977 (2)

On the Geometry of Abelian Schemes Over Arithmetic Varieties

A Dissertation presented

by

Troels Petersen

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

at

Stony Brook

August, 1983

STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

THE GRADUATE SCHOOL

Troels Petersen

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Michio Kuga, Professor of Mathematics
Dissertation Director

Chih-Han Sah, Professor of Mathematics
Chairman of Defense

Marie-Louise Michelsohn, Associate Professor of Mathematics

Max Dresden, Professor of Physics, Inst. for Theoretical Physics
Outside Member

This dissertation is accepted by the Graduate School.

Dean of the Graduate School

August 1983

Abstract of the Dissertation

On the Geometry of Abelian Schemes
Over Arithmetic Varieties

by

Troels Petersen

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1983

We present a Riemann-Roch formula for a line bundle L on a group theoretic Abelian scheme V over an arithmetic variety U. The formula gives $\chi(V,L)$ in terms of the weights of the symplectic representation defining V, the arithmetic genus of the base U and the polarization of the fiber. We also derive a vanishing condition for $H^k(V,L)$. Two special cases, the Abelian schemes of "Satake type" and of "Addington type", are considered.

Table of Contents

ACKNOWLEDGEMENTS

To Michio Kuga I give my warmest thanks. I am grateful to Estella Shivers for beautiful typing. For helpful discussions and encouragement I thank Susan Addington, Freddie Santiago, Aparna Dar and Julio Vidaurrazaga. Finally, I thank the members of the Kuga Seminar, Min Ho Lee, Salman Abdulali, e soprattutto, Monica Petri.

"The fact that the category H is very small, does not mean that H is a bad object of study. A wider category has lesser connotations comparatively to a <u>well chosen</u> smaller category, which has richy juicy connotations. Saying the same thing in a different way, a very complexed deeper theorem which involves much structures, can not have most grandeur domain of validity. This is a predestinative dilemma."

Michio Kuga

0. Introduction

A group theoretic Abelian scheme over an arithmetic variety is a family of Abelian varieties parametrized by a compact locally symmetric space and constructed from a symplectic representation of an algebraic group G. Such a scheme V is by construction a Hodge manifold. Kodaira's embedding theorem thus implies the existence of a line bundle $L \to V$ whose sections give a projective embedding of V. This paper is concerned with counting the number of independent sections of the line bundle L. We derive a Riemann-Roch formula for $\chi(V,L) = \Sigma (-1)^k \dim H^k(V,L)$ in terms of the weights of the symplectic representation defining V, the roots of G, the arithmetic genus of the base and the polarization of the fiber. Using Kodaira's vanishing theorem we obtain a condition for the vanishing of $H^k(V,L)$ for $k > 0$. Two special cases are considered: the Abelian schemes of "Satake type", arising from a group without compact factors, and those over quaternion Hilbert modular varieties, studied by Addington.

The general idea is to use the fibering of $V \to U$ to split bundles over V into horizontal and vertical parts. In this way we push the calculations on V down to the base U, where we can apply the results of Ise, Borel and Hirzebruch on homogeneous bundles over locally symmetric spaces.

We begin in Section 1 with a brief review of the construction of group theoretic Abelian schemes according to Kuga. Section 2 presents some results of Ise on locally homogeneous bundles over a compact locally symmetric space. These are vector bundles defined from a representation of a compact Lie group by an automorphy factor, or, equivalently, by an associated principal bundle. The characteristic classes of V are approached by splitting the tangent bundle into two parts, each the pullback of a locally homogeneous bundle on U. The Borel-Hirzebruch formula, discussed in Section 3, expresses these classes in terms of the weights of the representations defining the homogeneous bundles. The technicalities on roots and weights in this section can be skipped by the less radical reader. Section 3 also describes Ise's generalization of Hirzebruch's proportionality theorem. This theorem is used in the following section to shift the evaluation of the characteristic classes of V and L from $U = \Gamma \backslash X$ to an integral over the compact dual of X, independent of the arithmetic subgroup Γ. We then consider the two special types of symplectic representations defining Abelian schemes which have been classified. Section 5 is concerned with Satake type Abelian schemes, while, in Section 6, Addington's classification is used to obtain a Riemann-Roch formula and vanishing conditions for the non-Satake type schemes arising from quaternion algebras. Section 7 contains some concluding remarks.

1. Abelian schemes over arithmetic varieties

This section is a review of the construction of group theoretic Abelian schemes formulated by Kuga [6]. We begin by assembling the ingredients.

$\mathbb{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$ whose real points form a real semisimple Lie group G. K denotes a maximal compact subgroup of G. We assume that the quotient $X = G/K$ is a Hermitian symmetric domain. Let Γ be a discrete torsion free subgroup of G acting properly discontinuously on X. The quotient $U = \Gamma\backslash X$ is then a smooth complex manifold which we assume to be compact.

Let F denote a real symplectic vector space with symplectic form F, Sp(F,A) the symplectic group of F, and

$$\mathfrak{S}(F,A) = \{J \in GL(F) : J^2 = -1,\ AJ \text{ symmetric and positive definite}\}$$

the corresponding Siegel space. $\mathfrak{S}(F,A)$ is a smooth manifold on which Sp(F,A) acts transitively by $J \mapsto g^{-1}Jg$. For a given base point $J_0 \in \mathfrak{S}(F,A)$, the stabilizer $U(J_0) \subset Sp(F,A)$ is a maximal compact subgroup so $\mathfrak{S}(F,A) \cong {}^{Sp(F,A)}/_{U(J_0)}$ is a symmetric space. The parametrization $t \mapsto \exp(\frac{1}{2}tJ_0)$ of the one-dimensional center of $U(J_0)$ specifies one of two

$Sp(F,A)$ invariant complex structures on $\mathfrak{S}(F,A)$. With this complex structure $\mathfrak{S}(F,A)$ is a Hermitian symmetric space.

Let $\Lambda \subset F$ be a lattice on which the symplectic form takes integral values.

Let $\rho : G \to Sp(F,A)$ be a symplectic representation of G. We assume that $\rho(\Gamma)\Lambda \subset \Lambda$. We also assume that ρ admits a holomorphic "Eichler" map $\varphi : X \to \mathfrak{S}(F,A)$ compatible with the action of G, i.e. $\varphi(gx) = \rho(g)\varphi(x)$ for $g \in G$, $x \in X$.

Given a collection $(G,X,\Gamma,F,A,\Lambda,\rho,\varphi)$ satisfying our assumptions, we construct a family of Abelian varities as follows. Since the action of Γ on F preserves the lattice Λ, we can form the semidirect product group $\Gamma \ltimes \Lambda$. Define an action of $\Gamma \ltimes \Lambda$ on $X \times F$ by $(\gamma,\ell)\circ(x,w) = (\gamma x, \rho(\gamma)w+\ell)$. Let V be the quotient of $X \times F$ by this action:

$$X = \Gamma\ltimes\Lambda\backslash^{X\times F} .$$

The natural projection $V = \Gamma\ltimes\Lambda\backslash^{X\times F} \to \Gamma\backslash^{X} = U$ is a fibration with typical fiber the torus $G = \Lambda\backslash F$.

To each slice $\{x\} \times F$ we assign the complex structure corresponding to $\varphi(x) \in \mathfrak{S}(F,A)$. Together with the complex structure of X, this defines a G invariant complex structure on X × F. Passing to the quotient, we have a complex structure on V for which the projection $V \xrightarrow{\pi} U$ is holomorphic.

Let ds_0^2 be the G-invariant Bergman metric on X. The induced metric on U is then Hodge. On X × F we put the metric

$$ds^2 = ds_0^2 + {}^t d\xi\, S(X)\, d\xi$$

where $\xi = (\xi_1, \ldots, \xi_{2m})$ is a coordinate system on F and $S(X) = A\phi(X) \in GL(2m, \mathbb{R})$ is symmetric and positive definite This metric is invariant under $\Gamma \ltimes \Lambda$ and so defines a metric ds^2 on V. The condition $A(\Lambda, \Lambda) \subset \mathbb{Z}$ implies that ds^2 is Hodge. The Hodge form is $\theta = \pi^* \theta_U + \theta_A$ where θ_U is the Hodge form of U and $\theta_A = \frac{1}{2}\, {}^t d\xi \wedge A d\xi$ locally.

V is therefore a projective algebraic variety, and $V \xrightarrow{\pi} U$ is a smooth fiber bundle whose fibers are Abelian varieties.

We now review an equivalent but more practical description of the family of Abelian varieties $V \to U$. This description requires another realization of the Siegel space.

Choose a symplectic basis for the vector space F, i.e.

one which represents the symplectic form as $\begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$.

For a given complex structure $J \in \mathfrak{S}(F,A)$, let Ω be the matrix relating the real symplectic basis $\{e_k\}$, $k = 1,\ldots,2m$, to the complex basis $\{e_\alpha\}$, $\alpha = 1,\ldots,m$. Then

$$e_k = \sum_{\alpha=1}^{m} \Omega_{k\alpha} e_\alpha, \quad \Omega = \begin{pmatrix} 1 \\ Z \end{pmatrix},$$

and it can be shown that the m x m complex matrix Z is symmetric with positive definite imaginary part. Letting $\mathfrak{S} = \{Z \in M_m(\mathbb{C}) : {}^tZ = Z,\ \mathrm{Im}\, z \geqslant 0\}$, we have a map $\mathfrak{S}(F,A) \to \mathfrak{S}$ which can be shown to be bijective and holomorphic.

The choice of a symplectic basis identifies F with $\mathbb{R}^{2m}$ and $Sp(F,A)$ with $Sp(2m,\mathbb{R})$. The action of $Sp(F,A)$ on $\mathfrak{S}(F,A)$ corresponds to the action of $Sp(2m,\mathbb{R})$ on $\mathfrak{S}$ given by

$$g \cdot Z = (AZ+B)(CZ+D)^{-1}, \quad g \doteq \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m,\mathbb{R}).$$

Given a symplectic representation and Eichler map

$$G \xrightarrow{\rho} Sp(F,A) \cong Sp(2m,\mathbb{R})$$

$$X \xrightarrow{\varphi} \mathfrak{S}(F,A) \cong \mathfrak{S}$$

define a map

$$\alpha : X \times F \to X \times \mathbb{C}^m : (x,w) \mapsto (x,(-1,\varphi(x))w), \quad (w \in F \cong \mathbb{R}^{2m}).$$

With respect to the complex structure on $X \times F$ defined above, α is an isomorphism of complex vector bundles. The action of $\Gamma \ltimes \Lambda$ on $X \times F$ translates to an action on $X \times \mathbb{C}^m$ given by

$$(\gamma,\ell)\cdot(x,u) = (\gamma x, {}^t(CZ+D)^{-1}u + (-1,Z_{\gamma x})\ell),$$

where $\rho(\gamma) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m,\mathbb{R})$ and $Z_x = \varphi(x) \in \mathfrak{S}$.

To see this, apply α to $(\gamma,\ell)\cdot(x,w)$:

$$\begin{aligned}\alpha((\gamma,\ell)\cdot(x,w)) &= \alpha(\gamma x, \rho(\gamma)w+\ell) \\ &= (\gamma x, (-1,\varphi(\gamma x))(\rho(\gamma)w+\ell) \\ &= (\gamma x, (-1,Z_{\gamma x})\rho(\gamma)w + (-1,Z_{\gamma x})\ell)\end{aligned}$$

Since $\quad (-1,Z_{\gamma x}) = (Z_{\gamma x},1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and $\quad {}^t(Z_{\gamma x},1) = \begin{pmatrix} Z_{\gamma x} \\ 1 \end{pmatrix} = \begin{pmatrix} (AZ_x+B)(CZ_x+D)^{-1} \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} AZ+B \\ CZ+D \end{pmatrix}(CZ+D)^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} Z_x \\ 1 \end{pmatrix}(CZ_x+D)^{-1}$$

we have

$$(-1,Z_{\gamma x})\rho(\gamma)w = {}^t(CZ_x+D)^{-1}(Z_x,1)\,{}^t\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}w$$

$$= {}^t(CZ_x+D)^{-1}(Z_x,1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}w = {}^t(CZ_x+D)^{-1}u$$

for $u = (-1,\varphi(x))w$. We can therefore define V as the quotient $\Gamma\ltimes\Lambda\backslash^{X\times\mathbb{C}^m}$.

The fibering of V over U induces a bigrading of its cohomology ring defined as follows. For each integer N the map $X \times \mathbb{C}^m \to X \times \mathbb{C}^m : (x,u) \mapsto (x,Nu)$ normalizes the action of $\Gamma \times \Lambda$ and so defines a "stretching operator" $\Theta(N)^*: V \to V$. The eigenvalues of $\Theta(N)^*$ on $H^r(V,\mathbb{Q})$ are powers N^b, $b = 0,1,2,\ldots$ $H^r(V,\mathbb{Q})$ decomposes into eigenspaces

$$H^r(V,\mathbb{Q}) \quad \bigoplus_{a+b=r} \quad H^{\langle a,b\rangle}(V,\mathbb{Q})$$

where $H^{\langle a,b\rangle}(V,\mathbb{Q})$ is the eigenspace corresponding to the eigenvalue N^b. $H^{\langle a,b\rangle}(V,\mathbb{Q})$ can also be described as the $E_2 = E_\infty$ term

$$H^a(U,H^b(G,\mathbb{Q})) = H^a(U,\wedge^b F^*_{\mathbb{Q}})$$

of the collapsing spectral sequence for the fiber bundle $G \to V \to U$.

With respect to local coordinates $x_1,\ldots,x_n$ on X and coordinates $w_1,\ldots,w_m$ on $\mathbb{C}^m$, a cohomology class in $H^{\langle a,b\rangle}(V,\mathbb{R})$ is represented by a differential form on V of "$\langle a,b\rangle$ type", locally expressable as

$$\sum_{\substack{1\le i_1<\dots<i_a\le n\\ 1\le j_1<\dots<j_b\le m}} \varphi_{i_1\dots j_b}(x,u)\; dx^{i_1}\wedge\cdot\wedge dx^{i_a}\wedge dw^{j_1}\wedge\dots\wedge dw^{j_b},$$

It can be shown that the "horizontal" part $H^{\langle r,0\rangle}(V,\mathbb{Q})$ is isomorphic via π^* to $H^r(U,\mathbb{Q})$. The "vertical" part $H^{\langle 0,r\rangle}(V,\mathbb{Q})$ is isomorphic to the subspace $H^r(G,\mathbb{Q})^{\Gamma}$ of Γ-invariant elements of the cohomology of the typical fiber G.

Finally, consider the ring $A^*(X)^G$ of G-invariant differential forms on X. Each such form is closed and so defines a closed form on U. We therefore have a homomorphism

$$A^*(X)^G \to H^*(U,\mathbb{R})$$

which can be shown to be injective. The image is a subring which we will denote by $H^*(U,\mathbb{R})_G$.

2. Locally homogeneous bundles

In this section we show how the tangent bundle of V is obtained from two locally homogeneous bundles on the base U. This will enable us to determine characteristic class of V from the representations defining these two bundles.

The tangent bundle T(V) of V contains a vertical subbundle $T(V)'$ consisting of vectors tangent to the fibers of $V \to U$. The quotient of T(V) by $T(V)'$ is isomorphic to the pullback $\pi^*T(U)$ of the tangent bundle of U.

In the previous section, V was constructed by dividing $\Gamma\ltimes\Lambda\backslash^{X\times\mathbb{C}^m}$. We can take this quotient in two steps, constructing an intermediate bundle as follows.

Let $\quad j(g,x)$ denote ${}^t(CZ+D)^{-1} \in GL(m,\mathbb{C})$,

where $\quad \rho(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m,\mathbb{R})$, $Z = \varphi(x) \in \mathfrak{S}$.

Let $E = \Gamma\backslash^{X\times\mathbb{C}^m}$, where $\gamma(x,u) = (\gamma x, j(\gamma,x)u)$. Then E is a vector bundle over $U = \Gamma\backslash X$ with fiber $\mathbb{C}^m$. Recall the map $\alpha : X \times F \to X \times \mathbb{C}^m$ defined in the last section. Since the action of Γ is assumed to preserve the lattice Λ, the set $\alpha(X\times\Lambda) = \{(x,(1,\varphi(x))\ell : x \in X,\ \ell \in \Lambda\} \subset X \times \mathbb{C}^m$ is stable under $\Gamma \ltimes \Lambda$. Let $E_\Lambda = {}_{\Gamma\ltimes\Lambda}\backslash^{\alpha(X\times\Lambda)}$. E_Λ is a bundle of lattices contained in E and ${}_{E_\Lambda}\backslash^{E} \cong V$. The fibers of $E \xrightarrow{p} U$ are the universal covering spaces of the tori which form the fibers of $V \to U$. For each such torus we can identify its universal cover with its tangent space at any point. Therefore, we can identify the pullback π^*E with the vertical tangent bundle $T(V)'$. More explicitly, under the identification $T(X\times\mathbb{C}^m) \cong T(X)\times \mathbb{C}^m \times \mathbb{C}^m$ the action of $\Gamma \ltimes \Lambda$ is

$$(\gamma,\ell)\cdot(v,u,y) = (\gamma^*v, j(\gamma,x)u + (-1,\phi(\gamma x))\ell, j(\gamma,x)y)$$

for $v \in (TX)_x$, $u,y \in \mathbb{C}^m$, and $T(V) \cong {}_{\Gamma\ltimes\Lambda}\backslash^{T(X)\times\mathbb{C}^m\times\mathbb{C}^m}$. The vertical bundle is then $T(V)' \cong {}_{\Gamma\ltimes\Lambda}\backslash^{X\times\mathbb{C}^m\times\mathbb{C}^m}$. Denote by $(x,u)_V$ and $(x,u)_E$ the images of (x,u) under the projections

$$X \times \mathbb{C}^m \to {}_{\Gamma\ltimes\Lambda}\backslash^{X\times\mathbb{C}^m} \cong V \text{ and } X \times \mathbb{C}^m \to {}_{\Gamma}\backslash^{X\times\mathbb{C}^m} \cong E.$$

Define

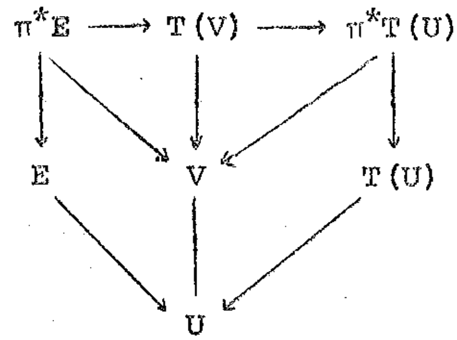
$$\beta : X \times \mathbb{C}^m \times \mathbb{C}^m \to \pi^*E = \{(v,e) \in V \times E : \pi(v) = p(e)\}$$

by

$$\beta(x,u,y) = ((x,u)_V, (x,u)_E).$$

This map is easily seen to be invariant under $\Gamma\ltimes\Lambda$ and to induce an isomorphism $T(V)' \cong \pi^*E$.

The picture is



where the top row is exact.

E is an example of the "locally homogeneous" vector bundles over a locally symmetric space studied by Ise [4]. Ise's description of such bundles is slightly different from ours in its use of the notion of an automorphy factor. We will summarize Ise's description and then verify its equivalence with ours.

First, recall the Harish-Chandra embedding of a symmetric domain $X = G/K$. The Lie algebra of $G^{\mathbb{C}}$ decomposes as $\underline{g}^{\mathbb{C}} = \underline{k}^{\mathbb{C}} \oplus \underline{m}^{\mathbb{C}} = \underline{k}^{\mathbb{C}} \oplus \underline{m}^{+} \oplus \underline{m}^{-}$, where $\underline{m}^{\pm}$ are the $\pm i$ eigenspaces of the complex structure. Let $M^{\pm}$ denote the subgroups of $G^{\mathbb{C}}$ with Lie algebras $\underline{m}^{\pm}$. The map $M^{+} \times K^{\mathbb{C}} \times M^{-} \to G^{\mathbb{C}}$ $: (n,k,m) \mapsto nkm$ is a holomorphic injection onto an open subset of $G^{\mathbb{C}}$ containing G. $GK^{\mathbb{C}}M^{-}$ is thus an open subset of $M^{+}K^{\mathbb{C}}M^{-}$. Dividing by $K^{\mathbb{C}}M^{-}$ and using the fact that $G \cap K^{\mathbb{C}}M^{-} = K$ we have an inclusion

$$\iota : X = G/K \cong GK^{\mathbb{C}}M^-/_{K^{\mathbb{C}}M^-} \hookrightarrow M^+K^{\mathbb{C}}M^-/_{K^{\mathbb{C}}M^-} \cong M^+.$$

Composing ι with $\log : M^+ \to \underline{m}^+$ gives the Harish-Chandra realization of X as a symmetric domain.

The embedding ι is G-equivalent. Therefore,

$$g \cdot \iota(x) \equiv \iota(gx)J(g,x) \bmod M^-$$

for some $J(g,x) \in K^{\mathbb{C}}$. This relation defines the "canonical automorphy factor" $J : G \times X \to K^{\mathbb{C}}$. J has the following properties:

1) For each $g \in G$ $x \mapsto J(g,x)$ is a holomorphic map $X \to K^{\mathbb{C}}$.

2) $J(gg',x) = J(g,g'x)J(g',x)$

3) $J(k,x) = k$ and $J(1,x) = 1$ for all $k \in K$, $x \in X$.

The locally homogeneous bundles discussed by Ise are defined as follows. Given a representation $\sigma : K^{\mathbb{C}} \to GL(m,\mathbb{C})$, denote the composition $\sigma \circ J : G \times X \to GL(m,\mathbb{C})$ by J_σ. As before, let $\Gamma \subset G$ be a discrete torsion free subgroup for which $U = \Gamma\backslash^X$ is a smooth compact complex manifold. Γ acts on $X \times \mathbb{C}^m$ by $\gamma(x,u) = (\gamma x, J_\sigma(\gamma,x)u)$. The quotient $E_{\sigma J} = \Gamma\backslash^{X\times\mathbb{C}^m}$ is a holomorphic vector bundle over U. The sections of $E_{\sigma J}$ are automorphic forms of "type σ" with respect to Γ.

In the case of the Siegel space $\mathfrak{S}$, $G = Sp(2m,\mathbb{R})$ and $K \cong U(m)$ is the subgroup

$$\{a\begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix}a^{-1} : g \in U(m)\}, \quad a = \begin{pmatrix} I_m & I_m \\ iI_m & -iI_m \end{pmatrix}.$$

It is easily seen that $M^+ = \{a\begin{pmatrix} I_m & b \\ 0 & I_m \end{pmatrix}a^{-1} : {}^t b = b\} \subset Sp(2m,\mathbb{C})$ and $\iota(Z) = a\begin{pmatrix} I_m & Z \\ 0 & I_m \end{pmatrix}a^{-1}$ for $Z \in \mathfrak{S}$. It follows that $J_{Sp}(g,Z) = a\begin{pmatrix} {}^t(CZ+D)^{-1} & 0 \\ 0 & (CZ+D) \end{pmatrix}a^{-1} \in Sp(2m,\mathbb{C})$ which we identify with ${}^t(CZ+D)^{-1} \in K^{\mathbb{C}} \cong GL(m,\mathbb{C})$.

The factor $j(g,x)$ used in constructing E is therefore equal to $J_{Sp}(\rho(g),\varphi(x))$.

If $\rho : G \to Sp(2m,\mathbb{R})$ is a symplectic representation, then ρ maps K to a maximal compact subgroup of $Sp(2m,\mathbb{R})$, which, by conjugating, we can assume to be

$$U(m) \cong \{a\begin{pmatrix} g & 0 \\ 0 & \bar{g} \end{pmatrix}a^{-1} : g \in U(m)\}.$$

Thus $\rho|K \sim \sigma \oplus \bar{\sigma}$, where $\sigma : K \to U(m)$ is a unitary representation of K. Let σ denote also the complex extension $K^{\mathbb{C}} \to GL(m,\mathbb{C})$. The following lemma says that $J_{Sp}(\rho(g),\varphi(x)) = \sigma J(g,x)$, so that Ise's bundle $E_{\sigma J}$ is the same as E.

<u>Lemma</u>. Let $X_i = G_i/K_i$ $(i=1,2)$ be Hermitian symmetric domains with canonical automorphy factors $J_i : G_i \times X_i \to K_i^{\mathbb{C}}$.

Let

$$G_1 \xrightarrow{\rho} G_2$$

$$X_1 \xrightarrow{\varphi} X_2$$

be compatible. Then

$$\begin{array}{ccc} G_1 \times X_1 & \xrightarrow{J_1} & K_1^{\mathbb{C}} \\ \Big\downarrow \rho\times\varphi & & \Big\downarrow \rho^{\mathbb{C}} \\ G_2 \times X_2 & \xrightarrow{J_2} & K_2^{\mathbb{C}} \end{array}$$

commutes.

Proof. Let $\iota_i : X_i \to M_i^+ \subset G_i^{\mathbb{C}}$ be the Harish-Chandra map. Now $\rho^{\mathbb{C}}(M_1^+) \subset M_2^+$ and in fact $\rho^{\mathbb{C}}$ restricted to $X_1 \subset M_1^+$ coincides with φ, i.e. $\rho^{\mathbb{C}}\iota_1(x) = \iota_2\varphi(x)$ [9-Prop. 8.1]. Applying $\rho = \rho^{\mathbb{C}}$ to the defining relation

$$g\ \iota_1(x) \equiv \iota_1(gx)J_1(g,x) \quad \text{mod } M_1^-,\ g \in G_1,\ x \in X_1$$

gives $\qquad \rho(g)\rho(\iota_1(x) \equiv \rho(\iota_1(gx))\rho(J_1(g,x)) \qquad \text{mod } M_1^-$

so $\qquad \rho(g)\iota_2(\varphi(x)) \equiv \iota_2(\varphi(gx))\rho(J_1(g,x))$

so $\qquad \rho(g)\iota_2(\varphi(x)) \equiv \iota_2(\rho(g)\varphi(x))\rho(J_1(g,x)).$

Comparing this to the relation defining $J_2(\rho(g),\varphi(x))$ shows

$$J_2(\rho(g),\varphi(x)) = \rho(J_1(g,x)).$$

Notice that the representation used here is σ and not the complexification of $\rho|K : K \to Sp(2m,\mathbb{R})$. $E_{\rho J}$ is a trivial complex symplectic vector bundle, of which $E_{\sigma J}$ is a nontrivial Lagrangian subbundle.

We now give a more geometric description of locally homogeneous bundles, which will be useful in the calculation of their characteristic classes.

The projection $G \to G/K = X$ is a principal K-bundle. Dividing by the action of Γ we have $\Gamma\backslash^G \to U$, also a principal K-bundle. Consider the complex vector bundle E_σ over U associated to $\Gamma\backslash^G \to U$ by the representation $\sigma : K \to GL(m,\mathbb{C})$.

$$E_\sigma = (\Gamma\backslash^G \times \mathbb{C}^m)/K \quad \text{where the action of K is}$$

$$(\Gamma g,u)k = (\Gamma gk,\sigma(k)^{-1}u), \quad g \in G,\ k \in K,\ \gamma \in \Gamma,\ u \in \mathbb{C}^m.$$

Lemma (Ise). E_σ and $E_{\sigma J}$ are isomorphic as C^∞ complex vector bundles.

It is easy to check that the isomorphism is induced by the smooth equivalent map

$$G \times \mathbb{C}^m \to G \times \mathbb{C}^m : (g,u) \mapsto (g,\sigma J(g,x_0)u)$$

where x_0 is the base point $K \in X$.

We have shown that the vertical tangent bundle $T(V)'$ of

V is isomorphic to the pullback of the locally homogeneous bundle over U associated to the representation σ. The horizontal tangent bundle is the pullback of the tangent bundle of U, which is also locally homogeneous. T(U) is associated to Ad $K|\underline{m}^+$, the adjoint representation of K on $\underline{m}^+$. We have therefore decomposed T(V) into two bundles, each the pullback of a locally homogeneous bundle over U.

3. Proportionality Theorem and Borel-Hirzebruch formula

We now review Ise's generalization of the Hirzebruch proportionality theorem. With this theorem we can replace the locally homogeneous bundles over U described in the last section by homogeneous bundles over the compact dual of X. We then derive the Borel-Hirzebruch formula for the Chern classes of these bundles.

Let $\check{G}$ be the compact real Lie subgroup of $G^{\mathbb{C}}$ with Lie algebra $\check{\mathfrak{g}} = \check{\mathfrak{k}} \oplus i\,\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$. Let $\check{X} = \check{G}/K$ be the compact Hermitian symmetric space dual to $X = G/K$. Let $j : X \to \check{X}$ denote the Borel embedding. We then have three principal K-bundles related by the diagram

$$\begin{array}{ccccc} \Gamma\backslash G & \longleftarrow & G & & \check{G} \\ \downarrow & & \downarrow & & \downarrow \\ U & \longleftarrow & X & \xrightarrow[j]{} & \check{X} \end{array}$$

Suppose now that $\sigma : K \to GL(m,\mathbb{C})$ is a representation. Let $\check{E}_\sigma \to \check{X}$ be the homogeneous complex vector bundle associated by σ to $\check{G} \to \check{X}$. This bundle is related to the bundle $E_\sigma \to U$ by a lemma of Ise.

Lemma (Ise). E_σ is isomorphic to $\Gamma\backslash j^*\check{E}_\sigma$.

The principal bundle $G \to G/K$ has a natural connection defined by left translation of the subspace $\underline{m} \subset \underline{g} \cong T_e G$ ($e \in G$ is the identity element). This connection is left G invariant so its curvature form is G invariant, given at the base point $x_0 = K \in G/K$ by $\Omega(v,w) = -\frac{1}{2}[v,w] \in \underline{k}$, $v,w \in \underline{m}$. The principal bundle $\Gamma\backslash G \to U$ therefore inherits a connection with curvature defined in the same way. Similarly, left translation of $i\underline{m} \subset \check{\underline{g}}$ defines an invariant connection on the principal bundle $\check{G} \to \check{X}$, with $\check{\Omega}(iv,iw) = \frac{1}{2}[v,w]$.

The Chern class of $E_\sigma \to U$, being a polynomial in the curvature Ω, lies in the "G-invariant" subring $H^*(U,\mathbb{R})_G$. Recall that $H^*(U,\mathbb{R})_G$ was defined in Section 1 as the image of the ring $A(X)^G$ of G invariant forms on X. Since any such form is determined by its restriction to $T_{x_0}(X) = \underline{m}$, this ring is isomorphic to the ring $\Lambda(\underline{m}^*)^K$ of Ad(K) invariant alternating forms on $\underline{m}$. Let $\eta = \Sigma\, \eta_k \in \Lambda(\underline{m}^*)^K$ be a form representing $c(E_\sigma)$. Since $\check{X}$ is a compact symmetric space its cohomology ring is isomorphic to $A(\check{X})^G$, which is identified with $\Lambda(i\underline{m}^*)^K$. The Chern class of $\check{E}_\sigma \to \check{X}$ is $\check{G}$-invariant and so is represented by a form $\check{\eta} = \Sigma\, \check{\eta}_k \in \Lambda(i\underline{m}^*)^K$. We have the relations

$$\eta_k(v_1,\ldots,v_{2k}) = (-1)^k\, \check{\eta}_k(iv_1,\ldots,iv_{2k}),\quad v_1,\ldots,v_{2k} \in \underline{m}.$$

Suppose that σ, μ are two representations of K. Let E_σ, E_μ be the associated vector bundles over U, $\check{E}_\sigma, \check{E}_\mu$ the associated bundles over $\check{X}$, and let $ch(E_\mu)$, $td(E_\sigma) \in H^*(U,\mathbb{R})_G$ and $ch(\check{E}_\mu)$, $td(\check{E}_\sigma) \in H^*(\check{X},\mathbb{R})$ denote their Chern characters and Todd classes. Ise has proved the following generalization of Hirzebruch's proportionality theorem.

Theorem (Ise-Hirzebruch).

$$\{ch(E_\mu)td(E_\sigma)\}[U] = \chi(U)\{ch(\check{E}_\mu)td(\check{E}_\sigma)\}[\check{X}].$$

Here $\chi(U)$ is the arithmetic genus of U; $\{\alpha\}[M]$ denotes evaluation on the fundamental class of compact m-dimensional manifold M of the degree m homogeneous part of differential form α.

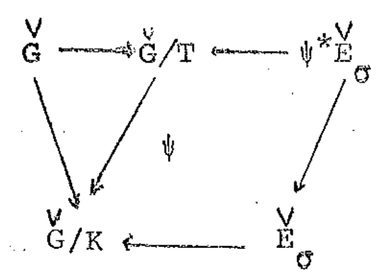
Actually, Ise stated his theorem in a slightly less general form. He considered the case when σ is the adjoint representation of K and $E_\sigma = T(U)$, $\check{E}_\sigma = T(\check{X})$. Using the Riemann-Roch formula, he stated the theorem as

$$\chi(U,E_\mu) = \chi(U)\chi(\check{X},\check{E}_\mu).$$

This result and the more general one both follows directly from the proportionality of the Chern numbers of associated bundles over U and $\check{X}$.

We now review the Borel-Hirzebruch formula expressing the Chern class of $\check{E}_\sigma$ as a polynomial in the weights of σ.

Let $T \subset K \subset \check{G}$ be a maximal torus of $\check{G}$ with Lie algebra $\underline{t}$ and let $\psi : \check{G}/T \to \check{G}/K$ be the natural map. We identify the weight lattice of T with $H^1(T,\mathbb{Z})$. (Writing $T = \underline{t}/L$, a weight $\mu \in 2\pi i L^*$ is identified with $\frac{1}{2\pi i}\mu \in L^* \cong H^1(T,\mathbb{Z})$). Let $\sigma : K \to GL(m,\mathbb{C})$ be a representation with weights $\lambda_1, \ldots, \lambda_m \in H^1(T,\mathbb{Z})$. The vector bundle over $\check{G}/T$ associated to σ/T is isomorphic to the pullback $\psi^*\check{E}_\sigma$ of the bundle $\check{E}_\sigma \to \check{G}/K$ associated to σ.



The real root decomposition $\check{\underline{g}} = \underline{t} \oplus \Sigma\, \underline{g}^\alpha = \underline{t} \oplus \underline{h}$ provides an Ad T invariant complementary subspace $\underline{h}$ to $\underline{t}$

and thus a $\check{G}$-invariant connection on the principal T-bundle $\check{G} \to \check{G}/T$. The curvature form is $\Omega_T(v,w) = -\frac{1}{2}[v,w]_{\underline{t}} \in t$, $v,w \in \underline{h}$ (the subscript denotes projection to $\underline{t}$). The curvature form of $\psi^* E_\sigma$ is therefore $\Omega_E(v,w) = -\frac{1}{2}\sigma([v,w]_{\underline{t}} \in \underline{g}(m,\mathbb{C})$. With respect to the weight space decomposition, $\sigma([v,w]_{\underline{t}})$ is a diagonal matrix whose j^{th} entry is $\lambda_j([v,w]_{\underline{t}})$.

Let τ denote the transgression $H^1(T,\mathbb{Z}) \to H^2(\check{G}/T,\mathbb{Z})$ which takes a weight λ to minus the Chern class of the associated line bundle $\check{E}_\lambda \to \check{G}/T$. $\tau(\lambda)$ is represented by the $\check{G}$-invariant 2-form $\eta_\lambda(v,w) = -\frac{1}{4\pi i}\lambda([v,w]_{\underline{t}})$ on h. We can write

$$\psi^* c(\check{E}_\sigma) = c(\psi^*\check{E}_\sigma) = \det(1 + \frac{i}{2\pi}\Omega_E) = \prod_j (1-\tau(\lambda_j)) \in H^*(\check{G}/T,\mathbb{R})$$

which

which is the formula of Borel-Hirzebruch [2].

To make sense of this formula it is useful to express the cohomologies of $\check{X}$ and $\check{G}/T$ in terms of polynomial rings. Let $\{\mu_1,\ldots,\mu_r\}$ be the basis of $\underline{t}^*$ given by the set of fundamental weights in $H^1(T,\mathbb{Z})$. The Weyl group $W(\check{G})$ acts on the polynomial ring $\mathbb{R}[\mu_1,\ldots,\mu_r]$ by permuting and changing signs of the generators. Let I_G^+ denote the ideal generated by the $W(\check{G})$ invariant polynomials of strictly positive degree. Let $\mathbb{R}[\mu_1,\ldots,\mu_r]^{W(K)}$ denote the subring of polynomials invariant under the Weyl group of K.

Proposition.

$$H^*(\check{G}/T,\mathbb{R}) \cong \mathbb{R}[\mu_1,\ldots,\mu_r]/I_G^+$$

$$H^*(\check{G}/K,\mathbb{R}) \cong \mathbb{R}[\mu_1,\ldots,\mu_r]^{W(K)}/I_G^+$$

Here the generators μ_i have degree 2. The old degree cohomology vanishes.

Outline of proof. Let H be a closed connected subgroup of $\check{G}$. Recall the Weil homomorphism

$$\mathcal{H} : S(\underline{h}^*)^H \to H^*(\check{G}/H,\mathbb{R})$$

for the principal H-bundle $\check{G} \to \check{G}/H$. Here $\underline{h}$ is the Lie algebra of H, $S(\underline{h}^*)$ is the symmetric algebra. Let Ω be the curvature of a connection on this bundle. Then $\mathcal{H}$ takes an H-invariant polynomial p to the cohomology class of the differential form on $\check{G}/H$ defined by the H-invariant horizontal closed form $p(\Omega,\ldots,\Omega)$ on $\check{G}$. If $\check{G}$ and H have equal rank then $\mathcal{H}$ is surjective. The kernel consists of the ideal generated by the polynomials on $\underline{h}$ of strictly positive degree which extend to $\check{G}$ invariant polynomials on $\check{\underline{g}}$. Let $\iota : S(\check{\underline{g}}^*)^G \to S(\underline{h}^*)^H$ be the restriction map, which is injective if $\mathrm{rk}\, H = \mathrm{rk}\, \check{G}$. The kernel of $\mathcal{H}$ is then generated by $\iota(S^+(\check{\underline{g}}^*)^{\check{G}})$. Letting $H = T$, it can be shown that the image of

$$\iota : S(\check{\underline{g}}^*)^{\check{G}} \to S(\underline{t}^*)^T = S(\underline{t}^*) \cong \mathbb{R}[\mu_1, \ldots, \mu_r]$$

is the subring $S(\underline{t}^*)^{W(G)}$ of $W(G)$-invariant polynomials on $\underline{t}$, so $\ker \mathcal{H} = I_G^+$. This gives the first formula. Similarly, taking $H = G$, we have $S(\underline{h}^*)^H \cong S(\underline{t}^*)^{W(H)}$. Now let $H = K$. Then

$$H^*(\check{G}/K, \mathbb{R}) \cong S(\underline{k}^*)^K/_{\ker \mathcal{H}} \cong S(\underline{t}^*)^{W(K)}/_{I_G^+}.$$

Returning to the Borel-Hirzebruch formula, it is clear that the map $\psi^* : H^*(\check{G}/K, \mathbb{R}) \to H^*(\check{G}/T, \mathbb{R})$ is induced from the inclusion of $\mathbb{R}[\mu_1, \ldots \mu_r]^{W(K)}$ in $\mathbb{R}[\mu_1, \ldots, \mu_r]$ and is therefore injective. The transgression τ sends the fundamental weight $\mu_j \in H^1(T, \mathbb{Z})$ to the generator $\mu_j \in H^2(\check{G}/T, \mathbb{R})$. Since the set of weights of $\sigma : K \to GL(m, \mathbb{C})$ is invariant under the Weyl group of K, the expression $\Pi(1-\tau(\lambda_j))$ is $W(K)$ invariant and so determines an element of $H^*(\check{G}/K, \mathbb{R})$.

We thus have formal expressions for the Todd class and Chern character of $\check{E}_\sigma$

$$td(\check{E}_\sigma) = \Pi\left(\frac{-\tau(\lambda_j)}{1-\exp(\tau(\lambda_j))}\right)$$

$$ch(\check{E}_\sigma) = \Sigma \exp(-\tau(\lambda_j)).$$

4. Riemann-Roch formula and vanishing theorem

V is a Hodge manifold with fundamental form $\theta = \theta_A + \pi^* \theta_U$. By the Kodaira Embedding Theorem there exists a holomorphic line bundle $L \to V$ with Chern class $C_1(L) = M\,\theta_A + N\,\pi^* \theta_U$, for suitable integers M, N, and a projective embedding of V into $P(H^0(V,L))$. In this section we begin the derivation of a Riemann-Roch formula for the line bundle L. We then use Kodaira's Vanishing Theorem to state a condition on L for the vanishing of $H^q(V,L)$, $q > 0$.

L can be written as a tensor product of two line bundles $L = L^v \otimes L^h$ with $C_1(L^v) = M\,\theta_A \in H^{\langle 0,2 \rangle}(V,\mathbb{Q})$ and $C_1(L^h) = \pi^* N\,\theta_U \in H^{\langle 2,0 \rangle}(V,\mathbb{Q})$. The "horizontal" part L^h is the pullback of a line bundle L_U on U. Recall that θ_U is the Hodge form induced from the Bergman metric on X and so belongs to the Chern class of the canonical bundle κ_U of U [5]. We can therefore take $L_U = \kappa_U^N$. L_U is a locally homogeneous bundle, associated to the character $k \mapsto \det(\mathrm{Ad}(k)^{-1}|\underline{m}^+)^N$ on K.

According to the Riemann-Roch-Hirzebruch Theorem,

$$\chi(V,L) = \{ch(L)td(V)\}[V].$$

We first decompose this expression into horizontal and vertical parts. By the multiplicative and functorial properties of the Chern character,

$$ch(L) = ch(L^v)ch(L^h) = ch(L^v)\pi^*ch(L_U).$$

From Section 2 we have the short exact sequence of bundles over V

$$0 \to \pi^*E \to T(V) \to \pi^*T(U) \to 0$$

where E is isomorphic to E_σ. Therefore,

$$td(V) = td(\pi^*E_\sigma \oplus \pi^*T(U)) = \pi^*(td(E_\sigma)td(U))$$

and so

$$ch(L)td(V) = ch(L^v)\pi^*(ch(L_U)td(E_\sigma)td(U))$$

where

$$ch(L^v) \in H^{\langle 0,*\rangle}(V,\mathbb{Q}),\ ch(L_U)td(E_\sigma)td(U) \in H^*(U,\mathbb{Q}).$$

Since $\quad H^{\langle r,0\rangle}(V,\mathbb{Q}) \cong H^r(U,\mathbb{Q}) = 0$ for $r > 2n$

and $\quad H^{\langle 0,s\rangle}(V,\mathbb{Q}) = 0$ for $s > 2m$,

$$[ch(L^v)\pi^*(ch(L_U)td(E_\sigma)td(U))]_{2(m+n)}$$

$$= [ch(L^v)]_{2m}[\pi^*(ch(L_U)td(E_\sigma)td(U))]_{2n}$$

($[\alpha]_k$ denotes the degree k homogeneous part of α.) To evaluate this form on V we integrate first along the fibers and then over U. The integral along the fibers gives the constant function

$$\{ch(L^v)\}[G] = \{\exp(M\theta_A)\}[G] = \chi(G, L^v) = \sqrt{\det M\theta_A}$$

$$= \sqrt{\det A}\ M^m$$

Using the proportionality theorem we can shift the integral over U to $\check{X}$. Let $\check{L} \to \check{X}$ be the line bundle associated to the character which defines $L_U \to U$. The tangent bundle $T(U)$ corresponds to the tangent bundle $T(\check{X})$; both are defined by the adjoint representation of K. The proportionality theorem gives

$$\{ch(L_U)td(E_\sigma)td(U)\}[U] = \chi(U)\{ch(\check{L})td(\check{E}_\sigma)td(\check{X})\}[X].$$

Therefore,

$$\chi(V, L) = \sqrt{\det A}\ M^m\ \chi(U)\{ch(\check{L})td(\check{E}_\sigma)td(\check{X})\}[\check{X}].$$

In principle, this formula combined with the Borel-Hirzebruch formula gives $\chi(V, L)$ in terms of the weights of σ, the polarization A and $\chi(U)$. The difficulty is that, except in special cases, it is impossible to carry out the integration. It is also difficult to calculate the arithmetic genus. To simplify the formula more information about the symplectic representation is needed. In the next two sections we will consider the special types of representations studied by Satake and Addington.

In order to equate $\chi(V,L)$ with the number of independent sections of L we need the vanishing theorem of Kodaira. Kodaira proved that if $L \otimes \kappa_V^*$ is a positive line bundle, then $H^q(V,L) = 0$ for $q > 0$. From Section 2 it follows that

$$\kappa_V^* = \Lambda^{2(n+m)}T(V) \cong \Lambda^{2(n+m)}\pi^*(T(U)\oplus E_\sigma)$$
$$\cong \pi^*(\Lambda^{2n}T(U) \otimes \Lambda^{2m}E_\sigma) \cong \pi^*(\kappa_U^* \otimes \Lambda^{2m}E_\sigma),$$

so

$$L \otimes \kappa_V^* \cong L^v \otimes \pi^*(L_U \otimes \kappa_U^* \otimes \Lambda^{2m}E_\sigma)$$
$$\cong L^v \otimes \pi^*(\kappa_U^{N-1} \otimes \Lambda^{2m}E_\sigma).$$

This bundle will be positive if L^v is positive as a bundle on the fiber G and $\kappa_U^{N-1} \otimes \Lambda^{2m}E_\sigma$ is positive over U. L^v is positive (for $M > 0$) since it is given by the polarization of G. The vanishing theorem therefore holds if $\kappa_U^{N-1} \otimes \Lambda^{2m}E_\sigma$ is positive. For a line bundle $\tilde{L} \to U$ associated to the weight λ, Ise has shown that $\tilde{L}$ is positive if $(\lambda,\alpha_j) < 0$ for α_j ranging over all positive complementary roots (i.e. those whose root spaces span $\underline{m}^+ \subset \underline{g}$). The weight of $\Lambda^{2m}E_\sigma$ is the sum $\sum \lambda_i$ of the weights of σ. The weight of κ_U is $-\delta_K$, where δ_K is the sum of the positive complementary roots.

Thus

$\kappa_U^{N-1} \otimes \Lambda^{2m} E_\sigma$ is positive if

$$((N-1)\delta_K - \Sigma\lambda_i, \alpha_j) > 0.$$

for all positive complementary roots α_j.

5. Schemes of Satake type

The problem of classifying group theoretical families of Abelian varieties was first studied by Satake [8]. In this section we will use his results to obtain a Riemann-Roch formula for the "Satake type" families.

For a given semisimple Lie group without compact factors, Satake considered the problem of classfying the symplectic representations which admit a holomorphic Eichler map. In other words, he considered pairs

$$G \xrightarrow{\rho} Sp(2m,\mathbb{R})$$

$$X \xrightarrow{\varphi} \mathfrak{S}$$

where $X = G/K$ is a Hermitian symmetric domain and φ is holomorphic and equivariant with respect to ρ. Satake assumed that φ commutes with the symmetries of X and $\mathfrak{S}$. Representations satisfying these conditions are called admissible and the families of Abelian varieties they define are said to be of "Satake type".

If ρ is admissible then each irreducible subrepresentation of ρ is admissible. It is therefore sufficient to consider irreducible representations. Writing G as a product of simple factors $G = G_1 x\ldots x G_s$, an irreducible representation is a tensor product $\rho = \otimes \rho_i$ where ρ_i is an irreducible representation of G_i. Satake showed that if ρ is admissible then all the factors G_i must be of classical type and all but one of the ρ_i must be trivial. The problem is therefore reduced to classifying the admissible representations of the noncompact simple classical groups of Hermitian type. Satake's results are listed along with a table of weights of the representations for the corresponding locally homogeneous bundles.

Satake's list:

Type	G	K		p
$I_{p,q}$	$SU(p,q)$	$S(U(p)\times U(q))$	$p\geq q\geq 2$	$id, \overline{id}$
			$p\geq q=1$	$\Lambda^k, \overline{\Lambda}^k$
II_n	$SU(n,n)\cap O(K,\mathbb{C})$	$U(n)$	$n\geq 5$	id
III_n	$Sp(2n,\mathbb{R})$	$U(n)$	$n\geq 1$	id
IV_p	$Spin(p,2)$	$Spin(n)\times SO(2)$	$p\geq 1$ odd	Δ
			$p\geq 4$ even	$\Delta^{\pm}$

Table of weights:

Type		weights of σ	$\dim_{\mathbb{C}}\sigma$	wts. of $\mathrm{Ad}\,K\vert\mathfrak{m}^+$
$I_{p,q}$	$p\geq q\geq 2$	$\{\lambda_1,\ldots,\lambda_p,-\lambda_{p+1},\ldots,-\lambda_{p+q}$	$(p+q)$	$\{\lambda_i-\lambda_{p+j}: 1\leq i\leq p, 1\leq j\leq q\}$
	$p\geq q=1$	$\left\{\begin{array}{l}\lambda_{i_1}+\ldots+\lambda_{i_k}, -(\lambda_{j_1}+\ldots+\lambda_{j_{k-1}}+\lambda_{p+1})\\ 1\leq i_1<\ldots<i_k\leq p, 1\leq j_1<\ldots j_{k-1}\leq p\end{array}\right\}$	$\binom{p+1}{k}$	$\{\lambda_i-\lambda_{p+i}; 1\leq i\leq p\}$
II_n	$n\geq 5$	$\{\lambda_1,\ldots,\lambda_n\}$	n	$\{\lambda_i+\lambda_j: 1\leq i<j\leq n\}$
III_n	$n\geq 1$	$\{\lambda_1,\ldots,\lambda_n\}$	n	$\{\lambda_i+\lambda_j: 1\leq i\leq j\leq n\}$
IV_p	$p=2n+1\geq 1$	$\{\frac{1}{2}(\pm\lambda_1\pm\ldots\pm\lambda_n+\lambda_{n+1})\}$	2^n	$\{\lambda_{n+1}, \lambda_i\pm\lambda_{n+1}: 1\leq i\leq n\}$
	$p=2n\geq 4$	$\left\{\frac{1}{2}(\pm\lambda_1\pm\ldots\pm\lambda_n+\lambda_{n+1})\right.$, with even (odd) no. of minus signs$\}$	2^{n-1}	$\{\lambda_i\pm\lambda_{n+1}: 1\leq i\leq n\}$

Notes on the table:

$I_{p,q}$ ($A\,III_{p,q}$). $p \geq q > 1$: The representation is "$id, \overline{id}$", by which is meant the injection

$$SU(p,q) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{\rho}{\longmapsto} \begin{pmatrix} \overline{D} & 0 & 0 & \overline{C} \\ 0 & A & B & 0 \\ 0 & C & D & 0 \\ \overline{B} & 0 & 0 & \overline{A} \end{pmatrix} \in SU(p+q, p+q).$$

$p \geq q = 1$: In this case the representation is not unique. For each k, $1 \leq k \leq p$, $\rho^{(k)}$ is defined by composing the alternating tensor representation $\Lambda^k : SU(p,1) \to SU(\binom{p}{k}, \binom{p}{k-1})$ with the map ρ given above.

II_n ($D\,III_n$). $G = SU(n,n) \cap O(K, \mathbb{C})$ when $K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$

and "id" : $G \to SU(n,n)$ is inclusion.

III_n ($C\,I_n$). $G = Sp(2n, \mathbb{R})$. The representation is the inclusion

$$Sp(2n, \mathbb{R}) \cong SU(n,n) \cap Sp(2n, \mathbb{C}) \to SU(n,n).$$

IV_p (BDI_p). $p = 2n+1$: ρ is the spin representation Δ with weights $\{\frac{1}{2}(\pm\lambda_1 \pm \ldots \pm\lambda_n \pm\lambda_{n+1})\}$. The Weyl group of $K = Spin(2n+1) \times SO(2)$ permutes and changes the signs of $\{\lambda_1, \ldots, \lambda_n\}$ and fixes λ_{n+1}. The weights of σ are therefore $\{\frac{1}{2}(\pm\lambda_1 \pm \ldots \pm\lambda_n + \lambda_{n+1})\}$. $p = 2n$: the weights of $\rho = \Delta^+$ (resp Δ^-)

are $\{\frac{1}{2}(\pm\lambda_1\pm\ldots\pm\lambda_n\pm\lambda_{n+1})$: with even (resp. odd) number of minus signs$\}$. $W(K)$ acts by permuting $\{\lambda_1,\ldots,\lambda_n\}$ and changing an even number of signs, so the weights of σ^+ (resp. σ^-) are $\{\frac{1}{2}(\pm\lambda_1\pm\ldots\pm\lambda_n+\lambda_{n+1})$: even (resp. odd) number of minus signs$\}$.

Suppose then that $V \to U$ is a Satake type family of Abelian varieties constructed from a symplectic representation and Eichler map

$$G \xrightarrow{\ \rho\ } Sp(2m,\mathbb{R})$$

$$X \xrightarrow{\ \varphi\ } \mathfrak{S}$$

such that $G = G_1 x \ldots x G_s$ is a product of simple noncompact Lie groups and φ commutes with symmetry. According to Satake, ρ has the form $\rho = \underset{\mu}{\oplus}\, \rho_\mu \circ pr_{i_\mu}$ where $pr_{i_\mu} : G \to G_{i_\mu}$ is the projection and ρ_μ is a representation of G_{i_μ} given in Satake's list.

Let $K = K_1 x \ldots x K_s \subset G$ be the maximal compact subgroup and let $X = X_1 x \ldots x X_s$ be the decomposition of X into irreducible components $X_i = {}^{G_i}/_{K_i}$. Let $\check{X} = \check{X}_1 x \ldots x \check{X}_s$ be the compact dual of X and let $p_i : \check{X} \to \check{X}_i$ be the projection. Then $\rho|K \sim \sigma \oplus \bar{\sigma}$ with $\sigma = \underset{\mu}{\oplus}\, \sigma_\mu \circ pr_{i_\mu}$.

Therefore, $\check{E}_\sigma = \bigoplus_\mu p^*_{i_\mu} \check{E}_{\sigma_\mu}$

so $\mathrm{td}(\check{E}_\sigma) = \prod_\mu p^*_{i_\mu} \mathrm{td}(\check{E}_{\sigma_\mu})$.

Let κ_i be the canonical line bundle of $\check{X}_i$. Then $\kappa = \otimes p^*_i \kappa_i$ is the canonical bundle of $\check{X}$ and so

$$\check{L} = (\otimes p^*_i \kappa_i)^N = \otimes p^*_i (\kappa_i)^N$$

so $\mathrm{ch}(\check{L}) = \prod p^*_i \mathrm{ch}(\kappa_i)^N$.

It is clear that

$$T(\check{X}) = \oplus p^*_i T(\check{X}_i) \quad \text{so} \quad \mathrm{td}(\check{X}) = \prod p^*_i \mathrm{td}(\check{X}_i).$$

Therefore,

$$\mathrm{ch}(\check{L})\mathrm{td}(\check{X})\mathrm{td}(\check{E}_\sigma) = \prod_i p^*_i(\mathrm{ch}(\kappa_i)^N \mathrm{td}(\check{X}_i)) \cdot \prod_\mu p^*_{i_\mu} \mathrm{td}(\check{E}_{\sigma_\mu})$$

$$= \prod_i p^*_i[\mathrm{ch}(\kappa_i)^N \mathrm{td}(\check{X}_i) \cdot \prod_{\mu : i_\mu = i} \mathrm{td}(\check{E}_{\sigma_\mu})]$$

where the second product is taken over those representations μ which are nontrivial on the i^{th} factor G_i. The integral over $\check{X}$ splits into a product of integrals over $\check{X}_i$ and so the Riemann-Roch formula becomes

$$\chi(V,L) = M^m\sqrt{\det A}\ \chi(U) \prod_i \{\mathrm{ch}(\kappa_i)^N \mathrm{td}(\check{X}_i) \prod_{\mu : i_\mu = i} \mathrm{td}(\check{E}_{\sigma_\mu})\}[\check{X}_i].$$

Suppose now that G has compact factors. Satake has shown that in this case, if ρ is assumed to split as $\rho = \rho_{nc} \oplus \rho_c$ with ρ_{nc} trivial on the compact part of G and ρ_c trivial on the noncompact part, then the same classification holds for ρ_{nc}. Families constructed from such representations are also known as "Satake type". The Riemann-Roch formula for these varieties is essentially the same. The representation σ splits as $\sigma = \sigma_{nc} \oplus \sigma_c$, so $\check{E}_\sigma = \check{E}_{\sigma_{nc}} \oplus \check{E}_{\sigma_c}$. Writing $G = G_{nx} \times G_c \supset K = K_{nc} \times G_c$, we have

$$\check{E}_{\sigma_c} = \check{G}_{nc} \times \check{G}_c \times \mathbb{C}^{m_c} / K_{nc} \times G_c \cong \left(\check{G}_{nc}/K_{nc}\right) \times \left(\check{G}_c \times \mathbb{C}^{m_c}\right) \Big/ \check{G}_c \cong \check{X} \times \mathbb{C}^{m_c}$$

since K_{nc} acts trivially on $\check{G}_c \times \mathbb{C}^{m_c}$ and G_c acts trivially on $\check{G}_{nc}$. Thus $\check{E}_{\sigma_c}$ is trivial. It follows that the Riemann-Roch formula depends only on ρ_{nc} (though of course, the dimension of ρ_c appears implicitly in m and A).

6. Schemes of Addington type

In this section we obtain a Riemann-Roch formula for Abelian schemes over quaternion Hilbert modular varieties. Representations defining such schemes have been classified by Susan Addington [1]. We begin by summarizing her results.

Let k be a totally real algebraic number field, $\Phi = \{\phi_1,\ldots,\phi_r\}$ the set of embeddings of k into $\mathbb{R}$ and $S = \{1,\ldots,r\}$ the set of indices of Φ. Let k' be the normal closure of k. The Galois group Gal(k':Q) acts on Φ and so on S. Let B be a quaternion algebra over k (i.e. a 4-dimensional central simple k-algebra) such that $B \not\cong M_2(k)$. For each $i \in S$, $B \otimes_{\phi_i} \mathbb{R}$ is isomorphic either to the matrix algebra $M_2(\mathbb{R})$ or to the algebra $\mathbb{H}$ of Hamiltonian quaternions. Let $S_0 \subset S$ be the subset of $i \in S$ for which $B \otimes_{\phi_i} \mathbb{R} \cong M_2(\mathbb{R})$ and let $S_1 = S - S_0$.

The group B^1 of norm one units of B defines an algebraic group over k. Let $\mathbb{G} = \mathrm{Res}_{k/\mathbb{Q}} B^1$ be the scalar restriction of this group. Then G is a semisimple algebraic group over $\mathbb{Q}$ with

$$G = G(\mathbb{R}) \cong \prod_i (B \otimes_{\phi_i} \mathbb{R})^1 \cong SL(2,\mathbb{R})^{r_0} \times SU(2)^{r_1}$$

where $r_j = |S_j|$. Let $K \subset G$ be the maximal compact subgroup

$SO(2)^{r_0} \times SU(2)^{r_1}$. The symmetric space G/K is then the product of upper-half planes $\mathfrak{h}^{r_0}$.

The representations considered by Addington are constructed as follows. For each so-called "atom" $i \in S$, let $\rho_i : G \to G_i \to SL(2,\mathbb{C})$ be the projection to the factor G_i ($=SL(2,\mathbb{R})$ or $SU(2)$) composed with inclusion in $SL(2,\mathbb{C})$. A subset $Y = \{i,j,\ldots,k\} \subset S$ (a set of atoms, hence a "molecule") defines the representation $\rho_Y = \rho_i \otimes \ldots \otimes \rho_k$ of G. A collection of molecules $P = \{Y_1,\ldots,Y_\ell\}$ (a "polymer") defines $\rho_P = \rho_{Y_1} \oplus \ldots \oplus \rho_{Y_\ell}$. A polymer is called "stable" if it is invariant under the action of $Gal(k':\mathbb{Q})$ and each molecule contains at most one atom from S_0. A stable polymer is called "rigid" if each molecule contains exactly one atom from S_0.

Addington showed that a representation of G which defines a family of Abelian varieties is equivalent to a polymer representation for some stable polymer. Conversely, for any stable polymer, some multiple (either 1, 2 or 4) of the corresponding representation will define a family of Abelian varieties. This means that there exist a nondegenerate symplectic form A on the representation space F for which $\rho(G) \subset Sp(F,A)$, a lattice $\Lambda \subset F$ with $A(\Lambda,\Lambda) \subset \mathbb{Z}$ a discontinuous subgroup $\Gamma \subset G$ such that $\Gamma\backslash\mathfrak{H}^{r_0}$ is compact and $\rho(\Gamma)\Lambda \subset \Lambda$, and a ρ-equivariant holomorphic map $\varphi : X = \mathfrak{H}^{r_0} \to \mathfrak{S}$ so that the octuple $(G,X,\Gamma,F,A,\Lambda,\rho,\varphi)$ defines a group theoretic family of Abelian varieties $V \to \mathfrak{H}^{r_0}$.

Using Addington's characterization of the representation defining $V \to \mathfrak{H}^{r_0}$, we can obtain an explicit formula for $\chi(V,L)$.

Consider first the case of $r = r_0 = 1$: $G = SL(2,\mathbb{R})$, $\check{G} = SU(2)$, $K = T = SO(2)$. $\rho : SL(2,\mathbb{R}) \to SL(2,\mathbb{C})$ is the inclusion, with weights $\pm 1 \in H^1(T,\mathbb{Z}) \cong \mathbb{Z}$, and $\rho|K \sim \sigma \oplus \bar{\sigma}$ where $\sigma : SO(2) \to U(1)$ is the natural isomorphism. $\check{E}_\sigma$ is therefore the universal bundle over $\check{G}/K = \mathbb{C}P^1$, and $c_1(\check{E}_\sigma) = -\frac{1}{2}e$, where e is the Euler class.

For the general case, let σ_j be the composition $K \to K_j \cong SO(2) \xrightarrow{\sim} U(1)$ and let e_j be the Euler class of the component $\check{X}_j = \check{G}_j/K_j \cong \mathbb{C}P^1 \subset \check{X}$.

Suppose $Y = \{j,\ldots,k\} \subset S$ is a molecule with $|Y|$ atoms and $Y \cap S_0 = \{j\}$. Then $\rho_Y = \rho_j \otimes \ldots \otimes \rho_k = \rho_j \otimes \rho_{Y-j}$, with $Y - j = Y \cap S_1$. Therefore

$$\rho_Y|K \sim (\sigma_j \oplus \overline{\sigma}_j) \otimes \rho_{Y-j}$$

so $\sigma_Y = \sigma_j \otimes \rho_{Y-j}$, and so the associated homogeneous bundle is $\check{E}_Y = \check{E}_j \otimes \check{E}_{Y-j}$, where $\check{E}_j$ is the line bundle given by σ_j and $\check{E}_{Y-j}$ is the rank $2^{|Y|-1}$ vector bundle given by ρ_{Y-j}. As remarked at the end of the last section, a representation that is trivial on the noncompact part of G produces a trivial bundle on $\check{X}$. Therefore, $\check{E}_{Y-j}$ is trivial. This implies

$$c(\check{E}_Y) = c(\check{E}_j)^{2^{|Y|-1}} = (1 - \frac{1}{2} e_j)^{2^{|Y|-1}} = 1 - 2^{|Y|-2} e_j$$

since $e_j^2 = 0$. The Todd class of $\check{E}_Y$ is therefore

$$td(\check{E}_Y) = 1 + \frac{1}{2} c_1(\check{E}_Y) = 1 - 2^{|Y|-3} e_j.$$

If Y is a molecule with $Y \cap S_0 = \phi$, then $\check{E}_Y$ is trivial, so $td(\check{E}_Y) = 1$.

Let $P = \{Y_1, \ldots, Y_\ell\}$ be a stable polymer and let $\check{E}_P \to \check{X}$ be the bundle associated to ρ_P. Then

$$td(\check{E}_P) = \prod_{k=1}^{\ell} td(\check{E}_{Y_k}) = \Pi(1-2^{|Y_k|-3} e_{k_0})$$

where $\{k_0\} = Y_k \cap S_0$ if $Y_k \cap S_0 \neq \phi$, and $e_{k_0} = 0$ if $Y_k \cap S_0 = \phi$. This expression can be rewritten as

$$\mathrm{td}(\check{E}_P) = \prod_{j\in S_0} \prod_{Y\ni j} (1-2^{|Y|-3}e_j)$$

where the second product is taken over all molecules Y in P for which $Y \cap S_0 = \{j\}$.

The other components of the formula to be calculated are the Todd class of $\check{X}$ and the Chern character of the line bundle. Since

$$T(\check{X}) = \bigoplus_{j\in S_0} p_j^* T(\check{X}_j) \quad \text{and} \quad c_1(\check{X}_j) = e_j$$

we have

$$\mathrm{td}(\check{X}) = \prod_{j\in S_0} p_j^* \mathrm{td}(\check{X}_j) = \prod p_j^*(1+\frac{1}{2}e_j).$$

Consider now the line bundle $L \to V$. We assume that L splits as $L = L^h \otimes L^v$ into a vertical part determined by the polarization A and a horizontal part which is the pullback of a line bundle L_U on U. Suppose that L_U is the locally homogeneous bundle defined by the automorphy factor

$$\prod_{j\in S_0} (c_j z_j + d_j)^{f_j},\ f_j \in \mathbb{Z},\ \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in G_j \cong SL(2,\mathbb{R}),\ z_j \in X_j \cong \mathfrak{h},$$

or, equivalently, by the weight $(-f_1,\ldots,-f_{r_0}) \in H^1(SO(2)^{r_0},\mathbb{Z}) \cong \mathbb{Z}^{r_0}$. The corresponding bundle $\check{L} \to \check{X}$ then has

$$c_1(\check{L}) = \Sigma\, \frac{1}{2} f_j e_j \quad \text{and} \quad \mathrm{ch}(\check{L}) = \exp(\Sigma\, \frac{1}{2} f_j e_j) = \prod(1+\frac{1}{2}f_j e_j).$$

Combining these calculations gives

$$\operatorname{ch}(\check{L})\operatorname{td}(\check{X})\operatorname{td}(\check{E}_p)$$

$$= \prod_{j\in S_0} [(1+\frac{1}{2}f_j e_j)(1+\frac{1}{2}e_j)\prod_{Y\ni j}(1-2^{|Y|-3}e_j)]$$

$$= \Pi[(1+\frac{1}{2}(f_j+1)e_j)(1-\sum_{Y\ni j}2^{|Y|-3}e_j)]$$

$$= \Pi[1+(\frac{1}{2}(f_j+1) - \Sigma\, 2^{|Y|-3})e_j].$$

The degree $2r_0$ homogeneous component evaluated on $\check{X}$ is

$$\Pi(\frac{1}{2}(f_j+1) - \Sigma\, 2^{|Y|-3})e_j[\check{X}_j].$$

Since $e_j[\check{X}_j] = 2$, we have

$$\Pi(f_j+1-\Sigma\, 2^{|Y|-2}).$$

Together with the calculation in Section 4, this gives

$$\chi(V,L) = \chi(U)\sqrt{\det \Lambda}\, M^m \prod_{j\in S_0}(f_j+1-\sum_{Y\ni j}2^{|Y|-2}).$$

We now consider the vanishing theorem for a line bundle over an Addington scheme. Let $\{\mu_1,\ldots,\mu_r\} \subset H^1(T,\mathbb{Z}) \cong \mathbb{Z}^r$ be the set of fundamental weights of G. The positive complementary roots are $\alpha_i = 2\mu_i$, $i = 1,\ldots,r_0$. L_U and κ_U^*

are associated to the weights

$$\sum_{j=1}^{r_0} - f_j \mu_j \quad \text{and} \quad \sum_{j=1}^{r_0} 2\mu_j \quad \text{respectively.}$$

For the molecule Y, the representation $\sigma_Y = \sigma_j \otimes \rho_{Y-j}$ has $2^{|Y|-1}$ weights, all of the form $\mu_j + \nu$, where ν is a "compact" weight (i.e. a weight of K). The sum of the weights of a polymer representation is therefore

$$\sum_{j \in S_0} \sum_{Y \ni j} 2^{|Y|-1} \mu_j + \{\text{compact weights}\}.$$

The bundle $L_U \otimes \kappa_U^* \otimes \Lambda^{2m} E_P$ is thus associated to

$$\sum_{j \in S_0} (-f_j + 2 + \sum_{Y \ni j} 2^{|Y|-1}) \mu_j + \{\text{compact weights}\}.$$

From the argument in Section 4 it follows that $L \otimes \kappa_V^*$ is positive if the innerproduct of this weight with each positive complementary root is negative. Since $(\mu_j, \alpha_i) = 2\,\delta_{ij}$, this means

$$(-f_j + 2 + \Sigma\, 2^{|Y|-1}) < 0 \text{ for each } j.$$

Thus the Kodaira vanishing theorem holds if the line bundle L is chosen so that

$$f_j > 2 + \sum_{Y \ni j} 2^{|Y|-1}, \quad j = 1, \ldots, r_0.$$

Note that this imples that each factor of the product in the Riemann-Roch formula above will be positive.

7. Concluding Remarks

We conclude with some remarks on questions raised by this work.

In order to write down explicit Riemann-Roch formulas for more general non-Satake, non-Addington type families of Abelian varieties, the representations defining such families need to be classified. The outlines of a general classification theory are visible, involving the combinatorial "chemistry" of semisimple algebras with involution.

Kodaira's theorem asserts that an embedding is given by a sufficiently large multiple of the positive line bundle corresponding to the Hodge form. The line bundle considered in Section 4 involved separate multiples, M and N, for the vertical and horizontal parts. From the theory of Abelian varieties, it is known that taking $M = 3$ is sufficient. It may be possible to determine N from work of Nakai and Moishezon.

The method of this paper depends on the fact that the line bundle splits into $L = L^V \otimes \pi^* L_U$. It is likely that in general every line bundle on V admits such a splitting, though L_U need not be locally homogeneous. This raises the problem of classifying the line bundle on V, a problem which is also relevant to the study of algebraic cycles on V.

References

1. S. Addington, Thesis, SUNY-Stony Brook, NY, 1981.

2. A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces I, Am. J. Math. 71 (1960) 472-507.

3. F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer, Berlin, 1978.

4. M. Ise, Generalized automorphic forms and certain holomorphic vector bundles, Am. J. Math. 86 (1964) 70-108.

5. K. Kodaira, On Kähler varieties of restricted type, Ann. Math. 60 (1954) 28-48.

6. M. Kuga, Fiber Varieties over a Symmetric Space whose Fibers are Abelian Varieties I, II, Lecture Notes, Univ. of Chicago, 1963-64.

7. H. Samelson, Notes on Lie Algebras, Van Nostrand, New York, 1969.

8. I. Satake, Holomorphic imbeddings of symmetric domains into a Siegel space, Am. J. Math. 87 (1965) 425-461.

9. __________, Algebraic Structures of Symmetric Domains, Publ. Math. Soc. Japan 14, Iwanami, Tokyo, and Princeton Univ. Press, 1980.