Conjugation of Group Theoretical Abelian Schemes over an Arithmetic Variety

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Abstract of the Dissertation

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Let $X = \Gamma\backslash D$ be an arithmetic variety. If $\rho: G_R \to Sp$ is a symplectic representation and $\tau: D \to h^n$ is a holomorphic map such that (ρ, τ) is equivariant, a group theoretical abelian scheme over X associated to (ρ, τ) is obtained by pulling back the universal family of abelian varieties $\pi': Y' \to X'$ by the map $\tau_X: X \to X'$ induced by τ, where Sp is a symplectic group and h^n is a Siegel upper half space.

If the symplectic representation ρ sends a symmetry to a symmetry, the equivariant pair (ρ, τ) is called

(H_2)-equivariant. In this thesis, we shall prove the following theorem:

Theorem. Let $\pi : Y \to X$ be a group theoretical abelian scheme over an arithmetic variety X associated to an equivariant pair (ρ, τ). Then, for each element σ in $\mathrm{Aut}(\mathbb{C})$, $\pi^\sigma : Y^\sigma \to X^\sigma$ is a group theoretical abelian scheme over X^σ associated to another equivariant pair. Furthermore, if ρ is a symplectic representation of a classical group containing no D_4 factors such that (ρ, τ) is (H_2)-equivariant, then $\pi^\sigma : Y^\sigma \to X^\sigma$ is a group theoretical abelian scheme over X^σ associated to another (H_2)-equivariant pair.

TABLE OF CONTENTS

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INTRODUCTION

Abelian schemes over an arithmetic variety, or what used to be called families of abelian varieties, have been proved to be of great significance in algebraic geometry and number theory. Given an arithmetic variety $X = \Gamma\backslash D$, if there exist a symplectic representation $\rho:G_R \to Sp$ and a holomorphic map $\tau:D \to h^n$ such that the pair (ρ,τ) is equivariant, a group theoretical abelian scheme over X associated to (ρ,τ) is obtained by pulling back the universal family of abelian varieties $\pi':Y' \to X'$ by the map $\tau_X:X \to X'$ induced by τ, where Sp is a symplectic group and h^n is a Siegel upper half space.

Let $\pi:Y \to X$ be a group theoretical abelian scheme over an arithmetic variety X. Then π is a morphism of complex projective varieties; hence, to each $\sigma \in Aut(C)$, there corresponds another morphism of projective varieties $\pi^\sigma:Y^\sigma \to X^\sigma$.

In the sixties, G. Shimura considered abelian schemes over arithmetic varieties associated to a so-called PEL type and showed (e.g. [9]) that, for each $\sigma \in Aut(C)$, if $\pi:Y \to X$ is a group theoretical abelian scheme over an arithmetic variety X associated to a PEL type, X^σ is an arithmetic variety and $\pi^\sigma:Y^\sigma \to X^\sigma$ is a group theoretical abelian scheme over X^σ. In 1970 D. A. Kajdan considered the base spaces of these fibre varieties and showed [3] that, for each element σ in Aut(C), if X is a compact arithmetic variety, X^σ is also an arithmetic variety.

The main purpose of this thesis is to show that a conjugation of a group theoretical abelian scheme over a compact arithmetic variety is also a group theoretical abelian scheme over an arithmetic variety. More precisely, given an element σ in Aut(C), if $\pi: Y \to X$ is a group theoretical abelian scheme over a compact arithmetic variety X, then $\pi^\sigma: Y^\sigma \to X^\sigma$ is a group theoretical abelian scheme over the arithmetic variety X^σ.

Another problem considered in this thesis is the one of conjugation of abelian schemes over an arithmetic variety corresponding to an (H_2)-equivariant pair. More precisely, we shall prove the following statement under the assumption that ρ is a symplectic representation of a classical group which contains no D_4 factors:

If $\pi : Y \to X$ is a group theoretical abelian scheme over an arithmetic variety X associated to an (H_2)-equivariant pair (ρ, τ), then $\pi^\sigma : Y^\sigma \to X^\sigma$ is a group theoretical abelian scheme over X^σ associated to another (H_2)-equivariant pair.

Notations. The letters Q, R, and C denote the rational numbers, the real numbers, and the complex numbers respectively.

CHAPTER I. STATEMENT OF MAIN THEOREMS.

Let G be a semisimple algebraic Q-group, G_R the group of real points of G, and K_R a maximal compact subgroup of G_R. We assume that the symmetric space $D = G_R/K_R$ has a G_R-invariant complex structure.

Let Γ be a cocompact arithmetic subgroup of G_R with no elements of finite order. Then $X = \Gamma\backslash D$ has the natural structure of a complex manifold. Such a complex manifold is called an arithmetic variety. We shall identify X with its embedded image in a complex projective space.

Let X be a complex projective variety determined by the equations

$$\{ \sum a_{i_0 \ldots i_N} X_0^{i_0} \cdots X_N^{i_N} \}$$

in the projective space $P^N(C)$. Then, to each $\sigma \in \mathrm{Aut}(C)$, there corresponds another complex projective variety X^σ determined by the following set of equations:

$$\{ \sum (a_{i_0 \ldots i_N})^\sigma X_0^{i_0} \cdots X_N^{i_N} \}$$

In fact, for each $\sigma \in \mathrm{Aut}(C)$, there is a functor from the category of complex projective varieties and morphisms of varieties to the same category, sending a variety X to a variety X^{σ} and a morphism of varieties $f:X \to Y$ to a morphism of varieties $f^{\sigma}:X^{\sigma} \to Y^{\sigma}$.

It is known [3] that, if X is an arithmetic variety, X^{σ} is also an arithmetic variety.

<u>Definition</u>. Let G_R(resp. G_R') be a semisimple Lie group with the associated symmetric domain D(resp. D′). Let $\rho:G_R \to G_R'$ be a homomorphism of Lie groups and $\tau:D \to D'$ a holomorphic map. Then the pair (ρ,τ) is called equivariant if and only if the following condition is satisfied:

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G_R$ and $z \in D$. Furthermore, if the additional condition

$$\tau \cdot S_z = S_{\tau(z)} \cdot \tau$$

is satisfied for all $z \in D$, the pair (ρ,τ) is called strongly equivariant, where S_z(resp. $S_{\tau(z)}$) is the symmetry of D(resp. D′) at the point $z \in D$(resp. $\tau(z) \in D'$).

<u>Theorem 1</u>. Let X(resp. X′) be an arithmetic variety, D(resp. D′) its universal covering space and G_R(resp. G_R') the associated semisimple Lie group. Let G_0^{σ}(resp. $G_0'^{\sigma}$)

be the connected component of the identity of $\mathrm{Aut}(D^{\sigma})$ (resp. $\mathrm{Aut}(D'^{\sigma})$). Let $\phi: X \to X'$ be a morphism of varieties, $\tilde{\phi}: D \to D'$ a lifting of ϕ, and $\rho: G_R \to G'_R$ a homomorphism of Lie groups such that $(\rho, \tilde{\phi})$ is equivariant. Then there exist a finite covering G_1^{σ} of G_0^{σ}, a homomorphism $\rho_1^{\sigma}: G_1^{\sigma} \to G_0'^{\sigma}$ of Lie groups and a lifting $\tilde{\phi}^{\sigma}: D^{\sigma} \to D'^{\sigma}$ of $\phi^{\sigma}: X^{\sigma} \to X'^{\sigma}$ such that the pair, $(\rho_1^{\sigma}, \tilde{\phi}^{\sigma})$, is equivariant.

Let X be an arithmetic variety and D its universal covering space and G_R a semisimple Lie group associated to D. Let Sp be a symplectic group and h^n the symmetric domain associated to Sp. If there are a homomorphism $\rho: G_R \to Sp$ and a holomorphic map $\tau: D \to h^n$ such that (ρ, τ) is equivariant, then, as is described in §3.2, we can construct a family of abelian varieties $\pi: Y \to X$ associated to (ρ, τ) called group theoretical abelian scheme over X. Since π is a morphism of complex projective varieties, to each $\sigma \in \mathrm{Aut}(C)$, there corresponds another morphism of complex projective varieties $\pi^{\sigma}: Y^{\sigma} \to X^{\sigma}$.

<u>Theorem 2</u>. Let $\pi: Y \to X$ be a group theoretical abelian scheme over an arithmetic variety X. Then $\pi^{\sigma}: Y^{\sigma} \to X^{\sigma}$ is a group theoretical abelian scheme over X^{σ}.

<u>Definition</u>. Let $\rho: G_R \to G'_R$ be a homomorphism of Lie groups and $\tau: D \to D'$ a holomorphic map such that (ρ, τ) is equivariant. Then the pair (ρ, τ) is called (H_2)-equivariant if and only if the following condition (H_2) is satisfied:

$$(H_2): \qquad \rho(S_z) = S_{\tau(z)} \quad \text{for all } z \in D.$$

From the definition, it follows easily that, if (ρ, τ) is (H_2)-equivariant, it is strongly equivariant.

<u>Theorem 3</u>. Let $\rho: G_R \to Sp$ be a homomorphism and $\tau: D \to h^n$ a holomorphic map such that (ρ, τ) is (H_2)-equivariant. Assume that G_R is a classical group with no D_4 factors. Let $\pi: Y \to X$ be a group theoretical abelian scheme over X associated to the (H_2)-equivariant pair (ρ, τ). Then $\pi^\sigma: Y^\sigma \to X^\sigma$ is a group theoretical abelian scheme over X^σ associated to another (H_2)-equivariant pair.

CHAPTER II. ARITHMETIC VARIETIES.

§2.1. Prouniversal Covering Manifolds.

Let X be an arithmetic variety, D the universal covering space and Γ the fundamental group of X. Consider the cofinal system $\{\Gamma_i\}$ of subgroups of finite index of Γ with

$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots\cdots$$

For each i, $X_i = \Gamma_i \backslash D$ is a finite unramified covering manifold of X and $\{X_i\}$ is a cofinal system of covering manifolds. We put $\hat{D} = \varprojlim X_i$. Then $\hat{D}$ has the natural structure of a complex manifold. We define

$$\hat{G} = \mathrm{Aut}(\hat{D})$$

$$G = \{g \in \mathrm{Aut}(D) \mid [\Gamma : g\Gamma g^{-1} \cap \Gamma] < \infty \text{ and } [\Gamma : g^{-1}\Gamma g \cap \Gamma] < \infty\}$$

Then the following is known [3]:

(a) D is isomorphic to a connected component of $\hat{D}$.

(b) If $\iota : D \to \hat{D}$ is a lifting of the covering maps $\pi_i : D \to X_i$ sending a point $d_0 \in D$ to $\hat{d}_0 = \varprojlim \pi_i(d_0)$, then

$$G = \{g \in \hat{G} \mid \hat{g}\hat{d}_0 \in \iota(D)\}.$$

(c) G is dense in $\hat{G}$.

§2.2. $\hat{\rho}:\hat{G} \to \hat{G}'$

Let $\phi:X \to X'$, $\tilde{\phi}:D \to D'$ and $\rho:G_R \to G_R$ be as in Theorem 1 with $\tilde{\phi}(gy) = \rho(g)\tilde{\phi}(y)$ for all $g \in G_R$ and $y \in D$; let Γ(resp. Γ') be the fundamental group of X(resp. X'). Consider the cofinal system of subgroups of finite index $\{\Gamma_i\}$(resp. $\{\Gamma_i'\}$) of Γ(resp. Γ') with $\rho(\Gamma_i) \subset \Gamma_i'$. We put $\hat{D} = \varprojlim X_k$ and $\hat{D}' = \varprojlim X_k'$, and define $\hat{\phi}:D \to D'$ by $\hat{\phi} = \varprojlim \phi_k$.

For each element $\hat{g}$ in $\hat{G}$ there is another cofinal system $\{X_{1_k}\}$ of covering manifolds of X and morphisms $g_k:X_{1_k} \to X_k$ such that $\hat{g} = \varprojlim g_k$. Let $\tilde{g}_k:D \to D$ be a lifting of $g_k:X_{1_k} \to X_k$ for each k. If $X_{1_k} = \Gamma_{1_k}\backslash D$, then we have $\tilde{g}_k(\Gamma_{1_k}) \subset \Gamma_k$. Furthermore, the liftings $\tilde{g}_k$ satisfy the following condition:

$$(*) \qquad \tilde{g}_k = \gamma_j^{\,k}\tilde{g}_j \quad \text{with} \quad \gamma_j^{\,k} \in \Gamma_j \quad \text{for } j < k$$

In general, each element $\hat{g}$ in $\hat{G}$ can be identified with a collection $\{\tilde{g}_k\}$ of liftings of g_k satisfying the condition (*). Applying ρ to these liftings, we obtain the maps $\rho(\tilde{g}_k):D' \to D'$ such that

$$\rho(\tilde{g}_k) = \rho(\gamma_j^{\,k}\tilde{g}_j) = \rho(\gamma_j^{\,k})\rho(\tilde{g}_j)$$

where $\gamma_j^{\,k} \in \Gamma_j$ for $j < k$.

Replacing the elements of $\{\Gamma_{1_k}\}$ by smaller subgroups if necessary, we can choose a cofinal system $\{\Gamma\acute{\imath}_k\}$ of subgroups of Γ' such that

$$\rho(\Gamma\imath_k) \subset \Gamma\acute{\imath}_k \quad \text{and} \quad \rho(\tilde{g}_k)\Gamma\acute{\imath}_k \subset \Gamma'_k \quad \text{for each } k.$$

Let $X\acute{\imath}_k = \Gamma\acute{\imath}_k \backslash D$ and define $p_k(\rho\tilde{g}_k): X\acute{\imath}_k \rightarrow X'_k$ to be the morphism induced by $\rho(\tilde{g}_k): D' \rightarrow D'$. Since $\rho(\tilde{g}_k) = \rho(\gamma_j{}^k)\rho(\tilde{g}_j)$ with

$$\rho(\gamma_j{}^k) \in \rho(\Gamma_j) \subset \Gamma'_j \quad \text{for } j < k,$$

$\{\rho(\tilde{g}_k)\}$ is a collection of liftings $\rho(\tilde{g}_k)$ of $p_k(\rho\tilde{g}_k)$ satisfying the condition (*) for D'. So $\varprojlim p_k(\rho\tilde{g}_k) \in \hat{G}'$. We define $\hat{\rho}: \hat{G} \rightarrow \hat{G}'$ by

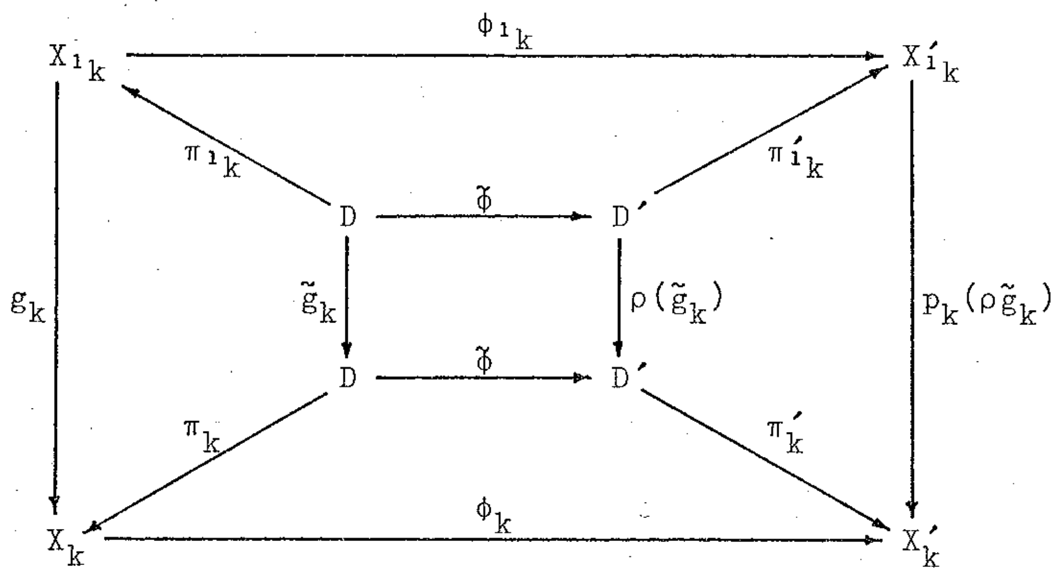
$$\hat{\rho}(\hat{g}) = \varprojlim p_k(\rho\tilde{g}_k).$$

<u>Proposition 1</u>. Let $\hat{\phi}: \hat{D} \rightarrow \hat{D}'$ and $\hat{\rho}: \hat{G} \rightarrow \hat{G}'$ be as above. Then

$$\hat{\phi}(\hat{g}\hat{y}) = \hat{\rho}(\hat{g})\hat{\phi}(\hat{y})$$

for all $\hat{g} \in \hat{G}$ and $\hat{y} \in \hat{D}$.

<u>Proof</u>. Let $\pi_k: D \rightarrow X_k$, $\pi\imath_k: D \rightarrow X\ {}_k$, $\pi'_k: D' \rightarrow X'_k$ and $\pi\acute{\imath}_k: D' \rightarrow X\acute{\imath}_k$ be the canonical covering maps. Then, for each k, we have the following commutative diagram:



From the commutativity of the diagram, we have

$$\pi'_k\tilde{\phi}(\tilde{g}_k y) = \phi_k(g_k \pi 1_k y) = \phi_k(g_k x1_k)$$
$$\pi'_k\rho(\tilde{g}_k)\tilde{\phi}(y) = (p_k(\rho\tilde{g}_k))\phi 1_k(\pi 1_k y)$$
$$= (p_k(\rho\tilde{g}_k))\phi 1_k(x1_k)$$

for all $y \in D$ and $x1_k = \pi 1_k y$. Since

$$\tilde{\phi}(\tilde{g}_k y) = \rho(\tilde{g}_k)\tilde{\phi}(y)$$

for all $y \in D$, we have

$$\phi_k(g_k x1_k) = (p_k(\rho\tilde{g}_k))\phi 1_k(x1_k)$$

for all $x1_k \in X1_k$. Thus it follows that

$$\hat{\phi}(\hat{g}\hat{y}) = \hat{\rho}(\hat{g})\hat{\phi}(\hat{y})$$

for all $\hat{y} = \varprojlim x1_k \in \hat{D}$.

§2.3. $\hat{\rho}^{\sigma}:\hat{G}^{\sigma} \to \hat{G}'^{\sigma}$

Let $\{X_k\}$ (resp. $\{X'_k\}$) be a cofinal system of finite unramified covering manifolds of X (resp. X'). Then $\{X_k^{\sigma}\}$ (resp. $\{X_k'^{\sigma}\}$) is a cofinal system of finite unramified covering manifolds of X^{σ} (resp. X'^{σ}). We put

$$\hat{D}^{\sigma} = \varprojlim X_k^{\sigma}, \qquad \hat{D}'^{\sigma} = \varprojlim X_k'^{\sigma}.$$

Define $\hat{\Phi}^{\sigma}:\hat{D}^{\sigma} \to \hat{D}'^{\sigma}$ by $\hat{\Phi}^{\sigma} = \varprojlim \phi_k^{\sigma}$, and put

$$\hat{G}^{\sigma} = \mathrm{Aut}(\hat{D}^{\sigma}), \qquad \hat{G}'^{\sigma} = \mathrm{Aut}(\hat{D}'^{\sigma}).$$

Let $\hat{h} = \varprojlim h_k \in \hat{G}^{\sigma}$. Then there exist cofinal systems of covering manifolds $\{Z_k\}$ and $\{W_k\}$ of X^{σ} such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Z_3 & \longrightarrow & Z_2 & \longrightarrow & Z_1 & \longrightarrow & X^{\sigma} \\
 & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \\
\cdots & \longrightarrow & W_3 & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & X^{\sigma}
\end{array}$$

Applying σ^{-1} to this diagram, we obtain

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Z_3^{\sigma^{-1}} & \longrightarrow & Z_2^{\sigma^{-1}} & \longrightarrow & Z_1^{\sigma^{-1}} & \longrightarrow & X \\
 & & \downarrow{\scriptstyle h_3^{\sigma^{-1}}} & & \downarrow{\scriptstyle h_2^{\sigma^{-1}}} & & \downarrow{\scriptstyle h_1^{\sigma^{-1}}} & & \\
\cdots & \longrightarrow & W_3^{\sigma^{-1}} & \longrightarrow & W_2^{\sigma^{-1}} & \longrightarrow & W_1^{\sigma^{-1}} & \longrightarrow & X
\end{array}$$

Define $\hat{\rho}^{\sigma}:\hat{G}^{\sigma} \to \hat{G}'^{\sigma}$ by

$$\hat{\rho}^{\sigma}(\hat{h}) = \varprojlim \left(p_k(\rho \tilde{h}_k^{\sigma^{-1}})\right)^{\sigma}$$

where $\tilde{h}_k^{\sigma^{-1}}:D \to D$ is a lifting of $h_k^{\sigma^{-1}}$.

<u>Proposition 2</u>. $\hat{\phi}^{\sigma}(\hat{g}^{\sigma}\hat{y}^{\sigma}) = \hat{\rho}^{\sigma}(\hat{g}^{\sigma})\hat{\phi}^{\sigma}(\hat{y}^{\sigma})$

for all $\hat{g}^{\sigma} \in \hat{G}^{\sigma}$ and $\hat{y}^{\sigma} \in \hat{D}^{\sigma}$.

<u>Proof</u>. For each k the relation $\phi(gy) = \rho(g)\phi(y)$ induces the following commutative diagram:

$$\begin{array}{ccc} X_{1_k} & \xrightarrow{\phi_{1_k}} & X'_{1_k} \\ \big\downarrow g_k & & \big\downarrow p_k(\rho\tilde{g}_k) \\ X_k & \xrightarrow{\phi_k} & X'_k \end{array}$$

Applying σ to this diagram, we obtain

$$\begin{array}{ccc} X_{1_k}^{\sigma} & \xrightarrow{\phi_{1_k}^{\sigma}} & X'^{\sigma}_{1_k} \\ \big\downarrow g_k^{\sigma} & & \big\downarrow (p_k(\rho\tilde{g}_k))^{\sigma} \\ X^{\sigma} & \xrightarrow{\phi_k^{\sigma}} & X'^{\sigma} \end{array}$$

which gives $\phi_k^{\sigma}(g_k^{\sigma}x_k^{\sigma}) = (p_k(\rho\tilde{g}_k))^{\sigma}\phi_{1_k}^{\sigma}(x_k^{\sigma})$ for all $x_k^{\sigma} \in X_{1_k}^{\sigma}$. Therefore it follows that

$$\hat{\phi}^{\sigma}(\hat{g}^{\sigma}\hat{y}^{\sigma}) = \hat{\rho}^{\sigma}(\hat{g}^{\sigma})\hat{\phi}^{\sigma}(\hat{y}^{\sigma}) \quad \text{for all } \hat{y}^{\sigma} \in \hat{D}^{\sigma}.$$

§2.4. $\rho^\sigma : G^\sigma \to G'^\sigma$

Let Γ^σ(resp. Γ'^σ) be the fundamental group of X^σ(resp. X'^σ) and D^σ(resp. D'^σ) the universal covering space of X^σ(resp. X'^σ). We define G^σ and G'^σ by

$$G^\sigma = \{g^\sigma \in \mathrm{Aut}(D^\sigma) \mid [\Gamma^\sigma : g^\sigma \Gamma^\sigma (g^\sigma)^{-1} \cap \Gamma^\sigma] < \infty \text{ and } [\Gamma^\sigma : (g^\sigma)^{-1} \Gamma^\sigma g^\sigma \cap \Gamma^\sigma] < \infty\}$$

$$G'^\sigma = \{g'^\sigma \in \mathrm{Aut}(D'^\sigma) \mid [\Gamma'^\sigma : g'^\sigma \Gamma'^\sigma (g'^\sigma)^{-1} \cap \Gamma'^\sigma] < \infty \text{ and } [\Gamma'^\sigma : (g'^\sigma)^{-1} \Gamma'^\sigma g'^\sigma \cap \Gamma'^\sigma] < \infty\}.$$

As in §2.1 we consider G^σ(resp. G'^σ) as a subgroup of $\hat{G}^\sigma$(resp. $\hat{G}'^\sigma$) and D^σ(resp. D'^σ) as a submanifold of $\hat{D}^\sigma$(resp. $\hat{D}'^\sigma$).

Proposition 3. $\rho^\sigma(G^\sigma)$ is contained in G'^σ.

Proof. Take an element $d_0^\sigma \in D^\sigma \subset \hat{D}^\sigma$. Without loss of generality we may assume that

$$\hat{\Phi}^\sigma(d_0^\sigma) \in D'^\sigma \subset D'^\sigma.$$

As is mentioned in §2.1, by Kajdan [3], we have

$$G^\sigma = \{\hat{g}^\sigma \in \hat{G}^\sigma \mid \hat{g}^\sigma d_0^\sigma \in D^\sigma\}$$
$$G'^\sigma = \{\hat{g}'^\sigma \in \hat{G}'^\sigma \mid \hat{g}'^\sigma \hat{\Phi}^\sigma(d_0^\sigma) \in D'^\sigma\}$$

If $g^\sigma \in G^\sigma$, then $g^\sigma d_0^\sigma \in D^\sigma$; hence, from the continuity of ϕ^σ, it follows that

$$\hat{\Phi}^\sigma(g^\sigma d_0^\sigma) \in \hat{\Phi}^\sigma(D^\sigma) \subset D'^\sigma.$$

This gives

$$\hat{\rho}^\sigma(g^\sigma)\hat{\Phi}^\sigma(d^\sigma) = \hat{\Phi}^\sigma(g^\sigma d_0^\sigma) \in D'^\sigma.$$

Therefore we have

$$\hat{\rho}^\sigma(g^\sigma) \in G'^\sigma \quad \text{for all} \quad g^\sigma \in G^\sigma.$$

We define

$$\rho^\sigma = \hat{\rho}^\sigma|_{G^\sigma}, \qquad \tilde{\Phi}^\sigma = \hat{\Phi}^\sigma|_{D^\sigma}.$$

Thus we obtain the maps $\rho^\sigma : G^\sigma \to G'^\sigma$ and $\tilde{\Phi}^\sigma : D^\sigma \to D'^\sigma$ satisfying the following relation:

$$\tilde{\Phi}^\sigma(g^\sigma d^\sigma) = \rho^\sigma(g^\sigma)\tilde{\Phi}^\sigma(d^\sigma)$$

for all $g^\sigma \in G^\sigma$ and $d^\sigma \in D^\sigma$.

§2.5. Proof of Theorem 1.

Let G_0^σ and $G_0'^\sigma$ be as in Theorem 1. Then, from the result of §2.4 and [3], G^σ is dense in G_0^σ and

$$\Phi^\sigma(hy) = \rho^\sigma(h)\Phi^\sigma(y)$$

for all $h \in G^\sigma$ and $y \in D^\sigma$. We put

$$G_2 = \{(g,g') \in G_0^\sigma \times G_0'^\sigma \mid \Phi^\sigma(gy) = g' \Phi^\sigma(y) \quad \text{for all} \quad y \in D^\sigma\}$$

Since the set $\{(g,\rho^\sigma(g)) \mid g \in G^\sigma\}$ is contained in G_2 and G^σ is dense in G_0^σ, the projection map $p_1 : G_2 \to G_0^\sigma$ is surjective.

Lemma 4. G_2 is a reductive Lie group.

Proof. Let K be the kernel of the projection map $p_1 : G_2 \to G_0^\sigma$. Then we have

$$K = \{(1,g') \in G_0^\sigma \times G_0'^\sigma \mid \Phi^\sigma(y) = g'\Phi^\sigma(y) \quad \text{for all} \quad y \in D^\sigma\}$$

$$\cong \{g' \in G_0'^\sigma \mid \Phi^\sigma(y) = g'\Phi^\sigma(y) \quad \text{for all} \quad y \in D^\sigma\}$$

$$= \bigcap_{y \in D^\sigma} \mathrm{Iso}(\Phi^\sigma(y))$$

where $\mathrm{Iso}(\Phi^\sigma(y))$ is the isotropy subgroup of $\Phi^\sigma(y)$ in

$G_0'^{\sigma}$. Thus K is a compact Lie group, and therefore a reductive Lie group. Since the sequence

$$1 \longrightarrow K \longrightarrow G_2 \xrightarrow{p_1} G_0^{\sigma} \longrightarrow 1$$

is an exact sequence of Lie groups with K and G_0^{σ} reductive, it follows that G_2 is also reductive.

Proof of Theorem 1.(cf. [1])

Decompose G_2 into a direct product of simple Lie groups and simple tori, and define G_1^{σ} to be the product of those simple factors of G_2 which map nontrivially to G_0^{σ}. Then the kernel of the map from G_1^{σ} to G_0^{σ} is finite, and hence G_1^{σ} is a finite covering Lie group of G_0^{σ}. Define the action of G_1^{σ} on D^{σ} and a homomorphism $\rho_1^{\sigma}: G_1^{\sigma} \rightarrow G_0'^{\sigma}$ by

$$(g,g')\cdot y = gy, \qquad \rho_1^{\sigma}(g,g') = g'$$

for all $(g,g') \in G_1^{\sigma} \subset G_2 \subset G_0^{\sigma} \times G_0'^{\sigma}$. Then we have

$$\Phi^{\sigma}((g,g')\cdot y) = \Phi^{\sigma}(gy) = g'\Phi^{\sigma}(y)$$
$$= \rho_1^{\sigma}(g,g')\Phi^{\sigma}(y)$$

for all $(g,g') \in G_1^{\sigma}$ and $y \in D^{\sigma}$. Thus we obtain a homomorphism of Lie groups ρ_1^{σ} from a finite covering

G_1^σ of G_0^σ to $G_0'^\sigma$ satisfying

$$\tilde{\Phi}^\sigma(h_1 Y) = \rho_1^\sigma(h_1)\tilde{\Phi}^\sigma(y)$$

for all $h_1 \in G_1^\sigma$ and $y \in D^\sigma$. Therefore the pair $(\rho_1^\sigma, \tilde{\Phi}^\sigma)$ is equivariant.

CHAPTER III. GROUP THEORETICAL ABELIAN SCHEMES OVER AN ARITHMETIC VARIETY.

§3.1. Universal Family of Abelian Varieties.

Let V be a Q-vector space of dimension $2n$, A an alternating bilinear form on V, and L a Lattice in V; let $Sp(V,A)$ be the symplectic group and put

$$Sp(L,A) = \{g \in Sp(V,A) \mid gL = L\}.$$

The symmetric domain associated to $Sp(V,A)$ is isomorphic to the Siegel upper half space h^n. Let Γ be a subgroup of $Sp(L,A)$ of finite index with no elements of finite order. Although the quotient space $\Gamma\backslash h^n$ is not compact, it is a quasiprojective variety, i.e. a Zariski open subset of a projective variety, by the theorem of Baily and Borel [2].

We can construct a fibre bundle over the quasi-projective variety $\Gamma\backslash h^n$, called universal family of abelian varieties, as described below.

Let $Sp(V,A) \ltimes V$ be the semidirect product of $Sp(V,A)$ and V. Then $Sp(V,A) \ltimes V$ acts on $h^n \times V$ by

$$(g,v)\cdot(z,u) = (gz, gu = v)$$

for all $(g,v) \in Sp(V,A) \ltimes V$ and $(z,u) \in h^n \times V$. This induces the action of $\Gamma \ltimes L$ on $h^n \times V$. We put

$$X' = \Gamma'\backslash h^n, \qquad Y' = (\Gamma' \times L)\backslash(h^n \times V).$$

Then we obtain a commutative diagram

$$\begin{array}{ccc} h^n \times V & \xrightarrow{\alpha} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi'} \\ h^n & \xrightarrow{\beta} & X' \end{array}$$

where α and β are obvious projections, $p(z,u) = z$, and π' is determined by $\beta p = \pi'\alpha$. Thus we have a fibre bundle $\pi':Y' \to X'$ whose fibres are complex tori isomorphic to $L\backslash V$. In fact, this fibre bundle has the structure of a family of abelian varieties associated to a PEL type which has been studied by G. Shimura extensively.

Proposition 5. (Shimura) Let Ω be a normal admissible PEL type (see [10] for definition). Then there is a family of abelian varieties $f:V \to U$, denoted by $F(\Omega)$, with the following properties:

(a) U and V are Zariski open subsets of projective varieties.

(b) U, V and f are defined over a number field $k(\Omega)$.

(c) For each $\sigma \in \mathrm{Aut}(C)$, there exists an isomorphism of

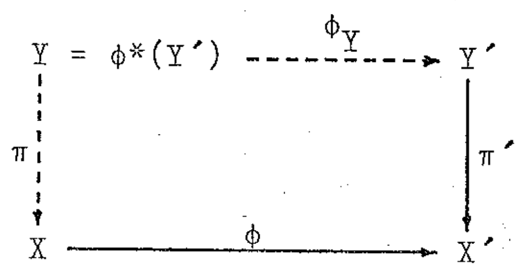
$F(\Omega)^\sigma$ to $F(\Omega^\sigma)$ defined over $k(\Omega^\sigma)$.

Proof. See [8] and [9].

Thus we have a fibre variety $\pi':Y' \to X'$ whose fibres are abelian varieties. It is called a universal family of abelian varieties.

§3.2. Group Theoretical Abelian Schemes over X.

Consider a morphism $\phi : X \to X'$ of arithmetic varieties and its lifting $\tilde{\phi} : D \to D'$ with $D' = h^n$ and $X' = \Gamma' \backslash h^n$ as in §3.1. If $\pi' : Y' \to X'$ is a universal family of abelian varieties, then we can consider the pullback bundle $\pi : Y \to X$ of $\pi' : Y' \to X'$ by $\phi : X \to X'$ as in the following diagram:



If there exist semisimple Lie groups G_R and G'_R associated to D and D′, and a homomorphism $\rho : G_R \to G'_R$ such that the pair $(\rho, \tilde{\phi})$ is equivariant, then it is known(e.g. [4],[7]) that $\pi : Y \to X$ is a morphism of projective varieties and that each fibre is an abelian variety. Thus we obtain a fibre variety whose fibres are abelian varieties. Such a fibre variety is called a group theoretical abelian scheme over X.

§3.3. Conjugation by σ and Proof of Theorem 2.

Let $\phi: X \to X'$ and $\tilde{\phi}: D \to D'$ be as in §3.1, i.e.

$$D' = h^n, \qquad X' = \Gamma \backslash h^n.$$

Then X' is a quasiprojective variety, and it is known (e.g. [7]) that X'^{σ} is an arithmetic variety for each $\sigma \in \mathrm{Aut}(C)$, more precisely, X'^{σ} is biregularly isomorphic to $\Gamma'^{\sigma} \backslash h^n$ for some arithmetic subgroup Γ'^{σ} of $Sp(V,A)$.

By applying the theorem of Baily-Borel[2], we can easily modify the proof of Theorem 1 for noncompact X', i.e. we consider the finite unramified covering manifolds of X' belonging to a cofinal system as embedded in projective varieties. Thus Theorem 1 is still true for $D' = h^n$ and $X' = \Gamma \backslash h^n$.

Consider $\phi^{\sigma}: X^{\sigma} \to X'^{\sigma}$ and its lifting $\tilde{\phi}^{\sigma}: D^{\sigma} \to D'^{\sigma} \cong h^n$ as in Chapter II. Then by Theorem 1 there are Lie groups G_1^{σ} and $G_0'^{\sigma}$ and a homomorphism $\rho_1^{\sigma}: G_1^{\sigma} \to G_0'^{\sigma}$ such that $(\rho_1^{\sigma}, \tilde{\phi}^{\sigma})$ is an equivariant pair. Therefore, as is mentioned in §3.2, there exists a group theoretical abelian scheme $\pi^{(\sigma)}: Y^{(\sigma)} \to X^{\sigma}$ which is a pullback of a universal family of abelian varieties $\pi'^{(\sigma)}: Y'^{(\sigma)} \to X'^{\sigma}$ by $\phi^{\sigma}: X^{\sigma} \to X'^{\sigma}$:

$$\begin{array}{ccc} Y^{(\sigma)} & \xrightarrow{\phi_Y^{(\sigma)}} & Y'^{(\sigma)} \\ \Big\downarrow{\scriptstyle \pi^{(\sigma)}} & & \Big\downarrow{\scriptstyle \pi'^{(\sigma)}} \\ X^{\sigma} & \xrightarrow{\phi^{\sigma}} & X'^{\sigma} \end{array}$$

On the other hand, applying $\sigma \in \mathrm{Aut}(C)$ to the diagram for $\pi: Y \to X$ in §3.2, we obtain the following diagram:

$$\begin{array}{ccc} Y^{\sigma} & \xrightarrow{\phi_Y^{\sigma}} & Y'^{\sigma} \\ \Big\downarrow{\scriptstyle \pi^{\sigma}} & & \Big\downarrow{\scriptstyle \pi'^{\sigma}} \\ X^{\sigma} & \xrightarrow{\phi^{\sigma}} & X'^{\sigma} \end{array}$$

By Proposition 5(c), a family of abelian varieties over an arithmetic variety associated to a PEL type is unique up to a biregular isomorphism. In particular, this is true for X'^{σ}. Therefore $Y'^{(\sigma)}$ is biregularly isomorphic to Y'^{σ}; hence, by uniqueness of a pullback, $Y^{(\sigma)}$ is also isomorphic to Y^{σ}. This proves that $\pi^{\sigma}: Y^{\sigma} \to X^{\sigma}$ is a group theoretical abelian scheme over X^{σ}.

CHAPTER IV. (H_2)-EQUIVARIANCE AND PROOF OF THEOREM 3.

§4.1. Strongly Equivariant Case.

As is described in Chapter III, if $\rho: G_R \to Sp(V,A)$ is a homomorphism of Lie groups and $\tau: D \to h^n$ a holomorphic map such that (ρ,τ) is equivariant, then we can construct a group theoretical abelian scheme over an arithmetic variety X associated to (ρ,τ). Thus the problem of classification of all group theoretical abelian schemes reduces to the problem of classification of all corresponding equivariant pairs.

Satake considered the problem of classifying all symplectic representations $\rho: G_R \to Sp(V,A)$ such that (ρ,τ) is strongly equivariant(cf. [5], [6] and [7]). In his proof, he reduced the problem to the one for (H_2)-equivariant case, i.e. he showed that to classify all symplectic representations ρ of G_R corresponding to a strongly equivariant pair (ρ,τ) it is enough to classify all such ρ corresponding to an (H_2)-equivariant pair.

The above observation leads us to believe that similar reduction should hold true for the problem of conjugation of group theoretical abelian schemes over an arithmetic variety. Thus, in this section, we shall prove the conjugation problem for (H_2)-equivariant case, which is stated in Chapter I as Theorem 3, and state the conjugation problem for strongly equivariant case as a conjecture:

Conjecture. Let $\pi:Y \to X$ is a group theoretical abelian scheme over X associated to a strongly equivariant pair. Then $\pi^{\sigma}:Y^{\sigma} \to X^{\sigma}$ is a group theoretical abelian scheme over X^{σ} associated to another strongly equivariant pair.

§4.2. Algebras with Involution.

Let $\mathbb{G}$ be a connected adjoint semisimple algebraic group defined over a field K of characteristic zero. Assume that $\mathbb{G}$ does not contain a factor isomorphic to either an exceptional group or D_4. Then it is known [12] that there is a semisimple algebra with involution (A,i) defined over K such that $\mathbb{G}$ is isomorphic to $Aut(A,i)_0$, the connected component of the identity of Aut(A,i).

Definition. Let A_R be a semisimple algebra over R. An involution α on A_R is called positive(resp. negative) if and only if

$$Tr(X^{\alpha}X) > 0 (resp. < 0)$$

for all nonzero $X \in A_R$. It is called definite if it is either positive or negative, and indefinite otherwise.

Proposition 6. Let $\mathbb{G}$ be a semi-simple algebraic group defined over Q and (A,i) a semisimple algebra with involution defined also over Q such that

$$G_R \cong Aut(A_R, i)_0$$

where G_R(resp. A_R) is the group(resp. algebra) of real points of $\mathbb{G}$(resp. A). Then there is a one to one correspondence between the set of positive involutions on A_R commuting with i and the set of maximal compact subgroups of G_R.

Proof. See [12].

Lemma 7. Let A_R be a semisimple algebra with involution α over R. Then there exists a positive involution on A_R over R commuting with α.

Proof. See [11, p.64].

Proposition 8. Let (A,i) be a semisimple algebra with involution defined over Q. Assume

$$(A_R, i) = \bigoplus_{k=1}^{m} (A_k, i_k)$$

where A_k are simple, $i_1, \ldots, i_h$ are indefinite and $i_{h+1}, \ldots, i_m$ are definite. Then there exists an involution α on A commuting with i such that α is defined over Q and $\alpha_1, \ldots, \alpha_h$ are positive where α_k is is an involution on A_k induced by α for each $k = 1, \ldots, m$.

Proof. Let α_Y be an involution on A given by

$$\alpha_Y(X) = Y^{-1} X^i Y$$

for all $X \in A$. We define

$$V_Q = \{Y \in A_Q^{\times} \mid \alpha_Y \text{ commutes with } i\}$$

$$V_R = \{Y \in A_R^{\times} \mid \alpha_Y \text{ commutes with } i\}$$

$$W_R^{+} = \{Y \in A_R^{\times} \mid \alpha_1 > 0, \ldots, \alpha_h > 0\}.$$

where α_k is an involution on A_k such that

$$\alpha_k(X_k) = Y_k^{-1} X_k^{i_k} Y_k$$

for all $X_k \in A_k$ and $Y_k \in A_k$. Then W_R^{+} is open in $A_R^{\times}$ and by Borel's density theorem V_Q is dense in V_R. By Lemma 7 $W_R^{+} \cap V_R$ is not empty; hence it is a nonempty open subset of $A_R^{\times}$. Therefore $W_R^{+} \cap V_Q$ is also a nonempty set and an element α in $W_R^{+} \cap V_Q$ satisfies the condition of the proposition.

§4.3. Density of Δ^{σ}.

Let $\mathbb{G}$ be a connected semisimple adjoint algebraic group defined over Q. Assume the decomposition of the semisimple algebraic group G_R is

$$G_R = G_1 \times \cdots \times G_m$$

where $G_1, \ldots, G_h$ are noncompact simple Lie groups and $G_{h+1}, \ldots, G_m$ are compact simple Lie groups. We consider a subgroup M of $\mathbb{G}$ such that

$$M_R = M_1 \times \cdots \times M_m$$

with $M_k \subset G_k$ for each k.

Definition. M_R is called a quasimaximal compact subgroup of G_R if and only if the following conditions are satisfied:

M_k: maximal compact subgroup of G_k $(1 \leq k \leq h)$

G_k/M_k: symmetric space $(h+1 \leq k \leq m)$.

Proposition 9. There exists a subgroup M of $\mathbb{G}$ defined over Q such that M_R is a quasimaximal compact subgroup of G_R.

Proof. Let (A,i) be the semisimple algebra with involution such that

$$\mathbb{G} = \mathrm{Aut}(A,i)_0, \quad G_R = \mathrm{Aut}(A_R,i)_0.$$

If $G_R = G_1 \times \cdots \times G_m$, then (A,i) can be decomposed as follows:

$$(A,i) = (A_1,i_1) \times \cdots \times (A_m,i_m)$$

with

$$G_k = \mathrm{Aut}(A_k,i_k)_0$$

for each k. By Proposition 8 there exists an involution $\alpha = (\alpha_1,\ldots,\alpha_m)$ defined over Q such that $\alpha_1,\ldots,\alpha_h$ are positive. We define the subgroup M of $\mathbb{G}$ by

$$M = \{g \in \mathbb{G} \mid g\alpha = \alpha g\}.$$

If $M_R = M_1 \times \cdots \times M_m$, then we have

$$M_k = \{g_k \in G_k \mid g_k\alpha_k = \alpha_k g_k\}$$

for each k. So, by Proposition 6, M_k is a maximal compact subgroup of G_k whenever $1 \leq k \leq h$. Therefore M_R is a quasimaximal subgroup of G_R.

Proposition 10. Let D be the symmetric domain associated to G_R. Then there is a symmetry S on D which is contained in G_Q, the group of rational points of $\mathbb{G}$.

Proof. Take a subgroup M of $\mathbb{G}$ defined over Q such that M_R is a quasimaximal compact subgroup of G_R. Consider the symmetric space

$$D_M = G_R/M_R = D_1 \times \cdots \times D_m$$

where $D_k = G_k/M_k$ for each k, and G_k, M_k are simple components of G_R, M_R respectively. Let C(M) be the center of M. Then, since M is defined over Q, C(M) is also defined over Q. $C(M)_R$, the group of real points of C(M), is the same as $C(M_R)$, the center of the Lie group M_R. Thus $C(M)_R$ can be decomposed as follows:

$$C(M)_R = C(M_R) = C(M_1) \times \cdots \times C(M_m)$$

where $C(M_k)$ is the center of M_k for each k. Since the center of a maximal compact subgroup of a simple classical group associated to a symmetric domain is a one dimensional torus, $C(M_k)$ contains an element of order two for $1 \leq k \leq h$. Without loss of generality we may assume that $\mathbb{G}$ is Q-simple. Then each M_j for $1 \leq j \leq m$ is a Galois conjugate of some M_k for $1 \leq k \leq h$; hence M_j for $h+1 \leq j \leq m$ also contains an element S_j of order two. Define an element S in $C(M)_R$ by

$$S = (S_1,\dots,S_m).$$

Then S is a symmetry on D_M and, if considered as an action on D, it is also a symmetry on D. Since C(M) is defined over Q, we have

$$S^{\sigma} \in C(M)_R^{\sigma} = C(M)_R$$

for all $\sigma \in \mathrm{Aut}(C)$. S^{σ} is also an element of order two, and hence it is a symmetry on D_M. But the symmetry on D_M contained in $C(M)_R$ is unique; so we have

$$S^{\sigma} = S \quad \text{for all} \quad \sigma \in \mathrm{Aut}(C).$$

Therefore S is contained in G_Q.

Proposition 11. Let G, D, G^{σ} and D^{σ} be as in Chapter 2 and S_z the symmetry on D at z contained in G. Then there exists an element w in D^{σ} such that S_w is the symmetry on D^{σ} at w and S_w is contained in G^{σ}.

Proof. Let X be an arithmetic variety as in Chapter 2. Consider a cofinal system $\{X_k\}$ of finite unramified covering manifolds of X where $X_k = \Gamma_k \backslash D$. We put

$$X_{k,z} = (S_z \Gamma_k S_z \cap \Gamma_k) \backslash D$$

for each k. Then $\{X_{k,z}\}$ is also a cofinal system of unramified covering manifolds of X and, since

$$\begin{aligned} S_z(\Gamma_k \cap S_z\Gamma_k S_z)x &= (S_z\Gamma_k \cap \Gamma_k S_z)x \\ &= (S_z\Gamma_k S_z \cap \Gamma_k)S_z x \end{aligned}$$

for each k and $x \in D$, S_z induces the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{3,z} & \longrightarrow & X_{2,z} & \longrightarrow & X_{1,z} & \longrightarrow & X \\ & & \Big\downarrow S_{3,z} & & \Big\downarrow S_{2,z} & & \Big\downarrow S_{1,z} & & \\ \cdots & \longrightarrow & X_{3,z} & \longrightarrow & X_{2,z} & \longrightarrow & X_{1,z} & \longrightarrow & X \end{array}$$

where each $S_{k,z}$ is a symmetry on $X_{k,z}$. Applying σ to this diagram, we obtain

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^{\sigma}_{3,z} & \longrightarrow & X^{\sigma}_{2,z} & \longrightarrow & X^{\sigma}_{1,z} & \longrightarrow & X^{\sigma} \\ & & \Big\downarrow S^{\sigma}_{3,z} & & \Big\downarrow S^{\sigma}_{2,z} & & \Big\downarrow S^{\sigma}_{1,z} & & \\ \cdots & \longrightarrow & X^{\sigma}_{3,z} & \longrightarrow & X^{\sigma}_{2,z} & \longrightarrow & X^{\sigma}_{1,z} & \longrightarrow & X^{\sigma} \end{array}$$

and each $S^{\sigma}_{k,z}$ is a symmetry of $X^{\sigma}_{k,z}$. Therefore there is a symmetry S_w on D^{σ} at a point w which is a common lifting of $S^{\sigma}_{k,z}$.

We define Δ and Δ^{σ} by

$$\Delta = \{z \in D \mid S_z \in G\}$$

$$\Delta^{\sigma} = \{w \in D^{\sigma} \mid S_w \in G^{\sigma}\}$$

Then, by Proposition 11, we easily obtain

<u>Corollary</u>. Δ is dense in D if and only if Δ^{σ} is dense in D^{σ}.

<u>Lemma 12</u>. If there exists a point z in D such that S_z is contained in G_Q, then Δ is dense in D.

<u>Proof</u>. Note that G_Q is contained in G and that it is dense in G_R. Let S_z be the symmetry contained in G_Q. Then, for each $g \in G_R$, we have

$$S_{(gz)} = gSg^{-1}$$

So, if g is an element of G_Q, $S_{(gz)}$ is also an element of $G_Q \subset G$; hence gz is contained in Δ. Since G_R acts transitively on D and G_Q is dense in G_R, the set $\{gz \mid g \in G\}$, which is a subset of Δ, is dense in D.

Therefore Δ is also dense in D.

<u>Proposition 13</u>. Assume that G_R is a classical group with no D_4 factors. Then Δ^σ is dense in D^σ.

<u>Proof</u>. By Proposition 10 and Lemma 12, Δ is dense in D. Therefore, by Corollary to Proposition 11, Δ^σ is dense in D^σ.

§4.4. <u>(H_2)- equivariance and Proof of Theorem 3</u>.

As in §2.5 we can construct a finite covering G_1^σ of G_0^σ and a homomorphism $\rho_1^\sigma : G_1^\sigma \to G_0'^\sigma$ such that $(\rho_1^\sigma, \Phi^\sigma)$ is equivariant, i.e.

$$\Phi^\sigma(h_1 y) = \rho_1^\sigma(h_1)\Phi^\sigma(y)$$

for all $h_1 \in G_1^\sigma$ and $y \in D^\sigma$. In this section, we shall show that $(\rho_1^\sigma, \Phi^\sigma)$ satisfies (H_2)-condition.

<u>Proposition 14</u>. Let Δ^σ and $\rho^\sigma : G^\sigma \to G'^\sigma$ be as in Chapter II. Then

$$\rho^\sigma(S_y) = S_{\Phi^\sigma(y)}$$

for all $y \in \Delta^\sigma$.

<u>Proof</u>. Let y be a point in Δ^σ such that S_y is contained in G^σ. Applying Proposition 11 for σ^{-1}, we obtain a symmetry S_z on D for some $z \in \Delta$. Since

$$\rho(S_z) = S_{\Phi(z)}, \qquad \rho(G) \subset G',$$

$\rho(S_z)$ is a symmetry $S_{\Phi(z)}$ on D' with $\Phi(z) \in \Delta'$. Applying Proposition 11 once again, we obtain a symmetry S_w on D'^σ for some $w \in \Delta'^\sigma$. From the definition of ρ^σ, it follows that

$$S_w = \rho^\sigma(S_y).$$

Since

$$S_w\varphi^\sigma(y) = \rho^\sigma(S_y)\varphi^\sigma(y) = \varphi^\sigma(S_y y) = \varphi^\sigma(y),$$

we have $S_w = S_{\varphi^\sigma(y)}$. Therefore

$$\rho^\sigma(S_y) = S_{\varphi^\sigma(y)} \quad \text{for all} \quad y \in \Delta^\sigma.$$

<u>Proposition 15</u>. The pair, $(\rho_1^\sigma, \varphi^\sigma)$, is strongly equivariant, i.e. it is equivariant and

$$\varphi^\sigma \cdot S_y = S_{\varphi^\sigma(y)} \cdot \phi$$

for all $y \in D^\sigma$.

<u>Proof</u>. Given a cofinal system $\{X_k\}$(resp. $\{X'_k\}$) of finite unramified covering manifolds of X(resp. X') and a point $z \in D$(resp. $z' \in D'$), we put

$$X_{k,z} = (S_z \Gamma_k S_z \cap \Gamma_k)\backslash D$$

$$X'_{k,z'} = (S_{z'} \Gamma'_k S_{z'} \cap \Gamma'_k)\backslash D'.$$

If S_y is a symmetry on D^σ at $y \in \Delta^\sigma$, then by Proposition 11 there is a symmetry S_z on D at $z \in \Delta$. For each k the relation

$$\varphi \cdot S_z = S_{\varphi(z)} \cdot \varphi$$

induces the following commutative diagram:

$$
\begin{array}{ccc}
X_{k,z} & \xrightarrow{\ \phi_{k,z}\ } & X'_{k,\tilde{\phi}(z)} \\
\Big\downarrow{\scriptstyle (S_z)_k} & & \Big\downarrow{\scriptstyle (S_{\tilde{\phi}(z)})_k} \\
X_{k,z} & \xrightarrow{\ \phi_{k,z}\ } & X'_{k,\tilde{\phi}(z)}
\end{array}
$$

Applying σ to this diagram, we obtain

$$
\begin{array}{ccc}
X^{\sigma}_{k,z} & \xrightarrow{\ \phi^{\sigma}_{k,z}\ } & X'^{\sigma}_{k,\tilde{\phi}(z)} \\
\Big\downarrow{\scriptstyle (S_z)^{\sigma}_k = (S_y)_k} & & \Big\downarrow{\scriptstyle (S_{\tilde{\phi}(z)})^{\sigma}_k = (S_{\tilde{\phi}^{\sigma}(y)})_k} \\
X^{\sigma}_{k,z} & \xrightarrow{\ \phi^{\sigma}_{k,z}\ } & X'^{\sigma}_{k,\tilde{\phi}(z)} .
\end{array}
$$

This induces

$$
\begin{array}{ccc}
D^{\sigma} & \xrightarrow{\ \tilde{\phi}^{\sigma}\ } & D'^{\sigma} \\
\Big\downarrow{\scriptstyle S_y} & & \Big\downarrow{\scriptstyle S_{\tilde{\phi}^{\sigma}(y)}} \\
D^{\sigma} & \xrightarrow{\ \tilde{\phi}^{\sigma}\ } & D'^{\sigma}
\end{array}
$$

which gives $\mathfrak{F}^{\sigma}\cdot S_y = S_{\mathfrak{F}^{\sigma}(y)}\cdot\mathfrak{F}^{\sigma}$ for all $y \in \Delta^{\sigma}$. From the density of Δ^{σ}, it follows that

$$\mathfrak{F}^{\sigma}\cdot S_y = S_{\mathfrak{F}^{\sigma}(y)}\cdot\mathfrak{F}^{\sigma} \quad \text{for all } y \in D^{\sigma}.$$

<u>Lemma 16</u>. Let $\rho : G_R \to G'_R$ be a homomorphism of Lie groups and $\tau : D \to D'$ a holomorphic map such that (ρ, τ) is is strongly equivariant. Let G_0(resp. G'_0) be the connected component of the identity of Aut(D)(resp. Aut(D')). Then G_1 is generated by products of even number of symmetries on D, and there is a homomorphism $\mu_1 : G_1 \to G'_0$ such that

$$\mu_1(S_{z_1}\cdots S_{z_{2k}}) = S_{\tau(z_1)}\cdots S_{\tau(z_{2k})}$$

for all products of even number of symmetries on D considered as elements of G_1.

<u>Proof</u>. See e.g. [5].

Applying Lemma 16 to $(\rho_1^{\sigma}, \mathfrak{F}^{\sigma})$, we obtain a finite covering G_2 of G_0^{σ} and a homomorphism $\rho_2 : G_2 \to G_0'^{\sigma}$ such that

$$\rho_2(S_{y_1}\cdots S_{y_{2k}}) = S_{\mathfrak{F}^{\sigma}(y_1)}\cdots S_{\mathfrak{F}^{\sigma}(y_{2k})}$$

for all products of even number of symmetries on D^σ. Since $\tilde{\phi}^\sigma \cdot S_y = S_{\tilde{\phi}^\sigma(y)} \cdot \tilde{\phi}^\sigma$ for all $y \in D^\sigma$ by Proposition 15, we obtain

$$\tilde{\phi}^\sigma(S_{y_1} \cdots S_{y_{2k}} y) = S_{\tilde{\phi}^\sigma(y_1)} \cdots S_{\tilde{\phi}^\sigma(y_{2k})} \tilde{\phi}^\sigma(y)$$

$$= \rho_2(S_{y_1} \cdots S_{y_{2k}}) \tilde{\phi}^\sigma(y).$$

So $(\rho_2, \tilde{\phi}^\sigma)$ is strongly equivariant.

<u>Lemma 17</u>. Let D, D', G_0 and G_0' be as in Lemma 16, and $\tau : D \to D'$ a holomorphic map such that

$$\tau \cdot S_z = S_{\tau(z)} \cdot \tau$$

for all $z \in D$. Let G_1 be a finite covering of G_0 and $\mu_1 : G_1 \to G_0'$ be a homomorphism such that (μ_1, τ) is equivariant. Then such μ_1 is unique.

<u>Proof</u>. (cf. [1, p.173]) Let $\mu_2 : G_2 \to G_0'$ be another homomorphism such that (μ_2, τ) is equivariant. Without loss of generality we may assume that $G_1 = G_2$. We put

$$K = \{ g' \in G_0' \mid \tau(x) = g'\tau(x) \text{ for all } x \in D \}.$$

Then K is compact. Defining β by

$$\beta(g) = \mu_1(g)(\mu_2(g))^{-1},$$

we have a homomorphism from G_1 to K. Since G_0 is a semi-simple Lie group with no compact factors, there are no nontrivial homomorphisms from G_1 to K; so we have $\mu_1 = \mu_2$.

<u>Proposition 18</u>. $(\rho_1^\sigma, \tilde{\Phi}^\sigma)$ is (H_2)-equivariant.

<u>Proof</u>. It follows from Lemma 17 that $\rho_1^\sigma = \rho_2$; hence we have

$$\rho_1^\sigma(S_{y_1} \cdots S_{y_{2k}}) = S_{\tilde{\Phi}^\sigma(y_1)} \cdots S_{\tilde{\Phi}^\sigma(y_{2k})}$$

for all products of even number of symmetries on D^σ. Let G_3 be the subgroup of G_0^σ generated by all products of even number of symmetries on D^σ contained in G^σ. Then G_3 is contained in G^σ and is dense in G_0^σ. Since $\rho^\sigma(S_y) = S_{\tilde{\Phi}^\sigma(y)}$ for all $S_y \in G^\sigma$, we have

$$\rho_1^\sigma|_{G_3} = \rho^\sigma|_{G_3}$$

which gives

$$\rho_1^\sigma(S_y) = S_{\tilde{\Phi}^\sigma(y)}$$

for all $S_y \in G_3$. Thus it follows from the density of G_3 in G_0^σ that

$$\rho_1^\sigma(S_y) = S_{\tilde{\Phi}^\sigma(y)}$$

for all symmetries on D^σ.

Proof of Theorem 3.

Let $\rho:G_R \to Sp(V,A)$ be a homomorphism of Lie groups and $\phi:D \to h^n$ a holomorphic map such that the pair, (ρ,ϕ), is (H_2)-equivariant. Let $\pi:Y \to X$ be the group theoretical abelian scheme over X associated to (ρ,ϕ). Then, as is proved in Chapter 3, $\pi^\sigma:Y^\sigma \to X^\sigma$ is the group theoretical abelian scheme over X^σ associated to $(\rho_1^\sigma,\phi^\sigma)$. The pair, $(\rho_1^\sigma,\phi^\sigma)$, is (H_2)-equivariant by Proposition 18 and this proves Theorem 3.

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