On Cohomology of Kleinian Groups

A Dissertation presented

by

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to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

December, 1984

STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

THE GRADUATE SCHOOL

Dipendra Chandra Sengupta

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This dissertation is accepted by the Graduate School.

Dean of the Graduate School

December 1984

Abstract of the Dissertation

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Let Γ be a non-elementary finitely generated Kleinian group with region of discontinuity Ω. Let q be an integer, $q \geq 2$. The group Γ acts on the right on the vector space Π_{2q-2} of polynomials of degree less than or equal to $2q-2$ via the Eichler action. The corresponding cohomology group is denoted by $H^1(\Gamma, \Pi_{2q-2})$. The subspace of parabolic cohomology classes is denoted by $PH^1(\Gamma, \Pi_{2q-2})$. Denote by $A_q(\Omega, \Gamma)$ the space of cusp forms for Γ of weight $(-2q)$. Bers introduced an anti-linear map

$$\beta_q^* : A_q(\Omega, \Gamma) \to PH^1(\Gamma, \Pi_{2q-2})$$

and proved that this mapping is injective.

We try to determine the class of Kleinian groups for which the Bers map is surjective. We show that the Bers map is surjective for geometrically finite function groups. Nakada proved this result for $q = 2$ in 1976. Our main theorem gives the following characterization of geometrically finite function groups:

Let Γ be a non-elementary finitely generated Kleinian group with an invariant component. Then Γ is geometrically finite if and only if the Bers map is surjective for one (hence all) $q \geq 2$ with the exception of $q = 3, 5$ or 7. For $q = 3, 5$ or 7 the Bers map is surjective for some Kleinian groups which are not geometrically finite.

We also prove $\dim H^1(\Gamma, \Pi_{2q-2}) = \dim PH^1(\Gamma, \Pi_{2q-2}) + N(q)$ for finitely generated function groups Γ (where $N(q)$, the number of inequivalent q-admissible parabolic fixed points for Γ). Kra proved the same equality for geometrically finite groups Γ.

As an application of the above equality, we reprove a theorem of Maskit on inequality involving the dimension of the space of cusp forms supported on an invariant component and the dimension of the space of cusp forms supported on the other components for finitely generated function groups.

To my mother and father.

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Acknowledgements

I am beholden to

my adviser Professor Irwin Kra for his patience, his encouragement and for his many contributions in the preparation of this thesis;

the members of the faculty of Mathematics in particular Professor Bernard Maskit for his uncommon patience to answer my many questions, as well as Professor Jeff Cheeger whose initial inspiration was the cause of much of this;

Jharna Dana for her extreme tolerence and for her help in both mathematics and non-mathematics;

Nimai Ghosh whose generosity knows no bounds, as well as to Ara Basmajian, Gadadhar Misra, Shiva Shankar and Tahir Sheikh for their concern;

my parents for enduring my wild childhood, as well as to my brothers and sisters for bearing my long silliness;

Estelle Shivers for her beautiful typing.

To all those gentlefolks, I offer my heartfelt thanks.

Introduction

There are a number of theorems on the structure of finitely generated Kleinian groups whose proofs depend on the cohomology theory introduced by Eichler for Fuchsian groups. Foremost among these are Ahlfors' finiteness theorem [1] and Bers' area theorem [3]. Recently, Sullivan [20] used cohomology to prove that there are finitely many conjugacy classes of maximal parabolic subgroups for finitely generated Kleinian groups. More recently, Kra [12] proved theorems on vanishing and bases for cusp forms using the cohomology machinery. Also, Gardiner and Kra [5] derived a cohomological condition for stability (to be defined in Section II). They proved that finitely generated Kleinian groups are both quasi-conformally stable and quasi-stable if the Bers map is surjective for $q = 2$. Sakan [19] proved that this condition is also necessary. Marden [14] showed, using 3-manifolds, that a finitely generated Kleinian group without torsion is both quasi-stable and quasi-conformally stable if it is geometrically finite. Then by Sakan's theorem, we conclude that the Bers map is surjective for $q = 2$ for torsion free geometrically finite groups. On the otherhand, the Bers map is not surjective for degenerate groups; these are finitely generated Kleinian groups which are not geometrically finite.

Now it is of interest to see whether the Bers map is surjective for all $q \geq 2$ for geometrically finite groups. The affirmative answer of this question will provide us with a new structure theorem for parabolic cohomology groups, and for $q = 2$, it will imply that all geometrically finite groups are quasi-conformally stable.

The study of cohomology theory are of interest, since it may reveal further information about Kleinian groups. In this dissertation we have mainly studied Eichler cohomology groups. We have shown that the Bers map is surjective for all $q \geq 2$ for geometrically finite function groups. Nakada [18] proved the same result for $q = 2$ in his study of quasi-conformal stability of such groups.

In Section 1, we give some preliminary definitions and statements of some known results concerning Kleinian groups and their automorphic forms. Section 2 is devoted to cohomology theory and the structure of these cohomology groups. In Section 3, we give a list of the dimensions of the cohomology groups for all elementary groups. In Section 4, we show the surjectivity of the Bers map for geometrically finite function groups and we also give a new characterization of such groups. We obtain a formula for the dimension of certain cohomology groups for finitely generated function groups. In Section V, we give a new proof of Maskit's theorem on inequalities involving the dimension of the space of cusp forms supported on an invariant component and the dimension of the space of cusp forms supported on the other components for finitely generated function groups by using our cohomology machinery.

SECTION I

Preliminaries.

In this section we summarize some basic facts in the theory of Kleinian groups and their automorphic forms. The material in this section is presented in details in Kra [10].

§1. We shall be studying groups Γ whose elements are Möbius transformations; that is, mappings of the form

$$\gamma : z \to \frac{az+b}{cz+d}, \quad ad-bc = 1.$$

Hence, the elements of Γ are conformal self-mappings of the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Let Γ be a subgroup of the group of all Möbius transformations. Then for $z \in \mathbb{C} \cup \{\infty\}$, we let Γ_z denote the stablizer of z; that is,

$$\Gamma_z = \{\gamma \in \Gamma;\ \gamma z = z\}.$$

We shall say that Γ is discontinuous at z if

(i) Γ_z is finite, and

(ii) there is a neighborhood U of z such that

$\gamma(U) = U$ for all $\gamma \in \Gamma_z$ and

$\gamma(U) \cap U$ is empty for $\gamma \in \Gamma - \Gamma_z$.

We set $\Omega = \Omega(\Gamma) = \{z \in \mathbb{C} \cup \{\infty\};\ \Gamma$ is discontinuous at $z\}$, and call Ω the region of discontinuity of Γ. The group Γ is called <u>discontinuous</u> if Ω is not empty. The <u>limit set</u> Λ is defined by

$$\Lambda = \mathbb{C} \cup \{\infty\} - \Omega.$$

Obviously, Ω is open, Γ-invariant ($\gamma\Omega = \Omega$, all γ).

It can be shown that card $\Lambda = 0,1,2$ or ∞. If card $\Lambda \leq 2$, then Γ is called <u>elementary</u>; otherwise it is called a (non-elementary) <u>Kleinian</u> group. For a Kleinian group, Λ is a closed, perfect, nowhere dense subset of $\mathbb{C} \cup \{\infty\}$.

All elementary groups are known. We shall give a complete list of all elementary groups in Section III. These are all the discontinuous groups with $\Omega = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}$, $\mathbb{C}\setminus\{0\} = \mathbb{C}^*$ up to conjugation. We can classify the elements of Γ according to the following scheme. Define

$\text{trace}^2\gamma = (a+d)^2$. If $\gamma \neq$ Identity, we call γ

$$\begin{aligned} &\underline{\text{elliptic}} &&\Leftrightarrow 0 \leq \text{trace}^2\gamma < 4 \\ &\underline{\text{parabolic}} &&\Leftrightarrow \text{trace}^2\gamma = 4 \\ &\underline{\text{loxodromic}} &&\Leftrightarrow \text{trace}^2\gamma \notin [0,4] \end{aligned}$$

Those loxodromic elements γ with $\text{trace}^2\gamma > 4$ are called hyperbolic. An element is parabolic if and only if it has one fixed point. It is easy to see that

γ is loxodromic $\Leftrightarrow$ γ is conjugate (in group of Möbius transformation) to $z \to \lambda z$, $|\lambda| > 1$,

γ is hyperbolic $\Leftrightarrow$ γ is conjugate to $z \to \lambda z$, $\lambda > 1$

γ is elliptic $\Leftrightarrow$ γ is conjugate to $z \to \lambda z$, $|\lambda| = 1$, $\lambda \neq 1$

and γ is parabolic $\Leftrightarrow$ γ is conjugate to $z \to z+1$.

A component Δ of Ω is called a <u>component</u> of Γ. Two distinct components, Δ and Δ_1, are called <u>equivalent</u> if $\Delta_1 = \gamma(\Delta)$ for some $\gamma \in \Gamma$. The <u>stabilizer</u> $\Gamma_\Delta = \{\gamma \in \Gamma; \gamma(\Delta) = \Delta\}$ of Δ in Γ is again a Kleinian group. If $\Gamma_\Delta = \Gamma$, Δ is called <u>invariant</u>. A Kleinian group with an invariant component is called a <u>function group</u>.

Assume Γ is Kleinian. The quotient $\Omega(\Gamma)/_\Gamma$ has a canonical complex structure so that the projection map $\pi : \Omega(\Gamma) \to \Omega(\Gamma)/_\Gamma$ is holomorphic and $\Omega(\Gamma)/_\Gamma$ is a disjoint union of Riemann surfaces $\Delta_i/\Gamma_{\Delta_i}$, where $\Delta_1, \Delta_2, \ldots$ is a maximal list of equivalent components of Γ.

We shall say that Γ is of <u>finite type</u> (or Ω/Γ is of <u>finite type</u>) if

(i) the list of inequivalent components of Ω/Γ is finite;

(ii) the mapping $\pi : \Omega \to \Omega/\Gamma$ is ramified over at most finitely many points.

(iii) for each i, there exists a compact Riemann surface $S_i = (\overline{\Delta_i/\Gamma_{\Delta_i}})$ of genus g_i such that $S_i - (\Delta_i/\Gamma_{\Delta_i})$ consists of a finite number of points.

We shall assign a ramification number ∞ to each of the points $p \in S_i - (\Delta_i/\Gamma_{\Delta_i})$.

Let $I = (g;\nu_1,\ldots,\nu_n)$, where g is an integer ≥ 0 and $2 \leq \nu_1 \leq \ldots \leq \nu_n \leq \infty$ (ν_j is an integer ≥ 2 or $\nu_j = \infty$). Let k be the largest integer $\leq n$ such that $\nu_k < \infty$. We shall say that Γ_{Δ_i} (or $\Delta_i/\Gamma_{\Delta_i}$) has signature I, if

(i) S_i has genus g,

(ii) the domain Δ_i contains exactly k inequivalent elliptic fixed points of order $\nu_1,\ldots,\nu_k$, respectively, and

(iii) $S_i - \Delta_i/\Gamma_{\Delta_i}$ consists of precisely n-k points.

The following is proven in Ahlfors [1].

<u>Lemma 1.1.</u> Let $\Delta_i/\Gamma_{\Delta_i} = S_i - \{p\}$ where S_i is a Riemann surface and $p \in S_i$. If there is a punctured neighborhood M of p such that π is unramified over M, then there exists a parabolic element $\gamma \in \Gamma_{\Delta_i}$ with fixed point $\zeta \in \Lambda$, and there is a Möbius transformation A with the following properties:

(i) $A(\infty) = \zeta$ and $A^{-1}\circ\gamma\circ A(z) = z+1$, $z \in \mathbb{C}$

(ii) $A^{-1}(\Delta_i)$ contains a half plane,

$U_c = \{z \in \mathbb{C};\ \operatorname{Im} z > c\}$ for some $c > 0$,

(iii) two points z_1 and z_2 of $A(U_c)$ are equivalent under Γ_{Δ_i} if and only if $z_2 = \gamma^n(z_1)$ for some integer n, and

(iv) the image of $A(U_c)$ under π is a deleted neighborhood of p, and is homeomorphic to a punctured disc.

The point p will be called a puncture on $\Delta_i/\Gamma_{\Delta_i}$. We shall also say that the parabolic element γ corresponds to or is determined by the puncture $p \in \overline{\Delta_i/\Gamma_{\Delta_i}}$.

One of the main results in the theory of Kleinian groups is Ahlfors' finiteness theorem [1]: A finitely generated Kleinian group is of finite type.

But there are also infinitely generated groups of finite type.

If Γ is Kleinian, and if there is a circle C in the extended complex plane (a straight line is a circle through ∞) such that the interior of C is fixed by Γ, then Γ is called Fuchsian. In this case $\Lambda \subset C$. If $\Lambda = C$, Γ is called of the first kind; of the second kind otherwise. Hence, by definition a Fuchsian group of first kind has two simply-connected invariant components.

A quasi-Fuchsian group is a quasi-conformal deformation of a Fuchsian group; that is, for a quasi-Fuchsian group G, there exists a Fuchsian group Γ and a quasi-conformal mapping $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $G = w \Gamma w^{-1}$.

§2. We now consider the action of the group of Möbius transformations from a different point of view. For this, let

$$\mathbb{H}^3 = \{(z,t);\ z \in \mathbb{C},\ t > 0\} \text{ and } z = x+iy,\ x,y \in \mathbb{R}.$$

We define the line element $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$. The hyperbolic metric ρ on $\mathbb{H}^3$ is given by

$$\rho(p,p') = \inf_{\gamma} \int_{\gamma} ds \quad \text{over all smooth curves } \gamma \text{ joining}$$

p to p' in $\mathbb{H}^3$.

It is well known that the group of Möbius transformation act on the hyperbolic 3-space $\mathbb{H}^3$ as the full group of orientation preserving isometries.

Let Γ be a Kleinian group. Let S be either $\Omega(\Gamma)$ or $\mathbb{H}^3$. By a <u>fundamental domain</u> w for Γ in S we mean an open subset of S such that

(i) whenever $\gamma z = \zeta$ for some $\gamma \in \Gamma$, $z \in w$, $\zeta \in w$, then $\gamma = \text{id}$,

(ii) for every point $\zeta \in S$, there is a $\gamma \in \Gamma$ and a $z \in c\ell w$ (the closure of w in S) such that $\gamma z = \zeta$, and

(iii) w is locally finite; that is, each compact subset of S intersects only a finite number of images of $c\ell w$.

It is well known that a fundamental domain exists for every Kleinian group. For reference, see Beardon [2].

Let Γ be a Kleinian group. We wish to construct a fundamental domain for the action of Γ on hyperbolic 3-space. There are various methods of such construction; one of the standard methods follows:

Let $x_0 \in \mathbb{H}^3$ be any point which is not fixed by any element of $\Gamma \setminus \{I\}$. Define the Dirichlet region for Γ in $\mathbb{H}^3$ with centre x_0 by

$$D_{x_0} = \{x \in \mathbb{H}^3; \rho(x,x_0) < \rho(x,\gamma(x_0)) \text{ for all } \gamma \in \Gamma \setminus \{I\}.$$

D_{x_0} is a fundamental domain for Γ in $\mathbb{H}^3$. Note that D_{x_0} is the intersection of the half spaces $\{x \in \mathbb{H}^3; \rho(x,x_0) < \rho(x,\gamma(x_0))\}$, $\gamma \in \Gamma \setminus \{I\}$. Thus D_{x_0} is hyperbolic convex.

A <u>convex polyhedron</u> P in $\mathbb{H}^3$ is defined as the intersection of countably many half spaces H_i with bounding hyperplanes S_i such that any compact subset of $\mathbb{H}^3$ meets only finitely many of the S_i. The intersection of S_i with $\overline{P}$ is called a <u>side</u> of P.

Let Γ be a Kleinian group. The convex polyhedron P is called a <u>fundamental polyhedron</u> for Γ in $\mathbb{H}^3$ if P is a fundamental domain for Γ in $\mathbb{H}^3$ and if the sides of P are pairwise identified by elements of Γ. Note that the Dirichlet region defined earlier is a fundamental polyhedron. We close this discussion with a definition. A Kleinian group Γ is called <u>geometrically finite</u> if Γ has a finite sided fundamental polyhedron in $\mathbb{H}^3$.

A geometrically finite Kleinian group Γ is finitely generated. But there are finitely generated Kleinian groups which are not geometrically finite.

§3. We introduce an operator on functions as follows. If A is a Möbius transformation and $2n, 2m \in \mathbb{Z}$ with $n+m \in \mathbb{Z}$, then for every function f on a domain D, we define a function on $f^{-1}(D)$ by

$$(A^*_{n,m} f)(z) = f(Az)A'(z)^n \overline{A'(z)}^m .$$

Abbreviate $A^*_{n,0}$ by A^*_n .

Let Γ be a (non-elementary) Kleinian group and let D be an invariant union of component of the region of discontinuity of Γ. Let $\lambda_D(z)dz$ be the <u>Poincaré metric</u> (that is, unique complete conformal Riemannian metric defined on each component of D with constant curvature $\equiv -1$. For example, for D the upper half plane, $\lambda(z) = (\operatorname{Im} z)^{-1}$, $z \in D$).

To define the analogue of the Poincaré metric on Ω for elementary groups, we let

$$\lambda(z)\,|dz|$$

be defined as follows. For $\Omega = \mathbb{C} \cup \{\infty\}$, we set $\lambda(z) = \dfrac{2}{1+|z|^2}$

and for $\Omega = \mathbb{C}$, $\lambda(z) = 1$, for all $z \in \mathbb{C}$. We require that $\lambda(z)|dz|$ be a conformal invariant. Thus for $\Omega = \mathbb{C}\setminus\{0\}$, $\lambda(z) = |z|^{-1}$, $z \in \mathbb{C}\setminus\{0\}$.

Let $q \geq 2$ be a fixed integer. A holomorphic function φ on D is called <u>automorphic form for Γ of weight</u> $(-2q)$ on D if

$$\gamma_q^*\varphi = \varphi \quad \text{for all } \gamma \in \Gamma.$$

An automorphic form φ of weight $(-2q)$ on D is called <u>integrable</u> if

$$\iint_{D/\Gamma} \lambda(z)^{2-q}|\varphi(z)dz\wedge d\bar{z}| < \infty.$$

We denote the Banach space of integrable automorphic forms on D by $A_q(D,\Gamma)$.

The form φ is called <u>bounded</u> if

$$\sup\{\lambda(z)^{-q}|\varphi(z)|;\ z \in D\} < \infty.$$

The Banach space of bounded automorphic forms on D is denoted by $B_q(D,\Gamma)$.

For $\varphi \in A_q(D,\Gamma)$ and $\Psi \in B_q(D,\Gamma)$, we define the <u>Petersson scalar product</u> by

$$\langle\varphi,\Psi\rangle_\Gamma = \iint_{D/\Gamma} \lambda(z)^{2-2q}\varphi(z)\overline{\Psi(z)}dz\wedge d\bar{z}.$$

It is well known (see, for example, Kra [10]) that the Petersson scalar product establishes an anti-linear topological isomorphism between $B_q(D,\Gamma)$ and the dual space of $A_q(D,\Gamma)$. If D/Γ is of finite type, and $\Delta_1, \Delta_2, \ldots, \Delta_k$ be the complete list of inequivalent components of D, then

$$A_q(D,\Gamma) = B_q(D,\Gamma),$$

$$B_q(D,\Gamma) = \bigoplus_{i=1}^{k} B_q(\Delta_i, \Gamma_{\Delta_i})$$

and $\dim B_q(\Delta_i, \Gamma_{\Delta_i}) = (2q-1)(g_i-1) + \sum_{p \in S_i} [q - \frac{q}{\ell(p)}]$,

where $[x]$ is the largest integer $\leq x$, $\ell(p)$ is ramification number of p and $[q - \frac{q}{\infty}] = q-1$.

Classically, these automorphic forms are called <u>cusp forms</u>. Whenever Γ is a finite type we don't distinguish between $A_q(D,\Gamma)$ and $B_q(D,\Gamma)$. We will, in this case, use the notation $A_q(D,\Gamma)$ for the space of cusp forms.

SECTION II

Eichler Cohomology of Kleinian Groups

In this section we will describe Eichler cohomology and the structure of these cohomology groups. Material of this section is well known. It can be found in Kra [10].

§1. Let $q \geq 2$ be an integer. Let Π_{2q-2} denote the vector space of polynomials in one complex variable of degree $\leq 2q-2$. Let Γ be a Kleinian or elementary group. The group Γ acts on the right on Π_{2q-2} via Eichler action

$$(2.1) \qquad p \cdot \gamma = \gamma^*_{1-q} p \in \Pi_{2q-2}, \quad \gamma \in \Gamma.$$

Thus to verify that (2.1) defines an action of Γ on Π_{2q-2}, it suffices to verify that for every Möbius transformation γ in Γ,

$$(2.1) \qquad \gamma^*_{1-q} : \Pi_{2q-2} \to \Pi_{2q-2}.$$

Let $\gamma(z) = \dfrac{az+b}{cz+d}$ with $ad-bc=1$, and let $p(z) = z^n$. Then

$$p \cdot \gamma(z) = (az+b)^n (cz+d)^{-n+2q-2}.$$

Thus $p \cdot \gamma \in \Pi_{2q-2}$ whenever $n \leq 2q-2$ (that is, whenever $p \in \Pi_{2q-2}$). A mapping

$$\chi : \Gamma \to \Pi_{2q-2}$$

is called a <u>cocycle</u> if $\chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1) \cdot \gamma_2 + \chi(\gamma_2)$, $\gamma_1, \gamma_2 \in \Gamma$.

Such a cocycle is <u>coboundary</u> if

$$\chi(\gamma) = p\cdot\gamma - p, \quad \gamma \in \Gamma$$

for some fixed $p \in \Pi_{2q-2}$.

The (first) <u>cohomology space</u> $H^1(\Gamma, \Pi_{2q-2})$ is defined as the vector space of cocycles, denoted by $Z^1(\Gamma, \Pi_{2q-2})$ modulo the vector space of coboundaries, denoted by $B^1(\Gamma, \Pi_{2q-2})$; that is,

$$Z^1(\Gamma, \Pi_{2q-2}) / B^1(\Gamma, \Pi_{2q-2}) = H^1(\Gamma, \Pi_{2q-2}).$$

Let B be a Möbius transformation, and $\hat{\Gamma} = B^{-1}\Gamma B$. Then conjugation by B^{-1} induces an isomorphism between $H^1(\Gamma, \Pi_{2q-2})$ and $H^1(\hat{\Gamma}, \Pi_{2q-2})$. The mapping is determined by sending the cocycle χ into the cocycle $\hat{\chi}$ where

$$(2.3) \qquad \hat{\chi}(B^{-1}\circ\gamma\circ B) = \chi(\gamma)\cdot B = B^*_{1-q}\chi(\gamma), \quad \gamma \in \Gamma.$$

<u>Lemma 2.1.</u> If Γ is non-elementary and generated by N elements, then $\quad \dim H^1(\Gamma, \Pi_{2q-2}) \leq (2q-1)(N-1) \quad$ for $q \geq 2$.

Equalities hold whenever Γ is a free group on N generators.

<u>Proof.</u> Since a cocycle is uniquely determined by its values on the generators of Γ, the dimension of the space of cocycles $\leq (2q-1)N$, and is exactly $N(2q-1)$ for a free group with N generators. To compute dimension of $B^1(\Gamma, \Pi_{2q-2})$ it suffices

to show that the map from Π_{2q-2} to $B^1(\Gamma, \Pi_{2q-2})$ is injective.

Assume that there is a $p \in \Pi_{2q-2}$ such that

$$(2.4) \qquad p(\gamma z)\gamma'(z)^{1-q} = p(z), \gamma \in \Gamma, \ z \in \mathbb{C}.$$

Clearly deg $p \geq 0$. If p vanishes at z_0, then by (2.4) p also vanishes at $\gamma(z_0)$ for all $\gamma \in \Gamma$. Thus p has infinitely many zeros (because Γ is non-elementary), and hence $p = 0$. Therefore, $\dim B^1(\Gamma, \Pi_{2q-2}) = 2q - 1$.

The above lemma is known and it can be found in Kra [10].

<u>Remark</u>: For elementary groups, $\dim B^1(\Gamma, \Pi_{2q-2})$ is not always $2q - 1$ as we will see in Section III.

We will see that the parabolic elements of Γ play a significant role in the cohomology theory. They determine distinguished cohomology classes.

<u>Definition 2.1</u>. Let $A \in \Gamma$ be a parabolic element. A cocycle χ is <u>parabolic with respect to A</u> provided there is a $v \in \Pi_{2q-2}$ such that

$$\chi(A) = v \cdot A - v.$$

A cocycle is called <u>parabolic</u> if it is parabolic with respect to all parabolic elements Γ; while it is called <u>Δ-parabolic</u> if it is parabolic with respect to every parabolic element in Γ determined by a puncture on Δ/Γ (where Δ is an invariant union of components of Γ). The space of parabolic cocycles (respectively, Δ-parabolic) is denoted by $PZ^1(\Gamma, \Pi_{2q-2})$ (respectively, $PZ^1_\Delta(\Gamma, \Pi_{2q-2})$. We denote by $PH^1(\Gamma, \Pi_{2q-2})$ (respectively, $PH^1_\Delta(\Gamma, \Pi_{2q-2})$), the space of parabolic cohomology (respectively, Δ-parabolic cohomology); that is, $PZ^1(\Gamma, \Pi_{2q-2})/B^1(\Gamma, \Pi_{2q-2})$ (respectively, $PZ^1_\Delta(\Gamma, \Pi_{2q-2})/B^1(\Gamma, \Pi_{2q-2})$). Note that (2.3) shows that parabolicity of cohomology classes is invariant under conjugation.

§2. Next, we describe an analytic way to obtain the elements of $H^1(\Gamma, \Pi_{2q-2})$. We assume that Γ is a finitely generated Kleinian group with at least one limit point. (The discussion for finite elementary groups is not necessary because of Proposition 3.1.)

For $\varphi \in A_q(\Omega, \Gamma)$, $\mu = \lambda^{2-2q}\overline{\varphi}$ is called a <u>canonical generalized Beltrami differential</u> for Γ; this means that

(2.5) $$\gamma^*_{1-q,1}\mu = \mu, \quad \text{all } \gamma \in \Gamma$$

and

$$|\mu| \leq \text{constant } \lambda^{2-q}.$$

A continuous function F on $\mathbb{C}$ is called a <u>potential</u> for μ provided

(2.6) $$F(z) = O(|z|^{2q-2}), \quad z \to \infty$$

and (2.7) $\frac{\partial F}{\partial \bar{z}} = \mu$

in the sense of generalized derivatives. Such a potential F is said to vanish at ∞ if $F(z) = o(|z|^{2q-2})$, $z \to \infty$.

The following is due to Bers [3] (Ahlfors [1] for $q = 2$).

Lemma 2.2. Let $q \geq 2$, and $\varphi \in A_q(\Omega, \Gamma)$. If $\{a_1, \ldots, a_{2q-1}\}$ are $(2q-1)$ distinct points in $\hat{\mathbb{C}}$, then

$$F(z) = \frac{(z-a_1)\ldots(z-a_{2q-1})}{2\pi i} \iint_{\Omega} \frac{\lambda^{2-2q}\overline{\varphi(\zeta)}}{(\zeta-z)(\zeta-a_1)\ldots(\zeta-a_{2q-1})} d\zeta \wedge d\bar{\zeta}$$

is the unique potential for $\lambda^{2-2q}\bar{\varphi}$ that vanishes at a_k, $k = 1, 2, \ldots, 2q-1$.

Convention: We may allow $a_j = \infty$ for some j. In that case we will omit terms of the form $(z - \infty)$ and $(\zeta - \infty)$.

For $\gamma \in \Gamma$, we define

(2.8) $\beta_q^*(\varphi)(z) = F(\gamma(z))\gamma'(z)^{1-q} - F(z) = \chi(\gamma)(z)$, $z \in \mathbb{C}$.

Using (2.5) and an application of chain rule; that is,

$$\frac{\partial}{\partial \bar{z}}(\gamma_{1-q}^* F) = \gamma_{1-q,1}^* \frac{\partial F}{\partial \bar{z}} = \gamma_{1-q,1}^* \mu$$

we see that $\chi(\gamma)$ is entire. By virtue of (2.6),

$$\chi(\gamma)(z) = O(|z|^{2q-2}), \quad z \to \infty.$$

Hence $\chi(\gamma) \in \Pi_{2q-2}$.

(2.8) defines a cocycle whose cohomology class $\beta_q^*(\varphi)$ depends not on F but only on μ. $\beta_q^*(\varphi)$ is a parabolic cohomology class for all $\varphi \in A_q(\Omega,\Gamma)$. To see this, let $A \in \Gamma$ be a parabolic element. Since parabolicity of cohomology classes is invariant under conjugation, we may assume that $A(z) = z+1$, $z \in \mathbb{C}$. Choose a potential F for $\mu = \lambda^{2-2q}\bar{\varphi}$ that vanishes at ∞. As we said earlier, a potential F is said to vanish at ∞ if $F(z) = o(|z|^{2q-2})$, $z \to \infty$. Thus we conclude that

$$\bar{\chi}(A)(z) = o(|z|^{2q-2}), \quad z \to \infty.$$

Hence $\chi(A)(z)$ has degree $\leq 2q-3$. But for every polynomial of degree $\leq 2q-3$, there is a polynomial $v \in \Pi_{2q-2}$ such that $v(z+1) - v(z) = \chi(A)(z)$, $z \in \mathbb{C}$. Hence $\beta_q^*(\varphi) \in PH^1(\Gamma, \Pi_{2q-2})$. So we obtain this way the Bers map

$$\beta_q^* : A_q(\Omega,\Gamma) \to PH^1(\Gamma, \Pi_{2q-2}).$$

The map β_q^* is antilinear and injective.

This result is due to Ahlfors [1] for $q=2$, and to Bers [3] for $q > 2$ for finitely generated (non-elementary) Kleinian groups. The extension to elementary groups with at least one limit point can be found in Kra [11].

We shall call $\beta^*_q(\varphi)$ the <u>Bers cohomology</u> class of $\varphi \in A_q(\Omega,\Gamma)$. We will see in Section IV (our main result) that every parabolic cohomology class is the Bers cohomology class of unique $\varphi \in A_q(\Omega,\Gamma)$ for geometrically finite function groups.

§3. We will now see that there is another way to construct cohomology classes. For this we need the

<u>Definition 2.2</u>. A holomorphic function F on Ω is called a <u>(holomorphic) Eichler integral</u> of order $1 - q$ (on Ω for the Kleinian or elementary group Γ) if

$$(2.9) \qquad \chi(\gamma) = \gamma^*_{1-q}F - F \in \Pi_{2q-2}, \quad \text{all } \gamma \in \Gamma.$$

Note that in the definition, for fixed $\gamma \in \Gamma$, $\chi(\gamma)$ is the <u>same</u> polynomial on each component of Γ.

Let D^{2q-1} denote differentiation $2q - 1$ times. As a consequence of identity

$$(2.10) \qquad D^{2q-1} \circ \gamma^*_{1-q} = \gamma^*_q \circ D^{2q-1}, \quad \gamma \in \Gamma,$$

due to I. Bol, we conclude that D^{2q-1} maps Eichler integrals of order $(1-q)$ into automorphic forms of weight $(-2q)$. A holomorphic Eichler integral F defined on Ω is called <u>bounded</u> if

$$(2.11) \qquad \varphi = D^{2q-1}F \in A_q(\Omega,\Gamma).$$

If F is a bounded integral, then the projection of φ to Ω/Γ has a pole of order $\leq q-1$ at each puncture of Ω/Γ. If instead of (2.11), we require that φ has a pole of order $\leq q$ at each puncture of Ω/Γ, then F is called a <u>quasi-bounded</u> (homomorphic) Eichler integral.

The space of bounded Eichler integrals modulo Π_{2q-2} will be denoted by $E^b_{1-q}(\Omega,\Gamma)$. Similarly, $E^c_{1-q}(\Omega,\Gamma)$ denotes the space of quasi-bounded Eichler integrals modulo Π_{2q-2}.

The construction of quasi-bounded (bounded) Eichler integrals is difficult. Recently, Sullivan has shown in [20] how to construct such integrals. Kra [11] incorporated Sullivan's construction into structure theorems for the Eichler cohomology groups $H^1(\Gamma,\Pi_{2q-2})$ and $PH^1(\Gamma,\Pi_{2q-2})$, and have shown that one can construct a basis for the space of quasi-bounded Eichler integrals, $E^c_{1-q}(\Omega,\Gamma)$ for a (non-elementary) finitely generated Kleinian group Γ if one starts with a group Γ for which the Bers map is surjective. This also suggests that one should determine the class of finitely generated Kleinian groups for which the Bers map is surjective. This may lead to a better understanding of Eichler integrals.

We use (2.9) to define a map

$$\alpha : E^c_{1-q}(\Omega,\Gamma) \to H^1(\Gamma,\Pi_{2q-2})$$

for Kleinian group Γ. Ahlfors (also Kra [8]) has proved that the map α is injective for Γ of finite type.

We shall call α the period map, and $\alpha(F)$ the Eichler cohomology class of $F \in E^c_{1-q}(\Omega,\Gamma)$.

Theorem 2.1. Let Γ be a Kleinian group, and Δ an invariant union of components of its region of discontinuity such that Δ/Γ is of finite type. There exists a canonical anti-linear surjective mapping

$$\beta_q : H^1(\Gamma,\Pi_{2q-2}) \to A_q(\Delta,\Gamma)$$

such that $\beta_q \circ \beta_q^* = \text{id}$ on $A_q(\Delta,\Gamma)$, $q \geq 2$.

Proof. Kra [8].

Now we can state the structure theorem for $H^1(\Gamma,\Pi_{2q-2})$ and $PH^1_\Delta(\Gamma,\Pi_{2q-2})$ due to Kra [8].

Theorem 2.2. Let Γ be a Kleinian group, and Δ an invariant union of components of its region of discontinuity. If Δ/Γ is of finite type, then for $q \geq 2$, the following is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^b_{1-q}(\Delta,\Gamma) & \xrightarrow{\alpha} & PH^1_\Delta(\Gamma,\Pi_{2q-2}) & \xrightarrow{\beta_q} & A_q(\Delta,\Gamma) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \text{id} & & \\
0 & \longrightarrow & E^c_{1-q}(\Delta,\Gamma) & \xrightarrow{\alpha} & H^1(\Gamma,\Pi_{2q-2}) & \xrightarrow{\beta_q} & A_q(\Delta,\Gamma) & \longrightarrow & 0.
\end{array}$$

<u>Corollary 1</u>: (Kra [8]). Under the hypothesis of the theorem, every cohomology class $p \in H^1(\Gamma,\Pi_{2q-2})$ can be written uniquely as

$$p = \beta_q^*(\varphi) + \alpha(f),$$

where $\varphi \in A_q(\Delta,\Gamma)$, and $f \in E^c_{1-q}(\Delta,\Gamma)$. Furthermore, $p \in PH^1_\Delta(\Gamma,\Pi_{2q-2})$ if and only if $f \in E^b_{1-q}(\Delta,\Gamma)$.

<u>Corollary 2</u>. (Bers [3] and Kra [8]). Let Γ be a finitely generated Kleinian group with an invariant component Δ. Then

$$\dim A_q(\Omega,\Gamma) \le \dim PH^1_\Delta(\Gamma,\Pi_{2q-2}) \le 2 \dim A_q(\Delta,\Gamma)$$

and $\dim A_q(\Omega,\Gamma) \le \dim H^1(\Gamma,\Pi_{2q-2}) \le 2\, A_q(\Delta,\Gamma) + n$, where n is the number of punctures on Δ/Γ. Furthermore, if Δ is connected and simply-connected, then

$$\dim PH^1_\Delta(\Gamma,\Pi_{2q-2}) = 2 \dim A_q(\Delta,\Gamma).$$

Remarks:

(1) If Γ is a finitely generated degenerate Kleinian group (with connected, simply connected region of discontinuity) then β_q^* is not surjective. In fact, in this case

$$\dim PH_{\Omega}^{1}(\Gamma, \Pi_{2q-2}) = 2 \dim A_q(\Omega, \Gamma).$$

Greenberg [6] showed that degenerate groups are not geometrically finite.

(2) We will improve the first inequality of Corollary 2 in Section V. Maskit [16] proved similar result.

Let Δ be a component of Ω for a Kleinian group Γ. An Eichler integral $F \in E_{1-q}^{b}(\Delta, \Gamma)$ for a Kleinian group Γ is called trivial if $D^{2q-1}F = 0$. The space of trivial Eichler integrals modulo Π_{2q-2} is denoted by $E_{1-q}^{o}(\Delta, \Gamma)$. The following theorem has been proved in Kra [8].

Theorem 2.3. Let Γ be a finitely generated Kleinian group with an invariant component Δ. If $\Omega - \Delta$ is non-empty, then for $q \geq 2$, every cohomology class $p \in PH_{\Omega-\Delta}^{1}(\Gamma, \Pi_{2q-2})$ can be written as the sum of a Bers cohomology class of some $\varphi \in A_q(\Omega, \Gamma)$ and an Eichler cohomology class of a trivial Eichler integral $F \in E_{1-q}^{o}(\Omega-\Delta, \Gamma)$.

Corollary. (Kra [8]). Let Γ be a finitely generated Kleinian group with two invariant components Δ and $\Omega - \Delta$. Then for $q \geq 2$,

$$PH^1(\Gamma, \Pi_{2q-2}) = PH^1_\Omega(\Gamma, \Pi_{2q-2}) = PH^1_\Delta(\Gamma, \Pi_{2q-2}) = \beta^*_q(A_q(\Omega, \Gamma)).$$

Hence by the corollary, the Bers map is surjective for a finitely generated Fuchsian or quasi-Fuchsian group of the first kind.

§4. We close this section by giving definition of stability of a finitely generated Kleinian group Γ due to Bers [3].

Let G be the group of all Möbius transformations. Let Γ be any subgroup of G. We denote by $Hom(\Gamma, G)$ the set of all homomorphisms of Γ into G. A homomorphism $\Theta : \Gamma \to G$ is called parabolic if $\text{trace}^2 \Theta(\gamma) = 4$ whenever $\gamma \in \Gamma$ is parabolic. We denote by $Hom_p(\Gamma, G)$ the set of all parabolic homomorphisms of Γ into G.

Let Γ be a Kleinian group, and let $w : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a quasiconformal self-mapping of the Riemann sphere. We say that w is compatible with a Kleinian group Γ if $w \Gamma w^{-1} \subset G$. If w is compatible with Γ, then the mapping

$$\Gamma \ni \gamma \to \Theta(\gamma) = w \circ \gamma \circ w^{-1} \in G$$

is an isomorphism of Γ into $w\Gamma w^{-1}$ and the isomorphism θ is a quasiconformal deformation of Γ. We denote by $\mathrm{Hom}_{qc}(\Gamma,G)$ the set of all quasiconformal deformations of the Kleinian group Γ into G. We have $\mathrm{Hom}_{qc}(\Gamma,G) \subset \mathrm{Hom}_{p}(\Gamma,G) \subset \mathrm{Hom}(\Gamma,G)$. Let Γ be a finitely generated Kleinian group with a system of generators $\{\gamma_1,\ldots,\gamma_N\}$. Then an element $\theta \in \mathrm{Hom}(\Gamma,G)$ is uniquely determined by $(\theta(\gamma_1),\ldots,\theta(\gamma_N)) \in G^N$, where G^N is the N times product space of G. We define the set $X(\Gamma;\gamma_1,\ldots,\gamma_N)$ in G^N such that

$$X(\Gamma;\gamma_1,\ldots,\gamma_N) = \{(\theta(\gamma_1),\ldots,\theta(\gamma_N)) \in G^N; \theta \in \mathrm{Hom}(\Gamma,G)\}.$$

Now we identify an element $\theta \in \mathrm{Hom}(\Gamma,G)$ with $(\theta(\gamma_1),\ldots,\theta(\gamma_N)) \in X(\Gamma;\gamma_1,\ldots,\gamma_N)$ and we regard $X(\Gamma;\gamma_1,\ldots,\gamma_N)$ as $\mathrm{Hom}(\Gamma,G)$. The corresponding spaces for $\mathrm{Hom}_p(\Gamma,G)$ and $\mathrm{Hom}_{qc}(\Gamma,G)$ in G^N are denoted by $X_p(\Gamma;\gamma_1,\ldots,\gamma_N)$ and $X_{qc}(\Gamma;\gamma_1,\ldots,\gamma_N)$, respectively. We define stability of Γ as follows: A finitely generated Kleinian group Γ is said to be quasiconformally stable if there exists a system of generators $\{\gamma_1,\ldots,\gamma_N\}$ of Γ and a neighborhood $U(\gamma_1,\ldots,\gamma_n)$ of $(\gamma_1,\ldots,\gamma_N) \in G^N$ in G^N such that

$$X_p(\Gamma;\gamma_1,\ldots,\gamma_N) \cap U(\gamma_1,\ldots,\gamma_N) = X_{qc}(\Gamma;\gamma_1,\ldots,\gamma_N) \cap U(\gamma_1,\ldots,\gamma_N).$$

We say Γ is <u>quasi-stable</u> if there exists an open neighborhood

$U(\gamma_1,\ldots,\gamma_N)$ of $(\gamma_1,\ldots,\gamma_N) \in G^N$ such that $U(\gamma_1,\ldots,\gamma_N) \cap X_{qc}(\Gamma;\gamma_1,\ldots,\gamma_N)$ is a complex analytic submanifold of dimension $\sigma(\Gamma)+3$ of U; where $\sigma(\Gamma) = \dim A_2(\Omega,\Gamma)$. With these definitions of stability the following two theorems will give connection between stability and surjectivity of the Bers map.

<u>Theorem 2.4.</u> (Kra and Gardiner [5]): Let Γ be a non-elementary finitely generated Kleinian group. If $\beta_2^*(A_2(\Omega,\Gamma)) = PH^1(\Gamma,\Pi_2)$, then Γ is both quasiconformally stable and quasi-stable.

<u>Theorem 2.5</u>. (Sakan [19]): Let Γ be a non-elementary finitely generated Kleinian group. If Γ is both quasiconformally stable and quasi-stable, then $\beta_2^*(A_2(\Omega,\Gamma)) = PH^1(\Gamma,\Pi_2)$.

SECTION III

Cohomology of Elementary Groups

In this section we will list the dimensions of $Z^1(\Gamma,\Pi_{2q-2})$, $PZ^1(\Gamma,\Pi_{2q-2})$ and $B^1(\Gamma,\Pi_{2q-2})$ for elementary groups Γ. These will be very useful in our discussion in the next section. The classification of elementary groups can be found in Kra and Farkas [13]. The computation of the dimensions of the above spaces by different methods also appears in Kra [11].

§1. For an arbitrary Γ, let

$$\Pi_{2q-2}(\Gamma) = \{v \in \Pi_{2q-2};\ v\cdot\gamma = v,\ \text{all } \gamma \in \Gamma\}$$

represent the fixed point space of Γ. If Γ is non-elementary, then we have proved in Section II that $\Pi_{2q-2}(\Gamma) = \{0\}$ for $q \geq 2$. It is clear that the fixed point space of Γ is same as the kernal of the map

$$\delta : \Pi_{2q-2} \rightarrow B^1(\Gamma,\Pi_{2q-2})$$

defined by $\delta(v) = v\cdot\gamma - v.$

So, $$\dim B^1(\Gamma,\Pi_{2q-2}) = (2q-1) - \dim \Pi_{2q-2}(\Gamma) \text{ for } q \geq 2\,.$$

For an elementary group Γ with the signature $(g;\nu_1,\nu_2,\ldots,\nu_n)$ we define the number $N_\Gamma(q)$ by $(2q-1)(q-1) + \sum_{j=1}^{n} [q - \frac{q}{\nu_i}]$.

Proposition 3.1 (Kra [11]. Let Γ be an elementary group and $q \in \mathbb{Z}$, $q \geq 2$. The dimensions of the cocycle spaces $Z^1(\Gamma, \Pi_{2q-2})$, $PZ^1(\Gamma, \Pi_{2q-2})$ and the dimension of the space of coboundaries $B^1(\Gamma, \Pi_{2q-2})$ are given in the following table:

Number of limit points	Name of the group	Dimension of $Z^1(\Gamma,\Pi_{2q-2})$		Dimension of $PZ^1(\Gamma,\Pi_{2q-2})$	Dimension of $B^1(\Gamma,\Pi_{2q-2})$	Signature of Γ	$N_\Gamma(q)$
0	$\mathbb{Z}_n$ $(2 \le n < \infty)$	$2[q-\frac{q}{n}]$			$2[q-\frac{q}{n}]$	$(0;n,n)$	$-(2q-1)+2[q-\frac{q}{n}]$
	dihedral n group	$2[q-\frac{q}{2}]+[q-\frac{q}{n}]$			$2[q-\frac{q}{2}]+[q-\frac{q}{n}]$	$(0;2,2,n)$	$-(2q-1)+2[q-\frac{q}{2}]+[q-\frac{q}{n}]$
	solid rotation groups	$[q-\frac{q}{2}]+[q-\frac{q}{3}]+[q-\frac{q}{e}]$			$[q-\frac{q}{2}]+[q-\frac{q}{3}]+[q-\frac{q}{e}]$	$(0;2,3,e)$ where $e=3,4$ or 5	$-(2q-1)+[q-\frac{q}{2}]+[q-\frac{q}{3}]+[q-\frac{q}{e}]$
1	parabolic group (rank 1)	2q-1		2q-2	2q-2	$(0;\infty,\infty)$	-1
	rank 2 parabolic	2q		2q-1	2q-2	(1;——)	0
	Euclidean triangle groups						
	$\mathbb{Z}_2$-extension of rank 1 parabolic	2q,	$q\equiv 0 \pmod 2$	2q-1	2q-1	$(0;2,2,\infty)$	0
		2q-2	otherwise	2q-2	2q-2		-1
	$\mathbb{Z}_6$-extension of rank 2 parabolic	2q+1	$q\equiv 0 \pmod 6$	2q	2q-1	$(0;2,3,6)$	1
		2q-2	$q\equiv 1 \pmod 6$	2q-2	2q-2		-1
		2q-1	otherwise	2q-1	2q-1		0
	$\mathbb{Z}_4$-extension of rank 2 parabolic	2q+1	$q\equiv 0 \pmod 4$	2q	2q-1	$(0;2,4,4)$	1
		2q-2	$q\equiv 1 \pmod 4$	2q-2	2q-2		-1
		2q-1	otherwise	2q-1	2q-1		0
	$\mathbb{Z}_3$-extension of rank 2 parabolic	2q+1	$q\equiv 0 \pmod 3$	2q	2q-1	$(0;3,3,3)$	1
		2q-2	$q\equiv 1 \pmod 3$	2q-2	2q-2		-1
		2q-1	otherwise	2q-1	2q-1		0

(continued)

Number of limit points	Name of the group	Dimension of $Z^1(\Gamma, \Pi_{2q-2})$		$PZ^1(\Gamma, \Pi_{2q-2})$	$B^1(\Gamma, \Pi_{2q-2})$	Signature of Γ	$N_\Gamma(q)$
	$\mathbb{Z}_2$-extension of rank 2 parabolic	$2q+1$	$q \equiv 0 \pmod 2$	$2q$	$2q-1$	$(0;2,2,2,2)$	1
		$2q-2$	otherwise	$2q-2$	$2q-2$		-1
2	loxodromic cyclic	$2q-1$			$2q-2$	(1;——)	0
	$\mathbb{Z} + \mathbb{Z}_n$	$2q-1$			$2q-2$	(1;——)	0
	$\mathbb{Z}_2 * \mathbb{Z}_2$ double-dihedral group	$2q$	$q \equiv 0 \pmod 2$		$2q-1$	$(0;2,2,2,2)$	1
		$2q-2$	otherwise		$2q-2$	$(0;2,2,2,2)$	-1

<u>Remark</u>. Observe that $\dim PZ^1(\Gamma,\Pi_{2q-2}) = (2q-1) + N_\Gamma(q)$ for all elementary groups Γ.

<u>Proof</u>. The proposition is established by a case by case examination of the list of elementary groups.

Let Γ be a finite group, and let χ be a cocycle for Γ.

From $\quad F = -\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\chi(\gamma)$, where $|\Gamma|$ is the cardinality of Γ.

Then $F \in \Pi_{2q-2}$ (because Γ is finite), and for $A \in \Gamma$,

$$\begin{aligned} F\cdot A - F &= -\frac{1}{|\Gamma|}\Big(\sum_{\gamma\in\Gamma}\chi(\gamma)\cdot A - \sum_{\gamma\in\Gamma}\chi(\gamma)\Big) \\ &= -\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(\chi(\gamma\circ A^{-1})\cdot A-\chi(\gamma)) \\ &= -\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(\chi(\gamma)-\chi(A)-\chi(\gamma)) \\ &= -\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(-\chi(A)) = \chi(A) . \end{aligned}$$

This shows that every cocycle is a coboundary. Therefore, $\dim Z^1(\Gamma,\Pi_{2q-2}) = \dim B^1(\Gamma,\Pi_{2q-2})$.

To compute the dimension $B^1(\Gamma,\Pi_{2q-2})$ for finite groups we may view an element $v \in \Pi_{2q-2}$ as a holomorphic $(1-q)$-differential $v(z)dz^{1-q}$ on $\mathbb{C} \cup \{\infty\}$. If v is Γ-invariant, $v(z)dz^{1-q}$ projects to a holomorphic $(1-q)$-differential on $\mathbb{C} \cup \{\infty\}/_\Gamma$. At the image of an elliptic fixed point of order ν, the projected differential has a zero of order

$$\geq -[(1-q)(1-\frac{1}{\nu})].$$

Hence the dimension of $\Pi_{2q-2}(\Gamma)$ for a finite group Γ of signature $(0;\nu,\nu)$ is

$$(2q-1) + 2[(1-q)(1-\frac{1}{\nu})],$$

and for a finite group Γ of signature $(0;\nu_1,\nu_2,\nu_3)$ is

$$(2q-1) + \sum_{j=1}^{3} [(1-q)(1-\frac{1}{\nu_j})]$$

Therefore, for the first case,

$$\begin{aligned}\dim B^1(\Gamma,\Pi_{2q-2}) &= -2[(1-q)(1-\frac{1}{\nu})] \\ &= (2q-2) - 2[\frac{q-1}{\nu}] \\ &= 2[q-\frac{q}{\nu}]\end{aligned}$$

and for the second case,

$$\dim B^1(\Gamma,\Pi_{2q-2}) = \sum_{j=1}^{3} [q-\frac{q}{\nu_j}].$$

Now let Γ be a rank 1 parabolic group then Γ is conjugate to a group generated by $A(z) = z+1$.

If $v \in \Pi_{2q-2}$ and is invariant under A then $v(z+1) = v(z)$. Thus v is periodic and hence constant. So $\dim \Pi_{2q-2}(\Gamma) = 1$. Let Γ be a $\mathbb{Z}_2$-extension of rank 1 parabolic group then by conjugation Γ consists of mapping $\{z \to \pm z + n\}$. We know that only polynomial invariant under $z \to z+1$ is a constant polynomial, say v_o. Hence v_o is invariant under Γ if and only if

$q \equiv 1 \pmod 2$. So

$$\dim B^1(\Gamma, \Pi_{2q-2}) = 2q-1,\ q \not\equiv 1 \pmod 2$$
$$= 2q-2,\ q \equiv 1 \pmod 2.$$

Every rank 2 parabolic group Γ consists of $\{z \to z + m + n\tau\}$, $\operatorname{Im} \tau > 0$ by conjugation. Clearly, v_0 is the only polynomial invariant under the group Γ. Hence $\dim \Pi_{2q-2}(\Gamma) = 1$. So $\dim B^1(\Gamma, \Pi_{2q-2}) = 2q-2$. Every $\mathbb{Z}_2$-extension of rank 2 parabolic group Γ consists of $\{z \to \pm z + m + n\tau\}$, $\operatorname{Im} \tau > 0$ by conjugation. Clearly v_0 is invariant under Γ if and only if $q \equiv 1 \pmod 2$. Hence $\dim \Pi_{2q-2}(\Gamma) = 1$ if $q \equiv 1 \pmod 2$, and $\dim \Pi_{2q-2}(\Gamma) = 0$ otherwise.

Every $\mathbb{Z}_n$-extension of a rank 2 parabolic group ($n = 3, 4$ or 6) is conjugate to a group Γ generated by

$$A(z) = e^{\frac{2\pi i}{n}} z, \quad B(z) = z + 1, \quad C(z) = z + e^{\frac{2\pi i}{n}}, \quad (n=3,4 \text{ or } 6).$$

Clearly v_0 is invariant under Γ if and only if it is invariant under A. But v_0 is invariant under A if and only if

$$q \equiv 1 \pmod n \ (n=3,4 \text{ or } 6).$$

Hence $\dim \Pi_{2q-2}(\Gamma) = 1$ if $q \equiv 1 \pmod n$, and $\dim \Pi_{2q-2}(\Gamma) = 0$ otherwise.

Finally, we consider a group Γ with two limit points.

Hence Γ contains a maximal cyclic subgroup generated by a loxodromic element. We can assume that $A : z \to \lambda z$, $|\lambda| \neq 1$ is the generator. If $p \in \Pi_{2q-2}$, $p(z) = \sum_{j=0}^{2q-2} a_j z^j$ is invariant under A if $\sum_{j=0}^{2q-2} a_j (1-\lambda^{j+1-q}) z^j = 0$. Thus an arbitrary polynomial invariant under A is given by $p(z) = a_{q-1} z^{q-1}$. So the fixed point space of A has dimension 1 and is generated by z^{q-1}. If Γ is not itself a loxodromic cyclic group, then the additional generators for other groups are

$$\mathbb{Z} + \mathbb{Z}_n : z \to e^{2\pi i/n} z \; (2 \leq n < \infty)$$

$$\mathbb{Z} * \mathbb{Z}_2 : z \to 1/z$$

double dihedral group: $z \to e^{2\pi i/n} z$, $z \to 1/z$ $(2 \leq n < \infty)$.

Clearly z^{q-1} is invariant by the additional generator for the group $\mathbb{Z} + \mathbb{Z}_n$. For the rest of the groups, z^{q-1} is invariant under the additional generators if and only if q is odd. Thus we have completed the computation of $\dim B^1(\Gamma, \Pi_{2q-2})$ for elementary groups Γ.

SECTION IV

Surjectivity of the Bers map

In this section we will show the surjectivity of the Bers map for geometrically finite function groups. This is the main result of our thesis. Nakada [18] proved this result for $q = 2$. The plan of the proof is same as Nakada's [18]. We have seen that the Bers map is surjective for finitely generated quasi-Fuchsian groups (of the first kind). Also, we have computed the dimension of the space of parabolic cohomology classes for elementary groups. Finally, we use the fact that a geometrically finite function group can be built up by combination theorems from quasi-Fuchsian and elementary groups to complete the proof of the main result.

§1. We begin this section by giving statements of combination theorems. For this we need the following definitions.

Definition 4.1. If Γ is a non-elementary Kleinian group with a simply connected invariant component Δ_o, then there is a conformal map φ from Δ_o onto the unit disc. A parabolic element $\gamma \in \Gamma$ is called accidental if $\varphi \circ \gamma \circ \varphi^{-1}$ is hyperbolic.

Definition 4.2. Let H be a cyclic (perhaps trivial) subgroup of a Kleinian group Γ. The interior B of a closed topological disc is called a <u>precisely invariant disc</u> under H if $h(\overline{B}-\Lambda(H)) = \overline{B}-\Lambda(H)$ for $h \in H$ and $\gamma((\overline{B}-\Lambda(H))) \cap (\overline{B}-\Lambda(H)))$ is empty for $\gamma \in \Gamma-H$, where $\overline{B}$ is the closure of B, $\Lambda(H)$ is the limit set of H and $\overline{B}-\Lambda(H) \subset \Omega(\Gamma)$.

The following two theorems of Maskit [16] will play an important role of our discussion in the rest of the thesis.

<u>Combination Theorem I</u>. Let H be a cyclic subgroup of both the Kleinian groups Γ_1 and Γ_2. For $i = 1,2$, let B_i be a precisely invariant disc under H. Assume that B_1 and B_2 have common boundary C, and $B_1 \cap B_2$ is empty. We also assume that if H is parabolic, then H is its own normalizer in either Γ_1 or Γ_2. Then

(i) Γ, the group generated by Γ_1 and Γ_2, is Kleinian;

(ii) Γ is the free product of Γ_1 and Γ_2 with amalgamated subgroup H;

(iii) $\Omega(\Gamma)/\Gamma = (\Omega(\Gamma_1)/\Gamma_1 - B_1/H) \cup (\Omega(\Gamma_2)/\Gamma_2 - B_2/H)$, where $\{\Omega(\Gamma_1)/\Gamma_1 - B_1/H\} \cap \{\Omega(\Gamma_2)/\Gamma_2 - B_2/H\} = C \cap \Omega(H)/H$;

(iv) every elliptic or parabolic element of Γ is conjugate in Γ to some element of either Γ_1 or Γ_2.

<u>Combination Theorem II</u>. Let H_1 and H_2 be cyclic subgroups of the Kleinian group Γ_1. For $i = 1,2$, let B_i be a precisely

invariant disc under H_i, and let C_i be the boundary of B_i. We assume that $\gamma(\overline{B}_1) \cap \overline{B}_2$ is empty for all γ in Γ_1. Let Γ_2 be cyclic, generated by f, where $f(C_1) = C_2$, $f(B_1) \cap B_2$ is empty and $f^{-1}H_1 f = H_2$. Let Γ be the group generated by Γ_1 and Γ_2. Then

(i) Γ, the group generated by Γ_1 and f is Kleinian;

(ii) every relation in Γ is a consequence of the relations in Γ_1 and the relations $f^{-1}H_1 f = H_2$;

(iii) $\Omega(\Gamma)/\Gamma = \Omega(\Gamma_1)/\Gamma_1 - (B_1/H_1 \cup B_2/H_2)$, where $(C_1 \cap \Omega(\Gamma))/H_1$ is identified in $\Omega(\Gamma)/\Gamma$ with $(C_2 \cap \Omega(\Gamma))/H_2$;

(iv) every elliptic or parabolic element of Γ is conjugate in Γ to some element of Γ_1.

In Combination Theorem II, the transformation f is necessarily loxodromic.

A <u>basic group</u> by definition, is finitely generated Kleinian group that has a simply connected invariant component and contains no accidental parabolic transformations. A basic group is either elementary, degenerate or quasi-Fuchsian (first kind). See Maskit [16] for details.

Let Γ be a non-elementary finitely generated function group. In [16], Maskit proved that there is a collection

$\Gamma_1, \ldots, \Gamma_s$ of basic subgroups of Γ, so that Γ is formed from $\Gamma_1, \ldots, \Gamma_s$ by $s - 1$ applications of Combination Theorem I, and say t applications of Combination Theorem II, where in each step the amalgamated subgroups or the conjugated subgroups are trivial, elliptic or parabolic cyclic.

Maskit [17] also proved that starting only with the elementary and quasi-Fuchsian groups (that is, excluding degenerate groups) as basic subgroups and using combination theorems as above, one obtains geometrically finite function groups. Every such group can be constructed in this manner. We finish these discussions by giving the following definitions and a theorem of Maskit for future reference.

A subgroup Γ' of Γ is called a <u>factor subgroup</u> if Γ' is a maximal subgroup of Γ with the following properties:

(i) The invariant component of Γ', which contains the invariant component of Γ, is simply connected.

(ii) If $\gamma \in \Gamma$ is parabolic and the fixed point of γ lies in $\Lambda(\Gamma')$, then $\gamma \in \Gamma'$.

(iii) Γ' contains no accidental parabolic elements.

With this definition the following has been proved in Maskit [16].

<u>Theorem 4.1.</u> Every factor subgroup Γ' of a finitely generated

function group Γ is conjugate in Γ to a unique basic subgroup of Γ.

Now we will state our main theorem as follows:

Let Γ be a (non-elementary) finitely generated function group and let Γ be constructed from basic subgroups $\Gamma_1,\ldots,\Gamma_s$ of Γ (with the exception of the signatures $(0;2,2,2,n)$, $3 \le n \le \infty$ or $(0;2,2,2,2,2)$, for degenerate basic subgroups), by using Combination Theorem I and II. Then Γ is geometrically finite if and only if

$$PH^1(\Gamma,\Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma),\Gamma) \text{ for one (hence all) } q \ge 2.$$

§2. First we derive a formula for dim $PZ^1(\Gamma,\Pi_{2q-2})$ in terms of dim $PZ^1(\Gamma_1,\Pi_{2q-2})$ and dim $PZ^1(\Gamma_2,\Pi_{2q-2})$ for a group Γ which is generated by its subgroups Γ_1 and Γ_2 by application of Combination Theorem I.

To this end, we need the following lemmas:

Lemma 4.1. Let Γ be a Kleinian group and Γ_o be an elliptic cyclic subgroup of the group Γ. Then the map

$$\Psi : Z^1(\Gamma,\Pi_{2q-2}) \mapsto Z^1(\Gamma_o,\Pi_{2q-2})$$

defined by $\Psi(p) = p|_{\Gamma_o}$ is surjective.

Proof. We have already shown in Section III that for finite elementary group every cocycle is a coboundary. Let p_o be a cocycle of Γ_o. Then there is a polynomial $v \in \Pi_{2q-2}$ such that

$$p_o(\gamma) = v\cdot\gamma - v \quad \text{for } \gamma \in \Gamma_o.$$

Then p_o can be extended to an element $\tilde{p}_o$ of $Z^1(\Gamma, \Pi_{2q-2})$ by

$$\tilde{p}_o(\gamma) = v\cdot\gamma - v \quad \text{for } \gamma \in \Gamma.$$

Hence the map Ψ is surjective.

Lemma 4.2. Let Γ be a Kleinian group and Γ_o be a parabolic cyclic subgroup of the group Γ. Then the map

$$\Psi : PZ^1(\Gamma, \Pi_{2q-2}) \to PZ^1(\Gamma_o, \Pi_{2q-2})$$

defined by $\Psi(p) = p|_{\Gamma_o}$ is surjective.

Proof. Since Γ_o is parabolic cyclic, we see that $PZ^1(\Gamma_o, \Pi_{2q-2}) = B^1(\Gamma_o, \Pi_{2q-2})$ by definition of $PZ^1(\Gamma_o, \Pi_{2q-2})$. Hence, we may proceed as in Lemma 4.1.

Theorem 4.2. If Γ is a (non-elementary) Kleinian group generated by finitely generated subgroups Γ_1 and Γ_2 by application of Combination Theorem I and if $H = \Gamma_1 \cap \Gamma_2$ is elliptic or parabolic cyclic or trivial, then

$$\dim PZ^1(\Gamma, \Pi_{2q-2}) = \dim PZ^1(\Gamma_1, \Pi_{2q-2}) + \dim PZ^1(\Gamma_2, \Pi_{2q-2}) - \dim PZ^1(H, \Pi_{2q-2}).$$

<u>Proof</u>. Define $\Phi : PZ^1(\Gamma, \Pi_{2q-2}) \to PZ^1(\Gamma_1, \Pi_{2q-2}) \times PZ^1(\Gamma_2, \Pi_{2q-2})$

by $\Phi(p) = (p_1, p_2)$, $p_i = p|_{\Gamma_i}$, $i = 1, 2$.

Observe, Φ is linear and injective.

Since Γ is generated by Γ_1 and Γ_2 we now consider the mapping

$$\tilde{\Phi} : [PZ^1(\Gamma_1, \Pi_{2q-2}) \times PZ^1(\Gamma_2, \Pi_{2q-2})]/\Phi(PZ^1(\Gamma, \Pi_{2q-2})) \mapsto PZ^1(H, \Pi_{2q-2})$$

defined by $\tilde{\Phi}(\{(p_1, p_2)\}) = p_1|_H - p_2|_H$, where $\{(p_1, p_2)\}$ is a representative of equivalence class. Clearly, the mapping $\tilde{\Phi}$ is well defined and linear.

Let $p \in PZ^1(H, \Pi_{2q-2})$, then by Lemma 4.1 and 4.2, p can be extended to a parabolic cocycle of $p_i \in PZ^1(\Gamma_i, \Pi_{2q-2})$ $(i=1,2)$ such that $p_i|_H = p$. Hence,

$$\tilde{\Phi}(\{(2p_1, p_2)\}) = 2p_1|_H - p_2|_H = 2p - p = p.$$

This shows that $\tilde{\Phi}$ is surjective.

Next, we shall show $\tilde{\Phi}$ is injective. Let $\tilde{\Phi}(\{(p_1, p_2)\}) = 0$. Then

$$p_1|_H = p_2|_H. \text{ We set } p = p_1|_H = p_2|_H.$$

Let $\Gamma_1 = H + \sum_\alpha H a_\alpha$ and $\Gamma_2 = H + \sum_\beta H b_\beta$ be the right H-coset decomposition of Γ_1 and Γ_2 respectively. Then for any element $\gamma \in \Gamma$, we have a unique representation

$$\gamma = h \circ \gamma_1 \circ \dots \circ \gamma_t,$$

where $h \in H$, and γ_i is some a_α or some b_β. Furthermore, γ_i and γ_{i+1} are not contained simultaneously in the same $\Gamma_i (i=1,2)$.

We define the mapping $\chi : \Gamma \to \Pi_{2q-2}$ by

$$\chi(\gamma) = p(h)\cdot(\gamma_1 \circ \cdots \circ \gamma_t) + p_{i_1}(\gamma_1)\cdot(\gamma_2 \circ \cdots \circ \gamma_t) + p_{i_2}(\gamma_2)\cdot(\gamma_3 \circ \cdots \circ \gamma_t) + \cdots + p_{i_t}(\gamma_t),$$

where $i_k = 1$ if $\gamma_k \in \Gamma_1$, and $i_k = 2$ if $\gamma_k \in \Gamma_2$.

We will show that χ, defined as above, belongs to $PZ^1(\Gamma, \Pi_{2q-2})$. This will show that

$$(p_1, p_2) = \Phi(\chi) \in \Phi(PZ^1(\Gamma, \Pi_{2q-2})); \text{ that is, } \{(p_1, p_2)\} = 0.$$

Thus the mapping $\tilde{\Phi}$ is injective. Therefore, $\tilde{\Phi}$ is bijective and consequently we have

$$\dim([PZ^1(\Gamma_1, \Pi_{2q-2}) \times PZ^1(\Gamma_2, \Pi_{2q-2})]/\Phi(PZ^1(\Gamma, \Pi_{2q-2}))) = \dim PZ^1(H, \Pi_{2q-2}).$$

The desired equality follows from injectivity of Φ. To prove $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$, first we will show that χ is a co-cycle of Γ.

Take one more $\gamma' \in \Gamma$ and let $\gamma' = h' \circ \gamma_1' \circ \ldots \circ \gamma_s'$ be a unique representation of γ'. By induction on t, we verify that $\chi(\gamma \circ \gamma') = \chi(\gamma) \cdot \gamma' + \chi(\gamma')$.

In fact, if t = 1, and if γ_1 and γ_1' are not contained simultaneously in the same Γ_i; say γ_1 is in Γ_1 and γ_1' is in Γ_2, then $h \circ \gamma_1 \circ h' = \tilde{h} \circ a_\alpha$ for some $a_\alpha \in \Gamma_1$ and $\tilde{h} \in H$. So

$$\gamma \circ \gamma' = \tilde{h} \circ a_\alpha \circ \gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s'.$$

Hence by definition of χ, we have

$$\chi(\gamma \circ \gamma') = p(\tilde{h}) \cdot (a_\alpha \circ \gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s') + p_1(a_\alpha) \cdot (\gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s')$$
$$+ p_2(\gamma_1') \cdot (\gamma_2' \circ \ldots \circ \gamma_s') + p_1(\gamma_2') \cdot (\gamma_3' \circ \ldots \circ \gamma_s') + \ldots + p_{i_1}(\gamma_s').$$

Since $a_\alpha = \tilde{h}^{-1} \circ h \circ \gamma_1 \circ h'$ and since $p_1 \in Z^1(\Gamma_1, \Pi_{2q-2})$ we have $p_1(a_\alpha) = -p(\tilde{h}) \cdot (\tilde{h}^{-1} \circ h \circ \gamma_1 \circ h') + p(h) \cdot (\gamma_1 \circ h')$

$$+ p_1(\gamma_1) \cdot h' + p(h').$$

Therefore, $\chi(\gamma \circ \gamma') = p(\tilde{h}) \cdot (\tilde{h}^{-1} \circ h \circ \gamma_1 \circ h' \circ \gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s')$

$$+ \{-p(\tilde{h}) \cdot (\tilde{h}^{-1} \circ h \circ \gamma_1 \circ h') + p(h) \cdot (\gamma_1 \circ h') + p_1(\gamma_1) \cdot h' + p(h')\} \cdot (\gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s')$$

$$+ p_2(\gamma_1') \cdot (\gamma_2' \circ \ldots \circ \gamma_s') + p_1(\gamma_2') \cdot (\gamma_3' \circ \ldots \circ \gamma_s') + \ldots + p_{i_1}(\gamma_s')$$

$$= p(h) \cdot (\gamma_1 \circ h' \circ \gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s') + p_1(\gamma_1) \cdot (h' \circ \gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s')$$
$$+ p(h') \cdot (\gamma_1' \circ \gamma_2' \circ \ldots \circ \gamma_s') + p_2(\gamma_1') \cdot (\gamma_2' \circ \ldots \circ \gamma_s') + p_1(\gamma_2') \cdot (\gamma_3' \circ \ldots \circ \gamma_s')$$
$$+ \ldots + p_{i_1}(\gamma_s')$$

$= p(h)\cdot(\gamma_1\circ\gamma') + p_1(\gamma_1)\cdot\gamma' + \chi(\gamma')$

$= \{p(h)\cdot\gamma_1 + p_1(\gamma_1)\}\cdot\gamma' + \chi(\gamma')$

$= \chi(\gamma)\cdot\gamma' + \chi(\gamma')$.

Similarly we can prove that $\chi(\gamma\circ\gamma') = \chi(\gamma)\cdot\gamma' + \chi(\gamma')$, when γ_1 and γ_1' are contained simultaneously in the same Γ_i. This step is left to the reader. Now we assume that $\chi(\gamma\circ\gamma') = \chi(\gamma)\cdot\gamma' + \chi(\gamma')$ holds for $\gamma = h\circ\gamma_1\circ\ldots\circ\gamma_t$ and $\gamma' = h'\circ\gamma_1'\circ\ldots\circ\gamma_s'$.
Let $\tilde{\gamma} = h\circ\gamma_1\circ\ldots\circ\gamma_{t+1}$ be a unique representation of $\tilde{\gamma}\in\Gamma$.

If γ_{t+1} and γ_1' are contained in the same Γ_i, say Γ_1, then $\gamma_{t+1}\circ h'\circ\gamma_1' = \tilde{h}\circ a_\alpha$ for some $a_\alpha\in\Gamma_1$ and $\tilde{h}\in H$, so $\tilde{\gamma}\circ\gamma' = h\circ\gamma_1\circ\ldots\circ\gamma_t\circ\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\ldots\circ\gamma_s'$.

Hence by induction hypothesis, we have

$$\chi(\tilde{\gamma}\circ\gamma') = \chi(h\circ\gamma_1\circ\ldots\circ\gamma_t)\cdot(\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\ldots\circ\gamma_s') + \chi(\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\ldots\circ\gamma_s').$$

So, by definition of χ, we have

$$\begin{aligned}\chi(\tilde{\gamma}\circ\gamma') = \{&\chi(h\circ\gamma_0\circ\ldots\circ\gamma_{t+1})\cdot\gamma_{t+1}^{-1} - p_1(\gamma_{t+1})\cdot\gamma_{t+1}^{-1}\}\\ &\cdot(\tilde{h}\circ a_\alpha\circ\gamma_2'\circ\ldots\circ\gamma_s') + p(\tilde{h})\cdot(a_\alpha\circ\gamma_2'\circ\ldots\circ\gamma_s')\\ &+ p_1(a_\alpha)\cdot(\gamma_2'\circ\ldots\circ\gamma_s') + p_2(\gamma_2')\cdot(\gamma_3'\circ\ldots\circ\gamma_s')\\ &+\ldots+p_{is}(\gamma_s').\end{aligned}$$

Since $a_\alpha = \tilde{h}^{-1}\circ\gamma_{t+1}\circ h'\circ\gamma_1'$ and since $p_1\in Z^1(\Gamma_1,\Pi_{2q-2})$

we have

$$p_1(a_\alpha) = -p(\tilde{h})\cdot(\tilde{h}^{-1}\circ\gamma_{t+1}\circ h'\circ\gamma_1') + p_1(\gamma_{t+1})\cdot(h'\circ\gamma_1') + p(h')\cdot\gamma_1' + p_1(\gamma_1').$$

Therefore,

$$\chi(\tilde{\gamma}\circ\gamma') = \chi(\tilde{\gamma})\cdot\gamma' + \chi(\gamma').$$

Similarly, we can prove the above if γ_{t+1} and γ_1' are not contained simultaneously in the same Γ_i. Therefore, $\chi(\gamma\circ\gamma') = \chi(\gamma)\cdot\gamma' + \chi(\gamma')$ for any γ and γ' in Γ.

So, χ defined as above belongs to $Z^1(\Gamma, \Pi_{2q-2})$. To prove $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$, let $\gamma \in \Gamma$ be any parabolic element. Then there exist a parabolic element $\gamma_i \in \Gamma_i$ ($i = 1$ or 2) and an element $\alpha_i \in \Gamma$ such that $\gamma = \alpha_i\circ\gamma_i\circ\alpha_i^{-1}$. From the definition of χ, we see that $\chi|\Gamma_i = p_i \in PZ^1(\Gamma_i, \Pi_{2q-2})$ and $\chi(\gamma_i) = v\cdot\gamma_i - v$ for some $v \in \Pi_{2q-2}$. Hence we see that

$$\begin{aligned}
\chi(\gamma) &= \chi(\alpha_i\circ\gamma_i\circ\alpha_i^{-1}) \\
&= \chi(\alpha_i)\cdot(\gamma_i\circ\alpha_i^{-1}) + p_i(\gamma_i)\cdot\alpha_i^{-1} + \chi(\alpha_i^{-1}) \\
&= \chi(\alpha_i)\cdot(\gamma_i\circ\alpha_i^{-1}) + (v\cdot\gamma_i - v)\cdot\alpha_i^{-1} + \chi(\alpha_i^{-1}) \\
&= -\chi(\alpha_i^{-1})\cdot(\alpha_i\circ\gamma_i\circ\alpha_i^{-1}) + (v\cdot\alpha_i^{-1})\cdot(\alpha_i\circ\gamma_i\circ\alpha_i^{-1}) - v\cdot\alpha_i^{-1} + \chi(\alpha_i^{-1}) \\
&= -\chi(\alpha_i^{-1})\cdot\gamma + (v\cdot\alpha_i^{-1})\cdot\gamma - v\cdot\alpha_i^{-1} + \chi(\alpha_i^{-1}) \\
&= w\cdot\gamma - w \quad \text{for } w = v\cdot\alpha_i^{-1} - \chi(\alpha_i^{-1}).
\end{aligned}$$

Therefore, we obtain $\chi \in PZ^1(\Gamma, \Pi_{2q-2})$.

Remarks:

(1) The above arguments also appear in Nakada [18].

(2) Note that $\dim PZ^1(H,\Pi_{2q-2}) = 2[q - \frac{q}{\nu}]$, where ν $(1 \le \nu \le \infty)$ is order of the cyclic group H. For a parabolic cyclic group H, ν is regarded as ∞ and $[q - \frac{q}{\infty}] = q-1$. When H is trivial, $\nu = 1$. It is clear that $\dim PZ^1(H,\Pi_{2q-2}) = 0$ in this case. We will make use of the above fact often in this section.

The following is a simple observation that follows from the proof of the Theorem 4.2. It will be needed for our discussion in the next section.

If Γ is a non-elementary Kleinian group which is generated by finitely generated subgroups Γ_1 and Γ_2 by application of Combination Theorem I and if $H = \Gamma_1 \cap \Gamma_2$ is elliptic or parabolic cyclic or trivial, then

$$\dim Z^1(\Gamma,\Pi_{2q-2}) \ge \dim Z^1(\Gamma_1,\Pi_{2q-2}) + \dim Z^1(\Gamma_2,\Pi_{2q-2}) - \dim Z^1(H,\Pi_{2q-2}).$$

The above inequality is equality if H is elliptic cyclic or trivial. Our argument of injectivity of $\tilde{\Phi}$ in Theorem 4.2 also holds for this setup. $\tilde{\Phi}$ is also surjective if H is elliptic or trivial.

Next we derive a relation between $\dim PZ^1(\Gamma,\Pi_{2q-2})$ and $\dim PZ^1(\Gamma_1,\Pi_{2q-2})$ for the group Γ which is generated by its subgroup Γ_1 and an element f by application of Combination Theorem II. We can state the following theorem in a more general manner.

Theorem 4.3. Let Γ be a (non-elementary) Kleinian group which is generated by its finitely generated subgroup Γ_1 and an element f. Let H_1 (or H_2) be an elliptic cyclic or parabolic cyclic or trivial subgroup conjugate in Γ but not in Γ_1. If every relation in Γ is a consequence of a relation in Γ_1 and the relation of $f^{-1} \circ H_2 \circ f = H_1$, and every elliptic or parabolic element in Γ is conjugate in Γ to an element in Γ_1, then $\dim PZ^1(\Gamma, \Pi_{2q-2}) = \dim PZ^1(\Gamma_1, \Pi_{2q-2}) + (2q-1) - \dim PZ^1(H_2, \Pi_{2q-2})$.

Proof. Since f is loxodromic, and since every cocycle χ of $PZ^1(\langle f \rangle, \Pi_{2q-2})$ can be uniquely determined by an arbitrary choice of $\chi(f)$, we conclude that $\dim PZ^1(\langle f \rangle, \Pi_{2q-2}) = 2q-1$.

Also observe if χ is a cocycle of Γ_1 and $v \in \Pi_{2q-2}$, then (χ, v) defines a cocycle $\tilde{\chi}$ for Γ with $\tilde{\chi}|\Gamma_1 = \chi, \tilde{\chi}(f) = v$ if and only if

$\chi(h_2) = v \cdot (h_1 \circ f^{-1}) + \chi(h_1) \cdot f^{-1} - v \cdot f^{-1}$; where h_1 and h_2 are generators of H_1 and H_2 respectively. See for example, Weil [21].

Moreover, if χ is parabolic cocycle of Γ_1 then $\tilde{\chi}$ is also parabolic by the same argument as in Theorem 4.2. Also, we can define a linear and injective mapping Φ as in the proof of

Theorem 4.2,

$$\Phi : PZ^1(\Gamma,\Pi_{2q-2}) \mapsto PZ^1(\Gamma_1,\Pi_{2q-2}) \times \Pi_{2q-2}$$

$\Phi(\tilde{\chi}) = (\chi,v)$, where $\chi = \tilde{\chi}|_{\Gamma_1}$ (that is, χ is restriction of $\tilde{\chi}$ to Γ_1) and $v = \tilde{\chi}(f)$.

Using the mapping Φ, we consider the mapping

$$\tilde{\Phi} : [PZ^1(\Gamma_1,\Pi_{2q-2}) \times \Pi_{2q-2}]/\Phi(PZ^1(\Gamma,\Pi_{2q-2})) \to PZ^1(H_2,\Pi_{2q-2})$$

defined by $\tilde{\Phi}\{(\chi,v)\} = p$, where

$$p(h_2) = v\cdot(h_1\circ f^{-1}) + \chi(h_1)\cdot f^{-1} - v\cdot f^{-1} - \chi(h_2) \quad \text{for}$$

$h_1 = f^{-1}\circ h_2\circ f \in H_1$. It is easy to see that $p \in PZ^1(H_2,\Pi_{2q-2})$ and $\tilde{\Phi}$ is well-defined and linear. To show that the mapping $\tilde{\Phi}$ is injective, we assume $\tilde{\Phi}\{(\chi,v)\} = 0$. Thus we have $p(h_2) = v\cdot(h_1\circ f^{-1}) + \chi(h_1)\cdot f^{-1} - v\cdot f^{-1} - \chi(h_2) = 0$. Hence, there exist an element $\tilde{\chi} \in PZ^1(\Gamma,\Pi_{2q-2})$ such that $\tilde{\chi}|_{\Gamma_1} = \chi$ and $\tilde{\chi}(f) = v$ which shows that $\{(\chi,v)\} \in \Phi(PZ^1(\Gamma,\Pi_{2q-2}))$; that is, $\tilde{\Phi}$ is injective.

Next we shall show the surjectivity of $\tilde{\Phi}$. Let p and χ_1 be arbitrary elements of $PZ^1(H_2,\Pi_{2q-2})$ and $PZ^1(\Gamma,\Pi_{2q-2})$ respectively.

For $h_i \in H_i (i=1,2)$ we see from Lemma 4.1 and 4.2 that

$$p(h_2) = u\cdot h_2 - u,\ \chi_1(h_1) = w_1\cdot h_1 - w_1,\ \chi_1(h_2) = w_2\cdot h_2 - w_2$$

for some polynomial $u, w_1, w_2 \in \Pi_{2q-2}^{-}$.

Now, for $h_i \in H_i (i=1,2)$ and $f \circ h_1 \circ f^{-1} = h_2$, we have

$$p(h_2)\cdot f - \chi_1(h_1) + \chi_1(h_2)\cdot f = (u\cdot h_2 - u)\cdot f - (w_1\cdot h_1 - w_1) + (w_2\cdot h_2 - w_2)$$

$$= u\cdot(h_2 \circ f) - u\cdot f - (w_1\cdot h_1 - w_1) + w_2\cdot(h_2\circ f) - w_2\cdot f$$

$$= u\cdot(f\circ h_1) - u\cdot f - (w_1\cdot h_1 - w_1) + w_2\cdot(f\circ h_1) - w_2\cdot f$$

$$= (u\cdot f - w_1 + w_2\cdot f)\cdot h_1 - (u\cdot f - w_1 + w_2\cdot f)$$

Set $\quad v = u\cdot f - w_1 + w_2\cdot f \in \Pi_{2q-2}^{-}$. Then we have,

$$p(h_2)\cdot f - \chi_1(h_1) + \chi_1(h_2)\cdot f = v\cdot h_1 - v.$$

Hence $p(h_2) = v\cdot(h_1\circ f^{-1}) - v\cdot f^{-1} + \chi_1(h_1)\cdot f^{-1} - \chi_1(h_2)$.

Thus, we have $\tilde{\Phi}(\{(\chi_1, v)\}) = p$; that is, Φ is surjective.

Consequently, $\dim([PZ^1(\Gamma_1, \Pi_{2q-2}^{-}) \times \Pi_{2q-2}^{-}]/\Phi(PZ^1(\Gamma, \Pi_{2q-2}^{-})))$

$$= \dim PZ^1(H_2, \Pi_{2q-2}^{-}).$$

From the injectivity of Φ, the equality of the theorem follows.

Remarks:

(1) The above argument is similar to the argument which appeared in Nakada [18].

(2) If Γ is a (non-elementary) Kleinian group which is generated by its finitely generated subgroup Γ_1 and an element f by application of Combination Theorem II,

and if the group H_1 (or H_2) is elliptic cyclic or parabolic cyclic or trivial, then

$$\dim Z^1(\Gamma,\Pi_{2q-2}) \geq \dim Z^1(\Gamma_1,\Pi_{2q-2}) + (2q-1) - \dim Z^1(H_2,\Pi_{2q-2}).$$

The above inequality is equality if H is elliptic cyclic or trivial.

§3. Let Γ be a non-elementary finitely generated Kleinian group. Let $\{\Delta_1,\ldots,\Delta_k\}$ be a maximal set of inequivalent components of Ω. Then $\Omega(\Gamma)/\Gamma = \bigoplus_{i=1}^{k} \Delta_i/\Gamma_{\Delta_i} = S_1+S_2+\ldots+S_k$, where $S_i = \Delta_i/\Gamma_{\Delta_i}$. For each i, let $(g_i;\nu_{i1},\ldots,\nu_{in_i})$ be the signature of S_i (that is, of Γ_{Δ_i}). Recall that $\dim A_q(\Omega,\Gamma) = \sum_{i=1}^{k}\{(2q-1)(g_i-1) + \sum_{j=1}^{n_i}[q - \frac{q}{\nu_{ij}}]\}$. For an elementary group Γ with the signature $(g;\nu_1,\ldots,\nu_n)$, we use $N_\Gamma(q)$ in place of $\dim A_q(\Omega,\Gamma)$. We recall from Section III that $N_\Gamma(q) = (2q-1)(q-1) + \sum_{i=1}^{n}[q - \frac{q}{\nu_i}]$ for an elementary group Γ with signature $(g;\nu_1,\nu_2,\ldots,\nu_n)$. Under this convention we have the following lemmas:

Lemma 4.3. If Γ is a Kleinian group (non-elementary) which is generated by its finitely generated subgroups Γ_1 and Γ_2 by application of Combination Theorem I as stated in the beginning, and if $H = \Gamma_1 \cap \Gamma_2$ is elliptic cyclic or parabolic cyclic or trivial, then

$$\dim A_q(\Omega(\Gamma),\Gamma) = \dim A_q(\Omega(\Gamma_1),\Gamma_1) + \dim A_q(\Omega(\Gamma),\Gamma_2) + (2q-1) - 2[q - \frac{q}{\nu}],$$

where $\nu(1 \le \nu \le \infty)$ is the order of H.

<u>Proof</u>. <u>Case I</u>: Γ_1 and Γ_2 are both non-elementary.

Let $\Omega(\Gamma_1)/\Gamma_1 = S_{11}+\dots+S_{1n}$ and $\Omega(\Gamma_2)/\Gamma_2 = S_{21}+\dots+S_{2m}$. Let $(g_{1i};\nu_{i1},\dots,\nu_{ik_i})$ be the signature of $S_{1i}(i=1,2,\dots n)$ and $(g_{2i};\mu_{i1},\dots,\mu_{ik_i'})$ be the signature of $S_{2i}(i=1,2,\dots m)$.

Since the precisely invariant disc $B_i(i=1,2)$ under H will be contained in some component of Γ_i, we assume first that $B_1/H \subset S_{11}$ and $B_2/H \subset S_{21}$. The other cases can be proved similarly. Assume that H is cyclic group of order ν. Then H is trivial or parabolic whenever $\nu = 1$ or ∞. Let $\nu = \nu_{1t} = \mu_{1s}$ for some $t(1 \le t \le k_1)$ and $s(1 \le s \le k_1')$. We may assume that $\nu = \mu_{11} = \nu_{11}$.

Then by Combination Theorem I,

$$\Omega(\Gamma)/\Gamma = S + S_{12}+\dots+S_{1n} + S_{22}+\dots+S_{2m}$$

where S is a surface constructed topologically from S_{11} by removing two punctured discs B_1/H and B_2/H (when H is parabolic) or two discs B_1/H and B_2/H (when H is trivial) and gluing along their boundaries. Thus we have,

$$\dim A_q(\Omega(\Gamma),\Gamma) = (2q-1)(g_{11}+g_{21}-1) + \sum_{j=2}^{k_1}[q - \frac{q}{\nu_{1j}}] + \sum_{j=2}^{k_1}[q - \frac{q}{\mu_{1j}}]$$

$$+\sum_{i=2}^{n}\{(2q-1)(g_{1i}-1)+\sum_{j=1}^{k_i}[q-\frac{q}{\nu_{ij}}]\}$$

$$+\sum_{i=2}^{m}\{(2q-1)(g_{2i}-1)+\sum_{j=1}^{k'_i}[q-\frac{q}{\mu_{ij}}]\}$$

$$=\sum_{i=1}^{n}\{(2q-1)(g_{1i}-1)+\sum_{j=1}^{k_i}[q-\frac{q}{\nu_{ij}}]\}$$

$$+\sum_{i=1}^{m}\{(2q-1)(g_{2i}-1)+\sum_{j=1}^{k'_i}[q-\frac{q}{\mu_{ij}}]\}+(2q-1)-2[q-\frac{q}{\nu}]$$

$$=\dim A_q(\Omega(\Gamma_1),\Gamma_1)+\dim A_q(\Omega(\Gamma_2),\Gamma_2)+(2q-1)-2[q-\frac{q}{\nu}].$$

<u>Case II</u>: Γ_1 is elementary and Γ_2 is non-elementary.

Let $\Omega(\Gamma_1)/\Gamma_1 = S_{11}$ and $\Omega(\Gamma_2)/\Gamma_2 = S_{21}+\dots+S_{2n}$.

Let $(g;\nu_1,\dots\nu_k)$ be the signature of S_{11} and $(g_{2i};\mu_{i1},\dots,\mu_{ik_i})$ be the signature of $S_{2i}(i=1,2\dots n)$.

Following Case I, we have

$$\dim A_q(\Omega(\Gamma),\Gamma) = (2q-1)(g+g_{21}-1)+\sum_{j=2}^{k}[q-\frac{q}{\nu_i}]+\sum_{j=2}^{k_1}[q-\frac{q}{\mu_{ij}}]$$

$$+\sum_{i=2}^{n}\{(2q-1)(g_{2i}-1)+\sum_{j=1}^{k_i}[q-\frac{q}{\mu_{ij}}]\}$$

$$=(2q-1)(g-1)+\sum_{i=1}^{k}[q-\frac{q}{\nu_i}]+\sum_{i=1}^{n}\{(2q-1)(g_{2i}-1)$$

$$+\sum_{j=1}^{k_i}[q-\frac{q}{\mu_{ij}}]\}+(2q-1)-2[q-\frac{q}{\nu}]$$

$$=N_{\Gamma_1}(q)+\dim A_q(\Omega(\Gamma_2),\Gamma_2)+(2q-1)-2[q-\frac{q}{\nu}].$$

Case III: Γ_1 and Γ_2 are both elementary.

In this case, H is either elliptic, cyclic or trivial. We may proceed as in the above cases.

Lemma 4.4. If Γ is a (non-elementary) Kleinian group which is generated by its finitely generated subgroup Γ_1 and an element f by application of Combination Theorem II as stated in the beginning, and if the subgroup H_1 (or H_2) of Γ_1 is elliptic cyclic or parabolic or trivial, then

$$\dim A_q(\Omega(\Gamma),\Gamma) = \dim A_q(\Omega(\Gamma_1),\Gamma_1) + (2q-1) - 2[q - \frac{q}{\nu}],$$

where $\nu(1 \le \nu \le \infty)$ is the order of H_1 (or H_2).

Proof. Γ_1 is either nonelementary or elementary. In case of elementary, $H_i(i=1,2)$ is either elliptic cyclic or trivial. Let $\Omega(\Gamma_1)/\Gamma_1 = S_{11}+\ldots+S_{1n}$.

Let $(g_{1i};\nu_{i1},\ldots,\nu_{ik_i})$ be the signature of $S_{1i}(i=1,\ldots n)$. Let B_1 and B_2 be the precisely invariant discs under H_1 and H_2 respectively. We assume that B_1/H_1 and B_2/H_2 both are subsets of S_{11}. The other cases can be done similarly.

Let $H_i(i=1,2)$ be a cyclic group of order $\nu(1 \le \nu \le \infty)$. Then $\nu = \nu_{1t} = \nu_{1s}$ for some t and s $(1 \le t \le k_1, 1 \le s \le k_1, t \ne s)$. We may assume that $\nu = \nu_{11} = \nu_{12}$. Then, by Combination Theorem II,

$$\Omega(\Gamma)/\Gamma = S+S_{12}+\ldots+S_{1n},$$

where S is a surface constructed topologically by removing two discs (possibly punctured) from S_{11} and gluing along the boundary curves (adding a handle). Thus we have,

$$\dim A_q(\Omega(\Gamma),\Gamma) = (2q-1)(g_{11}+1-1) + \sum_{i=3}^{k_1}[q - \frac{q}{\nu_{1i}}]$$

$$+ \sum_{i=2}^{n}\{(2q-1)(g_{1i}-1) + \sum_{j=1}^{k_i}[q - \frac{q}{\nu_{ij}}]$$

$$= \sum_{i=1}^{n}\{(2q-1)(g_{1i}-1) + \sum_{j=1}^{k_i}[q - \frac{q}{\nu_{ij}}]\}$$

$$+ (2q-1) - 2[q - \frac{q}{\nu}]$$

$$= \dim A_q(\Omega(\Gamma_1),\Gamma_1) + (2q-1) - 2[q - \frac{q}{\nu}].$$

Here, for our convenience, we record an observation made in Proposition 3.1.

Lemma 4.5. Let Γ_o be an elementary group. Then,

$$\dim PZ^1(\Gamma_o,\Pi_{2q-2}) = N_{\Gamma_o}(q) + (2q-1).$$

Next, we shall establish the following theorems which are essential parts of the proof of the main theorem.

Theorem 4.4. Let Γ be a Kleinian group which is generated by its finitely generated non-elementary subgroups Γ_1 and Γ_2 by application of Combination Theorem I. Assume that $H = \Gamma_1 \cap \Gamma_2$ be elliptic cyclic or parabolic cyclic or trivial. Then $PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma))$ if and only if $PH^1(\Gamma_i, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma_i), \Gamma_i)$, for $i = 1, 2$.

If Γ is formed by two elementary subgroups Γ_1 and Γ_2 by Combination Theorem I, then $PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma))$ If Γ is formed by a non-elementary subgroup Γ_1 and an elementary subgroup Γ_2 by Combination Theorem I, then $PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma))$ if and only if $PH^1(\Gamma_1, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma_1), \Gamma_1))$.

Proof. Case I: Γ_1 and Γ_2 are both non-elementary.

Let $\nu (1 \leq \nu \leq \infty)$ be the order of H. First, we assume that $PH^1(\Gamma_i, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma_i), \Gamma_i)$, for $i = 1, 2$. We have,

$$\dim PH^1(\Gamma, \Pi_{2q-2}) = \dim PH^1(\Gamma_1, \Pi_{2q-2}) + \dim PH^1(\Gamma_2, \Pi_{2q-2}) + (2q-1) - 2[q - \frac{q}{\nu}].$$

Since $\dim PH^1(\Gamma_i, \Pi_{2q-2}) = \dim A_q(\Omega(\Gamma_i), \Gamma_i)$, $(i=1,2)$, we have

$$\dim PH^1(\Gamma, \Pi_{2q-2}) = \dim A_q(\Omega(\Gamma_1), \Gamma_1) + \dim A_q(\Omega(\Gamma_2), \Gamma_2) + (2q-1) - 2[q - \frac{q}{\nu}].$$

Hence, from Lemma 4.3, we have $\dim PH^1(\Gamma, \Pi_{2q-2})$ $= \dim A_q(\Omega(\Gamma), \Gamma)$; that is, $PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma)$ by using the injectivity of the Bers map.

Conversely, we assume that $PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma))$. If $PH^1(\Gamma_i, \Pi_{2q-2}) \supsetneqq \beta_q^*(A_q(\Omega(\Gamma_i), \Gamma_i))$ for some i, say $i = 1$ then

$$\dim PH^1(\Gamma_1, \Pi_{2q-2}) > \dim A_q(\Omega(\Gamma_1), \Gamma_1).$$

Therefore, from Theorem 4.2, we have

$$\begin{aligned}\dim PH^1(\Gamma, \Pi_{2q-2}) &= \dim PH^1(\Gamma_1, \Pi_{2q-2}) + \dim PH^1(\Gamma_2, \Pi_{2q-2}) \\ &\quad + (2q-1) - 2[q - \tfrac{q}{\nu}] \\ &> \dim A_q(\Omega(\Gamma_1), \Gamma_1) + \dim A_q(\Omega(\Gamma_2), \Gamma_2) \\ &\quad + (2q-1) - 2[q - \tfrac{q}{\nu}]\end{aligned}$$

Again, from Lemma 4.3, right hand side of the above inequality is equal to $\dim A_q(\Omega(\Gamma), \Gamma)$. Hence

$$\dim PH^1(\Gamma, \Pi_{2q-2}) > \dim A_q(\Omega(\Gamma), \Gamma).$$

This contradicts our hypothesis. So $PH^1(\Gamma_i, \Pi_{2q-2})$ $= \beta_q^*(A_q(\Omega(\Gamma_i), \Gamma_i)$ for $i=1,2$.

<u>Case II.</u> Γ_1 and Γ_2 are both elementary.

Let $\nu (1 \leq \nu \leq \infty)$ be the order of H. By Theorem 4.2 we have

$$\begin{aligned}\dim PH^1(\Gamma, \Pi_{2q-2}) &= \dim PZ^1(\Gamma_1, \Pi_{2q-2}) + \dim PZ^1(\Gamma_2, \Pi_{2q-2}) \\ &\quad - 2[q - \tfrac{q}{\nu}] - (2q-1)\end{aligned}$$

But, by Lemma 4.5, the above inequality reads,

$$\dim PH^1(\Gamma,\Pi_{2q-2}) = N_{\Gamma_1}(q) + N_{\Gamma_2}(q) + (2q-1) - 2[q - \frac{q}{\nu}]$$

On the otherhand, $\dim A_q(\Omega(\Gamma),\Gamma) = N_{\Gamma_1} + N_{\Gamma_2} + (2q-1) - 2[q - \frac{q}{\nu}]$ by Lemma 4.3. Comparing the above two equalities we have

$$\dim A_q(\Omega(\Gamma),\Gamma) = \dim PH^1(\Gamma,\Pi_{2q-2});$$

that is, $PH^1(\Gamma,\Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma),\Gamma)$.

Case III. Γ_1 is non-elementary and Γ_2 is elementary.

Using Theorem 4.2 and Lemma 4.5 we have

$$\dim PH^1(\Gamma,\Pi_{2q-2}) = \dim PH^1(\Gamma_1,\Pi_{2q-2}) + N_{\Gamma_2}(q) + (2q-1) - 2[q - \frac{q}{\nu}].$$

On the otherhand, by Lemma 4.3, we have

$$\dim A_q(\Omega(\Gamma),\Gamma) = \dim A_q(\Omega(\Gamma_1),\Gamma_1) + N_{\Gamma_2}(q) + (2q-1) - [q - \frac{q}{\nu}].$$

Now, it is easy to see from the above two equalities that $\dim A_q(\Omega(\Gamma_1),\Gamma_1) = \dim PH^1(\Gamma_1,\Pi_{2q-2})$ if and only if $\dim PH^1(\Gamma,\Pi_{2q-2}) = \dim A_q(\Omega(\Gamma),\Gamma)$. This completes the proof.

Theorem 4.5. Let Γ be a Kleinian group which is generated by its finitely generated non-elementary subgroup Γ_1 and an element f by application of Combination Theorem II as stated in the beginning. Assume that H_1 (or H_2) be elliptic cyclic or parabolic cyclic or trivial, then

$$PH^1(\Gamma,\Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma),\Gamma)) \text{ if and only if}$$

$$PH^1(\Gamma_1,\Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma_1),\Gamma_1).$$

In the hypothesis, if Γ_1 is elementary subgroup then $PH^1(\Gamma,\Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma\ ),\Gamma\ ).$

Proof. Case I. Γ_1 is non-elementary

Let $\nu(1 \le \nu \le \infty)$ be the order of H_1. From Theorem 4.3, we have

$$\dim PH^1(\Gamma,\Pi_{2q-2}) = \dim PH^1(\Gamma_1,\Pi_{2q-2}) + (2q-1) - 2[q-\frac{q}{\nu}].$$

On the otherhand, by Lemma 4.4, we have

$$\dim A_q(\Omega(\Gamma),\Gamma) = \dim A_q(\Omega(\Gamma_1),\Gamma_1) + (2q-1) - 2[q-\frac{q}{\nu}].$$

It is clear from the above two equalities that $\dim A_q(\Omega(\Gamma_1),\Gamma_1) = \dim PH^1(\Gamma,\Pi_{2q-2})$, if and only if $\dim A_q(\Omega(\Gamma_1),\Gamma_1) = \dim PH^1(\Gamma_1,\Pi_{2q-2})$.

Case II. Γ_1 is elementary.

From Theorem 4.3, we have

$$\dim PH^1(\Gamma,\Pi_{2q-2}) = \dim PZ^1(\Gamma_1,\Pi_{2q-2}) - 2[q-\frac{q}{\nu}].$$

Hence, by Lemma 4.5, the above equality becomes

$$\dim PH^1(\Gamma,\Pi_{2q-2}) = N_{\Gamma_1}(q) + (2q-1) - 2[q-\frac{q}{\nu}].$$

On the otherhand, by Lemma 4.4, we have

$$\dim A_q(\Omega(\Gamma),\Gamma) = N_{\Gamma_1}(q) + (2q-1) - 2[q - \frac{q}{\nu}].$$

Comparing the above two equalities we have,

$$\dim A_q(\Omega(\Gamma),\Gamma) = \dim PH^1(\Gamma,\Pi_{2q-2}).$$

Hence, by injectivity of the Bers map, we have

$$\beta_q^*(A_q(\Omega(\Gamma),\Gamma) = PH^1(\Gamma,\Pi_{2q-2}).$$

§4. Let Γ be a (non-elementary) finitely generated function group. Then, as we stated earlier, we can construct Γ from basic subgroups in a finite number of steps by using Combination Theorems I and II where, in each step, the amalgamated subgroups and the conjugated subgroups are trivial, elliptic cyclic or parabolic.

Exceptional function groups: A finitely generated function group Γ whose all degenerate basic subgroups necessarily have signatures $(0;2,2,2,n)$, $3 \le n \le \infty$ or $(0;2,2,2,2,2)$ is called exceptional; otherwise, non-exceptional.

By definition, exceptional groups are not geometrically finite. Now, we can prove our main result stated in the beginning of the section which is formulated as follows:

Theorem 4.6. Let Γ be a non-elementary and non-exceptional finitely generated function group. Then Γ is geometrically finite if and only if $PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma))$ for one (hence all) $q \geq 2$.

If Γ is exceptional then $PH^1(\Gamma, \Pi_{2q-2}) \supsetneq \beta_q^*(A_q(\Omega(\Gamma), \Gamma))$ for all $q \geq 2$ except $q = 3$, 5 or 7.

<u>Proof</u>. First, assume that Γ is geometrically finite. Then Γ can be constructed from elementary and quasi-Fuchsian basic subgroups by using Combination Theorems I and II. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ be either elementary or quasi-Fuchsian basic subgroups of Γ from which Γ has been constructed.

For a quasi-Fuchsian group Γ_i, we have

$$\dim PH^1(\Gamma_i, \Pi_{2q-2}) = \dim A_q(\Omega(\Gamma_i), \Gamma_i)$$

(see Corollary of Theorem 2.3). So $PH^1(\Gamma_i, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma_i), \Gamma_i))$ by the injectivity of the Bers map.

As mentioned already, in each step of using Combination Theorems I and II, the amalgamated subgroups are trivial, elliptic cyclic or parabolic cyclic. Therefore, by Theorem 4.3 and Theorem 4.4, we have,

$$PH^1(\Gamma, \Pi_{2q-2}) = \beta_q^*(A_q(\Omega(\Gamma), \Gamma)) \quad \text{for all } q \geq 2.$$

On the otherhand, assume Γ_i is not geometrically finite. Then for some i, Γ_i is a degenerate basic group. Then

Γ_i has no accidental parabolic element. Hence we have,

$$\dim PH^1(\Gamma_i, \Pi_{2q-2}) = 2 \dim A_q(\Omega(\Gamma_i), \Gamma_i)$$

(see Section II).

For a Kleinian group Γ_i with signature $(g; \nu_1, \ldots, \nu_n)$ and $g \geq 1$, $\dim A_q(\Omega(\Gamma_i), \Gamma_i) > 0$ for all $q \geq 2$. Assume that Γ_i has signature $(0; \nu_1, \ldots, \nu_n)$. If $n \geq 6$, we observe that $\dim A_q(\Omega(\Gamma_i), \Gamma_i) = -(2q-1) + \sum_{i=1}^{6} [q - \frac{q}{\nu_i}]$ is always positive. Thus we consider only the cases when $n \leq 5$. But we know that a degenerate group cannot be a triangle group (that is, a Kleinian group with signature $(0; \nu_1, \nu_2, \nu_3)$) by a result of Kra [9]. Hence we reduce the case $n \leq 5$ to two cases $n = 4$ and $n = 5$. Since for a given n, dimension does not decrease when ramification number increases, we need to check only the lowest permissible signature in each case. The following table will clarify the rest of the cases:

Signature of the group Γ	$\dim A_q(\Omega, \Gamma)$
i) $(0;2,2,2,3)$	positive except for $q = 3,5$ or 7
ii) $(0;2,2,2,4)$	positive except for $q = 3,5$
iii) $(0;2,2,2,n)$ $(5 \leq n \leq \infty)$	positive except for $q = 3$
iv) $(0;2,2,2,2,2)$	positive except for $q = 3$

Hence, $\dim A_q(\Omega(\Gamma_i),\Gamma_i) > 0$ for a degenerate basic subgroup Γ_i of a nonexceptional group Γ.

Therefore, $\dim PH^1(\Gamma_i,\Pi_{2q-2}) > \dim A_q(\Omega(\Gamma_i),\Gamma_i)$ for all $q \geq 2$.

Hence by Theorem 4.4 and Theorem 4.5 we have,

$$\dim PH^1(\Gamma,\Pi_{2q-2}) > \dim A_q(\Omega(\Gamma),\Gamma) \text{ for a } q \geq 2.$$

So, $$PH^1(\Gamma,\Pi_{2q-2}) \supsetneq \beta_q^*(A_q(\Omega(\Gamma),\Gamma)) \qquad \text{for a } q \geq 2.$$

<u>This completes the proof.</u>

<u>Remark.</u> There are many examples of (non-function) geometrically finite groups for which the Bers map is surjective. See for example Kra [10]. But, at present, we do not have the answer whether the Bers map is surjective for all geometrically finite groups.

§5. Now, using Theorem 4.2, Theorem 4.3, Lemma 4.3 and Lemma 4.4 we shall obtain a formula for the dimension of the space of parabolic cohomology classes and a formula for the dimension of the space of cusp forms for non-elementary finitely generated function groups.

<u>Proposition 4.1.</u> Let Γ be a non-elementary finitely generated Kleinian group with an invariant component. Let Γ be

constructed from basic groups $\Gamma_1, \ldots \Gamma_s$ by $s-1$ applications of Combination Theorem I and say, t applications of Combination Theorem II, where in each step the amalgamated subgroups and the conjugated subgroups are trivial, elliptic or parabolic cyclic. Then

$$\dim PH^1(\Gamma, \Pi_{2q-2}) = \sum_{i=1}^{s} \dim PZ^1(\Gamma_i, \Pi_{2q-2}) + (2q-1)(t-1) - 2\sum_{i=1}^{s+t-1} [q - \frac{q}{\nu_i}]$$

and

$$\dim A_q(\Omega(\Gamma), \Gamma) = \sum_{i=1}^{s} \dim A_q(\Omega(\Gamma_i), \Gamma_i) + (2q-1)(s+t-1) - 2\sum_{i=1}^{s+t-1} [q - \frac{q}{\nu_i}].$$

In the second equality we replace $\dim A_q(\Omega(\Gamma_i), \Gamma_i)$ by N_{Γ_i}, when Γ_i's are elementary basic groups. The $\nu_i (1 \leq \nu_i \leq \infty)$ are the orders of cyclic subgroups used as an amalgamated and conjugated subgroup in the (s+t-1) application of Combination Theorem I and Combination Theorem II.

Proof. The first formula follows by using Theorem 4.2, $s-1$ times and Theorem 4.3 t, times. Similarly, the second formula follows by using Lemma 4.3, $s-1$ times and Lemma 4.4, t times.

<u>Remark:</u> Since $\Gamma_1,\ldots,\Gamma_s$ are basic subgroups of Γ, these are either elementary, degenerate or quasi-Fuchsian groups. So dim $PZ^1(\Gamma_i,\Pi_{2q-2})$ are all known for these groups in terms of the signature of Γ_i. Hence the dim $PH^1(\Gamma,\Pi_{2q-2})$ can be computed in terms of the signature of $\Gamma_1, \Gamma_2,\ldots,\Gamma_s$ by the above formula. Similarly, we can compute the dim $A_q(\Omega(\Gamma),\Gamma)$ in terms of the signature of $\Gamma_1, \Gamma_2,\ldots,\Gamma_s$.

SECTION V

Equalities and inequalities

The main purpose of this section is to show how to obtain some results of Maskit in [15] and [16] on inequalities involving the dimension of the space of the cusp forms supported on an invariant component, and the dimension of the space of the cusp forms supported on the other components of finitely generated function groups by using our cohomology machinery.

To this end, we first establish some equalities between the dimension of various cohomology spaces of finitely generated function groups. The Maskit inequalities are simple consequence of these equalities. We will see that the parabolic fixed points play an important role in proving these equalities. We begin this section by describing the structure of stabilizers of parabolic fixed points of a Kleinian group.

§1. Let Γ be a Kleinian group. Let $x \in \Lambda$ be a parabolic fixed point. We let

$$P_x = \{\gamma \in \Gamma;\ \gamma \text{ is parabolic and } \gamma x = x\} \cup \{I\},$$

$\{I\}$ denotes the identity element of Γ.

Recall, $$\Gamma_x = \{\gamma \in \Gamma;\ \gamma x = x\}.$$

Thus P_x is the parabolic stabilizer of x and Γ_x is the stabilizer of x. The group Γ_x is an elementary group with one limit point, and is therefore, a finite extension of the maximal rank one or rank two parabolic subgroup P_x. So Γ_x is generated by P_x and an elliptic element of order n (we say that Γ_x is a Z_n-extension of P_x). Here $n=1$ or $n=2$ if P_x is cyclic, and $n=1, 2, 3, 4$ or 6 if P_x has rank 2.

For $q \in \mathbb{Z}$, $q \geq 2$, we say that x is q-admissible provided $$q \equiv 0 \pmod{n}.$$

This definition is due to Kra. With this definition the following was proven in Kra [11]:

Theorem. Let Γ be a geometrically finite Kleinian group. Then for $q \geq 2$,

$$\dim H^1(\Gamma, \Pi_{2q-2}) = \dim PH^1(\Gamma, \Pi_{2q-2}) + N(q),$$

where $N(q)$ is the number of Γ-equivalence classes of q-admissible parabolic fixed points of Γ.

We will show that this also holds for some Kleinian groups that are not geometrically finite. As an example we will establish the following:

Theorem 5.1. Let Γ be a (non-elementary) finitely generated function group. Then for $q \geq 2$,

$$\dim H^1(\Gamma,\Pi_{2q-2}) = \dim PH^1(\Gamma,\Pi_{2q-2}) + N(q).$$

To prove the theorem we need the following lemmas:

Lemma 5.1. Let Γ be a finitely generated Kleinian group. Then for $q \geq 2$,

$$\dim H^1(\Gamma,\Pi_{2q-2}) \leq \dim PH^1(\Gamma,\Pi_{2q-2}) + N(q).$$

Proof. Kra [10].

Lemma 5.2. Let Γ be a finitely generated function group. Let Γ be constructed from the basic subgroups $\Gamma_1,\ldots,\Gamma_s$ by the application of Combination Theorem I and Combination Theorem II. Then every Z_2-extension of a rank 1 parabolic subgroup or every maximal Z_n-extension ($n=1,2,3,4$ or 6) of a rank 2 parabolic subgroup of Γ is conjugate to an elementary basic subgroup of Γ.

Proof. Let $\Gamma_1,\Gamma_2,\ldots\Gamma_s$ be the basic subgroups of Γ, and Γ is constructed from $\Gamma_1,\Gamma_2,\ldots\Gamma_s$ by application of Combination Theorem I and Combination Theorem II (as it was stated in Section IV), where the amalgamated subgroups and the conjugated subgroups are either elliptic cyclic, parabolic cyclic or trivial.

We recall the definition of a factor subgroup of Γ given in Section IV. Then it is easy to see that every Z_2-extension of a rank 1 parabolic subgroup or every maximal Z_n-extension (n=1,2,3,4 or 6) of a rank 2 parabolic subgroup of Γ is, by definition, a factor subgroup of Γ. Now by Theorem 4.1, each such subgroup is conjugate to a unique basic subgroup used in the construction of Γ. In fact this subgroup is conjugate to an elementary basic subgroup. This follows from the standard fact that quasi-Fuchsian groups and degenerate groups can not contain such as a subgroup.

Lemma 5.3. Let Γ_o be an elementary group. Then $\dim Z^1(\Gamma_o, \Pi_{2q-2}) = \dim PZ^1(\Gamma_o, \Pi_{2q-2})$ + number of q-admissible parabolic fixed points of Γ_o.

Proof. Without loss of generality, we can assume that Γ_o is an elementary group with one limit point.

Then the lemma follows from the list of the dimensions for elementary groups (see Proposition 3.1) and by the definition of a q-admissible parabolic fixed point.

Proof of Theorem 5.1. By Lemma 5.1, it suffices to prove that for a finitely generated function group Γ, $\dim H^1(\Gamma, \Pi_{2q-2}) \geq \dim PH^1(\Gamma, \Pi_{2q-2}) + N(q)$.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ be the basic subgroups of Γ from which Γ has been constructed. Then the $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ are either quasi-Fuchsian groups (of the first kind), degenerate groups (without accidental parabolic elements) or elementary groups.

Let us assume that Γ is formed from $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ by $s - 1$ applications of Combination Theorem I and t applications of Combination Theorem II. We reorder the basic subgroups $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ so that $\Gamma_1, \ldots, \Gamma_p$ are the quasi-Fuchsian and degenerate basic subgroups and $\Gamma_{p+1}, \ldots, \Gamma_s$ are the elementary basic subgroups.

Now we recall the following two facts which were observed at the end of the proof of the Theorem 4.2 and the Theorem 4.3.

(1) Let Γ be a non-elementary Kleinian group which is generated by its finitely generated subgroups Γ_1 and Γ_2 by application of Combination Theorem I and if $H = \Gamma_1 \cap \Gamma_2$ is elliptic, parabolic cyclic or trivial, then

$$\dim Z^1(\Gamma, \Pi_{2q-2}) \geq \dim Z^1(\Gamma_1, \Pi_{2q-2}) + \dim Z^1(\Gamma_2, \Pi_{2q-2}) - \dim Z^1(H, \Pi_{2q-2}).$$

Note that $\dim Z^1(H, \Pi_{2q-2}) = 2q - 1$ for H parabolic cyclic, $= 2[q - \frac{q}{\nu}]$ for H elliptic cyclic of order ν; $1 \leq \nu < \infty$.

(2) Let Γ be a non-elementary Kleinian group which is generated by its finitely generated subgroup Γ_1 and an element f by application of Combination Theorem II and the subgroup H_1 (or H_2) in Γ_1 is elliptic cyclic, parabolic cyclic or trivial then

$$\dim Z^1(\Gamma, \Pi_{2q-2}) \geq \dim Z^1(\Gamma_1, \Pi_{2q-2}) + (2q-1) - \dim Z^1(H_1, \Pi_{2q-2})$$

Now by using Fact (1) $s-1$ times and Fact (2) t times inductively we can write

$$(5.1) \qquad \dim Z^1(\Gamma, \Pi_{2q-2}) \geq \sum_{i=1}^{s} \dim Z^1(\Gamma_i, \Pi_{2q-2}) + (2q-1)t - (2q-1)r_1 - \sum_{i=1}^{r_2} 2[q - \frac{q}{\nu_i}],$$

r_1 is the number of times we applied Combination Theorem I and Combination Theorem II where the amalgamated subgroups and the conjugated subgroups are all parabolic cyclic, and r_2 is the number of times we applied Combination Theorem I and Combination Theorem II where the amalgamated subgroups and the conjugated subgroups are all elliptic cyclic of order $\nu_i \geq 1$, $i = 1, 2, \ldots, r_2$.

Similarly, we can write dimension of $PZ^1(\Gamma, \Pi_{2q-2})$ as follows (as we did in Proposition 4.1).

$$(5.2)\qquad \dim PZ^1(\Gamma,\Pi_{2q-2}) = \sum_{i=1}^{s} \dim PZ^1(\Gamma_i,\Pi_{2q-2}) + (2q-1)t$$

$$- r_1(2q-2) - \sum_{i=1}^{r_2} 2[q - \frac{q}{\nu_i}].$$

Now we write (5.1) as follows:

$$(5.3)\qquad \dim Z^1(\Gamma,\Pi_{2q-2}) \geq \sum_{i=1}^{p} \dim Z^1(\Gamma_i,\Pi_{2q-2})$$

$$+ \sum_{i=p+1}^{s} \dim Z^1(\Gamma_i,\Pi_{2q-2})$$

$$+ (2q-1)t - (2q-1)r_1 - \sum_{i=1}^{r_2} 2[q - \frac{q}{\nu_i}].$$

We know that (see, for example, Corollary 2 of Theorem 2.2 in Section 2)

$$(5.4)\qquad \dim Z^1(\Gamma_i,\Pi_{2q-2}) = \dim PZ^1(\Gamma_i,\Pi_{2q-2}) + n_i,\quad i = 1,2,\ldots,p$$

where n_i is the number of Γ_i-inequivalent parabolic fixed points for Γ_i, $i = 1,2,\ldots,p$.

Also, from Lemma 5.3 we have for elementary Γ_i

$$(5.5)\qquad \dim Z^1(\Gamma_i,\Pi_{2q-2}) = \dim PZ^1(\Gamma_i,\Pi_{2q-2}) + r_i(q)\ (i=p+1,\ldots,s)$$

where $r_i(q)$ is the number of q-admissible fixed points for Γ_i $(i=p+1,\ldots,s)$.

Using the result of (5.4), (5.5) and (5.2) we obtain from (5.3),

$$\dim Z^1(\Gamma,\Pi_{2q-2}) \geq \dim PZ^1(\Gamma,\Pi_{2q-2}) + \sum_{i=1}^{p} n_i + \sum_{i=p+1}^{s} r_i(q) - r_1. \tag{5.6}$$

Now we know

$$\dim H^1(\Gamma,\Pi_{2q-2}) = \dim Z^1(\Gamma,\Pi_{2q-2}) - \dim B^1(\Gamma,\Pi_{2q-2})$$

and $$\dim PH^1(\Gamma,\Pi_{2q-2}) = \dim PZ^1(\Gamma,\Pi_{2q-2}) - \dim B^1(\Gamma,\Pi_{2q-2}).$$

Also, we know that Γ is non-elementary, so the $\dim B^1(\Gamma,\Pi_{2q-2}) = 2q-1$. Hence (5.6) becomes

$$\dim H^1(\Gamma,\Pi_{2q-2}) \geq \dim PH^1(\Gamma,\Pi_{2q-2}) + \sum_{i=1}^{p} n_i + \sum_{i=p+1}^{s} r_i(q) - r_1.$$

Now we will show that

$$\sum_{i=1}^{p} n_i + \sum_{i=p+1}^{s} r_i(q) - r_1 \tag{5.7}$$

is exactly the number of Γ-inequivalent q-admissible parabolic fixed points.

We know by the conclusion of the Combination Theorems that every parabolic element of Γ lies in a conjugate of the basic subgroups $\Gamma_i (i=1,2,\ldots,s)$. Now it follows by Lemma 5.2 that the number of distinct conjugacy classes of maximal parabolic cyclic subgroups or rank 2 parabolic subgroup of Γ are atmost the sum of the number of distinct conjugacy classes of maximal parabolic subgroups of the basic subgroups Γ_i (i=1,2,...,p) and the number of elementary basic subgroups with one limit point.

We recall that n_i is the number of distinct conjugacy classes of maximal parabolic cyclic subgroups of Γ_i (i=1,2,...,p). It is clear by the conclusion of the Combination Theorems that each of these conjugacy classes is actually a conjugacy class of Γ if no cyclic subgroup of the corresponding conjugacy class is used as an amalgamated subgroup or conjugated subgroups in one of the combinations. But the conjugates of parabolic cyclic subgroups which are used in the application or Combination Theorem I are either contained in two non-elementary basic subgroups or in a non-elementary basic subgroup and a Z_2-extension of rank 1 parabolic basic subgroup. On the otherhand, two non-conjugate parabolic cyclic subgroups of a basic subgroup (necessarily non-elementary) used in the application of Combination Theorem II are conjugate in Γ by the Conclusion of the Combination Theorem.

So, the numbers of distinct conjugacy classes of maximal parabolic subgroups of Γ is exactly ($\sum_{i=1}^{p} n_i$ + number of elementary basic subgroups with one limit point - r_1).

Our proof now follows by observing that number of distinct conjugacy classes of maximal parabolic subgroups of Γ is equal to the number of Γ-inequivalent parabolic fixed points.

As an application of this theorem we can find the dimension of the space of quasi-bounded Eichler integrals for geometrically finite function groups. The following is a special case of Corollary 6.2 in Kra [11].

<u>Corollary</u>. Let Γ be a geometrically finite function group. Then for $q \geq 2$,

$$\dim E_{1-q}^{c}(\Omega,\Gamma) = N(q).$$

<u>Proof</u>. By structure theorem of cohomology group (see Theorem 2.2) we have for $q \geq 2$,

$$(5.8) \qquad \dim H^{1}(\Gamma,\Pi_{2q-2}) = \dim A_{q}(\Omega(\Gamma),\Gamma + \dim E_{1-q}^{c}(\Omega(\Gamma),\Gamma).$$

Since Γ is geometrically finite, we know from the result in Section IV

$$(5.9) \qquad \dim PH^{1}(\Gamma,\Pi_{2q-2}) = \dim A_{q}(\Omega(\Gamma),\Gamma).$$

The corollary follows from (5.7), (5.8) and the theorem.

Remark: From Kra [11] one can describe a basis for $E^c_{1-q}(\Omega,\Gamma)$ for this class of Kleinian groups.

§2. Let Γ be a finitely generated function group with an invariant component Δ. Now we want to observe some simple facts concerning dimensions of $PH^1_\Delta(\Gamma,\Pi_{2q-2})$, $PH^1(\Gamma,\Pi_{2q-2})$ and $H^1(\Gamma,\Pi_{2q-2})$.

To this end, we establish the following:

Lemma 5.4. Let $N_1(q)$ be the number of q-admissible Γ-equivalence classes of parabolic fixed points corresponding to the punctures of Δ/Γ.

Then for $q \geq 2$,

$$\dim H^1(\Gamma,\Pi_{2q-2}) \leq \dim PH^1_\Delta(\Gamma,\Pi_{2q-2}) + N_1(q).$$

We follow the proof of Lemma 5.1 given by Kra [11]. We will use the following two facts (established in Kra [12]) in the proof of Lemma 5.4.

Fact 1. (Kra [11]). Let Γ be a Kleinian group. Let χ be a Π_{2q-2}-cocycle. Let $A \in \Gamma$ be a parabolic element with the fixed point a. Then χ is parabolic with respect to A if and only if $\chi(A)(a) = 0$.

Fact 2. (Kra [12]). Let Γ_o be a rank 2 parabolic group of Möbius transformations. Let A and B generate Γ_o. Let χ be a cocycle for Γ_o. Then χ is parabolic with respect to A if and only if it is parabolic with respect to B.

Proof of Lemma 5.4. Let k be the number of Γ-equivalence classes of parabolic fixed points corresponding to the punctures of Δ/Γ.

We assume $k \geqslant 0$. We choose $b_1, b_2, \ldots, b_k$, a maximal set of inequivalent parabolic fixed points in Λ corresponding to the punctures on Δ/Γ.

We replace Γ by a conjugate group (if necessary) so that $b_j \in \mathbb{C}$, $j = 1,2,\ldots,k$. For each j, choose $B_j \in \Gamma$ so that B_j is parabolic and fixes b_j (that is, $B_j \in P_{b_j}$).

We construct a map

$$e : H^1(\Gamma, \Pi_{2q-2}) \to \mathbb{C}^k$$

as follows:

Let χ be a cocycle representing an element of $H^1(\Gamma, \Pi_{2q-2})$. We define

$$e(\chi) = (\chi(B_1)(b_1), \ldots, \chi(B_k)(b_k)).$$

We first observe that for a coboundary χ, $e(\chi) = 0$. Hence e is well defined on cohomology classes. Next, $e(\chi) = 0$ if and only if $\chi \in PH^1_\Delta(\Gamma, \Pi_{2q-2})$. (Consequences of the first fact stated in the beginning of this article.) Hence we proved

$$\dim H^1(\Gamma, \Pi_{2q-2}) \leq \dim PH^1_\Delta(\Gamma, \Pi_{2q-2}) + k.$$

To complete the proof of the lemma we recall the definition of q-admissible parabolic fixed points as well as the parabolic stabilizer P_x and stabilizer Γ_x of a parabolic fixed point $x \in \Lambda(\Gamma)$. Let χ be a cocycle for Γ. Then $\chi|_{\Gamma_x}$ defines a cohomology class in $H^1(\Gamma_x, \Pi_{2q-2})$. This later space is trivial (from the list of the dimensions of cohomology space for elementary groups in Section III), whenever x is not q-admissible. Hence we have strengthen the above inequality of the lemma.

Lemma 5.5. Let $N_o(q)$ be the numbers of q-admissible Γ-equivalence classes of parabolic fixed points which do not represent punctures on Δ/Γ.

Then for $q \geq 2$, $\dim PH^1_\Delta(\Gamma, \Pi_{2q-2}) \leq \dim PH^1(\Gamma, \Pi_{2q-2}) + N_o(q)$.

Proof. Let k be the number of Γ-equivalence classes of parabolic fixed points which do not represent punctures on Δ/Γ. Assume $k > 0$. Choose $b_1, \ldots, b_k$ as in Lemma 5.4.

We construct a map

$$e : PH^1_\Delta(\Gamma, \Pi_{2q-2}) \to \mathbb{C}^k$$

by following Lemma 5.4.

Let χ be a cocycle representing an element of $PH^1_\Delta(\Gamma, \Pi_{2q-2})$. We define

$$e(\chi) = (\chi(B_1)(b_1), \ldots, \chi(B_k)(b_k)).$$

Observe that $e(\chi) = 0$ if and only if $\chi \in PH^1(\Gamma, \Pi_{2q-2})$. The rest of the proof is exactly the same as the proof of Lemma 5.4.

We observe that Lemma 5.4 and Theorem 5.1 gives us

$$\dim PH^1_\Delta(\Gamma, \Pi_{2q-2}) \geq \dim PH^1(\Gamma, \Pi_{2q-2}) + N_o(q).$$

Combining Lemma (5.5) with the above inequality we have the following:

<u>Theorem 5.2</u>. Let Γ be a finitely generated Kleinian group with an invariant component Δ. Then for $q \geq 2$,

(5.10) $$\dim PH^1_\Delta(\Gamma, \Pi_{2q-2}) = \dim PH^1(\Gamma, \Pi_{2q-2}) + N_o(q).$$

We recall that $N_o(q)$ is the number of Γ-equivalence classes of parabolic fixed points which do not represent punctures on Δ/Γ. It now follows from the proof of Theorem 5.1 and Lemma 5.2 that these are q-admissible parabolic fixed points of the conjugacy classes of Z_n-extensions (n=1,2,3,4 or 6) of a rank 2 parabolic basic subgroup and q-admissible parabolic fixed points of a parabolic cyclic subgroup used as an amalgamated or conjugated subgroup in one of the combinations. We observe that the parabolic fixed points of Z_2-extensions of a rank 1 parabolic basic subgroup do not represent puncture on Δ/Γ if the parabolic cyclic group of this group used as an amalgamated subgroup in the application of Combination Theorem I, §3. Now we can prove the inequalities referred to in the beginning of the section which is stated as follows:

<u>Theorem 5.3</u>. (Maskit [15]). Let Γ be a finitely generated Kleinian group with an invariant component Δ. Let $\Delta, \Delta_1, \ldots \Delta_p$ be the complete list of inequivalent component Γ. Let Γ_{Δ_i} be the stabilizer of Δ_i for $i = 1,2,\ldots p$. Then

(a) for $q \geq 2$, $\dim A_q(\Delta,\Gamma) - \sum_{i=1}^{p} \dim A_q(\Delta_i,\Gamma_{\Delta_i}) \geq N_o(q)$.

(b) For $q = 2,4,6\ldots$, $\dim A_q(\Delta,\Gamma) - \sum_{i=1}^{p} \dim A_q(\Delta_i,\Gamma_{\Delta_i}) \geq k$,

where k is the number of Γ-equivalence classes of parabolic fixed points which come from the conjugacy classes of maximal parabolic cyclic subgroups or the conjugacy classes of Z_2-extensions of rank 1 parabolic cyclic subgroups and do not represent punctures in Δ/Γ.

(c) For $q = 3,5,7\ldots$,

$$\dim A_q(\Delta,\Gamma) - \sum_{i=1}^{p} \dim A_q(\Delta_i,\Gamma_{\Delta_i}) \geq k - k_1,$$

k_1 is the number of parabolic fixed points which come from the conjugacy classes of Z_2-extensions of a rank 1 parabolic cyclic group.

(d) For $q \geq 2$, $\text{Area}\,(\Delta/\Gamma) - \sum_{i=1}^{p} \text{Area}\,(\Delta_i/\Gamma_{\Delta_i}) \geq 0$.

All the inequalities are equalities whenever Γ is geometrically finite function group with a simply-connected invariant component Δ.

<u>Proof</u>. The proof is a simple application of the Theorem 5.2 and the structure theorem of cohomology group. We know that $\dim PH^1_\Delta(\Gamma,\Pi_{2q-2}) \leq 2\dim A_q(\Delta,\Gamma)$ by the corollary 2 of Theorem 2.2, and $\dim PH^1(\Gamma,\Pi_{2q-2}) \geq \dim A_q(\Omega,\Gamma)$ by injectivity of the Bers map.

With the help of these two facts and the equation (5.10) of the Theorem 5.2 it now follows that

$$\begin{aligned} 2\dim A_q(\Delta,\Gamma) &\geq \dim PH^1_\Delta(\Gamma,\Pi_{2q-2}) = \dim PH^1(\Gamma,\Pi_{2q-2}) + N_0(q) \\ &\geq \dim A_q(\Omega,\Gamma) + N_0(q) \\ &= \dim A_q(\Delta,\Gamma) + \sum_{i=1}^{p} \dim A_q(\Delta_i,\Gamma_{\Delta_i}) + N_0(q) \end{aligned}$$

Therefore (a) follows from above.

Observe that for a geometrically finite function group with a simply connected invariant component Δ, $\dim PH^1(\Gamma, \Pi_{2q-2}) = \dim A_q(\Omega, \Gamma)$ (from the result of Section IV), and $\dim PH^1_\Delta(\Gamma, \Pi_{2q-2}) = 2 \dim A_q(\Delta, \Gamma)$ (from the Corollary 2 of Theorem 2.2). So in this case, equality holds in (a).

For (b), we can prove more than we stated. It is clear that the left hand side of (b) is at least the number of Γ-equivalence classes of 2-admissible parabolic fixed points. But k is atmost the number of Γ-equivalence classes of 2-admissible parabolic fixed points.

When q is odd (c) follows from (b) by leaving out k_1 number of fixed points coming from the conjugacy classes of Z_2-extensions of a rank 1 parabolic.

Inequality (d) follows from either (b) or (c) by dividing by q and taking the limit.

Equality in (d) holds whenever equalities in (b) and (c) hold. But equalities in (b) and (c) hold whenever equality in (a) holds.

To see this we know that equality holds in (a) whenever Γ is geometrically finite function group with a simply connected invariant component. Then that Γ cannot contain a rank 2 parabolic subgroup, so every q-admissible parabolic fixed points are 2-admissible, either coming from rank 1 parabolic cyclic groups or Z_2-extensions of such groups. Hence for an even q, $N_o(q) = k$, and for an odd q, $N_o(q) = k - k_1$.

References

[1] L.V. Ahlfors, Finitely generated Kleinian groups, American Journal of Mathematics, 86 (1964), 413-429 and 87 (1965), 759.

[2] A. Beardon, The geometry of discrete groups, Discrete groups and automorphic functions, Academic Press, 1977, pp. 47-72.

[3] L. Bers, On Ahlfors' finiteness theorem, American Journal of Mathematics, 89 (1967), 1078-1082.

[4] ________, Spaces of Kleinian groups. Several Complex Variables, I (Proc. Conf. University of Maryland, College Park, Maryland, 1970), 9-34, Springer, Berlin, 1970.

[5] F. Gardiner and I. Kra, Stability of Kleinian groups, Indiana University Mathematics Journal, 21, No. 11 (1972), 1037-1059.

[6] L. Greenberg, Fundamental polyhedra for Kleinian groups, Ann. of Math., 84 (1966), 433-441.

[7] I. Kra, On cohomology of Kleinian groups, Ann. of Math., 89 (1969), 533-556.

[8] ________, On cohomology of Kleinian groups: II, Ann. of Math., 90 (1969), 575-589.

[9] ________, Deformations of Fuchsian groups: II, Duke Math. Journal, 38 (1971), 499-508.

[10] ________, Automorphic forms and Kleinian groups, Benjamin Reading, Massachusetts, 1972.

[11] ________, On cohomology of Kleinian groups: IV, The Ahlfors-Sullivan construction of holomorphic Eichler integrals, to appear.

[12] ________, On the vanishing of and spanning sets for Poincaré series for cusp forms, to appear.

[13] I. Kra and H. Farkas, Riemann surfaces, Springer-Verlag, New York.

[14] A. Marden, The geometry of finitely generated Kleinian groups, Annals of Mathematics 99 (1974), 383-462.

[15] B. Maskit, On boundaries of Teichmüller spaces and on Kleinian groups: II, Ann. of Math., 91 (1970), 607-639.

[16] ________, Decomposition of certain Kleinian groups, Acta. Math., 130 (1973), 243-263.

[17] ________, On the classification of Kleinian groups II-signatures, Acta Math., 138 (1977), 17-42.

[18] M. Nakada, Cohomology of finitely generated Kleinian groups with an invariant component, J. Math. Soc., Japan, 18, No.4 (1976).

[19] K. Sakan, On stability of finitely generated Kleinian groups, Osaka Journal of Mathematics, 17 (1980), 165-176.

[20] D. Sullivan, A finiteness theorem for cusps, Acta. Math., 147 (1981), 289-299.

[21] A. Weil, Remarks on cohomology of groups, Ann. of Math., 80 (1964), 149-157.