Invariant Subspaces of Shift Operators
for the Quarter Plane

A Dissertation presented

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Abstract of the Dissertation

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In this paper we decide when two shift operators on $H^2(D^2)$, the Hardy space, restricted to some invariant subspace, of finite co-dimension, are unitarily equivalent. To such pair of shift operators, there is a naturally associated hermitian holomorphic vector bundle. We use techniques of complex geometry introduced by Cowen and Douglas. Our associated hermitian holomorphic line bundle is holomorphically trivial. In finding a global holomorphic cross-section of the line bundle, we made critical use of a basis for $H^2(D^2)$, other than the usual one. Using this cross-section, the curvature of the associated line bundle was

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computed. We use a theorem of Cowen and Douglas to prove our result.
To my mother, Harbai. To my wife, Michele.
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I. **INTRODUCTION.**

In a beautiful paper [3], Beurling studied the invariant subspaces for the unilateral shift operator. He proved that a closed subspace $M$ of $H^2(\mathbb{D})$, the Hardy space, is invariant for $T_z$, multiplication by the co-ordinate function $z$ on the unit disc $\mathbb{D}$ in $\mathbb{C}$, if and only if $M = \varphi H^2(\mathbb{D})$ where $|\varphi| = 1$ a.e. on $T = \partial \mathbb{D}$, that is, $\varphi$ is an inner function. R.G. Douglas (c.f. [6]) has observed that the collection of operators obtained by restricting $T_z$ to its non-zero invariant subspaces are all unitarily equivalent to $T_z$ and has given a proof of Beurling's result based on his observation.

What are the invariant subspaces of $H^2(\mathbb{D}^2)$? Here invariant subspace means invariant under each $T_{z_i}$, multiplication by the co-ordinate function $z_i$ on the bi-disc $\mathbb{D}^2$ in $\mathbb{C}^2$, for $i = 1, 2$. The obvious generalization of Beurling's theorem for $H^2(\mathbb{D}^2)$ fails, that is, it is known (c.f. [8]) that there is an invariant subspace which is not of the form $\varphi H^2(\mathbb{D}^2)$ for any inner function $\varphi$. An explicit description of or determining the invariant subspaces of $H^2(\mathbb{D}^2)$ is, it seems, a difficult problem. However, seeking a model for the operators $T_{z_i}$ on $H^2(\mathbb{D}^2)$ restricted to its non-zero invariant subspaces may help to understand the nature of the invariant subspace. To be explicit, let $\mathfrak{A}$ be a subalgebra
of \( \mathcal{L}(\mathcal{H}) \), the algebra of bounded linear operators on a Hilbert space \( \mathcal{H} \) and let \( \text{Lat}(G) \) be the lattice of invariant subspaces for \( G \). One is interested in determining \( \mathfrak{m}(\text{Lat}(G)) \), the space of equivalent representations, that is, algebra homomorphisms from \( G \) to \( G|_M \) which maps \( T \) in \( G \) to \( T|_M \), the restriction of \( T \) to \( M \), for \( M \) in \( \text{Lat}(G) \). In this generality, it is unlikely to get a usable model for \( \mathfrak{m}(\text{Lat}(G)) \). However, for natural classes of operators, it is not unreasonable to expect a good model for restriction operators. This is evidenced by Douglas' observation of Beurling's theorems in this case for \( G = G(T_z) \), the subalgebra generated by \( T_z \) in \( \mathcal{L}(H^2(\mathbb{D})) \), the space \( \mathfrak{m}(\text{Lat}(G)) \) is given by a point.

In seeking models for the operators \( T_{z_1} \), on \( H^2(\mathbb{D}^2) \), restricted to its invariant subspace, one possibility is to consider ideals \( I \) in \( \mathcal{A}[z_1, z_2] \), the algebra of polynomials in two complex variables. If \([I]\) denotes its closures in the Hardy space \( H^2(\mathbb{D}^2) \), then \([I]\) is invariant for multiplication by \( \mathcal{P}(\mathbb{D}^2) \), the algebra of polynomials in \( \mathbb{D}^2 \), then \( \mathcal{P}(\mathbb{D}^2)|_{[I]} \) is a restriction representation of \( \mathcal{P}(\mathbb{D}^2) \). In this case ideals in \( \mathcal{A}[z_1, z_2] \) provide a model for the restriction of \( \mathcal{P}(\mathbb{D}^2) \) to some invariant subspace. However, not all restriction representation of \( \mathcal{P}(\mathbb{D}^2) \) arise from ideals. This follows from the fact that invariant subspaces
arising from ideals are all finitely generated and $H^2(D^2)$ has an invariant subspace which is not finitely generated (c.f. [8]). Which invariant subspaces arise from ideals? In this direction, Abern and Clark [1] proved: If $M$ is an invariant subspace of $H^2(D^2)$, of finite co-dimension, then there is an ideal $I$ in $\mathbb{C}[z_1, z_2]$ such that $M = [I]$. Hence for invariant subspaces of $H^2(D^2)$, of finite co-dimension, the model for the restriction representation of $\mathbb{C}(D^2)$ is given by ideals in $\mathbb{C}[z_1, z_2]$.

It is not known when different ideals give rise to inequivalent restriction representation of $\mathbb{C}(D^2)$. However, in a few cases this is known. For example, let $0 < p_1 < p_2 < \ldots < p_r$ and $0 < q_x < q_{x-1} < \ldots < q_1$ be integers, and let $A$ be a finite subset of $D^2$, and let

$$I^A_{p, q} = \{ f \in \mathbb{C}[z_1, z_2] : \frac{\partial^{1+j_0}}{\partial z_1^{1+j_0}} (\lambda) = 0 \text{ for each } \lambda \text{ in } A; \ 1 \leq p_k, j \leq q_k, 1 \leq k \leq n \}.$$

Note that $V(I^A_{p, q})$, the set of common zeros of polynomials in $I^A_{p, q}$, is equal to the set $A$. In the case when the set $A$ consists of just the origin; Berger, Coburn and Lebow [2] showed that all the restriction representations are inequivalent, that is, the representation $\mathbb{C}(D^2) \rightarrow \mathbb{C}(D^2)_{[I^A_{p, q}]}$ is unitarily equivalent to the representation $\mathbb{C}(D^2)_{[I^A_{p, q}]}$ if and only if $p_x = 0$ and $q_x = 0$. In [4]
arising from ideals are all finitely generated and $H^2(\mathbb{D}^2)$ has an invariant subspace which is not finitely generated (c.f. [8]). Which invariant subspaces arise from ideals? In this direction, Ahern and Clark [1] proved: If $M$ is an invariant subspace of $H^2(\mathbb{D}^2)$, of finite co-dimension, then there is an ideal $I$ in $\mathbb{A}[z_1, z_2]$ such that $M = \{I\}$. Hence for invariant subspaces of $H^2(\mathbb{D}^2)$, of finite co-dimension, the model for the restriction representation of $\mathbb{P}(\mathbb{D}^2)$ is given by ideals in $\mathbb{A}[z_1, z_2]$.

It is not known when different ideals give rise to inequivalent restriction representation of $\mathbb{P}(\mathbb{D}^2)$. However, in a few cases this is known. For example, let $0 \leq p_1 < p_2 < \ldots < p_r$ and $0 \leq q_r < q_{r-1} < \ldots < q_1$ be integers, and let $A$ be a finite subset of $\mathbb{D}^2$, and let

$$I^A_{p,q} = \{ f \in \mathbb{A}[z_1, z_2] : \frac{\partial^{i+j} f}{\partial z_1^i \partial z_2^j}(\lambda) = 0 \text{ for each } \lambda \text{ in } A; \quad i \leq p_k, j \leq q_k, 1 \leq k \leq n \}.$$  

Note that $V(I^A_{p,q})$, the set of common zeros of polynomials in $I^A_{p,q}$, is equal to the set $A$. In the case when the set $A$ consists of just the origin; Berger, Coburn and Lebow [2] showed that all the restriction representations are inequivalent, that is, the representation $\mathbb{P}(\mathbb{D}^2) \to \mathbb{P}(\mathbb{D}^2)_{I^A_{p,q}}$ is unitarily equivalent to the representation $\mathbb{P}(\mathbb{D}^2)_{I^A_{p,q}}$ if and only if $p_1 = \tilde{p}_i$ and $q_1 = \tilde{q}_i$. In [4]
Cowen and Douglas gave an alternate proof of this result based on their techniques of complex geometry. In this thesis we generalize this result to the case where the set A consists of one non-zero point. We prove that the representation $\mathcal{P}(\mathbb{D}^2) \to \mathcal{P}(\mathbb{D}^2)$ is unitarily equivalent to the representation $\mathcal{P}(\mathbb{D}^2) \to \mathcal{P}(\mathbb{D}^2)$ if and only if

$$\lambda = \beta, \quad p_i = \tilde{p}_i, \quad q_i = \tilde{q}_i \quad i = 1, \ldots, r$$

where $\beta$ is in $\mathbb{D}^2$, that is, all restriction representation of $\mathcal{P}(\mathbb{D}^2)$ are inequivalent. Some of our results generalize to polydisc in $\mathbb{D}^n$. We were unable to prove that the restriction representation of $\mathcal{P}(\mathbb{D}^2)$ are inequivalent if the set A contains more than one point.
CHAPTER I.

In this section we state some of the known facts we need for our purposes. Let \( \mathcal{H} \) be a separable, infinite dimensional, complex Hilbert space. Let \( \mathcal{L}(\mathcal{H}) \) denote the Banach algebra of all bounded linear operators on \( \mathcal{H} \).

**Definition 1.1:** Let \( \Omega \) be an open connected set in \( \mathbb{C}^m \), and let \( T_1, \ldots, T_m \) be operators in \( \mathcal{L}(\mathcal{H}) \). Given an integer \( n \geq 1 \), we say that \( T = (T_1, \ldots, T_m) \) is in \( \mathfrak{S}_n(\Omega) \) if the following conditions are satisfied:

1. \( \{T_i\}_{i=1}^m \) are pairwise commuting.
2. ran \( D_{T-\lambda} \) is closed for \( \lambda \) in \( \Omega \) where \( D_T : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) defined by \( m \)-times
   \[ D_T x = T_1 x \oplus \cdots \oplus T_m x. \]
3. \( \text{span}(\ker D_{T-\lambda} : \lambda \text{ is in } \Omega) \) is dense in \( \mathcal{H} \).
4. \( \dim \ker D_{T-\lambda} = n \) for all \( \lambda \) in \( \Omega \).

The class \( \mathfrak{S}_n(\Omega) \) for \( m = 1 \) was introduced and studied by Cowen and Douglas in [4] and for \( m \geq 2 \) by the same authors in a subsequent paper [5], and more recently by Curto and Salinas in [7].

**Definition 1.2:** Let \( \Omega \) be a complex manifold and let \( n \) be an integer \( \geq 1 \). A holomorphic vector bundle of rank \( n \) consists
of a complex manifold \( E \) with a holomorphic map \( \pi \) from \( E \) onto \( \Omega \) such that each fibre \( E_\lambda = \pi^{-1}(\lambda) \) is isomorphic to \( \mathbb{C}^n \) and such that for each \( \lambda_0 \) in \( \Omega \) there is an open set \( U \) containing \( \lambda_0 \) and holomorphic functions \( s_1, \ldots, s_n \) from \( U \) to \( E \) such that \( \{s_1(\lambda), \ldots, s_n(\lambda)\} \) forms a basis for \( E_\lambda \) for all \( \lambda \) in \( U \). A holomorphic cross-section of \( E \) is a holomorphic map \( s : \Omega \to E \) such that \( s(\lambda) \) is in \( E_\lambda \) for each \( \lambda \) in \( \Omega \). For \( T = (T_1, \ldots, T_m) \) in \( \mathfrak{s}_n(\Omega) \), let \((E_T, \pi)\) denote the subbundle of the trivial bundle \( \Omega \times \mathbb{H} \) defined by \( E_T = \{ (\lambda, x) \in \Omega \times \mathbb{H} : x \in \text{Ker } D_{T-\lambda} \}, \pi(\lambda, x) = \lambda \). That \( E_T \) is a holomorphic vector bundle of rank \( n \) follows from the following:

**Lemma 1.3:** Let \( \Omega \subset \mathbb{C}^m \) be an open connected set and let \( \mathbb{H}_1, \mathbb{H}_2 \) be Hilbert spaces. Let \( X : \Omega \to \mathcal{B}(\mathbb{H}_1, \mathbb{H}_2) \) be holomorphic, that is, it can be defined locally by a power series, with coefficients in \( \mathcal{B}(\mathbb{H}_1, \mathbb{H}_2) \), which converges in norm. Let \( \lambda_0 \in \Omega \) be such that \( \text{ran } X(\lambda_0) \) is closed and \( \dim \text{Ker } X(\lambda) = n \) for \( \lambda \) near \( \lambda_0 \). Then there exist holomorphic \( \mathbb{H}_1 \)-valued functions \( s_1, \ldots, s_n \) defined in some neighborhood \( \Omega_0 \) of \( \lambda_0 \) such that \( \{s_1(\lambda), \ldots, s_n(\lambda)\} \) forms a basis for \( \text{Ker } X(\lambda) \) for each \( \lambda \) in \( \Omega_0 \).

**Proof:** See Cowen and Douglas [5], page 16 or Curto and Salinas [7], page 8.
In order to study simultaneous unitary equivalence we need some more notions from complex geometry.

**Definition 1.4:** A hermitian holomorphic vector bundle $E$ over $\Omega$ is a holomorphic vector bundle such that each fibre $E_\lambda$ is an inner product space. The bundle is said to have smooth (real analytic) metric if $\lambda \rightarrow \|s(\lambda)\|^2$ is smooth (real analytic) for each holomorphic cross-section of $E$.

1.5: Let $E$ be a hermitian holomorphic vector bundle over $\Omega$. A connection on $E$ is a first order differential operator $D : \mathcal{E}(\Omega, E) \rightarrow \mathcal{E}^1(\Omega, E)$ such that $D(f\sigma) = df \otimes \sigma + f D\sigma$ for $f$ in $\mathcal{E}(\Omega)$ and $\sigma$ in $\mathcal{E}(\Omega, E)$, where $\mathcal{E}(\Omega)$ denotes the algebra of complex valued $C^\infty$-functions on $\Omega$ and $\mathcal{E}^p(\Omega, E)$ denotes the spaces of smooth differential $p$-forms with coefficients in $E$, that is, $\mathcal{E}^p(\Omega, E) = \mathcal{E}(\Omega, \wedge^p T^*(\Omega) \otimes E)$. Now given a connection $D$ on a hermitian holomorphic vector bundle $E$ over $\Omega$, we define an operator $D : \mathcal{E}^p(\Omega, E) \rightarrow \mathcal{E}^{p+1}(\Omega, E)$ by using Leibnitz’s rule

$$D(f \otimes \sigma) = df \otimes \sigma + (-1)^p f \wedge D\sigma$$

for $f$ in $\mathcal{E}^p(\Omega) = \mathcal{E}(\Omega, \wedge^p T^*(\Omega))$, a $p$-form on $\Omega$ and $\sigma$ in $\mathcal{E}(\Omega, E)$. An easy calculation shows that $D^2(f\sigma) = f(D^2\sigma)$ for $f$ in $\mathcal{E}(\Omega)$ and $\sigma$ in $\mathcal{E}(\Omega, E)$.
Thus \( D^2 \) is a bundle map from \( E \) to \( \wedge^2 T^* (\Omega) \otimes E \) and we define the curvature \( K(E,D) = K \) as the \( C^\infty \)-section of
\[ \text{Hom}(E, \wedge^2 T^* (\Omega) \otimes E) \] by \( K = K(E,D) = D^2 \).

For more complete treatment see Wells [9].

How is simultaneous unitary equivalence between two \( m \)-tuples of operators \( T = (T_1, \ldots, T_m) \) and \( \hat{T} = (\hat{T}_1, \ldots, \hat{T}_m) \) in \( \mathcal{B}_n(\Omega) \) related to the associated hermitian holomorphic vector bundle \( E_T \) and \( \hat{E}_T \)? The relation is given by the following:

**Proposition 1.6:** Let \( T = (T_1, \ldots, T_m) \) and \( \hat{T} = (\hat{T}_1, \ldots, \hat{T}_m) \) be in \( \mathcal{B}_n(\Omega) \). Then \( T \) and \( \hat{T} \) are simultaneously unitarily equivalent if and only if \( E_T \) and \( \hat{E}_T \) are holomorphically and isometrically equivalent, that is, there exists an isometric holomorphic bundle map from \( E_T \) onto \( \hat{E}_T \).

**Proof:** See Cowen and Douglas [5], page 16.

For operators in \( \mathcal{B}_1(\Omega) \), the simultaneous unitary equivalence is related to the curvature of the associated line bundles as the following proposition shows.

**Proposition 1.7:** Let \( T = (T_1, \ldots, T_m) \) and \( \hat{T} = (\hat{T}_1, \ldots, \hat{T}_m) \) be in \( \mathcal{B}_1(\Omega) \). Then \( T \) and \( \hat{T} \) are simultaneously unitarily equivalent if and only if the curvatures of the associated line bundles are equal.

**Proof:** See Cowen and Douglas [5], page 16–17.
CHAPTER II.

In this section we state and prove our main result.

2.1. Let \( D^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_i| < 1 \ i = 1, 2\} \) be the bi-disc in \( \mathbb{C}^2 \). We let \( H^2(D^2) \) denote the class of holomorphic functions on \( D^2 \) which satisfy the following condition:

\[
\sup_{0 < r < 1} \int_{\mathbb{T}^2} |f_r|^2 \, dm_2 < \infty \quad \text{where } \mathbb{T}^2 \text{ is the distinguished boundary}
\]

of \( D^2 \), \( dm_2 \) is the normalized Lebesgue measure on \( \mathbb{T}^2 \) and \( f_r(z) = f(rz_1, rz_2) \) for \( z = (z_1, z_2) \in \mathbb{T}^2 \).

**Proposition 2.2:** For \( f \) in \( H^2(D^2) \), \( f^*(z) = \lim_{r \to 1} f_r(z) \) exists a.e. on \( \mathbb{T}^2 \) and the following are true:

(a) \( f^* \) is in \( L^2(\mathbb{T}^2) \) and \( f_r \to f \) in \( L^2(\mathbb{T}^2) \)

(b) If \( f(z) = \sum_{m,n \geq 0} c_{mn} z_1^m z_2^n \) is the Taylor expansion of \( f \) in \( H^2(D^2) \) and

\[
f^*(e^{i\theta_1}, e^{i\theta_2}) = \sum_{m,n \in \mathbb{Z}^2} a_{mn} e^{im\theta_1} e^{in\theta_2}
\]

is the Fourier expansion of \( f^* \) in \( L^2(\mathbb{T}^2) \) then

\[
c_{mn} = a_{mn} \quad \text{for } m,n \geq 0 \text{ and } a_{m,n} = 0 \quad \text{otherwise}.
\]

**Proof:** See Rudin [8].
Definition 2.3: Let

\[ H^2(\mathbb{D}^2) = \{ f \in L^2(\mathbb{D}^2) \} \]

\[ : a_{m,n} = \frac{1}{(2\pi)^2} \int_{\mathbb{D}^2} \int_{\mathbb{D}^2} f(e^{i\theta_1}, e^{i\theta_2}) e^{-im\theta_1 - in\theta_2} \, dm \]

\[ = 0 \text{ for } m < 0 \text{ or } n < 0. \]

Note that \( H^2(\mathbb{D}^2) \) is a closed subspace of \( L^2(\mathbb{D}^2) \) and hence a Hilbert space.

Proposition 2.4: The map from \( H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2) \) given by \( f \rightarrow f^* \) is an isometrical onto isomorphism.

Proof: See Rudin [8].

Under this identification we treat \( H^2(\mathbb{D}^2) \) as a closed subspace of \( L^2(\mathbb{D}^2) \). For more detailed study of these concepts see Rudin [8].

Let \( 0 \leq p_1 < p_2 < \ldots < p_r \) and \( 0 \leq q_r < q_{r-1} < \ldots < q_1 \) be integers and let \( \lambda = (\lambda_1, \lambda_2) \) be a point in \( \mathbb{D}^2 \).

Definition 2.5: We denote \( m_{\lambda}^{(p,q)} = \{ f \in H^2(\mathbb{D}^2) : \frac{\partial^{p+q} f}{\partial z_1^{p} \partial z_2^{q}}(\lambda) = 0 \} \]

for \( 1 \leq p_k, j \leq q_k \)

all \( k, 1 \leq k \leq r \)

Observe that \( m_{\lambda}^{(p,q)} \) is a closed subspace of \( H^2(\mathbb{D}^2) \).
Definition 2.6: We define $S_i$ on $m^\lambda_{p,q}$ by

$$S_i f = P \left( \overline{z}_i f \right), \quad i = 1,2$$

for $f$ in $m^\lambda_{p,q}$, where $P(p,q)$ is the orthogonal projection on $H^2(D^2)$ onto $m^\lambda_{p,q}$ and $z_1, z_2$ are independent variables.

Note that $S_1$ and $S_2$ are bounded linear operators on $m^\lambda_{p,q}$ and depend not only on the point $\lambda$ but also on $p$'s and $q$'s.

2.7: Let $\mathcal{H}$ be a functional Hilbert space, that is, $\mathcal{H}$ is Hilbert space of complex valued functions on a non-empty set $X$ such that the evaluation map $f \to f(y)$ is a bounded linear functional for each $y$ in $X$. Consequently, by the Riesz Representation Theorem, there exists, for each $y$ in $X$, an element $K_y$ in $\mathcal{H}$ such that $f(y) = \langle f, K_y \rangle$, where $\langle \rangle$ denotes the inner product in $\mathcal{H}$. The function $K$ on $X \times X$ defined by $K(x,y) = K_y(x)$ is called the kernel function for $\mathcal{H}$.

Observe that $H^2(D^2)$ is a functional Hilbert space. Its kernel function is given by $K_w(z) = \frac{1}{(1 - \overline{w}_1 z_1)(1 - \overline{w}_2 z_2)}$ for each $w \in D^2$. $P_{m^\lambda_{p,q}}K_w$ is the kernel function for $m^\lambda_{p,q}$ as can be seen quite easily.

In order to study the pair $(S_1, S_2)$, we require a basis for $H^2(D^2)$ other than the usual one.
Proposition 2.8: For \( \lambda \) in \( \mathbb{D} \), the unit disc in \( \mathbb{C} \), the functions defined by \( e_m(z) = \frac{\sqrt{1-|\lambda|^2}(z-\lambda)^m}{(1-\lambda z)^{m+1}} \) form a complete orthonormal basis for \( H^2(\mathbb{D}) \).

Proof: Suppose \( m > n \). Then

\[
\langle e_m, e_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left( e^{i\theta} - \frac{\lambda}{1-\lambda e^{i\theta}} \right)^m \left( e^{i\theta} - \frac{\lambda}{1-\lambda e^{i\theta}} \right)^n \frac{1-|\lambda|^2}{1-\lambda e^{i\theta}} \frac{1}{1-\lambda e^{-i\theta}} \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( e^{i\theta} - \frac{\lambda}{1-\lambda e^{i\theta}} \right)^{m-n} \frac{1}{1-\lambda e^{i\theta}} \, d\theta
\]

\[
= \langle f, K_\lambda \rangle = f(\lambda) = 0
\]

where

\[
f(z) = \frac{(z-\lambda)^{m-n}(1-|\lambda|^2)}{1-\lambda z} \text{ is in } H^2(\mathbb{D}).
\]

This shows that \( \{e_m\} \) is an orthogonal family. Now

\[
\|e_m\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-\lambda e^{i\theta}} \left| \frac{e^{i\theta} - \lambda}{1-\lambda e^{i\theta}} \right|^2 \, d\theta = 1 \text{ since } \left| \frac{e^{i\theta} - \lambda}{1-\lambda e^{i\theta}} \right| = 1
\]

and \( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-\lambda e^{i\theta}} \, d\theta = \frac{1}{1-|\lambda|^2} \) where \( K_\lambda \) is the kernel function for \( H^2(\mathbb{D}) \) defined by \( K_\lambda(z) = \frac{1}{1-\lambda z} \). This shows that \( \{e_m\} \) is an orthonormal family. It remains to show that this family is complete, that is, if \( \langle f, e_m \rangle = 0 \) for all \( m \geq 0 \) then \( f = 0 \). To show this we prove that such an \( f \) has a zero of infinite order at \( \lambda \); and since \( f \) is holo-
morphic on the open connected set $\mathbb{D}$, $f$ is identically equal to zero. Now we claim that if $\langle f, e_j \rangle = 0$ for $j = 0, 1, \ldots, n$ then $f$ has a zero, of order at least $n+1$, at $\lambda$. We use induction. This is obviously true for $n = 0$ since $\langle f, e_0 \rangle = 0$, then $0 = \langle f, e_0 \rangle = \sqrt{1-|\lambda|^2} f(\lambda)$ and hence $f$ has a zero, of order $\geq 1$, at $\lambda$. Assume $\langle f, e_j \rangle = 0$ for $j = 0, \ldots, n$, then $f$ has a zero, of order $\geq n+1$, at $\lambda$. Suppose $\langle f, e_j \rangle = 0$ for $j = 0, 1, \ldots, n+1$.

Then

$$0 = \langle f, e_j \rangle = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) (1-\bar{\lambda}z)^j}{(1-\lambda z)^{j+1}} \frac{d\theta}{(\bar{\lambda} - \lambda z)^j},$$

for $j = 0, 1, \ldots, n+1$.

$$= \frac{(1-|\lambda|^2)}{2\pi i} \int_{\gamma} \frac{f(z) (1-\bar{\lambda}z)^j}{(1-\lambda z)^{j+1}} \frac{dz}{(\bar{\lambda} - \lambda z)^j} = \frac{(1-|\lambda|^2)}{j!} g(j)(\lambda)$$

by the Cauchy integral formula,

for $j = 0, 1, \ldots, n+1$, where

$$g(z) = f(z) (1-\bar{\lambda}z)^j$$

$$= \frac{(1-|\lambda|^2)}{j!} \sum_{k=0}^{j} \binom{j}{k} f(k)(\lambda) h(j-k)(\lambda)$$

by Leibnitz's rule, where

$$h_j(z) = (1-\bar{\lambda}z)^j$$ for $j = 0, 1, \ldots, n+1$.

But by the induction hypothesis, $f$ has a zero, of order $\geq n+1$, at $\lambda$, that is, $f^{(k)}(\lambda) = 0$ for $k = 0, 1, \ldots, n$. 
morphic on the open connected set $\mathbb{D}$, $f$ is identically equal to zero. Now we claim that if $\langle f,e_j \rangle = 0$ for $j = 0,1,...,n$ then $f$ has a zero, of order at least $n+1$, at $\lambda$. We use induction. This is obviously true for $n = 0$ since $\langle f,e_0 \rangle = 0$, then $0 = \langle f,e_0 \rangle = \sqrt{1-|\lambda|^2} f(\lambda)$ and hence $f$ has a zero, of order $\geq 1$, at $\lambda$. Assume $\langle f,e_j \rangle = 0$ for $j = 0,...,n$, then $f$ has a zero, of order $\geq n+1$, at $\lambda$. Suppose $\langle f,e_j \rangle = 0$ for $j = 0,1,...,n+1$.

Then

$$0 = \langle f,e_j \rangle = \frac{1}{2\pi i} \int_0^{2\pi} T(e^{i\theta})(\frac{e^{i\theta}-\lambda}{1-\lambda e^{i\theta}})^j \frac{1}{1-\lambda e^{i\theta}} \, d\theta$$

$$j = 0,1,...,n+1$$

$$= \frac{(1-|\lambda|^2)^j}{2\pi i} \int T' \frac{f(z)(1-\overline{z})^j}{(z-\lambda)^{j+1}} \, dz = \sqrt{1-|\lambda|^2} g(j)(\lambda)$$

by the Cauchy integral formula, for $j = 0,1,...,n+1$, where

$$g(z) = f(z)(1-\overline{z})^j$$

$$= \sqrt{1-|\lambda|^2} \sum_{k=0}^j \binom{j}{k} f(k)(\lambda) h_{j-k}(\lambda)$$

by Leibnitz's rule, where

$$h_j(z) = (1-\overline{z})^j$$

for $j = 0,1,...,n+1$.

But by the induction hypothesis, $f$ has a zero, of order $\geq n+1$, at $\lambda$, that is, $f^{(k)}(\lambda) = 0$ for $k = 0,1,...,n$. 
Hence \[ 0 = \langle f, e_{n+1} \rangle = \frac{\sqrt{1-|\lambda|^2}}{(n+1)!} \sum_{k=0}^{n+1} f^{(k)}(\lambda) h^{(n+1-k)}(\lambda) \]

\[ = \frac{\sqrt{1-|\lambda|^2}}{(n+1)!} f^{(n+1)}(\lambda) h_{n+1}(\lambda) \]

which implies \( f^{(n+1)}(\lambda) = 0 \)

since \( h_{n+1}(\lambda) = (1-|\lambda|^2)^{n+1} \neq 0 \), proving what was required.

**Corollary 2.9:** For \( \lambda = (\lambda_1, \lambda_2) \) in \( \mathbb{D}^2 \), the family \( \{e_{mn}\}_{m,n \geq 0} \) is a basis for \( H^2(\mathbb{D}^2) \) where

\[ e_{mn}(z) = \frac{\sqrt{1-|\lambda_1|^2}(1-|\lambda_2|^2)}{\lambda_1^m \lambda_2^n (1-\lambda_1 \overline{z}_1 z_1)^{m+1} (1-\lambda_2 \overline{z}_2 z_2)^{n+1}} \lambda_1^m \lambda_2^n (z_1 - \lambda_1)^m (z_2 - \lambda_2)^n \]

**Proof:** Proposition 2.8. shows that the family \( \{e_m\} \) where

\[ e_m(z) = \frac{\sqrt{1-|\lambda_1|^2}(z-\lambda_1)^m}{(1-\lambda_1 z)^{m+1}} \]

is a basis for \( H^2(\mathbb{D}) \) and hence \( \{f_m\} \) is also a basis for \( H^2(\mathbb{D}) \) where

\[ f_m(z) = \frac{\lambda_1^m (z-\lambda_1)^m}{\lambda_1^m (1-\lambda_1 z)^{m+1}} = \frac{\lambda_1^m}{\lambda_1} e_m(z) \]

It follows that \( e_{mn}(z) = f_m(z_1) f_n(z_2) \) is a basis for \( H^2(\mathbb{D}^2) \).

**Corollary 2.10:** \( \{e_{mn}\}_{m \geq p_k+1, 0 \leq n \leq q_k+1} \) for all \( k, l \leq k \leq r \) is an orthonormal basis for \( \mathbb{H}_\lambda^{(p_k, q_k)} \) and \( \{e_{mn}\}_{m \leq p_k, n \leq q_k} \) for all \( k, l \leq k \leq r \) is an orthonormal basis for \( \mathbb{H}_\lambda^{(p_k, q_k)} \).
orthonormal basis for $m_{\lambda}(p,q)$ and $\dim m_{\lambda}(p,q)$

$$
= \sum_{k=1}^{r} (q_k + 1)(p_k - p_{k-1}) \quad \text{where} \quad p_0 = -1.
$$

**Proof:** This follows from the definition of $m_{\lambda}(p,q)$ and corollary 2.9.

**Proposition 2.11:** The pair $(S_1, S_2)$ is in $\mathcal{A}_1(\mathbb{D}^2 \setminus \{\lambda\})$.

**Proof:** The map $\eta(z) = \left(\frac{z_1 - \lambda_1}{1 - \overline{\lambda}_1 z_1}, \frac{z_2 - \lambda_2}{1 - \overline{\lambda}_2 z_2}\right)$ is a biholomorphic map from $\mathbb{D}^2$ onto itself. This map $\eta$ induces a unitary operator $U : m_0^{(p,q)} \to m_{\lambda}^{(p,q)}$ defined by

$$
(U\eta)(z) = \frac{1}{\overline{\eta}(z)} \eta(\overline{\eta}(z)) \quad \text{where}
$$

$$
\eta'(z) = \det\left(\frac{\partial \tau_i}{\partial z_j}\right)_{i, j = 1, 2} \quad \tau_i(z) = \frac{z_i - \lambda_i}{1 - \overline{\lambda}_i z_i} \quad i = 1, 2.
$$

We get the following commutative diagram:

$$
\begin{array}{ccc}
m_0^{(p,q)} & \xrightarrow{D_S} & m_0^{(p,q)} \oplus m_0^{(p,q)} \\
U & & U \oplus U \downarrow \\
m_{\lambda}^{(p,q)} & \xrightarrow{D_S} & m_{\lambda}^{(p,q)} \oplus m_{\lambda}^{(p,q)} \\
\end{array}
$$

where $D_S f = S_1 f \oplus S_2 f$ acting on $m_0^{(p,q)}$ and $D_S f = S_1 f \oplus S_2 f$ acting on $m_{\lambda}^{(p,q)}$. Hence $D_S$ acting on $m_{\lambda}^{(p,q)}$ is unitarily.
equivalent to $D_S$ acting on $m_0^{(p,q)}$. But the pair $(S_1, S_2)$ acting on $m_0^{(p,q)}$ is in $\mathfrak{A}(\mathbb{D}^2(\{0\}))$ (see Cowen and Douglas [5] page 20). Hence the pair $(S_1, S_2)$ acting on $m_0^{(p,q)}$ is in $\mathfrak{A}(\mathbb{D}^2(\{\lambda\}))$.

**Proposition 2.12:** Let $\Omega_0 \subset \Omega \subset \mathbb{D}^m$, $\Omega_0$ connected bounded, then $\mathfrak{A}(\Omega) \subset \mathfrak{A}(\Omega_0)$.

**Proof:** See Cowen and Douglas [4], page 193.

We want to calculate the curvature of the associated line bundle $E_S$, for $S = (S_1, S_2)$ in $\mathfrak{A}(\mathbb{D}^2(\{\lambda\}))$.

**Proposition 2.13:** $K_S(w)$, the curvature of the associated bundle $E_S$, for $S = (S_1, S_2)$ in $\mathfrak{A}(\mathbb{D}^2(\{\lambda\}))$, is given by

$$K_S(w) = \overline{\partial} \partial \log \|K_w\|^2 + \overline{\partial} \partial \log F_{p,q,\lambda}(w)$$

where

$$F_{p,q,\lambda}(w) = \sum_{k=1}^{r+1} \left[ \frac{w_1 - \lambda_1}{1-\lambda_1 w_1} \right]^{2p_{k-1}+2} \left[ \frac{w_2 - \lambda_2}{1-\lambda_2 w_2} \right]^{2q_{k}+2}$$

and $q_{r+1} = -1$

**Proof:** A holomorphic cross-section for the line bundle $E_S$ is given by $P_{m_\lambda^{(p,q)}} K_w$. Hence the curvature for the
bundle \( E_S \) is given by \( K_S(w) = -\delta \log \| P_{m}^{(p,q)} K_w \|^2 \)

\[ \frac{\delta}{\delta \lambda} \log \| P_{m}^{(p,q)} K_w \|^2 \]. We want to compute the norm \( \| P_{m}^{(p,q)} K_w \|^2 \). Now by Corollary 2.10 a basis for \( m^{(p,q)} \)

is given by \( \{ e_{ij} \} i \leq p_k, j \leq q_k, 1 \leq k \leq r \) where \( e_{ij} \) is as in Corollary 2.9.

Hence \( \| P_{m}^{(p,q)} K_w \|^2 \)

\[ = \Sigma_{k=1}^{r} \Sigma_{i=p_{k-1}+1}^{p_k} \Sigma_{j=0}^{q_k} \left( \langle P_{m}^{(p,q)} K_w, e_{ij} \rangle \right)^2 \]

\[ = \Sigma_{k=1}^{r} \Sigma_{i=p_{k-1}+1}^{p_k} \Sigma_{j=0}^{q_k} |e_{ij}(w)|^2 \text{ since } K_w \]

is the kernel function and \( e_{ij} \) are in \( m^{(p,q)} \).

\[ = \Sigma_{k=1}^{r} \Sigma_{i=p_{k-1}+1}^{p_k} \Sigma_{j=0}^{q_k} \frac{1}{\| K_w \|^2} \left( \frac{1}{1 - \lambda_1 w_1} \right)^{2i} \left( \frac{1}{1 - \lambda_2 w_2} \right)^{2j} \frac{1}{|1 - \lambda_1 w_1|^2 |1 - \lambda_2 w_2|^2} \]

\[ = \frac{1}{\| K_w \|^2} \left( \frac{1}{1 - \lambda_1 w_1} \right)^{2p_{k-1}} \left( \frac{1}{1 - \lambda_2 w_2} \right)^{2p_k} \]

\[ \times \left( \frac{1}{1 - \lambda_1 w_1} \right)^{2q_{k+2}} \]

\[ \times \left( \frac{1}{1 - \lambda_2 w_2} \right)^{2q_{k+2}} \]
bundle $E_S$ is given by $K_S(w) = -\partial \log \| P_{\mu_\lambda}(p,q) K_w \|^2$

$= \partial \log \| P_{\mu_\lambda}(p,q) K_w \|^2$. We want to compute the norm

$\| P_{\mu_\lambda}(p,q) K_w \|^2$. Now by Corollary 2.10 a basis for $\mathbb{m}_\lambda(p,q)$

is given by $\{ e_{ij} \} i \leq p_k, j \leq q_k, 1 \leq k \leq r$ where $e_{ij}$ is

as in Corollary 2.9.

Hence

$\| P_{\mu_\lambda}(p,q)^T K_w \|^2 = \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} \langle P_{\mu_\lambda}(p,q)^T K_w, e_{ij} \rangle^2$

$= \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} \| e_{ij}(w) \|^2$ since $K_w$

is the kernel function and $e_{ij}$ are in $\mathbb{m}_\lambda(p,q)$.

$= \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} \frac{1}{\| K_\lambda \|^2} \frac{1}{1-\lambda_1 w_1} \frac{1}{1-\lambda_2 w_2} \frac{2i}{|1-\lambda_1 w_1|^2} \frac{2j}{|1-\lambda_2 w_2|^2}$

$= \frac{1}{\| K_\lambda \|^2} \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} \left( \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \right)^{2p_{k-1}+2} \left( \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \right)^{2p_k+2} \left( \frac{w_2-\lambda_2}{1-\lambda_2 w_2} \right)^{2q_k+2}$

$\times \left( \frac{1}{1-\lambda_2 w_2} \right)^{2q_k+2}$
by summing the geometric sequence

\[
\sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{1}w_{1}} \right)^{2p_{k-1}+2} - \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2} \right) \left( 1 - \frac{w_{2-k+1}^{-\lambda_{2}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2q_{k}+2}
\]

\[
= \frac{\sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{1}w_{1}} \right)^{2p_{k-1}+2} - \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2}}{\| K_{\lambda} \|^{2} \left| 1-\bar{\lambda}_{1}w_{1} \right|^{2} \left| 1-\bar{\lambda}_{2}w_{2} \right|^{2} \left( 1 - \frac{w_{2-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2} \left( 1 - \frac{w_{2-k+1}^{-\lambda_{2}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2}}
\]

\[
= \frac{\sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{1}w_{1}} \right)^{2p_{k-1}+2} - \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2}}{\| K_{\lambda} \|^{2} \left( \left| 1-\bar{\lambda}_{1}w_{1} \right|^{2} - \left| w_{1-k+1}^{-\lambda_{1}} \right|^{2} \right) \left( \left| 1-\bar{\lambda}_{2}w_{2} \right|^{2} - \left| w_{2-k+1}^{-\lambda_{2}} \right|^{2} \right)}
\]

Simplifying both the numerator and the denominator we get

\[
1 - \sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{1}w_{1}} \right)^{2p_{k-1}+2} - \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2} + \sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2q_{k}+2} \left( \frac{w_{2-k+1}^{-\lambda_{2}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2}
\]

\[
= \frac{1 - \sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{1}w_{1}} \right)^{2p_{k-1}+2} - \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2} + \sum_{k=1}^{\infty} \left( \frac{w_{1-k+1}^{-\lambda_{1}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2q_{k}+2} \left( \frac{w_{2-k+1}^{-\lambda_{2}}}{1-\bar{\lambda}_{2}w_{2}} \right)^{2p_{k}+2}}{(1-|w_{1}|^{2})(1-|w_{2}|^{2})}
\]

\([q_{r+1} = -1]\)
by summing the geometric sequence

\[
\sum_{k=1}^{r} \left( \frac{\lambda_{1} - \lambda_{k}}{1 - \lambda_{1} w_{1}} \right)^{2p_{k-1} + 2} \left( \frac{\lambda_{1} - \lambda_{k}}{1 - \lambda_{1} w_{1}} \right)^{2p_{k}} (1 - \frac{\lambda_{2} - \lambda_{k}}{1 - \lambda_{2} w_{2}})^{2} \right) \left( 1 - \frac{\lambda_{2} - \lambda_{k}}{1 - \lambda_{2} w_{2}} \right)^{2} \left( 1 - \frac{\lambda_{2} - \lambda_{k}}{1 - \lambda_{2} w_{2}} \right)^{2} \left( 1 - \frac{\lambda_{2} - \lambda_{k}}{1 - \lambda_{2} w_{2}} \right)^{2} \right) \right) \right) \left( 1 - \frac{\lambda_{2} - \lambda_{k}}{1 - \lambda_{2} w_{2}} \right)^{2} \left( 1 - \frac{\lambda_{2} - \lambda_{k}}{1 - \lambda_{2} w_{2}} \right)^{2} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \r...
\[ = \|K_w\|^{2}(1-F_{P, q, \lambda}(w)) \]

where
\[ F_{P, q, \lambda}(w) = \sum_{k=1}^{r+1} \frac{|w_1 - \lambda_1|}{1 - \lambda_1 w_1} + \frac{|w_2 - \lambda_2|}{1 - \lambda_2 w_2} \]

and \[ q_{r+1} = -1. \]

Now
\[ \|P_{m_\lambda}(p, q)K_w\|^{2} = \|K_w\|^{2} - \|P_{m_\lambda}(p, q) - K_w\|^{2} \]

\[ = \|K_w\|^{2} - \|K_w\|^{2}(1-F_{P, q, \lambda}(w)) = \|K_w\|^{2}F_{P, q, \lambda}(w) \]

which implies, after taking logarithms of both sides
\[ \log\|P_{m_\lambda}(p, q)K_w\|^{2} = \log\|K_w\|^{2} + \log F_{P, q, \lambda}(w) \]

Hence
\[ K_{\theta}(w) = \frac{\theta}{\theta} \log\|K_w\|^{2} + \frac{\theta}{\theta} \log F_{P, q, \lambda}(w). \]

2.14: Let \( S(p, q, \lambda) = (S_1, S_2) \) and \( S(p, q, \beta) = (\bar{S}_1, \bar{S}_2) \) be two pairs of operators on \( m_\lambda(p, q) \) and \( m_\beta(\bar{p}, \bar{q}) \), respectively for \( \lambda \neq \beta \) in \( \mathbb{D}^{2} \) and let \( 0 < \bar{\delta}_1 < \ldots < \bar{\delta}_n, 0 < \bar{\gamma}_1 < \bar{\gamma}_2 < \ldots < \bar{\gamma}_1 \) be integers. Then by Proposition 2.11 \( S(p, q, \lambda) \) is in \( S_{1}(\mathbb{D}^{2} \setminus \{\lambda\}) \) and \( S(p, q, \beta) \) is in \( S_{1}(\mathbb{D}^{2} \setminus \{\beta\}) \). By Proposition 2.12 we obtain \( S(p, q, \lambda) \) and \( S(p, q, \beta) \) both are in \( S_{1}(\mathbb{D}^{2} \setminus \{\lambda, \beta\}) \). Now we
\[ \|K_w\|^2 (1 - F_{p, q, \lambda}(w)) \]

where
\[
F_{p, q, \lambda}(w) = \sum_{k=1}^{r+1} \left| \frac{w_1 - \lambda_1}{1 - \lambda_1 w_1} \right|^{2p_{k-1} + 2} \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k + 2}
\]

\[
- \sum_{k=1}^{r+1} \left| \frac{w_1 - \lambda_1}{1 - \lambda_1 w_1} \right|^{2p_k + 2} \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k + 2}
\]

and \( q_{r+1} = -1 \).

Now
\[
\|P_{m(\lambda)}(p, q)K_w\|^2 = \|K_w\|^2 - \|P_{m(\lambda)}K_w\|^2
\]

\[
= \|K_w\|^2 - \|K_w\|^2 (1 - F_{p, q, \lambda}(w)) = \|K_w\|^2 F_{p, q, \lambda}(w)
\]

which implies, after taking logarithms of both sides

\[
\log \|P_{m(\lambda)}(p, q)K_w\|^2 = \log \|K_w\|^2 + \log F_{p, q, \lambda}(w)
\]

Hence
\[
K_\beta(w) = \overline{\partial} \log \|K_w\|^2 + \overline{\partial} \log F_{p, q, \lambda}(w).
\]

2.14: Let \( S_{\lambda}(p, q, \lambda) = (S_1, S_2) \) and \( S_{\beta}(\tilde{p}, \tilde{q}, \beta) = (\tilde{S}_1, \tilde{S}_2) \) be two pairs of operators on \( m_{\lambda}(p, q) \) and \( m_{\beta}(\tilde{p}, \tilde{q}) \), respectively for \( \lambda + \beta \in \mathbb{D}^2 \) and let \( 0 \leq \tilde{p}_1 < \ldots < \tilde{p}_s \), \( 0 \leq \tilde{q}_s < \tilde{q}_{s-1} < \ldots < \tilde{q}_1 \) be integers.

Then by Proposition 2.11 \( S_{\lambda}(p, q, \lambda) \) is in \( \mathfrak{A}_1(\mathbb{D}^2 \setminus \{\lambda\}) \) and \( S_{\beta}(\tilde{p}, \tilde{q}, \beta) \) is in \( \mathfrak{A}_1(\mathbb{D}^2 \setminus \{\beta\}) \). By Proposition 2.12 we obtain \( S_{\lambda}(p, q, \lambda) \) and \( S_{\beta}(\tilde{p}, \tilde{q}, \beta) \) both are in \( \mathfrak{A}_1(\mathbb{D}^2 \setminus \{\lambda, \beta\}) \). Now we
state and prove our main result:

**Theorem 2.15:** Let \((p, q)\) and \((\tilde{p}, \tilde{q})\) and \(\lambda, \beta\) be as before. If the pair \(S(p, q, \lambda) = (S_1, S_2)\) is simultaneously unitarily equivalent to the pair \(S(\tilde{p}, \tilde{q}, \beta) = (S_1, S_2)\), then \(\lambda = \beta\).

**Proof:** By the discussion preceding the theorem we see that both \(S(p, q, \lambda)\) and \(S(\tilde{p}, \tilde{q}, \beta)\) are in \(\mathcal{B}(\mathbb{D}^2\setminus\{\lambda, \beta\})\) and by Proposition 1.7 the curvatures of the associated line bundles are the same. But by Proposition 2.13 \(K_{S(p, q, \lambda)}(w)\), the curvature, is given by \(K_{S(p, q, \lambda)}(w) = \bar{\partial} \bar{\partial} \log \|K_w\|^2 + \bar{\partial} \bar{\partial} \log F_{p, q, \lambda}(w)\) and the corresponding curvature for \(S(\tilde{p}, \tilde{q}, \beta)\) has a similar expression. Now the equality of \(K_{S(p, q, \lambda)}(w)\) with \(K_{S(\tilde{p}, \tilde{q}, \beta)}(w)\) on \(\mathbb{D}^2\setminus\{\lambda, \beta\}\) implies \(\bar{\partial} \bar{\partial} \log F_{p, q, \lambda}(w) = \bar{\partial} \bar{\partial} \log F_{\tilde{p}, \tilde{q}, \beta}(w)\) on \(\mathbb{D}^2\setminus\{\lambda, \beta\}\) and hence equality holds on \(\mathbb{T}^2\) since \(F_{p, q, \lambda}\) and \(F_{\tilde{p}, \tilde{q}, \beta}\) are both real analytic in a neighborhood of \(\text{CL} \mathbb{D}^2\setminus\{\lambda, \beta\}\). Now

\[
\bar{\partial} \bar{\partial} \log F_{p, q, \lambda}(w) = \sum_{i, j=1}^{2} \frac{\partial^2 \log F_{p, q, \lambda}(w)}{\partial \bar{w}_i \partial w_j} \, d\bar{w}_i \wedge dw_j
\]

and hence we have

\[
\frac{\partial^2 \log F_{p, q, \lambda}(w)}{\partial \bar{w}_i \partial w_j} = \frac{\partial^2 \log F_{\tilde{p}, \tilde{q}, \beta}}{\partial \bar{w}_i \partial w_j} \quad \text{on } \mathbb{T}^2.
\]

Recall that
\[ f_{p,q,\lambda}(w) = \sum_{k=1}^{r+1} \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_{k-1}+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2} - \sum_{k=1}^{r} \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_k+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2} \]

Rewriting, we get

\[ f_{p,q,\lambda}(w) = \frac{w_2-\lambda_2}{1-\lambda_2 w_2} \cdot \frac{2q_k+2}{\sum_{k=1}^{r} \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_k+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2}} - \frac{w_2-\lambda_2}{1-\lambda_2 w_2} \cdot \frac{2q_k+2}{\sum_{k=1}^{r} \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_k+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2}} \]

Differentiating \( f_{p,q,\lambda} \) with respect to \( \overline{w}_1 \), we get

\[ \frac{\partial f_{p,q,\lambda}}{\partial \overline{w}_1} = \sum_{k=1}^{r} \frac{\partial}{\partial \overline{w}_1} \left( \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_k+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2} \right) - \sum_{k=1}^{r} \frac{\partial}{\partial \overline{w}_1} \left( \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_k+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2} \right) \]

\[ = \frac{(1-|\lambda_1|^2)(w_1-\lambda_1)}{(1-\lambda_1 \overline{w}_1)^2(1-\lambda_1 w_1)} \sum_{k=1}^{r} \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2p_k}{\overline{w}_1} \sum_{k=1}^{r+1} \frac{w_1-\lambda_1}{1-\lambda_1 w_1} \cdot \frac{2q_k+2}{w_2-\lambda_2} \cdot \frac{2q_k+2}{1-\lambda_2 w_2} \]

Differentiating, once more, \( \frac{\partial f_{p,q,\lambda}}{\partial \overline{w}_1} \) with respect to \( w_2 \)
\[ F_{p,q,\lambda}(w) = \sum_{k=1}^{r+1} \left| \frac{w_1 - \lambda_1}{1 - \lambda_1 w_1} \right|^{2p_k+2} \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} \]

Rewriting we get

\[ F_{p,q,\lambda}(w) = \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} + \sum_{k=1}^{r} \left| \frac{w_1 - \lambda_1}{1 - \lambda_1 w_1} \right|^{2p_k+2} \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} \left( \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_{k+1}+2} - \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} \right) \]

Differentiating \( F_{p,q,\lambda} \) with respect to \( w_1 \), we get

\[ \frac{\partial F_{p,q,\lambda}}{\partial w_1} = \sum_{k=1}^{r} \frac{\partial}{\partial w_1} \left| \frac{w_1 - \lambda_1}{1 - \lambda_1 w_1} \right|^{2p_k+2} \left( \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} - \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} \right) \]

\[ = \frac{(1 - |\lambda_1|^2)(w_1 - \lambda_1)}{(1 - \lambda_1 w_1)^2} \sum_{k=1}^{r} (p_k+1) \left| \frac{w_1 - \lambda_1}{1 - \lambda_1 w_1} \right|^{2p_k} \left( \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} - \left| \frac{w_2 - \lambda_2}{1 - \lambda_2 w_2} \right|^{2q_k+2} \right) \]

Differentiating, once more, \( \frac{\partial F_{p,q,\lambda}}{\partial w_1} \) with respect to \( w_2 \)
we get

\[ \frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial w_1} = \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)(w_{1}-\lambda_1)(\bar{w}_2-\bar{\lambda}_2)}{(1-\lambda_1 w_1)^2(1-\lambda_2 w_2)(1-\bar{\lambda}_1 w_1)(1-\bar{\lambda}_2 w_2)^2} \]

\[ \sum_{k=1}^{r} \left( \frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial w_1} \right)^{2p_k} \left( \frac{\partial F_{p,q,\lambda}}{\partial w_2} \right)^{2q_k+1} \]

\[ \left( \frac{\partial F_{p,q,\lambda}}{\partial w_1} \right)^{2q_{k+1}} \left( \frac{\partial F_{p,q,\lambda}}{\partial w_2} \right)^{2q_k} \]

Now \( \frac{\partial F_{p,q,\lambda}}{\partial \bar{w}_1} = \frac{\partial F_{p,q,\lambda}}{\partial w_2} = 0 \) on \( \mathbb{T}^2 \)

as \( \frac{\partial F_{p,q,\lambda}}{\partial w_1} = 1 \) when \( |w_1| = 1, \quad 1 = 1,2 \)

and \( F_{p,q,\lambda} = 1 \) on \( \mathbb{T}^2 \) for the same reason.

Differentiating \( \log F_{p,q,\lambda} \) first with respect to \( \bar{w}_1 \) and then with respect to \( w_2 \) we get

\[ \frac{\partial^2 \log F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} = \frac{1}{(F_{p,q,\lambda})^2} \left( \frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} \right) \left( \frac{\partial F_{p,q,\lambda}}{\partial w_2} \right) \left( \frac{\partial F_{p,q,\lambda}}{\partial \bar{w}_1} \right) \]

\[ = \frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} \quad \text{on} \quad \mathbb{T}^2 \quad \text{since} \quad F_{p,q,\lambda} = 1, \]
\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial \overline{w}_i \partial \overline{w}_j} = \frac{\partial^2 F_{p,q,\lambda}}{\partial \overline{w}_i \partial \overline{w}_j} = 0 \quad \text{on } \mathbb{T}^2.
\]

Since
\[
\frac{\partial^2 \log F_{p,q,\lambda}}{\partial \overline{w}_i \partial \overline{w}_j} = \frac{\partial^2 \log F_{\tilde{p},\tilde{q},\tilde{\lambda}}}{\partial \overline{w}_i \partial \overline{w}_j} \quad \text{for } i, j = 1, 2 \quad \text{on } \mathbb{T}^2
\]

we have
\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial \overline{w}_1 \partial \overline{w}_i} = \frac{\partial^2 F_{\tilde{p},\tilde{q},\tilde{\lambda}}}{\partial \overline{w}_2 \partial \overline{w}_1} \quad \text{on } \mathbb{T}^2 \quad (1)
\]

But
\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial \overline{w}_2 \partial \overline{w}_1} = \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)(w_1-\lambda_1)(\overline{w}_2-\overline{\lambda}_2)}{(1-\lambda_1 \overline{w}_1)^2(1-\lambda_2 \overline{w}_2)^2(1-\overline{\lambda}_1 w_1)(1-\overline{\lambda}_2 w_2)} \sum_{k=1}^{r} (p_{k+1}(q_{k+1}+q_k)
\]

and
\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial \overline{w}_2 \partial \overline{w}_1} = \frac{(1-|\beta_1|^2)(1-|\beta_2|^2)(w_1-\beta_1)(\overline{w}_2-\overline{\beta}_2)}{(1-\beta_1 \overline{w}_1)^2(1-\beta_2 \overline{w}_2)^2(1-\overline{\beta}_1 w_1)(1-\overline{\beta}_2 w_2)} \sum_{k=1}^{s} (\tilde{p}_{k+1}(\tilde{q}_{k+1}+\tilde{q}_k)
\]

on \( \mathbb{T}^2 \).

Hence from (1) and using \(|w_i| = 1\) for \( i = 1, 2 \) we get
\[
\frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)w_1 \overline{w}_2 \sum_{k=1}^{r} (p_{k+1}(q_{k+1}+q_k)}}{|1-\lambda_1 \overline{w}_1|^2 |1-\lambda_2 \overline{w}_2|^2}
\]

and
\[
\frac{(1-|\beta_1|^2)(1-|\beta_2|^2)w_1 \overline{w}_2 \sum_{k=1}^{s} (\tilde{p}_{k+1}(\tilde{q}_{k+1}+\tilde{q}_k)}}{|1-\beta_1 \overline{w}_1|^2 |1-\beta_2 \overline{w}_2|^2}
\]

on \( \mathbb{T}^2 \).
\[
\frac{\partial F_{p,q,\lambda}}{\partial w_1} = \frac{\partial F_{p,q,\lambda}}{\partial w_2} = 0 \quad \text{on } \mathbb{T}^2.
\]

Since

\[
\frac{\partial^2 \log F_{p,q,\lambda}}{\partial w_i \partial w_j} = \frac{\partial^2 \log F_{p,q,\lambda}}{\partial w_i \partial \bar{w}_j} \quad \text{for } i,j = 1,2 \quad \text{on } \mathbb{T}^2
\]

we have

\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} = \frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} \quad \text{on } \mathbb{T}^2 \quad (1)
\]

But

\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} = \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)(w_1-\lambda_1)(\bar{w}_2-\bar{\lambda_2})}{(1-\lambda_1 \bar{w}_1)^2(1-\bar{\lambda}_2 w_2)^2(1-\bar{\lambda}_1 w_1)(1-\lambda_2 \bar{w}_2)} \sum_{k=1}^{r} (p_{k+1})(q_{k+1}-q_k)
\]

and

\[
\frac{\partial^2 F_{p,q,\lambda}}{\partial w_2 \partial \bar{w}_1} = \frac{(1-|\beta_1|^2)(1-|\beta_2|^2)(w_1-\beta_1)(\bar{w}_2-\bar{\beta_2})}{(1-\beta_1 \bar{w}_1)^2(1-\bar{\beta}_2 w_2)^2(1-\bar{\beta}_1 w_1)(1-\beta_2 \bar{w}_2)} \sum_{k=1}^{s} (q_{k+1})(q_{k+1}-q_k)
\]

on \( \mathbb{T}^2 \).

Hence from (1) and using \(|w_i| = 1\) for \(i = 1,2\) we get

\[
\frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)w_1 \bar{w}_2 \sum_{k=1}^{r} (p_{k+1})(q_{k+1}-q_k)}{|1-\bar{\lambda}_1 w_1|^2|1-\bar{\lambda}_2 w_2|^2} \quad \text{on } \mathbb{T}^2
\]

and

\[
\frac{(1-|\beta_1|^2)(1-|\beta_2|^2)w_1 \bar{w}_2 \sum_{k=1}^{s} (q_{k+1})(q_{k+1}-q_k)}{|1-\bar{\beta}_1 w_1|^2|1-\bar{\beta}_2 w_2|^2} \quad \text{on } \mathbb{T}^2
\]
from which it follows that
\[ c(1-|\lambda_1|^2)(1-|\lambda_2|^2)|1-\bar{\beta}_1 w_1|^2|1-\bar{\beta}_2 w_2|^2 = \mathcal{E}(1-|\beta_1|^2)(1-|\beta_2|^2)|1-\lambda_1 w_1|^2|1-\lambda_2 w_2|^2 \]
\[ \text{on } \mathbb{T}^2 \ldots (2) \]
where
\[ c = c(p, q) = \sum_{k=1}^{r} (p_{k+1})(q_{k+1}-q_k) \neq 0 \]
and \[ \mathcal{E} = \mathcal{E}(\bar{p}, \bar{q}) = \sum_{k=1}^{r} (\bar{p}_{k+1})(\bar{q}_{k+1}-\bar{q}_k) \neq 0 . \]

Now
\[ |1-\bar{\lambda}_1 w_1|^2|1-\bar{\lambda}_2 w_2|^2 = \left( (1+|\lambda_1|^2)-\bar{\lambda}_1 w_1-\lambda_1 \bar{w}_1 \right) \left( (1+|\lambda_2|^2)-\bar{\lambda}_2 w_2-\lambda_2 \bar{w}_2 \right) \]
\[ = (1+|\lambda_1|^2)(1+|\lambda_2|^2)-\bar{\lambda}_1 (1+|\lambda_2|^2)\bar{w}_1 - \bar{\lambda}_2 (1+|\lambda_1|^2)\bar{w}_2 \]
\[ - \bar{\lambda}_1 (1+|\lambda_2|^2)\bar{w}_2 - \lambda_2 (1+|\lambda_1|^2)w_2 + \bar{\lambda}_1 (1+|\lambda_2|^2)w_1 + \lambda_2 (1+|\lambda_1|^2)\bar{w}_1 \]
\[ \text{ (using } |w_1| = 1) \]

Hence from (2) we get
\[ c(1-|\lambda_1|^2)(1-|\lambda_2|^2)|(1+|\beta_1|^2)(1+|\beta_2|^2)-\bar{\beta}_1 (1+|\beta_2|^2)\bar{w}_1 - \bar{\beta}_2 (1+|\beta_1|^2)\bar{w}_2 \]
\[ - \bar{\beta}_2 (1+|\beta_1|^2)\bar{w}_2 - \beta_2 (1+|\beta_1|^2)w_2 + \bar{\beta}_1 (1+|\beta_2|^2)w_1 + \beta_1 (1+|\beta_2|^2)\bar{w}_1 \]
\[ = \mathcal{E}(1-|\beta_1|^2)(1-|\beta_2|^2) \left( (1+|\lambda_1|^2)(1+|\lambda_2|^2)-\bar{\lambda}_1 (1+|\lambda_2|^2)w_1 + \lambda_2 (1+|\lambda_1|^2)\bar{w}_1 \right) \]
from which it follows that
\[ c(1-|\lambda_1|^2)(1-|\lambda_2|^2) |1-\beta_1 w_1|^2 |1-\beta_2 w_2|^2 = \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2) |1-\lambda_1 w_1|^2 |1-\lambda_2 w_2|^2 \quad \text{on } \mathbb{R}^2 \ldots (2) \]

where
\[ c = c(p,q) = \sum_{k=1}^{r} (p_{k+1} + 1)(q_{k+1} - q_k) \neq 0 \]

and \[ \tilde{c} = \tilde{c}(\tilde{p},\tilde{q}) = \sum_{k=1}^{s} (\tilde{p}_{k+1} + 1)(\tilde{q}_{k+1} - \tilde{q}_k) \neq 0 \]

Now
\[ |1-\lambda_1 w_1|^2 |1-\lambda_2 w_2|^2 = \{(1+|\lambda_1|^2) - \lambda_1 \overline{w_1} - \lambda_1 w_1\}(1+|\lambda_2|^2) - \lambda_2 \overline{w_2} - \lambda_2 w_2 \]
\[ = (1+|\lambda_1|^2)(1+|\lambda_2|^2) - \lambda_1 (1+|\lambda_2|^2) w_1 \]
\[ - \lambda_2 (1+|\lambda_1|^2) w_2 - \lambda_1 \lambda_2 \overline{w_2} - \lambda_2 \lambda_1 \overline{w_1} + \lambda_1 \lambda_2 w_1 w_2 \]
\[ + \lambda_1 \lambda_2 w_1 \overline{w_2} + \lambda_1 \lambda_2 \overline{w_1} w_2 + \lambda_1 \lambda_2 \overline{w_1} w_2 \]
\[ \quad (\text{using } |w_1| = 1). \]

Hence from (2) we get
\[ c(1-|\lambda_1|^2)(1-|\lambda_2|^2)((1+|\beta_1|^2)(1+|\beta_2|^2) - \beta_1 (1+|\beta_2|^2) w_1 - \beta_1 (1+|\beta_2|^2) \overline{w_2} \]
\[ - \beta_2 (1+|\beta_1|^2) w_2 - \beta_2 (1+|\beta_1|^2) \overline{w_2} - \beta_1 \beta_2 w_1 w_2 + \beta_1 \beta_2 \overline{w_1} w_2 + \beta_1 \beta_2 \overline{w_1} w_2 \]
\[ = \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2) (1+|\lambda_1|^2)(1+|\lambda_2|^2) - \lambda_1 (1+|\lambda_2|^2) w_1 \]
\[-\lambda_1(1+|\lambda_2|^2)\bar{w}_1 - \bar{\lambda}_2(1+|\lambda_1|^2)w_2 - \lambda_2(1+|\lambda_1|^2)w_2 + \bar{\lambda}_1\bar{\lambda}_2w_1w_2 \]

\[+ \lambda_1\lambda_2\bar{w}_1\bar{w}_1 + \lambda_1\lambda_2\bar{w}_1\bar{w}_2 + \lambda_1\lambda_2\bar{w}_1w_2 \].

Since these polynomials in \(w_1, \bar{w}_1, w_2\) and \(\bar{w}_2\) are equal on \(\mathbb{P}^2\), the coefficients of these polynomials must be equal.

So we have constant term:

\[c(1-|\lambda_1|^2)(1-|\lambda_2|^2)(1+|\beta_1|^2)(1+|\beta_2|^2) \]

\[= \bar{c}(1-|\lambda_1|^2)(1-|\lambda_2|^2)(1+|\lambda_1|^2)(1+|\lambda_2|^2) \ldots (3) \]

coefficient of \(\bar{w}_1\): \[c(1-|\lambda_1|^2)(1-|\lambda_2|^2)\beta_1(1+|\beta_2|^2) \]

\[= \bar{c}(1-|\beta_1|^2)(1-|\beta_2|^2)\lambda_1(1+|\lambda_2|^2) \ldots (4) \]

Dividing \((4)\) by \((3)\) we get \[\frac{\beta_1}{1+|\beta_1|^2} = \frac{\lambda_1}{1+|\lambda_1|^2} \ldots (5) \]

Taking the absolute value and cross-multiplying we obtain

\[|\beta_1|(1+|\lambda_1|^2) = |\lambda_1|(1+|\beta_1|^2) \]

\[= |\beta_1| + |\beta_1||\lambda_1|^2 = |\lambda_1|-|\lambda_1||\beta_1|^2 = 0 \]

\[\Rightarrow (|\beta_1|-|\lambda_1|)(1-|\lambda_1||\beta_1|) = 0 \Rightarrow |\lambda_1| = |\beta_1| \]

since \(|\lambda_1| < 1\) and \(|\beta_1| < 1\).

Hence from \((5)\) we get \(\lambda_1 = \beta_1\). Similarly equating the
\[
- \lambda_1 (1+|\lambda_2|^2) \overline{w}_1 - \lambda_2 (1+|\lambda_1|^2) w_2 - \lambda_2 (1+|\lambda_1|^2) \overline{w}_2 + \lambda_1 \overline{\lambda}_2 w_1 \overline{w}_2.
\]

\[
+ \lambda_1 \lambda_2 \overline{w}_1 \overline{w}_2 + \lambda_1 \lambda_2 w_1 \overline{w}_2 + \lambda_1 \lambda_2 w_1 w_2.
\]

Since these polynomials in \(w_1, \overline{w}_1, w_2, \overline{w}_2\) are equal on \(\mathbb{T}^2\), the coefficients of these polynomials must be equal.

So we have constant term:

\[
c(1-|\lambda_1|^2)(1-|\lambda_2|^2)(1+|\beta_1|^2)(1+|\beta_2|^2)
\]

\[
= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)(1+|\lambda_1|^2)(1+|\lambda_2|^2) \ldots \tag{3}
\]

coefficient of \(\overline{w}_1\): \(c(1-|\lambda_1|^2)(1-|\lambda_2|^2)|\beta_1|(1+|\beta_2|^2)\)

\[
= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)|\lambda_1|(1+|\lambda_2|^2) \ldots \tag{4}
\]

Dividing (4) by (3) we get \(\frac{\beta_1}{1+|\beta_1|^2} = \frac{\lambda_1}{1+|\lambda_1|^2} \ldots \tag{5}\)

Taking the absolute value and cross-multiplying we obtain

\[
|\beta_1|(1+|\lambda_1|^2) = |\lambda_1|(1+|\beta_1|^2)
\]

\[
= |\beta_1| + |\beta_1||\lambda_1|^2 = |\lambda_1| - |\lambda_1||\beta_1|^2 = 0
\]

\[
= (|\beta_1| - |\lambda_1|)(1-|\lambda_1||\beta_1|) = 0 = |\lambda_1| = |\beta_1|
\]

since \(|\lambda_1| < 1\) and \(|\beta_1| < 1\).

Hence from (5) we get \(\lambda_1 = \beta_1\). Similarly equating the
coefficients of $\bar{w}_2$ and dividing by the constant term we get
\[
\frac{\beta_2}{1+|\beta_2|^2} = \frac{\lambda_2}{1+|\lambda_2|^2}
\]
which implies $\lambda_2 = \beta_2$. Hence $\lambda = \beta$ what we are required to show.

**Theorem 2.16**: If the pair $S(p, q, \lambda) = (S_1, S_2)$ on $\mathfrak{m}_\lambda^{p, q}$ is simultaneously unitarily equivalent to the pair $S(\tilde{p}, \tilde{q}, \lambda) = (S_1, S_2)$ on $\mathfrak{m}_\lambda^{(\tilde{p}, \tilde{q})}$, then $r = s$, $p_i = \tilde{p}_i$ and $q_i = \tilde{q}_i$ for $i = 1, \ldots, r = s$.

**Proof**: The complete unitary invariants for the pair $S(p, q, \lambda) = (S_1, S_2)$ on $\mathfrak{m}_\lambda^{(p, q)}$ are
\[
\frac{\partial^2 \log \|p\|_{\mathfrak{m}_\lambda^{(p, q)}}}{\partial w_i \partial w_j} K \| \frac{w}{w} \|_f^2
\]
for $i, j = 1, 2$.

Let
\[
W_{ij}(p, q)(w_1, w_2) = \frac{\partial^2 \log F_{p, q, \lambda}(w_1, w_2)}{\partial w_i \partial w_j}
\]
where $F_{p, q, \lambda}$ is given by Proposition 2.13. Thus the complete unitary invariants are $W_{1i}(p, q) + (1-|w_i|^2)^{-2}$ for $i = 1, 2$ and $W_{12}(p, q)$, $W_{21}(p, q)$. Let $\tau_i(w_1, w_2) = \frac{w_i - \lambda_1}{1 - \lambda_1 w_i}$ for $i = 1, 2$ and
\[
\psi_{p, q}(z_1, z_2) = \sum_{k=1}^{r+1} |z_1|^{2p_{k-1}+2} |z_2|^{2q_{k+2}} - \sum_{k=1}^{r} |z_1|^{2p_{k+2}} |z_2|^{2q_{k+2}}.
\]
Observe that $\psi_{p, q}$ is bi-circularly symmetric. Now
\[
\log F_{p,q,\lambda}(w_1, w_2) = \log \psi_{p,q}(r_1(w_1, w_2), r_2(w_1, w_2)).
\]

Hence, by chain rule, we get

\[
W_{jj}(p,q)(w_1, w_2) = \frac{\partial}{\partial w_j} \psi_{p,q}(r_1(w_1, w_2), r_2(w_1, w_2)) \quad j = 1, 2.
\]

where

\[
W_{jj}(p,q)(z_1, z_2) = \frac{\partial^2 \log \psi_{p,q}(z_1, z_2)}{\partial z_j \partial \bar{z}_j}.
\]

Let \( z = re^{i\theta} \), \( r > 0 \). Then \( \frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta} \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \right) \). Thus

\[
W_{jj}(p,q)(r_1, r_2) = \frac{1}{4} \left( \frac{\partial^2}{\partial r_j^2} + \frac{1}{r_j} \frac{\partial}{\partial r_j} \right) \log \psi_{p,q}(r_1, r_2).
\]

Fix \( r_2 \neq 0 \) and let \( G(r_1, r_2) = \log \psi_{p,q}(r_1, r_2) \). Then we have

\[4r_1 W_{11}(p,q) = \frac{\partial}{\partial r_1} (r_1 \frac{\partial G}{\partial r_1}), \text{ so}\]

\[(*) \quad 4 \int_0^{r_1} w_{11}(s, r_2) ds = r_1 \frac{\partial G}{\partial r_1} \text{ and hence}\]

\[(**) \quad C(r_1, r_2) = 4 \int_0^{r_1} \frac{1}{2} \int_0^{r_2} w_{11}(s, r_2) ds dt + G(0, r_2), \]

Using the formula similar to \((*)\) for \( W_{22}(p,q) \) when \( r_1 \neq 0 \) and taking the limit as \( r_1 \to 0 \) we get, again with \( r_2 \neq 0 \),

\[r_2 \frac{\partial G}{\partial r_2} = 4 \lim_{r_1 \to 0} \int_0^{r_2} W_{22}(r_1, t) dt. \text{ Hence using the}\]

\[\text{fact that } G(0, r_2) = 0 \text{ we have}\]

\[G(0, r_2) = 4 \int_0^{r_2} \frac{1}{2} \lim_{r_1 \to 0} \int_0^{r_1} W_{22}(p,q)(r_1, t) dt ds.\]
\[ \log F_{p,q,\lambda}(w_1,w_2) = \log \psi_{p,q}(\tau_1(w_1,w_2),\tau_2(w_1,w_2)). \]

Hence, by chain rule, we get

\[ W_{jj}(p,q)(w_1,w_2) = \frac{\partial \psi_{p,q}(z_1,z_2)}{\partial z_j} = \frac{\partial^2 \log \psi_{p,q}(z_1,z_2)}{\partial z_j \partial \bar{z}_j} \]

where

\[ \tilde{W}_{jj}(p,q)(z_1,z_2) = \frac{\partial^2 \log \psi_{p,q}(z_1,z_2)}{\partial z_j \partial \bar{z}_j} \]

Let \( z = re^{i\theta}, \ r > 0. \) Then \( \frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta} \left( \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right). \) Thus

\[ \tilde{W}_{jj}(p,q)(r_1,r_2) = \frac{1}{4} \left( \frac{\partial^2}{\partial r_j^2} + \frac{1}{r} \frac{\partial}{\partial r_j} \right) \log \psi_{p,q}(r_1,r_2). \]

Fix \( r_2 \neq 0 \) and let \( G(r_1,r_2) = \log \psi_{p,q}(r_1,r_2). \) Then we have

\[ 4r_1 \tilde{W}_{11}(p,q) = \frac{\partial}{\partial r_1} (r_1 \frac{\partial G}{\partial r_1}), \]

so

\[ (*) \quad 4 \int_0^{r_1} \tilde{W}_{11}(p,q)(s,r_2) ds = r_1 \frac{\partial G}{\partial r_1} \quad \text{and hence} \]

\[ (**) \quad G(r_1,r_2) = 4 \int_0^{r_1} \frac{1}{t} \int_0^t \tilde{W}_{11}(p,q)(s,r_2) ds \ dt + G(0,r_2). \]

Using the formula similar to (*) for \( \tilde{W}_{22}(p,q) \) when \( r_1 \neq 0 \) and taking the limit as \( r_1 \to 0 \) we get, again with \( r_2 \neq 0, \)

\[ r_2 \frac{\partial G}{\partial r_2}(0,r_2) = 4 \lim_{r_1 \to 0} \int_0^{r_2} \tilde{W}_{22}(p,q)(r_1,t) dt. \]

Hence using the fact that \( G(0,1) = 0 \) we have

\[ G(0,r_2) = 4 \int_1^{r_2} \frac{1}{s} \lim_{r_1 \to 0} \int_0^s \tilde{W}_{22}(p,q)(r_1,t) dt \ ds. \]
Combining with (**) we get

\[
\text{(***)} \quad \mathcal{C}(\ell_1,\ell_2) = \int_0^1 \int_0^t s^{\tilde{\omega}}_{11}(s,\ell_2)ds dt + \int_0^1 \int_{\ell_1}^{\ell_2} \lim_{r_1 \to 0} \frac{1}{r_1} \int_0^r s^{\tilde{\omega}}_{22}(r_1,\ell_2)dr_1 ds.
\]

Now the pair \( \tilde{\mathcal{S}}(p,q,\lambda) = (\tilde{\mathcal{S}}_1,\tilde{\mathcal{S}}_2) \) on \( \mathcal{W}(p,q) \) is unitarily equivalent to the pair \( \tilde{\mathcal{S}}(\tilde{p},\tilde{q},\lambda) = (\tilde{\mathcal{S}}_1,\tilde{\mathcal{S}}_2) \) on \( \mathcal{W}(\tilde{p},\tilde{q}) \).

Imply \( \mathcal{W}_{1j}(p,q)(w_1,w_2) = \mathcal{W}_{1j}(\tilde{p},\tilde{q})(w_1,w_2) \) which in turn implies

\[
\tilde{\mathcal{W}}_{1j}(p,q)(\ell_1,\ell_2) |_{\nu_j} = \mathcal{W}_{1j}(p,q)(\ell_1,\ell_2) |_{\nu_j}
\]

and hence

\[
\tilde{\mathcal{W}}_{1j}(p,q)(\ell_1,\ell_2) = \mathcal{W}_{1j}(\tilde{p},\tilde{q})(\ell_1,\ell_2)
\]

since \( |\nu_j| > 0 \). From this

and (**) we obtain

\[
\log \psi_{p,q}(\ell_1,\ell_2) = \log \psi_{\tilde{p},\tilde{q}}(\ell_1,\ell_2)
\]

which implies \( \psi_{p,q}(\ell_1,\ell_2) = \psi_{\tilde{p},\tilde{q}}(\ell_1,\ell_2) \).

and hence \( \tilde{x} = x \), \( \tilde{v}_1 = \tilde{v}_1 \), \( \tilde{q}_1 = \tilde{q}_1 \) for \( i = 1,2,\ldots, n \).

\( \psi_{p,q} \) and \( \psi_{\tilde{p},\tilde{q}} \) are real analytic, in fact polynomials in \( \ell_1 \) and \( \ell_2 \).
Combining with (**) we get

\[(***) \quad G(r_1, r_2) = 4 \int_0^{r_1} \int_0^{r_2} \tilde{w}_{11}(p, q)(s, r_2)ds \, dt
+ \int_{r_1}^{r_2} \frac{1}{s} \lim_{r_1 \to 0} \int_0^{s} \tilde{w}_{22}(p, q)(r_1, t) \, dt \, ds.\]

Now the pair \(s(p, q, \lambda) = (s_1, s_2)\) on \(m(p, q)\) is unitarily equivalent to the pair \(s(\tilde{p}, \tilde{q}, \lambda) = (s_1, s_2)\) on \(m(p, q)\),
implies \(w_{1j}(p, q)(w_1, w_2) = w_{1j}(\tilde{p}, \tilde{q})(w_1, w_2)\) which in turn implies

\[\tilde{w}_{jj}(p, q)(z_1, z_2) \frac{\partial \tau}{\partial w_j} = w_{1j}(p, q)(w_1, w_2) = \tilde{w}_{jj}(\tilde{p}, \tilde{q})(z_1, z_2) \frac{\partial \tau}{\partial w_j}\]

and hence

\[\tilde{w}_{jj}(p, q)(z_1, z_2) = \tilde{w}_{jj}(\tilde{p}, \tilde{q})(z_1, z_2)\]

since \(\left|\frac{\partial \tau}{\partial w_j}\right|^2 > 0\). From this and (***) we obtain

\[\log \psi_{\tilde{p}, \tilde{q}}(r_1, r_2) = \log \psi_{\tilde{p}, \tilde{q}}(r_1, r_2)\]

which implies \(\psi_{\tilde{p}, \tilde{q}}(r_1, r_2) = \psi_{\tilde{p}, \tilde{q}}(r_1, r_2)\).

and hence \(r = s, p_i = \tilde{p}_i, q_i = \tilde{q}_i\) for \(i = 1, 2 \ldots r = s\) as \(\psi_{\tilde{p}, \tilde{q}}\) and \(\psi_{\tilde{p}, \tilde{q}}\) are real analytic, in fact polynomials in \(r_1\) and \(r_2\).
Corollary 2.17: If the pair \( s(p,q,\lambda) = (S_1,S_2) \) on \( m_{\lambda}(p,q) \)
is simultaneously unitarily equivalent to the pair \( s(\tilde{p},\tilde{q},\beta) = (S_1,S_2) \) on \( m_{\beta}(\tilde{p},\tilde{q}) \) then

\[
\lambda = \beta, \quad r = s, \quad p_i = \tilde{p}_i, \quad q_i = \tilde{q}_i \quad i = 1, \ldots, r=s.
\]

Proof: Combine Theorems 2.15 and 2.16
References


