Intersection R-Torsion and Analytic Torsion for Pseudomanifolds

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Abstract of the Dissertation

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In the papers by Cheeger [3] and Müller [14] it is shown that for a closed Riemannian manifold, the Reidemeister torsion τ equals the analytic torsion T. This thesis represents the initial portion of an attempt to generalize this result to singular spaces (i.e. pseudomanifolds) equipped with suitable metrics. The starting point for this generalization is based on the work of Goresky and Macpherson [9] and Cheeger [4], [5], [6]. The steps carried out so far (and described below) indicate that such a generalization should indeed hold.

For any pseudomanifold X^n we define an invariant $I\tau^{\bar{p}}$ parametrized by the perversity $\bar{p}$. This invariant is analogous to Whitehead/Reidemeister torsion and is a combinatorial

invariant. When X^n is a manifold $I\tau^{\bar{p}} = \tau$ for any perversity $\bar{p}$. Let $\bar{p}, \bar{q}$ be complementary perversities. When n is even, $\ell n I\tau^{\bar{p}} + \ell n I\tau^{\bar{q}} = 0$; when n is odd $\ell n I\tau^{\bar{p}} = \ell n I\tau^{\bar{q}}$ (Duality). However, examples (the simplest of which is $S(P^3 \times P^3)$) show that $I\tau^{\bar{m}}$ is distinct from the usual Reidemeister torsion and is a finer invariant.

We begin our study of the analytic torsion T by considering manifolds with isolated metrically conical singularities, $X^{m+1} = M^{m+1} \cup C_{0,1}(N^m)$. Since for such spaces the asymptotic expansion of the trace of the heat kernel can contain logarithmic terms it is necessary to show that T is actually well defined (i.e. finite). It is then a formal consequence that T satisfies duality, i.e. when $m+1 = 2k$, $\ell n T = 0$.

These results indicate that the analytic torsion $\ell n T$ should equal the Intersection R-Torsion

$$\frac{\ell n I\tau^{\bar{m}} + \ell n I\tau^{\bar{n}}}{2}$$

in this case also.

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To my mother.

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0. Introduction

Let K be a finite connected CW-complex and ε a flat orthogonal bundle over K. Let $C(K,\varepsilon)$ denote the cellular cochain complex of k with coefficients in the bundle ε (as in [17]), and let $h^i(\varepsilon)$ be a basis for the cohomology group $H^i(K,\varepsilon)$. Associated with the cochain complex $C(K,\varepsilon)$ and the preferred choice of basis $h^*(\varepsilon)$ in cohomology, there is defined a topological invariant, the Reidemeister Torsion (briefly R-Torsion) denoted $\tau(K,\varepsilon,h^*(\varepsilon))$.

In particular, let $K = M^n$ be a closed manifold. With respect to any triangulation or cell structure of M the torsion $\tau(M,\varepsilon,h^*(\varepsilon))$ is defined and satisfies a duality condition analogous to Poincaré duality, i.e. when n is even $\ln\tau(M,\varepsilon) = 0$.

For a closed Riemannian manifold M^n, by Hodge theory of the Laplace operator Δ_i on i-forms, we have that the space of harmonic i-forms with coefficients in the flat bundle ε is isomorphic to the de Rham cohomology of closed i-forms modulo exact i-forms which is in turn isomorphic to the cohomology $H^i(M,\varepsilon)$ with coefficients in the bundle ε. Using the global inner product on the space of harmonic i-forms, we can choose a preferred basis for $H^i(M,\varepsilon)$ coming from an orthonormal basis of harmonic i-forms. With these preferred choice of bases $h^*(\varepsilon)$, it was shown in Cheeger [3] and Müller [14] that the R-torsion $\tau(M,\varepsilon,h^*(\varepsilon))$ equals the analytic torsion $T(M,\varepsilon)$.

The analytic torsion $T(M,\varepsilon)$ is an analytic invariant constructed out of the eigenvalues of the Laplacian as follows. By the functional calculus for elliptic operators on compact manifolds (as in Gilkey [8]) we can form powers of the Laplacian Δ_i^{-s}. The zeta function on i-forms is then defined as $\zeta_i(s) = \text{trace}(\Delta_i^{-s})$. The zeta function is a meromorphic function which has an analytic continuation to the whole complex plane, and is well defined at zero. The analytic torsion is then defined to be the alternating sum $T(M,\varepsilon) = \sum_{i=0}^{n} (-1)^i \, i \, \zeta_i'(0)$. Since the $*$ operator is an isomorphism from i-forms to n-i forms we have that the analytic torsion satisfies a duality condition analogous to Poincaré duality, i.e. when n is even $\ln T(M,\varepsilon) = 0$.

An extension of the theory of the Laplace operator to singular spaces (i.e. pseudomanifolds) equipped with suitable metrics, was made by J. Cheeger in [4], [5], [6]. An n-dimensional pseudomanifold is a compact space X for which there exists a closed subspace Σ with $\dim(\Sigma) \leq n-2$ such that $X-\Sigma$

is an n-dimensional oriented manifold which is dense in X. In particular, the functional calculus for the Laplace operator was developed in [4], [6]. For a manifold with isolated metrically conical singularity, the asymptotic expansion of the trace of the heat kernel $e^{-\Delta_i t}$ contains a logarithmic term which implies that the zeta function on i-forms has a pole at s = 0. However, when we take the alternating sum $\sum_{i=0}^{n} (-1)^i i \zeta_i(s)$ the contribution from the logarithmic terms cancels out and we obtain a finite, well defined expression for the analytic torsion $T(X, \varepsilon)$. Since the * operator is an isomorphism from i-forms to n-i forms the analytic torsion satisfies a duality condition analogous to Poincaré duality in this case also, i.e. when n is even $\ln T(X, \varepsilon) = 0$.

Furthermore it was shown in [5] that the space of closed and co-closed L^2 harmonic forms on X with coefficients in a flat bundle ε over X is isomorphic to the L^2 cohomology of X with coefficients in ε. However, the L^2 cohomology groups of X were seen to be isomorphic not to the usual simplicial cohomology groups of X but rather to the dual of the middle Intersection Homology Groups $(IH^{\bar{m}})^*(X)$ of Goresky and Macpherson [9].

In Goresky and Macpherson [9], for any P.L. pseudomanifold X^n, Intersection Homology Groups $IH_i^{\bar{p}}$ are defined, parametrized by a multi-index $\bar{p}$ denoted the perversity. The Intersection Homology Groups are defined to be the homology groups of a subcomplex $IC^{\bar{p}}(X)$ of the simplicial chain complex of X (over all triangulations of X) where $IC_i^{\bar{p}}(X)$ are the 'i-allowable chains' with 'i-1 allowable boundary.' The allowability condition is a restriction on the dimension of the intersection of the chain with the singularities of X. For the middle perversity $\bar{m} = (0,0,1,1,2,2,...)$ the dual of the Intersection Homology Groups $(IH_i^{\bar{m}}(X))^*$ are isomorphic to the L^2-cohomology groups. For complementary perversities $\bar{p},\bar{q}$ the Intersection Homology Groups satisfy a 'generalized Poincaré duality' i.e. the Intersection Homology Groups $IH^{\bar{p}}$, $IH^{\bar{q}}$ are paired in complementary dimensions.

Since the usual R-torsion for a CW-complex does not satisfy duality for pseudomanifolds, it was reasonable to expect that the analytic torsion $T(X,\varepsilon)$ should be related to a combinatorial/topological invariant analogous to the R-torsion which would satisfy duality for pseudomanifolds, based not on the usual simplical/cellular chain complex of X but rather on the subcomplex $IC^{\overline{m}}(X)$ of allowable chains with allowable boundary.

In fact, we use a finitely generated chain complex $\mathcal{R}^{\overline{p}}(X)$ analogous to $IC^{\overline{p}}(X)$ to define the Intersection R-torsion $I\tau^{\overline{p}}(X)$. In particular for the middle perversities $\overline{m} = (0,0,1,1,2,2,\ldots)$, $\overline{n} = (0,1,1,2,2,\ldots)$ the dual of the Intersection Homology Groups $(IH^{\overline{m}}(X,\varepsilon))^*$, $(IH^{\overline{n}}(X,\varepsilon))^*$ are isomorphic to the L^2-cohomology groups which are in turn isomorphic to the space of closed and coclosed L^2-harmonic forms. We can then make a preferred choice of basis for $(IH_i^{\overline{m}}(X,\varepsilon))^*$, $(IH_i^{\overline{n}}(X,\varepsilon))^*$ coming from an orthonormal basis of L^2-harmonic forms and we are led to expect that with this preferred choice of basis the analytic torsion $\ln T(X,\varepsilon)$ should equal the Intersection R-torsion $\dfrac{\ln I\tau^{\overline{m}} + \ln I\tau^{\overline{n}}}{2}$.

The main difficulty that arose in working with the chain complex $\mathcal{R}^{\overline{p}}(X)$ was the following. Let ξ_i be an i-chain of $\mathcal{R}_i^{\overline{p}}(X)$, i.e. ξ_i is a $(\overline{p},i)$ allowable chain with $(\overline{p},i-1)$ allowable boundary. Then in proving combinatorial invariance for $I\tau^{\overline{p}}$ it is crucial to be able to express ξ_i as a sum of ξ_{ij}, i.e. $\xi_i = \Sigma\,\xi_{ij}$ where ξ_{ij} is a $(\overline{p},i)$ allowable chain with $(\overline{p},i-1)$ allowable boundary which has simply connected support. Our initial attempt was to break up ξ_i into such pieces by restricting ξ_i to simply connected pieces N_j of the pseudomanifold X. The restriction $\xi_i \cap N_j$ is certainly an i allowable chain. $\partial\xi_i = \Sigma\, \partial(\xi_i \cap N_j)$. We know that $\partial\xi_i$ is i-1 allowable. However, this does not mean that $\partial(\xi_i \cap N_j)$ is i-1 allowable, because it could happen that $\partial(\xi_i \cap N_j)$ contains some i-1 chain ξ_{i-1} which is not allowable but which cancels with the boundary of some other piece $\partial(\xi_i \cap N_k), k \neq j$. This is the reason that the attempt to express ξ_i as a sum of i-allowable chains with i-1 allowable boundary which have simply connected support, fails.

However, there are two distinct methods by which we can resolve this difficulty. The first method, following the approach of J. Cheeger in [5] is to construct the torsion $I\tau^{\bar{p}}(X,\varepsilon)$ inductively using a handle decomposition and Mayer-Vietoris argument.

Let $h_n = C(L^{n-1})$ be an n-dimensional n-handle of X^n (as in [5]) where L^{n-1} is the link of a vertex of X^n and $C(L^{n-1})$ denotes the cone on L^{n-1}. Then $X^n = h^n \cup Z^{n-1}$ where $h^n \cap Z^{n-1} \cong L^{n-1}$. Let ξ_i be an i-allowable chain of X^n with i-1 allowable boundary. Then we have that $\xi_i \cap h_n$ is i-allowable and that $\partial(\xi_i \cap h_n)$ is i-1 allowable; similarly $\xi_i \cap Z^{n-1}$ is i-allowable and $\partial(\xi_i \cap Z^{n-1})$ is i-1 allowable; so also

$\xi_i \cap L^{n-1}$ is i-allowable and $\partial(\xi_i \cap L^{n-1})$ is i-1 allowable. Thus we have that the following Mayer-Vietoris-sequence is a short exact sequence

$$0 \to R^{\bar{p}}(L^{n-1}) \to R^{\bar{p}}(h^n) \oplus R^{\bar{p}}(Z^{n-1}) \to R^{\bar{p}}(X) \to 0$$

and therefore $I\tau^{\bar{p}}(X) = I\tau^{\bar{p}}(Z^{n-1}) + I\tau^{\bar{p}}(h^n) - I\tau^{\bar{p}}(L_n) + \tau(\mathcal{H}_n)$ where $\mathcal{H}$ is the associated long exact sequence in homology. Similarly $I\tau^{\bar{p}}(Z^{n-1})$ can now be expressed in terms of $I\tau^{\bar{p}}(Z^{n-2})$, $I\tau^{\bar{p}}(h^{n-1})$, $I\tau^{\bar{p}}(L_{n-1})$ and $\tau(\mathcal{H}_{n-1})$. We may continue inductively until we get to Z^0 which is just a disjoint union of pieces isomorphic to I^n. Thus we get a formula for $I\tau^{\bar{p}}$ in terms of $I\tau^{\bar{p}}$ on the links; $I\tau^{\bar{p}}$ on the handles and the torsion of the associated long exact sequences in homology. The handles are simply connected and therefore the torsion has a purely homological interpretation. The links are pseudomanifolds of lower dimension, and the torsion of the long exact sequence is also purely homological. Therefore, using the handle decomposition, and using inductive arguments we are able to prove combinatorial invariance, duality and independence of stratification for $I\tau^{\bar{p}}$.

The second method is to use a subcomplex of $\mathcal{R}^{\bar{p}}(X)$ using the family of basic sets $\{Q_i^{\bar{p}}\}$ defined in Goresky and Macpherson

[9]. We construct the chain complex $S^{\bar{p}}(X)$ where $S_i^{\bar{p}}(X)$ is the i-chains of $Q_i^{\bar{p}}$ with boundary in $Q_{i-1}^{\bar{p}}$. The torsion of $S^{\bar{p}}(X)$ also gives the Intersection R-Torsion $I\tau^{\bar{p}}(X)$.

The basic sets are analogous to the Poincaré dual cells. If ξ_i is an i-chain of $Q_i^{\bar{p}}$ with boundary in $Q_{i-1}^{\bar{p}}$ then it is in fact possible to express ξ_i as the sum $\xi_i = \Sigma\, \xi_{ij}$, where ξ_{ij} is an i-chain in $Q_i^{\bar{p}}$, $\partial\xi_{ij}$ is an i-1 chain in $Q_{i-1}^{\bar{p}}$ and ξ_{ij} has simply connected support. It is then possible to prove combinatorial invariance, duality and independence of stratification for $I\tau^{\bar{p}}$ directly, using the chain complex $S^{\bar{p}}(X)$.

1. Review of Whitehead/Reidemeister Torsion

We begin with the required algebraic theory. We then recall the definition of Whitehead/Reidemeister torsion for finite CW complexes and smooth, compact manifolds respectively and describe its characteristic properties. Our basic reference is Milnor [12].

Definition 1.1 The Whitehead group of a ring A. GL(n,A) denotes the group of all nonsingular n x n matrices over A. Identifying each $M \in GL(n,A)$ with the matrix $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1,A)$ we obtain inclusions $GL(1,A) \subset GL(2,A) \subset \ldots$. The union is called the infinite general linear group GL(A).

A matrix is elementary if it coincides with the identity matrix except for one off-diagonal element. The subgroup $E(A) \subset GL(A)$ generated by all elementary matrices is the commutator subgroup of GL(A). E(A) is a normal subgroup of GL(A) with commutative quotient group, the Whitehead group $K_1A = GL(A)/E(A)$.

Let $[-1] \in K_1A$ denote the element of order two corresponding to the unit $(-1) \in GL(1,A)$. The quotient $K_1A/\{0,[-1]\}$ is denoted $\bar{K}_1A$ and called the reduced Whitehead group of A.

Examples

1. For the ring of integers Z, $\bar{K}_1Z$ is zero.

2. For the real numbers R, $\overline{K}_1 R$ is isomorphic to the multiplicative group R^+ of positive reals.

A specific isomorphism is given by $(a_{ij}) \to |\det(a_{ij})|$.

<u>Definition 1.2</u> <u>The Whitehead group Wh(π) of a group.</u> Let π be a multiplicative group and $Z\pi$ the integral group ring of π. Then π is contained in the group of units $U(Z\pi) = GL(1,Z\pi) \subset GL(Z\pi)$. There are homomorphisms $\pi \to K_1(Z\pi) \to \overline{K}_1 Z\pi$. The cokernel $\overline{K}_1(Z\pi)/\text{image}(\pi)$ is called the Whitehead group of π.

Let F be a free A-module and let $b = (b_1, \ldots\ldots, b_k)$, $c = (c_1, \ldots\ldots, c_k)$ be two different bases for F. Then $c_i = \Sigma a_{ij} b_j$ and $(a_{ij}) \in GL(k,A)$. The corresponding element of $\overline{K}_1 A$ is denoted $[c/b]$.

<u>Definition 1.3</u> <u>The torsion of a chain complex of free A-modules.</u> Let $C_n \to C_{n-1} \to \ldots\ldots \to C_1 \to C_0$ be a chain complex of free A-modules such that the homology modules $H_i(C)$ are free. Let c_i be a preferred basis for C_i and h_i a preferred basis for $H_i C$. Let B_i be the image of the boundary homomorphism $\partial : C_{i+1} \to C_i$ and let Z_{i+1} be its kernel. We assume that each B_i is free, and choose a basis b_i for each B_i. Using the inclusions $0 \subset B_i \subset Z_i \subset C_i$ where $Z_i/B_i \cong H_i$, $C_i/Z_i \cong B_{i-1}$ we see that the bases b_i, h_i, b_{i-1} combine to give a new basis

$b_i h_i b_{i-1}$ for C_i. We define the torsion $\tau \in \overline{K}_1(A)$ of the chain complex C as $\tau = \sum_{i=0}^{n} (-1)^i [b_i h_i b_{i-1}/c_i]$. This does not depend on the choice of the b_i since choosing different bases $\overline{b}_i$ we have

$$\sum_{i=0}^{n} (-1)^i [\overline{b}_i h_i \overline{b}_{i-1}/c_i] = \sum_{i=0}^{n} (-1)^i ([b_i h_i b_{i-1}/c_i] + [\overline{b}_i/b_i] + [\overline{b}_{i-1}/b_{i-1}])$$

where the last two terms sum up to zero.

We state the Algebraic Subdivision Theorem which is a basic tool for comparing the torsion of a chain complex C with the torsion of a "filtered" chain complex $\overline{C}$ obtained from C as follows. The proof may be found in Milnor [12], Section 5.

Let $C^{(0)} \subset C^{(1)} \subset \ldots . C^{(n)} = C$ be a filtration of the chain complex C by subcomplexes such that $H_i(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ for $i \neq \lambda$. The inclusion $C^{(\lambda-2)} \subset C^{(\lambda-1)} \subset C^{(\lambda)}$ gives rise to the exact sequence $0 \to C^{(\lambda-1)}/C^{(\lambda-2)} \to C^{(\lambda)}/C^{(\lambda-2)} \to C^{(\lambda)}/C^{(\lambda-1)} \to 0$, and the corresponding homology exact sequence $\ldots . . 0 \to H_\lambda(C^{(\lambda)}/C^{(\lambda-1)}) \xrightarrow{\partial_\lambda} H_{\lambda-1}(C^{(\lambda-1)}/C^{(\lambda-2)}) \to 0 \ldots$ We define a new chain complex $\overline{C}$ by setting $C_\lambda = H_\lambda(C^{(\lambda)}/C^{(\lambda-1)})$, where the boundary homomorphism $\partial_\lambda : \overline{C}_\lambda \to \overline{C}_{\lambda-1}$ is obtained from the homology exact sequence as above.

The homology groups $H_i\overline{C}$ are canonically isomorphic to the groups H_iC. Suppose each $C_i^{(\lambda)}/C_i^{(\lambda-1)}$ is free with preferred basis c_i^λ, such that C_i is free with basis $c_i = c_i^0 c_i^1 c_i^2 \ldots c_i^n$ and suppose $\overline{C}_\lambda$ is free with preferred basis $\overline{c}_\lambda$. Then we have

<u>Theorem 1.4</u> <u>Algebraic Subdivision Theorem</u>. If each quotient complex $C^{(k)}/C^{(k-1)}$ has torsion $\tau(C^{(k)}/C^{(k-1)})$ equal to zero then $\tau(C) = \tau(\overline{C})$.

We will also use the following result. Consider a short exact sequence $0 \to C' \to C \to C'' \to 0$ in the category of chain complexes and chain mappings. We will assume that the modules C_i', C_i, C_i'' are free with preferred bases c_i', c_i, c_i'' which are compatible, in the sense that $c_i = c_i' c_i''$. Then we have

<u>Theorem 1.5</u>. $\tau(C) = \tau(C') + \tau(C'') + \tau(\mathcal{H})$ where $\mathcal{H}$ is the long exact sequence in homology associated with the short exact sequence of chain complexes.

The proof may be found in Milnor [12], Section 3. We also give here a useful formula, the proof of which can be found in Cheeger [3], Section 1.

Let F

$$F_n \to F_{n-1} \to \cdots \to F_0$$

be a complex of free abelian groups and set $K_i = R_i \otimes R$. We then obtain a complex of real vector spaces K and with a choice of preferred basis for $K_i, H_i(K)$ coming from integral classes, the torsion of the complex K is given by

$$\tau = \pi \frac{O_{2k+1}}{O_{2k}} \qquad (1.6)$$

where O_i represents the order of the torsion subgroup of the homology group $H_i(F)$.

Definition 1.7 Whitehead/Reidemeister torsion for a finite CW complex. Let K be a finite CW-complex and let C(K) denote the cellular chain complex associated with K, i.e. $C_i(K)$ is the group of i-cells e_i of K. Let $\hat{K}$ denote the universal covering complex of K, and π the fundamental group of K. π acts on $\hat{K}$ as the group of covering transformations and this action is cellular. Thus each $\sigma \in \pi$ determines a mapping $\sigma : \hat{K} \to \hat{K}$.

Next, let $C(\hat{K})$ denote the cellular chain complex associated with $\hat{K}$, i.e. $C_i(\hat{K})$ is the group of i-cells of $\hat{K}$. With respect to the action of π, each chain group $C_i(\hat{K})$ is a free $Z\pi$-module, generated by the i-cells e_i of the complex K.

We pick a preferred basis for $C_i(\hat{K})$ as follows. Let $e_i^1, \ldots, e_i^{\alpha}$ denote the i-cells of K. For each e_i^j choose a representative cell $\hat{e}_i^j$ for $\hat{K}$ lying over e_i^j. Then $(\hat{e}_i^1, \hat{e}_i, \ldots, \hat{e}_i^{\alpha})$ is a preferred basis for $C_i(\hat{K})$. We <u>assume</u> that the homology groups $H_iC(\hat{K})$ are $Z\pi$-free. Then with a choice of preferred basis $\hat{h}_i$ for $H_iC(\hat{K})$ the torsion of the chain complex $C(\hat{K})$ is defined to be the <u>Whitehead torsion</u> <u>$\tau_W(K) \in \overline{K}_1(Z\pi)$</u>.

When the homology groups $H_iC(\hat{K})$ are not all $Z\pi$-free the Whitehead torsion is not defined. However, by a change of rings we may work with an alternative form of torsion which is always defined, the <u>Reidemeister torsion</u>.

View $C(\hat{K})$ as a chain complex of $R\pi$-modules. Let $\varepsilon : \pi \to O(n)$ be an orthogonal representation of the fundamental group. Then ε makes R^n into an $R\pi$-module and the chain complex defined by setting $C_i(K, \varepsilon) = C_i(\hat{K}) \underset{R\pi}{\otimes} R^n$ is a chain complex of real vector spaces. We have a preferred choice of basis for each vector space $C_i(\hat{K}, \varepsilon)$ given by $\{x_k \otimes \hat{e}_i^j\}$ where $\{x_k\}$ is an orthonormal basis for R^n and $\{\hat{e}_i^j\}$ the preferred basis for $C_i(\hat{K})$. With a choice of preferred basis in homology denoted $h_*(\varepsilon)$ the torsion of the complex of real vector spaces $C_i(K, \varepsilon)$ is a real number and will be denoted $\tau(K, \varepsilon, h_*(\varepsilon))$.

Equivalently we can use the complex of dual vector spaces $C^i(K, \varepsilon)$ and given a preferred basis in cohomology denoted $h^*(\varepsilon)$ the torsion $\tau(K, \varepsilon, h^*(\varepsilon))$ is defined.

The Whitehead/Reidemeister torsion is a combinatorial invariant, i.e. invariant under subdivision of the CW-complex K. The proof may be found in Milnor [12]. In Chapman [2] it is shown that Whitehead torsion is a topological invariant for compact, connected CW-complexes.

Definition 1.8 Whitehead/Reidemeister torsion for a compact Riemannian manifold. Let M be a compact Riemannian manifold.

We define the Whitehead/Reidemeister torsion to be the torsion of the cell complex determined by a cell structure of M.

In particular for the cochain complex $C^i(M, \varepsilon)$ by Hodge theory we can identify the cohomology groups $H^i(M, \varepsilon)$ with the space of harmonic forms with coefficients in the flat bundle determined by the representation ε. We can then choose a preferred basis in cohomology $h^*(\varepsilon)$ coming from an orthonormal basis of harmonic forms. With this choice of preferred basis in cohomology it was shown in Cheeger [3] and Muller [14] that $\tau(M, \varepsilon, h^*(\varepsilon))$ equals the analytic torsion

$T(M, \varepsilon)$.

The Reidemeister torsion satisfies the following characteristic property. Proofs may be found in Cheeger [3], Milnor [13].

Theorem 1.9 Duality. Let M^n be a closed even dimensional orientable manifold. Let $\{h_i\}$ be a basis for $\{H^i(M,\varepsilon)\}$ and $\{h_i^*\}$ the induced basis for $\{H^i(M,\varepsilon)^*\}$. Then if $\lambda_i : (H^i(M,\varepsilon))^* \to H^{n-i}(M,\varepsilon)$ is the isomorphism of Poincaré duality and $\lambda_i^*(h_{n-i}) = h_i^*$ we have $\ell n\,\tau(M,\varepsilon,h_i) = 0$.

Remark 1.10. When X is not a manifold, the R-torsion $\tau(X,\varepsilon)$ does not satisfy duality. In Section 2, we construct a generalization of the R-torsion for a class of singular spaces, i.e. pseudomanifolds, denoted Intersection R-torsion $I\tau$. $I\tau$ recovers duality for the class of pseudomanifolds.

Remark 1.11. When X is a manifold $I\tau = \tau$. In Section 2 we will prove combinatorial invariance, and duality for $I\tau$. Since $I\tau = \tau$ for manifolds, the proofs in Section 2 can be regarded as proofs of combinatorial invariance, and duality for the R-torsion $\tau(M,\varepsilon)$ for manifolds as well.

Remark 1.12. The Whitehead torsion τ_W is invariant under "simple homotopy equivalence." For details we refer to

Rourke and Sanderson [16]. For example C(X) (the cone on a complex X) has the simple homotopy type of a point. Therefore, $\tau_W(C(X)) = 0$, $\tau(C(X)) = 0$.

2. Intersection R-Torsion for Pseudomanifolds

We begin by recalling some facts from the Intersection Homology Theory of Goresky and Macpherson [9] and then proceed to define the Intersection R-torsion $I\tau^{\bar{p}}$. We then prove combinatorial invariance, duality and independence of stratification for $I\tau^{\bar{p}}$ using a handle decomposition of the pseudomanifold following the approach of J. Cheeger in [5]. We then give a different proof of combinatorial invariance, duality and independence of stratification using the family of basic sets $\{Q_i^{\bar{p}}\}$ defined in Goresky and Macpherson [9].

Definition 2.1. A pseudomanifold of dimension n is a compact P.L. space X for which there exists a closed subspace Σ with $\dim(\Sigma) \leq n-2$ such that $X - \Sigma$ is an n-dimensional oriented manifold which is dense in X. (Equivalently X is the closure of the union of the n-simplices in any triangulation of X and each n-1 simplex is a face of exactly two n-simplices.)

A stratification of a pseudomanifold is a filtration by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} = \Sigma \supset X_{n-3} \supset \ldots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1}$ there is a filtered space $V = V_n \supset V_{n-1} \supset \ldots \supset V_i$ = a point and a mapping

$V \times B^i \to X$ which for each j takes $V_j \times B^i$ p.ℓ. homeomorphically to a neighborhood of p in X_j (where B^i denotes the i-ball). Thus if $X_i - X_{i-1}$ is not empty it is a manifold of dimension i and is called the i-dimensional <u>stratum</u> of the stratification.

<u>Definition 2.2</u> <u>Piecewise linear chains</u>. If T is a triangulation of X let $C_*^T(X)$ denote the chain complex of simplicial chains of X with respect to T. A P.L. geometric chain is an element of $C_*^T(X)$ for some triangulation T; however, we identify two P.L. chains $c \in C_*^T(X)$ and $c' \in C_*^{T'}(X)$ if their canonical images in $C_*^{T''}$ coincide for some common refinement T'' of T and T'. The group $C_i(X)$ of all P.L. geometric chains is thus the direct limit under refinement of the $C_*^T(X)$ over all triangulations of X.

If $\xi_i \in C_i^T(X)$ define $|\xi_i|$ (the <u>support</u> of ξ_i) to be the union of the closures of those i-simplices σ for which the coefficient of σ in ξ_i is non-zero.

<u>Definition 2.3</u>. A <u>perversity</u> is a sequence of integers $\bar{p} = (p_2, p_3, \ldots, p_n)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$. If i is an integer and $\bar{p}$ is a perversity, a subspace $Y \subset X$ is called <u>$(\bar{p}, i)$ allowable</u> if $\dim(Y) \le i$ and $\dim(Y \cap X_{n-k}) \le i - k + p_k$

for $k \geq 2$. Define $IC_i^{\bar{p}}(X)$ to be the subgroup of $C_i(X)$ consisting of those chains ξ_i such that $|\xi_i|$ is $(\bar{p},i)$ allowable and $|\partial\xi|$ is $(\bar{p},i-1)$ allowable.

The i^{th} Intersection Homology Group of perversity $\bar{p}$ denoted $IH_i^{\bar{p}}(X)$ is the i^{th} homology group of the chain complex $IC_*^{\bar{p}}(X)$.

In order to define the Intersection R-torsion we need to work with finitely generated chain groups. To do this we use the basic sets $R_i^{\bar{p}}$, referred to in Goresky and Macpherson [9], Section 3.

Definition 2.4 Basic Sets $R_i^{\bar{p}}$. Let X^n be a pseudomanifold with a fixed stratification. Let T be a triangulation of X subordinate to the stratification, i.e. such that each X_k is a subcomplex of T. Let $R_i^{\bar{p}}$ be the subcomplex of T' (first barycentric subdivision of T) consisting of all simplices which are $(\bar{p},i)$ allowable. We have that $IH_i^{\bar{p}} \cong \text{Image } H_i(R_i^{\bar{p}}) \to H_i(R_{i+1}^{\bar{p}})$.

Definition 2.5 Intersection R-Torsion $I\tau^{\bar{p}}$. Let $\mathcal{R}^{\bar{p}}(X)$ be the chain complex defined by setting $\mathcal{R}_i^{\bar{p}} = H_i(R_i^{\bar{p}}, R_{i-1}^{\bar{p}})$. This i^{th} chain group is in one-to-one correspondence with simplicial chains e_i such that $|e_i| \subset R_i^{\bar{p}}$ and $|\partial e_i| \subset R_{i-1}^{\bar{p}}$. It is a free abelian group generated by finitely many chains

$\{e_i^j\}$. The boundary map $\partial_i : \mathcal{R}_i^{\bar{p}} \to \mathcal{R}_{i-1}^{\bar{p}}$ is obtained from the homology exact sequence $\cdots \to H_i(R_i^{\bar{p}}, R_{i-1}^{\bar{p}}) \xrightarrow{\partial_i} H_{i-1}(R_{i-1}^{\bar{p}}) \cdots \to$. The homology group $H_i\mathcal{R}^{\bar{p}}(X)$ is canonically isomorphic to $IH_i^{\bar{p}}$.

Let $\hat{X}$ denote the universal covering complex of X, and let $\hat{R}_i^{\bar{p}}$ denote the lift of $R_i^{\bar{p}}$ to $\hat{X}$. The fundamental group of X, denoted π, acts on $\hat{X}$ as the group of covering transformations. Thus each $\sigma \in \pi$ determines a mapping $\sigma: \hat{R}_i^{\bar{p}} \to \hat{R}_i^{\bar{p}}$. Next let $\mathcal{R}^{\bar{p}}(\hat{X})$ be the chain complex defined by setting $\mathcal{R}_i^{\bar{p}} = H_i(\hat{R}_i^{\bar{p}}, \hat{R}_{i-1}^{\bar{p}})$. With respect to the action of π each chain group $\mathcal{R}_i^{\bar{p}}(\hat{X})$ is a free $Z\pi$-module generated by the lifts of the chains $\{e_i^j\}$, and $\mathcal{R}_i^{\bar{p}}(\hat{X})$ is a chain complex of free $Z\pi$-modules. When the homology groups $H_i\mathcal{R}^{\bar{p}}(\hat{X})$ are all $Z\pi$-free we can define the Intersection Whitehead torsion $I\tau_W^{\bar{p}}$ (analogous to Whitehead torsion).

We pick a preferred basis for $\mathcal{R}_i^{\bar{p}}(\hat{X})$ as follows. Let $e_i^1, \ldots, e_i^j$ denote the i-chains of $R_i^{\bar{p}}$ with boundary in $R_{i-1}^{\bar{p}}$. For each e_i^k choose a representative chain $\hat{e}_i^k$ of $\hat{R}_i^{\bar{p}}$ lying over e_i^k. Then $(\hat{e}_i^1, \hat{e}_i^2, \ldots, \hat{e}_i^j)$ is a preferred basis for $\mathcal{R}_i^{\bar{p}}(\hat{X})$. With a choice of preferred basis $\hat{h}_i$ for the homology groups $H_i(\mathcal{R}^{\bar{p}}(\hat{X}))$ the torsion of the complex $\mathcal{R}^{\bar{p}}(\hat{X}, \hat{e}_i, \hat{h}_i)$ is defined to be the Intersection Whitehead Torsion $I\tau_W^{\bar{p}}$ of X.

$I\tau_W^{\bar{p}}(X, \hat{e}_i, \hat{h}_i) \in Wh(\pi)$.

When the homology groups $H_i \mathcal{R}^{\bar{p}}(\hat{X})$ are not all $Z\pi$-free the Intersection Torsion $I\tau_W^{\bar{p}}$ is not defined. However, by a change of rings we may work with an alternative form of torsion which is always defined.

View $\mathcal{R}^{\bar{p}}(\hat{X})$ as a chain complex of $R\pi$-modules. Let $\varepsilon : \pi \to O\ O(n)$ be an orthogonal representation of the fundamental group. Then ε makes R^n into an $R\pi$-module and the chain complex defined by setting $\mathcal{R}_i^{\bar{p}}(X, \varepsilon) = \mathcal{R}_i^{\bar{p}}(\hat{X}) \underset{R\pi}{\otimes} R^n$ is a chain complex of real vector space $\mathcal{R}_i^{\bar{p}}(X, \varepsilon)$ given by $\{X_k \otimes \hat{e}_i^j\}$ where $\{X_k\}$ is an orthonormal basis for R^n, and $\{\hat{e}_i^j\}$ the preferred basis for $R_i^{\bar{p}}(\hat{X})$. With a choice of preferred basis in homology denoted $h_*(\varepsilon)$ the torsion of the chain complex $\mathcal{R}^{\bar{p}}(X, \varepsilon)$ is a real humber and will be denoted $I\tau^{\bar{p}}(X, \varepsilon, h*(\varepsilon))$, the Intersection R-Torsion of X.

Equivalently we can use the complex of dual vector spaces ${}^i\mathcal{R}^{\bar{p}}(X, \varepsilon)$ and given a choice of preferred basis $h^*(\varepsilon)$, $I\tau^{\bar{p}}$ is defined. In particular by Hodge theory for pseudomanifolds (the reference is Cheeger [5]), we identify the cohomology groups ${}^{\bar{m}}H^i(X, \varepsilon)$ with the space of L^2-harmonic forms with coefficients in the flat bundle determined by the representation ε. We can then choose a preferred basis for ${}^{\bar{m}}H^i(X, \varepsilon)$ coming from an orthonormal basis of L^2-harmonic forms. With this choice of preferred basis in cohomology the Intersection R-Torsion $I\tau^{\bar{m}}$ is related to the analytic torsion $T(X, \varepsilon)$.

Next we describe the handle decomposition of a pseudomanifold X^n and obtain a formula for $I\tau^{\bar{p}}(X, \varepsilon)$ in terms of the Intersection R-Torsion of the handles, links and the associated long exact sequences in homology. This formula will then be used to prove combinatorial invariance, duality and independence of stratification for $I\tau^{\bar{p}}(X, \varepsilon)$.

For $i > 1$, by an n-dimensional i-handle we mean a pseudomanifold which is homeomorphic to $I^{n-i} \times C(N^{i-1})$ where N^{i-1} is a pseudomanifold of dimension $i-1$, $C(N^{i-1})$ denotes the cone on N^{i-1} and I denotes the closed interval. 0 and 1 handles are homeomorphic to I^n.

Let T be a triangulation of X^n and T', T'' the first and second barycentric subdivisions of T respectively. Let x_0 be the barycentre of a zero dimensional simplex σ_0 of T, i.e. x_0 is a vertix of T. Let $L_{x_0}^{n-1}$ be a star of x_0 in T''. Then $x_0 * L_{x_0}^{n-1}$ (where $*$ denotes join) is a neighborhood of x_0 homeomorphic to $C(L_{x_0}^{n-1})$ and is an n-handle denoted $h_{x_0}^n$. Similarly for any other vertex x_0' of T we have the associated n-handle $h_{x_0'}^n$. The set of n-handles is denoted $\{h_j^n \mid j = 1,2,\ldots,k_n\}$. Next, let x_1 be a barycentre of a one domensional simplex σ_1

of T. Then x_1 has a neighborhood in T" which is homeomorphic to $I \times C(L_{x_1}^{n-2})$ where $L_{x_1}^{n-2}$ is a pseudomanifold of dimension n-2 in T" and is by definition the link of the simplex σ_1. $I \times C(L_{x_1}^{n-2})$ is an n-1 handle denoted $h_{x_1}^{n-1}$. The set of n-1 handles is denoted $\{h_j^{n-1} | j = 1,2,\ldots,k_{n-1}\}$. Similarly let x_i be a barycentre of an i dimensional simplex σ_i of T. Then x_i has a neighborhood in T" which is homeomorphic to $I^i \times C(L_{x_i}^{n-i-1})$ where $L_{x_i}^{n-i-1}$ is a pseudomanifold of dimension n-i-1 and is by definition the link of the simplex σ_i in T". $I^i \times C(L_{x_i}^{n-i-1})$ is an n-i handle and denoted $h_{x_i}^{n-i}$. The set of n-i handles is denoted $\{h_j^{n-i} | j = 1,2,\ldots,k_{n-i}\}$.

Let $Z^n = X^n$. Let $Z^{n-1} = X^n - \operatorname{int}(\bigcup_{j=1}^{k_n} h_j^n)$ (where int h_j^n denotes the interior of the handle h_j^n).

Let $Z^{n-2} = X^n - \operatorname{int}(\bigcup_{j=1}^{k_n} h_j^n \bigcup_{j=1}^{k_{n-1}} h_j^{n-1})$

and similarly

$$Z^i = X^n - \operatorname{int}(\bigcup_{j=1}^{k_n} h_j^n \bigcup_{j=1}^{k_{n-1}} h_j^{n-1} \bigcup_{j=1}^{k_{n-2}} h_j^{n-2} \ldots \bigcup_{j=1}^{k_{n-i+1}} h_j^{n-i+1}).$$

Then we have that $h_j^n \cap Z^{n-1}$ is homeomorphic to L_j^{n-1} for each j. Similarly $h_j^{n-1} \cap Z^{n-2}$ is homeomorphic to $I \times L_j^{n-2}$

for each j. So also $h_j^{n-i} \cap Z^{n-i-1}$ is homeomorphic to $I^i \times L^{n-i-1}$.

Now let X^n be a pseudomanifold with a fixed stratification and let T be a triangulation subordinate to the stratification, T' and T" first and second barycentric subdivisions of T. Let X^n have a handle decomposition as described above, the n-handles are $\{h_j^n | j = 1,2,\ldots,k_n\}$ with associated links $\{L_j^{n-1} | j = 1,2,\ldots,k_n\}$; then n-1 handles are $\{h_j^{n-1} | j = 1,2,\ldots,k_{n-1}\}$ with associated links $\{L_j^{n-2} | j = 1,2,\ldots,k_{n-1}\}$ and similarly the i handles are $\{h_j^i | j = 1,2,\ldots,k_i\}$ with associated links $\{L_j^{i-1} | j = 1,2,\ldots,k_i\}$. $Z^i = X^n - \mathrm{int}(\bigcup_{j=1}^{k_n} h_j^n \bigcup_{j=1}^{k_{n-1}} h_j^{n-1} \bigcup_{j=2}^{k_{n-2}} \ldots \bigcup_{j=1}^{k_{n-i+1}} h^{n-i+1})$ and $h^i \cap Z^{i-1}$ is homeomorphic to $I^{n-i} \times L^{i-1}$.

Let $R_i^{\bar{p}}$ be the basic sets with respect to the triangulation T' of X^n. Let $\xi_i \in H_i(R_i^{\bar{p}}, R_{i-1}^{\bar{p}})$, i.e. ξ_i is a $(\bar{p},i)$ allowable chain of X^n whose boundary is $(\bar{p},i-1)$ allowable. Consider the restriction of ξ_i to the interior of an n-handle $h_{x_0}^n$, i.e. $\xi_i \cap \mathrm{int}\, h_{x_0}^n$. Then handle $h_{x_0}^n$ has an induced stratification from the stratification of X^n which is the cone stratification, therefore $\xi_i \cap \mathrm{int}\, h_{x_0}^n = x_0 * \eta_{i-1}$ where η_{i-1} is a $(\bar{p},i-1)$ allowable chain. Thus $\xi_i \cap \mathrm{int}\, h_{x_0}^n$ is a $(\bar{p},i)$ allowable chain with $(\overline{p},i-1)$ allowable boundary.

A one-dimension pseudomanifold L' is a disjoint union of circles. The restriction of ξ_i to a set of the type $L' \times I^{n-2}$ is therefore $(\overline{p},i)$ allowable with $(\overline{p},i-1)$ allowable boundary.

Suppose that the restriction of ξ_i to a set of the type $L^j \times I^{n-j-1}$ is $(\overline{p},i)$ allowable with $(\overline{p},i-1)$ allowable boundary. Then on a set of the type $L^{j+1} \times I^{n-j}$, on the j+1 handle h^{j+1} $\xi_i \cap h^{j+1}$ is of the type $x*\eta_{i-1}$ and therefore is $(\overline{p},i)$ allowable with $(\overline{p},i-1)$ allowable boundary, and the rest of the set is of the type $L^j \times I^{n-j-1}$ and therefore by the induction hypothesis on this piece also the restriction of ξ_i is $(\overline{p},i)$ allowable with $(\overline{p},i-1)$ allowable boundary. Thus the restriction of ξ_i to $L^{j+1} \times I^{n-j}$ is $(\overline{p},i)$ allowable with $(\overline{p},i-1)$ allowable boundary. This implies in particular that $\xi_i \cap L^{n-1}_{x_0}$ is $(\overline{p},i)$ allowable with $(\overline{p},i-1)$ allowable boundary.

Let $Z^{n-1}_j = X^n - \bigcup_{k=1}^{j} h^n_k$. Then $Z^{n-1}_1 = X^n - h^n_1$; $(h^n_1 = h^n_{x_0})$.

Now $\xi_i = \xi_i \cap \operatorname{int} h^n_{x_0} + \xi_i \cap L^{n-1}_{x_0} + \xi_i \cap Z^{n-1}_{x_0}$

and $\partial\xi_i = \partial(\xi_i \cap \operatorname{int} h^n_{x_0}) + \partial(\xi_i \cap L^{n-1}_{x_0}) + \partial(\xi_i \cap Z^{n-1}_{x_0})$.

Therefore $\partial(\xi_i \cap Z^{n-1})$ is $(\overline{p},i-1)$ allowable, and therefore

$$0 \to \mathbb{R}^{\overline{p}}(X^n) \to \mathbb{R}^{\overline{p}}(h^n_1) \oplus \mathbb{R}^{\overline{p}}(Z^{n-1}_1) \to \mathbb{R}^{\overline{p}}(L^{n-1}_1) \to 0$$

is a short exact sequence of chain complexes which implies by formula (1.6) that

$$\ell n\, I\tau^{\overline{P}}(X^n,\varepsilon) = \ell n\, I\tau^{\overline{P}}(h_1^n,\varepsilon) + \ell n\, I\tau^{\overline{P}}(Z_1^{n-1},\varepsilon) - \ell n\, I\tau^{\overline{P}}(L_1^{n-1}) + \ell n\, \tau(\mathcal{H}_1^n)$$

where $\mathcal{H}_1^n$ is the associated long exact sequence in homology.

By exactly the same argument we obtain that

$$\ell n\, I\tau^{\overline{P}}(Z_1^{n-1},\varepsilon) = \ell n\, I\tau^{\overline{P}}(h_2^n,\varepsilon) + \ell n\, I\tau^{\overline{P}}(Z_2^{n-1},\varepsilon) - \ell n\, \tau^{\overline{P}}(L_2^{n-1}) + \ell n\, \tau(\mathcal{H}_2^n).$$

and so on.

Therefore we can write

$$\ell n\, I\tau^{\overline{P}}(X^n,\varepsilon) = \sum_{i=1}^{k_n} \ell n\, I\tau^{\overline{P}}(h_i^n,\varepsilon) - \sum_{i=1}^{k_n} \ell n\, I\tau^{\overline{P}}(L_i^n,\varepsilon) + \sum_{i=1}^{k_n} \ell n\, \tau(\mathcal{H}_i^n)$$
$$+ \ell n\, I\tau^{\overline{P}}(Z^{n-1},\varepsilon)$$

We can now repeat the argument with the n-1 handles $\{h_1^{n-1}, h_2^{n-1}, \ldots, h_{k_{n-1}}^{n-1}\}$ to obtain the result that

$$\ell n\, I\tau^{\overline{P}}(Z^{n-1},\varepsilon) = \sum_{i=1}^{k_{n-1}} \ell n\, I\tau^{\overline{P}}(h_i^{n-1},\varepsilon) - \sum_{i=1}^{k_{n-1}} \ell n\, I\tau^{\overline{P}}(L_i^{n-1},\varepsilon) + \sum_{i=1}^{k_{n-1}} \ell n\, \tau(\mathcal{H}_i^{n-1})$$
$$+ \ell n\, I\tau^{\overline{P}}(Z^{n-2},\varepsilon).$$

Similarly

$$\ell n\, I\tau^{\overline{P}}(Z^{n-2},\varepsilon) = \sum_{i=1}^{k_{n-2}} \ell n\, I\tau^{\overline{P}}(h_i^{n-2},\varepsilon) - \sum_{i=1}^{k_{n-2}} \ell n\, I\tau^{\overline{P}}(L_i^{n-2},\varepsilon) + \sum_{i=1}^{k_{n-2}} \ell n\, \tau(\mathcal{H}_i^{n-2})$$
$$+ \ell n\, I\tau^{\overline{P}}(Z^{n-3},\varepsilon)$$

and so on.

Thus we arrive at the following formula for the Intersection R-Torsion $I\tau^{\bar{p}}(X, \varepsilon)$.

<u>Formula 2.6.</u> $$\ln I\tau^{\bar{p}}(X^n, \varepsilon) = \sum_{i=1}^{k_n} \ln I\tau^{\bar{p}}(h_i^n, \varepsilon) + \sum_{i=1}^{k_{n-1}} \ln I\tau^{\bar{p}}(h_i^{n-1}, \varepsilon)$$
$$+ \sum_{i=1}^{k_{n-2}} \ln I\tau^{\bar{p}}(h_i^{n-2}, \varepsilon) + \ldots + \sum_{i=1}^{k_0} \ln I\tau^{\bar{p}}(h_i^0, \varepsilon)$$
$$-\Big(\sum_{i=1}^{k_n} \ln I\tau^{\bar{p}}(L_i^n, \varepsilon) + \sum_{i=1}^{k_{n-1}} \ln I\tau^{\bar{p}}(L_i^{n-1}, \varepsilon) + \sum_{i=1}^{k_{n-2}} \ln I\tau^{\bar{p}}(L_i^{n-2}, \varepsilon)$$
$$+ \ldots + \sum_{i=1}^{k_0} \ln I\tau^{\bar{p}}(L_i^0, \varepsilon)\Big)$$
$$+ \sum_{i=1}^{k_n} \ln \tau(\mathcal{H}_i^n) + \sum_{i=1}^{k_{n-1}} \ln \tau(\mathcal{H}_i^{n-1}) + \sum_{i=1}^{k_{n-2}} \ln \tau(\mathcal{H}_i^{n-2}) + \ldots \sum_{i=1}^{k_0} \ln \tau(\mathcal{H}_i^0) + \ln I\tau^{\bar{p}}(Z^0)$$

<u>Theorem 2.7</u> <u>Combinatorial Invariance for $I\tau^{\bar{p}}(X, \varepsilon)$</u>. Let X^n be a pseudomanifold with triangulation T and T', T" first and second barycentric subdivisions of T. Let $\mathbb{R}^{\bar{p}}(\hat{X}, T', \varepsilon)$ be the chain complex with respect to T', with torsion $I\tau^{\bar{p}}(X, T', \varepsilon)$. Let S be a subdivision of T'. Then $I\tau^{\bar{p}}(X, S, \varepsilon) = I\tau^{\bar{p}}(X, T', \varepsilon)$.

<u>Proof</u>. The proof is by induction, using formula 2.6. For n=1, any closed pseudomanifold is a disjoint union of circles so that $\tau = I\tau^{\bar{p}}$ is defined and is a combinatorial invariant. Suppose that for any closed pseudomanifold of dimension $\leq$ n-1 $I\tau^{\bar{p}}$ is a combinatorial invariant.

Let X^n be an n-dimensional pseudomanifold with triangulation T, and a handle decomposition as described above. Then by formula 2.6

$$\ln I\tau^{\overline{p}}(X_n, \varepsilon) = \sum_{i=1}^{k_n} \ln I\tau^{\overline{p}}(h_i^n, \varepsilon) + \sum_{i=1}^{k_{n-1}} \ln I\tau^{\overline{p}}(h_i^{n-1}, \varepsilon)$$

$$+ \sum_{i=1}^{k_{n-2}} \ln I\tau^{\overline{p}}(h_i^{n-2}, \varepsilon) + \ldots + \sum_{i=1}^{k_0} \ln I\tau^{\overline{p}}(h_i^0, \varepsilon)$$

$$-\Big(\sum_{i=1}^{k_n} \ln I\tau^{\overline{p}}(L_i^n, \varepsilon) + \sum_{i=1}^{k_{n-1}} \ln I\tau^{\overline{p}}(L_i^{n-1}, \varepsilon) + \sum_{i=1}^{k_{n-2}} \ln I\tau^{\overline{p}}(L_i^{n-2}, \varepsilon)$$

$$+ \ldots + \sum_{i=1}^{k_0} \ln I\tau^{\overline{p}}(L_i^0, \varepsilon)\Big)$$

$$+ \sum_{i=1}^{k_n} \ln \tau(\mathcal{H}_i^n) + \sum_{i=1}^{k_{n-1}} \ln \tau(\mathcal{H}_i^{n-1}) + \sum_{i=1}^{k_{n-2}} \ln \tau(\mathcal{H}_i^{n-2}) + \ldots + \sum_{i=1}^{k_0} \ln \tau(\mathcal{H}_i^0)$$

$$+ \ln I\tau^{\overline{p}}(Z^0).$$

The handles h_i^j are simply connected. Therefore, ε restricted to h_i^j is simply the trivial flat line bundle R and $I\tau^{\overline{p}}(h_i^j, R)$ can be written (by formula (1.6)) as

$$\ln I\tau^{\overline{p}}(h_i^j, R) = \Sigma(-1)^k \ln O_k$$

where O_k is the order of the torsion subgroup of $IH_k(h_i^j, Z)$. Since the Intersection Homology Groups $IH^{\overline{p}}$ are combinatorial invariants so also is $\ln I\tau^{\overline{p}}(h_i^j, R)$. By the induction hypothesis

$I_\tau^{\bar{p}}(L_i^j, \varepsilon)$ is a combinatorial invariant. $\tau(H_i^j)$ is purely homological and therefore is a combinatorial invariant. Therefore, using formula (2.6) we observe that $\ln I_\tau^{\bar{p}}(X^n, \varepsilon)$ is also a combinatorial invariant.

<u>Theorem 2.8</u> <u>Duality for $I_\tau^{\bar{p}}$</u>. Let X^n be an n dimensional pseudomanifold. Let $\{h_i^{\bar{p}}\}$ be a basis for $IH_i^{\bar{p}}(X, \varepsilon)$. Let $* : IH_i^{\bar{p}} \to {}^{\bar{q}}IH^{n-i}$ denote the isomorphism of generalized Poincaré duality. Suppose $* : \{h_i^{\bar{p}}\} \to \{(h_{n-i}^{\bar{q}})^*\}$. Then we have the following

1) When n is even $\ln I_\tau^{\bar{p}}(X, \varepsilon) + \ln I_\tau^{\bar{q}}(X, \varepsilon) = 0$.

2) When n is odd $\ln I_\tau^{\bar{p}}(X, \varepsilon) = \ln I_\tau^{\bar{q}}(X, \varepsilon)$.

<u>Proof</u>. The proof is by induction using formula 2.6. For $n = 1$ we have that a closed pseudomanifold is a disjoint union of circles, $I_\tau^{\bar{p}} = \tau$ and duality is true.

Suppose duality is true for pseudomanifolds of dimension $\leq n-1$. Let X^n be a pseudomanifold of dimension n. We have that Z^0 is a disjoint union of sets homeomorphic to I^n so that duality is true; i.e.

$$\ln I_\tau^{\bar{p}}(Z^0, \varepsilon) + \ln I_\tau^{\bar{q}}(Z^0, \partial Z^0, \varepsilon) = 0, \text{ n even}$$

$$\ln I_\tau^{\bar{p}}(Z^0, \varepsilon) = \ln I_\tau^{\bar{q}}(Z^0, \partial Z^0, \varepsilon) \text{ n odd.}$$

Suppose duality is true for Z^i; i.e.

$$\ln I\tau^{\overline{p}}(Z^i,\varepsilon) = -\ln I\tau^{\overline{q}}(Z^i,\partial Z^i,\varepsilon),\ n \text{ even}$$

$$\ln I\tau^{\overline{p}}(Z^i,\varepsilon) = \ln I\tau^{\overline{q}}(Z^i,\partial Z^i,\varepsilon),\ n \text{ odd.}$$

Consider $h_j^{i+1} \cup Z^i$

$$\ln I\tau^{\overline{p}}(h_j^{i+1}\cup Z^i,\varepsilon) = \ln I\tau^{\overline{p}}(Z^i,\varepsilon) + \ln I\tau^{\overline{p}}(h_j^{i+1},\varepsilon) - \ln I\tau^{\overline{p}}(L_j^i,\varepsilon) + \ln\tau(\mathcal{H}_j^i)$$

$$\ln I\tau^{\overline{q}}(h_j^{i+1}\cup Z^i,\partial(h_j^{i+1}\cup Z^i),\varepsilon) = \ln I\tau^{\overline{q}}(h_j^{i+1}\cup Z^i,\partial Z^i - L_j^i,\varepsilon)$$

$$= \ln I\tau^{\overline{q}}(h^{i+1}\cup Z^i,\partial Z^i,\varepsilon) + \ln I\tau^{\overline{q}}(L_j^i,\varepsilon)$$

$$= \ln I\tau^{\overline{q}}(Z^i,\partial Z^i),\varepsilon) + \ln I\tau^{\overline{q}}(h_j^{i+1},L_j^i,\varepsilon) + \ln I\tau^{\overline{q}}(L_j^i,\varepsilon) + \ln\tau(\tilde{\mathcal{H}}_j^i).$$

We will consider the case of n odd and n even separately.

1) <u>n even</u>

$$\ln I\tau^{\overline{p}}(Z^i,\varepsilon) = -\ln I\tau^{\overline{q}}(Z^i,\partial Z^i,\varepsilon) \quad \text{(by induction hypothesis)}$$

$$\ln I\tau^{\overline{p}}(h_j^{i+1},\varepsilon) = -\ln I\tau^{\overline{q}}(h_j^{i+1},L_j^i) \quad \text{(by duality for } IH_*\text{)}$$

$$\ln I\tau^{\overline{p}}(L_j^i,\varepsilon) = \ln I\tau^{\overline{q}}(L_j^i,\varepsilon) \quad \text{(by induction hypothesis)}$$

$$\ln\tau(\mathcal{H}_j^i) = -\ln\tau(\tilde{\mathcal{H}}_j^i). \quad \text{(by duality for } IH_*\text{)}$$

Therefore, $\ln I\tau^{\overline{p}}(Z^{i+1},\varepsilon) = -\ln I\tau^{\overline{q}}(Z^{i+1},\partial Z^{i+1},\varepsilon)$ and for $i = n-1$ $\ln I\tau^{\overline{p}}(X^n,\varepsilon) = -\ln I\tau^{\overline{q}}(X^n,\varepsilon)$, i.e. $\ln I\tau^{\overline{p}}(X^n,\varepsilon) + \ln I\tau^{\overline{q}}(X^n,\varepsilon) = 0$.

2) n odd

$$\ell n\ I\tau^{\bar{p}}(Z^i,\varepsilon) = \ell n\ I\tau^{\bar{q}}(Z^i,\partial Z^i,\varepsilon) \quad \text{(by induction hypothesis)}$$

$$\ell n\ I\tau^{\bar{p}}(h_j^{i+1},\varepsilon) = \ell n\ I\tau^{\bar{q}}(h_j^{i+1},L_j^i,\varepsilon) \quad \text{(by duality for } IH_*\text{)}$$

$$\ell n\ I\tau^{\bar{p}}(L_j^i,\varepsilon) = -\ell n\ I\tau^{\bar{q}}(L_j^i,\varepsilon) \quad \text{(by induction hypothesis)}$$

$$\ell n\ \tau(\mathcal{H}_j^i) = \ell n\ \tau(\tilde{\mathcal{H}}_j^i)\ . \quad \text{(by duality for } IH_*\text{)}$$

Therefore, $\ell n\ I\tau^{\bar{p}}(Z^{i+1},\varepsilon) = \ell n\ I\tau^{\bar{q}}(Z^{i+1},\partial Z^{i+1},\varepsilon)$ and for $i = n-1$ $\ell n\ I\tau^{\bar{p}}(X^n,\varepsilon) = \ell n\ I\tau^{\bar{q}}(X^n,\varepsilon)$.

<u>Theorem 2.9</u> <u>Independence of Stratification for $I\tau^{\bar{p}}(X,\varepsilon)$</u>.
$I\tau^{\bar{p}}(X,\varepsilon)$ is independent of the stratification used to define it.

<u>Proof</u>. Again the proof is by induction using formula 2.6. For $n = 1$ any closed pseudomanifold is a disjoint union of circles so that $\tau = I\tau^{\bar{p}}$ is independent of stratification. Suppose that for any closed pseudomanifold of dimension $\leq n-1$. $I\tau^{\bar{p}}$ is independent of stratification. Let X^n be an n-dimensional pseudomanifold with handle decomposition. Then formula (2.6) is applicable for $I\tau^{\bar{p}}(X,\varepsilon)$.

On the handles h_i^j, $\ell n\ I\tau^{\bar{p}}(h_i^j) = \sum_{k=0}^{n}(-1)^k \ell n\ O_k$ where O_k is the order of the torsion subgroup of $IH_k^{\bar{p}}$. Since the Intersection Homology Groups are independent of stratification so also is $I\tau^{\bar{p}}(h_i^j,R)$.

By the induction hypothesis $I\tau^{\overline{p}}(L_i^j, \varepsilon)$ is independent of stratification. $\tau(H_i^j)$ is independent of stratification because the Intersection Homology groups are independent of stratification.

Therefore, using formula (2.6) we observe that $I\tau^{\overline{p}}(X^n, \varepsilon)$ is also independent of stratification.

Remark 2.10. In Goresky and Macpherson [10] it is shown that the Intersection Homology Groups $IH_i^{\overline{p}}(X)$ are topological invariants. In the same spirit it is reasonable to expect that the Intersection R-Torsion $I\tau^{\overline{p}}$ is also a topological invariant although we do not prove it here.

Theorem 2.11. When X^n is a manifold $I\tau_W^{\overline{p}} = \tau_W$, $I\tau^{\overline{p}} = \tau$ for every perversity $\overline{p}$, where τ_W, τ is the usual Whitehead/Reidemeister torsion for compact manifolds as in Definition 1.6.

Proof. When X^n is a manifold it has a stratification with one strata, the whole of X^n. Let T be a triangulation of X^n. Then T is subordinate to X^n and the complex $R_i^{\overline{p}}$ of $(\overline{p}, i)$ allowable simplices consists of the i-skeleton T_i' of the first barycentric subdivision T' of T. Thus the chain complex $\mathcal{R}^{\overline{p}}(\hat{X})$ coincides with the simplicial chain complex of (X^n, T') as in Definition 1.6 and $I\tau_W^{\overline{p}}(X) = \tau_W(X)$, $I\tau^{\overline{p}}(X) = \tau(X)$.

We now construct another chaim complex $S^{\bar{p}}$ using an alternative family of basic sets $\{Q_i^{\bar{p}}\}$ defined in Goresky and Macpherson [9], Section 3. We will show that the torsion of the complex $S^{\bar{p}}$ also gives the Intersection R-Torsion $I\tau^{\bar{p}}$. The complex $S^{\bar{p}}$ is used to give another proof of combinatorial invariance, duality and independence of stratification for $I\tau^{\bar{p}}(X)$.

<u>Definition 2.8</u> <u>Basic Sets $Q_i^{\bar{p}}$.</u> X^n any pseudomanifold. T any triangulation of X^n. T' the first barycentric subdivision of T. For each perversity $\bar{p}$ and integer $i \geq 0$ define the function $L_i^{\bar{p}}$ as follows.

$$L_i^{\bar{p}}(0) = i, \; L_i^{\bar{p}}(1) = i - 1, \; L_i^{\bar{p}}(n+1) = -1,$$

and if $2 \leq c \leq n$ set

$$L_i^{\bar{p}}(c) = \begin{cases} -1 & \text{if } i - c + p_c \leq -1 \\ n - c & \text{if } i - c + p_c \geq n - c \\ i - c + p_c & \text{otherwise} \end{cases}$$

$\Delta L_i^{\bar{p}}(c) = L_i^{\bar{p}}(c) - L_i^{\bar{p}}(c)$ (which is 0 or 1). Define $Q_i^{\bar{p}}$ to be the subcomplex of T' spanned by the set of barycentres of simplices

$$\{\hat{\sigma} \mid \sigma \in T \text{ and } \Delta L_i^{\bar{p}}(n\text{-dim}\,\sigma) = 1\}.$$

Then we have that $Q_i^{\bar{p}}$ is an i-dimensional subcomplex of T' such that

$$X = Q_n^{\bar{p}} \supset Q_{n-1}^{\bar{p}} \supset \ldots \supset Q_i^{\bar{p}} \supset Q_{i-1}^{\bar{p}} \supset \ldots \supset Q_0^{\bar{p}} .$$

We also have that $IH_i^{\bar{p}} = \text{Image } H_i(Q_i^{\bar{p}}) \to H_i(Q_{i+1}^{\bar{p}})$.

Example 2.13. Basic sets $Q_i^{\bar{m}}$ for the middle perversity $\bar{m} = (0,0,1,1,2,2,\ldots)$

Let X^n be a pseudomanifold of even dimension $n = 2k$. Let T be a triangulation of X^n, and let T' be its first barycentric subdivision. Then $Q_i^{\bar{m}}$ is the i-dimensional subcomplex of T' obtained as follows:

	Span of barycentres of simplices of dimension
$Q_0^{\bar{m}}$	n
$Q_1^{\bar{m}}$	n, n-1
$Q_2^{\bar{m}}$	n, n-1, n-2
$Q_3^{\bar{m}}$	n, n-1, n-2, n-4
$Q_4^{\bar{m}}$	n, n-1, n-2, n-4, n-6
⋮	
$Q_k^{\bar{m}}$	n, n-1, n-2, n-4, n-6,............ 6,4,2
$Q_{k+1}^{\bar{m}}$	n, n-1, n-2, n-4, n-6,............ 6,4,2,0

$Q^{\bar{m}}_{k+2}$	n, n-1, n-2, n-4, n-6,............ 6,4,2,1,0
$Q^{\bar{m}}_{k+3}$	n, n-1, n-2, n-4, n-6,............ 6,4,3,2,1,0
$Q^{\bar{m}}_{k+4}$	n, n-1, n-2, n-4, n-6,............ 6,5,4,3,2,1,0
.	
.	
.	
$Q^{\bar{m}}_{n-1}$	n, n-1, n-2, n-4, n-5, n-6,....... 6,5,4,3,2,1,0
$Q^{\bar{m}}_{n}$	n, n-1, n-2, n-3, n-4, n-5, n-6,.. 6,5,4,3,2,1,0

Let X^n be a pseudomanifold of odd dimension $n = 2k+1$. Let T be a triangulation of X^n and let T' be its first barycentric subdivisi n. Then $Q^{\bar{m}}_i$ is the i-dimensional subcomplex of T' obtained as follows:

	Span of barycentres of simplices of dimension
$Q^{\bar{m}}_0$	n
$Q^{\bar{m}}_1$	n, n-1
$Q^{\bar{m}}_2$	n, n-1, n-2
$Q^{\bar{m}}_3$	n, n-1, n-2, n-4
$Q^{\bar{m}}_4$	n, n-1, n-2, n-4, n-6
.	
.	
.	
$Q^{\bar{m}}_k$	n, n-1, n-2, n-4, n-6,............ 5,3

$Q^{\bar{m}}_{i+1}$	n, n-1, n-2, n-4, n-6,.............5,3,1
$Q^{\bar{m}}_{k+2}$	n, n-1, n-2, n-4, n-6,.............5,3,1,0
$Q^{\bar{m}}_{k+3}$	n, n-1, n-2, n-4, n-6,.............5,3,2,1,0
$Q^{\bar{m}}_{k+4}$	n, n-1, n-2, n-4, n-6,.............5,4,3,2,1,0
⋮	
$Q^{\bar{m}}_{n-1}$	n, n-1, n-2, n-4, n-5, n-6,........5,4,3,2,1,0
$Q^{\bar{m}}_{n}$	n, n-1, n-2, n-3, n-4, n-5, n-6,...5,4,3,2,1,0

Lemma 2.14. Let X^n be an n-dimensional pseudomanifold with fixed stratification. Let T be a triangulation of X^n subordinate to the stratification and T' the first barycentric subdivision of T. Let the $Q^{\bar{p}}_i$ be defined with respect to a perversity $\bar{p}$. Let $\xi_i \in H_i(Q^{\bar{p}}_i, Q^{\bar{p}}_{i-1})$, i.e. ξ_i is an i-chain supported in $Q^{\bar{p}}_i$ such that $\partial\xi_i$ is supported in $Q^{\bar{p}}_{i-1}$. Then $\xi_i = \sum_{j=0}^{k} \xi_{i,j}$ where $\xi_{i,j}$ is an i-chain of $Q^{\bar{p}}_i$ with boundary in $Q^{\bar{p}}_{i-1}$ and $|\xi_{i,j}|$ is simply connected.

Proof. $Q^{\bar{p}}_{i-1}$ is the i-1 dimensional subcomplex of T' spanned by barycentres of simplices of T of dimension

n, n-1, $q_2, q_3, q_4, \ldots q_{i-1}$ and $Q^{\bar{p}}_i$ is the i dimensional subcomplex of T' spanned by barycentres of simplices of T of dimension n, n-1, $q_2, q_3, q_4, \ldots, q_{i-1}, q_i$. Let $\xi_i \in H_i(Q_i^{\bar{p}}, Q_{i-1}^{\bar{p}})$, i.e. ξ_i is an i-chain of $Q_i^{\bar{p}}$ with boundary in $Q_{i-1}^{\bar{p}}$. Let $x_1, \ldots, x_j$ be the barycentres of simplices of T of dimension q_i such that $\xi_i \cap x_k \neq \emptyset$. Let N_{x_k} be the star of x_k in $Q_i^{\bar{p}}$ and let $x_k * N_{x_k}$ (where * denotes join) be a neighborhood of x_k in Q_i. Then $\xi_i = \sum_{k=1}^{j} \xi_i \cap x_k * N_{x_k}$. Consider $\xi_i \cap x_k * N_{x_k}$ and denote it $\xi_{i,k}$. Then since ξ_i is i allowable, certainly its restriction $\xi_{i,k}$ is i allowable.

<u>Claim</u>. $\partial \xi_{i,k}$ is contained in $Q_{i-1}^{\bar{p}}$ and therefore is i-1 allowable.

For <u>suppose not</u>. Then $\partial \xi_{i,k}$ contains an i-1 simplex of the form $x_i p$ where p is an i-2 simplex in N_{x_i}. But $\xi_i = \sum_{k=1}^{j} \xi_{i,k}$ and $\partial \xi_i = \sum_{k=1}^{j} \partial \xi_{i,k}$. But if $\partial \xi_{i,k}$ contains a simplex of the type $x_i p$ then $x_i p$ cannot be cancelled by the boundary of any other piece of the chain $\xi_{i,\ell}$, $\ell \neq k$ because $\partial \xi_{i,\ell}$ is disjoint from px_i. Therefore, if $\partial \xi_{i,k}$ contains a simplex of type px_i then $\partial \xi_i$ also contains the simplex px_i, $px_i \not\subset Q_{i-1}^{\bar{p}}$. But this contradicts the fact that ξ_i is an i-chain in $Q_i^{\bar{p}}$ whose boundary is in $Q_{i-1}^{\bar{p}}$. The claim follows.

Definition 2.15. The complex $S^{\overline{p}}(X)$. Let $S^{\overline{p}}(X)$ be the chain complex defined by setting $S_i^{\overline{p}}(X) = H_i(Q_i^{\overline{p}}, Q_{i-1}^{\overline{p}})$. This i-th chain group is in one-to-one correspondence with simplicial chains f_i such that $|f_i| \subset Q_i^{\overline{p}}$ and $|\partial f_i| \subset Q_{i-1}^{\overline{p}}$. It is a free abelian group generated by finitely many chains $\{f_i^j\}$ with contractible support as in Lemma 2.14. The boundary map $\partial_i : S_i^{\overline{p}} \to S_{i-1}^{\overline{p}}$ is obtained from the homology exact sequence $\cdots \to H_i(Q_i^{\overline{p}}, Q_{i-1}^{\overline{p}}) \xrightarrow{\partial_i} H_{i-1}(Q_{i-1}^{\overline{p}}) \to \cdots$. The homology group $H_i S^{\overline{p}}(X)$ is canonically isomorphic to $IH_i^{\overline{p}}$.

Let $\hat{X}$ denote the universal covering complex of X, and let $\hat{Q}_i^{\overline{p}}$ denote the lift of $Q_i^{\overline{p}}$ to $\hat{X}$. The fundamental group of X, denoted π, acts on $\hat{X}$ as the group of covering transformations. Thus each $\sigma \in \pi$ determines a mapping $\sigma : \hat{Q}_i^{\overline{p}} \to \hat{Q}_i^{\overline{p}}$. Next let $S^{\overline{p}}(\hat{X})$ be the chain complex defined by setting $S_i^{\overline{p}}(\hat{X}) = H_i(\hat{Q}_i^{\overline{p}}, \hat{Q}_{i-1}^{\overline{p}})$. With respect to the action of π each chain group $S_i^{\overline{p}}(\hat{X})$ is a free $Z\pi$-module generated by the lifts of the chains $\{f_i^j\}$; and $S^{\overline{p}}(\hat{X})$ is a chain complex of free $Z\pi$-modules. When the homology groups $H_i S^{\overline{p}}(\hat{X})$ are all $Z\pi$ free we have

Theorem 2.16. $\tau(S^{\overline{p}}(\hat{X})) = \tau(\mathcal{R}^{\overline{p}}(\hat{X}))$.

Proof. We use the formalism of the Algebraic Subdivision Theorem 1.4. Let C denote the chain complex $\mathcal{R}^{\overline{p}}(\hat{X})$. We will construct a filtration of C by subcomplexes $C^{(0)} \subset C^{(1)} \subset C^{(2)} \subset \ldots \subset C^{(n)} = C$ such that the associated chain complex $\overline{C}$ given by $\overline{C}_\lambda = H_\lambda(C^{(\lambda)}/C^{(\lambda-1)})$ can be identified with the chain complex $S^{\overline{p}}(\hat{X})$. We will then show that the hypotheses of the Algebraic Subdivision Theorem are all satisfied. We can then conclude that

$\tau(C) = \tau(\overline{C})$ i.e. $\tau(\mathbb{R}^{\overline{p}}) = \tau(S^{\overline{p}}) = I\tau^{\overline{p}}$. The construction follows.

Recall that $C = C^{(n)} = \mathbb{R}^{\overline{p}}(\hat{X})$ is the chain complex given by

$$H_n(\hat{R}^{\overline{p}}_n, \hat{R}^{\overline{p}}_{n-1}) \to H_{n-1}(\hat{R}^{\overline{p}}_{n-1}, \hat{R}^{\overline{p}}_{n-2}) \to \cdots \to H_0(\hat{R}^{\overline{p}}_0)$$

Next we introduce the following notation. Let $R^{\overline{p}}_{i,j}$ denote the subcomplex of T' consisting of all $(\overline{p},j)$ allowable simplices of T' contained inside $|Q^{\overline{p}}_i|$, $j \leq i$, where $|Q^{\overline{p}}_i|$ denotes the underlying topological space of the complex $Q^{\overline{p}}_i(T)$. Let $\hat{R}^{\overline{p}}_{i,j}$ denote the corresponding subcomplex of $\hat{X}$. Now we construct the filtration of $C = C^{(n)}$ as follows.

$C^{(n-1)}$ is the subcomplex of $C^{(n)}$ given by

$$H_{n-1}(\hat{R}^{\overline{p}}_{n-1,n-1}, \hat{R}^{\overline{p}}_{n-1,n-2}) \to H_{n-2}(\hat{R}^{\overline{p}}_{n-1,n-2}, \hat{R}^{\overline{p}}_{n-1,n-3}) \to \cdots \to H_0(\hat{R}^{\overline{p}}_{n-1,0})$$

and similarly $C^{(i)}$ is the subcomplex given by

$$H_i(\hat{R}^{\bar{p}}_{i,i}, \hat{R}^{\bar{p}}_{i,i-1}) \to H_{i-1}(\hat{R}^{\bar{p}}_{i,i-1}, \hat{R}^{\bar{p}}_{i,i-2}) \to \cdots \to H_0(\hat{R}^{\bar{p}}_{i,0})$$

for $0 \leq i \leq n$. We claim that this is the required filtration.

We begin with the observation that $H_i(C^{(\lambda-1)}) = H_i(C^{(\lambda-1)})$ for $i \leq \lambda - 2$. Therefore, using the long exact sequence in homology associated with the short exact sequence of chain complexes

$$0 \to C^{(\lambda-1)} \to C^{(\lambda)} \to C^{(\lambda)}/C^{(\lambda-1)} \to 0$$

we see that $H_i(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ for $i \leq \lambda - 2$.

In the case $i = \lambda - 1$ the map $H_{\lambda-1}C^{(\lambda-1)} \to H_{\lambda-1}C^{(\lambda)}$ is surjective so that $H_{\lambda-1}(C^{(\lambda)}/C^{(\lambda-1)}) = 0$

We also have that $H_\lambda(C^{(\lambda)}/C^{(\lambda-1)}) = H_\lambda(\hat{Q}^{\bar{p}}_\lambda, \hat{Q}^{\bar{p}}_{\lambda-1})$ so that the chain complex $\bar{C}$ can be canonically identified with the complex $S^{\bar{p}}(\hat{X})$.

We are now required to show that $\tau(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ for all λ. This uses the same basic idea as in Milnor [12], Lemma 7.2.

We begin with the observation that when we delete the support of the $(\lambda-1)$ cycles of $Q^{\bar{p}}_{\lambda-1}$, we divide $Q^{\bar{p}}_\lambda$ into simply connected components (i.e. each component is precisely a λ-chain $f_\lambda^{\bar{p}}$ with contractible support as in Lemma 2.14. Every chain of the complex $C^{(\lambda)}/C^{(\lambda-1)}$ is the lift to $\hat{X}$ of a chain which lies inside one of these contractible components $f_\lambda^{\bar{p}}$.

Choose a representative component $\hat{f}_\lambda$ of f_λ. Then $\hat{f}_\lambda$ projects homeomorphically onto f_λ. For each chain f such that $|f| \subset f_\lambda$ choose the representative chain $\hat{f}$ contained in $\hat{f}_\lambda$. No representative chain $\hat{f}_k$ in $\hat{f}_\lambda$ can be incident to a proper translate $\sigma \hat{f}_{k-1}$, $\sigma \neq 1$ because $\sigma \hat{f}_{k-1}$ must be contained in a component $\sigma \hat{f}_\lambda$ which is disjoint from $\hat{f}_\lambda$. This means that the boundary $\partial \hat{f}_k$ can be expressed as a linear combination of $k-1$ chains with coefficients which are integers (rather than group ring elements). Thus in computing the torsion $Z \subset Z\pi$. Then $\tau C^{(\lambda)}/C^{(\lambda-1)} \in \bar{K}_1 Z = 0$. Therefore, $\tau(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ and the proof is complete.

Theorem 2.16 Combinatorial Invariance for $I\tau_W^{\bar{p}}$. Let T be a triangulation of X^n and let W be a subdivision of T'. Then we have $I\tau^{\bar{p}}(X,T) = I\tau^{\bar{p}}(X,W)$.

Proof. The proof is analogous to the proof of the combinatorial invariance of Whitehead Torsion as in Milnor [12] and uses the algebraic formalism of the Algebraic Subdivision Theorem 1.4.

Let $C = S^{\bar{p}}(\hat{X},W)$ be the chain complex associated with the subdivision W. We will construct a filtration of C by subcomplexes

$$C^{(0)} \subset C^{(1)} \subset C^{(2)} \subset \ldots \subset C^{(n)} = C$$

such that the associated chain complex $\bar{C}$ given by $\bar{C}_\lambda = H_\lambda(C^{(\lambda)}/C^{(\lambda-1)})$ can be identified with the chain complex $S^{\bar{p}}(\hat{X},T)$. We then check that all the hypotheses of the Algebraic Subdivision Theorem are satisfied. We can then conclude that $\tau(C) = \tau(\bar{C})$, i.e. $I\tau^{\bar{p}}(X,W) = I\tau^{\bar{p}}(X,T)$.

The construction follows. To begin with, recall that $C = C^{(n)} = S^{\bar{p}}(\hat{X},W)$ is the chain complex given by

$$H_n(\hat{Q}^{\bar{p}}_n(W),\ \hat{Q}^{p}_{n-1}(W)) \to H_{n-1}(\hat{Q}^{\bar{p}}_{n-1}(W),\ \hat{Q}^{\bar{p}}_{n-2}(W)) \to \ldots \to H_0(\hat{Q}^{\bar{p}}_0(W))$$

Next we introduce the following notation. Let $R^{\bar{p}}_{i,j}(W)$ denote the subcomplex of W' consisting of all $(\bar{p},j)$ allowable simplices of W' contained inside $|Q^{\bar{p}}_i(T)|$ $j \leq i$ where $|Q^{\bar{p}}_i(T)|$ denotes the underlying topological space of the complex $Q^{\bar{p}}_i(T)$. Let $\hat{R}^{\bar{p}}_{i,j}$ denote the corresponding subcomplex of $\hat{X}$. Now we construct the filtration of $C = C^{(n)}$ as follows.

$C^{(n-1)}$ is the subcomplex of $C^{(n)}$ given by

$$H_{n-1}(\hat{R}^{\bar{p}}_{n-1,n-1}(W),\hat{R}^{\bar{p}}_{n-1,n-2}(W)) \to H_{n-2}(\hat{R}^{\bar{p}}_{n-1,n-2}(W),\hat{R}^{\bar{p}}_{n-1,n-3})$$
$$\to \ldots \to H_0(\hat{R}^{\bar{p}}_{n-1,0}(W))$$

and similarly $C^{(i)}$ is the subcomplex given by

$$H_i(\hat{R}^{\bar{p}}_{i,i}(W), \hat{R}^{\bar{p}}_{i,i-1}(W)) \to H_{i-1}(\hat{R}^{\bar{p}}_{i,i-1}(W), \hat{R}^{\bar{p}}_{i,i-2}(W)) \to \cdots \to H_0(\hat{R}^{\bar{p}}_{i,0}(W)) \quad \text{for } 0 \leq i \leq n.$$

We claim that this is the required filtration.

We begin with the observation that $H_i(C^{(\lambda)}) = H_i(C^{(\lambda-1)})$ for $i \leq \lambda - 2$ (since the Intersection Homology groups are combinatorial invariants). Therefore using the long exact sequence in homology associated with the short exact sequence of chain complexes

$$0 \to C^{(\lambda-1)} \to C^{(\lambda)} \to C^{(\lambda)}/C^{(\lambda-1)} \to 0$$

we see that $H_i(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ for $i \leq \lambda - 2$.

In the case $i = \lambda - 1$ the map $H_{\lambda-1}C^{(\lambda-1)} \to H_{\lambda-1}C^{(\lambda)}$ is surjective so that $H_{\lambda-1}(C^{(\lambda)}/C^{(\lambda-1)}) = 0$.

We also have that $H_\lambda(C^{(\lambda)}/C^{(\lambda-1)} = H_\lambda(\hat{Q}^{\bar{p}}_\lambda(T), \hat{Q}^{\bar{p}}_{\lambda-1}(T))$. Thus the chain complex $\bar{C}$ can be canonically identified with $S^{\bar{p}}(\hat{X},T)$.

We are now required to show that $\tau(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ for all λ. This uses the same basic idea as in Milnor [12], Lemma 7.2.

We begin with the observation that when we delete the support of the $(\lambda-1)$ cycles of $Q^{\bar{p}}_{\lambda-1}$, we divide $Q^{\bar{p}}_{\lambda}$ into simply connected components (i.e. each component is precisely a contractible λ-chain as in Lemma 2.14. Every chain of the complex $C^{(\lambda)}/C^{(\lambda-1)}$ is the lift of a chain which lies inside one of these contractible components e_λ.

Choose a representative component $\hat{e}_\lambda$ of e_λ. Then $\hat{e}_\lambda$ projects homeomorphically onto e_λ. For each chain e such that $|e| \subset e_\lambda$ choose the representative chain $\hat{e}$ contained in $\hat{e}_\lambda$. No representative chain $\hat{e}_k$ in $\hat{e}_\lambda$ can be incident to a proper translate $\sigma\hat{e}_{k-1}$, $\sigma \neq 1$ because $\sigma\hat{e}_{k-1}$ must be contained in a component $\sigma\hat{e}_\lambda$ which is disjoint from $\hat{e}_\lambda$. This means that the boundary $\partial\hat{e}_k$ can be expressed as a linear combination of $k-1$ chains with coefficients which are integers (rather than group ring elements). Thus in computing the torsion of the complex $C^{(\lambda)}/C^{(\lambda-1)}$ we need only work with the subring $Z \subset Z\pi$. Then $\tau C^{(\lambda)}/C^{(\lambda-1)} \in \bar{K}_1 Z = 0$. Therefore $\tau(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ and the proof of Combinatorial Invariance is complete.

<u>Theorem 2.17</u> <u>Duality for $I\tau^{\bar{p}}$</u>. Let X^n be an n-dimensional oriented pseudomanifold such that its universal covering space $\hat{X}$ is compact. Let $\{h^{\bar{p}}_i\}$ be a basis for $IH^{\bar{p}}_i(X, \varepsilon)$. Let $* : IH^{\bar{p}}_i \to {}^{\bar{q}}IH^{n-i}$ denote the isomorphism of generalized Poincaré

duality. Suppose $* : \{h_i^{\bar{p}}\} \to \{(h_{n-i}^{\bar{q}})^*\}$. Then we have the following:

1) When m is even $\ln I\tau^{\bar{p}}(X,\varepsilon) + \ln I\tau^{\bar{q}}(X,\varepsilon) = 0$

2) When m is odd $\ln I\tau^{\bar{p}}(X,\varepsilon) = \ln I\tau^{\bar{q}}(X,\varepsilon)$.

<u>Proof</u>. Recall that $S^{\bar{p}}(X)$ is the chain complex given by

$$H_n(Q_n^{\bar{p}},Q_{n-1}^{\bar{p}}) \to H_{n-1}(Q_{n-1}^{\bar{p}},Q_{n-2}^{\bar{p}}) \to \cdots \to H_1(Q_1^{\bar{p}},Q_0^{\bar{p}}) \to H_0(Q_0^{\bar{p}}).$$

This is isomorphic to the dual complex

$$H^n(Q_n^{\bar{p}},Q_{n-1}^{\bar{p}}) \leftarrow H^{n-1}(Q_{n-1}^{\bar{p}},Q_{n-2}^{\bar{p}}) \leftarrow \cdots \leftarrow H^1(Q_1^{\bar{p}},Q_0^{\bar{p}}) \leftarrow H^0(Q_0^{\bar{p}}).$$

We have from Goresky and Macpherson [9], Section 3 the isomorphisms (for $i+j=n$)

$$H_i(Q_i^{\bar{p}}) \cong H^j(Q_{j+1}^{\bar{q}}) \qquad (1)$$

and

$$\operatorname{Im} H_i(Q_i^{\bar{p}}) \to H_i(Q_{i+1}^{\bar{p}}) \cong \operatorname{Im} H^j(Q_{j+1}^{\bar{q}}) \to H^j(Q_j^{\bar{q}}) \qquad (2)$$

(1) and (2) together imply that $H_i(Q_i^{\bar{p}},Q_{i-1}^{\bar{p}}) \cong H^j(Q_j^{\bar{q}},Q_{j-1}^{\bar{q}})$. Thus the complexes

$$H_n(Q_n^{\bar{p}},Q_{n-1}^{\bar{p}}) \to H_{n-1}(Q_{n-1}^{\bar{p}},Q_{n-2}^{\bar{p}}) \to \cdots \to H_1(Q_1^{\bar{p}},Q_0^{\bar{p}}) \to H_0(Q_0^{\bar{p}})$$

and

$$H^n(Q_n^{\bar{q}},Q_{n-1}^{\bar{q}}) \leftarrow H^{n-1}(Q_{n-1}^{\bar{q}},Q_{n-2}^{\bar{q}}) \leftarrow \cdots \leftarrow H^1(Q_1^{\bar{q}},Q_0^{\bar{q}}) \leftarrow H^0(Q_0^{\bar{q}})$$

are dual to each other.

Since $\hat{X}$ is compact we have that the complexes

$$H_n(\hat{Q}_n^{\bar{p}},\hat{Q}_{n-1}^{\bar{p}}) \to H_{n-1}(\hat{Q}_{n-1}^{\bar{p}},\hat{Q}_{n-2}^{\bar{p}}) \to \cdots \to H_1(\hat{Q}_1^{\bar{p}},\hat{Q}_0^{\bar{p}}) \to H_0(\hat{Q}_0^{\bar{p}}) \qquad (3)$$

$$H^n(\hat{Q}_n^{\bar{q}},\hat{Q}_{n-1}^{\bar{q}}) \leftarrow H^{n-1}(\hat{Q}_{n-1}^{\bar{q}},\hat{Q}_{n-2}^{\bar{q}}) \leftarrow \cdots \leftarrow H^1(\hat{Q}_1^{\bar{q}},\hat{Q}_0^{\bar{q}}) \leftarrow H^0(\hat{Q}_0^{\bar{q}}) \qquad (4)$$

are also dual to each other (when regarded as Z-modules).

Since the action of the fundamental group commutes with the deformation retractions of generalized Poincaré duality as in [9], (3) and (4) are dual even when regarded as $Z\pi$-modules.

The theorems 1) and 2) now follow by the argument of duality for the torsion of dual chain complexes as in Milnor [13], Cheeger [3].

<u>Theorem 2.18</u> <u>Independence of Stratification</u>. $I\tau_W^{\bar{p}}$, $I\tau^{\bar{p}}$ are independent of the stratification used to define it.

<u>Proof</u>. Given two stratifications S_1, S_2 of X^n there is a triangulation T subordinate to both. Then we may define the chain complex $S^{\bar{p}}(T)$ and by Theorem 2.18 we have that

$$\tau(\mathcal{R}^{\bar{p}},S_1) = \tau(S^{\bar{p}}(T)) = \tau(\mathcal{R}^{\bar{p}},S_2) = I\tau^{\bar{p}}.$$

<u>Example 2.19</u>. If ε is the trivial flat line bundle R over a pseudomanifold X^n, 1.6 is applicable for $I\tau^{\bar{p}}(X,R)$ and we have

$$\ell n\, \tau^{\bar{p}} = \Sigma (-1)^{i} \ell n\, O_i \qquad (1.16)$$

where O_i is the order of the torsion subgroup of $IH_i^{\bar{p}}$. We will use formula (1.16) to calculate $I\tau^{\bar{p}}$ and to show that $I\tau^{\bar{p}}$ is distinct from the usual Reidemeister torsion in general.

Let M^n be any even dimensional orientable manifold and let $S(M^n)$ be the suspension on M^n. Then by Poincaré duality using formula (2.12) we have $\ell n\, \tau(S(M^n)) = 0$. However, $\ell n\, I\tau^{\bar{m}} \neq 0$ in general as the following example shows.

<u>$S(P^3 x P^3)$</u>

$H_0(P^3) = Z$	$H_0(P^3 x P^3) = Z$	$H_0(S(P^3 x P^3)) = Z$
$H_1(P^3) = Z_2$	$H_1(P^3 x P^3) = Z_2 \oplus Z_2$	$H_1(S(P^3 x P^3)) = Z$
$H_2(P^3) = 0$	$H_2(P^3 x P^3) = 0$	$H_2(S(P^3 x P^3)) = Z_2 \oplus Z_2$
$H_3(P^3) = Z$	$H_3(P^3 x P^3) = Z \oplus Z$	$H_3(S(P^3 x P^3)) = 0$
	$H_4(P^3 x P^3) = Z_2 \oplus Z_2$	$H_4(S(P^3 x P^3)) = Z \oplus Z$
	$H_5(P^3 x P^3) = 0$	$H_5(S(P^3 x P^3)) = Z_2 \oplus Z_2$
	$H_6(P^3 x P^3) = Z$	$H_6(S(P^3 x P^3)) = 0$
		$H_7(S(P^3 x P^3)) = Z$

$IH_0^{\bar{m}}(S(P^3 x P^3)) = H_0(S(P^3 x P^3)) = Z$

$IH_1^{\bar{m}}(S(P^3 x P^3)) = H_1(S(P^3 x P^3)) = Z$

$IH_2^{\bar{m}}(S(P^3 x P^3)) = H_2(S(P^3 x P^3)) = Z_2 \oplus Z_2$

$$IH_3^{\bar{m}}(S(P^3 \times P^3)) = H_3(S(P^3 \times P^3)) = 0$$

$$IH_4^{\bar{m}}(S(P^3 \times P^3)) = \operatorname{Im} H_4(P^3 \times P^3)) \to H_4(S(P^3 \times P^3)) = 0$$

$$IH_5^{\bar{m}}(S(P^3 \times P^3)) = H_5(P^3 \times P^3) = 0$$

$$IH_6^{\bar{m}}(S(P^3 \times P^3)) = H_6(P^3 \times P^3) = Z$$

$$IH_7^{\bar{m}}(S(P^3 \times P^3)) = H_7(P^3 \times P^3) = 0$$

$$\ln \tau(S(P^3 \times P^3)) = 0$$

$$\ln I\tau^{\bar{m}}(S(P^3 \times P^3)) = \ln 4.$$

This example shows that even when $\ln \tau = 0$, $\ln I\tau^{\bar{m}} \neq 0$; i.e. $I\tau^{\bar{m}}$ is a finer invariant than τ.

Remark 2.20. The Intersection R-torsion is not invariant under "simple homotopy equivalence." For example $C(P^3 \times P^3)$ (the cone on $P^3 \times P^3$) has the simple homotopy type of a point. However, $I\tau^{\bar{m}}(C(P^3 \times P^3)) \neq 0$.

3. Review of Analytic Torsion for Compact Riemannian Manifolds.

Let M^n be a compact orientable Riemannian manifold without boundary of dimension n. Let $\hat{M}$ be the universal covering space of M and let π be the fundamental group of M. Let $\varepsilon : \pi \to O(n)$ be an orthogonal representation of the fundamental group. Then π acts on $\hat{M} \times R^n$ as follows: Let $\sigma \in \pi$. Then $\sigma(m,x) = (\sigma(m), \varepsilon(\sigma)x)$. The quotient $\hat{M}xR^n/\pi$ will also be denoted ε and is a flat bundle over M.

Let $w = (w_1, \ldots\ldots, w_n)$ where w_i is a differential form on $\hat{M}$. For any $\sigma \in \pi$ we define $\sigma^*(w)$ by $\sigma^*(w) = (\sigma^*(w_1), \sigma^*(w_2), \ldots\ldots, \sigma^*(w_n))$. If w satisfies $\sigma^*(w) = \varepsilon(\sigma)w$ then we say that w is an ε-valued differential form or equivalently a differential form with coefficients in the flat bundle ε.

For any $w = (w_1, \ldots\ldots, w_n)$ we define $dw = (dw_1, \ldots\ldots, dw_n)$. It is easy to check that if $\sigma^*(w) = \varepsilon(\sigma)w$ then $\sigma^*(dw) = \varepsilon(\sigma)dw$ so that if w is a differential form with coefficients in ε then so is dw. We let $\Lambda^i(\varepsilon)$ denote the i-forms on M with coefficients in the flat bundle ε. Then $d : \Lambda^i(\varepsilon) \to \Lambda^{i+1}(\varepsilon)$ and we can form the de Rham cohomology group

$H^i(\varepsilon) = \dfrac{\ker d : \Lambda^i(\varepsilon) \to \Lambda^{i+1}(\varepsilon)}{\text{Image } d : \Lambda^{i-1}(\varepsilon) \to \Lambda^i(\varepsilon)}$. The de Rham cohomology with coefficients in the flat bundle is canonically isomor to the simplicial homology groups of M with coefficients in ε as described in Definition 1.6.

We define $\delta = (-1)^{n(i+1)+1} * d *$ and the Laplace operator $\Delta_i : \Lambda^i(\varepsilon) \to \Lambda^i(\varepsilon)$ by $\Delta_i = d\delta + \delta d$. The Laplace operator Δ_i is then a positive semi-definite elliptic operator with spectrum $0 \leq {}_i\lambda_0 \leq {}_i\lambda_1 \ldots\ldots \to \infty$ and corresponding eigenforms ϕ_λ. The space of harmonic forms, the kernel of Δ_i is isomorphic to $H^i(\varepsilon)$ by Hodge theory.

By the functional calculus for elliptic operators on compact manifolds we can form functions of the Laplace operator; in particular the heat operator $e^{-t\Delta}$ and the operator Δ^{-s}. $e^{-t\Delta}$ has a smooth kernel function, $E(x,y,t) = \Sigma\, e^{-\lambda_j t} \phi_j(x) \otimes \phi_j(y)$, i.e. $(e^{-t\Delta}f)(x) = \int E(x,y,t) f(y) dy$. The trace of the heat kernel $tr(E(t)) = \Sigma\, e^{-\lambda_j t}$ may be computed as $tr(E(t)) = \int_M E(x,x,t)$.

As $t \to 0$, the trace of the heat kernel $\Sigma\, e^{-\lambda_j t} \to \infty$ in a manner which can be described quite explicitly. The pointwise trace of the heat kernel has an asymptotic expansion as $t \to 0$ $E_i(x,x,t) \sim \sum_{j=0}^{\infty} {}_i a_j(x,x) t^{-n/2+j}$ where $a_j(x,x)$ are locally computable, i.e. in any coordinate system they depend in a universal way on the coefficients $g_{ij}(x)$ of the metric and a finite number of their partial derivatives. When we integrate over M we get

$$trE_i(t) = \sum_{j=0}^{N} \int_M {}_ia_j(x,x)t^{-n/2+j} + O(t^{-n/2+N+\frac{1}{2}}).$$

The trace of the heat kernel $tr(e^{-t\Delta_i})$ and the trace of the operator Δ_i^{-s}, $tr(\Delta_i^{-s})$ are related by the Mellon transform as follows:

$$\zeta_i(s) = tr(\underline{\Delta}_i^{-s}) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} tr(e^{-\underline{\Delta}_i t})dt \qquad [3.1]$$

$\underline{\Delta}_i$ denotes Δ_i projected on nonzero eigenspace. Setting $e^{-\underline{\Delta}_i t} = \sum_{\lambda_j > 0} e^{-{}_i\lambda_j t}$ and $u = {}_i\lambda_j t$, $\int_0^{\infty} t^{s-1} e^{-{}_i\lambda_j t} dt = \int_0^{\infty} \frac{u^{s-1}}{{}_i\lambda_j^{s-1}} e^{-u} \frac{du}{{}_i\lambda_j}$. $= {}_i\lambda_j^{-s}(s)$.

For $s > \frac{n}{2}$ the integral in (3.1) converges since the integral decreases faster than $Ke^{-\lambda t}$ at infinity and is bounded above by $Kt^{s-n/2-1}$ as $t \to 0$ (where λ denotes the smallest nonzero eigenvalue of $\underline{\Delta}_i$).

Let $\mu(t) = tr(E(t)) - \sum_{j=0}^{N} \int_M a_j(x,x)t^{-n/2+j}$. For $\epsilon > 0$.

Since $\mu(t) = O(t^{-n/2+N+1/2})$ the expression

$$\zeta_i(s) = \frac{1}{\Gamma(s)}[\int_{\epsilon}^{\infty} t^{s-1} tr(e^{-\underline{\Delta}_i t}) + \int_0^{\epsilon} t^{s-1} \mu_i(t) - \frac{b_i}{s} - \epsilon^s$$
$$+ \sum_{j=0}^{N} {}_ia_j \frac{\epsilon^{s-n/2+j}}{s-n/2+j} \qquad [3.2]$$

makes sense for $s > n/2 - N - 1/2$ and agrees with $\zeta_i(s)$ for $s > n/2$. One sees that $\zeta_i(s)$ has only simple poles at points $n/2 - j$ which are not nonpositive integers and the residues are the locally computable values $\frac{{}_ia_{j/2}}{\Gamma(n/2-j)}$. For $n/2 - j$

a nonpositive integer the pole inside the bracket in (3.2) is cancelled by the simple zero of $\frac{1}{\Gamma(s)}$ and $\zeta_i(s)$ has finite value at such points. $\zeta_i(0) = {}_ia_{n/2} - b_i$, n even.

Now $\zeta_i'(0)$ can be computed by differentiating under the integral sign and setting s = 0. Since $\Gamma(s)s = \Gamma(s+1)$ and $\Gamma'(1) = -\gamma$ where γ is Euler's constant

$$\zeta_i'(0) = \int_\epsilon^\infty t^{-1}\operatorname{tr}(e^{-\Delta_i t})dt + \int_0^\epsilon t^{-1}\mu_i(t) + \sum_{j\neq 1} {}_ia_j \frac{\epsilon^{-n/2+j}}{-n/2+j}$$

$$+ ({}_ia_{n/2}-b_i)\gamma + ({}_ia_{n/2}-b_i)\ln\epsilon.$$

For $s > n/2$ $\zeta_i'(s) = \sum_{\lambda_i > 0} -\ln {}_i\lambda_j \; {}_i\lambda_j^{-s}$ so that $-\zeta_i'(0)$ may be thought of as a generalization of

$$\sum_{\lambda_j>0} \ln {}_i\lambda_j \; {}_i\lambda_j^{0} = \sum_{\lambda_j>0} \ln\lambda_j = \ln\det$$

in the finite dimensional case. We then define the analytic torsion $T(M,\epsilon)$ as

$$\ln T(M,\epsilon) = \frac{1}{2}\sum_{i=0} (-1)^i\, i\,\zeta_i'(0) \qquad [3.3]$$

When M is even dimensional, orientable and $\partial M = \phi$, then $\ln T = 0$. We also have the product formula

$$\ln T(M_1\times M_2, \pi_1^*(\epsilon_1)\otimes\pi_2^*(\epsilon_2)) = \lambda(\epsilon_2)\ln T(M_1,\epsilon_1) + \lambda(\epsilon_1)\ln T(M_2,\epsilon_2)$$

In Cheeger [3], it is shown that when M^n is a closed Riemannian manifold, the analytic torsion $T(M^n, \varepsilon)$ equals the Reidemeister torsion $\tau(M^n, \varepsilon, h^*(\varepsilon))$, where $h^*(\varepsilon)$ is a basis of orthonormal harmonic forms. In particular when there are no harmonic forms $T(M^n, \varepsilon)$ is independent of the metric, i.e. a manifold invariant.

4. Analytic Torsion for manifolds with isolated conical singularity

We will begin by recalling the required parts of the theory of the Laplace operator on manifolds with isolated conical singularities as developed by J. Cheeger in [4], [5] and [6].

Let N^m be a closed Riemannian manifold of dimension m with metric g. By the cone C(N) we mean the space $(0,\infty) \times N$ equipped with the metric $dr \otimes dr + r^2 g$ where $r \in (0,\infty)$. Set

$$C_{0,u}(N) = \{(r,x) \in C(N), \quad 0 < r \leq u\} \text{ and}$$

$$\overline{C_{0,u}(N)} = \{(r,x) \in C(N), \quad 0 \leq r \leq u\}$$

Definition 4.1. X^{m+1} is called a space with conical singularities if there exists $p_j \in X^{m+1}$ $j = 1,2,\ldots\ldots,k$ such that $X^{m+1} \setminus \bigcup_{j=1}^{k} \{p_j\}$ is a smooth Riemannian manifold and each p_j has a neighborhood U_j such that $U_j \setminus \{p_j\}$ is isometric to $C_{0,u_j}(N_j^m)$ for some u_j and N_j^m.

Without loss of generality we assume that k = 1 and $u_j \geq 1$. We write $X^{m+1} = \overline{C_{0,1}(N^m)} \cup M^{m+1}$ where $N = \partial M$ and the union is along the boundary.

By difinition analysis on X^{m+1} means analysis on the smooth part $X \setminus \{p\}$. Since the manifold is incomplete the situation is different from the compact or complete case.

In particular, the Laplacian Δ_i on i-forms is not essentially self adjoint and we have to choose a self adjoint extension.

Since analysis on M^{m+1} is well understood we restrict our attention to $C_{0,1}(N^m)$. We begin by recalling the functional calculus for the Laplacian Δ_i on $C_{0,1}(N^m)$, as developed in Cheeger [4] [6]. The Laplacian on functions on $C(N^m)$ is given by

$$\Delta = \frac{-\partial^2}{\partial r^2} - \frac{m}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\tilde{\Delta}$$

where $\tilde{\Delta}$ denotes the Laplace operator on the base N^m. Let ϕ_j, μ_j denote the eigenfunctions and eigenvalues of $\tilde{\Delta}$ and set

$$\alpha = \frac{1-m}{2}$$

$$\nu_j = (\mu_j + \alpha^2)^{1/2} .$$

Let $g(r,x) \in L^2(C(N^m))$. If we restrict $g(r,x)$ to $\{r\} \times N$ then by the standard theory of the eigenfunction expansion of Δ on compact manifolds we can write

$$g(r,x) = \Sigma\, g_i(r)\phi_i(x) \qquad [4.2]$$

$\{\phi_i\}$ are the eigenfunctions of $\tilde{L} = L|_{\{r\}\times N}$ which can be identified with those of $\tilde{L} = L|_{\{1\}\times N}$ by parallel translation along the radial geodesic $R \times \{x\}$. The sum [4.2] converges in the L^2-norm.

The eigenfunctions of Δ with the property that they together with their differentials when restricted to $C_{0,1}(N^m)$ are in L^2, are just the functions

$$r^{-\frac{(m-1)}{2}} J_{\nu_j}(\lambda r)\phi_j = r^{\alpha} J_{\nu_j}(\lambda r)\phi_j .$$

The associated eigenvalue is λ^2.

If $g(r)$ is a smooth function with compact support in $(0,\infty)$ its Hankel transform $H_\nu(g)$ is defined by

$$H_\nu(g)(\lambda) = \int_0^\infty g(r) J_\nu(\lambda r) r\, dr.$$

The Hankel inversion formula states that

$$g(r) = H_\nu(H_\nu(g))(r).$$

Let $\Delta_\mu = \dfrac{-\partial^2}{\partial r^2} - \dfrac{m}{r} + \dfrac{\mu}{r^2}$

Then

$$\begin{aligned} H_\nu(r^{-\alpha}\Delta_\mu g) &= \int_0^\infty r^{-\alpha}[\Delta_\mu g] J_\nu(\lambda r) r\, dr \\ &= \int_0^\infty [\Delta_\mu g] r^{\alpha} J_\nu(\lambda r) r^m\, dr \\ &= \int_0^\infty g\ \Delta_\mu[r^{\alpha} J_\nu(\lambda r)] r^m\, dr \\ &= \lambda^2 H_\nu(r^{-\alpha} g). \end{aligned}$$

Then the map defined by

$$g(r,x) \to (H_{\nu_0}(r^{-\alpha}g_0),\ H_{\nu_1}(r^{-\alpha}g_1),\ldots\ldots)$$

provides an isometry of $L^2(C(N^m))$ with $L^2(R^+,\lambda d\lambda,\ell^2)$ wuch that Δ is carried into multiplication by λ^2. This provides the spectral representation for the Laplacian.

Now we describe the situation for i-forms on $C(N^m)$. Operations on the cross-section are denoted by a tilda. The coclosed eigenforms of $\tilde{\Delta}$ in dimension i are denoted by ϕ_j and the corresponding eigenvalues by μ_j. We set

$$\alpha(i) = \frac{1+2i-m}{2}$$

$$\nu_j(i) = \sqrt{\mu_j+\alpha^2(i)}$$

$$a_j^{\pm}(i) = \alpha(i) \pm \nu_i(i).$$

If $\theta(r,x) = \phi(r,x) + dr\wedge\omega(r,x)$ is an i-form on $C(N^m)$ and $\frac{\partial\omega}{\partial r} = \omega'$, and so on we have

$$*\theta = r^{m-2i+2}\,\tilde{*}\omega + (-1)^i r^{m-2i}\,dr\wedge\tilde{*}\phi$$

$$\delta\theta = r^{-2}\,\tilde{\delta}\phi - r^{-2}\,dr\wedge\tilde{\delta}\omega - (\omega'+(m-2i+2)r^{-1}\omega$$

$$\Delta\theta = \phi'' - (m-2i)r^{-1}\phi' + r^{-2}\tilde{\Delta}\phi - 2r^{-3}dr\wedge\tilde{\delta}\phi + dr\wedge$$

$$[-\omega''-(m-2i+2)r^{-1}\omega' + (m-2i)r^{-2}\omega + r^2\tilde{\Delta}\omega] - 2r^{-1}\tilde{d}w$$

Let $\phi_i(r,x)$ be an i-form such that for each r, $\phi_i(r,x) \in L^2(N^m)$. Let $\phi^i = {}_{\tilde{H}}\phi^i + {}_{\tilde{e}}\phi^i + {}_{\tilde{ce}}\phi^i$ denote the Hodge decomposition of ϕ. Then forms of type 1,2,3 and 4

$r^{\alpha(i)} \underset{\widetilde{ce}}{\phi^{i}}$

$r^{\alpha(i-1)}\tilde{d}\underset{\widetilde{ce}}{\phi^{i-1}} + dr \wedge (r^{\alpha(i-1)} \underset{\widetilde{ce}}{\phi^{i}})'$

$r^{\alpha(i-1)+1}(r^{-\alpha(i-1)}\tilde{d}\underset{\widetilde{ce}}{\phi^{i-1}})' + r^{\alpha(i-1)-1}dr \wedge \tilde{\delta}\tilde{d}\underset{\widetilde{ce}}{\phi^{i}}$

$r^{\alpha(i-2)+1}dr \wedge \tilde{d}\underset{\widetilde{ce}}{\phi^{i}}$.

As before the Hankel transform gives the map

$$O = \Sigma\, g_j\phi_j \to (H_{\nu_0}(r^{-\alpha(i)}g_0),\ H_{\nu_1}(r^{-\alpha(i)}g_1), \ldots\ldots)$$

which provides an isometry of L^2 type 1 forms onto $L^2(R^+, \ell^2, \lambda d\lambda)$. This isometry carries Δ into multiplication by λ^2 and thus provides a spectral representation of Δ on type 1 forms, which extends to forms of type 2, 3 and 4. We can now form functions of the Laplace operator using the Hankel inversion formula

$$f(\Delta) = (r_1r_2)^{\alpha(i)} \sum_j \int_0^\infty f(\lambda^2)J_{\nu_j(i)}(\lambda r_2)\lambda d\lambda\, \phi_j(x_1) \otimes \phi_j(x_2).$$

In particular for the functions $e^{-\Delta t}$, Δ^{-s} we can explicitly calculate the kernels by evaluating various classical integrals.

<u>Example 1</u>. The heat kernel $e^{-\Delta_i t}$ on forms of type 1 is given by

$$E_i(x_1,r_1,x_2,r_2,t) = (r_1r_2)^{\alpha(i)} \sum_j \int_0^\infty e^{-\lambda^2 t} J_{\nu_j(i)}(\lambda r_2)\lambda d\lambda\, \phi_j(x_1) \otimes \phi_j(x_2)$$

$$= (r_1r_2)^{\alpha(i)} \sum_j \frac{1}{2t} e^{-(r_1^2+r_2^2)/4t} I_{\nu_j}\left(\frac{r_1r_2}{2t}\right) \phi_j(x_1) \otimes \phi_j(x_2)$$

where I_{ν_j} is the modified Bessel function.

Example 2. $\Gamma(s)\Delta^{-s}$ is represented by the kernel function for forms of type 1

$$(r_1r_2)^{\alpha(i)} \Sigma\,\Gamma(s) \int_0^\infty \lambda^{1-2s} J_{\nu_i}(\lambda r_1) J_{\nu_j}(\lambda r_2) d\lambda\ \phi_j(x_1) \otimes \phi_j(x_2)$$

$$= \sum_j \frac{r_1^{\nu_j} r_2^{-\nu_j+2(s-1)}\ \Gamma(\nu_j-s+1)}{2^{2s-1}\Gamma(\nu_j+1)} F\left(1-s+\nu_j, s+1, \nu_j+1, \frac{r_1^2}{r^2}\right) \phi_j(x_1) \otimes \phi_j(x_2).$$

Let $\psi(x,s) = \dfrac{1}{2\sqrt{\pi}} \dfrac{\Gamma(x-s+1)}{\Gamma(x+s)} \Gamma(s-\tfrac{1}{2})$

We can write the complete trace of $\Gamma(s)\Delta^{-s}$ which is the zeta function $\Gamma(s)\zeta_i(s)$ as follows.

$$\Gamma(s)\zeta_i(s) = \sum_{j=1}^{b_i} \psi(|\alpha(i)|,s)\,dr\wedge h_j^i \wedge \tilde{*}h_j^i$$

$$+ \sum_{j=1}^{b_{i-1}} \{\psi(|\alpha(i-1)|,s) + 2[s+\alpha(i-1)]\psi(|\alpha(i-1)|,s+1)\}\, dr\wedge h_j^{i-1} \wedge \tilde{*}h_j^{i-1}$$

$$+ \sum_j \psi(\nu_j(i),s)\,dr\wedge \phi_j^i \wedge \tilde{*}\phi_j^i$$

$$+ \sum_j \{\psi(\nu_j(i-1),s) + 2[s+\alpha(i-1)]\psi(\nu_j(i-1),s+1)dr \wedge \phi_j^{i-1} \wedge \tilde{*}\phi_j^{i-1}$$

$$+ \sum_j \{\psi(\nu_j(i-1),s) + 2[s-\alpha(i-1)]\psi(\nu_j(i-1),s+1) \wedge \frac{\wedge \tilde{*} d\phi_j^{i-1}}{\sqrt{\mu_j}}$$

$$+ \sum_j \psi(\nu_j(i-2),s)dr \wedge \frac{d\phi_j^{i-2}}{\sqrt{\mu_j}} \wedge \tilde{*}$$

$\Gamma(s)\zeta_i(s)$ and $tr(E(t))$ are related as follows.

Theorem 4.2 [Cheeger [4], Section 4].

1) The pointwise relation $\int_0^\infty t^{s-1} trE_{\le b}(t) = \Gamma(s)\zeta_{\le b}(s)$ holds in the strip $\frac{1}{2} < Res < \nu_0 + 1$.

2) The pointwise relation $\int_0^\infty t^{s-1} trE_{>b}(t) = \Gamma(s)\zeta_{>b}(s)$ holds in the strip $\frac{m+1}{2} < Res < b + 1$.

3) $\Gamma(s)\zeta(s)$ has an analytic continuation to a meromorphic function in all of $\mathbb{C}$ with possible poles at

   a) $s = \frac{m+1}{2} - \frac{j}{2}$

   b) $s = \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \ldots$

   c) s such that $\nu_j - s = 0, -1, \ldots$ for some ν_j

4) For $\frac{m+1}{2} - \frac{j}{2} \neq \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}$,

$$a_{j/2}(x) = \operatorname*{Res}_{s = \frac{m+1}{2} - \frac{j}{2}} \Gamma(s)\, \zeta_{>\frac{m+1}{2}}(s)$$

5) For $\frac{m+1}{2} - \frac{j}{2} = -\frac{1}{2}, -\frac{3}{2}, \ldots$

$$a_{j/2}(x) = \operatorname*{Res}_{s = \frac{m+1}{2} - \frac{j}{2}} \Gamma(s)\zeta(s).$$

In Cheeger [4] Section 4, the analytic continuation of $\Gamma(s)\zeta_i(s)$ is calculated directly. In particular the following result is obtained

$$\operatorname*{Res}_{s=0} \psi(\nu(s)) = -\operatorname{Res} \nu .$$

As in Cheeger [4] Section 2, we demonstrate the form of the asymptotic expansion of the trace of the heat kernel on $C_{0,1}(N^m)$.

Let $E_i(r_1, x_1, r_2, x_2, t)$ denote the heat kernel on $C(N^m)$. Let $w(r,x)$ denote the volume form and let $T_k : C(N^m) \to C(N^m)$ be the homothetic transformation defined by $T_k((r,x)) = (kr,x)$. Set

$$\operatorname{tr}(E_i(r,x,r,x,t)) = f(r,x,t)w$$

where w is the volume form. Then because of the conformal homogeneity of $C(N^m)$

$$w(r,x) = r^{m+1} w(1,x) \qquad (1)$$

$$f(r,x,t) = r^{-(m+1)} f(1,x,{}^{t}/_{r^2}) \qquad (2)$$

If the pointwise asymptotic expansion of $\operatorname{tr} E_i(r,x,t)$ is given by

$$\Sigma\, \bar{a}_{j/a}(r,x)t^{-\frac{m+1}{2}+\frac{j}{2}}$$

then (1) and (2) imply that

$$\bar{a}_{j/2}(r,x) = r^{-j}\bar{a}_{j/2}(1,x).$$

In particular if $X_u = X \setminus C_{0,u}(N)$ then

$$\int_{X_u} \bar{a}_{j/2}(r,x)w = c_{j/2} + \begin{cases} \int_N \bar{a}_{j/2}(x,1)\, \frac{\beta u^{m+1-j}}{m+1-j} & j \neq m+1 \\ \int_N \bar{a}_{\frac{m+1}{2}}(x,1)\, \beta \log u & j = m+1 \end{cases} \quad (3)$$

for some constant $c_{j/2}$.

If $j \geq m+1$ the integral on the right hand side of (3) diverges as $u \to 0$. We <u>define</u> its finite part

$$\text{p.f.} \int_X \bar{a}_{j/2}(r,x)w = c_{j/2} \qquad \text{if } j \geq m+1$$

$$\text{p.f.} \int_X \bar{a}_{j/2}(r,x)w = \int_X \bar{a}_{j/2}w \qquad \text{if } j < m+1$$

Since the integral on the right hand side of (3) does not converge for $j \geq m+1$ it is clear that we cannot obtain the coefficients of the integrated trace of the heat kernel by integrating the pointwise coefficients of the pointwise trace of the heat kernel. However, instead of using the integrated trace (which does not exist) we use its finite part p.f. $a_{j/2}(x)$. This gives the following form for the asymptotic expansion of integrated trace of the heat kernel.

Theorem 4.3 [Cheeger [4], Section 2]

$$\operatorname{tr}E(t) \sim \sum_{j=0}^{k} (\text{p.f.} \int_X \bar{a}_{j/2})\, t^{-\frac{m+1}{2}+\frac{j}{2}} - \frac{1}{2} a_{\frac{m+1}{2}}(1) \log t$$

$$+ \frac{1}{2}[\int_1^\infty \int_N u^{-1} f(1,x,u)\beta \, du + \int_0^1 u^{-1} \mu_k(u) du + \sum_{j\neq m+1} \frac{a_{j/2}(1)}{-\frac{m+1}{2}+\frac{j}{2}}]$$

We are now ready to begin the study of analytic torsion for manifolds with isolated conical singularities. We begin by recalling that for compact manifolds the analytic continuation of $\zeta_i(s)$ to s=0 was obtained by using the asymptotic expansion for the trace of the heat kernel as in (3.2).

$$\zeta_i(s) = \frac{1}{\Gamma(s)}[\int_\epsilon^\infty t^{s-1} \operatorname{tr}(e^{-\Delta_i t}) + \int_0^\epsilon t^{s-1}\mu(t) - \frac{b_i \epsilon^s}{s} + \sum_{j=0}^{N} a_j \frac{\epsilon^{s-n/2+j}}{s-n/2+j}]$$

For manifolds with isolated conical singularities the asymptotic expansion of the trace if the heat kernel contains logarithemic terms, as in Theorem 4.3. This leads to a pole of $\zeta_i(s)$ at s=0 which can be seen by the following simple calculation. The log term contributes a factor $\int_0^\epsilon t^{s-1} a_{\frac{m+1}{2}} \log t$

which we calculate by integration by parts.

$u = \log t;\ \frac{dv}{dt} = t^{s-1}$ gives $\frac{t^s}{s}\log t - \int \frac{t^s}{st}dt = \frac{t^s}{s}\log t - \frac{t^s}{s^2}$.

For s=0 the pole of $\frac{t^s}{s}\log t$ is cancelled by the simple zero of $\Gamma(s)$ but since the term $\frac{t^s}{s^2}$ has a pole of order two it is not cancelled by the simple zero of $\frac{1}{\Gamma(s)}$, and the expression for $\zeta_i(s)$ given by

$$\zeta_i(s) = \frac{1}{\Gamma(s)}[\int_\epsilon^\infty t^{s-1}\mathrm{tr}(e^{-\Delta_i t}) + \int_0^\epsilon t^{s-1}\mu(t) - \frac{b_i}{s}\epsilon^s$$

$$+ \Sigma\, a_{j/2}\frac{\epsilon^{s-\frac{m+1}{2}+\frac{j}{2}}}{s-\frac{m+1}{2}+\frac{j}{2}} - a_{\frac{m+1}{2}}\frac{\epsilon^2}{s}\log t - a_{\frac{m+1}{2}}\frac{\epsilon^s}{s^2}$$

$$+ \frac{1}{2}[\int_1^\infty \int_N u^{-1}f(1,x,u)\beta\, du + \int_0^1 u^{-1}\mu_k(u)du + \sum_{j\neq m+1}\frac{a_{j/2}(1)}{-\frac{m+1}{2}+\frac{j}{2}}]\frac{\epsilon^s}{s}.$$

has a pole at s=0, so that $\zeta_i(0)$ is undefined. We now make the following observation

<u>Theorem 4.4</u> $\sum_{i=0}^{m-1} (-1)^i i\ {}_i a_{\frac{m+1}{2}} = 0$

<u>Proof</u>. We have seen that

$$\begin{aligned} {}_i a_{\frac{m+1}{2}} &= \mathrm{Res}\ \Gamma(s)\zeta_i(s) \\ &= \sum_j \mathrm{Res}\ \psi(\nu_j(i)) + 2\psi(\nu_j(i-1)) + \psi(\nu_j(i-2)) \\ &= \mathrm{Res}\ \nu(i) - 2\,\mathrm{Res}\ \nu(i-1) + \mathrm{Res}\ \nu(i-2) \end{aligned}$$

Therefore, $\sum_{i=0}^{m-1} (-1)^i i \; {}_i a_{\frac{m+1}{2}} = (-1)^i i \sum_{i=0}^{m-1} \text{Res } \nu(i) + 2 \text{ Res } \nu(i-1) + \text{Res } \nu(i-2)$

The contribution from each Res $\nu(i)$ is

$$(-1)^i i \text{ Res } \nu(i) + (-1)^{i+1}(i+1)\, 2 \text{ Res } \nu(i) + (-1)^{i+2}(i+2) \text{ Res } \nu(i)$$

$$= [(-1)^i(2i+2) + (-1)^{i+1}(2i+2)] \text{ Res } \nu(i) = 0.$$

Therefore $\sum_{i=0}^{m-1} (-1)^i i \; {}_i a_{\frac{m+1}{2}} = 0.$

Thus even though $\zeta_i(s)$ has a pole at $s=0$, the alternating sum $(-1)^i i\zeta_i(s)$ is finite because the contribution from the logarithmic term drops out. We have then

$$\sum_{i=0}^{m-1} (-1)^i i\zeta_i(s) = \frac{1}{\Gamma(s)} \sum_{i=0}^{m-1} (-1)^i i[\int_\epsilon^\infty t^{s-1} \text{tr}(e^{-\Delta_i t}) + \int_0^\epsilon t^{s-1} \mu_i(t) - \frac{b_i}{s}\epsilon^s$$
$$+ \sum a_{j/2} \frac{\epsilon^{s-\frac{m+1}{2}+\frac{j}{2}}}{s-\frac{m+1}{2}+\frac{j}{2}} + \frac{1}{2}[\int_1^\infty \int_N u^{-1} f(1,x,u)\,\beta du + \int_0^1 u^{-1}\mu_k(u)du$$
$$+ \sum_{j\neq m+1} \frac{a_{j/2}^{(1)}}{\frac{-m+1}{2}+\frac{j}{2}}] \frac{\epsilon^s}{s}]$$

which is a well defined expression.

We then have a well defined expression for the analytic torsion T on $C(N^m)$ as follows.

$$\ln T(C(N^m)) = \frac{1}{2}\sum_{i=0}^{m-1}(-1)^i i\zeta_i'(0)$$

$$\sum_{i=0}^{m-1}\frac{1}{2}(-1)^i i[\int_\epsilon^\infty t^{-1}\mathrm{tr}(e^{-\Delta_i t}) + \int_0^\epsilon t^{-1}\mu_i(t) + \sum_{j\neq\frac{m+1}{2}} {}_i a_{j/2}\frac{\epsilon^{-\frac{m+1}{2}+\frac{j}{2}}}{-\frac{m+1}{2}+\frac{j}{2}}$$

$$+ [-b_i + \frac{1}{2}\int_1^\infty\int_N u^{-1}f(1,x,u)\beta\, du + \int_0^1 u^{-1}\mu_k(u)du + \sum\frac{a_{j/2}(1)}{-\frac{m+1}{2}+\frac{j}{2}}]\gamma$$

$$+ [-b_i + \frac{1}{2}\int_1^\infty\int_N u^{-1}f(1,x,u)\beta\, du + \int_0^1 u^{-1}\mu_k du + \sum\frac{a_{j/2}(1)}{-\frac{m+1}{2}+\frac{j}{2}}]\ln\epsilon.$$

The analytic torsion for a manifold with isolated conical singularity $X^{m+1} = H^{m+1} \cup C(N^m)$ is now obtained as follows. Let Δ_i^M denote the Laplace operator on i-forms on the manifold M^{m+1} and let μ_i^M, a_i^M have the usual meaning on the manifold M. Then we define the analytic torsion $T(X^{m+1},\epsilon)$ as

$$\ln T(X^{m+1},\epsilon) = \sum_{i=0}^{m-1}\frac{1}{2}(-1)^i i[\int_\epsilon^\infty t^{-1}\mathrm{tr}(e^{-\Delta_i t}) + \int_0^\epsilon t^{-1}\mu_i(t) + \sum_{j\neq\frac{m+1}{2}} {}_i a_{j/2}\frac{\epsilon^{-\frac{m+1}{2}+\frac{j}{2}}}{-\frac{m+1}{2}+\frac{j}{2}}$$

$$+ [-b_i + \frac{1}{2}\int_\epsilon^\infty\int_N u^{-1}f(1,x,u)\beta\, du + \int_0^1 u^{-1}\mu_k(u)du + \sum\frac{a_{j/2}(1)}{-\frac{m+1}{2}+\frac{j}{2}}](\gamma+\ln\epsilon)$$

$$+ \int_\epsilon^\infty t^{-1}\mathrm{tr}(e^{-\Delta_i^M t}) + \int_0^\epsilon t^{-1}\mu_i^M(t) + \sum_{j\neq m+1} a_j^M\frac{\epsilon^{-\frac{m+1}{2}+j}}{-\frac{m+1}{2}+j} + ({}_i a^M_{\frac{m+1}{2}} - b_i^M)$$

$$(\gamma+\ln\epsilon)]$$

If $m = 2k$, and $H^k(N^{2k},R) \neq 0$, then in order to ensure self-adjointness of the Laplacian on k-forms we must choose ideal boundary conditions for L^2 k-forms as discussed in Cheeger [6], Section 1.

Then by exactly the same argument as in Ray and Singer [15], Section 2 we have the following property for the analytic torsion $T(X,\varepsilon)$ for a manifold with isolated conical singularity.

Theorem 4.5. Let $X^{m+1} = M^{m+1} \cup C(N^m)$ be a closed even dimensional manifold with isolated metrically conical singularity, and let ε be a flat orthogonal bundle over X. Then $\ln T(X^{m+1},\varepsilon) = 0$.

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