Surfaces of Constant Mean Curvature One
in Hyperbolic Space

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Gary Lynn Kerbaugh

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

H. Blaine Lawson, Professor of Mathematics
Dissertation Director

Detlef Gromoll, Professor of Mathematics
Chairman of Defense

Claude LeBrun, Assistant Professor of Mathematics

William I. Weisberger, Professor,
Institute of Theoretical Physics
Outside member

This dissertation is accepted by the Graduate School.

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Abstract of the Dissertation

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This paper investigates surfaces of constant mean curvature one in hyperbolic space. By a correspondence of H. B. Lawson, these surfaces are strongly related to minimal surfaces in Euclidean Space.

The study begins by defining for each mean curvature one surface, a pair of Gauss maps. In the minimal surface case, these Gauss maps are defined but are coincident. Complex analytic techniques are also introduced by means of R. Bryant's holomorphic lifting of these surfaces to null curves in $SL(2,\mathbb{C})$. Here the twistor techniques of N. Hitchin are employed to obtain an explicit "Weierstrass representation" of these curves in terms of two holomorphic functions (the

Gauss maps). This representation is then used to examine the global behavior of these surfaces, which differs drastically from that of minimal surfaces. Interestingly, finite total curvature is shown to be no restriction on the asymptotic behavior of these surfaces, nor is it quantized, as in the case of minimal surfaces. It is also shown that the only polynomial null curves are horospheres.

Finally, numerous examples are constructed and studied. Surfaces of finite total curvature are generated that have an arbitrary finite set of points in their asymptotic boundary.

This paper is dedicated to my mother.

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The author delights in taking this opportunity to express his gratitude to the faculty and staff of the Mathematics Department of Stony Brook for providing an incomparable atmosphere for research. A special expression of gratitude is due Professor H. Blaine Lawson, whose unfailing wisdom and support have made this work not only possible, but quite pleasurable. Also, special thanks are due Professor Claude R. LeBrun, for his sage consultation and support, and to Mrs. Lisa Koch-Sen for assistance in the refinement of vague ideas. The author would like to express respect and gratitude to Mrs. Estella Shivers for transforming this manuscript into intelligible form. Finally, the author expresses thanks for things too numerable to mention to Mrs. Mildred A. Kerbaugh.

Introduction

One of the most useful and widely applied endeavors of mathematics is the study of extremal problems, that is, the search for configurations of a system on which some important scaler quantity assumes a critical value. Hamilton founded his theory of classical mechanics on extremal principles and it was once remarked to the author by a physicist, that all of physics could be couched in these terms. A corresponding branch of engineering is optimal control theory, and the name alone suggests the potential value of such techniques. The theory of smooth problems in optimal control is even sometimes expressed using the notation of Hamiltonian classical mechanics, so techniques in this area have some potential for direct applicability.

We are concerned with a multi-dimensional version of this subject for which the scalar quantity is area, i.e., the study of surfaces in space on which the area functional assumes an extreme value over some restricted class of similarly defined surfaces. The simplest problem of this type is that of two-dimensional surfaces in three-dimensional euclidean space. A beautiful resolution of this problem was given by Weierstrass in the mid-nineteenth century who transformed the problem to a particular complex analytic setting where the minimal surface equations reduce to the Cauchy-Riemann equations. Such an approach to problem solving is the central theme of a school of mathematics founded by Roger Penrose called twistor theory. There is an account of Weierstrass' theory in twistor terms and this paper essentially applies that theory to a closely related problem, resolving it in a similar way. The original move in this direction was by Robert Bryant and the results of this paper are an extension of his work.

It has long been known that a particular perturbation of the minimal surface equations in euclidean space were the equations of surfaces of constant mean curvature in spaces of constant sectional curvature. The cases of constant negative sectional curvature are rescalings of hyperbolic space and it is the problem of representing surfaces of constant mean curvature one in hyperbolic space that this paper addresses.

We begin with an expository account of hyperbolic space and then develop a theory of surfaces in euclidean space and hyperbolic space that is tailored to our particular needs. In these terms we give, in Section 3, a brief account of the work of Weierstrass. That account centers around the application of techniques of complex analysis to the problem and thereby motivates a change of setting.

In Section 4 we present some of the main tools of twistor theory and use them to describe three-dimensional twistor theory. Then in Section 5 we again present the work of Weierstrass, this time in twistor terms.

In the next two sections we specialize to hyperbolic space, first describing the association between minimal surfaces in euclidean space and mean curvature one surfaces in hyperbolic space, and then presenting Robert Bryant's results. We follow with a theorem on polynomial solutions and then, in Section 9, describe the Gauss maps on which the twistor theory is based, in Bryant's setting.

In Section 10 we present, in three different ways, the main result, which generates a surface of mean curvature one in hyperbolic space, from a pair of arbitrary holomorphic Gauss maps. Then in Section 11 we use a version of this result to show that one of the above Gauss maps, when defined at a singularity, describes the asymptotic behavior of the surface at that singularity and then we examine examples which illustrate some of the important properties of these surfaces.

Hyperbolic Space

The central setting of this paper is three-dimensional Hyperbolic space, which will be denoted $\mathbb{H}^3$. The name was given by Poincaré, who studied Hyperbolic space as one of the first examples of non-euclidean geometry. Since complex analysts consider straight lines in the complex plane $\mathbb{C}$, to be simply a special kind of circle, that is, one containing infinity, it seems natural to consider a geometry where circles replace straight lines in Euclid's axioms (weakening the parallel axiom, of course). The result is called Hyperbolic geometry and, as the literature concerning it is extensive, we shall be content with a brief description noting pertinent objects.

In modern mathematics, $\mathbb{H}^3$ is the three-dimensional space form of constant sectional curvature -1. If we denote $\mathbb{R}(1,3)$ as the 4-dimension real vector space equipped with the Lorentz norm:

$$\|\vec{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

and the Lorentz metric, which is the polorization of the Lorentz norm, then $\mathbb{H}^3$ can be represented as the submanifold of points, $\vec{x}$, such that

$$\|\vec{x}\| = 1, \ x_0 > 0,$$

with the induced metric. Because, like the sphere, this submanifold is the level set of a non-degenerate quadratic form, it is often referred to as the pseudo-sphere. It is a Riemannian manifold because the tangent space is space-like; that is, the induced metric is definite. The induced connection has constant sectional curvature -1, and in this connection the geodesics also resemble those of the sphere in that they are given, as point sets, by intersection of the pseudo-sphere with two-planes containing the origin. Unlike the sphere, however, this manifold is complete but not compact. The vectors in Minkowski space, $\mathbb{R}(1,3)$, whose Lorentz norm vanishes, are called null, and the set of such points is referred to as the light cone. The pseudo-sphere approaches the light cone asymptotically; indeed from a sufficient distance from the origin they would appear indistinguishable. This can be made more precise. In a paper, [Eberlein-O'Neill], Eberlein and O'Neill define the asymptotic boundary of a hyperbolic manifold to be the set of equivalence classes of geodesic rays, where two rays are identified if they remain a bounded distance from one another. From the above definition of geodesics, it is not difficult to see that the two-planes associated with some equivalence class would all contain a single line in the light cone. Hence the projectivized light cone can be identified with the asymptotic boundary.

There is another important difference between the sphere and the pseudo-sphere, that is, the existence of submanifolds of the pseudo-sphere which are flat in the induced metric.

These are given as point sets by intersection of the pseudo-sphere with affine null hyperplanes and are called horospheres. Such a null hyperplane defines a unique null direction and it is possible to show that the geodesics defined by the unit normal vector to the horosphere at each of its points are all contained in the equivalence class defined by that direction. Hence the horosphere has a unique point in its asympotic boundary. This characterizes them completely among embedded mean curvature one hypersurfaces of $\mathbb{H}^3$ [Lawson, DoCarmo].

Another representation of hyperbolic space, one of two actually due to Poincaré, is the upper half-space. This model consists of points in $\mathbb{R}^3$ such that $x_3 > 0$, equipped with the metric induced by the norm

$$\|\vec{x}\|^2 = \frac{(x_1^2+x_2^2+x_3^2)}{x_3^2} = ds^2/_{x_3^2} .$$

From the obvious symmetries of the metric, it is clear that vertical lines are geodesics and horizontal planes are horo-spheres. These, however, correspond to the family of horo-spheres and equivalence class of geodesics associated with just one point in the asymptotic boundary: the vertical one. The remaining ones can be gotten by "rotating" the picture by an element of the group of isometries. For a detailed treat-ment of this the reader is referred to [Ahlfors, 1].

Just as SO(4) restricts to the sphere as the orientation-preserving group of motions, similarly the connected component of the identity in the Lorentz group, called the proper Lorentz group and denoted $SO(1,3)^{\uparrow}$, restricts to the pseudosphere as the orientation-preserving group of motions. As with the sphere, the stabilizer of a point is conjugate to SO(3), the subgroup of spacial rotations. For details of this and the following representation of the action of the isometry group on $\mathbb{H}^3$, the reader is referred to [Ahlfors, 1]. To display these motions, it is necessary to consider the upper half-space as being imbedded in the quaternions such that

$$x_3 > 0 \quad \text{and} \quad x_4 = 0.$$

Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon \ SL(2,\mathbb{C})$, z ε quaternions such that $x_3 > 0$, $x_4 = 0$

$$\gamma(z) = (az+b)\cdot(cz+d)^{-1} = (cz+d)^{-1}(az+b).$$

Although the representation of the group of motions as $PSL(2,\mathbb{C})$ is the one we will be interested in, we will not use the above action, but will let $PSL(2,\mathbb{C})$ act on Minkowski space directly via the spin representation. The above representation does however, best display the action induced on the boundary. Since the boundary, minus the vertical point at infinity, is the complex numbers sitting in the quaternions, the above action restricts to the boundary as a Möbius transformation. In particular, the boundary inherits a conformal structure, that of $\mathbb{CP}_1$. We may also restrict our attention to vertical planes, in particular, one whose $x_1 x_2$ coordinates are zero and which contains the x_3 axis. This plane is not fixed by the full action above but is invariant under the action of $SL(2,\mathbb{R})$. Under the action of the real group, the quaternion j behaves like the complex i and we can again understand this action in terms of Möbius transformations. Furthermore, this picture can be rotated around the x_3 axis, so from this and the fact that fixed-point sets of isometries are totally geodesic, we see that hemispheres meeting the boundary orthogonally are totally geodesic. Also, since a Möbius transformation takes vertical straight lines to circles and because horospheres have a unique point in their asymptotic boundary, it follows that horospheres are euclidean spheres that meet the boundary tangentially.

Finally, to give an idea of how complete a picture these structures give of hyperbolic space, we note that the above objects can be used to define the map relating these two representations. Horospheres can be grouped into families according to their point at infinity. For instance, the horizontal planes in the upper half-space are the family associated with the vertical point at infinity. In Minkowski space, horospheres come from intersections of $\mathbb{H}^3$ with affine null hyperplanes and their point at infinity is determined by the null direction. Hence a family of horospheres corresponds to a family of affine null hyperplanes, all having the same null direction and intersecting the time axis on the positive side. Given this picture it is possible to visualize the following map. If we choose the null direction to be $\vec{n} = (1,1,0,0)$, then the following is an isometry from the pseudosphere to the upper half plane which associates the two families of horospheres:

$$\vec{X} : \{(x_0,x_1,x_2,x_3) \mid x_0^2-x_1^2-x_2^2-x_3^2=1\} \mapsto (\langle\vec{x},\vec{n}\rangle,x_2,x_3)$$

where $\langle , \rangle$ is the Lorentz inner-product.

Surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$

We now establish notation and describe the geometric setting in which we will be working. For calculations, we will find the representation of $\mathbb{H}^3$ as the pseudo-sphere in Minkowski space to be the most tractable representation, and we will also be talking about minimal surfaces in euclidean space, so our first order of business is to describe the geometry of the linear spaces $\mathbb{L}^4 = \mathbb{R}(1,3)$ and $\mathbb{R}^3$.

$\mathbb{L}^4$, Minkowski space, is the vector space $\mathbb{R}^4$ with coordinates that we will denote (x_o, x_1, x_2, x_3) and a pseudo-Riemannian metric induced by the norm

$$\|\vec{x}\|^2 = x_o^2 - x_1^2 - x_2^2 - x_2^2.$$

We will also consider it to be oriented by requiring $dx_o \wedge dx_1 \wedge dx_2 \wedge dx_3$ to be positive, and time oriented by letting

$$dx^\circ > 0$$

correspond to the positive time direction. We can therefore specify a subbundle of the principal bundle of pseudo-orthonormal frames with structure group $O(1,3)$. This bundle has connected fiber and will be called the principal bundle of oriented, time-oriented pseudo-orthonormal frames and will be denoted $SO(\mathbb{L}^4)^\circ$. It has fiber $\{\vec{e}_o, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$, such that

$$\langle \vec{e}_\alpha, \vec{e}_\beta \rangle = \begin{cases} 1 & \alpha = \beta = 0 \\ -1 & \alpha = \beta > 0 \\ 0 & \alpha \neq \beta \end{cases}$$

$$\vec{e}_o \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 > 0 \text{ and } dx^o(\vec{e}_o) > 0,$$

and structure group which is the identity component of the full Lorentz group, $O(1,3)$, and is reduced from $O(1,3)$ by the requirement that transformations preserve the above orientations. This structure group is often referred to as the Lorentz group and will be denoted by

$$SO(1,3)^{\uparrow}$$

where the arrow refers to "time orientation preserving."

This action is free and transitive (by Witt's theorem) so the frames can be identified with elements of the group in a one-to-one fashion given a choice of pseudo-orthonormal coordinates, x_α, on Minkowski space. One associates the frame $\{e_\alpha\}$ with the unique element g of $SO(1,3)^{\uparrow}$ such that

$$\partial/\partial x_\alpha \cdot g = \vec{e}_\alpha.$$

We remark at this time that we will follow the standard convention of denoting all four coordinates of Minkowski space with Greek letters and the subset of three spatial coordinates with Roman letters.

Because $SO(1,3)^{\uparrow}$ is a Lie group and because $\mathbb{L}^4$ is flat, from Maurer-Cartan equations, we have the left invariant forms

$$\{\omega^\beta_\alpha \mid d\vec{e}_\alpha = \vec{e}_\beta \omega^\beta_\alpha \ , \ \omega^\alpha_o = \omega^o_a \ , \ \text{ and } \ \omega^b_a = -\omega^a_b\},$$

which satisfy the structural equations

$$d\omega_{\alpha}^{\beta} = -\omega_{\gamma}^{\beta} \wedge \omega_{\alpha}^{\gamma} .$$

Since points in the pseudo-sphere correspond to time-like unit vectors, $\vec{e}_o$, we can write the metric on $\mathbb{H}^3$ as

$$ds^2 = \sum_a \omega_o^a \otimes \omega_o^a .$$

We have more structure associated with this group. The group $SO(1,3)^{\uparrow}$ has a 2-fold covering, called the spin covering, by the complex group $SL(2,\mathbb{C})$. To describe this covering we must represent Minkowski space as 2×2 Hermitian matrices, the correspondence being

$$(x^\circ, x^1, x^2, x^3) \mapsto \begin{pmatrix} x^\circ + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^\circ - x^3 \end{pmatrix}.$$

The square of the Lorentz norm corresponds to determinant, and the induced inner product is

$$\frac{1}{2}\, \mathrm{tr}(AB^{co}) \qquad A,B \text{ Hermitian.}$$

B^{co} refers to the cofactor matrix of B. Therefore, for every $g \in SL(2,\mathbb{C})$,

gAg^*, A Hermitian and $g^* = \bar{g}^t$, is Hermitian conjugation, is Hermitian. Because $\det(g) = 1$, the cofactor matrix is the inverse, so inner products are preserved. Hence each g corresponds to a Lorentz transformation and it can be shown that the kernel is $\pm$Id. While the explicit form of the covering is inconsequential to us, we observe in passing that the covering is quadratic.

Of significant consequence, however, is the relationship between Lie algebras. They are isomorphic and the left invariant connection form of $SL(2,\mathbb{C})$ is

$$\begin{pmatrix} \omega_o^3 + i\omega_1^2 & (\omega_o^1+i\omega_2^3) + i(\omega_o^2-i\omega_1^3) \\ (\omega_o^1+i\omega_2^3) - i(\omega_o^2-i\omega_1^3) & -(\omega_o^3+i\omega_1^2) \end{pmatrix},$$

given in terms of the connection forms defined above for $SO(1,3)^{\uparrow}$. We therefore have an almost complex structure on $SO(1,3)^{\uparrow}$ which is integrable.

Given the spin representation, we can make more precise our description of the action induced on the boundary by an isometry. Let us denote the future-pointing nappe of the light cone by

$$\mathbb{N}^3 = \{\vec{y} \in \mathbb{L}^4 | \langle\vec{y},\vec{y}\rangle = 0,\ y^\circ > 0\},$$

In the above representation $\mathbb{N}^3$ corresponds to 2×2 Hermitian matrices with determinant zero. Because the column vectors are dependent, such a matrix can be written as $\vec{\zeta}\cdot\vec{\zeta}^*$ where $\vec{\zeta}^t = (\zeta_1,\zeta_2)$ $\zeta_1,\zeta_2 \in \mathbb{C}$, is a spinor. Such a vector in $\mathbb{C}^2$ is determined by an element of $\mathbb{N}^3$ up to a factor of $e^{i\theta}$ and therefore an element of the asymptotic boundary, $\mathbb{N}^3/_{\mathbb{R}^+}$, determines a spinor up to complex multiple. This identifies the asymptotic boundary with $\mathbb{CP}^1$ and represents the action induced on the boundary as the action of Möbius transformations on $\mathbb{CP}^1$. If we normalize $y \in \mathbb{N}^3$ such that $dx^\circ(y) = 1$, then each element of that unit sphere determines the class of spinors

$$[1, g_2]$$

where g_2 is obtained from that sphere by projecting sterographically from the point (1,0,0,1). The above spin representation will play a major role in our development of hyperbolic surfaces of mean curvature one.

This development will parallel that for euclidean minimal surfaces, so we wish to present that theory in terms that will elucidate the relationship. For that we will require a frame bundle on Euclidean space comparable to the restriction to the pseudo-sphere of the bundle $SO(\mathbb{L}^4)^\circ$ over Minkowski space. A comparable way to view $\mathbb{R}^3$ is as an affine space and the comparable bundle is the bundle of oriented affine orthonormal frames, $ASO(\mathbb{R}^3)$, with structure group $ASO(3)$, the group of euclidean motions on $\mathbb{R}^3$. This group is well-known to be a semidirect product of $SO(3)$ and $\mathbb{R}_3$, and the Lie algebra is referred to as the semidirect sum of the corresponding Lie algebras. A frame in this bundle is a position vector in $\mathbb{R}^3$, $\vec{v}_o = (v_o^1, v_o^2, v_o^3)$ and an orthonormal frame $\{\vec{v}_i\}$ $i = 1,2,3$ such that $\vec{v}_1 \wedge \vec{v}_2 \wedge \vec{v}_3 > 0$, where the frame is considered to be based at the point $\vec{v}_o$. We also have the connection one-forms

$$d\vec{v}_o = \vec{v}_i \theta^i_o \qquad i,j = 1,2,3$$

$$d\vec{v}_i = \vec{v}_j \theta^j_i$$

and structural equations

1) $d\theta^j_o = -\theta^j_o \wedge \theta^k_o \quad i,j = 1,2,3$

2) $d\theta^j_i = -\theta^j_k \wedge \theta^k_i$

where the latter equations are the structural equations for SO(3). We find, therefore, that the bundle of orthonormal frames is simply the subbundle corresponding to $v_o \equiv 0$.

We will extend the notion of adapted frame (see Kobayashi and Nomizu, vol. 2) to this affine bundle. Given an immersion of a two-manifold into $\mathbb{R}^3$,

$$\vec{W} : M^2 \hookrightarrow \mathbb{R}^3,$$

and projection π,

$$\pi(\{\vec{v}_o, \vec{v}_1, \vec{v}_2, \vec{v}_3\}) \mapsto \{\vec{v}_1, \vec{v}_2, \vec{v}_3\},$$

reducing the bundle of affine frames to the ordinary orthonormal frame bundle, we define an affine orthonormal frame u, to be adapted along $\vec{W}$ if

3) $W(p) = \vec{v}_o(p) \qquad p \in M$

4) $\pi(u)$ is adapted in the usual sense.

This is a principal bundle on M with structure group

$O(1,\mathbb{R}) \times O(2,\mathbb{R})$. If we denote the pullback to M of the connection one forms by the same symbols, we have

$$ds^2 = \theta_o^1 \otimes \theta_o^1 + \theta_o^2 \otimes \theta_o^2 ,$$

because from 4) we know that

5) $\vec{W}^*(\theta_o^3) = 0.$

Therefore, T^*M is spanned by $\{\theta_o^1, \theta_o^2\}$ and the induced area element is $dA_{\vec{W}} = \theta_o^1 \wedge \theta_o^2$ so we have the following equations for the second fundamental form of $\vec{W}$:

6) $$\begin{matrix} \theta_1^3 \\ \theta_2^3 \end{matrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{matrix} \theta_o^1 \\ \theta_o^2 \end{matrix} .$$

The mean curvature H, of the immersion is the trace of the second fundamental form, hence, $H = \beta_{11}+\beta_{22}$. Minimal surfaces are surfaces such that $H \equiv 0$.

Further, by the structural equations and 4) we have

$$d\theta_1^2 = -\theta_3^2 \wedge \theta_1^3 = \theta_1^3 \wedge \theta_2^3 = -\det\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \theta_o^1 \wedge \theta_o^2 = -KdA,$$

where the last equation is Gauss equation for hypersurfaces. It is also known (Chern 1) that locally one can find isothermal coordinates and a complex parameter z, so that, possibly after passing to a 2-fold cover, M is a Riemann surface with local parameter z, and

$$ds^2 = 2F|dz|^2.$$

Given this, we define a one-form $\theta = \theta_o^1 + i\theta_o^2$ and a function $f = \beta_{11}-H-i\beta_{12}$. θ is holomorphic because of the isothermal coordinates and the structural equations. If H is constant, f is holomorphic by the Codazzi-Mainardi equations for hypersurfaces. Because Minimal surfaces satisfy $\beta_{11}+\beta_{22} \equiv 0$ and of course, (by $\theta_o^3 = d\theta_o^3 = 0$) $\beta_{12} = \beta_{21}$ so $f = \beta_{11}-i\beta_{12}$ and we define $\eta = \theta_3^1-i\theta_3^2 = f\theta$ so η is holomorphic. It is worth noting for later reference that SO(3) has a spin covering by SU(2) and the connection one-forms 1) for SO(3) lift to an SU(2)-valued form

$$\mathbf{\Theta} = \begin{pmatrix} i\theta_1^2 & -\bar{\eta} \\ \eta & -i\theta_1^2 \end{pmatrix}.$$

Now because Ad_h, where $h \in SO(3)$, preserves the Lie algebra of the translation group, a connection on ASO(3) restricts to one on SO(3) by Proposition 6.4 II of Kobayashi and Nomizu. Thus the distribution defined by $\mathbf{\Theta}$ is integrable in SO(3), although this can be computed directly from the structural equations. The integral curve is the adapted frame of the surface defined by $\vec{v}_o$. The form $\mathbf{\Theta}$ will appear in the verification of Robert Bryant's theorem.

From the structural equations we have

$$\begin{aligned} d\theta &= -i\theta_1^2 \wedge \theta \\ d\eta &= i\theta_1^2 \wedge \eta \\ d\theta_1^2 &= -K(^i/_2)\theta\wedge\bar{\theta} = (^i/_2)\bar{\eta}\wedge\bar{\eta} . \end{aligned}$$

We can now prove the following proposition:

Proposition. If $\vec{v}_3 : M \to \mathbb{R}^3$ is conformal for a frame, $\{\vec{v}_i\}$, adapted along $\vec{W} : M \hookrightarrow \mathbb{R}^3$, then $\vec{W}$ is minimal or totally umbilic.

Proof. Conformality of this map for an adapted frame implies

$$<d\vec{v}_3(e_1),\ d\vec{v}_3(e_2)> = 0$$

$$= 2F\beta_{12}(\beta_{11}+\beta_{22}) = 0 \ .$$

If $\vec{W}$ is not minimal then $\beta_{12} = 0$ and

$$<d\vec{v}_3(\vec{e}_1),\ d\vec{v}_3(\vec{e}_1)> = <d\vec{v}_3(\vec{e}_2),\ d\vec{v}_3(\vec{e}_2)>$$

implies $\beta_{11} = \beta_{22} = -c$ for real c.

Then $\qquad \bar{\eta} = c\theta$

and $\qquad d\bar{\eta} = i\theta_1^2 \wedge \bar{\eta} = -ic\theta_1^2 \wedge \theta = dc \wedge \omega - ic\theta_1^2 \wedge \theta$

so $dc \wedge \omega = 0$ and c = constant on connected components of M.

We remark here that $\vec{v}_3$ is the traditional Gauss map for a surface in $\mathbb{R}^3$ and is anti-conformal for minimal surfaces because of the negative sign in η. The search for a Gauss map that was anti-conformal on surfaces of constant mean curvature one initiated this research.

We conclude our discussion of surfaces in euclidean space with a result that will be needed in the discussion of the generalized Gauss map of minimal surface. We will need some background.

<u>Definition</u>. Given an immersed hypersurface in $\mathbb{R}^n$, we define the second fundamental form of this immersion to be the normal component of the covariant derivative.

Specifically if the immersion of an n-1 manifold in $\mathbb{R}^n$ is given by

$$\vec{W} : M \to \mathbb{R}^n$$

and $\vec{X}$ is a section of

$$\vec{W}^*(T\,\mathbb{R}^n) ,$$

then the standard metric on $\mathbb{R}^n$ gives a splitting of this bundle into tangent and normal bundles

$$\vec{W}^*(T\,\mathbb{R}^n) = TM \oplus (\vec{W}^*(T\,\mathbb{R}^n))^N$$

with the corresponding projections denoted

$$\vec{X} = \vec{X}^T + \vec{X}^N.$$

Therefore, given vector fields $\vec{x},\vec{y}$ on M

$$II(\vec{x},\vec{y}) = (\nabla_{\vec{x}}\vec{y})^N.$$

It is a well-known fact that if the immersion is isometric then, denoting the induced connection by $\overline{\nabla}$, and the standard euclidean connection by ∇, we have

$$\overline{\nabla}_{\vec{x}}\vec{y} = (\nabla_{\vec{x}}\vec{y})^T.$$

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We will also need the following:

Definition. If $\{\vec{e}_i\}$ is an orthonormal frame on M then we define the Laplace-Beltrami operator as

$$\Delta f = \Sigma_{i=1}^{n-1} \vec{e}_i \vec{e}_i f - (\overline{\nabla}_{\vec{e}_i} \vec{e}_i) f.$$

Finally, the mean curvature is, of course, the trace of the second fundamental form. We can now give the following:

Proposition. If we consider the individual coordinates of $\vec{W} : M \hookrightarrow \mathbb{R}^n$ as functions, $(v_o^1, v_o^2, \cdots, v_o^n)$, then

$$-\Delta\vec{W} = \text{the mean curvature vector}$$

Proof.
$$\begin{aligned} \text{tr II} &= \Sigma_{i=1}^{n-1} \text{II}(\vec{e}_i, \vec{e}_i) \\ &= \Sigma_{i=1}^{n-1} (\nabla_{\vec{e}_i} \vec{e}_i)^N \\ &= \Sigma_{i=1}^{n-1} \nabla_{\vec{W}_* \vec{e}_i} W_* \vec{e}_i - \vec{W}_* (\overline{\nabla}_{\vec{e}_i} \vec{e}_i) \\ &= \Sigma_{i=1}^{n-1} \vec{e}_i \vec{e}_i (\vec{v}_o) - \nabla_{\vec{e}_i} \vec{e}_i (\vec{v}_o) \\ &= \Delta \vec{v}_o . \end{aligned}$$

This shows that the coordinate functions of a minimal surface are harmonic. This will be useful when n is three because we can find a local complex coordinate z on M such that

$$ds^2 = 2F|dz|^2$$

and
$$\Delta = \frac{2}{F} \frac{d}{dz} \frac{d}{d\overline{z}} .$$

This will show that the Gauss map for minimal surfaces is holomorphic with respect to the complex structure associated with z.

As promised, we will now give the parallel theory of surfaces in the Minkowski pseudosphere. We have already discussed the pseudo-orthonormal frame bundle and have given its structural equations. We now define adapted exactly as before: Given an immersion of a 2-manifold M into the pseudosphere,

$$\vec{X} : M \hookrightarrow \mathbb{H}^3 \subset \mathbb{R}(1,3),$$

we call a pseudo-orthonormal frame $\{\vec{e}_\alpha\}$ adapted along M

$$\vec{X}(p) = \vec{e}_o(p) \qquad p \in M$$

and $$\vec{X}_* T_p M = \text{span}\{\vec{e}_1, \vec{e}_2\},$$

where we identify the tangent space of each point of $\mathbb{R}(1,3)$ with that of the origin. Again we denote the pullback of the connection forms by the same symbol and note that for adapted frames

$$ds^2 = \omega_o^1 \otimes \omega_o^1 + \omega_o^2 \otimes \omega_o^2$$

and $$\vec{X}^*(\omega_o^3) = 0.$$

Further equations follow as before;

$$dA_{\vec{x}} = \omega_o^1 \wedge \omega_o^2 = \text{area form},$$

and since T^*M is spanned by $\{\omega_o^1, \omega_o^2\}$ we have

$$7) \quad \begin{matrix}\omega_1^3 \\ \omega_2^3\end{matrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{matrix}\omega_o^1 \\ \omega_o^2\end{matrix} \quad \text{, where, by } d\omega_o^3 = 0,\ h_{12} = h_{21}.$$

So

$$d\omega_1^2 = -\omega_3^2 \wedge \omega_1^3 - \omega_o^2 \wedge \omega_o^1 = (1+h_{12}^2 - h_{11}h_{22})\,\omega_o^1 \wedge \omega_o^2 \;,$$

and

$$K = -(1+h_{12}^2 - h_{11}h_{22}) = \text{Gauss curvature.}$$

Also as before, we will regard M^2 as a Riemann surface with induced orientation and conformal structure, and denote the compatable local coordinate by

$$z = x+iy$$

with

$$ds^2 = 2F|dz|^2.$$

We define $\omega = \omega_o^1 + i\omega_o^2$, which is holomorphic when the adapted frame agrees with the complex parameter, and we write $\tilde{f} = h_{11} - 1 - ih_{22}$, which is holomorphic when $\vec{X}$ has constant mean curvatue 1, again by the Codazzi-Mainardi equations. Hence the form

$$\tilde{\eta} = \tilde{f}\omega = (\omega_o^1 + \omega_3^1) - i(\omega_o^2 + \omega_3^2)$$

is of type (1,0), that is, holomorphic. We must then interpret its meaning.

For hypersurfaces in Euclidean space we know that the second fundamental form is a section

$$II \in H^{\circ}(M, T^*M \otimes T^*M \otimes NM),$$

where NM is the normal bundle pulled back from the adapted frame of $\mathbb{R}^3$ by $\vec{W}$. Because this bundle is a line bundle, with a metric pulled back from $\mathbb{R}^3$, we can raise and lower indices. Therefore, in the presence of an orientation, which exists globally on at worst the 2-fold cover of M, we can define the map

$$A : TM \to TM$$

by $$A(\vec{x}) = \nabla_{\vec{x}} \vec{v}_3, \quad \vec{x} \in TM$$

where $\vec{v}_3$ is a well defined choice of unit normal. We have mentioned that we can consider M as a Riemann surface and, using the complex parameter, we can consider the map A, to be a complex valued one-form. If $\{\vec{v}_i\}$ are adapted and $\vec{v}_1 \wedge \vec{v}_2$ agree with the orientation, then this form is exactly

$$\bar{\eta} = \theta^1_3 + i\theta^2_3.$$

The beginning of my research on this problem, and the reason that I felt that it would have a nice resolution was the discovery of a conformal Gauss map for constant mean curvature 1 immersions in $\mathbb{H}^3$. That map is similar to the above map but is induced from the covariant derivative of $\vec{e}_0 + \vec{e}_3$ and is exactly

$$\bar{\tilde{\eta}} = (\omega_o^1+\omega_3^1) + i(\omega_o^2+\omega_3^2),$$

and from what we have shown this map is anti-holomorphic. The alert reader should immediately object to the confusion caused by redundant use of symbols, but we will show in the verification of the immersion theorem (which asserts that either of the above constant mean curvature immersions induces an isometric immersion of the other type) that, for these pairs of maps the above functions and forms coincide when pulled back to M, where similarly named.

To substantiate this similarity, we verify a series of equations analogous to those satisfied by η. From the Gauss equation for hypersurfaces we have

$$\tilde{\eta}\wedge\bar{\tilde{\eta}} = -K\omega\wedge\bar{\omega}$$

and from the structural equations

$$d\omega = -i\omega_1^2\wedge\omega$$

$$d\tilde{\eta} = i\omega_1^2\wedge\tilde{\eta}$$

$$d\omega_1^2 = -K(i/2)\omega\wedge\bar{\omega} = (i/2)\eta\wedge\bar{\tilde{\eta}}.$$

We now have enough show that conformality of the hyperbolic Gauss map, $\vec{e}_o + \vec{e}_3$, is nearly equivalent to the immersion having mean curvature one. We note that it also parallels the case for minimal surfaces (see Lawson's thesis). We pull back the Lorentz metric via $\vec{e}_o + \vec{e}_3$ to

$$d\sigma^2 = \langle d(\vec{e}_o+\vec{e}_3), d(\vec{e}_o+\vec{e}_3)\rangle$$

$$= (\omega_o^1+\omega_3^1)^2+(\omega^2+\omega_3^2)^2$$

$$= (\omega_o^1-h_{11}\omega_o^1-h_{12}\omega_o^2)+(\omega_o^2-h_{12}\omega_o^1-h_{22}\omega_o^2)^2$$

$$=(1-2h_{11}+h_{11}^2+h_{12}^2)\omega_o^1\otimes\omega_o^1+2h_{12}(h_{11}+h_{22}-2)\omega_o^1\otimes\omega_o^2$$

$$+(1-2h_{22}+h_{22}^2+h_{12}^2)\omega_o^2\otimes\omega_o^2,$$

so by Gauss' equation and the definition of mean curvature,

$$8)\quad d\sigma^2 = [2(H^2-H) - K+(h_{11}-h_{22})(H-1)]\omega_o^1\otimes\omega_o^1 + 4h_{12}(h-1)\omega_o^1\otimes\omega_o^2$$

$$+ [2(H^2-H) - K+(h_{22}-h_{11})(H-1)]\omega_o^2\otimes\omega_o^2$$

$$= [2(H^2-H)-K]ds^2 + (H-1)\{(h_{11}-h_{22})[(\omega_o^1\otimes\omega_o^2) - (\omega_o^2\otimes\omega_o^2]$$

$$+ 4h_{12}\omega_o^2\otimes\omega_o^2\}.$$

Hence we have the following proposition:

<u>Proposition</u>. The map $\vec{e}_o + \vec{e}_3$ is conformal if and only if $\vec{X}$ is totally umbilic or has mean curvature 1. For $H = 1$ the result is clear. When $H \neq 1$ we have that $h_{12} = 0$ and $h_{11} = h_{22} = c$ for a real function c and

$$\bar{\tilde{\eta}} = (1+c)\omega.$$

Therefore, $\quad d\bar{\tilde{\eta}} = i\omega_1^2\wedge\bar{\tilde{\eta}} = -i(1+c)\omega_1^2\wedge\omega = dc\wedge\omega \ -i(1+c)\omega_1^2\wedge\omega$

so $dc = 0$ and c is a constant. Also, from

$$(\omega^1+\omega_3^1)\wedge(\omega^2+\omega_3^2) = (H-1)^2\omega^1\wedge\omega^2 > 0$$

we see that $\vec{e}_o + \vec{e}_3$ preserves orientation in the umbilic case.

<u>Note</u>. Not only is this in complete analogy to the case of minimal surfaces, but we have also shown that $K \leq 0$ because $H = 1$ above implies $d\sigma^2 = -Kds^2 \geq 0$. However, once we have established that $\eta = \tilde{\eta}$ and $f = \tilde{f}$, we will have most local results from minimal immersions automatically. We will need the above formalism to verify Robert Bryant's theorem, but we will take a little more advantage of the equivalence of the above forms. The author would like to express his gratitude to Bryant for the notation of this section which spares the reader exposure to the author's original coordinate dependent computations.

Weierstrass Representation

Let M be a connected two manifold and define a minimal immersion of M into $\mathbb{R}^n$ as an immersion whose mean curvature vanishes. We have shown that this implies harmonic coordinate functions. It is also known [Chern 1] that there exist isothermal local coordinates (x,y) such that the metric induced on the manifold M by a minimal submersion,

$$\vec{X} : M \to \mathbb{R}^n$$

has the form

$$ds^2 = 2F|dz|^2$$

where $z - x + iy$. Since transformations between such coordinates are either conformal or anti-conformal, it follows that orientability is the only obstruction to covering M by conformal atlas. We will therefore pass to the oriented double cover and henceforth assume that M is a Riemann surface, conformal immersed in $\mathbb{R}^3$ by $\vec{X}$.

In these coordinates, the Laplace-Beltrami operator has a particularly nice form given by

$$\Delta = \frac{2}{F}\frac{d}{dz}\frac{d}{d\bar{z}} \quad \text{where} \quad \frac{d}{dz} = \frac{1}{2}(\partial/\partial x - i\ \partial/\partial y).$$

We can express the fact that the coordinates are harmonic by

2) $$\frac{d}{d\bar{z}}\frac{d}{dz}\vec{X} = 0 .$$

This implies that the map into $\mathbb{C}^3$ given by

3) $$\Phi = \frac{d}{dz}\vec{X} = \frac{1}{2}\left(\vec{X}_x - i\vec{X}_y\right)$$

is holomorphic. This map determines the induced metric in the usual way by

$$g_{11} = |\vec{X}_x|^2 = 2F$$

$$g_{22} = |\vec{X}_y|^2 = 2F$$

$$g_{12} = \langle\vec{X}_x, \vec{X}_y\rangle = 0.$$

These equations are used to show two important properties possessed by this map:

4) $$\Phi^2 \stackrel{\text{def}}{=} \Sigma_{i=1}^{n} \Phi_i^2 = 0$$

5) $$\Phi^2 = F$$

Equation 4) follows from

$$|\Phi|^2 = |\vec{X}_x|^2 - |\vec{X}_y|^2 - 2i\langle\vec{X}_x, \vec{X}_y\rangle.$$

To describe the range of this map it is necessary to determine the effect of coordinate changes. If w represents a holomorphic change of parameter so that

$$\vec{X}(w) = \vec{X}(w(z)) \quad \text{then}$$

6) $$\Phi(z) = \frac{d}{dz}\vec{X}(w(z)) = \frac{d}{dw}\vec{X}(w)\cdot\frac{dw}{dz} = \Phi(w)\frac{dw}{dz}$$

so $\Phi(z)$ is a well-defined map into $\mathbb{CP}^{n-1}$. If we consider Φdz then we have a section of the cotangent bundle.

In either case, equation 4) is independent of coordinates, so Φ takes values in the projective quadric $\emptyset_{n-2} \subset \mathbb{P}_{n-1}$ defined in homogeneous coordinates by

$$\Sigma_{i=1}^{n} z_i^2 = 0.$$

This quadric is the base manifold of the bundle which will be our twistor space, so a better understanding of its geometry is in order.

The quadric, here, represents the Grassman manifold of oriented 2-planes in $\mathbb{R}^n$. This interpretation is represented by a construction similar to the above. Given a 2-plane in $\mathbb{R}^n$, choose 2 orthogonal vectors of equal length that span the plane, say $\vec{X}$ and $\vec{Y}$, and consider the vector $\vec{X} - i\vec{Y} \varepsilon \emptyset_{n-2}$. Any similar basis is a linear combination of these given by right multiplication by the matrix

$$r\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin & \cos\theta\end{bmatrix}.$$

This multiplies the complex vector by $re^{i\theta}$ and the map is therefore well-defined. It can be shown to be a diffeomorphism, with the Grassmannian having the differential structure of the homogeneous space

$$\frac{SO(n)}{SO(2) \times SO(n-2)}.$$

The base manifold of our twistor space for $\mathbb{R}^3$ will be the space of positive directions, i.e., the 2-sphere. This is clearly dual to the Grassmannian of oriented 2-planes in $\mathbb{R}^3$, with respect to the standard metric. Actually because of the minus in our definition ($\Phi = \frac{d\vec{X}}{dz}$) we change orientation but holomorphic is easier to write and we only mention the traditional Gauss map so we shall adopt the above definition for the Gauss map of a minimal surface in $\mathbb{R}^n$. Also, because the Gauss map is holomorphic and the above duality is an anti-conformal diffeomorphism for $\mathbb{R}^3$, we have that

Proposition. A surface in $\mathbb{R}^3$ is minimal if and only if its traditional Gauss map is anti-conformal.

We will give a different proof later. More can be said however by looking at $\frac{d\vec{X}}{dz}\,dz$, a section of the cotangent bundle.

Proposition. Given a set of non-vanishing holomorphic differentials on M, satisfying $\Sigma(\Phi dz)^2 = 0$ and having purely imaginary periods, there exists a minimal immersion into $\mathbb{R}^n$ with this section of the cotangent bundle as the generalized Gauss map.

Proof. The map $\vec{X}(z) = 2\mathrm{Re}\{\int_0^z \phi dz\}$ is well-defined and has harmonic coordinates, hence is minimal.

This integral gives a useful way to construct minimal surfaces in $\mathbb{R}^n$. This can be coupled with a nice characterization of curves in $\emptyset_1$ to produce a powerful tool to construct minimal immersions in $\mathbb{R}^3$, due to Weierstrass.

If $\phi_1^2(z) + \phi_2^2(z) + \phi_2^2(z) = 0$ then

$$\frac{\phi_1+i\phi_2}{-\phi_3} = \frac{\phi_3}{\phi_1-i\phi_2} = g(z)$$

which imply

$$\phi_1 : \phi_2 : \phi_3 = \frac{1-g^2}{2} : \frac{i(1+g^2)}{2} : g.$$

Letting ${}^{\phi_3}/g$ be h(z), we obtain

$$x = \text{Re}\{\int\phi_1 dz\} = \text{Re}\{\int\frac{1-g^2}{2}\, h\, dz\}$$
$$y = \text{Re}\{\int\phi_2 dz\} = \text{Re}\{\int\frac{i(1+g^2)}{2}\, h\, dz\}$$
7) $$z = \text{Re}\{\int\phi_3 dz\} = \text{Re}\{\int gh\, dz\}.$$

This is referred to as the Weierstrass representation. However, he obtained a lesser known representation by considering g as a local parameter. This is possible away from points where $g'(z) = 0,\infty$ which is, except on planes, a discrete set of points. Also, let f be the third integral of $h(g^{-1}(u))$, that is,

$$f''' = h(g^{-1}(u)).$$

Then integration of 7) by parts yields the formulas

$$x = \text{Re}\{\tfrac{1}{2}(1+u^2)f''(u) + uf'(u) - f(u)\}$$
8) $$y = \text{Re}\{\tfrac{i}{2}(1+u^2)f''(u) - iuf'(u) + if(u)\}$$
$$z = \text{Re}\{uf''(u) - f'(u)\}$$

Finally, more than providing a good local parameter for the minimal surface, the function g has geometric significance, which will be of fundamental importance in the twistor theory. First, we combine equation 5) with the derivative of the Weierstrass representation, which in light of the conformality of the immersion, implies

$$F = (1+|g|^2)^2|h|^2 = \frac{1}{2}\|\vec{X}_x \times \vec{X}_y\|.$$

The traditional Gauss map is given by $(\vec{X}_x\times\vec{X}_y)\cdot\|\vec{X}_x\times\vec{X}_y\|^{-1}$, which we will call N. From the definition of Φ in 3) we find

$$\begin{aligned}\vec{X}_x \times \vec{X}_y &= 4\ \mathrm{IM}\{\phi_2\overline{\phi}_3, \phi_3\overline{\phi}_1, \phi_1\overline{\phi}_2\}\\ &= 4\ \mathrm{Im}\{\tfrac{i}{2}(\overline{g}+g|g|^2)|h|^2, \tfrac{1}{2}(g-\overline{g}|g|^2)|h|^2, \tfrac{i}{4}(1-|g|^4)|h|^2\\ &= 2[\tfrac{1}{2}(\overline{g}+g|g|^2+g+\overline{g}|g|^2)|h|^2, \tfrac{i}{2}(g-\overline{g}|g|^2-\overline{g}+g|g|^2)|h|^2,\\ &\qquad \tfrac{1}{2}(1-|g|^4)|h|^2]\\ &= (1+|g|^2)|h|^2(2\ \mathrm{Re}\{g\}, 2\ \mathrm{Im}\{g\}, |g|^2-1)\end{aligned}$$

so $$N = \frac{1}{1+ g^2}(2\ \mathrm{Re}\{g\}, 2\ \mathrm{Im}\{g\}, |g|^2-1),$$

which shows that g is stereograph projection of the traditional Gauss map. We must notice from the third coordinate that this stereographic projection is anti-conformal. Therefore, our Weierstrass representation 8) has a standard Euclidean coordinate of $\mathbb{CP}^1$ as its parameter. This will be the point of departure for our twistor theory.

Twistor Theory

It is clear that complex analysis plays a central role in the theory of minimal surfaces. In fact this represents one of the earliest interactions between complex analysis and differential geometry, two fields which, until recent years, have not enjoyed extensive interaction. That, of course, has changed drastically, culminating in the solution by Atiyah, Hitchin, and Singer, of the self-dual Einstein equations in four dimensions. In as much as geometry evolved from contemplation of the world around us, it should not be at all surprising that these deep results have their roots in a recent exciting interaction between complex analysis and physics: twistor theory.

For a long time the major role of complex analysis in physics was confined to quantum mechanics, with most other mathematical descriptions of space-time being based on real manifolds. However, the different disciplines of physics are inexorably bound together through the world they attempt to describe, and it has long been the ultimate goal of all who endeavor in this art that the whole of physics be described in a unified way. Toward this end, it was the feeling of Roger Penrose, and now many others, that any unification of quantum mechanics and space-time geometry should be based in an essential way on a complex structure. Indeed, his move in this direction seems quite natural in light of the aforementioned ultimate aim, for in the "fuzzy" realm of quantum mechanics, the most difficult mathematical definition to justify, the "point," is at the very foundation of the theory.

If we are ever to understand the world that we see, we cannot lose sight of the fact that our experience is an interaction. Penrose bases his construction on the thing that we depend on to bring us information, the light ray, and his construction impresses us as the only one that incorporates our mode of perception into its basic foundation. In Penrose's description, the base space is like a web of information carriers, with each point being a light ray, or a null geodesic. The "points" of Minkowski space become "viewpoints," the projective sphere of light rays passing through that point. The geometry of space-time can then be described in terms of change of viewpoint, deformations of that celestial sphere.

Though we hope to have piqued the interest of all readers with any philosophical bent, we cannot hope to convey any more of the foundation of the theory that has blossomed into an extensive sector of mathematics. We must instead proceed with a development of a geometric side of twistor theory, and we feel that it will serve as an excellent example to improve intuition in this area, as it takes place in three dimensions. Out of the study of time-invariant self-dual Yang Mills equations, there has grown a twistor theory for three dimensions. Indeed the twistor space of $\mathbb{R}^3$ (whose Cauchy-Riemann equations can be shown to be equivalent to the minimal surface equations in $\mathbb{R}^3$) is the quotient of Penrose's original twistor space by the action induced by the complexification of time translation. The twistor space for three-dimensional Einstein-Weyl complex manifolds is a complex surface on which there is a complete system of rational curves, "celestial spheres," having normal bundle O(2). In order to see how they generate the geometry of the three manifold, we will need a couple of theorems due to LeBrun. To a geometer they might seem backward: they describe how the solutions to certain equations generate those equations, but that simply reflects the difference of the twistor viewpoint. To develop those theorms, which are quite elegant, we will need a couple of theorems by Kodaira, which are the source of the power underlying that elegance.

We begin with a brief glossary of symbols:

$\mathbb{C}E$ will mean $E \otimes_{\mathbb{R}} \mathbb{C}$ where E is a real bundle

$T'M$ will refer to the holomorphic tangent bundle, which is the subbundle of complex valued derivations of complex valued functions that vanishes on anti-holomorphic functions.

$\mathbb{P}E$ is the bundle is the bundle E minus the zero section, modulo multiplication by non-zero complex numbers.

$\mathcal{O}$ is the sheaf of germs of holomorphic functions on a complex manifold.

$O(m)$ is the line bundle of degree m on $\mathbb{P}_k$ or possibly restricted to a holomorphic projective subvariety

$\mathcal{O}(E)$ denotes the sheaf of germs of holomorphic sections of the holomorphic vector bundle E

$\mathcal{O}(m)$ denotes $\mathcal{O}(E)$ for a holomorphic vector bundle $E \to M$ whose fiber over $p \in M$ is $O(m)$ over a $\mathbb{P}_k$ or projective subvariety thereof.

The first Kodaira theorem is one on which much twistor theory is based but we can only sketch the proof here. (c.f. [Kodaira 1].)

<u>Theorem</u>. Let $V \subset W$ be a compact complex submanifold of W whose normal bundle N, satisfies $H^1(V,\mathcal{O}(N)) = 0$. Then V belongs to a locally complete family of such manifolds $(V_p, p\in M)$ for a complex manifold M and there is a canonical isomorphism $T'_p M \cong H^\circ(V_p,\mathcal{O}(N))$.

<u>Sketch of proof</u>. Kodaira works in local coordinates on W and uses coordinates of $H^\circ(V_o,\mathcal{O}(N))$ as coordinates on M. He assumes the defining functions of nearby compact submanifolds to be expanded in power series in coordinates of M with coefficients in holomorphic functions on V. Hence differences of these Taylor coefficients on overlaps are 1-cocycles on the nerve of the coordinate neighborhoods with coefficients in $\mathcal{O}(N)$. The vanishing condition guarantees correction functions on each neighborhood, for each order, producing

power series that agree on overlaps. A previous result of Kodaira and Spencer (see [Kodaira and Spencer 1]) verifies that those corrections can be made on a compact manifold so as to ensure convergence near V. The choice of coordinates for M also shows that the above family is maximal.

The second theorem is on the rigidity of complex structures under deformations. By a deformation of a compact complex mainfold $M \subset W$, we mean a proper regular holomorphic map between complex manifolds, $\tilde{\omega} : W \to U$ such that M is biholomorphically equivalent to $\tilde{\omega}^{-1}(x)$ for some x in U. By proper we mean that the preimage of a compact set is compact and regular implies that the Jacobian maintains maximal rank. We seek a criterion for determing the existence of a neighborhood V of x such that

$$\omega^{-1}(V) \cong V \times M \text{ biholomorphically.}$$

In that case the deformation is said to be trivial. The important point for us here is that this condition is local in U but global on M. Also worth noting is the fact that the corresponding condition in the differential category always holds, hence the terminology "deformation of complex structures."

Kodaira and Spencer denote by Θ the sheaf $\mathcal{O}(T'M)$ and obtain the following result (see [Kodaira and Spencer 2]).

Theorem. If $H^1(\Theta) = 0$ then small deformations are trivial, that is, for any deformation, $\tilde{\omega} : W \to U$ of $M = \tilde{\omega}(x) \subset W$, then there exists a neighborhood V of x in U such that $\tilde{\omega}^{-1}(V) \cong V \times M$ biholomorphically.

Sketch of proof. Kodaira and Spencer start with the exact sequence defining the normal bundle and then consider the corresponding sequence in which the normal bundle is replaced by the canonical lift of $T'U$. They then consider the sheaves of germs of holomorphic sections of these bundles and consider the corresponding long exact sequence of sheaves. By passage to the direct limit over x, the above vanishing condition is seen to imply the vanishing of the limit coboundary operator. Because all sheaves considered are finite dimensional modules of $\mathcal{O}_{\tilde{V}}$ for open $\tilde{V} \subset U$, the authors infer the vanishing of the coboundary operator on germs of sections over some open set V. There is then a short exact subsequence for holomorphic sections over V and it is then a simple matter to lift exp to a function from $T'_xV \times M$ to $\tilde{\omega}(V)$ to parameterize the trivial deformation.

At this point I feel it worthwhile to quote the sage words of LeBrun on this theorem.

"One can think of a Cech cochain with coefficients in Θ as an infinitesimal change in the transition functions which define M as a complex manifold, and factoring out by coboundaries just removes the ambiguity of infinitesimal changes in holomorphic coordinates; thus the theorem can be thought of as saying that there are no small deformations of M if there are no infinitesimal ones."

Kodaira and Spencer also develop a similar theory on deformations of complex vector bundles but in the sequel we shall only need to consider deformations of the bundle structure over a fixed base. In this case the deformation is defined by

$$E \to W \xrightarrow{\tilde{\omega}} U$$

where

$$E \to \tilde{\omega}^{-1}(y), \; y \in U$$

are the bundles to be deformed and

$$W \cong M \times U \text{ biholomorphically.}$$

The latter fact is crucial, as the tangent bundle of W splits canonically, and deformations of the base can be assumed to vanish. Kodaira and Spencer work with principal bundles since derivatives of the bundle structure take values in the Lie algebra of the structure group, and in this setting the coboundary map of the previous theorem, which essentially mods out by coordinate changes, lifts to a map taking values in the sheaf of holomorphic sections of a bundle whose fiber is the Lie algebra of the structure group. In the case of deformations of vector bundles, then, the triviality condition specializes to the vanishing of $H^1(E \otimes E^*)$, since the structure group is $GL(n,\mathbb{C})$. This and the previous vanishing condition will be referred to as rigidity. Let us now verify a few vanishing results.

We will be working with line bundles on $\mathbb{P}_1$, which are of the type $0(m)$ where the $m + 1$ global sections of $0(m)$ $(m>0)$ can be represented as the homogeneous polynomials of degree m. For instance, consider $T\mathbb{P}_1$, the tangent bundle of $\mathbb{P}_1$. If v is a global holomorphic vector field then

$$v = a(z)\frac{d}{dw} \text{ where } w = \frac{1}{z}.$$

Let $$a(z) = \Sigma_{i=0}^{\infty} a_i z^i \text{ and } b(w) = \Sigma_{i=0}^{\infty} b_i w^i,$$

then $$a = b\,\frac{dz}{dw} = b\,\frac{1}{w^2},$$

so $$\Sigma_{i=0}^{\infty} a_i z^i = z^2\,\Sigma\, b_i\,\frac{1}{z^i},$$

implying $$a = a_o + a_1 z + a_2 z^2 = b_2 + b_1 z_1 + b_o z^2$$

and $T\mathbb{P}_1 = 0(2)$. We will also need a portion of <u>Bott's rule</u>:

$$\dim H^{\circ}(\mathbb{P}_n, 0(m)) = \binom{n+m-1}{m} \quad m \geq 0$$

$$\dim H^{p}(\mathbb{P}_n, 0(m)) = 0 \qquad n > p > 0$$

Clearly then, a normal bundle isomorphic to $0(m)$ for $m \geq 0$ satisfies the hypothesis of Kodaira's theorem. We will eventually be concerned with normal bundles isomorphic to $0(2)$.

From Bott's rule, we have the following:

Corollary. If $Q \subset \mathbb{P}_{n+1}$ is a non-degenerate quadric, then $H^1(Q,\mathcal{O}(m)) = 0$ if $n \geq 2$ $m \geq 0$.

Proof. This follows directly from the long exact sequence induced by the following short exact sequence

$$1) \quad 0 \to \mathcal{O}(m-2) \underset{g}{\to} \mathcal{O}(m) \underset{\rho}{\to} \mathcal{O}_Q(m) \to 0$$

where g is multiplication by the defining function and ρ is restriction to Q. This implies $H^1(Q,\mathcal{O}(m)) \subset H^2(\mathbb{P}_{n+1},\mathcal{O}(m-2))$.

Now we need to verify the rigidity of two bundles.

I) $T'\mathbb{P}_n$. This bundle is described by the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_n} \xrightarrow{\lambda} (n+1)\mathcal{O}(1) \to T'\mathbb{P}_n \to 0$$

where λ multiplies a holomorphic function (constant) by the $n + 1$ homogeneous coordinates on $\mathbb{P}_n$. Bott's rule implies that $H^1(\mathbb{P}_n,\mathcal{O}(1)) = H^2(\mathbb{P}_n\mathcal{O}) = 0$ so from the induced long exact sequence $H^1(\mathbb{P}_n,\mathcal{O}(T'\mathbb{P}_n)) = 0$.

II) $T'Q$ where Q is a quadric hypersurface in $\mathbb{P}_n$. Let $T = \mathcal{O}(T'\mathbb{P}_n|_Q)$, and let $N = {}^T/T'Q \cong \mathcal{O}(2)$ be the sheaf of holomorphic sections of the normal bundle of Q. The short exact sequence defining N induces the following long exact sequence:

$$0 \to H^\circ(T'Q) \to H^\circ(T) \underset{\mu}{\to} H^\circ(N) \to H^1(T'Q) \to H^1(T) \to \cdots$$

As in the above example T is given by

$$0 \to \mathcal{O} \xrightarrow[\lambda]{} (n+1)\mathcal{O}(1) \to T \to 0$$

so by the corollary to Bott's rule, $H^1(T) = 0$. From the fact that $SL(n+1,\mathbb{C})$ acts effectively on $\mathbb{P}_n$, $H^\circ(\mathbb{P}_n,T) \cong \mathfrak{sl}(n+1,\mathbb{C})$. On the other hand $SO(n+1,\mathbb{C})$ acts effectively on Q, so $H^\circ(Q,T'Q) = \mathfrak{so}(n+1,\mathbb{C})$ and the image of μ has dimension

$$(n+1)^2 - 1 - \binom{n+1}{2} = \binom{n+2}{2} - 1$$

while by 1) $H^\circ(N)$ can be identified with the symmetric $(n+1) \times (n+1)$ matrices modulo the defining function of Q and also has dimension $\binom{n+2}{2} - 1$. Therefore μ is onto and $H^1(T'Q) = 0$.

We mention that this verification of the above well-known results, and the following two theorems come from the thesis of LeBrun.

We will now define two structures on complex manifolds that, although common, will not be given the standard definitions due to the manner they are arrived at in twistor theory. The first is the conformal structure.

The conformal structure of a complex manifold M is an equivalence class of holomorphic metrics on T'M under the equivalence relation of multiplication by non-vanishing holomorphic functions. Such a metric determines in each tangent space a non-degenerate quadric cone of null vectors, that is, vectors $\vec{v}$, such that $g(\vec{v},\vec{v}) = 0$. This is clearly independent of the choice of representative metric, but what is not clear is that such a holomorphic prescription of quadric cones in the tangent space of M defines a conformal structure.

Here we define a conformal structure to be a holomorphic quadric subbundle Q, of $\mathbb{P}T'M$. By this we mean a complex submanifold of $\mathbb{P}T'M$ that intersects each fiber in a non-degenerate quadric.

<u>Proposition</u>. Q defines a holomorphic line subbundle of $(T'^{*}M)^{\odot 2}$, $\odot$ being the symmetric tensor product, hence defining a conformal structure in the usual sense.

<u>Proof</u>. By the rigidity of the quadric, Q has the structure of a holomorphic fiber bundle, i.e., local trivalizations. Over such a region V, consider the exact sequence of sheaves on $\mathbb{P}T'V$

$$0 \to \mathcal{I}_Q(2) \to \mathcal{O}(2) \underset{\rho}{\to} \mathcal{O}_Q(2) \to 0$$

which describes the ideal sheaf of Q of homogeneity 2. ρ here restricts to Q; that is, annihilates stalks over points not in Q. Now since $H^1(Q,\mathcal{O}) = 0$, the restriction of the Hopf bundle to Q over V is biholomorphically the Hopf bundle restricted to $\emptyset \times V$, so by the Kunneth formula

$$H^\circ(\mathcal{O}_Q(2)) = H^\circ(\emptyset \times V, \mathcal{O}(2)) = H^\circ(V,\mathcal{O}) \otimes_{\mathbb{C}} H^\circ(\emptyset,\mathcal{O}(2))$$
$$= (\frac{(n+1)n}{2} - 1) H^\circ(V,\mathcal{O}) \ ,$$

while for similar reasoning for $\mathbb{P}_n$ instead of Q

$$H^\circ(\mathcal{O}(2)) = (\frac{n(n+1)}{2}) H^\circ(V,\mathcal{O}).$$

Therefore ρ has a kernel acting on global sections over V, any element of which is a local representative metric of the conformal structure. Hence a conformal structure in the usual sense is equivalent to a holomorphically varying prescription of non-degenerate quadric cones in the tangent bundle of a complex manifold. We wish now to examine projective structures.

Projective structure usually refers to an equivalence class of connections, where two such are associated if they define the same geodesics. In the same spirit as before we might hope to define the structure in terms of the geodesic. Here we will define a projective structure on M, as a collection L of complex curves (inextendable immersed connected complex 1-manifolds) such that over each point $(p,\vec{v}) \in \mathbb{P}T'M$ there is a unique curve with $\vec{v}$ as its tangent direction, and such that the resulting foliation is holomorphic. We will refer to the curves of L as geodesics.

Proposition. On a complex manifold M with a projective structure, over any coordinate neighborhood one can find Christoffel symbols such that solutions of the geodesic equations parameterize the curves of L.

Proof. We first claim that the equivalence classes of such

Christoffel symbols are described by the exact sequence.

$$2)\qquad 0 \to \mathcal{O}(1) \xrightarrow{\alpha} n\mathcal{O}(2) \xrightarrow{\beta} C \to 0$$

where $\alpha(f)(y_i\,{}^{\partial}/\partial x_i) = (y_i f(y_i\,{}^{\partial}/\partial x_i), \cdots, y_n f(y_i\,{}^{\partial}/\partial x_i))f \in \mathcal{O}(1)$. Given a Christoffel symbol and a point $(p,\vec{v}) \in T'M$, find the corresponding geodesic, $\ell(s)$, and lift it to the tangent bundle. The Christoffel symbol would equal $\frac{d^2\ell}{ds^2}$. A different Christoffel symbol giving that curve lifted to the same point would differ from the first by

$$\frac{d^2\ell}{ds^2} - \frac{d^2\ell}{dt^2} = \frac{d^2\ell}{ds^2} - \frac{d}{dt}\left(\frac{d\ell}{ds}\cdot\frac{ds}{dt}\right) = \frac{d\ell}{dt}\cdot\frac{d^2 s}{dt^2} = \alpha(\ell^*(dt))\left(\frac{d\ell}{dt}\right)$$

where the use of α implies passage to $\mathbb{P}T'M$ which is justified by the linearity of ℓ^*.

Finding a holomorphic section of $n\mathcal{O}(2)$ on an open set in $\mathbb{P}T'M$ is known to be possible because it is possible to straighten out the foliation, L, in local coordinates such that z_1 parameterizes the curves and $\frac{d^2\ell}{dz_1{}^2}$, $\ell \in L$ works, but there is no guarantee that this is even an algebraic function of tangent directions of M. However, we know by our vanishing results, that over a Stein coordinate patch V in M

$$H^1(\mathbb{P}TV, \mathcal{O}(1)) = H^1(V \times \mathbb{P}_{n-1}, \mathcal{O}(1)) = H^1(\mathbb{P}_{n-1}, \mathcal{O}(1) \otimes_{\mathbb{C}} \mathcal{O}_V = 0$$

so by the long exact sequence associated with 2),

$$0 \to \Gamma\mathcal{O}(1) \xrightarrow[\alpha_*]{} n\Gamma\mathcal{O}(2) \xrightarrow[\beta_*]{} \Gamma C \to 0 \to \cdots,$$

the above local sections of $n\mathcal{O}(2)$ can be patched together to form a global section of $n\mathcal{O}(2)$ over V. This is then quadratic on the tangent space and can be uniquely associated with a symmetric bilinear form whose components are Christoffel symbols which locally, in M, define the system of curves L.

Now that we have shown how to construct the standard analytic structures defining conformal structure and projective structure from the geometric objects they define, let us look at the way in which twistor theorists arrive at these geometric objects. We will be concerned with three-dimensional twistor theory which, in our cases, describes the geometry of certain three-dimensional spaces in terms of rational curves on a surface. Let V be a compact holomorphic complex one-submanifold of a complex two-manifold, W, whose normal bundle N is isomorphic to $\mathcal{O}(2)$. Kodaira's theorem then applies and V belongs to a locally complete family, V_p, parameterized by a 3-dimensional complex manifold M such that

$$T'_p M \cong H^\circ(V_p, N_p).$$

We first define the conformal structure on M by specifying the null cone of $T'_p M$ as elements of $H^\circ(V_p, N_p)$ that vanish at some point with multiplicity 2. Since $N_p \cong 0(2)$ a section is given in terms of homogeneous coordinates (z_1, z_2) on V_p by $az_o^2 + bz_o z_1 + cz_1^2$ so our vanishing condition is equivalent to the vanishing of the discriminant

$$b^2 - 4ac$$

which defines a quadric cone and therefore a conformal structure.

Now pick a point $p \varepsilon M$ and a direction in

$$T'_p M \cong H^\circ(V_p, N_p).$$

By the isomorphism that direction is a one-dimensional space of quadratic polynomials and can therefore be specified by the two roots y_1, y_2. To find a corresponding curve in M we use the algebraic geometric technique of blowing up a line bundle at points (see Griffiths and Harris). The blow-up of the line bundle $0(m)$ at a point produces the line bundle $0(m-1)$ so in the case where $y_1 \neq y_2$, blowing up the surface W at points y_1 and $y_2 \subset V_p \subset W$ produces a new curve $\tilde{V}_p$ which is the lift of V_p in a new surface $\tilde{W}$, such that $\tilde{V}_p$ now has a trivial normal bundle in $\tilde{W}$. Kodaira's theorem now guarantees a 1-parameter family of such lines which, for homological reasons, all intersect both exceptional divisors, that is, projected back down to W, they all pass through y_1 and y_2. Therefore, the tangent vector to this family p corresponds to a section of the normal bundle that vanished at y_1 and y_2, hence is the prescribed direction. If $y_1 = y_2$ we blow up W at y_1 to get a new curve $\tilde{V}_p$ in $\tilde{W}$ and then blow up W at the intersection of $\tilde{V}_p$, with the exceptional divisor. The new curve $\tilde{\tilde{V}}_p \subset \tilde{\tilde{W}}$ again has trivial normal bundle, thus defining a 1-parameter family of curves whose projection back to W to curve that all meet V_p <u>tangentially</u> at y. Hence they meet each other tangentially so the tangent vector to the curve at every point q, corresponds to a section of the normal bundle of V_q which has a double root. This shows that the tangent vector to the curve is everywhere null.

Having now generated a distinguished curve in M at all points p and for every direction, we use our previous result to assert the existence of a projective structure on M for which these curves are geodesics. Further, because curves starting in a null direction remain null, we have compatability between projective and conformal structures on M. Hitchin goes on to show that there is a distinguished affine connection with the projective class which preserves the conformal structure by showing the three manifold M has a Weyl geometry. However, our examples are the simplest, being the only two corresponding to rational curves on compact surface and the choice of the above structures will be obvious.

Both of our examples are quadric cones in $\mathbb{P}_3$. The first example is the degenerate cone

$$W = \{(z_0, z_1, z_2, z_3) \in \mathbb{P}_3 | z_1^2 + z_2^2 + z_3^2 = 0\}.$$

Hyperplanes omitting the vertex (1,0,0,0) then cut the cone in a plane conic. Because two such planes intersect in a line which meets the cone at two points these curves have self-intersection 2 and hence a normal bundle 0(2). If we parameterize the hyperplane sections by the dual space, $\mathbb{P}_3^*$, then by the omission of the vertex, we may find euclidean coordinates on the entire parameter sapce M, such that $M \cong \mathbb{C}^3$. Indeed the corresponding conformal structure is flat. We will go further into this example in a later section.

The example which is our twistor space is the non-singular quadric $\emptyset_2$, in $\mathbb{P}_3$:

$$W \cong \emptyset_2 \cong \{(z_{11}, z_{12}, z_{21}, z_{22}) \in \mathbb{P}_3 \mid z_{11}z_{22} - z_{12}z_{21} = 0\},$$

so $W \cong \mathbb{P}_1 \times \mathbb{P}_1$ and we put coordinates on this quadric via the Segrè imbedding,

$$(1, g_1) \times (1, g_2) \to (-g_1, g_1 g_2, -1, g_2)$$

and again choose the rational curves to be non-degenerate hyperplane sections, which we parameterize by $\mathbb{P}_3^* \setminus \emptyset_2^*$. If we use the isomorphism provided by the above quadratic form to relate $\mathbb{P}_3$ and its dual, our rational curves are given by,

3) $\{(g_1, g_2) \mid z_{11}g_2 + z_{12} - z_{21}g_1 g_2 - z_{22}g_1 = 0,\ \det(z_{ij}) \neq 0\}$

or $\{(g_1, g_2) \mid g_1 - \dfrac{z_{11}g_2 + z_{12}}{z_{21}g_2 + z_{22}} = 0,\ \det(z_{ij}) \neq 0\}.$

By a reasoning similar that above, these curves have normal bundle 0(2). The non-null geodesics are described in terms of two null directions, not orthogonal to each other, as the set of hyperplane sections vanishing at exactly those null directions. With the above isomorphism these can be identified with the pencil of directions orthogonal to the span of the given null directions. If one identifies $\mathbb{P}_3 \setminus \emptyset_2$ with

$$SL(2,\mathbb{C}) = \{(z_{11}, z_{12}, z_{21}, z_{22}) \mid z_{11}z_{22} - z_{12}z_{21} = 1\},$$

then the holomorphic metric induced by the quadratic form associated with $\det(z_{ij})$, is the Cartan-Killing form for $SL(2,\mathbb{C})$. Since this metric is bivariant (left and right invariant) the group homomorphism exp and the geodesic flow exp coincide. Because $SL(2,\mathbb{C})$ is an affine quadric, its geodesics are intersections of the affine quadric with 2-planes containing the origin. Indeed, because a matrix satisfies its own characteristic polynomial (quadratic for two-by-two matrices), the power series for exp reduces to holomorphic combinations of position and direction. In fact, for null vectors the Cayley-Hamilton Theorem implies that exp reduces to an exponential function times a null vector plus a constant position vector. By the definition of the tangent space for an affine quadric, these two vectors are orthogonal and the defining quadratic form induces a degenerate conformal structure on their span; that is, they span a null plane.

Hitchin describes another structure that exists in three dimensions and has a nice realization in the present example. Take V_o, a curve in a complex surface W, with normal bundle $0(2)$, as before. Choose on V_o a distinguished point v and blow up W at v. V_o then lift to a curve $\tilde{V}_o$ with normal bundle $0(1)$. By Kodaira's theorem we have a net of curves, containing v, corresponding to a surface S, in the parameter space of deformations. A tangent vector to S at the point corresponding to V_o, corresponds to section of the normal bundle of V_o that vanishes at v and some other point v_1. The geodesic in that direction will, by construction, correspond to curves passing through these two points, so the geodesic remains in S. This reasoning, applied to other points of S, shows that it is totally geodesic. Further, since all tangent directions of S correspond to sections of a normal bundle that vanish at V, the quadratic form representing such a section, factors into

$$(z_o - vz_1)(\alpha z_o + \beta z_1).$$

Therefore, the conformal structure degenerates on this plane so that there is a unique null direction in the tangent space to S corresponding to sections of a normal bundle having a double zero at v. In the case of $SL(2,\mathbb{C})$, each such surface is the intersection of $SL(2,\mathbb{C})$ with a null hyperplane; hence the degenerate conformal structure. Each surface is foliated by a pencil of null geodesics corresponding to a Schubert cycle of null 2-planes containing the null direction and contained in the dual null hyperplane. These 2-planes osculate the null quadric in $\mathbb{P}_3 \subset \mathbb{C}^4$ to first order, so in the large, the space of null geodesics on $SL(2,\mathbb{C})$ is parameterized by excising verticle and horizontal directions from $\mathbb{P}T'(\mathbb{P}_1 \times \mathbb{P}_1)$; i.e., by $\mathbb{P}T'(\mathbb{P}_1 \times \mathbb{P}_1) \setminus (\mathbb{P}_1 \amalg \mathbb{P}_1)$.

In a sense, the above observations constitutes half of our problem, but a closer look at the twistor resolution of the minimal surface equations in $\mathbb{R}^3$ will be required to motivate the remainder of the solution.

Twistor Theory in $\mathbb{R}_3$

The interaction between complex analysis and geometry has taken tremendous leaps in recent years. In fact, Nigel Hitchin has said that the aim of twistor theory is "to encode as much of mathematical physics as possible into holomorphic form and there to rely on the geometry, supported only by the constraints of the Cauchy-Riemann equations, to provide a description of the universe" [Hitchin 2]. He likes to point out, however, that this approach is by no means new. Weierstrass did essentially this when he gave a construction of minimal surfaces in terms of one holomorphic function. It turns out that this solution, in twistor theoretic terms, is the flat version of ours so we present here, Hitchin's version of the Weierstrass representation.

The space of oriented geodesics in $\mathbb{R}^3$ can be parameterized by a unit tangent vector $\vec{u}$ and a position vector $\vec{v}$ which is the point of the geodesic closest to the origin. The latter requirement not only determines $\vec{v}$ uniquely, but forces $\vec{v}$ to be perpendicular to $\vec{u}$. These conditions describe the tangent bundle of the two sphere. We denote a parameterized geodesic by $\gamma(t)$ and a variation of this by $\gamma(t,s)$. If we consider $\vec{u}$ and $\vec{v}$ to be functions of s and denote $\partial/\partial s$ as "." then

$$\gamma = t\vec{u} + \vec{v}$$

and the normal component of the Jacobi field is

$$(\partial/\partial s\gamma)^{\perp} = t\dot{\vec{u}} + \dot{\vec{v}} - (\dot{\vec{v}}\cdot\vec{u})\vec{u}.$$

If we, on the other hand, consider $\gamma(s)$ to be a curve in $T(S^2)$, given by $(\vec{u},\vec{v})$, then the tangent vector to that curve is

$$(\dot{\vec{u}},\dot{\vec{v}} - (\dot{\vec{v}}\cdot\vec{u})\vec{u})$$

where the above splitting of $TT(S^2)$ is that induced by the flat connection on $\mathbb{R}^3$. We define an almost complex structure for TS^2 by

$$J(\dot{\vec{u}},\dot{\vec{v}} - (\dot{\vec{v}},\dot{\vec{u}})\vec{u}) = (\dot{\vec{u}}\times\vec{u},\dot{\vec{v}}\times\vec{u} - (\dot{\vec{v}},\vec{u})\vec{u}).$$

With the benefit of this splitting it is not hard to see that the above almost complex structure is that of $T'\mathbb{P}_1$, as it is well known that

$$\mathbb{P}_1 \cong S^2.$$

Henceforth, we will describe our space of geodesics as $T'\mathbb{P}_1$.

We will now describe the global holomorphic sections of this bundle. We denote by $P_{\vec{x}}$ the section of $T'\mathbb{P}_1$ corresponding to the set of lines through the point $\vec{x} \in \mathbb{R}^3$. Because there is a unique line for each direction this defines a section and because the nearest point to the origin varies smoothly with direction, this is a smooth section. Further, the second component of the velocity of a curve in $T'\mathbb{P}_1$, contained in this section, is the velocity of a curve obtained by intersecting the corresponding lines with a fixed plane orthogonal to the line at which the velocity is taken. Therefore, the section is easily seen to be holomorphic. In general, if ζ is an euclidean coordinate on $\mathbb{P}^1$, then we showed in the last section that $T'\mathbb{P}_1$ is the line bundle $0(2)$ so sections can be represented by

$$s(\zeta) = (a\zeta^2+b\ +c)\ \frac{d}{d\zeta}\ .$$

To characterize sections of the form $P_{\vec{x}}$ for some $\vec{x}$, we notice that such sections are fixed by the map that reverses orientation of the lines, which is given by

$$\tau(\vec{u},\vec{v}) \mapsto (-\vec{u},\vec{v})\ .$$

This lifts the action of the negative of the antipodal map α, which is given in terms of ζ by

$$\alpha(\zeta) = -\bar{\zeta}^{-1}.$$

Therefore τ is an antiholomorphic involution on the sections of $T'\mathbb{P}_1$ and so defines a real structure that identifies sections of the form $P_{\vec{x}}$ as real sections. The constraints this imposes on coordinates (a,b,c) are

$$b = \bar{b} \quad \text{and} \quad a = -\bar{c}\ .$$

We may then represent $P_{\vec{x}}$ for $\vec{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ as a real section by:

$$s(\zeta) = ((x_1+ix_2) - 2x_3\zeta - (x_1-ix_2)\zeta^2)\,{}^d/d\zeta.$$

In the last section we found the conformal structure to be given by the cone of sections that have a double zero at some point. This is equivalent to the vanishing of the descriminant, $b^2 - 4ac$, which is a quadratic form and hence a representative metric of the conformal class. In terms of the complexification of the above coordinates $\vec{x} \in \mathbb{C}^3$, this form is simply the extension of the standard euclidean metric to $\mathbb{C}^3$ holomorphically, so our conformal class is flat. We remark that the above representation of $\mathbb{C}^3$ is the complexification of the Lie algebra of SU(2).

We can now prove the main result of this section.

Theorem (Hitchin). Minimal surfaces in $\mathbb{R}^3$ are in one-to-one correspondence with holomorphic curves in $T\mathbb{P}$.

Proof. Given a minimal surface, we add i times the conjugate harmonic function of each coordinate to that coordinate, which by the Cauchy-Riemann equations and the formula for the generalized Gauss map produces a holomorphic null curve in $\mathbb{C}^3$. Through each point in the curve passes a unique tangent null hyperplane orthogonal to the tangent direction. If we denote the null curve by $\vec{X}(z)$, where $z = x+iy$, then the null direction is

$$(\mathrm{Re}\{\vec{X}\})_x + i(\mathrm{Re}\{\vec{X}\})_y$$

and the unique real direction orthogonal to this is simply given by $(\mathrm{Re}\{\overline{X}\})_x \times (\mathrm{Re}\{\overline{X}\})_y$. Therefore, the intersection of this null plane with the real slice is a geodesic whose direction is given by the traditional Gauss map so we have a holomorphic curve in the twistor space.

Conversely, given such a curve, we may represent it in local coordinates, away from points where tangent to the curve is vertical, by a holomorphic function

$$\eta = f(\zeta).$$

The equations for a section that osculates this curve to second order at ζ are

$$\begin{aligned} a + b\zeta + c\zeta^2 &= f(\zeta) \\ b + 2c\zeta &= f'(\zeta) \\ 2c &= f''(\zeta). \end{aligned}$$

The corresponding curve in $\mathbb{C}^3$ is given by

$$(f-\zeta f' + \tfrac{1}{2}\zeta^2 f'',\ f' - \zeta f'', \tfrac{1}{2} f'').$$

In standard euclidean coordinates given above, these equations become

$$\begin{aligned} x_1 &= \mathrm{Re}(\tfrac{1}{2}(1-\zeta^2) f'' + \zeta f' - f) \\ x_2 &= \mathrm{Re}(-\tfrac{i}{2}(1+\zeta^2) f'' + i\zeta f' - if) \\ x_3 &= \mathrm{Re}(\zeta f'' - f') \end{aligned}$$

which, from a previous section, is exactly Weierstrass' later representation of minimal surface.

Immersion Theorem

We give a local, intrinsic characterization of the metrics on Riemann surfaces that can be realized as metrics induced by a minimal immersion of the surface into $\mathbb{R}^3$. This is the well known Ricci condition, but we present part of a generalization that shows that such surfaces can also be immersed as constant mean curvature one surfaces in Hyperbolic space. For the full generalization see [Lawson, 3].

<u>Theorem</u>. Let ds^2 be a C^3 Riemannian metric defined over a simply-connected Riemann surface S. Suppose that the Gauss curvature of this metric is negative and suppose further that the metric satisfies the Ricci condition, that is,

$$d\tilde{s}^2 = \sqrt{-K}\, ds^2$$

is a flat metric. Then for $c = 0,1$, there is a 2π-periodic family of isometric immersions of constant mean curvature c, into the three-dimensional space form of constant sectional curvature $-c$. Moreover, given such a surface, its maximal extension is contained in the family for parameter value θ, $0 < \theta < \pi$, up to rigid motions.

<u>Proof</u>. We assume, for now, that S is not $\mathbb{CP}^1$. By the Koebe Uniformization Theorem, then we can assume S to be the plane or the disk, and we can assume isothermal coordinates, the metric being given by

$$ds^2 = Fdz.$$

The Gauss curvature of $d\tilde{s}^2$ vanishing implies

$$\Delta \log(-KF^2) \equiv 0,$$

which, in turn implies existence of a holomorphic function, unique up to rotation and globally defined, with

$$|f|^2 = KF^2.$$

The important observation is that this function is essentially the second fundamental form. Specifically, let

$$\Pi_{ij} = \begin{pmatrix} \mathrm{Re}\{e^{i\theta}f\} + cF & \mathrm{Im}\{e^{i\theta}f\} \\ \mathrm{Im}\{e^{i\theta}f\} & -\mathrm{Re}\{e^{i\theta}f\} + cF \end{pmatrix}.$$

Immediately we observe

1) $\det[\Pi] = (c-K)F^2,$

and some computation verifies

2) $\Pi_{ij;k} = \Pi_{ik;j},$

where semi-colon refers to covariant differentiation.

Equations 1) and 2) are Gauss curvature and Codazzi-Mainardi equations for hypersurfaces in a space of constant curvature. These are well known to be integrability conditions for the existence of a surface realizing the given

What we have done to the second fundamental form is to add the metric to it so,

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} = \begin{pmatrix} 1+\beta_{11} & \beta_{12} \\ \beta_{12} & 1-\beta_{11} \end{pmatrix}.$$

In particular, we have shown

4) $$\tilde{\eta} = \eta \qquad f = \tilde{f}.$$

Concluding, we have shown that a simply-connected Riemann surface with a metric satisfying the Ricci condition can be isometrically immersed both as a minimal surface in $\mathbb{R}^3$ or as a surface of constant mean curvature one in hyperbolic space. Lawson actually shows considerably more and we refer the interested reader to [Lawson, 3]. Also, we note that because we assumed the above metrics to have negative curvature our exclusion of $\mathbb{CP}^1$ was not restrictive. Further, from equation 4) we have identified the important holomorphic forms in both settings. These forms will be central in the verification of the results of Robert Bryant.

metric and second fundamental form (see [Spivak]). Because we are dealing with hypersurfaces, the metric and second fundamental form contain all of the information about the ambient connection, so for each radial direction in a global coordinate system, we define a curve in the Lie algebra of the structure group for the bundle of pseudo-orthonormal frames. This gives rise to a system of equations

$$3) \qquad X'(t^{i\phi}) = A(t^{i\phi})X(te^{i\phi}),$$

where the column vectors of X are an adapted pseudo-orthonormal frame. In particular, the first column of $X, \vec{X}_o = \vec{e}_o$, is the desired immersion.

Because $\mathbb{H}^3$ is homogeneous, this frame bundle is a group and equation 3) can be expressed in terms of connection one forms. When c = 0 we get equations of a minimal surface in $\mathbb{R}^3$ lifted to the bundle of affine orthonormal frames. In this case $\theta_o^1 + i\theta_o^2 = \theta$ such that

$$\theta \otimes \overline{\theta} = 2F|dz|^2,$$

$$\theta_o^3 = 0$$

θ_1^2 is determined intrinsically from Christoffel symbols, and θ_2^3 come from the second fundamental form. When c = 1, we have the same metric so

$$\omega = \theta$$

and $$\omega_1^2 = \theta_1^2 .$$

The Holomorphic Lifting

This is the appellation modestly suggested by Robert Bryant in response to our references to the Bryant Lifting. His reason was to avoid the physicists' tendency to name everything that moves after someone. Maybe, in the spirit of Banach, he should call it lifting B.

In the last section we verified the existence of the desired immersion by demonstrating a map of the surface into the structure group of the bundle of pseudo-orthonormal frames which correspond to adapted frames. It is the marvelous observation of Robert Bryant that there is a lifting to the same group which is not adapted but holomorphic. In fact, we can find complex coordinates on this group by lifting to the spin double cover, $SL(2,\mathbb{C})$. There one has, as with minimal surfaces, a holomorphic resolution to the problem. The resolution for minimal surfaces is a holomorphic null curve in $\mathbb{C}^3$, the complexified translation group. The translation group moves an initial frame around by parallel translation and a rotation is required to make the frame adapted at other points of the minimal surface. Interestingly, exactly the same rotation is the difference between the holomorphic and adapted frames for the hyperbolic surface. Although this is not fully accounted for yet, we present Bryant's proof in terms that emphasize the similarity. As his paper is quite beautiful the interested reader should refer to it.

As this result is true up to isometry we will take the liberty of fixing an initial point. In terms of the representation of $\mathbb{L}^4$ as two-by-two Hermitian matrices and the spin action of $SL(2,\mathbb{C})$ described previously, we will consider a projection from $SL(2,\mathbb{C})$ to $\mathbb{H}^3$

$$\begin{array}{ccc} SU(2) & \rightarrow & SL(2,\mathbb{C}) \\ & & \downarrow \\ & & \mathbb{H}^3 \end{array}$$

corresponding to the orbit under the spin action of the point $(1,0,0,0)$ = identity matrix. Hence the projection is given simply by

$$g \mapsto gg^*, \quad g \in SL(2,\mathbb{C}).$$

<u>Theorem</u> (Bryant). Let M be a simply-connected Riemann surface endowed with a metric satisfying the hypothesis of the immersion theorem of the previous section, and let

$$\overline{X}_o : M \rightarrow \mathbb{H}^3,$$

be the conformal immersion of constant mean curvature 1 generated by that theorem. Then there is a lifting to a holomorphic curve in $SL(2,\mathbb{C})$ which is null with respect to the Cartan-Killing form (given on $SL(2,\mathbb{C})$ by determinant) such that the following diagram commutes.

$$\begin{array}{ccc} & & SL(2,\mathbb{C}) \\ & \overset{Y}{\nearrow} & \downarrow \; gg^* \\ M & \xrightarrow{\;\overline{X}_o\;} & \mathbb{H} \end{array}$$

Conversely any such holomorphic null curve in $SL(2,\mathbb{C})$ projects to a surface of constant mean curvature one in hyperbolic space.

<u>Proof</u>. We first prove the converse. This projection is known to be conformal on null curves so we indeed obtain a smooth surface $\vec{X}_o$, in the pseudosphere. Let $\vec{X}_o = \vec{e}_o$ where $\{\vec{e}_\alpha\}$ is an adapted frame. In a calculation similar to that done previously for $\mathbb{R}^3$

$$\Delta\vec{X}_o = \vec{e}_o + H\vec{e}_3$$

where H is mean curvature. However, because Y is holomorphic, with respect to a complex parameter z,

$$\Delta\vec{X}_o(z) = \frac{2}{F}\frac{d}{dz}\frac{d}{dz}(YY^*) = \frac{2}{F}\,Y'Y'^*,$$

which is a null vector in $\mathbb{L}^4$. Because the frame is pseudo-orthonormal this implies $H \equiv 1$.

Given a constant mean curvature one immersion $\vec{X}_o$, of M into $\mathbb{H}^3$, we seek a holomorphic curve which then necessarily must project conformally into $\mathbb{H}^3$. It can be shown that for the projection to restrict to the curve conformally, the lifting must be null. For such curves, if we pull back the $\mathbb{H}^3$ metric to the Lie algebra, $\mathfrak{sl}(2,\mathbb{C}) \cong \mathbb{C}^3$ we get the standard Hermitian norm on $\mathbb{C}^3$. The left invarient one-form restricted to a holomorphic curve is holomorphic so by the rigidity of holomorphic maps into $\mathbb{C}^n$ with prescribed Hermitian norm, the curve in $\mathfrak{sl}(2,\mathbb{C})$

corresponding to a holomorphic lifting would necessarily be the generalized Gauss map of the associated minimal surface. However, we wish to present Bryant's proof of the theorem in terms that will emphasize the relationship between the holomorphic and adapted frames.

In the notation used previously, where the pullback of the connection forms on the bundle of frames, to the Riemann surface is denoted by the same symbol, the Lie algebra-valued one-form corresponding to an adapted frame is given by

$$\Omega = \frac{1}{2}\begin{pmatrix} i\omega_1^2 & 2\omega - \overline{\tilde{\eta}} \\ \tilde{\eta} & -i\omega_1^2 \end{pmatrix}.$$

From the identities, $\tilde{\eta} = \eta$, $\tilde{f} = f$ and $\theta = \omega$, obtained from the immersion theorem, (see that section) we have that

$$1) \qquad \Omega = \boldsymbol{\Theta} + 2\omega\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \boldsymbol{\Theta} + 2\theta\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let us denote the solution of the equations that these forms generate by

$$\mathbf{X} : M \to SL(2,\mathbb{C}),$$

which corresponds to an adapted frame on the surface in $\mathbb{H}^3$. This is actually the solution to the equations described in the section of the immersion theorem. In order that $\mathbf{X}$ and $\mathbf{Y}$ project to the same surface, it is necessary that

$$\mathbf{X} = \mathbf{Y}h^{-1}$$

for

$$h : M \to SU(2).$$

It was Robert Bryant's beautiful observation that h solves

$$2) \qquad h^{-1}h' = -\Theta$$

which means that h is the lift to SU(2) of an adapted frame of the associated minimal surface in $\mathbb{R}^3$.

Therefore by 2)

$$\begin{aligned} \Omega = \Theta + 2\theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= X^{-1}X' \\ &= (Yh^{-1})^{-1}(Yh^{-1})' \\ &= hY^{-1}Y'h^{-1} + \Theta \end{aligned}$$

and so

$$\begin{aligned} Y^{-1}Y' &= 2\theta h^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h \\ &= 2\theta h^{*}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h \ . \end{aligned}$$

This is the spin action of SU(2) on $u(2) \cong \mathbb{R}^3$, which is isomorphic to the representation of SO(3) on $\mathbb{R}^3$, and is extended to $\mathbb{C}^3$ via multiplication by a complex-valued one-form. But the curve in SO(3) is just the turning of the adapted frame of the associated minimal surface and the one-form is one such that

$$ds^2 = \theta \otimes \overline{\theta}$$

hence this is exactly the generalized Gauss map of the minimal surface. In particular, this is a holomorphic curve in the Lie algebra of SL(2,$\mathbb{C}$) which is null, since the initial vector is, so **Y** is a holomorphic null curve having the original immersion as a projection.

We have shown that premultiplication by a rotation, translation by a null curve in a complex Lie group, and then projecting onto the real slice conformally produces an adapted frame in two different settings. As much as is known about this situation, we feel that there is more to learn. The role of h in particular is not well understood but the twistor theory sheds much light on the rest; indeed, Nigel Hitchin has conjectured a generalization.

We conclude this section with a couple of remarks, both of which are due to Robert Bryant. First, the holomorphic lifting is unique up to a constant, for the kernel of the projection is SU(2), so the difference of two such liftings would be a holomorphic curve in a purely imaginary slice, hence constant. Secondly, we assumed simple connectivity but in absence of this, by continuation we still get a holomorphic lifting, defined on the universal cover and giving a representation of the fundamental group in SU(2) in the natural way.

Polynomial Null Curves

It was shown by Lawson and Do Carmo [Lawson and Do Carmo 1] that a complete embedded hypersurface in $\mathbb{H}^n$ of constant mean curvature 1 with exactly one point in its asymptotic boundary, must be a horosphere. We have a similar result in that direction and we believe that more can be shown. A polynomial null curve would have a single point in its asymptotic boundary but would not necessarily be embedded or even immersed. Let us give definitions.

Definition. By a polynomial null curve we will mean a two-by-two matrix of polynomials, $P(z)$,

$$P(z) = \begin{pmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{pmatrix} \text{ such that } \det(P_{ij}) = 1$$

in $SL(2,\mathbb{C})$, such that the restriction of the Cartan-Killing on $SL(2,\mathbb{C})$ to the curve vanishes. In other words, $\det(P'_{ij}(z)) = 0$. We will view the matrix as a pair of row vectors and will expand these in powers. Therefore, if a_i and b_j denote constant row vectors, we have

$$P(z) = \begin{pmatrix} a_o + a_1 z + a_2 z^2 + \cdots + a_n z^k \\ b_o + b_1 z + b_2 z^2 + \cdots b_n z^n \end{pmatrix}$$

and

$$P'(z) = \begin{pmatrix} a_1 + 2a_2 z + \cdots + n a_n z^{n-1} \\ b_1 + 2b_2 z + \cdots + n b_n z^{n-1} \end{pmatrix}.$$

Now because the nullity condition extends across the singularity and because the addition of a multiple of one row to another is an isometry in the Cartan-Killing metric, we may assume, without loss of generality, that $b_n = 0$. In order to examine the determinant in more depth, we will denote

$$\det\binom{a_i}{b_i} \text{ by } \delta_{ij}.$$

In those terms we have

$$\det(P(z)) = \Sigma_{ij=0}^{n} \delta_{ij} z^{i+j} = \Sigma_{k=0}^{2n-1} \Sigma_{j=\max(0,k-n)}^{\max(k,n-1)} \delta_{k-j,j} \quad z^k$$

and

$$\det(P'z) = \Sigma_{ij=1}^{n} ij\, \delta_{ij} z^{i+j-2} = \Sigma_{k=2}^{2n-1} \Sigma_{j=\max(1,k-n)}^{\max(k-1,n-1)} (k-j)j\delta_{k-j,j} \; z^{k-2}$$

which, by the conditions on determinant, imply:

$$\delta_{\infty} = 1, \quad \Sigma_{j=\max(0,k-n)}^{\max(k,n-1)} \delta_{k-j,j} = 0, \quad k = 1,\cdots,2n-1,$$

and

$$\Sigma_{j=\max(1,k-n)}^{\max(k-1,n-1)} (k-j)j\delta_{k-j,j} = 0 \quad \text{for} \quad k = 2,\cdots,2n-1.$$

Therefore, when $k = 2n-1$ we can infer that

$$b_{n-1} = \beta_1 a_n,$$

when $k = 2n-2$ we find:

$$-\delta_{n-1,n-1} = \delta_{n,n-2} \text{ and } n(n-2)\delta_{n,n-2} = (n^2-2n)\delta_{n,n-2}$$
$$= (n^2-2n+1)\delta_{n-1,n-1}$$

so

$$\delta_{n-1,n-1} = \delta_{n,n-2} = 0,$$

which implies $b_{n-2} = \beta_2 a_n$ and $a_{n-1} = \gamma_1 b_{n-1} = \alpha_1 a_n$ if $\beta_1 \neq 0$. We are now prepared for the induction step.

Suppose now that all δ terms in equations corresponding to $k = 2n - \ell$ are zero for $\ell \leq i - 1$, that $b_{n-\ell} = \beta_\ell a_n$ for $\ell \leq i - 1$, and that $a_{n-i+m+1} = \alpha_m a_n$ if $\beta_1, \beta_2, \cdots,$ or $\beta_m \neq 0$ for $m \leq i - 2$. Also assume $i < n$. Then $K = 2n - i$ implies

$$\Sigma_{j=n-i}^{n-1} \delta_{k-j,j} = 0 = \Sigma_{j=n-i}^{n-1} (k-j) j \delta_{k-j,j}.$$

Let β_s denote the first non-zero β. Assuming $s < i$ implies

$$a_{n-r} = \alpha_r a_n \quad \text{for } r < i - s - 1$$

and $\delta_{k-j,j} = 0$ for $j = n+1 - s, \cdots, n - 1$ by the definition of δ and s. Also $\delta_{k-j,j} = 0$ for $j = n-i+1, \cdots, n-s-1$ since

$$k-j = n-r \geq n-i+s+1 \text{ implies that } j \leq k-n+i-s-1 = n-s-1$$

and $a_{n-r} = \alpha_r a_n$ for these j's while $b_{n-0} = \beta_\ell a_n$ for $\ell \leq i - 1$. Therefore, in the equation for $K = 2n-i$, we have that

$$(k-n+s)(n-s)\delta_{k-n+s,n-s} + (k-n+i)(n-i)\delta_{k-n+i,n-i} = 0$$

$$= 0 = \delta_{k-n+s,n-s} + \delta_{k-n+i,n-i}.$$

This shows that these δ's are zero. If $i = s$ there is only one term, which is then zero, and if $i < s$ then δ's are zero by definition. Further, $\delta_{k-n+i,n-i} = \delta_{n,n-i}$ implies

$$b_{n-i} = \beta_i a_n$$

The Gauss Maps

With the lifting of a surface of constant mean curvature one to $SL(2,\mathbb{C})$ as a holomorphic null curve, we have a tremendously powerful tool which we will now exploit. We first need to find a representation of our hyperbolic Gauss map in this new setting.

In the last section we gave a representation of this Gauss map that used the holomorphic lifting, specifically

$$\vec{e}_o + \vec{e}_3 = \Delta\vec{e}_o = \frac{2}{F}\frac{d}{dz}\frac{d}{d\bar{z}} YY = \frac{2}{F} Y'Y'^*.$$

Recalling that this projection is simply the orbit of the vector (1,0,0,0) under the spin action of $SL(2,\mathbb{C})$ on $\mathbb{L}^4$, we are led to consider the projection of Y' in terms of an action on $\mathbb{L}^4$ of the group $\mathfrak{gl}(2,\mathbb{C})$. Specifically, we will let a constant null Y' act on $\mathbb{L}^4$ by the natural extension of the spin action to $\mathfrak{gL}(2,\mathbb{C})$ and determine the image. Because the Lorentz metric on 2×2 matrices is given by determinant, which vanishes on Y', we see that the image lies in the null cone. Because it is a linear subspace and since we know the image for one point, we conclude that the image of this map is $\mathbb{R}^+(\vec{e}_o+\vec{e}_3)$ or

$$[\vec{e}_o+\vec{e}_3] \in {}^{\mathbb{N}}/_{\mathbb{R}^+} .$$

In particular, we see that the projection of $Y'Y^{-1}$ determines

$$[Y'Y^{-1})(Y'Y^{-1})^*] = [Y'(Y^{-1}Y^{-1*})Y'] = [\vec{e}_o+\vec{e}_3].$$

and $\delta_{k-n+s,n-s} = \delta_{n-i+s,n-s} = 0$ implies

$$a_{n-r} = \alpha_r a_n \quad \text{for } r \leq i-s,$$

that is, $a_{n-i+m+1} = \alpha_m a_n$ if $\beta_1, \beta_2, \cdots,$ or $\beta_m \neq 0$ for $m \leq i-1$.

We have, therefore, verified the induction hypothesis for i. The process stops when max(0,k-n) = 0, for then det(P) has a $\delta_{n,0}$ term that does not appear in det(P'(z)). However, if we consider it as being there with a coefficient of 0, then the above machinery works on more time, giving $\delta_{n,0} = 0$ which implies $b_o = \beta_n a_n$ and we have then shown that

$$P(z) = \begin{pmatrix} a_o + (\Sigma_{r=1}^{n} \alpha_{n-r} z^r) a_n \\ (\Sigma_{r=0}^{n-1} \beta_{n-r} z^r) a_n \end{pmatrix}.$$

Finally, $\det(P(z)) = 1 = (\Sigma_{r=0}^{n-1} \beta_{n-r} z^r) \det\binom{a_o}{a_n}$ implies that

$$\beta_{n-r} = 0 \quad \text{for} \quad r = 1, \cdots, n-1$$

so if we change basis so that $\beta_n a_n = (1,0)$ then we see that

$$P(z) = \begin{pmatrix} \sigma(z) & 1 \\ 1 & 0 \end{pmatrix},$$

which is a horosphere.

Now we look at the Lie algebra of right invariant vector fields on $SL(2,\mathbb{C})$, which is also isomorphic as an algebra to $\mathbb{C}^3$ with a vector cross-product. The preimage of a line in the light cone under the restriction of the above projection to the Lie algebra is complex line in $\mathfrak{sl}(2,\mathbb{C}) \cong \mathbb{C}^3$ which is clearly null. If we projectivize

$$\mathbb{P}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathbb{P}_2$$

then the preimage of the projectivized light cone is the null quadric $Q_1 \subset \mathbb{P}_2$. Both these spaces are diffeomorphically two-spheres and from the explicit representation we will obtain, the projection is an isometry with respect to the Fubini-Study metric on $\mathbb{P}_2$ and a spherical metric on the projectivized light cone.

First we give the map between $\mathbb{P}_1$ and $Q_1 \subset \mathbb{P}\,\mathfrak{sl}(2,\mathbb{C})$,

$$[1,g_2] \mapsto \begin{bmatrix} -g_2 & g_2^2 \\ -1 & g_2 \end{bmatrix},$$

which is the rational normal curve of degree two and hence a biholomorphism. If we represent the projectivized light cone as a unit sphere

$$(1,x_1,x_2,x_3)$$

then the composition of the above map with the projection can be expressed by

$$g_2 = \frac{x_1 - ix_2}{1-x_3},$$

which is simply a stereographic projection. So the projection is a biholomorphism from $\mathcal{Q}_2 \subset \mathbb{P}(\mathfrak{sl}(2,\mathbb{C}))$ to the projectivized light cone represented as a sphere with the usual complex structure. Therefore, the velocity vector of the lifted null curve in "space" coordinates (right invarient coordinates, see [Abraham and Marsden]) is exactly the hyperbolic Gauss map. Now let us examine the velocity vector in "body" coordinates (left invarient coordinates).

The Lie algebra of left invarient vector fields is also obviously isomorphic to $\mathbb{C}^3$. Further, because SL(2,$\mathbb{C}$) is acting on $\mathbb{L}^4$ on the left by isometries, the metric on M induced by the immersion in $\mathbb{H}^3$ is left invarient. Explicitly, we represent an element of the null cone with respect to the Cartan-Killing form on SL(2,$\mathbb{C}$) as metric on M induced by the

$$f_1\begin{pmatrix} -g_1 & (g_1)^2 \\ -1 & g_1 \end{pmatrix}.$$

immersion into $\mathbb{H}^3$. If we denote the induced metric on M by

$$ds^2 = 2F|dz|^2$$

then a calculation shows

$$2F = |f_1|^2(1+|g_1|^2)^2$$

which is simply the matrix norm on $Y^{-1}Y'$. The alert reader will also notice that this is exactly the formula given for the metric of a minimal surface in terms of the functions in the Weierstrass representation. This is of course no accident for as we have pointed out several times previously these are the same functions! Specifically

$$f_1 = h \text{ and } g_1 = g,$$

where h and g are the functions given in the Weierstrass representation of the associated isometric minimal surface, up to rotation.

We would now have as powerful a tool as the Weierstrass representation but for the fact that "integration" of a curve in the Lie algebra of a group is not quite as easy as integration of $\mathbb{R}^3$-valued forms. We can however, almost reduce the problem to vector integration in $\mathbb{C}^4$ with what we have. If we express the right invarient one-form on $SL(2,\mathbb{C})$ restricted to the curve (space coordinates) by

$$Y'Y^{-1} = f_2\begin{pmatrix} -g_2 & (g_2)^2 \\ -1 & g_2 \end{pmatrix}$$

then a calculation shows that we can represent Y' by

$$Y' = f_3\begin{pmatrix} -g_2 & g_1g_2 \\ -1 & g_1 \end{pmatrix}.$$

If we projectivize this vector we obtain a curve in the quadric $Q_2 \subset \mathbb{P}_3$. This quadric is well-known to be isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and indeed the above representation in terms of g's is exactly the Segre imbedding of $\mathbb{P}_1 \times \mathbb{P}_1$ into $\mathbb{P}_3$ as Q_2.

Further extensive calculations yield the relations

$$f_1^2 = f_3^2 \cdot \frac{g_2'(g_1)^2}{g_1'(g_2)^2} \quad \text{and} \quad f_2^2 = f_3^2 \frac{g_1'}{g_2'}$$

which imply that

$$f_2 g_2' = f_1 g_1' = f$$

where this f is KF^2, the function that determines the second fundamental form. Unfortunately this is still one equation short of determining f_3 and allowing us to generate the curve by integration. For determination of the unknown factor we will need the twistor theory we have developed.

Main Theorem

It has long been known, (see [Lawson 1]) that minimal surfaces of finite total curvature have interestingly well-behaved Gauss maps. It is a theorem of Osserman that a Riemann surface that is complete with respect to a metric of finite total curvature is conformally equivalent to a compact Riemann surface punctured at a finite number of points. He also showed that the generalized Gauss map of a minimal immersion into Euclidean space that induces a complete metric of finite total curvature is algebraic and therefore has removable singularities at the singularities of the metric. It was at one time the hope of Lawson, Bryant, and the author that a similar theorem could be proved for hyperbolic space. It is a consequence of the main result of this paper that quite the opposite is true. Finite total curvature imposes no restriction whatever on the hyperbolic Gauss map. Indeed, the Gauss map can be specified completely independently of curvature.

We need now to establish some formalism. We will prove the main theorem in the complex vector space $\mathbb{C}^4$, equipped with nondegenerate quadratic form Δ,

$$\{(\mathbb{C}^4,\Delta)\,|\,(z_{11},z_{12},z_{21},z_{22}) = \vec{z} \in \mathbb{C}^4,\ \Delta(\vec{z},\vec{z}) = z_{11}z_{22}-z_{12}z_{21}\}.$$

The solutions will be contained in the affine quadric:

$$SL(2,\mathbb{C}) = \{\vec{z} \in \mathbb{C}^4\,|\,\Delta(\vec{z},\vec{z}) = 1\}.$$

Derivatives of the solutions will be curves in the null cone

$$\mathbb{N} = \{\vec{z} \in \mathbb{C}^4 | \Delta(\vec{z},\vec{z}) = 0\}.$$

The theorem will be verified by examining the projectivization of the above spaces:

$$\mathbb{P}_1 \times \mathbb{P}_1 \cong Q_2 \cong \mathbb{P}(N) \subset \mathbb{P}(\mathbb{C}^4) = \mathbb{P}_3.$$

To use the projectivized results we need the following:

<u>Note</u>. Given a holomorphic curve $\Psi : M \to \mathbb{P}_3 \setminus Q_2$, we can lift uniquely to a holomorphic curve Φ, in $PSL(2,\mathbb{C}) = {}^{SL(2,\mathbb{C})}/\mathbb{Z}_2$. For $|z| < \varepsilon$ we represent Ψ by

$$\Psi(z) = \begin{bmatrix} \Psi_{11}(z) & \Psi_{12}(z) \\ \Psi_{21}(z) & \Psi_{22}(z) \end{bmatrix},$$

where $\Delta(\Psi,\Psi) \neq 0$. Then $\Phi(z) \equiv \left(\frac{\pm 1}{\sqrt{\Delta(\Psi,\Psi)}}\right)\Psi(z)$ is holomorphic, since $\sqrt{\Delta(\cdot,\cdot)}$ is holomorphic. We recall the definition of null curve in $\mathbb{C}^4$ as one whose tangent lines are all null, i.e., in term of a local parameter z

$$\Delta(\Phi'(z),\Phi'(z)) \equiv 0.$$

If we lift Δ to the exterior algebra $\Lambda^*\mathbb{C}^4$ as the linear extension of

$$\Delta(\vec{v}_1 \wedge \cdots \wedge \vec{v}_k, \vec{w}_1 \wedge \cdots \wedge \vec{w}_k) = \det(\Delta(\vec{v}_i,\vec{w}_j)),$$

defined on simple vectors, we can define the projectivization of a curve $[\Psi(z)] \subset \mathbb{P}_3$ to be <u>null</u> by requiring that

1) $\Delta(\Psi \wedge \Psi', \Psi \wedge \Psi') \equiv \Delta(\Psi,\Psi)\Delta(\Psi',\Psi') - \Delta(\Psi,\Psi')^2 \equiv 0.$

One easily checks that this condition is independent of local parameter and homogeneous representation, Ψ.

Proposition 1. Where $\Delta(\Psi,\Psi) \neq 0$, $\Phi(z)$ is defined and $\Phi(z)$ being null is equivalent to $[\Psi(z)]$ being null.

Proof. Taking the derivative of $\Delta(\Phi,\Phi) = 1$ implies $\Delta(\Phi,\Phi') = 0$ and the claim then follows from 1).

We remark that if $\Delta(\Psi,\Psi) \equiv 0$, $[\Psi(z)]$ is null while no such concept is defined for a homogeneous representation, Ψ.

For this nondegenerate quadratic form we can define a "Hodge duality operator," $*$ by

$$* : \Lambda^k \mathbb{C}^n \to \Lambda^{n-k}\mathbb{C}^n$$

$$\vec{v} \wedge * \vec{w} = (\vec{v},\vec{w})\omega \qquad \vec{v},\vec{w} \in \Lambda^k \mathbb{C}^n,$$

where $\omega \in \Lambda^n \mathbb{C}^n$ is $\vec{e}_1 \wedge \cdots \wedge \vec{e}_n$ for any oriented orthonormal basis of $\mathbb{C}^n$. That this map is well-defined is verified exactly as in the real case. We also define the span:

$$\text{span}(\vec{w}) = \{\vec{v} \in \mathbb{C}^n | \vec{v} \wedge \vec{w} = 0\} \quad \vec{w} \in \Lambda^k \mathbb{C}^n.$$

Note. span$(\vec{w})$ is a linear subspace of $\mathbb{C}^n$ of dimension k or less. For simple vectors $\vec{w}_1 \wedge \cdots \wedge \vec{w}_k$,

$$\text{span}(\vec{w}_1 \wedge \cdots \wedge \vec{w}_k) = \text{span}(\{\vec{w}_i\}),$$

the span defined here coincides with the usual definition.

Lemma 1. Let $S \subset \mathbb{C}^n$ be a k-dimension subspace of $\mathbb{C}^n$. There is a basis $\{\vec{e}_i\}$ of $\mathbb{C}^n$ s.t. $\mathrm{span}(\vec{e}_1,\cdots,\vec{e}_k) = S$ and the columns of $\Delta(\vec{e}_i,\vec{e}_j)$ are a permutation of those of the identity matrix $1 \le i,j \le n$.

Proof. If Δ is nondegenerate on S then one diagonalizes Δ on S and extends to an orthonormal basis via Witt's theorem. If Δ is degenerate, one again diagonalizes Δ on S with respect to $\{\vec{e}_1,\cdots,\vec{e}_k\}$ such that $\mathrm{span}(\vec{e}_1,\cdots,\vec{e}_i)$ is isotropic and $\{\vec{e}_{i+1},\cdots,\vec{e}_k\}$ are orthonormal. Let

$$\mathbb{C}^n = T \oplus_\perp \mathrm{span}(\vec{e}_{i+1},\cdots,\vec{e}_k)$$

and note that by Witt's theorem Δ is nondegenerate on T. Therefore, because an isotropic plane in S, has dimension less than or equal to half the dimension of S there is a null vector, $\vec{e}_{k+1}$, such that $\Delta(\vec{e}_1,\vec{e}_{k+1}) = 1$. Because Δ is nondegenerate on $\mathrm{span}(\vec{e}_1,\vec{e}_{k+1})$, it is nondegenerate on T' such that $T = T' \oplus_\perp \mathrm{span}(\vec{e}_1,\vec{e}_{k+1})$ so we repeat this procedure until the $\{\vec{e}_1,\cdots,\vec{e}_i\}$ are exhausted. Then

$$T = R \oplus_\perp \mathrm{span}(\vec{e}_1,\cdots,\vec{e}_i,\vec{e}_{k+1},\cdots,\vec{e}_{k+i}),$$

such that Δ is nondegenerate on R so by diagonalizing it there, we are done.

Lemma 2. $\mathrm{span}(*(\vec{v}_1\wedge\cdots\wedge\vec{v}_k)) \perp \mathrm{span}(\vec{v}_1\wedge\cdots\wedge\vec{v}_k)$.

Proof. Let $\{\vec{e}_1,\cdots,\vec{e}_n\}$ be a basis given by lemma 1 for $S = \mathrm{span}(\vec{v}_1\wedge\cdots\wedge\vec{v}_k)$. Note that

$$\mathrm{span}(*\vec{v}_1\wedge\cdots\wedge\vec{v}_k)) = \mathrm{span}(*\vec{e}_1\wedge\cdots\wedge\vec{e}_k) = \{\vec{u}\in\mathbb{C}^n \mid \vec{u}\wedge(*\vec{e}_1\wedge\cdots\wedge\vec{e}_k)=0\}$$

$$= \{\vec{u}\in\mathbb{C}^n \mid \Delta(\vec{u}\wedge\vec{q},\vec{e}_1\wedge\cdots\wedge\vec{e}_k) = 0 \quad \forall\vec{q}\in\Lambda^{k-1}\mathbb{C}^n\}.$$

Because of the definition of the $\vec{e}$'s,

$$\vec{q} = \vec{e}_{k+1}\wedge\cdots\wedge\vec{e}_{k+i}\wedge\vec{e}_{i+1}\wedge\cdots\wedge\hat{\vec{e}}_m\wedge\cdots\wedge\vec{e}_k$$

implies $\vec{u}\perp\vec{e}_m$ if $\Delta(\vec{e}_m,\vec{e}_m) = 1$ and letting

$$\vec{q} = \vec{e}_{k+1}\wedge\cdots\wedge\hat{\vec{e}}_{k+m}\wedge\cdots\wedge\vec{e}_{k+i}\wedge\vec{e}_{i+1}\wedge\cdots\wedge\vec{e}_k$$

shows $\vec{u}\perp\vec{e}_m$ if $\Delta(\vec{e}_m,\vec{e}_m) = 0$.

Proposition 2. For $\vec{w}\in\Lambda^k\mathbb{C}^n$ $\mathrm{span}(*\vec{w}) \subset (\mathrm{span}(\vec{w}))^\perp$.

Proof. Let $\{\vec{e}_1,\cdots,\vec{e}_k\}$ for $0 \le 1 \le k$ be a basis for $\mathrm{span}(\vec{w})$ of the sort guaranteed by Lemma 1. With respect to such a basis, w is a sum of simple vectors which are pairwise orthogonal with respect to the lifting of Δ. Therefore, since

$$\mathrm{span}(*\vec{w}) = \{\vec{u}\in\mathbb{C}^n \mid \vec{u}\wedge *\vec{w} = 0\}$$

$$= \{\vec{u}\in\mathbb{C}^n \mid \Delta(\vec{u}\wedge\vec{q},\vec{w}) = 0 \quad \forall q\in\Lambda^{k-1}\mathbb{C}^n\}$$

if $\vec{u}\wedge\vec{q}$ is orthogonal to $\vec{w}$ it is orthogonal to each component. Then the same reasoning used to finish Lemma 2 shows $\vec{u}$ is orthogonal to the span of each component.

The geometric interpretation is clear; we can express a nonsimple element in $\Lambda^k\mathbb{C}^n$ as a sum of k-planes so that span, as defined here, is the intersection of those planes and the span of the dual is the subspace orthogonal to those planes. We could then define a span that would be the span of all the vectors in these planes as the orthogonal space to the span of the dual. However, because we will be concerned with associated curves, which are simple, we will henceforth use the term dual to mean both the dual form and the orthogonal space that it determines.

We will continue to denote a holomorphic null curve by $Y : M \to SL(2,\mathbb{C})$, noting now that the projective class of this curve $[Y]$, is also a null curve. Let $\vec{Y}$ be the projection of Y into $\mathbb{H}^3$. We will use an euclidean coordinate on $\mathbb{P}_1$ and thereby refer to the Gauss maps as g_1 and g_2 determined by

$$[Y^{-1}Y'] = \begin{bmatrix} -g_1 & (g_1)^2 \\ -g & g_1 \end{bmatrix}$$

$$[Y'Y^{-1}] = \begin{bmatrix} -g_2 & (g_2)^2 \\ -1 & g_2 \end{bmatrix}$$

projectively. We can now state the first version of the main theorem.

<u>Theorem 1</u>. Given a Riemann surface and two meromorphic functions having at worst simple poles, there is a null curve in $SL(2,\mathbb{C})$ which is holomorphic where ${}^{dg_1}/_{dg_2} \neq 0, \infty$ and having the projective class

$$\begin{bmatrix} -g_2 & g_1 g_2 \\ -1 & g_1 \end{bmatrix}$$

as its tangent direction. Further, the curve is locally an immersion near points satisfying

$$\{g_1 \circ g_2^{-1}, z\} \neq 0$$

where $\{f(z),z\}$ is the Schwartzian derivative, $(\ddot{f}/\dot{f})^{\cdot} - \frac{1}{2}(\ddot{f}/\dot{f})^2$, and z is the coordinate of the range of g_1.

<u>Proof</u>. The curve is given by

$$\begin{pmatrix} \frac{-g_2}{\dot{g}_2}\sqrt{\frac{\dot{g}_2}{\dot{g}_1}}^{\,\cdot} + \sqrt{\frac{\dot{g}_2}{\dot{g}_1}} & \frac{\dot{g}_1 g_2 - \dot{g}_2 g_1}{\dot{g}_2}\sqrt{\frac{\dot{g}_2}{\dot{g}_1}} + \frac{g_1 g_2}{\dot{g}_2}\sqrt{\frac{\dot{g}_2}{\dot{g}_1}}^{\,\cdot} \\ \frac{-1}{\dot{g}_2}\sqrt{\frac{\dot{g}_2}{\dot{g}_1}}^{\,\cdot} & \frac{\dot{g}_1}{\dot{g}_2}\sqrt{\frac{\dot{g}_2}{\dot{g}_1}} + \frac{g_1}{\dot{g}_2}\sqrt{\frac{\dot{g}_2}{\dot{g}_1}}^{\,\cdot} \end{pmatrix}.$$

Verification is left to the reader.

<u>Remark</u>. We have shown previously that the ratio of spherical area of g_1 and the induced metric area on the Riemann surface is the curvature of that metric. Although a similar theorem would be nice for the hyperbolic Gauss map, as we mentioned the sphere at infinity inherits only a conformal structure. Now a choice of origin in $\mathbb{H}^3$ determines a representative metric and then the theorem has meaning but is only true at the origin. However, there is a bound on the degree to which the theorem fails, which is an increasing function of distance to the origin. Knowledge of the rate at which a function goes

to infinity can give some control on total curvature. Conversely, knowledge of both Gauss maps can give information about distance to the origin. For instance, if $\vec{y}_0$ is the projection of Y into the pseudosphere and X° the time coordinate on the ambient Minkowski space then

$$dx^\circ(\vec{y}) \geq \left\| \frac{1}{\dot{g}_2} \sqrt{\frac{\dot{g}_2}{\dot{g}_1}} \right\|^2 .$$

We will show the existence of the null curve via twistor theory so we now describe our twistor space in more detail. As we have mentioned the twistor surface for a three-dimensional space of constant curvature is the space of geodesics on the three-manifold. For $\mathbb{H}^3$ represented as the pseudosphere in $\mathbb{L}^4$, geodesics correspond to intersections of the pseudo-sphere with time-like 2-planes in $\mathbb{L}^4$. We orient the geodesics and parameterize the space of geodesics by specifying the geodesic by its ordered "endpoints," that is, the intersection of the 2-plane with the null cone. The space of geodesics is then parameterized by the product of spheres, minus the diagonal, and the conformal structure inherited by the boundary is exactly that which makes the parameter space conformally $\mathbb{P}_1 \times \mathbb{P}_1$. This can be seen more directly in the complexification, $\mathbb{L}^4 \otimes_{\mathbb{R}} \mathbb{C}$. The 2-planes are the intersections of null hyperplanes with the real slice, where the null direction dual to the hyperplane is not real. There the parameter space $Q_2 \backslash (\mathbb{P}_1 \amalg \mathbb{P}_1) \cong \mathbb{P}_1 \times \mathbb{P}_1 \backslash (\mathbb{P}_1 \amalg \mathbb{P}_1)$. These considerations osculating hyperplane. If we lift the given curve to $\vec{v}$ the curve is given irrespective of lifting by

$$[*\vec{v} \wedge \vec{v}' \wedge \vec{v}''] .$$

In $\mathbb{C}^4$ the derivative of this dual curve is tangent to

$$*(\vec{v} \wedge \vec{v}' \wedge \vec{v}'' + t\vec{v} \wedge \vec{v}' \wedge \vec{v}''') .$$

This is a simple vector for all t which, by the nature of the Hodge dual, is for every t, orthogonal to $\vec{v} \wedge \vec{v}'$. The first order osculating plane is then $*(\vec{v} \wedge \vec{v}')$, which can be written as $\vec{v} \wedge (*(\vec{v} \wedge \vec{v}' \wedge \vec{v}''))$.

<u>Theorem 2</u>. Given a holomorphic curve in

$$\mathbb{P}_1 \times \mathbb{P}_1 \cong Q_2 \subset \mathbb{P}_3 \cong \mathbb{P}(\mathfrak{sl}(2,\mathbb{C}))$$

then, away from points where the original curve is tangent to one of the factors of $\mathbb{P}_1 \times \mathbb{P}_1$, there is a holomorphic curve in PSL(2,$\mathbb{C}$) whose derivative, where non-zero, is the given null direction.

<u>Proof</u>. Using the notation of Proposition 3, by what we showed at the beginning of this section, where $*\vec{v} \wedge \vec{v}' \wedge \vec{v}''$ is not null, the desired curve is

$$\pm(\Delta(*\vec{v} \wedge \vec{v}' \wedge \vec{v}'', *\vec{v} \wedge \vec{v}' \wedge \vec{v}''))^{-\frac{1}{2}} \cdot (*\vec{v} \wedge \vec{v}' \wedge \vec{v}'') .$$

If $*\vec{v} \wedge \vec{v}' \wedge \vec{v}''$ is null then we have shown $\vec{v} \wedge \vec{v}' \wedge \vec{v}''$ is. The determine the complex structures that we will examine but not the representation we will give them.

Specifically we will examine $\mathbb{C}^4 \cong \mathfrak{gl}(2,\mathbb{C})$ with non-degenerate quadratic form Δ which is the polarization of determinant on $\mathfrak{gl}(2,\mathbb{C})$. $SL(2,\mathbb{C})$ is the affine quadric determined by $\Delta(g,g) = 1$ and $\mathbb{H}^3$ is the real slice determined by the anti-holomorphic involution, Hermitian transpose. $\mathbb{H}^3$ is the image under exp of the real part of the Lie algebra $SL(2,\mathbb{C})$ and such elements of $SL(2,\mathbb{C})$ are described in terms of the corresponding Lorentz transformations as "boosts." We will consider the projectivization $\mathbb{P}(\mathfrak{gl}(2,\mathbb{C})) \cong \mathbb{P}_3$ with null quadric $Q_2 \cong \mathbb{P}_1 \times \mathbb{P}_1$, of Δ, with the biholomorphism being given by the Segré imbedding

$$[1,g_1] \times [1,g_2] \to \begin{bmatrix} -g_2 & g_1 g_2 \\ -1 & g_1 \end{bmatrix}.$$

The use of g_1 and g_2 here is not accidental and connects the two main theorems of this section.

<u>Proposition 3</u>. Given a holomorphic curve in

$$\mathbb{P}_1 \times \mathbb{P}_1 \cong Q_2 \subset \mathbb{P}_3 \cong \mathbb{P}(\mathfrak{gL}(2,\mathbb{C}))$$

there is a holomorphic null curve in $\mathbb{P}_3$ whose first order osculating plane is at each point spanned by position and the null direction determined by the given curve at that point.

<u>Proof</u>. The curve is given by the dual of the second order

determinant defined by the above lifting of Δ implies $\Delta(\vec{v}',\vec{v}') = 0$ or $\Delta(\vec{v},\vec{v}'') = 0$. By differentiating $\Delta(\vec{v},\vec{v}) = 0$ we find that $\Delta(\vec{v}',\vec{v}') = \Delta(\vec{v},\vec{v}'')$ so we have that $\Delta(\vec{v}',\vec{v}') = 0$. Therefore the first order osculating plane, $\mathrm{span}(*(v\wedge v'))$ is isotropic and the original curve is tangent to a factor as promised.

It is possible to compute this curve explicitly. From equations 3) in our twistor section, describing the restriction of hyperplane sections of $0(1)$ on $\mathbb{P}_3$ restricted to $\emptyset_2$, it is obvious that the intersection of a non-null hyperplane in $\mathbb{P}_3$ with $\emptyset \cong \mathbb{P}_1 \times \mathbb{P}_1$ is the graph of a Möbius transformation from $\mathbb{P}_1$ to $\mathbb{P}_1$. Where the given curve in $\emptyset_2 \cong \mathbb{P}_1 \times \mathbb{P}_1$ is not tangent to one of the factors, it can also be locally represented as the graph of some function η, from $\mathbb{P}_1$ to $\mathbb{P}_1$. The second order osculating hyperplane is then simply the graph of the Möbius transformation that agrees with η to second order at each point. This generates a curve in $PSL(2,\mathbb{C})$ which is the same as that found above. It is given locally by

$$\pm \frac{1}{\sqrt{\eta'}} \begin{pmatrix} \dfrac{-\eta\cdot\eta''}{2\eta'} + \eta' & \eta + z\,\dfrac{\eta\cdot\eta''}{2\eta'} - \eta' \\[2ex] \dfrac{-\eta''}{2\,'} & 1 + z\,\dfrac{\eta''}{2\eta'} \end{pmatrix}$$

where z is the argument of $\eta(z)$. A simple but exceedingly messy computation shows this to be equivalent to our preceding representation for $\eta = g_1 \circ g_2^{-1}$.

Finally, we verify the claim made in Theorem 1 concerning the vanishing of the derivative of this curve. Interestingly enough, as with every other tool of this section, we find another way of generating all holomorphic null curves in $SL(2,\mathbb{C})$. The Schwartzian derivative is a known invariant of Möbius transformations, which of course is zero for a Möbius transformation. Hence, one might ask if it in some way measures the deviation of a function from the osculating Möbius transformation. Since the derivative of our curve is exactly this deviation, we are in a position to make that characterization.

As previously mentioned, the Schwartzian derivative of $\eta(z)$, with respect to z, is given by

$$\{\eta,z\} = \left(\frac{\eta''}{\eta'}\right)' - \frac{1}{2}\left(\frac{\eta''}{\eta'}\right)^2 = \frac{1}{\sqrt{\eta'}}\frac{d^2}{dz^2}(\sqrt{\eta'})$$

In our case $\eta = g_1 \circ g_2^{-1}$ and we will denote $\{\eta,z\} = 2q$. To the fanciers of the Schwartzian it is well known that (see [Goluzin 1]) if w_1, w_2 are two independent solutions of an associated second order differential equation

2) $$w'' + qw = 0$$

then $\eta = \dfrac{w_1}{w_2}$ satisfies $\{\eta,z\} = 2q$.

By Abel's formula the Wronskian of solutions to 2) are constant, which we normalize to be 1. This implies

$$w_1 = \frac{\eta}{\sqrt{\eta'}} \text{ and } w_2 = \frac{1}{\sqrt{\eta'}}.$$

Therefore, the Wronskian matrix of these solutions is a holomorphic curve in $SL(2,\mathbb{C})$. Though this curve is not null itself, nullity is surprisingly easy to obtain. Consider

$$W = \begin{pmatrix} w_1' & w_1 - zw_1' \\ w_2' & w_2 - zw_2' \end{pmatrix}.$$

This alteration clearly does not affect the determinant but observe what it does to the derivative:

$$W' = \begin{pmatrix} w_1'' & w_1' - w_1' - zw_1'' \\ w_2'' & w_2' - w_2' - zw_2'' \end{pmatrix}$$

$$= q\begin{pmatrix} -w_1 & zw_1 \\ -w_2 & zw_2 \end{pmatrix}.$$

Since the Wronskian of solutions does not vanish, the derivative of W is non-zero when $\{\eta,z\}$ is non-zero. Another simple but messy calculation shows that W is exactly the curve presented previously. Thus the derivative of our null curve vanishes precisely when

$$\{g_1 \circ g_2^{-1}, z\} = 0.$$

Examples

We have shown that restrictions of curvature impose no restrictions on the hyperbolic Gauss map, which is possibly the most marked difference between the hyperbolic and Euclidean cases. So now, after having spent most of the paper exploiting the similarities between these situations, we conclude by using the results to generate examples that demonstrate some of the differences. We also generate branched surfaces with any finite number of points in their asymptotic boundaries.

We develop some formalism that will allow us to talk about the asymptotic boundary. At the end of the last section, we showed that two independent normalized solutions, w_1, w_2 of the equation

$$w'' + qw = 0,$$

generated holomorphic null curves of the form

$$1)\quad \begin{pmatrix} w_1' & w_1 - zw_1' \\ w_2' & w_2 - zw_2' \end{pmatrix}.$$

When projected to $\mathbb{H}^3$, by gg^*, the result is

$$(1+z\bar{z})(|w_1'|^2+|w_2'|^2)\left(\frac{1}{|w_1'|^2+|w_2'|^2}{}^2\begin{pmatrix} w_1' \\ w_2' \end{pmatrix}(\bar{w}_1', \bar{w}_2')+R(z)\right)$$

where the matrix $R(z)$ goes to 0 in two cases. The first case is curves for which

$$\frac{|w_1|^2 + |w_2|^2}{|w_1'|^2 + |w_2'|^2}$$

is bounded as z goes to infinity. Then R(z) consists of terms of order z^{-1}. The other case involves poles of w_1 and w_2. Then w' terms dominate w terms by one order and R contains terms on the order of

$$\frac{(|w_1|^2 + |w_2|^2)^{\frac{1}{2}}}{(|w_1'|^2 + |w_2'|^2)^{\frac{1}{2}}}$$

which goes to zero at a pole.

In these cases the limit of the projection in the asymptotic boundary of $\mathbb{H}^3$, given in terms of homogeneous coordinates on $\mathbb{P}_1$ described earlier, is

$$[w_1', w_2'] .$$

It is worth noting that while we will find that in most cases the limit is given by the value of the hyperbolic Gauss map, that map is always given by

$$[w_1, w_2],$$

which, a priori, would seem to have little chance of being the asymptotic boundary.

We now consider two solutions that are natural in our setting but are due to Robert Bryant. The first is generated by e^{2z} and corresponds to a constant Schwartzian. It is given by

$$\begin{pmatrix} e^z/_2 & (1-z)e^z \\ -e^{-z}/_2 & (1+z)e^{-z} \end{pmatrix}$$

The left invariant one-form restricted to the curve is

$$2) \qquad \begin{pmatrix} z & -2z^2 \\ \frac{1}{2} & -z \end{pmatrix} dz$$

and the hyperbolic Gauss map is given by

$$[1, e^{2z}].$$

From 2) we see that this immersion is isometric to Enneper's surface but the hyperbolic Gauss map clearly has an essential singularity at the boundary. This solution corresponds to $w_1 = e^z$ and $w_2 = e^{-z}$, which is an example of case 1 in our discussion of asymptotic boundary. Hence, the curve itself comes arbitrarily close to $[1, e^{2z}]$ for large z, so the entire null cone is in the asymptotic boundary of this surface. Clearly this is not an embedding.

The second example is generated by $z^r = z^{2\mu+1}$, $r > 0$, $r \neq 1$ and is given by

$$\frac{1}{\sqrt{2\mu+1}} \begin{pmatrix} (\mu+1)z^{\mu} & \mu z^{-(\mu+1)} \\ \mu z^{\mu+1} & (\mu+1)z^{-\mu} \end{pmatrix}.$$

The curve for negative r is a reflection of the curve for positive r and r = 1 generates a constant. These are beautiful examples, all being surfaces of revolution, and Robert Bryant does a thorough analysis of them so we quote here just the results. We feel them to be quite necessary to include here because we feel that their behavior at infinity might be a model for all surfaces that approach a single boundary point at a pole.

The projection of these surfaces can be written in the form

$$\frac{1}{2\mu+1}\begin{pmatrix} \mu+1 & \mu/z \\ \mu z & \mu+1 \end{pmatrix}\begin{pmatrix} (z\bar{z})^{\mu} & 0 \\ 0 & (z\bar{z})^{-\mu} \end{pmatrix}\begin{pmatrix} \mu+1 & \mu\bar{z} \\ \mu/\bar{z} & \mu+1 \end{pmatrix}$$

which shows that the projection is single-valued even though the lifting is multivalued. Bryant computes the metric and finds that the total curvature is $-4\pi(2\mu+1)$, which is clearly not quantized as it is for minimal surfaces. Bryant also finds that these surfaces of revolution have profile curves that are embedded for $\frac{-1}{2} < \mu < 0$ and have a single self-intersection for $\mu > 0$.

The next most logical examples to consider are solutions generated by polynomials. We will consider only polynomials whose derivatives have simple zeroes; that is, the branch points of the map all have multiplicity one. We will denote the polynomial by $\eta(z)$ and its Schwartzian derivative by $2q = \{\eta(z),z\}$. In order to describe asymptotic behavior we will need to consider solutions of

$$w'' + qw = 0,$$

which are given by

3) $$w_1 = {}^1/\sqrt{\eta'}, \quad w_2 = {}^{\eta}/\sqrt{\eta'}$$

and $$w_1' = \frac{-1}{\sqrt{\eta'}}({}^{\eta''}/{}_{2\eta'}), \quad w_2' = \frac{-1}{\sqrt{\eta'}}\left(\frac{\eta\eta'' - 2(\eta')^2}{2\eta'}\right)$$

We now ask how the solution behaves at zeroes of η'. Because this implies that the corresponding graph is horizontal, the null curve and its projection go to infinity. From the discussion at the beginning of the section, we must determine which terms in 3) dominate. However, we can ignore the common factor and otherwise w_1 and w_2 are finite, so the point in the asymptotic boundary is

$$[w_1', w_2'] = [1, {}^{w_2'}/{}_{w_1'}] = [1, \quad \eta - \frac{2(\eta')^2}{\eta''}]$$

$$= [1, \quad \eta].$$

From the same formula we see that at the poles of η', w_1' and w_2' again dominate and the limit in the asymptotic boundary is [0, 1]. However, if η' has a pole then η goes to infinity so we again have the limit in the boundary to be $[1, \eta]$. In particular, we have shown:

<u>Proposition</u>. Given a holomorphic null generated via our technique from a meromorphic function on $\mathbb{P}_1$, the limits in the asymptotic boundary of singularities of the projection to $\mathbb{H}^3$

are given by the values of the hyperbolic Gauss map at the singular points of η'.

By the addition of a small linear term it is possible to have the polynomial η take distinct values at the singular points of η' so we have produced branched surfaces of constant mean curvature one which have an arbitrary finite number of distinct points in the asymptotic boundary. Also, because we are not concerned with location of the singular points in the domain of η, we have considerable freedom in locating these points in the boundary.

In an effort to eliminate this branching, we have sought functions whose Schwartzian derivative does not vanish. To this end we have found functions that generate immersions of $\mathbb{P}_1$, punctured at an arbitrary finite number of points. However, the behavior of the immersions is not yet well understood because we know η' but not η. These surfaces are generated in the fashion used at the beginning of the Main Theorem section, using two Gauss maps. They are

$$g_1 = \int \frac{dz}{(z^n+1)^2} \text{ and } g_2 = z^{2n-1}.$$

The integral does have periods so the behavior of this function is hard to characterize. This integral is of the type in the Schwartz-Christoffel formula, which describe maps from the disc to circular polygons, but the polygon corresponding to this example has all of its vertices at infinity.

The symmetry of the situation suggests that the periods of g_1 give rise to representations of the fundamental group in SO(3) that are cyclic but this is not yet known. However, it is known that the Schwartzian derivative does not vanish. To compute it we use a composition rule

$$\{g_1 \circ g_2^{-1}, x\} = \left. \frac{[\{g_1, z\} - \{g_2, z\}]}{(g_2')^2} \right|_{x = g_2(z)}$$

We note that from this formula, the Schwartzian derivative takes value in the line bundle $0(4)$ on $\mathbb{P}_1$. The Schwartzian of the above functions is

$$\left. \frac{-\dfrac{4\binom{n}{2} z^{n-2}}{(z^n + 1)} - \dfrac{(1 - 4n^2 - 4n - 1)}{2z^2}}{(2n - 1)^2 z^{4n-4}} \right|_{x^{\frac{1}{(2n-1)}}}$$

$$= -4\binom{n}{2}\left[\frac{z^{n-2}}{(2n-1)^2} \quad \frac{1}{x^2(x^{n/(2n-1)} + 1)}\right]$$

which, on $0(4)$, does not vanish, even at singularities. We mention that similar functions were used by Meeks [Meeks (1)] to generate new minimal surfaces.

Finally, we close by elaborating on the statement by Bryant that numerous examples may be found via the techniques of algebraic geometry. One technique is as follows: Given holomorphic null curves in $\mathbb{C}^3$, which correspond to minimal surfaces, we can map them to null curves in the affine quadric of a one higher dimensional space via stereographic projection, which is conformal in the standard holomorphic metric.

Specifically: compactify $\mathbb{C}^3$ as the image of a euclidean coordinate chart on $\mathbb{P}_3$, then embed this as a hyperplane in $\mathbb{P}_4$. One then stereographically projects onto a nondegenerate quadric minus two lines and then the image of this quadric by a euclidean coordinate chart onto $\mathbb{C}^4$ is exactly the affine quadric. The drawback here is that most null curves in $\mathbb{C}^3$ are generated by the Weierstrass representation and have imaginary periods. It is not yet clear exactly what conditions guarantee that surface would be well-defined after the projection into $\mathbb{H}^3$. Clearly, however, there should be more examples, so this area of research should still have considerable room for expansion.

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