsion of quotients of the isotopy spaces. Here we compute explicitly $\tau_\rho$ and $\tau_g$ for two examples directly from the definition and from the aforementioned formulas.

5.1

For our first example let $X$ = disc = $\{(x,y) | x^2+y^2 \le 1\}$ with $E$ the trivial $\mathbb{R}^2$ bundle over X. We give X the cell decomposition consisting of q two-cells $\{\sigma_0,\cdots,\sigma_{q-1}\}$ 2q one-cells $\{\alpha_{0_1},\cdots,\alpha_{2_q}\}$ and q+1 zero-cells $\{\beta_0,\cdots,\beta_{q+1}\}$. Take $\sigma_\ell = \{(r_1\theta)\ 0 \le r \le 1\ \frac{2\pi\ell}{q} \le \theta \le \frac{2\pi(\ell+1)}{q}\}$. The following diagram will serve to define the $\alpha_i$, $\beta_i$.




Let g be rotation by $\frac{2\pi}{q}$. Then $G = \langle g \rangle = \mathbb{Z}_q$. Let $\tilde{g}$ denote the extension of g to $E$ and define $\tilde{g}(\sigma_\ell \otimes e) = \sigma_{\ell+1} \otimes R_{\frac{1}{q}}e$, $\tilde{g}(\alpha_\ell \otimes e) = \alpha_{\ell+1} \otimes R_{\frac{1}{q}}e$, and $\tilde{g}(\beta_{\ell+1} \otimes e) = \beta_{\ell+1} \otimes R_{\frac{1}{q}}e$ where $\sigma_1 \otimes e$ denotes the horizontal lift of a two-cell, $\alpha_1 \otimes e$ the lift of a one-cell,.and $\beta_\ell \otimes e$ the lift of a zero-cell.

Taking $\{v_0, v_{-0}, \cdots, v_{q-1}, v_{-q-1}\}$, $\{u_0, u_{-0}, \cdots, u_{2(q-1)}, u_{-2(q-1)}\}$ $\{w_0, w_{-0}, w_1, w_{-1}, w_q, w_{-q}, w_2, w_{-2}, \cdots, w_{q-1}, w_{-q-1}\}$ as bases for $C_2(X,E)$, $C_1(X,E)$ and $C_0(X,E)$ respectively, with

$$(5.1.1)\qquad \begin{aligned} v_\ell &= \sum_{j=0}^{q-1} \sigma_j \otimes R_{-\frac{j\ell+j}{q}}\, e_1 \\ v_{-\ell} &= \sum_{j=0}^{q-1} \sigma_j \otimes R_{-\frac{j\ell+j}{q}}\, e_2 \end{aligned}$$

$$(5.1.2)\qquad \begin{aligned} u_{2\ell} &= \overset{q-1}{\underset{j=0}{}}\; 2j \otimes R_{-\frac{j\ +j}{q}}\, e_1 \\ u_{-2\ell} &= \sum_{j=0}^{q-1} \alpha_{2j} \otimes R_{-\frac{j\ell+j}{q}}\, e_2 \\ u_{2\ell+1} &= \sum_{j=0}^{q-1} \alpha_{2j+1} \otimes R_{-\frac{j\ell+j}{q}}\, e_1 \\ u_{-2\ell+1} &= \sum_{j=0}^{q-1} \alpha_{2j+1} \otimes R_{-\frac{j\ell+j}{q}}\, e_2 \end{aligned}$$

$$(5.1.3)\qquad \begin{aligned} w_\ell &= \sum_{j=0}^{q-1} \beta_\ell \otimes R_{-\frac{j\ell+j}{q}}\, e_1 \\ w_{-\ell} &= \sum_{j=0}^{q-1} \beta_\ell \otimes R_{-\frac{j\ell+j}{q}}\, e_2 \\ w_q &= \beta_{q+1} \otimes e_1 \\ w_{-q} &= \beta_{q+1} \otimes e_2 \end{aligned}$$

where $0 \le \ell \le q-1$ and $-j\ell+j$ is taken mod $q$. Then with respect to these bases $\tilde{g}$ has matrix representations given by

$$\begin{bmatrix} I & & & & \\ & R_{\frac{1}{q}} & & & \\ & & \ddots & & \\ & & & & R_{\frac{q-1}{q}} \end{bmatrix} \quad \text{on } C_2(X,E)$$

$$\begin{bmatrix} I & & & & & & & \\ & I & & & & & & \\ & & R_{\frac{1}{q}} & & & & & \\ & & & R_{\frac{1}{q}} & & & & \\ & & & & \ddots & & & \\ & & & & & R_{\frac{q-1}{q}} & \\ & & & & & & R_{\frac{q-1}{q}} \end{bmatrix} \quad \text{on } C_1(X,E)$$

and

$$\begin{bmatrix} I & & & & \\ & R_{\frac{1}{q}} & & & \\ & & R_{\frac{2}{q}} & & \\ & & & \ddots & \\ & & & & R_{\frac{q-1}{q}} \end{bmatrix} \quad \text{on } C_0(X,E)$$

as seen in a manner similar to Example 2.4.7.

We now compute the boundary maps $\partial_2 : C_2(X,E) \to C_1(X,E)$ and $\partial_1 : C_1(X,E) \to C_0(X,E)$.

$$\partial_2(v_\ell) = \sum_{j=0}^{q-1} \partial_2(\sigma_j) \otimes R_{-\frac{j\ell+j}{q}} e_1 .$$

$$= \sum_{j=0}^{q-1} (\alpha_{2j+1}+\alpha_{2j+2}-\alpha_{2j}) \otimes R_{-\frac{j\ell+j}{q}} e_1$$

$$= \sum_{j=0}^{q-1} \alpha_{2j+1} \otimes R_{-\frac{j\ell+j}{q}} e_1 + \sum_{j=0}^{q-1} \alpha_{2j+2} \otimes R_{-\frac{j\ell+j}{q}} e_1$$

$$- \sum_{j=0}^{q-1} \alpha_{2j} \otimes R_{-\frac{j\ell+j}{q}} e_1$$

$$= u_{2\ell+1} + \sum_{j=0}^{q-1} \alpha_{2j+2} \otimes R_{-\frac{j\ell+j}{q}} e_1 - u_{2\ell}.$$

Observe that

$$\sum_{j=0}^{q-1} \alpha_{2j+2} \otimes R_{-\frac{j\ell+j}{q}} e_i = \sum_{j=0}^{q-1} \alpha_{2j+2} \otimes R_{-\frac{1}{q}} R_{-\frac{j\ell+j+1}{q}} e_i$$

$$= R_{-\frac{1}{q}} \sum_{j=0}^{q-1} \alpha_{2j+2} \otimes R_{-\frac{j\ell+j+1}{q}} e_i = \begin{cases} R_{-\frac{1}{q}} \tilde{g}(u_{2\ell}) & i=1 \\ R_{-\frac{1}{q}} \tilde{g}(u_{-2\ell}) & i=2 \end{cases}$$

$$= \begin{cases} \cos \frac{2\pi(\ell-1)}{q} u_{2\ell} + \sin \frac{2\pi(\ell-1)}{q} u_{-2\ell} & i=1 \\ -\sin \frac{2\pi(\ell-1)}{q} u_{2\ell} + \cos \frac{2\pi(\ell-1)}{q} u_{-2\ell} & i=2 \end{cases}$$

and hence

$$\partial_2(v_\ell) = (\cos \frac{2\pi(\ell-1)}{q} -1)u_{2\ell} + \sin \frac{2\pi(\ell-1)}{q} u_{-2\ell} + u_{2\ell+1}.$$

A similar computation shows

$$\partial_2(v_{-\ell}) = -\sin\frac{2\pi(\ell-1)}{q}\, u_{2\ell} + (\cos\frac{2\pi(\ell-1)}{q} - 1)u_{-2\ell} + u_{2\ell+2}$$

$$\partial_1(u_{2\ell}) = \sum_{j=0}^{q-1} \partial_1(\alpha_{2j}) \otimes R_{-\frac{j\ell+j}{q}}\, e_1$$

$$= \sum_{j=0}^{q-1}(\beta_{q+1}-\beta_\ell) \otimes R_{-\frac{j\ell+j}{q}}\, e_1$$

$$= -\sum_{j=0}^{q-1} \beta_\ell \otimes R_{-\frac{j\ell+j}{q}}\, e_1 + \sum_{j=0}^{q-1} \beta_{q+1} \otimes R_{-\frac{j\ell+j}{q}}\, e_1$$

$$= \begin{cases} -w_\ell & \ell \neq 1 \\ -w_\ell + qw_q & \ell = 1 \end{cases} .$$

Similarly one has

$$\partial_1(u_{-2\ell}) = \begin{cases} -w_{-\ell} & \ell \neq 1 \\ -w_{-\ell} + qw_q & \ell = 1 \end{cases}$$

$$\partial_1(u_{2\ell+1}) = \sum_{j=0}^{q-1} \partial_1(\alpha_{2j+1}) \otimes R_{-\frac{j\ell+j}{q}}\, e_1$$

$$= \sum_{j=0}^{q-1}(\beta_{\ell+1}-\beta_\ell) \otimes R_{-\frac{j\ell+j}{q}}\, e_1$$

$$= \sum_{j=0}^{q-1} \beta_{\ell+1} \otimes R_{-\frac{j\ell+j}{q}}\, e_1 - \sum_{j=0}^{q-1} \beta_\ell \otimes R_{-\frac{j\ell+j}{q}}\, e_1$$

$$= \begin{cases} (\cos\frac{2\pi(\ell-1)}{q} - 1)w_\ell - \sin\frac{2\pi(\ell-1)}{q}\, w_{-\ell} & \ell \neq 1 \\ 0 & \ell = 1 \end{cases} .$$

Again a similar computation yields

$$\partial_1(u_{-2\ell+1}) = \begin{cases} -\sin\frac{2\pi(\ell-1)}{q}\, w_\ell + \left(\cos\frac{2\pi(\ell-1)}{q} - 1\right) w_{-\ell} & \ell \neq 1 \\ 0 & \ell = 1 \end{cases}.$$

Let $C^*_{\rho_\ell}(X,E)$ denote the direct sum of the subcomplexes on which G acts irreducibly by $R_{\frac{\ell}{q}}$. Comparing $\tau_{\rho_\ell}(X,E) = \tau(C^*_{\rho_\ell}(X,E))$ splits into two cases, $\ell \neq 1$ and $\ell = 1$. Explicitly one has the cochain complexes and their boundary maps below.

<u>Case $\ell \neq 1$</u>

(5.1.4) $\{w^*_\ell, w^*_{-\ell}\} \xrightarrow{d_0^\ell} \{u^*_{2\ell}, u^*_{-2\ell}, u^*_{2\ell+1}, u^*_{-2\ell+1}\} \xrightarrow{d_1^\ell} \{v^*_\ell, v^*_{-\ell}\}$

$$\partial_1^t\big|_{C_{\rho_\ell}} = d_0^\ell = \begin{bmatrix} -I \\ (R_{\frac{\ell-1}{q}} - I)^t \end{bmatrix} \text{ and } \partial_2^t\big|_{C_{\rho_\ell}} = d_1^\ell = \begin{bmatrix} (R_{\frac{\ell-1}{q}} - I)^t & I \end{bmatrix}$$

<u>Case $\ell = 1$</u>

(5.1.5) $\{w^*_1, w^*_{-1}, w^*_q, w^*_{-q}\} \xrightarrow{d_0^1} \{u^*_2, u^*_{-2}, u^*_3, u^*_{-3}\} \xrightarrow{d_1^1} \{v^*_1, v^*_{-1}\}$

$$\partial_1^t\big|_{C_{\rho_1}} = d_0^1 = \begin{bmatrix} -I & qI \\ 0 & 0 \end{bmatrix} \quad \partial_2^t\big|_{C_{\rho_1}} = d_1^1 = \begin{bmatrix} 0 & I \end{bmatrix}$$

In the case $\ell \neq 1$ choose $\omega_0 = w_\ell \wedge w_{-\ell} \in \Lambda^2(C^0_{\rho_\ell}(X,E))^*$, $\omega_1 = u_{2\ell} \wedge u_{-2\ell} + u_{2\ell+1} \wedge u_{-2\ell+1} \in \Lambda^4(C^1_{\rho_\ell}(X,E))^*$ and $\omega_2 = v_\ell \wedge v_{-\ell} \in \Lambda^2(C^2_{\rho_\ell}(X,E))^*$. Then taking $\rho_2 = \omega_2$,

$\rho_1 = u_2 \wedge u_{-2}$, $\rho_0 = 1$ we have

$$\rho_0 \wedge d_0^{\ell *}(\rho_1) = d_0^{\ell *}(u_2 \wedge u_{-2}) = w_\ell \wedge w_{-\ell} = m_0 \omega_0$$

$$\rho_1 \; d_1^{\ell *}(\rho_2) = u_2 \wedge u_{-2} \wedge d_1^{*}(v_\ell \wedge v_{-\ell}) = u_2 \wedge u_{-2} \wedge u_3 \wedge u_{-3} = m_1 \omega_1$$

$$\rho_2 \; d_1^{\ell *}(\rho_3) = \rho_2 = w_2 = m_2 \omega_2 .$$

This gives $m_0 = m_1 = m_2 = 1$ and $\tau_{\rho_\ell}(X,E) = 1$. An entirely similar analysis shows that $\tau_{\rho_1}(X,E) = 1$ and hence $\tau_g(X,E) = 1$.

In Section 4 we derived the formula

$$(5.1.6) \qquad \tau_\rho(X,E) = \prod_{H \in I} \tau(X\{H\}, E_\rho)\tau_\rho(H_H^r).$$

It is our intention to use (5.1.6) to recompute $\tau_{\rho_\ell}(X,E)$ for the above example. In this example ther are only two isotropy spaces $X(e)$ and $X(\mathbb{Z}_q) = F(\mathbb{Z}_q)$, consequently (6) reduces to

$$\tau_{\rho_\ell}(X,E) = \tau(X\{e\}, E_{\rho_\ell})\tau(F(\mathbb{Z}_q),E)\tau_\rho(H_e^r).$$

Since $F(\mathbb{Z}_q)$ equals a point $\tau(F(\mathbb{Z}_q),E) = 1$. Thus it remains to examine the terms $\tau_\rho(H_e^r)$ and $\tau(X\{e\}, E_{\rho_\ell})$. We begin with $\tau_\rho(H_e^r)$. The exact sequence $H_e^r$ is

$$0 \to H^0_{\rho_\ell}(X,pt,E) \to H^0_{\rho_\ell}(X,E) \to H^0_{\rho_\ell}(pt,E) \to H^1_{\rho_\ell}(X,pt,E) \to$$

$$H^1_{\rho_\ell}(X,E) \to H^1_{\rho_\ell}(pt,E) \to H^2_{\rho_\ell}(X,pt,E) \to H^2_{\rho_\ell}(X,E) \to H^2_{\rho_\ell}(pt,E) \to 0.$$

There are two cases to consider $\ell=1$ and $\ell \neq 1$. Examination of the complexes (5.1.4) and (5.1.5) shows

$$H^j_{\rho_\ell}(X,E) = \begin{cases} 0 & j=0 \\ 0 & j=1 \\ 0 & j=3 \end{cases} \qquad H^j_{\rho_\ell}(pt,E) = \begin{cases} 0 & j=0 \\ 0 & j=1 \\ 0 & j=3 \end{cases}$$

for $\neq 1$ and

$$H^j_{\rho_\ell}(X,E) = \begin{cases} \mathbb{R}^2 & j=0 \\ 0 & j=1 \\ 0 & j=3 \end{cases} \qquad H^j_{\rho_\ell}(pt,E) = \begin{cases} \mathbb{R}^2 & j=0 \\ 0 & j=1 \\ 0 & j=3 \end{cases}$$

for $\ell=1$.

The relative complexes $C^*_{\rho_\ell}(X,pt,E)$ are given below along with their appropriate boundary maps

<u>Case $\ell\neq 1$</u>

(5.1.7) $\{w^*_\ell, w^*_{-\ell}\} \xrightarrow{{}_r d^\ell_0} \{u^*_{2\ell}, u^*_{-2\ell}, u^*_{2\ell+1}, u^*_{-2\ell+1}\} \xrightarrow{{}_r d^\ell_1} \{v^*_\ell, v^*_{-\ell}\}$

$${}_r d^\ell_0 = \begin{bmatrix} -I \\ \\ (R_{\frac{\ell-1}{q}} - I)^t \end{bmatrix} \quad \text{and} \quad {}_r d^\ell_1 = \begin{bmatrix} (R_{\frac{\ell-1}{q}} - I)^t & I \end{bmatrix}$$

<u>Case $\ell=1$</u>

(5.1.8) $\{w^*_1, w^*_{-1}\} \xrightarrow{{}_r d^1_0} \{u^*_2, u^*_{-2}, u^*_3, u^*_{-3}\} \xrightarrow{{}_r d^1_1} \{v^*_1, v^*_{-1}\}$

$${}_r d^1_0 = \begin{bmatrix} -I \\ \\ 0 \end{bmatrix} \qquad {}_r d^1_1 = [0 \quad I]$$

This implies

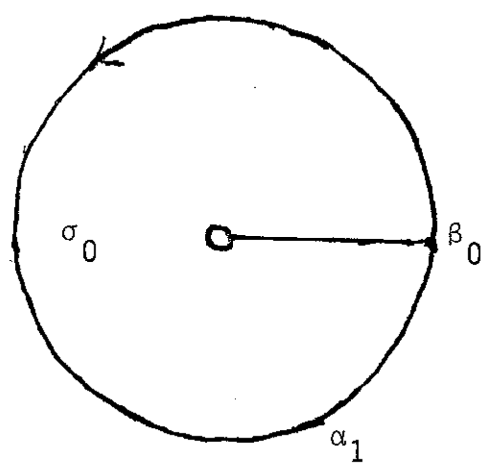
$$H^j_{\rho_\ell}(X,pt,E) = \begin{cases} 0 & j=0 \\ 0 & j=1 \\ 0 & j=2 \end{cases} \quad \ell \neq 1 \qquad H^j_{\rho_1}(X,pt,E) = \begin{cases} 0 & j=0 \\ 0 & j-1 \\ 0 & j-2 \end{cases} .$$

For $\ell \neq 1$ $\tau_{\rho_\ell}(H^r_e) = 1$. For $\ell = 1$ we have the long exact sequence

$0 \to 0 \to \mathbb{R}^2 \xrightarrow{i} \mathbb{R}^2 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0 \to 0$ where

the homomorphism i is induced by inclusion ind equals the identity matrix which gives $\tau_{\rho_1}(H^r_e) = 1$.

Finally we are reduced to computing $\tau(X\{e\},E_{\rho_\ell})$. X(e) equals the disc minus its center point and $X\{e\} = {}^{X(e)}/G$ is also equal to the disc minus a point; however, the resulting chain complex consists of 1 two-cell, 2 one-cells, and 1 zero-cell.




The bundles $E_{\rho_\ell}$ are $\mathbb{R}^2$ bundles with holonomy $R_{\frac{\ell-1}{q}}$. To see this recall $\overleftarrow{\pi}(E)_y = \bigoplus_{x \in \pi^{-1}(y)} E_x$ where $\pi : X(e) \to X\{e\}$ and $y \in X\{e\}$. Then $\overleftarrow{\pi}(E)_{\rho_\ell} = E_{\rho_\ell}$ is the subbundle of $\overleftarrow{\pi}(E)$ whose sections transform by $\rho_\ell$. If $s(x) = (x,v) \varepsilon \Gamma(E)$ then $S(s(x)) = (y,v,R_{\frac{1}{q}}v,\cdots,R_{\frac{q-1}{q}}v)$ where $\pi(x) = y$. Thus $\overline{g}(S(s(x)) = (y,R_{\frac{1}{q}}v,R_{\frac{2}{q}}v,\cdots R_{\frac{q-1}{q}}v)$ and setting this equal to $R_{\frac{\ell}{g}}S(s(x))$ gives $R_{\frac{1}{q}}S(s(x)) = R_{\frac{\ell}{q}}S(s(x))$ and hence $S(s(x)) = R_{\frac{\ell-1}{q}}S(s(x))$. Therefore, $E_{\rho_\ell}$ is the flat $\mathbb{R}^2$ bundle over $X\{e\}$ with holonomy $R_{\frac{\ell-1}{q}}$.

So we consider the two cases $\ell=1$ and $\ell\neq 1$. For $\ell=1$ let $\{v_1,v_2\} = \{\sigma_0\otimes e_1,\sigma_0\otimes e_2\}$, $\{u_1,u_2,u_3,u_4\} = \{\alpha_2\otimes e_1,\alpha_2\otimes e_2,\alpha_1\otimes e_1,\alpha_1\otimes e_2\}$, and $\{w_1,w_2\} = \{\beta_0\otimes e_1,\beta_0\otimes e_2\}$ be bases for $C_2(X\{e\},E\rho_\ell)$, $C_1(X\{e\},E_{\rho_\ell})$, and $C_0(X\{e\},E_{\rho_\ell})$ respectively. We have

$$\begin{aligned}\partial_2(v_1) &= \partial_2(\sigma_0\otimes e_1)\\ &= \alpha_1 \otimes e_1 - \alpha_2 \otimes e_1 + \alpha_2 \otimes R_{\frac{\ell-1}{q}} e_1\\ &= u_3 + (\cos(\frac{2\pi(\ell-1)}{q}) - 1)u_1 + \sin \frac{2\pi(\ell-1)}{q} u_2\end{aligned}$$

and in a similar manner obtain

$$\partial_2(v_2) = u_4 - \sin(\frac{2\pi(\ell-1)}{q})u_1 + (\cos \frac{2\pi(\ell-1)}{q} -1)u_2.$$

$$\partial_1(u_1) = -\beta_0 \otimes e_1 = w_1$$

$$\partial_1(u_2) = -\beta_0 \otimes e_2 = w_2$$

$$\partial_1(u_3) = \beta_0 \otimes R_{\frac{\ell-1}{q}} e_1 - \beta_0 \otimes e_1 = (\cos \frac{2\pi(\ell-1)}{q} -1)w_1 + \sin\frac{2\pi(\ell-1)}{q}w_1$$

$$\partial_1(u_3) = \beta_0 \otimes R_{\frac{\ell-1}{q}} e_2 - \beta_0 \otimes e_2 = \sin \frac{2\pi(\ell-1)}{q} w_1 + \cos \frac{2\pi(\ell-1)}{q} w_2$$

The above gives the following cochain complex

$$\{w_1^*, w_2^*\} \xrightarrow{d_0^\ell} \{u_1^*, u_2^*, u_3^*, u_4^*\} \xrightarrow{d_1^\ell} \{v_1^*, v_2^*\}$$

$$d_0^\ell = \begin{bmatrix} -I \\ \\ (R_{\frac{\ell-1}{q}} - I)^t \end{bmatrix} \quad \text{and} \quad d_1^\ell = \begin{bmatrix} (R_{\frac{\ell-1}{q}} - I)^t & I \end{bmatrix}$$

which we observe to be the same complex obtained in (5.1.4). This gives then $\tau(X\{e\}, E_{\rho_\ell}) = 1$ and combined with our previous observation that $\tau_{\rho_\ell}(H_e^r) = 1$ that $\tau_{\rho_\ell}(X,E) = 1$. A similar computation for $\ell=1$ shows $\tau_{\rho_1}(X,E) = 1$.

## 5.2

Our next example will be for the case $X = S^3$ and $E$ the flat $\mathbb{R}^2$ bundle over $X$. Let $(z_0, z_1)$ be complex coordinates for $\mathbb{C}^2$, then $S^3 = \{(z_0,z_1) | (z_0)^2 + (z_1)^2 = 1\}$. Define g by $g(z_0,z_1) = (e^{\frac{2\pi i}{q}} z_0, z_1)$, then $G = \langle g \rangle = \mathbb{Z}_q$ and $F(\mathbb{Z}_q) = S^1$. To get a cell structure on $S^3$ for which G acts cellularly first stereographically project $S^3$ onto $\mathbb{R}^3$ from a point (say (0,i)) on $F(\mathbb{Z}_q)$. Letting $z_j = x_{2j} + ix_{2j+1}$ and identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ via sending $(z_0,z_1)$ to $(x_0,x_1,x_2,x_3)$ we have the stereographic projection map $\phi((x_0,x_1,x_2,x_3)) = (\frac{x_0}{1-x_3}, \frac{x_1}{1-x_3}, \frac{x_2}{1-x_3})$. Thus

$$\phi(g(z_0,z_1)) = \left(\frac{\cos\frac{2\pi}{q} x_0 - \sin\frac{2\pi}{q} x_1}{1-x_3}, \frac{\cos\frac{2\pi}{q} x_1 + \sin\frac{2\pi}{q} x_0}{1-x_3}, \frac{x_2}{1-x_3}\right)$$

$$\begin{bmatrix} \cos\frac{2\pi}{q} & -\sin\frac{2\pi}{q} & 0 \\ \sin\frac{2\pi}{q} & \cos\frac{2\pi}{q} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x_0}{1-x_3} \\ \frac{x_1}{1-x_3} \\ \frac{x_2}{1-x_3} \end{bmatrix}$$

and the induced action of g on $\mathbb{R}^3$ is rotation by $\frac{2\pi}{q}$ about the z axis. Using cylindrical coordinates $(r,\theta,z)$ on $\mathbb{R}^3$, define $p_K = \{(r,\theta,z) | \frac{2\pi k}{q} \leq \theta \leq \frac{2\pi(k+1)}{q}\}$ for $0 \leq k \leq q-1$. Taking $\phi^{-1}$ of the $p_K$ gives a cell decomposition of $S^3$ consisting of q three-cells, q two-cells, 1 one-cell, and 1 zero-cell for which G acts cellularly. Denote the three-cells as $\Sigma_0,\cdots,\Sigma_{q-1}$, the two-cells by $\sigma_0,\cdots,\sigma_{q-1}$, the one-cell by $\beta_0$, and the zero-cell by $\alpha_0$.

Taking bases of horizontal lifts $\{v_0,v_{-0},\cdots,v_{q-1},v_{-q-1}\}$, $\{u_0,u_{-0},\cdots,u_q,u_{-q}\}$, $\{q_1,w_{-1}\}$, and $\{y_1,y_{-1}\}$ for $C_3(S^3,E)$ $C_2(S^3,E)$, $c_1(S^3,E)$ and $C_0(S^3,E)$ respectively where

$$v_\ell = \sum_{j=0}^{q-1} \Sigma_j \otimes R_{\frac{-j\ell+j}{q}} e_1$$

$$v_{-\ell} = \sum_{j=0}^{q-1} \Sigma_j \otimes R_{\frac{-j\ell+j}{q}} e_2$$

$$u_\ell = \sum_{j=0}^{q-1} \sigma_j \otimes R_{\frac{-j\ell+j}{q}} e_1$$

$$u_{-\ell} = \sum_{j=0}^{q-1} \sigma_j \otimes R_{\frac{-j\ell+j}{q}} e_2$$

$$w_1 = \beta_0 \otimes e_1,\ w_{-1} = \beta_0 \otimes e_2$$

$$y_1 = \alpha_0 \otimes e_1,\ y_{-1} = \alpha_0 \otimes e_2.$$

With respect to these bases $\tilde{g}$ has matrix representations

$$\begin{bmatrix} I & & & & \\ & R_{\frac{1}{q}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & R_{\frac{q-1}{q}} \end{bmatrix}$$

on $C_3(S^3,E)$ and $C_2(S^3,E)$ and $R_{\frac{1}{q}}$ on $C_1(S^3,E)$ and $C_0(S^3,E)$. We now compute the boundary maps with respect to the above bases

$$\partial_3(v_\ell) = \sum_{j=0}^{q-1} \partial_3(\Sigma_j) \otimes R_{\frac{-j\ell+j}{q}} e_1$$

$$= \sum_{j=0}^{q-1} \sigma_{j+1} \otimes R_{\frac{-j\ell+j}{q}} e_1 - \sum_{j=0}^{q-1} \sigma_j \otimes R_{\frac{-j\ell+j}{q}} e_1$$

$$= (\cos \frac{2\pi(\ell-1)}{q} - 1)u_\ell - \sin \frac{2\pi(\ell-1)}{q} u_{-\ell}$$

$$\partial_3(v_{-\ell}) = \sin \frac{2\pi(\ell-1)}{q} u_\ell + (\cos \frac{2\pi(\ell-1)}{q} - 1)u_{-\ell}$$

$$\partial_2(u_\ell) = \sum_{j=0}^{q-1} \partial_2(\sigma_j) \otimes R_{\frac{-j\ell+j}{q}} e_1$$

$$= \sum_{j=1}^{q-1} \beta_0 \otimes R_{\frac{-j\ell+j}{q}} e_1$$

$$= \begin{cases} q\ \beta_0 \otimes e_1 & \ell=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\partial_2(u_{-\ell}) = \begin{cases} q\ \ \ {}_0 \otimes e_2 & \ell=1 \\ 0 & \text{otherwise} \end{cases}$$

From the above relations we have $m_0=m_2=m_3=1$ and $m_1 = \det\begin{pmatrix} q & 0 \\ 0 & q\end{pmatrix} = q^2$. Thus $\tau_{\rho_1}(S^3,E) = \frac{1}{q^2}$.

Compiling the above yields

$$(4.2.3) \quad \tau_g(S^3,E) = \left(\frac{1}{q^2}\right)^{2\cos\frac{2\pi}{q}} \prod_{\substack{\ell=0 \\ \ell\neq 1}}^{q-1} \left(4\sin^2\left(\frac{\pi(\Pi-1)}{q}\right)\right)^{2\cos\frac{2\pi\ell}{q}} .$$

To compute $\tau_{\rho_\ell}(S^3,E)$ using (5.1.6) we notice as before there are only two isotropy spaces $X(e)$ and $X(\mathbb{Z}_q) = F(\mathbb{Z}_q)$ and therefore

$$(5.2.4) \quad \tau_{\rho_\ell}(S^3,E) = \tau(S^3\{e\},E_{\rho_\ell})\tau(S^1,E)\tau_{\rho_\ell}(H_e^r).$$

We begin with the $\tau_{\rho_\ell}(H_e^r)$ term. The exact sequence $H_e^r$ is

$$0 \to H^0_{\rho_\ell}(S^3,S^1,E) \to H^0_{\rho_\ell}(S^3,E) \to H^0_{\rho_\ell}(S^1,E) \to H^1_{\rho_\ell}(S^3,S^1,E) \to$$
$$H^1_{\rho_\ell}(S^3,E) \to H^1_{\rho_\ell}(S^1,E) \to H^2_{\rho_\ell}(S^3,S^1,E) \to H^2_{\rho_\ell}(S^3,E) \to H^2_{\rho_\ell}(S^1,E) \to$$
$$H^3_{\rho_\ell}(S^3,S^1,E) \to H^3_{\rho_\ell}(S^3,E) \to H^3_{\rho_\ell}(S^1,E) \to 0$$

Examination of the complexes (5.2.1) and (5.2.2) shows

$$\underset{\ell\neq 1}{H^j_{\rho_\ell}(S^3,E)} \begin{cases} 0 & j=0 \\ 0 & j=1 \\ 0 & j=2 \\ 0 & j=3 \end{cases} \qquad H^j_{\rho_1}(S^3,E) \begin{cases} \mathbb{R}^2 & j=0 \\ 0 & j=1 \\ 0 & j=2 \\ \mathbb{R}^2 & j=3 \end{cases}$$

For $H^j_{\rho_\ell}(S^1,E)$ we note that since $E$ is the trivial $\mathbb{R}^2$ bundle and $\tilde{g}$ acts $E|_{S^1}$ via $\rho_1$ that

$$H^j_{\rho_1}(S^1,E) = \begin{cases} \mathbb{R}^2 & j=0 \\ \mathbb{R}^2 & j=1 \end{cases}$$

and $H^j_{\rho_\ell}(S^1,E) = 0$, $j=1,2$, $\ell \neq 1$.

The relative cochain complex has the form

$\{0\} \to \{0\} \to \{u^*_\ell, u^*_{-\ell}\} \xrightarrow{{}_r d_2} \{v^*_\ell, v^*_{-\ell}\}$ where ${}_r d^\ell_2 = (R_{\frac{\ell-1}{q}} - I)^t$. Thus

$$H^j_{\rho_\ell}(S^3,S^1,E) \underset{\ell\neq 1}{=} \begin{cases} 0 & j=0 \\ 0 & j=2 \\ 0 & j=1 \\ 0 & j=0 \end{cases} \qquad H^j_{\rho_\ell}(S^3,S^1,E) = \begin{cases} 0 & j=0 \\ 0 & j=1 \\ \mathbb{R}^2 & j=1 \\ \mathbb{R}^2 & j=3 \end{cases}$$

It is immediate that $\tau_{\rho_\ell}(H^r_e) = 1$ for $\ell \neq 1$. In the case $\ell \neq 1$ we have the complex

$$0 \to 0 \to \mathbb{R}^2 \xrightarrow{i^0_*} \mathbb{R}^2 \to 0 \to 0 \to \mathbb{R}^2 \xrightarrow{\partial_*} \mathbb{R}^2 \to 0 \to 0 \to \mathbb{R}^2 \xrightarrow{i^3_*} \mathbb{R}^2 \to 0 \to 0.$$

Since $i^0_*$ and $i^3_*$ are induced by inclusion $i^0_* = i^3_* = I$. The map $\partial_*$ is the connecting homomorphism between $H^1_{\rho_1}(S^1,E)$ and $H^2_{\rho_\ell}(S^3,S^1,E)$. Since $\partial_*$ is an isomorphism and $\partial_2(u_\ell) = q\,\beta_0 \otimes e_1$ and $\partial_2(u_{-\ell}) = q\,\beta_0 \otimes e_2$ we have $\partial_* = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$. This gives that $\tau_{\rho_1}(H^r_e) = \frac{1}{q^2}$.

Since we remarked earlier that $\tau(S^1,E) = 1$ it remains to compute $\tau(X\{e\},E_{\rho_\ell})$. $\phi(X(e))$ is equal to $\mathbb{R}^3 - \{z \text{ axis}\}$ and when we quotient by G we again get $\mathbb{R}^3 - \{z \text{ axis}\}$. Therefore the resulting chain complex consists of 1 three-cell $(\Sigma_0)$ and 2 two-cells $(\sigma_0, \sigma_1)$. The same arguement used in the previous example shows that $E_{\rho_\ell}$ is a flat $\mathbb{R}^2$ bundle over X{e} with holonomy $R_{\frac{\ell-1}{q}}$.

Setting $\{v_1, v_{-1}\} = \{\Sigma_0 \otimes e_1, \Sigma_0 \otimes e_2\}$ and $\{u_1, u_{-1}\} = \{\sigma_0 \otimes e_1, \sigma_0 \otimes e_2\}$ as bases for the horizontal lifts, the boundary map with respect to these bases is

$$\partial_3^\ell(v_1) = (\cos \frac{2\pi(\ell-1)}{q} - 1)u_1 + \sin \frac{2\pi(\ell-1)}{q}\, u_{-1}$$

$$\partial_3^\ell(v_{-1}) = -\sin \frac{2\pi(\ell-1)}{q}\, u_1 + (\cos \frac{2\pi(\ell-1)}{q} - 1)u_{-1}.$$

For $\ell \neq 1$ and

$$\partial_3^1(v_1) = 0$$

$$\partial_3^1(v_{-1}) = 0$$

for $\ell = 1$.

The cochain complexes are then

$$\{u_1^*, u_{-1}^*\} \xrightarrow{{}_r d_2^\ell} \{v_1^*, v_{-1}^*\}$$

where ${}_r d_2^\ell = (R_{\frac{\ell-1}{q}} - 1)^t$ and ${}_r d_2^1 = 0$. Thus for $\ell \neq 1$ this is the same complex as (5.2.1) and $\tau(X\{e\}, E_{\rho_\ell}) = 4\sin^2(\frac{\pi(\ell-1)}{q})$. For $\ell=1$ clearly $\tau(X\{e\}, E_{\rho_\ell}) = 1$. Using (5.2.4) gives $\tau_{\rho_\ell}(S^3, E) = 4\sin^2(\frac{\pi(\ell-1)}{q})$ for $\ell \neq 1$ and $\tau_{\rho_\ell}(S^3, E) = \frac{1}{q^2}$ for $\ell = 1$. This agrees with our previous computations.

5.3

For our final example we will take $X = S^5$ and $E$ to be the flat $\mathbb{R}^2$ bundle over X. In Section 4 we derived two formulas for the torsion and by comparing the two noticed that the product of the Mayer-Vietor is terms $\tau_\rho(H_H^M)$ equals one. We will check this cancellation explicitly in this example.

Let $(z_0,z_1,z_2)$ be complex coordinates of $\mathbb{C}^3$, then $S^5 = \{(z_0,z_1,z_2) \in \mathbb{C}^3 |\ |z_0|^2 + |z_1|^2 + |z_2|^2 = 1\}$. Define $g(z_0,z_1,z_2) = (e^{\frac{2\pi i}{q}} z_0, e^{\frac{2\pi i}{q}} z_1, z_2)$ where p and q, are assumed to be relatively prime, then $G = \langle g\rangle = \mathbb{Z}_q \oplus \mathbb{Z}_p$ and
$F(\mathbb{Z}_q) = \{(z_0,z_1,z_2) |\ |z_1|^2 + |z_2|^2 = 1\} = S^3$,
$F(\mathbb{Z}_p) = \{(z_0,z_1,z_2) |\ |z_0|^2 + |z_2|^2 = 1\} = S^3$, and
$F(\mathbb{Z}_q \oplus \mathbb{Z}_p) = \{(z_0,z_1,z_2) |\ |z_2|^2 = 1\} = S^1$. To get a cell structure for which G acts cellularly first stereographically project $S^5$ onto $\mathbb{R}^5$ from a point (say $(0,0,i)$) on $F(\mathbb{Z}_q \oplus \mathbb{Z}_p)$. Letting $z_j = x_{2j} + ix_{2j+1}$ and identifying $\mathbb{C}^3$ with $\mathbb{R}^6$ by sending $(z_0,z_1,z_2)$ to $(x_0,\cdots,x_5)$ we have the stereographic projection map $\phi(x_0,\cdots,x_5) = (\frac{x_0}{1-x_5},\cdots,\frac{x_4}{1-x_5})$. Thus $\phi(g(z_0,z_1,z_2))$

$$\begin{bmatrix} \cos\frac{2\pi}{q} & -\sin\frac{2\pi}{q} & 0 & 0 & 0 \\ \sin\frac{2\pi}{q} & \cos\frac{2\pi}{q} & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{p} & -\sin\frac{2\pi}{p} & 0 \\ 0 & 0 & \sin\frac{2\pi}{p} & \cos\frac{2\pi}{q} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{x_0}{1-x_5} \\ \\ \vdots \\ \\ \frac{x_4}{1-x_5} \end{bmatrix}$$

Letting $(y_1,\cdots,y_5)$ denote coordinates for $\mathbb{R}^5$ we see the induced action of g on $\mathbb{R}^5$ is rotation by $\frac{2\pi}{q}$ in the $y_1y_2$ plane, rotation by $\frac{2\pi}{p}$ in the $y_3y_4$ plane, and the $g_5$ axis is fixed by the action of G.

Now take coordinates $(r_1,\theta_1,r_2,\theta_2,y_5)$ for $\mathbb{R}^5$ where $y_1 = r_1\cos\theta_1$, $y_2 = r_1\sin\theta_1$, $y_3 = r_2\cos\theta_2$ and $y_4 = r_2\sin\theta_2$. Set

$$P_{i,\ell} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \frac{2\pi k}{q} \le \theta_1 \le \frac{2\pi(k+1)}{q},\ \frac{2\pi\ell}{p} \le \theta_2 \le \frac{2\pi(\ell+1)}{p}\}$$

$$Q_{k,\ell} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \theta_1 = \frac{2\pi k}{q},\ \frac{2\pi\ell}{p} \le \theta_2 \le \frac{2\pi(\ell+1)}{p}\}$$

$$\overline{Q}_{k,} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \theta_2 = \frac{2\pi\ell}{p},\ \frac{2\pi k}{q} \le \theta_1 \le \frac{2\pi(k+1)}{q}\}$$

$$R_{k,\ell} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \theta_1 = \frac{2\pi k}{q},\ \theta_2 = \frac{2\pi\ell}{p}\}$$

$$R_{0,\ell} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \frac{2\pi\ell}{p} \le \theta_2 \le \frac{2\pi(\ell+1)}{p},\ r_1 = \theta_1 = 0\}$$

$$\overline{R}_{k,0} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \frac{2\pi k}{q} \le \theta_1 \le \frac{2\pi(k+1)}{q},\ r_2 = \theta_2 = 0\}$$

$$S_{0,\ell} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \theta_2 = \frac{2\pi\ell}{p},\ r_1 = \theta_1 = 0\}$$

$$\overline{S}_{k,0} = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \theta_1 = \frac{2\pi k}{q},\ r_2 = \theta_2 = 0\}$$

$$T = \{(r_1,\theta_1,r_2,\theta_2,y_5) \mid \theta_1 = r_1 = \theta_2 = r_2 = 0\}$$

and let ${}_5\sigma_{k,\ell} = \phi^{-1}(P_{k,\ell})$, ${}_4\sigma_{k,\ell} = \phi^{-1}(Q_{k,\ell})$, ${}_4\overline{\sigma}_{k,\ell} = \phi^{-1}(\overline{Q}_{k,\ell})$

${}_3\sigma_{k,\ell} = \phi^{-1}(R_{k,\ell})$, ${}_3\sigma_{0,\ell} = \phi^{-1}(R_{0,\ell})$, ${}_3\overline{\sigma}_{k,0} = \phi^{-1}(\overline{R}_{k,0})$

${}_2\sigma_{k,\ell} = \phi^{-1}(S_{0,\ell})$, ${}_2\overline{\sigma}_{k,0} = \phi^{-1}(\overline{S}_{k,0})$, ${}_1\sigma_0 = \phi^{-1}(T)$, and let

${}_0\sigma_0$ denote the zero-cell. The above is a cell decomposition on $S^5$ for which G acts cellularly.

Taking $\tilde{g}$ to be the extension to $E$ given by $\tilde{g}({}_j\sigma_{k,\ell} \otimes e) = {}_j\sigma_{k+1,\ell+1} \otimes R_{\frac{1}{pq}} e$ where $0 \leq j \leq 5$ and ${}_j\sigma_{k,\ell} \otimes e$ denotes a horizontal lift of the cell ${}_j\sigma_{k,\ell}$. In order to write down bases for the $C_j^{\rho_m}(S^5,E)$ it will be convenient to re-index the cells ${}_5\sigma_{k,\ell}$, ${}_4\sigma_{k,\ell}$, ${}_4\sigma_{k,\ell}$ and ${}_3\sigma_{k,\ell}$. Instead of indexing the ${}_j\sigma_{k,\ell}$ over $\mathbb{Z}_q \oplus \mathbb{Z}_p$ we would like to index them over $\mathbb{Z}_{pq}$ such that if ${}_j\sigma_{k,\ell}$ goes to ${}_j\sigma_i$ then ${}_j\sigma_{k+1,\ell+1}$ goes to ${}_j\sigma_{i+1}$. In other words we seek a homomorphism $\psi : \mathbb{Z}_q \otimes \mathbb{Z}_p \to \mathbb{Z}_{pq}$ such that

$$
\begin{array}{ccc}
\mathbb{Z}_q \otimes \mathbb{Z}_p & \xrightarrow{\phi} & \mathbb{Z}_q \oplus \mathbb{Z}_p \\
\psi \downarrow & & \downarrow \psi \\
\mathbb{Z}_{pq} & \xrightarrow{\phi'} & \mathbb{Z}_{pq}
\end{array}
$$

he above diagram commutes where $\phi(k,\ell) = (k+1,\ell+1)$ and $\phi'(i) = i+1$. By the Chinese Remainder Theorem there exists integers a and b such that $a \equiv 1 \bmod q$, $a \equiv 0 \bmod p$ and $b \equiv 0 \bmod q$, $b \equiv 1 \bmod p$ where a and b are unique mod pq. Taking $\psi(k,\ell) = ak + b\ell$ we have desired homomorphism. The inverse to $\psi$ is simply given by taking i to (i,i) where in the first coordinate i is taken mod q and in the second mod p which we will write as $(i_q, i_p)$.

Before giving bases for the various $C_j(S^5,E)$ we give the boundary maps for the various cells

$$\partial({}_5\sigma_{k,\ell}) = {}_4\sigma_{k+1,\ell} - {}_4\sigma_{k,\ell} + {}_4\bar{\sigma}_{k,\ell+1} - {}_4\bar{\sigma}_{k,\ell}$$

$$\partial({}_4\bar{\sigma}_{k,\ell}) = -{}_3\sigma_{k+1,\ell} + {}_3\sigma_{k,\ell} + {}_3\bar{\sigma}_{k,0}$$

$$\partial({}_4\sigma_{k,\ell}) = {}_3\sigma_{k,\ell+1} - {}_3\sigma_{k,\ell} + {}_3\sigma_{0,\ell}$$

$$\partial({}_3\sigma_{k,\ell}) = {}_2\bar{\sigma}_{k,0} - {}_2\sigma_{0,\ell}$$

$$\partial({}_3\sigma_{0,\ell}) = {}_2\sigma_{0,\ell+1} - {}_2\sigma_{0,\ell}$$

$$\partial({}_3\bar{\sigma}_{k,0}) = {}_2\bar{\sigma}_{k+1,0} - {}_2\bar{\sigma}_{k,0}$$

$$\partial({}_2\sigma_{0,}\ ) = {}_1\sigma_0$$

$$\partial({}_2\bar{\sigma}_{k,0}) = {}_1\sigma_0$$

$$\partial({}_1\sigma_0) = 0$$

$$\partial({}_0\sigma_0) = 0$$

Take as bases for the horizontal lifts $\{{}_5v_m, {}_5v_{-m}\}$ $\{{}_4v_m, {}_4v_{-m}, {}_4\bar{v}_m, {}_4\bar{v}_{-m}\}$, $\{{}_3v_m, {}_3v_{-m}, {}_3v^q_r, {}_3v^q_{-r}, {}_3v^p_s, {}_3v^p_{-s}\}$, $\{{}_2v^q_r, {}_2v^q_{-r}, {}_2v^p_s, {}_2v^p_{-s}\}$, $\{{}_1v_1, {}_1v_{-1}\}$, $\{{}_0v_1, {}_0v_{-1}\}$, for $C_j(S^5,E)$ $5 \geq j \geq 0$ respectively and where $0 \leq m \leq pq-1$, $0 \leq r \leq q-1$, $0 \leq s \leq p-1$, and

$$\left.\begin{aligned} {}_jv_m &= \sum_{i=0}^{pq-1} {}_j\sigma_i \otimes R_{-\frac{im+i}{pq}}\, e_1 \\ {}_j\bar{v}_{-m} &= \sum_{i=0}^{pq-1} {}_j\bar{\sigma}_i \otimes R_{-\frac{im+i}{pq}}\, e_2 \end{aligned}\right\} \quad 3 \leq j \leq 5$$

$${}_4\bar{v}_m = \sum_{i=0}^{pq-1} {}_4\bar{\sigma}_i \otimes R_{-\frac{im+i}{pq}}\, e_1$$

$${}_4\bar{v}_{-m} = \sum_{i=0}^{pq-1} {}_4\bar{\sigma}_i \otimes R_{-\frac{im+i}{pq}}\, e_2$$

$$ {}_j v_r^q = \sum_{k=0}^{q-1} {}_j\bar{\sigma}_{k,0} \otimes R_{-\frac{pkr}{pq}} e_1 $$

$$ {}_j v_{-r}^q = \sum_{k=0}^{q-1} {}_j\bar{\sigma}_{k,0} \otimes R_{-\frac{pkr}{pq}} e_2 $$

$$ {}_j v_s^p = \sum_{\ell=0}^{p-1} {}_j\sigma_{0,\ell} \otimes R_{-\frac{q\ell s}{pq}} e_1 \qquad j = 2,3 $$

$$ {}_j v_{-s}^p = \sum_{\ell=0}^{p-1} {}_j\sigma_{0,\ell} \otimes R_{-\frac{q\ell s}{pq}} e_2 $$

$$ {}_1 v_1 = {}_1\sigma_0 \otimes e_1 $$

$$ {}_1 v_{-1} = {}_1\sigma_0 \otimes e_2 $$

$$ {}_0 v_1 = {}_0\sigma_0 \otimes e_1 $$

$$ {}_0 v_{-1} = {}_0\sigma_0 \otimes e_2. $$

Setting

$$ A = \begin{bmatrix} I & & & & \\ & R_{\frac{1}{pq}} & & & \\ & & R_{\frac{m}{pq}} & & \\ & & & \ddots & \\ & & & & R_{\frac{pq-1}{pq}} \end{bmatrix} \quad B = \begin{bmatrix} R_{\frac{1}{pq}} & & & & \\ & \ddots & & & \\ & & R_{\frac{1+rp}{pq}} & & \\ & & & \ddots & \\ & & & & R_{\frac{1+(q-1)p}{pq}} \end{bmatrix} $$

$$C = \begin{bmatrix} R_{\frac{1}{pq}} & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & R_{\frac{1+qs}{pq}} & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & R_{\frac{1+q(p-1)}{pq}} \end{bmatrix}$$

g has matrix representations

$A$ on $C_5(S^5,E)$

$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ on $C_4(S^5,E)$

$\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$ on $C_3(S^5,E)$

$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ on $C_2(S^5,E)$

and $R_{\frac{1}{pq}}$ on $C_1(S^5,E)$ and $C_0(S^5,E)$. We now compute the boundary map with respect to the above bases.

$$\partial({}_5V_m) = \partial\left(\sum_{i=0}^{pq-1} {}_5\sigma_i \otimes R_{\frac{-im+i}{pq}} e_1\right)$$

$$= \partial\left(\sum_{i=0}^{pq-1} {}_5\sigma_{i,i} \otimes R_{\frac{-im+i}{pq}} e_1\right)$$

$$= \sum_{i=0}^{pq-1} ({}_4\sigma_{i+1,i} - {}_4\sigma_{i,i} + \bar{\sigma}_{i,i+1} - \bar{\sigma}_{i,i}) \otimes R_{-\frac{im+i}{pq}} e_1$$

$$= \sum_{i=0}^{pq-1} {}_4\sigma_{i+1,i} \otimes R_{-\frac{im+i}{pq}} + \sum_{i=0}^{pq-1} {}_4\bar{\sigma}_{i,i+1} \otimes R_{-\frac{im+i}{pq}} - {}_4v_m - {}_4\bar{v}_m$$

$$(20) \qquad = \sum_{i=0}^{pq-1} {}_4\sigma_{i+a} \otimes R_{-\frac{im+i}{pq}} + \sum_{i=0}^{pq-1} {}_4\bar{\sigma}_{i+b} \otimes R_{-\frac{im+i}{pq}} - {}_4v_m - {}_4\bar{v}_m$$

where the last step follows since (1,0) = a and (0,1) = b via $\psi$. From (20) we have

$$\partial({}_5v_m) = \sum_{i=0}^{pq-1} {}_4\sigma_{i+a} \otimes R_{-\frac{(i+a)m+(i+a)}{pq}} R_{\frac{a(m-1)}{pq}} e_1 - {}_4v_m$$

$$+ \sum_{i=0}^{pq-1} {}_4\bar{\sigma}_{i+h} \otimes R_{-\frac{(i+b)m+(i+b)}{pq}} R_{\frac{b(m-1)}{pq}} e_1 - {}_4\bar{v}_m$$

$$= \left(\cos \frac{2\pi a(m-1)}{pq} - 1\right) {}_4v_m + \sin \frac{2\pi a(m-1)}{pq} \, {}_4v_{-m}$$

$$+ \left(\cos \frac{2\pi b(m-1)}{pq} - 1\right) {}_4\bar{v}_m + \sin \frac{2\pi b(m-1)}{pq} \, {}_4\bar{v}_{-m}.$$

Similarly one gets

$$\partial({}_5v_{-m}) = -\sin \frac{2\pi a(m-1)}{pq} \, {}_4v_m + \left(\cos \frac{2\pi a(m-1)}{pq} - 1\right) {}_4v_{-m}$$

$$-\sin \frac{2\pi b(m-1)}{pq} \, {}_4\bar{v}_m + \left(\cos \frac{2\pi b(m-1)}{pq} - 1\right) {}_4\bar{v}_{-m}.$$

$$\partial({}_4v_m) = \partial\left(\sum_{i=0}^{pq-1} {}_4\sigma_i \otimes R_{-\frac{im+i}{pq}} e_1\right)$$

$$= \partial\left(\sum_{i=0}^{pq-1} {}_4\sigma_{i,i} \otimes R_{-\frac{im+i}{pq}} e_1\right)$$

$$= \sum_{i=0}^{pq-1} ({}_3\sigma_{i,i+1} - {}_3\sigma_{i,i} + {}_3\sigma_{0,\ell}) \otimes R_{-\frac{im+i}{pq}} e_1$$

$$= \sum_{i=0}^{pq-1} {}_3\sigma_{i,i+1} \otimes R_{-\frac{im+i}{pq}} e_1 - \sum_{i=0}^{pq-1} {}_3\sigma_{i,i} \otimes R_{-\frac{im+i}{pq}} e_1$$

$$+ \sum_{i=0}^{pq-1} {}_3\sigma_{0,i} \otimes R_{-\frac{im+i}{pq}} e_1$$

$$= \sum_{i=0}^{pq-1} {}_3\sigma_{i+b} \otimes R_{-\frac{im+i}{pq}} e_1 - \sum_{i=0}^{pq-1} {}_3\sigma_i \otimes R_{-\frac{im+i}{pq}} + \sum_{i=0}^{pq-1} {}_3\sigma_{0,i} \otimes R_{-\frac{im+i}{pq}} e_1$$

$$= (\cos \frac{2\pi b(m-1)}{pq} - 1)\, {}_3v_m + \sin \frac{2\pi b(m-1)}{pq}\, {}_3v_{-m} + \sum_{i=0}^{pq-1} {}_3\sigma_{0,i} \otimes R_{-\frac{im+i}{pq}} e_1$$

Examining the last term in the above expression we get

$$\sum_{i=0}^{pq-1} {}_3\sigma_{0,i} \otimes R_{-\frac{im+i}{pq}} = \sum_{\ell=0}^{p-1} {}_3\sigma_{0,\ell} \otimes \left(\sum_{j=0}^{q-1} R_{-\frac{(pj+\ell)(m-1)}{pq}} e_1\right).$$

To evaluate the sum $\sum_{j=0}^{q-1} R_{-\frac{(pj+\ell)(m-1)}{pq}}$ we need to consider two cases. First we suppose that $m = 1+qs$. Then

$$\sum_{j=0}^{q-1} R_{-\frac{(pj+\ell)(m-1)}{pq}} = \sum_{j=0}^{q-1} R_{-\frac{(pj+\ell)(qs)}{pq}}$$

$$= R_{-\frac{q\ell s}{pq}} \left(\sum_{j=0}^{q-1} R_{-j}\right)$$

$$= R_{-\frac{q\ell s}{pq}}$$

If $m \neq 1+qs$ we have

$$\sum_{j=0}^{q-1} R_{-\frac{(pj+\ell)(m-1)}{pq}} = R_{-\frac{\ell(m-1)}{pq}} \left(\sum_{j=0}^{q-1} R_{-\frac{j(m-1)}{q}}\right) = 0.$$

Thus

$$\partial({}_4v_m) = \begin{cases} (\cos \frac{2\pi bs}{p} - 1)_3\ {}_{1+qs} + \sin \frac{2\pi bs}{p}\ {}_3v_{-(1+qs)} + q({}_3v_s^p) & m = 1+qs \\ (\cos \frac{2\pi b(m-1)}{pq} - 1)_3 v_m + \sin \frac{2\pi b(m-1)}{pq}\ {}_3v_{-m} & \text{otherwise.} \end{cases}$$

Similarly

$$\partial({}_4v_{-m}) = \begin{cases} -\sin \frac{2\pi bs}{p}\ {}_3v_{1+qs} + (\cos \frac{2\pi bs}{p} - 1)_3 v_{-(1+qs)} + q({}_3v_{-s}^p) & m = 1+qs \\ -\sin \frac{2\pi b(m-1)}{pq}\ {}_3v_m + (\cos \frac{2\pi b(m-1)}{pq} - 1)_3 v_{-m} & \text{otherwise.} \end{cases}$$

$$\partial({}_4\bar{v}_m) = \begin{cases} (-\cos \frac{2\pi ar}{q} + 1)_3 v_{1+pr} - \sin \frac{2\pi ar}{q}\ {}_3v_{-(1+pr)} + p({}_3v_r^q) & m = 1+pr \\ (-\cos \frac{2\pi a(m-1)}{pq} + 1)_3 v_m - \sin \frac{2\pi a(m-1)}{pq}\ {}_3v_{-m} & \text{otherwise} \end{cases}$$

$$\partial_4(\bar{v}_{-m}) = \begin{cases} \sin \frac{2\pi ar}{q}\ {}_3v_{1+pr} + (-\cos \frac{2\pi ar}{q} + 1)_3 v_{-(1+pr)} + p({}_3v_{-r}^q) & m = 1+pr \\ \sin \frac{2\pi a(m-1)}{pq}\ {}_3v_m + (-\cos \frac{2\pi a(m-1)}{pq} + 1)_5 v_{-m} & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \partial({}_3v_m) &= \partial\Big(\sum_{i=0}^{pq-1} {}_3\sigma_i \otimes R_{\frac{-im+i}{pq}}\, e_1\Big) \\ &= \sum_{i=0}^{pq-1} {}_3\sigma_{i,i} \otimes R_{\frac{-im+i}{pq}}\, e_1 \\ &= \sum_{i=0}^{pq-1} ({}_2\bar{\sigma}_{i,0} - {}_2\sigma_{0,i}) \otimes R_{\frac{-im+i}{pq}}\, e_1 \\ &= \sum_{i=0}^{pq-1} {}_2\bar{\sigma}_{i,0}\, R_{\frac{-im+i}{pq}}\, e_1 - \sum_{i=0}^{pq-1} {}_2\sigma_{0,i} \otimes R_{\frac{-im+i}{pq}}\, e_1 \\ &= \sum_{k=0}^{q-1} {}_2\bar{\sigma}_{k,0} \otimes \Big(\sum_{j=0}^{p-1} R_{\frac{-qj+k)(m-1)}{pq}}\, e_1\Big) - \sum_{\ell=0}^{p-1} {}_2\sigma_{0,\ell} \otimes \Big(\sum_{j=0}^{q-1} R_{\frac{-pj+\ell)(m-1)}{pq}}\, e_1\Big) \end{aligned}$$

$$= \begin{cases} p \sum_{k=0}^{q-1} {}_2\bar{\sigma}_{k,0} \otimes R_{-\frac{pkr}{pq}} e_1 & m = 1+pr \\ -q \sum_{\ell=0}^{p-1} {}_2\sigma_{0,\ell} \otimes R_{-\frac{q\ell s}{pq}} e_1 & m = 1+qs \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} p \; {}_2v_r^q & m = 1+pr \\ -q \; {}_2v_s^q & m = 1+qs \\ 0 & \text{otherwise .} \end{cases}$$

Similarly one has

$$\partial({}_3v_{-m}) = \begin{cases} p \; {}_2v_{-r}^q & m = 1+pr \\ -q \; {}_2v_{-s}^p & m = 1+qs \\ 0 & \text{otherwise.} \end{cases}$$

$$\partial({}_3v_r^q) = \partial\left(\sum_{k=0}^{q-1} {}_3\bar{\sigma}_{k,0} \otimes R_{-\frac{pkr}{pq}} e_1\right)$$

$$= \sum_{k=0}^{q-1} ({}_2\bar{\sigma}_{k+1,0} - {}_2\bar{\sigma}_{k,0}) \otimes R_{-\frac{pkr}{pq}} e_1$$

$$= \sum_{k=0}^{q-1} {}_2\bar{\sigma}_{k+1,0} \otimes R_{-\frac{pkr}{pq}} e_1 - \sum_{k=0}^{q-1} {}_2\bar{\sigma}_{k,0} \otimes R_{-\frac{pkr}{pq}} e_1$$

$$= \sum_{k=0}^{q-1} {}_2\bar{\sigma}_{k+1,0} \otimes R_{-\frac{p(k+1)r}{pq}} R_{\frac{r}{q}} e_1 - {}_2v_r^q$$

$$= (\cos \tfrac{2\pi r}{q} - 1)\, {}_2v_r^q + \sin \tfrac{2\pi r}{q} \; {}_2v_{-r}^q$$

Similarly one has

$$\partial({}_3v^q_{-r}) = -\sin\frac{2\pi r}{q}\ {}_2v^q_r + (\cos\frac{2\pi r}{q} - 1)\,{}_2v^q_{-r}$$

$$\partial({}_3v^p_s = (\cos\frac{2\pi s}{p} - 1)\,{}_2v^p_s + \sin\frac{2\pi s}{p}\ {}_2v^p_{-s}$$

$$\partial({}_3v^p_{-s}) = -\sin\frac{2\pi s}{p}\ {}_2v^p_s + (\cos\frac{2\pi s}{p} - 1)\,{}_2v^p_{-s}.$$

$$\partial({}_2v^q_r) = \partial(\sum_{k=0}^{q-1} {}_2\bar{\sigma}_{k,0} \otimes R_{-\frac{pkr}{pq}} e_1)$$

$$= \sum_{k=0}^{q-1} {}_1\sigma_0 \otimes R_{-\frac{pkr}{pq}} e_1$$

$$= {}_1\sigma_0 \otimes (\sum_{k=0}^{q-1} R_{-\frac{kr}{q}} e_1)$$

$$= \begin{cases} q\ {}_1\sigma_0 \otimes e_1 & r=0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} q\ {}_1v_1 & r=0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly one has

$$\partial({}_2v^q_{-r}) = \begin{cases} q\ {}_1v_{-1} & r=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\partial({}_2v^p_s) = \begin{cases} p\ {}_1v_1 & s=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\partial({}_2v^p_{-s}) = \begin{cases} p\ {}_1v_{-1} & s=0 \\ 0 & \text{otherwise} \end{cases}$$

Finally $\partial({}_1v_1) = \partial({}_1v_{-1}) = \partial({}_0v_1) = \partial({}_0v_{-1}) = 0$.

To compute the torsion $\tau_{\rho_m}(S^5,E)$ we will consider four cases; $m = 1$, $m = 1+qs$, $1 \leq s \leq p-1$, $m = 1+pr$, $1 \leq r \leq q\leq 1$ and $m$ not any of the above. For $R_{\frac{1}{pq}}$, ie. where $m = 1$ or $r = s = 0$, we have the cochain complex

$$(5.3.1) \qquad \{{}_0v_1^*, {}_0v_{-1}^*\} \xrightarrow{d_0^1} \{{}_1v_1^*, {}_1v_{-1}^*\} \xrightarrow{d_1^1} \{{}_2v_0^{q^*}, {}_2v_{-0}^{q^*}, {}_2v_0^{p^*}, {}_2v_{-0}^{p^*},\} \xrightarrow{d_2^1}$$

$$\{{}_3v_1^*, {}_3v_{-1}^*, {}_3v_0^{q^*}, {}_3v_{-0}^{q^*}, {}_3v_0^{p^*}, {}_3v_{-0}^{p^*}\} \xrightarrow{d_3^1} \{{}_4v_1^*, {}_4v_{-1}^*, {}_4\overline{v}_1^*, {}_4\overline{v}_{-1}^*\} \xrightarrow{d_4^1}$$

$$\{{}_5v_1^*, {}_sv_{-1}^*\} \xrightarrow{d_5^1} \{0\}$$

with

$$d_5^1 = 0 \qquad d_4^1 = [0 \quad 0] \qquad d_3^1 = \begin{bmatrix} 0 & 0 & qI \\ 0 & pI & 0 \end{bmatrix}$$

$$d_2^1 = \begin{bmatrix} pI & -qI \\ 0 & 0 \end{bmatrix} \qquad d_1^1 = \begin{bmatrix} qI \\ pI \end{bmatrix}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ except in $d_5^1$.

$\tau_{\rho_1}(S^5,E) = \dfrac{m_0m_2m_4}{m_1m_3m_5}$ where the $m_i$ are defined by the folloqing equations.

$$\rho_0 \Lambda d_0^{1^*}(\rho_1) \Lambda \pi^*(\mu_0) = m_0\omega_0$$

$$\rho_1 \Lambda d_1^{1^*}(\rho_2) = m_1\omega_1$$

$$\rho_2 \wedge d_2^{1*}(\rho_3) = m_2\omega_2$$

$$\rho_3 \wedge d_3^{1*}(\rho_4) = m_3\omega_3$$

$$\rho_4 \wedge d_4^{1*}(\rho_5) = m_4\omega_4$$

$$\rho_5 \wedge \pi^*(\mu_5) = m_5\omega_5.$$

With the preferred volume forms $\omega_0 = {}_0v_1 \wedge {}_0v_{-1} = \pi^*(\mu_0)$
$\omega_1 = {}_1v_1 \wedge {}_1v_{-1}$, $\omega_2 = {}_2v_0^q\ {}_2v_{-0}^q \wedge {}_2v_0^p \wedge {}_2v_{-0}^p$,
$\omega_3 = {}_3v_1 \wedge {}_3v_{-1} \wedge {}_3v_0^q, \wedge {}_3v_{-0}^q \wedge {}_3v_0^p \wedge {}_3v_{-0}^p$, $\omega_4 = {}_4v_1 \wedge_4 {}_{-1} \wedge {}_4\overline{v}_1 \wedge {}_4\overline{v}_{-1}$,
$\omega_5 = {}_5v_1 \wedge {}_5v_{-1} = \pi^*(\mu_5)$ and taking $\rho_5 = 1$, $\rho_4 = \omega_4$,
$\rho_3 = {}_3v_1 \wedge {}_3v_{-1}$, $\rho_2 = {}_2v_0^q \wedge {}_2v_{-0}^q$, $\rho_1 = 1$, and $\rho_0 = 1$ we get
$m_0 = 1$, $m_1 = q^2$, $m_2 = q^2$, $m_3 = p^2q^2$, $m_4 = 1$, and $m_5 = 1$. Thus
$\tau_{\rho_1}(S^5, E) = \dfrac{1}{p^2q^2}$.

Now suppose $m = 1+pr \quad 1 \le r \le q-1$, then we have the cochain complex

$$\{0\} \xrightarrow{d_0^r} \{0\} \xrightarrow{d_1^r} \{{}_2v_r^{q*}, {}_2v_{-r}^{q*}\} \xrightarrow{d_2^r} \{{}_3v_{1+pr}^*, {}_3v_{-(1+3r)}^*, {}_3v_r^{q*}, {}_3v_{-r}^{q*}\} \xrightarrow{d_3^r}$$

$$\{{}_4v_{1+pr}^*, {}_4v_{-(1+pr)}^*, {}_4\overline{v}_{1+pr}^*, {}_4\overline{v}_{-(1+pr)}^*\} \xrightarrow{d_4^r} \{{}_5v_{1+pr}^*, {}_5v_{-(1+pr)}^*\} \xrightarrow{d_5^r} 0$$

with

$$d_5^r = 0 \qquad d_4^r = [(R_{\frac{r}{q}} - I)^t \qquad 0]$$

$$d_3^r = \begin{bmatrix} 0 & 0 \\ (-R_{\frac{r}{q}} + I)^t & pI \end{bmatrix} \qquad d_2^r = \begin{bmatrix} pI \\ (R_{\frac{r}{q}} - I)^t \end{bmatrix}$$

where we have used the fact that a = 1 mod q and b = 0 mod q.

$$\tau_{\rho_{1+pr}}(S^5,E) = \frac{m_0 m_2 m_4}{m_1 m_3 m_5} \quad \text{where the } m_i \text{ are given by}$$

$$\rho_2 \wedge d_2^{r*}(\rho_3) = m_2\omega_2$$

$$\rho_3 \wedge d_3^{r*}(\rho_4) = m_3\omega_3$$

$$\rho_4 \wedge d_4^{r*}(\rho_5) = m_4\omega_4$$

$$\rho_5 = m_5\omega_5$$

and $m_0 = m_1$ are trivially equal to one.

With the preferred volume forms $\omega_2 = {}_2v_r^q \wedge {}_2v_{-r}^q$,

$\omega_3 = {}_3v_{1+pr} \wedge {}_3v_{-(1+pr)} \wedge {}_3v_r^q \wedge {}_3v_r^q$, $\omega_4 = {}_4v_{1+pr} \wedge {}_4v_{-(1+pr)} \wedge {}_4\bar{v}_{(1+pr)} \wedge {}_4\bar{v}_{-(1+pr)}$,

$\omega_5 = {}_5v_{1+pr} \wedge {}_5v_{-(1+pr)}$ and taking $\rho_5 = \omega_5$,

$\rho_4 = {}_4\bar{v}_{1+pr} \wedge {}_4\bar{v}_{-(1+pr)}$, $\rho_3 = {}_3v_{1+pr} \wedge {}_3v_{-(1+pr)}$, $\rho_2 = 1$ we get

$m_5 = 1$, $m_4 = \det(R_{\frac{r}{q}} - I)$, $m_3 = p^2$, and $m_2 = p^2$. Thus

$$\tau_{\rho_{1+3r}}(S^5,E) = \det(R_{\frac{r}{q}} - I).$$

For m = 1+qs we have the cochain complex

$$\{0\} \xrightarrow{d_0^s} \{0\} \xrightarrow{d_1^s} \{{}_2v_s^{p*}, {}_2v_{-s}^{p*}\} \xrightarrow{d_2^2} \{{}_3v_{1+qs}^*, {}_3v_{-(1+qs)}^*, {}_3v_s^{p*}, {}_3v_{-s}^{p*}\} \xrightarrow{d_3^s}$$

$$\{{}_4v_{1+qs}^*, {}_4v_{-(1+qs)}^*, {}_4\bar{v}_{1+qs}^*, {}_4\bar{v}_{-(1+qs)}^*\} \xrightarrow{d_4^s} \{{}_5v_{1+qs}^*, {}_5v_{-(1+qs)}^*\} \xrightarrow{d_5^s} 0$$

with

$$d_5^s = 0 \qquad d_4^s = [0 \quad (R_{\frac{s}{p}} - I)^t]$$

$$d_3^s = \begin{bmatrix} (R_{\frac{s}{p}} - I)^t & qI \\ 0 & 0 \end{bmatrix} \qquad d_2^s = \begin{bmatrix} -qI \\ (R_{\frac{s}{p}} - I)^t \end{bmatrix} .$$

Then a computation similar to the previous one shows that
$\tau_{\rho_{1+qs}}(S^5,E) = \det(R_{\frac{s}{p}} - I)$.

Finally for m not one of the previous cases we have the cochain complex

$$\{0\} \xrightarrow{d_0^m} \{0\} \xrightarrow{d_1^m} \{0\} \xrightarrow{d_2^m} \{{}_3v_m^*,\ {}_3v_{-m}^*\} \xrightarrow{d_3^m} \{{}_4v_m^*,\ {}_4v_{-m}^*,\ {}_4\overline{v}_m^*,\ {}_4\overline{v}_{-m}^*\} \xrightarrow{d_4^m}$$

$$\{{}_5v_m,\ {}_5v_{-m}^*\} \xrightarrow{d_5^m} 0$$

with

$$d_5^m = 0 \qquad d_4^m = [(R_{\frac{a(m-1)}{pq}} - I)^t \ (R_{\frac{b(m-1)}{pq}} - I)^t]$$

$$d_3^m = \begin{bmatrix} (R_{\frac{b(m-1)}{pq}} - I)^t \\ (-R_{\frac{a(m-1)}{pq}} + I)^t \end{bmatrix} .$$

With preferred volume forms $\omega_3 = {}_3v_m \Lambda {}_3v_{-3}$, $\omega_4 = {}_4v_m \Lambda {}_4v_{-m} \Lambda {}_4\overline{v}_m \Lambda {}_4\overline{v}_{-m}$, and $\omega_5 = {}_5v_m \Lambda {}_5v_{-m}$ and taking $\rho_5 = \omega_5$, $\rho_4 = {}_4v_m \Lambda {}_4v_{-m}$, and $\rho_3 = 1$ yields $m_0 = m_1 = m_2 = m_5 = 1$ and $m_4 = m_3 = \det(R_{\frac{b(m-1)}{pq}} - I)$ and thus $\tau_{\rho_m}(S^5,E) = 1$.

In summary we have

We begin by computing $\tau_{\rho_m}(S^5, F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E)$. The relative cochain complex $C^*_{\rho_m}(S^5, F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E)$ is identical to the complex in (5.3.1) for $m \neq 1$ hence $\tau_{\rho_m}(S^5, F(\mathbb{Z}_q) \cup F(\mathbb{Z}_p), E) = 1$ for $m \neq 1$. For m=1 we have the cochain complex

$$(5.3.4) \qquad \{{}_3v_1^*,\ {}_3v_{-1}^*\} \xrightarrow{d_3^1} \{{}_4v_1^*,\ {}_4v_{-1}^*,\ {}_4\bar{v}_1^*,\ {}_4\bar{v}_{-1}^*\} \xrightarrow{d_4^1} \{{}_5v_1^*,\ {}_5v_{-1}^*\}$$

where $d_3^1 = 0$ and $d_4^1 = 0$ and hence $\tau_{\rho_1}(S^5, F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) = 1$.

The relative cochain complex $C^*_{\rho_1}(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E)$ is

$$(5.3.5) \qquad \{{}_2v_r^{q^*},\ {}_2v_{-r}^{q^*}\} \xrightarrow{d_2^r} \{{}_3v_r^{q^*},\ {}_3v_{-r}^{q^*}\}$$

for $m = 1+pr$, $r \neq 0$,

$$(5.3.6) \qquad \{{}_2v_s^{p^*},\ {}_2v_{-s}^{p^*}\} \xrightarrow{d_2^s} \{{}_3v_s^{q^*},\ {}_3v_{-s}^{q^*}\}$$

for $m = 1+qs$, $s \neq 0$,

and

$$(5.3.7) \qquad \{{}_0v_1^*,\ {}_0v_{-1}^*\} \xrightarrow{d_0^1} \{{}_1v_1^*,\ {}_1v_{-1}^*\} \xrightarrow{d_1^1} \{{}_2v_0^{q^*},\ {}_2v_{-0}^{q^*},\ {}_2v_0^{p^*},\ {}_2v_{-0}^{p^*}\} \xrightarrow{d_2^1}$$
$$\{{}_3v_0^{q^*},\ {}_3v_{-0}^{q^*},\ {}_3v_0^{p^*},\ {}_3v_{-0}^{p^*}\}$$

for m=1, where $d_2^r = (R_{\frac{r}{q}} - I)^t$, $d_2^s = (R_{\frac{s}{p}} - I)^t$,

$d_0^1 = 0$, $d_1^1 = \begin{bmatrix} qI \\ qI \end{bmatrix}$, and $d_2^1 = 0$.

As in (5.2.1) we have $\tau_{1+pr}(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) = 4 \sin^2(\frac{\pi r}{q})$ and $\tau_{1+ps}(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) = 4 \sin^2(\frac{\pi s}{p})$ where $r, s \neq 0$. For m=1

$$(5.3.2)\quad \tau_{\rho_m}(S^5,E) = \begin{cases} \dfrac{1}{p^2q^2} & m=1 \quad r=0 \quad s=0 \\ 4\sin^2(\frac{r}{q}) & m = 1+pr \quad 1 \le r \le q-1 \\ 4\sin^2(\frac{s}{p}) & m = 1+qs \quad 1 \le s \le p-1 \\ 1 & \text{otherwise.} \end{cases}$$

In Section 4 we derived the formula $\tau_\rho(X,E) = \prod_{H\in I} \tau_\rho(F(H),F^H,E)\tau_\rho(H_H^r)\tau_\rho(H_H^M)$. We will use this formula to recompute the torsions $\tau_{\rho_m}(S^5,E)$ and show directly that $\tau_{\rho_m}(H_e^m) = 1$.

By the formula in the previous paragraph we have

$$\tau_{\rho_m}(S^5,E) = \tau_{\rho_m}(S^5,F(\mathbb{Z}_p)\cup F(\mathbb{Z}_q),E)\tau_{\rho_m}(H_e^r)\tau_{\rho_m}(F(\mathbb{Z}_q),F(\mathbb{Z}_p\oplus\mathbb{Z}_q),E)$$
$$\tau_{\rho_m}(H_{\mathbb{Z}_q}^r)\tau_{\rho_m}(F(\mathbb{Z}_p),F(\mathbb{Z}_p\oplus\mathbb{Z}_q),E)\tau_{\rho_m}(H_{\mathbb{Z}_p}^r)\tau_{\rho_m}(F(\mathbb{Z}_p\oplus\mathbb{Z}_q),E)\tau_{\rho_m}(H_e^M).$$

To reduce the number of different computation we use Lemma 4.6 applied to $X = F(\mathbb{Z}_p)\cup F(\mathbb{Z}_q)$. This gives

$$\tau_{\rho_m}(F(\mathbb{Z}_q),F(\mathbb{Z}_p\oplus\mathbb{Z}_q),E)\tau_{\rho_m}(H_{\mathbb{Z}_q}^r)\tau_{\rho_m}(F(\mathbb{Z}_p),F(\mathbb{Z}_p\oplus\mathbb{Z}_q),E)\tau_{\rho_m}(H_{\mathbb{Z}_p}^r).$$

$\tau_{\rho_m}(F(\mathbb{Z}_p\oplus\mathbb{Z}_q),E) = \tau_{\rho_m}(F(\mathbb{Z}_p)\cup F(\mathbb{Z}_q),E)$. Thus we can rewrite the above expression for $\tau_{\rho_m}(S^5,E)$ as

$$(5.3.3)\qquad \tau_{\rho_m}(S^5,E) = \tau_{\rho_m}(S^5,F(\mathbb{Z}_p)\cup F(\mathbb{Z}_q),E)\tau_{\rho_m}(F(\mathbb{Z}_p)\cup F(\mathbb{Z}_q),E)\cdot$$
$$\tau_{\rho_m}(H_e^r)\tau_{\rho_m}(H_e^M).$$

we have preferred volume forms $\omega_0 = {}_0v_1 \wedge {}_0v_{-1}$,

$\omega_1 = {}_1v_1 \wedge {}_1v_{-1}$, $\omega_2 = {}_2v_0^q \wedge {}_2v_{-0}^q \; {}_2v_0^p \wedge {}_2v_{-0}^p$, $\omega_3 = {}_3v_0^q \wedge {}_3v_{-0}^q$, $\wedge {}_3v_0^p \wedge {}_3v_{-0}^p$

$\pi^*(\mu_0) = \omega_0$, $\pi^*(\mu_2) = (-p \; {}_2v_0^q + q \; {}_2v_0^r) \wedge (-p \; {}_2v_{-0}^q + q \; {}_2v_{-0}^p)$,

$\pi^*(\mu_3) = {}_3v_0^q \wedge {}_3v_{-0}^q \wedge {}_3v_0^p \wedge {}_3v_{-0}^p$ and taking $\rho_3 = 1$, $\rho_2 = {}_2v_0^q \wedge {}_2v_{-0}^q$, $\rho_1 = 1$, and $\rho_0 = 1$ gives $m_0 = m_3 = 1$ and $m_2 = m_1 = q^2$. Thus $\tau_{\rho_1}(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) = 1$.

Since the complexes $C^*_{\rho_m}(S^5, E)$, $C^*_{\rho_m}(S^5, F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E)$ and $C^*_{\rho_m}(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E)$ are acyclic for $m \neq 1$ we have $\tau_{\rho_m}(H_e^m) = 1$ for $m \neq 1$. From the complexes (5.3.1), (5.3.4), and (5.3.7) we have

$$H^j_{\rho_1}(S^5, E) = \begin{cases} \mathbb{R}^2 & j=0 \\ 0 & j=1 \\ 0 & j-2 \\ 0 & j=3 \\ 0 & j=4 \\ \mathbb{R}^2 & j=5 \end{cases} \qquad H^j_{\rho_1}(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) = \begin{cases} \mathbb{R}^2 & j=0 \\ 0 & j=1 \\ \mathbb{R}^2 & j=2 \\ \mathbb{R}^4 & j=3 \\ 0 & j=4 \\ 0 & j=5 \end{cases}$$

$$H^j_{\rho_1}(S^5, F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) = \begin{cases} 0 & j=0 \\ 0 & j=1 \\ 0 & j=2 \\ \mathbb{R}^2 & j=3 \\ \mathbb{R}^4 & j=4 \\ \mathbb{R}^2 & j=5 \end{cases} .$$

Therefore $H_e^r$ is

$$0 \to \mathbb{R}^2 \xrightarrow{i_*} \mathbb{R}^2 \to 0 \to 0 \to 0 \to 0 \to 0 \to \mathbb{R}^2 \xrightarrow{\partial 1_*} \mathbb{R}^2 \to 0 \to \mathbb{R}^4 \xrightarrow{\partial 2_*} \mathbb{R}^4$$

$\to 0 \to 0 \to \mathbb{R}^2 \xrightarrow{i_*} \mathbb{R}^2 \to 0$ where $i_* = I$ and

$$\partial_2^* = \begin{bmatrix} 0 & qI \\ pI & 0 \end{bmatrix} \qquad \partial_1^* = -I$$

since $\partial_4({}_4v_1) = q\ {}_3v_0^p$, $\partial_4({}_4v_{-1}) = q\ {}_3v_{-0}^p$, $\partial_4({}_4\bar{v}_1) = p\ {}_3v_0^q$, $\partial_4({}_4\bar{v}_{-1}) = p\ {}_3v_0^q$, and $\partial({}_3v_1) = p\ {}_2v_0^q - q\ {}_2v_0^p$, $\partial({}_3v_{-1}) = p\ {}_2v_{-0}^q - q\ {}_2v_{-0}^q$. Thus $\tau_{\rho_1}(H_e^r) = \frac{1}{p^2q^2}$ .

It remains to compute $\tau_{\rho_1}(H_e^M)$. From the complexes

(5.3.8) $\{{}_0v_1^*, {}_0v_{-1}^*\} \xrightarrow{d_0^1} \{{}_1v_1^*, {}_1v_{-1}^*\{ \xrightarrow{d_1^1} \{{}_2v_0^{q*}, {}_2v_{-0}^{q*}\} \xrightarrow{d_2^1} \{{}_3v_0^{q*}, {}_0v_{-0}^{q*}\}$

and

(5.3.9) $\{{}_0v_1^*, {}_0v_{-1}^*\} \xrightarrow{d_0^1} \{{}_1v_1^*, {}_1v_{-1}^*\} \xrightarrow{d_1^1} \{{}_2v_0^{p*}, {}_2v_{-0}^{p*}\} \xrightarrow{d_2^1} \{{}_3v_0^{pI}, {}_3v_{-0}^{p*}\}$

where

$d_0^1 = 0$, $d_1^1 = qI$ for (28) and $pI$ for (29), $d_2^1 = 0$ we have

$$H_{\rho_1}^j(F(\mathbb{Z}_q), E) = H_{\rho_1}^j(F(\mathbb{Z}_p, E) = \begin{cases} \mathbb{R}^2 & j=0 \\ 0 & j=1 \\ 0 & j=2 \\ \mathbb{R}^2 & j=3 \\ 0 & j=4 \\ 0 & j=5 \end{cases}$$

$H_e^M$ is the long exact in cohomology given by

$$\to H^i(F(\mathbb{Z}_p) \cup F(\mathbb{Z}_q), E) \to H^i(F(\mathbb{Z}_p), E) \oplus H^i(F(\mathbb{Z}_q), E) \to H^i(F(\mathbb{Z}_p \oplus \mathbb{Z}_q), E) \to 0$$

and hence by the above $H_e^M$ is

$$\mathbb{R}^2 \xrightarrow{\partial_*} \mathbb{R}^4 \xrightarrow{\partial_*} \mathbb{R}^2 \to 0 \to 0 \to \mathbb{R}^2 \xrightarrow{\partial_*} 0 \to 0 \to 0 \to \mathbb{R}^4 \xrightarrow{i_*} \mathbb{R}^4 \to 0 \to 0 \to 0$$

where

$$i_* = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \qquad _* = 0, \qquad j_* = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad r_* = [0 \quad I].$$

and hence $\tau_{\rho_1}(H_e^M) = 1$.

Thus by (5.3.3) we have

$$\tau_{\rho_m}(S^5, E) = \begin{cases} \dfrac{1}{p^2q^2} & m=1 \quad r=0 \quad s=0 \\ 4\sin^2\left(\frac{\pi r}{q}\right) & m = 1+pr \quad r\neq 0 \\ 4\sin^2\left(\frac{\pi s}{p}\right) & m = 1+qs \quad s\neq 0 \\ 1 & \text{otherwise} \end{cases}$$

which agrees with our previous computation.

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