GAUSSIAN BEAMS

A Dissertation presented

by

Li-yeng Sung

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August 1983

STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

THE GRADUATE SCHOOL

Li-yeng Sung

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Michael Taylor, Professor of Mathematics
Dissertation Director

David Ebin, Professor of Mathematics
Chairman of Defense

Daryl Geller, Assistant Professor of Mathematics

Robert Schrader, Visiting Professor, Institute for Theoretical Physics
Outside member

This dissertation is accepted by the Graduate School.

Dean of the Graduate School

August 1983

Abstract of the Dissertation

Gaussian Beams

by

Li-yeng Sung

Doctor of Philosophy

in

Department of Mathematics

State University of New York at Stony Brook

1983

There are two main results in this thesis. The first one is the construction of Gaussian beam solutions to perfectly reflecting boundary value problems in section 3.4. It is a generalization of a result of Ralston (cf.[10]). The second one is the construction of diffracted beams in chapter 4. It is based on the grazing ray parametrix of Melrose and Taylor. The width of the diffracted beams is found to be $\rho^{-1/8+\varepsilon}$ whereas the width of the Gaussian beams is $\rho^{-\frac{1}{2}+\varepsilon}$.

There are also three other new results. Section 1.4

contains a new construction of the phase function of a Gaussian beam by a generalized method of characteristics. We prove in lemma 3.2.2 that a system (P, B_j) is perfectly reflecting iff its adjoint system (P^*, C_k) is perfectly reflecting. The classical result on coercive boundary conditions is obtained as a corollary. Theorem 3.4.2 contains a new proof of a result of Majda and Osher (cf. [3]).

Applications to the propagation of singularities of hyperbolic equations, microlocal regularity of elliptic coercive boundary value problems, reflection and diffraction of singularities are also given in the thesis.

To the memory of my father.

Acknowledgements

I would like to thank my advisor Michael Taylor for his guidance and encouragement throughout the research of this thesis.

My sincere thanks go to the Mathematics Department of Rice University for its support, especially for the support I received in the years 1980-82 when I was away from Rice. I am also grateful to the Mathematics Department at Stony Brook for the support of the last year of my graduate study.

Without the understanding and support of the members of my family I would have never persued a career in mathematics. My gratitude to each of them is unmeasurable.

Last but not least, I want to thank Miss Susanne Brenner for typing up part of the thesis and for correcting the numerous gramatical errors in it.

Table of Contents

Chapter 0 Introduction

Section 1 Gaussian Beams

A Gaussian beam is a function of the form

$$u(x,\rho) = e^{i\rho\psi(x)}(a_0(x) + a_1(x)\rho^{-1} + \cdots + a_N(x)\rho^{-N}).$$

$\psi(X), a_0(X), \ldots, a_N(x)$ are smooth functions on R^n. ρ is a positive parameter. Im $\psi(x)$ satisfies the following condition:

There exists a smooth curve Γ in R^n such that

$$\operatorname{Im} \psi(x) \geq \gamma \cdot \operatorname{dist}(x,\Gamma)^2, \text{ for some } \gamma > 0. \qquad (1)$$

Typically, Γ is the characteristic curve (i.e. the projection of the null-bicharacteristic on R^n) of a differential operator P and u is an asymptotic solution of $Pu = 0$.

It follows from (1) that $u = O(\rho^{-\infty})$ away from Γ. We can therefore multiply the a_j's by cut-off functions so that u is supported in an arbitrarily small neighborhood of Γ. This is why they are called beams.

The construction of Gaussian beam solutions to strictly hyperbolic equations will be carried out in chapter 1. The presentation basically follows Ralston ([10]) except for the solution of the eikonal equation. There we consider the Gaussian beam as an oscillatory function corresponding to a strictly positive almost analytic Lagrangian submanifold of order one (cf. section 0.2) and solve the eikonal equation by a generalization of the method of characteristics.

In chapter 2, asymptotic solutions to elliptic equations similar to (1) are constructed. A similar construction can be found in Ralston ([10]) under more restrictive conditions.

Via a Green's formula, an understanding of the asmptotic solutions to the adjoint operator will lead to results about propagations of singularities of the original operator. Thus propagation of singularities of hyperbolic initial value problems and microlocal regularity of elliptic boundary value problems are proved as applications in chapters 1 and 2.

In chapter 3, the construction of chapters 1 and 2 are combined to treat the mixed initial-boundary value problems. As application, various results in the reflection of singularities due to Lax-Nirenberg ([8]) and Majda-Osher ([3]) are proved. The Gaussian beam construction also leads to a new proof that the perfect reflecting condition is essential for the result in reflection of singularities to hold (cf. theorem 3.4.2).

In order to treat the grazing ray problem, we introduce a more complicated ansatz in chapter 4 which will be called the diffracted beam. These diffracted beams are obtained by using the Fourier-Airy integral operators of Melrose and Taylor.

For Gaussian beams, it follows from (1) that given any $\varepsilon > 0$, $u(x,\rho) = O(\rho^{-\infty})$ for $x \in \{x: \mathrm{dist}(x,\Gamma) \geq \rho^{-\frac{1}{2}+\varepsilon}\}$. The corresponding result for the diffracted beam is $u(x,\rho) = O(\rho^{-\infty})$ for $x \in \{x: \mathrm{dist}(x,\Gamma) \geq \rho^{-1/8+\varepsilon}\}$.

Using these diffracted beams, the result of Melrose and Taylor on diffraction of singularities are proved in chapter 4.

The rest of this chapter will record a collection of basic definitions and results that will be needed in the following chapters.

Section 2 Almost Analytic Machinery

Since the Gaussian beams have complex phase functions, we will be forced to go from R^n into C^n all the time. The appropriate machinery will be recorded in this section. We will follow Melin and Sjöstrand ([5]) and refer the readers to that paper for the proof of the assertions.

Definition 2.1

Let $\Omega \subset C^n$ be an open set. If $f \in C^\infty(\Omega)$, we say that f is almost analytic if $\bar{\partial} f$ vanishes to infinite order on $\Omega_R = \Omega \cap R^n$.

It is easy to prove that every $f \in C^\infty(\Omega_R)$ has an almost analytic extension, uniquely determined up to equivalence.

Definition 2.2

Let f_1 and $f_2 \in C^\infty(\Omega)$, $\Omega \subset C^n$. We say that f_1 and f_2 are equivalent, denoted by $f_1 \sim f_2$, if $f_1 - f_2$ vanishes to infinite order at Ω_R.

The technical tool in obtaining asymptotic expansions for Gaussian beams is the complex stationary phase method.

Let $a(x,w)$, $x \in R^n$, $w \in R^k$ be a C^∞ function defined in a neighborhood of $(0,0)$. We suppose that $d_x a(0,0)=0$, $\det\left(\frac{\partial^2 a}{\partial x_i \partial x_j} \right) \neq 0$ and that $\operatorname{Im} a \geq 0$ with equality at $(0,0)$. $\therefore$ we have $\operatorname{Im} \left(\frac{\partial^2 a}{\partial x_i \partial x_j}\right) \geq 0$.

Theorem 2.1

Let a be as above. Then there are neighborhoods U and V of the origin in R^n and R^k respectively and differential operators $C_{\nu,w}(D)$ of order $\leq 2\nu$ which are C^∞ functions of $w \in V$ such that we have the asymptotic expansion

$$\int e^{i\rho a(x,w)} u(x)\, dx \sim \sum_{\nu=0}^{\infty} \rho^{-\nu-n/2}\, e^{i\rho \tilde{a}(z(w),w)} \cdot (C_{\nu,w}(D)\tilde{u})(z(w)) \qquad (1)$$

Here u is a C^∞ function with compact support in U. $\tilde{a}$ and $\tilde{u}$ are almost analytitic extensions of a and u. z(w) is the solution of $d_z a(z,w) = 0$.

We shall also need the concept of an almost analytic positive Lagrangian submanifold. First of all, we have the following definition.

<u>Definition 2.3</u>

Let $\Omega \subset C^n$ be an open set and let $M \subset \Omega$ be a real submanifold of real dimension 2k. M is called an almost analytic submanifold if for each real point z_0 of M there exists a neighborhood U of z_0 and C^∞ functions $f_{k+1}, \ldots, f_n$ such that M is given by $f_{k+1}(z) = \ldots = f_n(z) = 0$ in U and

$|\bar{\partial} f_j(z)| \leq C_N(|\text{Im } z|^N + \max_\nu |f_\nu(z)|^N)$, $z \in U$, for all $N \in Z_+$ and the complex linear differentials $df_{k+1}(z), \cdots df_n(z)$ are linearly independent over C.

<u>Definition 2.4</u>

Let M_1 and M_2 be two real submanifolds of an open set $\Omega \subset C^n$. We say that M_1 and M_2 are equivalent (and we write $M_1 \sim M_2$) if they have the same intersection

with R^n and the same real dimension and if for every $\Omega' \subset\subset \Omega$ and $N \in Z_+$ we have
$\mathrm{dist}\,(z,M_2) \leq C_{N,\Omega'}\,|\mathrm{Im}\, z|^N$, $z \in \Omega' \cap M_1$ for some constant $C_{N,\Omega'}$.

Let $\omega = d\xi_1 \wedge dx_1 + \cdots + d\xi_n \wedge dx_n$ be the symplectic two form on R^{2n}. $\tilde{\omega} = d\Sigma_1 \wedge dz_1 + \cdots + d\Sigma_n \wedge dz_n$ is the extension of ω to C^{2n}.

Let U be an open subset of R^{2n} and $f: U \longrightarrow R^{2n}$ be a diffeomorphism.

Definition 2.5

If $f^*\omega = \omega$, then f is called a symplectomorphism. If $\tilde{f}$ is an almost analytic extension of the symplectomorphism f to a neighborhood of U in C^{2n}, then $\tilde{f}$ is called an almost analytic symplectomorphism.

Definition 2.6

Let M be an almost analytic submanifold of C^{2n}. Assume that for any real point $(x_0,\xi_0) \in M$ there exists a neighborhood U of (x_0,ξ_0) in M such that by an almost analytic symplectomorphism, (x_0,ξ_0) is mapped to (y_0,η_0) and U is equivalent to $\{(d_y h(y),y)$: y belongs to an open neighborhood of (y_0,η_0) in C^{2n} and h is an almost analytic function with $\mathrm{Im}\, h \geq 0$ and $\mathrm{Im}\, h(y_0) = 0\}$. Then M is called an almost analytic positive Lagrangian submanifold.

There is an important special case.

Definition 2.7

An almost analytic submanifold $M \subset C^n$ is called a strictly positive Lagrangian submanifold if

(i) $\dim_R M = 2n$,

(ii) $M_R = M \cap R^{2n}$ is a submanifold of M ,

(iii) $\tilde{\omega}|_M \sim 0$ (i.e. $\tilde{\omega}$ vanishes to infinite order on M_R)

(iv) $\frac{1}{i}\tilde{\omega}(v,\bar{v}) > 0$ for all $v \in T_p(M) \setminus T_p(M_R)$, $p \in M_R$. Here $\tilde{\omega}$ is regarded as a bilinear form on T_pC^{2n} and by $T_p(M_R)$ we denote the complexification of $T_p(M_R)$.

The order of M is defined to be the (real) dimension of M_R.

It is easy to see that a strictly positive Largrangian submanifold is indeed positive in the sense of definition 2.6.

Definition 2.8

Let $\psi \in C^\infty(R^n)$. ψ is said to be strictly positive at 0 if $\mathrm{Im}\,\psi(0) = 0$ and $\mathrm{Im}\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right) > 0$.

Section 3 Pseudodifferential Operators

In this section we will record some basic definitions of pseudodifferential operators. We will follow Taylor's book ([12]).

Definition 3.1

Let Ω be an open subset of $R^n, m, \rho, \delta \in R$, and suppose $0 \leq \rho, \delta \leq 1$. We define the symbol class $S^m_{\rho,\delta}(\Omega)$ to consist of the set of $p \in C^\infty(\Omega \times R^n)$ with the property that for any compact $K \subset \Omega$, any multi-indices α, β, there exists a constant $C_{K,\alpha,\beta}$ such that

$$|D^\beta_x D^\alpha_\xi \; p(x,\xi)| \leq C_{K,\alpha,\beta} \; (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for all $x \in K$, $\xi \in R^n$. We drop the Ω and use $S^m_{\rho,\delta}$ when the context is clear.

Most of the time we will only use the classical symbols S^m.

Definition 3.2

The symbol $p(x,\xi)$ belongs to $S^m(\Omega)$ if $p \in S^m_{1,0}(\Omega)$ and there are smooth $p_{m-j}(x,\xi)$, homogeneous of degree $m-j$ in ξ, for $|\xi| \geq 1$, i.e. $p_{m-j}(x,r\xi)=r^{m-j}p_{m-j}(x,\xi)$, $|\xi| \geq 1$, $r \geq 1$, such that

$$p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$$

where the asymptotic condition means that

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S_{1,0}^{m-N-1}(\Omega)$$

Definition 3.3

Let $p(x,\xi) \in S^m_{\rho,\delta}$. The operator $p(x,D)\colon C_c^\infty(\Omega) \to C^\infty(\Omega)$ is defined by the following formula.

$$p(x,D)u(x) = \int p(x,\xi)\cdot\hat{u}(\xi)e^{ix\cdot\xi}d\xi,$$

where $\hat{u}(\xi) = \frac{1}{(2\pi)n}\int u(x)e^{-ix\cdot\xi}dx.$

We say that $p(x,D) \in OPS^m_{\rho,\delta}$.

In order to extend $p(x,D)$ to a map between $C^\infty(\Omega)$ and itself, we need the concept of properly supported pseudodifferential operators. Since $p(x,D)$ induces a distribution on $\Omega \times \Omega$, we consider the following more general definition.

Definition 3.4

A distribution $A \in \mathcal{D}'(\Omega x \Omega)$ is said to be properly supported if supp A has compact intersection with $K \times \Omega$ and with $\Omega \times K$ for any compact $K \subset \Omega$.

For every $p(x,D) \in OPS^m_{\rho,\delta}$, we can construct a properly supported $q(x,D) \in OPS^m_{\rho,\delta}$ such that $p(x,D) - q(x,D) \in OPS^{-\infty}$. We can therefore assume all the pseudodifferential operators to be properly supported from now on.

There is a symbol calculus for the class of properly supported pseudodifferential operators with nice symbols.

We refer the readers to Taylor's book for a detailed discusstion. The following theorem due to Melin and Sjöstrand is more relevant to our purpose.

Theorem 3.1

Let $P \in OPS^m_{1-\delta,\delta}(R^n)$, $\delta > \frac{1}{2}$ and let $\psi \in C\,(R^n)$ satisfy $\mathrm{Im}\,\psi \geq 0$ and $d\psi \neq 0$ where $\mathrm{Im}\,\psi = 0$. Then we have

$$P(u\cdot e^{i\rho\psi}) \sim \sum_{\alpha=0}^{\infty} e^{i\rho\psi(x)}\tilde{p}^{(\alpha)}(x,\rho\cdot d\psi(x))\cdot$$

$$\frac{1}{\alpha!} D_y^{\alpha}(u(y)e^{i\rho\theta(x,y)})|_{y=x} \qquad (1)$$

with asympotic convergence in $S^m_{0,1}(R^n \times R_+)$. Here $u \in C_o^{\infty}(R^n)$ and $\tilde{p}$ is an almost analytic extension of p to a conic open neighborhood of $R^n \times R^n$ in $C^n \times C^n$. $\theta(x,y) = \psi(y)-\psi(x)-\langle y-x, d\psi(x)\rangle$.

This asymptotic expansion is of course a generalization of the well-known result in the case that ψ is real. We refer the readers to [5] for a proof. Note that at points where $d\psi(x)$ is real, we can use p in stead of $\tilde{p}$ on the right hand side of (1). This will be the case when we apply theorem 3.1 to Gaussian Beams.

Section 4 Wave Front Sets

In order to study the propagation of singularities, we need the concept of wave front sets. First of all, we recall the definition of the singular support of a distribution. Ω will denote an open subset of R^n.

Definition 4.1

Let $u \in \mathcal{D}'(\Omega)$. We define the singular support of u, denoted by sing supp u, to be the complement of the set $\{x: u \text{ is } C^\infty \text{ in a neighborhood of } x\}$.

It is easy to see that $x \notin$ sing supp u iff there exists a $\zeta \in C_0^\infty(\Omega)$ such that $\zeta(x) \neq 0$ and the Fourier transform of ζu is rapidly decreasing.

Let $u \in \mathcal{D}'(\Omega)$. The wave front set of u, denoted by WF(u), is a subset of $T^*(\Omega) \sim 0$ defined in the following way.

Definition 4.2

$(x_0, \xi_0) \notin WF(u)$ iff there is a $\zeta \in C_0^\infty(\Omega)$, $\zeta(x_0) \neq 0$, and a conic neighborhood Γ of ξ_0, such that, for every N,

$$|(\widehat{\zeta u})(\xi)| \leq C_N(1 + |\xi|)^{-N}, \ \xi \in \Gamma. \qquad (1)$$

Here C_N is some constant depending on N.

By the following lemma, we actually have a lot of freedom in choosing the function ζ.

Lemma 4.1

Let u, ζ, Γ and (x_0, ξ_0) be as in definition 4.2. If $\eta \in C_0^\infty(\Omega)$ and $v = \zeta\eta u$, then there exists a conic neighborhood Γ' of ξ_0, $\Gamma' \subset \Gamma$, such that, for every N,

$$|\hat{v}(\xi)| \leq C_N'(1 + |\xi|)^{-N}, \ \xi \in \Gamma.$$

The proof of lemma 4.1 can be found in [2]. By lemma 4.1 and the observation following definition 4.1, we immediately obtain the next proposition.

Proposition 4.1

$\pi(WF(u)) =$ sing supp u, where π is the canonical projection from $T^*(\Omega)$ onto Ω.

The concept of wave front sets can be defined for distributions on manifolds. In fact, we have the following lemma.

Lemma 4.2

Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$, then $(x_0, \xi_0) \notin WF(u)$ iff given any function $\psi \in C^\infty(\Omega)$ with $d\psi(x_0) = \xi_0$, there exists $\zeta \in C_0^\infty(\Omega)$, $\zeta(x_0) \neq 0$, such that, for every N,

$$\left|\int u(x) \cdot \zeta(x) e^{i\rho\psi(x)} dx\right| \leq C_N \cdot \rho^{-N}.$$

A proof of lemma 4.2 can be obtained by adapting the argument in the proof of lemma 4.4.

If P is a pseudodifferential operator, we can describe WF(Pu) in terms of WF(u).

Proposition 4.2

Let $P \in OPS^m_{\rho,\delta}, \rho > 0$, then $WF(Pu) \subset WF(u)$.

Proposition 4.3

If $P \in OPS^m_{\rho,\delta}$ is elliptic, $\rho > \delta$, then $WF(Pu) = WF(u)$.

We refer the readers to [12] for the proof of propositions 4.2 and 4.3.

We will also need the following result, the proof of which can be found in [8].

Lemma 4.3

Let u be a C^∞ function in t with values in the space of distributions in the x-variables. Suppose $(x_0,t_0,\xi_0,\tau_0) \notin WF(u)$. If $A(x,D_x)$ is a pseudodifferential operator in the x-variables, $A \in OPS^m_{1,0}$, then $(x_0,t_0,\xi_0,\tau_0) \notin WF(Au)$.

Let u be as in lemma 4.3. There is a relation between WF(u) and $WF(u|_{t=0})$.

Proposition 4.4

Let $i: \{(x,t): t=t_0\} \longrightarrow R^{n+1}$ be the natural injection. If $(x_o,0,t_o,1) \notin WF(u)$, then $WF(u|_{t=o}) \subset i^*(WF(u))$.

We refer the readers to [1] for a proof of proposition 4.4. Actually it is shown in [1] that if S is a submanifold and the conormal bundle of S is disjoint from WF(u), then the restriction of u to S is well defined and $WF(u|_S) \subset i^*(WF(u))$.

For the interior of Ω, we have a theorem due to Hörmander on propagation of singularities. We need another definition before we can state the theorem.

Definition 4.3

Let $p(x,\xi) \in S^m(\Omega)$ has principal symbol $p_m(x,\xi)$ (homogeneous of degree m in ξ). The characteristic set of $P = p(x,D)$ is defined by

$$\text{Char } P = \{(x,\xi) \in T^*(\Omega)\setminus 0 : p_m(x,\xi) = 0\}.$$

Theorem 4.1

Let $P \in OPS^m(\Omega)$ have real principal symbol and u, $f \in \mathcal{D}'(\Omega)$. Suppose that $Pu = f$. Then $WF(u)\setminus WF(f)$ is contained in Char P and is invariant under the flow of the Hamiltonian vector field H_{p_m}.

A proof of a more general version of theorem 4.1 can be found in [12].

We now turn to the relation between wave front sets and Gaussian beams. The following lemma under the assumption that $u \in L^2$ can be found in [10].

<u>Lemma 4.4</u>

Assume $d\psi(x_0) = \xi_0 \neq 0$ and $(x_0, \xi_0) \notin WF(u)$, $u \in \mathcal{D}'(\Omega)$. Assume $a \in C_0^\infty(\Omega)$ and $\operatorname{Im} \psi \geq \gamma\|x - x_0\|^2 (\gamma > 0)$ on the support of a. Then there are constants C_N such that $|\int a \cdot u \cdot e^{i\rho\psi(x)} dx| \leq C_N \rho^{-N}$, for $\rho > 1$ and $N \in Z_+$. Here the integral is evaluated in the sense of distributions.

<u>Proof</u>: By assumption, there exists $\zeta_1 \in C_0^\infty(\Omega)$, $\zeta_1(x) = 1$ in a neighborhood of x_0, such that, given any positive integer N,

$|\widehat{\zeta_1 u}(\xi)| \leq C_N'(1 + |\xi|)^{-N}$, for ξ in a conic neighborhood Γ of ξ_0. (2)

$$\int a \cdot u \cdot e^{i\rho\psi} dx$$

$$= \int a \cdot \zeta_1 \cdot u \cdot e^{i\rho\psi} dx + \int (1 - \zeta_1) \cdot a \cdot u \cdot e^{i\rho\psi} dx$$

$$= H + J$$

Since $\operatorname{Im} \psi \geq \gamma\|x - x_0\|^2$ on supp ψ, $J = O(\rho^{-\infty})$. $\therefore$ we only have to estimate H.

Let $\zeta_2 \in C_0^\infty(\Omega)$, $\zeta_2 = 1$ in a neighborhood of x_0 be such that $\|d\psi(x)\| > \frac{1}{2}\|\xi_0\|$ and $\|d\psi(x) - \xi\| > \delta > 0$ for $x \in$ supp ζ_2 and $\xi \in \Gamma^c$.

$$H = \int a \cdot \zeta_1 \cdot \zeta_2 \cdot u \cdot e^{i\rho\psi} dx + \int a \cdot \zeta_1 \cdot (1 - \zeta_2) \cdot u \cdot e^{i\rho\psi} dx$$

$= I + G$

Again, $G = O(\rho^{-\infty})$. To estimate I, let $f = a \cdot \zeta_1 \cdot u$ and then we can write

$$I = \rho^n \iint \hat{f}(\rho\xi) \cdot \zeta_2(x) \cdot e^{i\rho x \cdot \xi + i\rho\psi(x)} d\xi dx, \quad \text{where}$$

$\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int f(x) \cdot e^{-ix \cdot \xi} dx$ is the Fourier transform of f.

$\therefore I = I_1 + I_2 + I_3$. The domains of integration for I_1, I_2 and I_3 are $\{(x,\xi): |\xi| > \|\xi_0\|/4\}$, $\{(x,\xi): |\xi| \geq \|\xi_0\|/4$ and $\xi \in \Gamma\}$ and $\{(x,\xi): |\xi| \geq \|\xi_0\|/4$ and $\xi \notin \Gamma\}$ respectively.

Integration by part with respect to x shows that I_1 and I_3 are $O(\rho^{-\infty})$. I_3 is $O(\rho^{-\infty})$ because of (2).

Q.E.D.

The proof of lemma 4.4 can be generalized to prove the next lemma.

Lemma 4.5

Let $v(x,\rho)$ be a Gaussian beam along a smooth curve $C \subset \Omega$ and $u \in \mathcal{D}'(\Omega)$ with supp $u \subset\subset \Omega$. Assume that the phase function ψ of v satisfies the following conditions:

$0 \neq d\psi(x) \notin WF(u)$ for $x \in C$.

Then for any positive integer N,

$$\left|\int u(x) \cdot \overline{v(x,\rho)}\, dx\right| = O(\rho^{-N}).$$

In order to deal with the case in which supp u is not away from $\partial\Omega$, we need the following concept. From now on Ω will be a domain in R^{n+1} with smooth boundary.

Let $u \in \mathcal{D}'(\Omega)$ and $p \in \partial\Omega$. Assume that in some local coordinates $p = (0,0)$ and $\Omega = \{(x,t): x \in R^n \text{ and } t > 0\}$.

Definition 4.4

Let $(x_0,\xi_0) \in T^*(\partial\Omega) \setminus 0$. We say that u is microlocally smooth at (x_0,ξ_0) if u is a C^∞ function of $t \geq 0$ in the space of distributions in the x-variables and $\exists$ $\delta > 0$, $\zeta \in C_0^\infty(\Omega)$, $\zeta(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0, such that, for any positive integer N,

$$|\widehat{\zeta \cdot u}(t,\xi)| \leq C_N(1 + |\xi|)^{-N}, \ \xi \in \Gamma \text{ and } 0 \leq t \leq \delta. \quad (3)$$

Here $\widehat{\zeta \cdot u}$ is the Fourier transform of $\zeta \cdot u$ in the x - variables.

Using lemmas 4.1 and 4.2, it is easy to see that definition 4.4 does not depend on the choice of local coordinates.

Definition 4.5

If u is also defined for $t \leq 0$, then we say that u is microlocally smooth at (x_0,ξ_0) with respect to the hypersurface $t = 0$ if (3) holds for $|t| \leq \delta$.

Definition 4.6

We say that u is microlocally smooth along a curve

$(x(s),\xi(s),t(s),\tau(s))$ in $T^*(\Omega)\setminus 0$ if u is microlocally smooth at $(x(s),\xi(s))$ with respect to the hypersurfaces $t = t(s)$, for all s.

It follows from definition 4.5 that if u is microlocally smooth at (x_0,ξ_0) with respect to $t = 0$, then $(x_0,\xi_0,0,\tau) \notin WF(u)$ for any τ.

The following lemma is an immediate consequence of lemma 4.4 and definition 4.4

<u>Lemma 4.6</u>

Let $v(x,\rho)$ be a Gaussian beam along a smooth curve $C \subset \Omega$, $p \in \partial\Omega$ and C have a contact of order at most 2 with $\partial\Omega$ at p. Let ψ be the phase function of v and $\alpha = (p,d\psi|_{\partial\Omega}(p)) \in T^*(\partial\Omega)\setminus 0$. Suppose that $u \in \mathcal{D}'(\Omega)$ is supported in a neighborhood of p, $0 \neq d\psi(x) \notin WF(u)$ for $x \in C\setminus\{p\}$ and u is microlocally smooth at α with respect to $\partial\Omega$. Then given any positive integer N,

$$\left| \int_\Omega u(x)\cdot\overline{v(\rho,x)}\, dx \right| = O(\rho^{-N}).$$

Section 5 Airy Functions

Airy functions are solutions to the Airy equation

$$y'' = xy. \tag{1}$$

We will need the Airy functions in our treatment of the grazing ray problem in chapter 4. We record the basic properties of Airy functions in this section. We refer the readers to [9] for details.

A solution to (1) is given by

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos(t^3/3 + xt)\, dt, \quad x \in R, \tag{2}$$

where the integral is evaluated as $\lim_{t \to \infty} \int_0^\infty$.

We can of course extend Ai to the complex plane. There is an asymptotic expansion of Ai(z).

$$Ai(z) \sim \frac{e^{-\xi}}{2\pi^{\frac{1}{2}} z^{1/4}} \sum_{s=0}^{\infty} (-1)^s \frac{u_s}{\xi^s} \tag{3}$$

as $z \to \infty$ in the sector $|ph\ z| \le \pi - \delta$ $(\delta > 0)$. Here $\xi = 2/3 \cdot z^{3/2}$, $u_0 = 1$,

$$u_s = \frac{2^s}{3^{3s}(2s)!} \cdot \frac{\Gamma(3s + \frac{1}{2})}{\Gamma(\frac{1}{2})} \tag{4}$$

The following two functions will be used in chapter 4.

$$A_\pm(z) = Ai(-e^{2\pi i/3} \cdot z) \tag{5}$$

It is clear that $A_\pm$ satisfy the following equation:

$y'' = -xy$ (6)

It follows from (3) that we have formulae for the asymptotic expansion of $A_{\pm}(z)$ in the sector $|\mathrm{ph}\ z| \leq 2\pi/3 - \delta$ $(\delta > 0)$.

We should also mention the non-trivial fact that the zeros of $Ai(z)$ and $Ai'(z)$ are all real and negative. In particular, $A_{\pm}(t) \neq 0$ for $t \in R$.

Section 6 Two Lemmas

We will prove in this section two lemmas that are needed later.

The first lemma is concerned with the following problem in O.D.E.

Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_o$ be a polynomial. b_i, $1 \leq i \leq l$ $(l \geq n)$, is a set of polynomials.

Consider the problem of solving

$p(D_t)\ u = 0,$ (1)

and $b_i(D_t)u(0) = 0.$ (2)

We want to find an algebraic condition under which (1) and (2) will have only trivial solution.

If the roots $\lambda_1, \dots, \lambda_n$ of $p(z) = 0$ are all distinct, then it is easy to write down the condition as

rank of the matrix $(b_i(\lambda_j))_{\substack{1\le i\le l \\ 1\le j\le n}} = n.$ (3)

If the roots $\lambda_1,\dots,\lambda_k$ of $p(z) = 0$ have multiplicities $m_1,\dots,m_k$, we define B to be the following matrix.

$$B = \begin{bmatrix} b_1(\lambda_1) & b_1'(\lambda_1) & \cdots & b_1^{(m_1-1)}(\lambda_1) \cdots & b_1(\lambda_k) \cdots & b_1^{(m_k-1)}(\lambda_k) \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ b_l(\lambda_1) & b_l'(\lambda_1) & \cdots & b_l^{(m_1-1)}(\lambda_1) \cdots & b_l(\lambda_k) \cdots & b_l^{(m_k-1)}(\lambda_k) \end{bmatrix}$$

(4)

Then the condition can be written as

rank of B = n. (5)

What we want is a condition that is insensitive to the multiplicities of the roots of $p(z) = 0$.

Let C be a curve in the complex plane encircling all the roots of $p(z) = 0$ such that the winding number of C with respect to each root equals 1.

Lemma 6.1

(1) and (2) have only trivial solution iff the rank of the matrix $\tilde{B} = n$, where

$$\tilde{B} = \left(\frac{1}{2\pi i} \int_C \frac{b_i(z)}{p(z)} z^{j-1} \right)_{\substack{1 \le i \le l \\ 1 \le j \le n}} \qquad (6)$$

<u>Proof</u>: Let $\vec{u} = \begin{bmatrix} b_1^{(r)}(\lambda_s) \\ \vdots \\ b_l^{(r)}(\lambda_s) \end{bmatrix}$, $0 \le r \le m_s - 1$, be one of the column vectors in B(see(4)).

Denote $p(z)/(z-\lambda_s)^{r+1}$ by $f(z)$. Then we have

$$\vec{u} = \begin{bmatrix} \frac{1}{2\pi i} \int_C \frac{b_1(z) f(z)}{p(z)} \, dz \\ \vdots \\ \frac{1}{2\pi i} \int_C \frac{b_l(z) f(z)}{p(z)} \, dz \end{bmatrix} .$$

$\therefore \vec{u}$ belongs to the column space of $\tilde{B}$. Conversely, let $\vec{v} = \begin{bmatrix} \frac{1}{2\pi i} \int_C \frac{b_1(z)}{p(z)} z^{r-1} \, dz \\ \vdots \\ \frac{1}{2\pi i} \int_C \frac{b_l(z)}{p(z)} z^{r-1} \, dz \end{bmatrix}$ be a column vector of $\tilde{B}$.

It follows from partial fractions and Cauchy's theorem that $\vec{v}$ belongs to the column space of B.

Q.E.D.

<u>Remark</u>: In the case of $l = n$, then

$$p(D_t)\, u = 0, \qquad (1')$$

and $b_i(D_t)u(o) = \beta_j \qquad (2')$

have a unique solution for any β_j, $1 \le j \le l$, iff the matrix $\tilde{B} = (\frac{1}{2\pi i} \int_C \frac{b_i(z)}{p(z)} z^{j-1} \, dz)$, $1 \le i,j \le n$, is nonsingular.

Also, in the definition of $\tilde{B}$ we can replace $\{z^{j-1},$ $1 \le j \le n\}$ by any basis of $C[z]/(p(z))$.

The second lemma is concerned with nonsingular matrices.

Let A be a nonsingular n by n matrix and B be the inverse of A. Let I be a subset of $\{1,2,\ldots,n\}$ and J be the complement of I in $\{1,2,\ldots,n\}$. Assume that $|I| = m$ and $|J| = n-m$.

Let $\tilde{A} = (a_{ik})_{\substack{i\in I \\ 1\le k\le m}}$ and $\tilde{B} = (b_{lj})_{\substack{j\in J \\ m+1\le l\le n}}$.

Lemma 6.2

$\tilde{A}$ is nonsingular iff $\tilde{B}$ is nonsingular.

Proof: $\tilde{A}$ is singular iff $\exists$ a vector $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \neq \vec{0}$ such

that $u_{m+1}=\cdots=u_n=0$ and $A\vec{u} = \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $(\neq \vec{0})$, with

$v_i = 0$ for $i \in I$. This is true iff $\tilde{B}\vec{u} = \vec{v}$, which in turn is equivalent to $\tilde{B}$ being singular.

Q.E.D.

Section 7 Some Notational Conventions

We will adopt the following conventions in this work.

(1) Let i, j, and k be natural numbers. Theorem(lemma, proposition, formula) i.j. means theorem j in section i of the same chapter. Theorem (lemma, proposition, formula) i.j.k. means theorem (lemma, proposition, formula) k in section j of chapter i.

(2) The summation convention $a_i b_i = \sum_{i=1}^{n} a_i b_i$ will be used whenever there is no ambiguity about n.

(3) df will denote the gradient of f.

(4) If $v(x,\rho)$ is a function depending on a positive parameter ρ, then, for any natural number N, $v(x,\rho) = O(\rho^{-N})$ means $|\rho^N v(x,\rho)| \le C_N$ for all x in some domain . . (1)

$v(x,\rho) = O(\rho^{-\infty})$ means (1) being true for all N.

(5) $A \subset\subset \Omega$ means that A is a compact subset of Ω.

(6) We say that a contour C encircles complex numbers $\lambda_1, \ldots, \lambda_k$ if the winding number of C with respect to $\lambda_i (i=1,\ldots,k)$ equals 1.

(7) $\mathcal{D}'(\Omega)$ is the space of distributions in Ω.

Chapter I Initial Value Problems

Section 1 Strictly Hyperbolic Operators

Throughout this chapter, P will be an operator of the form

$P(x,t,D_x,D_t) = D_t^m + A_1(x,t,D_x)D_t^{m-1} + \dots + A_m(x,t,D_x)$,

where $t \in [0,T]$ for some $T > 0$, $A_i(x,t,D_x) \in OPS^i(R^n)$ (def. 0.3.3) is properly supported and $A_i(x,t,D_x)$ depends smoothly on t, for $1 \leq i \leq m$.

Let $a_i(x,t,\xi) \in S^i(R^n)$(def. 0.3.1) be the principal symbols of A_i, for $1 \leq i \leq m$. The pricipal symbol of P is defined to be

$p_m(x,t,\xi,\tau) = \tau^m + a_1(x,t,\xi)\tau^{m-1} + \dots + a_m(x,t,\xi)$.

p_m is assumed to be real and to satisfy the following strictly hyperbolic condition: $p_m(x,t,\xi,\tau) = 0$ has m distinct real roots for any non-zero real ξ.

We shall write $x_{n+1} = t$ and $\xi_{n+1} = \tau$ whenever we do not want to emphasize the "time variable" t. In this notation we shall write

$p_m(x,\xi)$, $(x,\xi) \in R^{n+1} \times R^{n+1}$, for the principal symbol of P. $p_{m-1}(x,\xi)$ will denote the part of the symbol of P that is homogeneous of degree m-1.

Section 2. Initial Value Problems

Let $v_k = \sum_{j\le 0} a_{kj}(x)\rho^{k+j}e^{i\rho\psi_0(x)}$, $0 \le k \le m-1$ be oscillatory functions with strictly positive complex phase function ψ_0. v_k is supported near 0.

We want to solve asymptotic initial value problems of the form:

$$Pu = O(\rho^{-N}) \qquad (1a)$$

$$D_t^k u|_{t=0} = v_k + O(\rho^{-N}) \qquad (1b)$$

where $N = 0,1,2,\ldots$, $t \in [0,T]$ and $0 \le k \le m-1$.

Section 3 The Ansatz

In order to solve (1a), we set

$u = \sum_{j\le 0}^{M} u_j(x,t)\rho^j e^{i\rho\psi(x,t)}$, where M depends on the N in (1a).

By (0.3.1)

$$Pu = \sum_{j\le m} c_j\rho^j e^{i\rho\psi}$$

$$c_m(x) = p_m(x, d\psi(x))\, u_o(x) \qquad (1)$$

$$c_{m-1}(x) = (Lu_o)(x) + p_m(x, d\psi(x))u\ (x) \tag{2}$$

where $L = \frac{1}{i}\frac{\partial p_m}{\partial \xi_j}(x, d\psi(x))\frac{\partial}{\partial x_j} + [\frac{1}{2i}\frac{\partial^2 p_m}{\partial \xi_j \partial \xi_k}(x, d\psi(x))\ \cdot$

$$\frac{\partial^2 \psi}{\partial x_j \partial x_k} + p_{m-1}(x, d\psi(x))\ ].$$

In general, $c_{m-r-1}(x) = (Lu_r)(x) + p_m(x, d\psi(x))u_{r+1}(x) +$

$$g_r(x) \tag{3}$$

where g_r is a function of ψ, u_o, ... , u_{r-1} and their derivatives.

Setting (1) and (3) equal to zero, we obtain the eikonal equation

$$p_m(x, d\psi(x)) = 0 \tag{4}$$

and the transport equations

$$Lu_r + g_r = 0,\ r = 0, -1, \ldots \tag{5}$$

We shall solve (4) in two ways. Once we have the phase function ψ, solutions of (5) are easy to construct.

Section 4 Solution of the Eikonal Equation by the Method of Characteristics

Let $d\psi_0(0) = \xi_0$ and $\tau_1, \dots, \tau_m$ be the m distinct roots of the equation $p_m(0,0,\xi_0,\tau) = 0$. It follows from strict hyperbolicity of P that

$$p_m(x,\xi) = 0 \text{ implies } \frac{\partial p_m}{\partial \xi_{n+1}}(x,\xi) = 0 \qquad (1)$$

Let $\tilde{\psi}_0$ be any almost analytic extension (def.0.2.1) of $\psi_0(x)$ to C^n in a neighborhood of 0 and $\tilde{p}_m$ be any almost analytic extension of p_m to a neighborhood of $R^{n+1} \times R^{n+1}$ in $C^{n+1} \times C^{n+1}$.

We have the following Hamiltonian equations corresponding to $\tilde{p}_m$, namely,

$$\left.\begin{aligned} \frac{dZ_i}{ds} &= \frac{\partial \tilde{p}_m}{\partial \Sigma_i}(Z, \Sigma) \\ \frac{d\Sigma_i}{ds} &= -\frac{\partial \tilde{p}_m}{\partial Z_i}(Z, \Sigma) \end{aligned}\right\} \quad 1 \le i \le n+1 \qquad (2)$$

For $Z(0) = 0$, $\Sigma(0) = (\xi_0,\tau_1)$ the solution to (2) is just the null-bicharacteristic of p_m through $(0,(\xi_0,\tau_1))$. We shall denote this null-bicharacteristic by b. By (1), the projection of b on R^{n+1} is a smooth curve c.

Many properties of the Hamiltonian vector fields on real symplectic manifolds can be generalized to the almost analytic category. Here we only deal with two of them.

Let f_s be the flow generated by (2), i.e. $f_s(z,\theta) = (Z(s),\Sigma(s)) \in C^{n+1} \times C^{n+1}$ is the solution of (2) with initial conditions $(Z(0),\Sigma(0)) = (z,\theta)$.

Since p_m is real, $f_s: R^{2n+2} \to R^{2n+2}$

$$\therefore \ |\mathrm{Im}\, f_s(z,\theta)| = O(|\mathrm{Im}(z,\theta)|) \qquad (3)$$

where the estimate is uniform for (z,θ,s) in any compact subset of $C^{n+1} \times C^{n+1} \times R$. We shall use the symbol O in the same sense for the following two propositions.

Let $\omega = d\xi_1 \wedge dx_1 + \dots + d\xi_{n+1} \wedge dx_{n+1}$

$\tilde{\omega} = d\Sigma_1 \wedge dZ_1 + \dots + d\Sigma_{n+1} \wedge dZ_{n+1}$ is the extension of ω to $T^*(C^{n+1})$.

<u>Proposition 4.1</u>

$$f_s^* \tilde{\omega}_{f_s(z,\theta)} = \tilde{\omega}_{(z,\theta)} + O(|\mathrm{Im}(z,\theta)|^N) \qquad (4)$$

for $N = 0, 1, 2, \dots$.

<u>Proof</u>: Let w_1, w_2 be any two of $z_1, \dots, z_{n+1}, \theta_1, \dots, \theta_{n+1}$.

$$\frac{d}{ds}\left(\frac{\partial \Sigma_k}{\partial w_1} \cdot \frac{\partial Z_k}{\partial w_2} - \frac{\partial \Sigma_k}{\partial w_2} \cdot \frac{\partial Z_k}{\partial w_1} \right) = O(|\mathrm{Im}(Z(s),\Sigma(s))|^N)$$

by (2) and the almost analyticity of $\tilde{p}_m$.

(4) then follows from the fundamental theorem of calculus and (3).

Q.E.D.

In particular, if $(x,\xi) \in R^{n+1} \times R^{n+1}$, then

$$f_s^* \tilde{\omega}_{f_s(x,\xi)} = \tilde{\omega}_{(x,\xi)} \qquad (5)$$

Let $(\bar{\ })$ be the canonical conjugation on C^{2n+2}. It induces a canonical conjugation on the tangent bundle of C^{2n+2}.

Given $(x,\xi) \in R^{2n+2}$, since $f_{s*} : T_{(x,\xi)}C^{2n+2} \longrightarrow T_{f_s(x,\xi)}C^{2n+2}$ is real, it follows from (5) that for any $u, v \in T_{(x,\xi)}C^{2n+2}$

$$\tilde{\omega}_{(x,\xi)}(v,\bar{u}) = \tilde{\omega}_{f_s(x,\xi)}(f_{s*}v, \overline{f_{s*}u}) \qquad (6)$$

<u>Proposition 4.2</u>

$$\frac{d}{ds}\tilde{p}_m(f_s(z,\theta)) = O(|\mathrm{Im}(z,\theta)|^N) \qquad (7)$$

for $N = 0, 1, 2, \ldots$

<u>Proof</u>: (2) and the almost analyticity of $\tilde{p}_m$ implies L.H.S. $= O(|\mathrm{Im}\, f_s(z,\theta)|^N)$. (7) then follows from (3).

Q.E.D.

The complex phase function ψ_0 being strictly positive at 0 means that

$\operatorname{Im} \psi_o(0) = 0$ and the matrix

$$\left(\frac{\partial^2 \operatorname{Im} \psi_o}{\partial x_i \partial x_j} (0) \right) \text{ is positive definite.} \qquad (8)$$

Therefore there exists a neighborhood U_1 of 0 in C^n such that (8) holds for $U_1 \cap R^n$ and

$$d\tilde{\psi}_o : U_1 \longrightarrow C^n \text{ is a diffeomorphism.} \qquad (9)$$

(1) implies that there exists a neighborhood U_2 of 0 in C^n such that

$\tilde{p}_m(w, 0, d\tilde{\psi}_o(w), \tau) = 0$ has a smooth solution $\tau_1(w)$ for $w \in U_2$ and $\tau_1(0) = \tau_1$.

Define U to be $U_1 \cap U_2$.

Let $F(w,s) \in C^{2n+2}$, $w \in U$ and $s \in [0,s]$, be the solution of (2), i.e.

$$\frac{dF_i}{ds} = \frac{\partial \tilde{p}_m}{\partial \Sigma_i} (F(w,s))$$

$$\frac{dF_{n+1+i}}{ds} = - \frac{\partial \tilde{p}_m}{\partial Z_i} (F(w,s)) \qquad i = 1, \ldots, n+1 \qquad (10)$$

with initial conditions

$$F_i(w,0) = w \qquad i = 1,\ldots,n$$

$$F_{n+1+i}(w,0) = \frac{\partial \tilde{\psi}_o(w)}{\partial z_i} \qquad i = 1,\ldots,n$$

$$F_{n+1}(w,0) = 0$$

$$F_{2n+2}(w,0) = \tau_1(w) \qquad (11)$$

and $F(0,s)$, $0 \le s \le S$ is the null bicharacteristic b with $F_{n+1}(0,S) = T$.

Proposition 4.3

$F: U \times [0,T] \longrightarrow T^*(C^{n+1})$ is an imbedding.

Proof: Follows from (1), (9) and the fundamental theorem of O.D.E.

Q.E.D.

We will denote the image of F by $\tilde{F}$. Let $\tilde{L}_o = \{ (w, d\tilde{\psi}_o(w)) : w \in U \}$, then $\tilde{L}_o$ is a positive almost analytic Lagrangian submanifold (def.0.2.3) of C^{2n} and $\tilde{F}$ is just the flow out of $\tilde{L}_o$ by the Hamiltonian vector field (2).

Proposition 4.4

$\tilde{F} \cap T^*(R^{n+1}) = b$

Proof: $F(w_o,s_o) \in T^*(R^{n+1})$ for some $(w_o,s_o) \in U \times [0,S]$ implies $F(w_o,s) \in T^*(R^{n+1})$ for all $s \in [0,S]$ because the Hamiltonian equations (2) are real on $T^*(R^{n+1})$. In particular, $F(w_o,0) = (w_o, 0, d\tilde{\psi}_o(w_o), 0)$ is real, i.e. $w_o \in R^n$ and $d\tilde{\psi}_o(w_o) = d\psi_o(w_o) \in R^n$. (8) implies $w_o = 0$ and hence $F(w_o,s_o) = F(0,s_o) \in b$.

Q.E.D.

It follows that we can use $|w|$ to measure the distance of $F(w,s)$ from $F \cap T^*(R^{n+1}) = b$.

For the rest of this section, the symbol 0 always denotes estimates uniform for $s \in [0,S]$ and for w belonging to some neighborhood $V \subset U$ of 0 in C^n.

Lemma 4.1

$$|\mathrm{Im}\, F(w,s)| = O(|w|)$$

Proof: Follows from proposition 4.4 and the fundamental theorem of O.D.E.

Q.E.D.

Proposition 4.5

$$\frac{\partial F}{\partial w_j}(w,s) = O(|w|^N) \qquad 1 \leq j \leq n \tag{12}$$

$$\tilde{p}_m(F(w,s)) = O(|w|^N) \tag{13}$$

for N = 0, 1, 2, ...

Proof: To prove (12), observe that if we differentiate (10), by almost analyticity of $\tilde{p}_m$, we get

$$\frac{d}{ds}\left(\frac{\partial F}{\partial W_j}(w,s)\right) = O(\mathrm{Im}\,|F(w,s)|^N)$$
$$= O(|w|^N), \text{ by lemma 4.1}$$

Similarly, in order to prove (13), we observe that

$$\frac{d}{ds}\tilde{p}_m(F(w,s)) = \frac{\partial \tilde{p}_m}{\partial Z_i} \cdot \frac{dF_i}{ds} + \frac{\partial \tilde{p}_m}{\partial \Sigma_i} \cdot \frac{dF_{i+1}}{ds} + O(|w|^N)$$
$$= O(|w|^N), \text{ by (10)}$$

Q.E.D.

Proposition 4.6

$\tilde{\omega}$ is flat on $\tilde{F}$, i.e. $\tilde{\omega}|_{\tilde{F}}$ vanishes to infinite order on b.

Proof: By proposition 4.1 and lemma 4.1, it suffices to prove that $\tilde{\omega}_{(w,0)} = O(|w|^N)$. But this follows easily from (11) and the almost analyticity of $\tilde{\Psi}_0$.

Q.E.D.

In summary, we have shown that F is a 2n + 1 dimensional real submanifold of $T^*(C^{n+1})$ whose intersection with $T^*(R^{n+1})$ is b and $\tilde{\omega}$ is flat on F.

It turns out that the canonical projection $\pi: \tilde{F} \longrightarrow C^{n+1}$ is an imbedding in a neighborhood of b. To verify this assertion, we proceed in the following manner.

Lemma 4.2

$$\frac{1}{i}\,\tilde{\omega}_{F(0,s)}(v,\bar{v}) > 0 \text{ for } v \in T_{F(0,s)}\tilde{F} \setminus T_{F(0,s)}b.$$

Proof: By (6), it suffices to check this for s = 0. But when s = 0, this follows from (11) and (8).

Q.E.D.

Lemma 4.3

π is an immersion along b.

Proof: $u \in T_{F(0,s)}(\tilde{F})$ implies that $u = v_1 + v_2$ where $v_1 \in T_{F(0,s)}\tilde{F} \setminus T_{F(0,s)}b$ and $v_2 \in T_{F(0,s)}b$.

Since b is real, v_2 is real and $\tilde{\omega}(u,\bar{u}) = \tilde{\omega}(v_1,\bar{v}_1)$ by proposition 4.6.

$\pi_*(u) = 0$ implies $\tilde{\omega}(u,\bar{u}) = 0$ and hence $\tilde{\omega}(v_1,\bar{v}_1) = 0$. Lemma 4.2 then implies $v_1 = 0$.

$\therefore \pi(v_2) = 0$ and we have $v_2 = 0$ by (1).

Q.E.D.

Proposition 4.7

$\pi: \tilde{F} \longrightarrow C^{n+1}$ is an imbedding in a neighborhood of b.

Proof: Follows from lemma 4.3 and the fact that $\pi: b \longrightarrow c$ is an imbedding.

Q.E.D.

Since $\tilde{p}_m$ is real along b, there exists n+1 linearly independent real vectors in $T_{F(0,s)}(\tilde{F}), s\in[0,S]$. By the implicit function theorem, we obtain the following corollary to proposition 4.7.

Corollary:

$\pi(\tilde{F}) \cap R^{n+1}$ contains a neighborhood of c in R^{n+1}.

We shall denote this neighborhood of c by G. Then there exists $H: G \longrightarrow C^{n+1}$ such that $\pi(x,H(x)) = x$ for all $x \in G$.

To make the discussion simpler, by a change of coordinates, we can assume that

$c = \{ (x_1, \ldots, x_{n+1}) : x_1 = \ldots = x_n = 0, 0 \le x_{n+1} \le T \}$

and $G = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_n^2 \le r^2, r > 0$ and $0 \le x_{n+1} \le T \}$.

Note that in this coordinate system

$$H(x',0) = \frac{\partial \psi_0}{\partial x_i}(x') \text{ for } i = 1, \ldots, n. \tag{14}$$

We define the function ψ by the following formula.

$$\psi(x',x_{n+1}) = \sum_{i=1}^{n} \int_0^1 H_i(tx')x_i dt + \int_0^{x_{n+1}} H_{n+1}(0,t)dt + \psi_0(0) \tag{15}$$

where $x' = (x_1, \ldots, x_n)$, $|x'| < r$ and $0 \le x_{n+1} \le T$.

<u>Lemma 4.4</u>

$$\frac{\partial H_i}{\partial x_j}(x',x_{n+1}) = \frac{\partial H_j}{\partial x_i}(x',x_{n+1}) + O(|x'|^N)$$

for $N = 0, 1, 2, \ldots$ and the estimates are uniform on G.

<u>Proof</u>: Follows from proposition 4.6 and the fact that $|x'|$ measures the distance of $(x, H(x))$ to b on $\tilde{F}$.

Q.E.D.

<u>Lemma 4.5</u>

For $N = 0, 1, 2, \ldots, i = 1, \ldots, n+1$

$$\frac{\partial \psi}{\partial x_i}(x',x_{n+1}) - H_i(x',x_{n+1}) = O(|x'|^N), \text{ uniformly on G.}$$

<u>Proof</u>: Follows from (15) and lemma 4.4. Q.E.D.

<u>Theorem 4.1</u>

There exists a complex phase function ψ in a neighborhood G of c which has the following properties:

(i) $\psi|_{G \cap \{x_{n+1} = 0\}} = \psi_0|_{G \cap \{x_{n+1} = 0\}}$

(ii) $\frac{\partial\psi}{\partial x_{n+1}}(0) = \tau_1$

(iii) $p_m(x, d\psi(x))$ is flat on c, i.e.

$p_m(x, d\psi(x)) = O(\ d(x,c)^N\ ),\ N = 0, 1, 2, \ldots$

uniformly for $x \in G$

($d(x,c)$ = distance from x to c)

(iv) $d\psi \neq 0$ along c

(v) along c, $\left(\frac{\partial^2\psi}{\partial x_i \partial x_j}\right)_{1 \leq i,\ j \leq n+1}$ is positive definite on any n-dimensional subspace of the tangent space transversal to c.

<u>Proof</u>: (i) and (ii) are clear from (15). (iii) follows from proposition 4.5 and lemma 4.5. (iv) follows from the fact that $(x, d\psi(x))$, $x \in c$, describes $b \subset T^*(R^{n+1}) \setminus$ zero section. (v) is just lemma 4.2.

Q.E.D.

We have thus solved the eikonal equation (3.4) in the sense of (i) to (v) in theorem 4.1.

Remark: Let $\tilde{\psi}$ be an almost analytic extension of ψ to a neighborhood of c in C^{n+1}, then $\tilde{L} = \{(z, d\psi(z))\}$ is a strictly positive almost analytic Lagrangian submanifold of order one. (def0.2.7) The flow out F of $\tilde{L}_o = \{(w, d\psi_o(w))\}$ is equivalent (def.0.2.4) to a (2n+1) dimensional real submanifold of $\tilde{L}$.

Section 5 Solution of the Eikonal Equation via Ricatti Equation

In this section we will directly attack the problem of solving the equation

$p_m(x, d\psi(x)) = 0$ to infinite order on c, with initial conditions $\psi(x',0) = \psi_o(x')$

$$\frac{\partial\psi}{\partial x_{n+1}}(0,0) = \tau_1$$

where $x' \in R^n$. (1)

Recall that c is the projection of the null bi-characteristic b of p_m through the point $(0,0,\xi_o,\tau_1)$.

Again, for simplicity, we assume that c is given by $x_1(s) = \ldots = x_n(s) = 0$, $x_{n+1}(s) = s$ and $0 \le s \le T$.

We shall follow [10] closely in our treatment.

We have the Hamiltonian equations

$$\frac{dx_i}{ds} = \frac{\partial p_m}{\partial \xi_i}$$

$$\frac{d\xi_i}{ds} = -\frac{\partial p_m}{\partial x_i} \qquad 1 \leq i \leq n+1 \tag{2}$$

$b(s) = ((0,s), \Sigma(s))$ is the solution of (2) with initial condition $(0,0,\xi_o,\tau_1)$.

If we differentiate $p_m(x, d\psi(x)) = 0$ with respect to x_j, we get

$$\frac{\partial p_m}{\partial x_j}(x, d\psi(x)) + \frac{\partial p_m}{\partial \xi_k}(x, d\psi(x))\frac{\partial^2 \psi}{\partial x_k \partial x_j}(x) = 0 \tag{3}$$

$$1 \leq j \leq n+1$$

If we restrict (3) to c, we get

$$\frac{\partial p_m}{\partial x_j}(0,s,d\psi(0,s)) + \frac{\partial p_m}{\partial \xi_k}(0,s,d\psi(0,s))\frac{\partial^2 \psi}{\partial x_k \partial x_j}(0,s) = 0 \tag{3'}$$

By (2), (3') will be satisfied if we set

$$d\psi(0,s) = \Sigma(s) \tag{4}$$

and we have solved (1) up to first order on c.

Note that by our assumption on c, we have

$$\frac{\partial p_m}{\partial \xi_j}(0,s,\Sigma(s)) = \delta_{j,n+1} \tag{5}$$

If we differentiate (3) with respect to x_i, we get

$$\frac{\partial^2 p_m}{\partial x_i \partial x_j} + \frac{\partial^2 p_m}{\partial \xi_l \partial x_j} \frac{\partial^2 \psi}{\partial x_l \partial x_i} + \frac{\partial^2 p_m}{\partial x_i \partial \xi_k} \frac{\partial^2 \psi}{\partial x_k \partial x_j} +$$

$$\frac{\partial^2 p_m}{\partial \xi_k \partial \xi_l} \frac{\partial^2 \psi}{\partial x_l \partial x_i} \frac{\partial^2 \psi}{\partial x_k \partial x_j} + \frac{\partial p_m}{\partial \xi_k} \frac{\partial^3 \psi}{\partial x_k \partial x_j \partial x_i} = 0 \qquad (6)$$

If we restrict (6) to c, by (5), we have

$$A + MB + B^t M + MCM + \dot{M} = 0 \qquad (6')$$

where for i,j = 1,2,...,n+1,

$$M(s)_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} (b(s))$$

$$A(s)_{ij} = \frac{\partial^2 p_m}{\partial x_i \partial x_j} (b(s))$$

$$B(s)_{ij} = \frac{\partial^2 p_m}{\partial \xi_i \partial x_j} (b(s))$$

$$C(s)_{ij} = \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_j} (b(s)) \qquad (7)$$

and $\dot{} = \frac{d}{ds}$.

Note that A, B, C are real matrices and

$$A = A^t, \quad C = C^t \qquad (8)$$

(6')is a matrix Ricatti equation. For a detailed discussion of matrix Ricatti equations see [11].

We want to solve (6') with the following initial conditions

$$M(0)_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} o(0) \qquad 1 \le i,j \le n$$

$$M(0)_{k(n+1)} = M(0)_{(n+1)k} = \frac{\partial p_m}{\partial x_k}(0,\xi_0,\tau_1)$$

$$1 \le k \le n+1 \qquad (9)$$

We consider the following system of linear equations associated with (6'):

$$\dot{y} = By + C\eta$$
$$\dot{\eta} = -Ay - B^t\eta \qquad (10)$$

with initial conditions

$$y(0) = Id$$
$$\eta(0) = M(0) \qquad (11)$$

<u>Lemma 5.1</u>

If y, η satisfy (10) and y is non-singular, then $M = \eta \cdot y^{-1}$ is a solution of (6').

<u>Proof</u>: Direct substitution.

Q.E.D.

Recall that $\tilde{\omega}$ is the canonical 2-form on C^{2n+2}; if $(u_i,v_i) \in C^{n+1} \times C^{n+1}$, $i = 1,2$, then $\tilde{\omega}((u_1,v_1),(u_2,v_2)) = v_1 \cdot u_2 - u_1 \cdot v_2$.

<u>Lemma 5.2</u>

If $(u_i(s),v_i(s))$, $i = 1,2$, are solutions of (10), then $\tilde{\omega}((u_1,v_1),(u_2,v_2))$ is independent of s.

Proof: $\frac{d}{ds}\tilde{\omega}((u_1,v_1),(u_2,v_2))$

$$= (-Au_1 - B^t v_1)\cdot u_2 + v_1\cdot(Bu_2 + Cv_2)$$
$$-(Bu_1 + Cv_1)v_2 - u_1\cdot(-Au_2 - B^t v_2), \text{ by (10)}$$
$$= 0 \quad \text{, by (8)}$$

Q.E.D.

Since A, B and C are real matrices, we can deduce by the same proof the following lemma:

Lemma 5.3

Under the same assumptions as in lemma 5.2, $\omega((u_1,v_1),(\bar{u}_2,\bar{v}_2))$ is also independent of s.

Lemma 5.4

If y(s) and $\eta(s)$, $0 \le s \le T$, satisfy (10) and (11), then $(y_{n+1}(s), \eta_{n+1}(s))^t = \dot{b}(s) = (0,1,\dot{\Sigma}(s))$, where y_{n+1} and η_{n+1} are the $n+1^{st}$ column vectors of y and η.

Proof: Differentiating (2) with respect to s we obtain

$$\frac{d\dot{x}_i}{ds} = \frac{\partial^2 p_m}{\partial x_k \partial \xi_i}\dot{x}_k + \frac{\partial^2 p_m}{\partial \xi_l \partial \xi_i}\dot{\xi}_l$$

$$\frac{d\dot{\xi}_i}{ds} = -\frac{\partial^2 p_m}{\partial x_k \partial x_j}\dot{x}_k - \frac{\partial^2 p_m}{\partial \xi_l \partial x_i}\dot{\xi}_l$$

By (7) and (11), we see that $(y_{n+1},\eta_{n+1})^t$ and $\dot{b}$ satisfy the same differential equation with the same initial conditions. Q.E.D.

<u>Lemma 5.5</u>

If $y(s)$ and $\eta(s)$, $0 \leq s \leq T$, satisfy (10) and (11) then $y(s)$ is non-singular for $0 \leq s \leq T$.

<u>Proof</u>: Let $y_1, \ldots, y_{n+1}$ and $\eta_1, \ldots \eta_{n+1}$ be the column vectors of y and η.

Suppose that $a_i y_i(s_0) = 0$ for some $s_0 \in [0,T]$. Let $u = a_i y_i$ and $v = a_i \eta_i$, then $\tilde{\omega}((u(s_0),v(s_0)),\overline{u(s_0)},\overline{v(s_0)}) = 0$. By lemma 5.3,

$\tilde{\omega}((u(0),v(0)),(\overline{u(0)},\overline{v(0)})) = 0$, i.e.

$a_i \cdot \overline{M(0)_{ij} a_j} - \bar{a}_i \cdot M(0)_{ij} a_j = 0$ or $(2i) a_i \cdot \mathrm{Im}\, M(0)_{ij} \bar{a}_j = 0$.

By (9), $\mathrm{Im}\, M(0)_{k(n+1)} = \mathrm{Im}\, M(0)_{(n+1)k} = 0$, $1 \leq k \leq n+1$ and $(\mathrm{Im}\, \eta(0)_{ij}) = (\mathrm{Im}\, \frac{\partial^2 \psi_0}{\partial x_i \partial x_j}(0))$ is positive definite.

$\therefore a_i = 0$ for $1 \leq i \leq n$. But then $a_{n+1} y_{n+1} = 0$ which implies $a_{n+1} = 0$ by lemma 5.4.

Q.E.D.

<u>Proposition 5.1</u>

Equation (6') with initial conditions (9) is solvable for $0 \leq s \leq T$.

<u>Proof</u>: Let $(y(s),\eta(s))$, $0 \leq s \leq T$, be the solution of (10) and (11). $y(s)$ is non-singular by lemma 5.5. By lemma 5.1, $M = \eta \cdot y^{-1}$ will solve (6'). It is clear that (11) implies (9). Q.E.D.

So far we have solved (1) to order two along c. If we differentiate (6) again and then restrict it to c, we will obtain non-homogeneous linear differential equations. We can solve them inductively with initial conditions compatible with (1).

(1) is therefore completely solved. By Whitney's theorem (see [4]), we can extend ψ to a neighborhood of c. This extension of ψ will satisfy all the properties stated in theorem 4.1. We have thus reproved theorem 4.1.

Suppose χ is another function satisfying properties (i), (ii) and (iii) of theorem 4.1, then

$\left(\frac{\partial^2 \chi}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n+1}$ will satify (6) and (9) along c. But there is a uniqueness theorem for matrix Ricatti equations (see[11]). The uniqueness theorem for O.D.E. will take care of the other derivatives. χ and ψ therefore coincide to infinite order on c. We have thus proved the following theorem:

Theorem 5.1

If χ satisfies properties (i) to (iii) of theorem 4.1, then $\chi - \psi$ is flat on c. In particular, χ will also satisfy (iv) and (v) in theorem 4.1.

The solution to the eikonal equation (3.4) is therefore essentially unique.

Remark: If ψ_0 depends continuously on a parameter, then all the estimates obtained in sections 4 and 5 will also be uniform with respect to the same parameter by the fundamental theorem of O.D.E.

Section 6 Solution of the Transport Equation

We shall follow the notation in section 3.

Theorem 6.1

Given any positive integer N and $\varepsilon > 0$, there exist functions $u_0, \ldots, u_{-N}$ supported in c_ε such that $Lu_r + g_r$ is flat on c, where

$$c_\varepsilon = \{(x', x_{n+1}): \quad 0 \le x_{n+1} \le T \text{ and } d(x,c) \le \varepsilon\}$$

Proof: If we differentiate $Lu_0 = 0$ with respect to x_j and then restrict it to c, we will get a sequence of non-homogeneous linear equations of the derivatives of u_0 along c. We can solve these equations inductively with arbitrary initial conditions on $\{x_{n+1} = 0\}$. If we extend u_0 to a neighborhood of c, then Lu_0 is flat on c.

Since g_r only depends on $u_0, \ldots, u_{r+1}$, we can solve inductively $Lu_r + g_r = 0$ to infinite order on c in exactly the same manner.

By using cut-off functions, we can assume that $\operatorname{supp} u_j \subset c_\varepsilon$ for $j = 0,\dots,-N$.

Q.E.D.

Remark: If the initial conditions of $u_1,\dots,u_{-N}$ depend continuously on a parameter, then all the estimates involved will also be uniform with respect to the same parameter by the fundamental theorem of O.D.E.

Section 7 Solution of the Initial-Value Problem

We will follow the notation of sections 1,2 and 3.

Recall that $\tau_1,\dots,\tau_m$ are the distinct solutions of the equation $p_m(0,0,\xi_0,\tau) = 0$. Let $b^1,\dots,b^m$ be the null-bicharacteristic of p_m with initial conditions $b^i(0) = (0,0,\xi_0,\tau_i)$, $i = 1,\dots,m$, and $c^1,\dots,c^m$ be the projections of b to R^{n+1}.

By theorem 4.1, we have complex phase functions $\psi_1,\dots,\psi_m$ defined on neighborhoods of $c^1,\dots,c^m$, with $\frac{\partial\psi_i}{\partial x_{n+1}}(0) = \tau_i$, $i = 1,\dots,m$.

In what follows, l,k will range from 0 to m-1 and j will range from -N-m to 0.

Let $u(x) = u_{kj}(x)\ \rho^{j}\ e^{i\rho\psi_k(x)}$ (1)

then $D^{l}_{x_{n+1}} u(x',0) = a^{l}_{j}(x')\ \rho^{l+j}\ e^{i\rho\psi_o(x)}$ (2)

with $a^{l}_{o}(x') = \tau^{l}_{k}\ u_{ko}(x',0)$ (3)

and in general

$$a^{l}_{j} = \tau^{l}_{k}\ u_{kj}(x',0) + h_{lj} \tag{4}$$

where h_{lj} only depends on $u_{k\,r}$, $j+1 \le r \le 0$.

If we compare (2) to the initial conditions

$$v_l(x') = \alpha_{lj}(x')\ \rho^{l+j}\ e^{i\rho\psi_o(x')} \tag{5}$$

then we obtain the following equations

$$\alpha_{lj}(x') = \tau^{l}_{k}\ u_{kj}(x',0) + h_{lj} \tag{6}$$

which can be solved inductively for $u_{kj}(x',0)$ because the matrix (τ^{l}_{k}) is non-singular.

By theorem 6.1, we can construct $u_{kj}(x)$ supported in c^{k}_{ε}, for $0 \le k \le m-1$ and $-N-m \le j \le 0$, with initial data $u_{kj}(x',0)$.

Theorem 7.1

Given any positive integer N and $\varepsilon > 0$, there exists a solution to (2.1) in the form of a collection of Gaussian beams $u = u_{kj}\ \rho^{j}\ e^{i\rho\psi_k}$ $0 \le k \le m-1$, $-N-m \le j \le 0$, with supp $u_{kj} \subset c^{k}_{\varepsilon}$.

Proof: Follows from the above construction, theorems 4.1, 6.1 and the equations (3.1, 3.2). Q.E.D.

By the uniqueness theorem for all the differential equations involved, if $u' = u'_{kj}\, \rho^{j}\, e^{i\rho\psi'_k}$ is another solution, then the corresponding quantities in u and u' will coincide to some high order (depending on N) on c, i.e. the Gaussian beam solution to (2.1) is essentially unique.

Remark: If the initial conditions (2.1 b) depend continuously on some parameters, then the estimates in (2.1a, 2.1b) will be uniform with respect to the same parameters.

Section 8 An Application

As an application of the theory developed in the previous sections, we shall prove a result about the propogation of singularities of strictly hyperbolic equations of the first order.

We consider a first order hyperbolic operator $P = D_t - A(x,t,D_x)$, where $A \in OPS^1(R^n)$ is properly supported and $a(x,t,\xi) \in S^1(R^n)$, the principal symbol of A is assumed to be real. A also depends smoothly on t. Also, $a(t,x,\xi) \neq 0$ for $\xi \neq 0$.

Let v be a C^∞ function of $t \in [0,T]$ with value in the space of distributions in the x-variable. We assume that v is a solution of the following initial value problem:

$$Pv = g, \quad t \in [0,T] \tag{1}$$

$$v(x,0) = v_0(x) \tag{2}$$

where $g \in C^\infty(R^n \times [0,T])$.

Let $(x_0,\xi_0) \in T^*(R^n)$ and $b(t) = (x(t),t,\xi(t),\tau(t))$, $0 \le t \le T$, be the null bicharacteristic of $\tau - a(x,t,\xi)$ through the point $(x_0,0,\xi_0,a(x_0,0,\xi_0))$. $c(t) = (x(t),t)$ is the projection of b to R^{n+1}.

Theorem 8.1

If $(x_0,\xi_0) \notin WF(v_0)$(def.0.4.2) then v is microlocally smooth along b. (def.0.4.6) Consequently, $(x(t'),\xi(t')) \notin (D^i v_{|t=t'})$ for $i = 0,1,2,\dots,$ $0 \le t' \le T$ and $b(t) \notin WF(v)$ for $0 < t < T$.

Proof: Let $t' \in [0,T]$ and

$$u_{(t',x',\xi')}(x) = \zeta(x)\, e^{i\rho(x\cdot\xi' + i\frac{|x-x|^2}{2})} \tag{3}$$

where $\zeta \in C_c^\infty(R^n)$, $\zeta = 1$ near $x(t')$ and the parameters (x',ξ') vary in a neighborhood of $(x(t'),\xi(t'))$.

By theorem 7.1, given any positive integer N, there exists a Gaussian beam

$$u(x,t) = u_j(x,t;\, x',t',\xi')\, \rho^j\, e^{i\rho\psi(x,t;\, x',t',\xi')}$$

$$-N-1 \le j \le 0, \quad 0 \le t \le t' \tag{4}$$

such that

$$P^* u = O(\rho^{-N})$$

$$u_{|t=t'} = u_{(t',x',\xi')} + O(\rho^{-N}) \qquad (5)$$

We have the following Green's formula

$$\iint_{[0,T]\times R^n} (v\overline{P^* u} - Pv\bar{u})\, dx\, dt$$

$$= \int_{R^n} (v(t')\overline{u(t')} - v(0)\overline{u(0)})\, dx \qquad (6)$$

If the parameters (x',ξ') are close to $(x(t'),\xi(t'))$, then we can assume that supp u is contained in an arbitrarily small neighborhood of c. In particular, $u_{(t',x',\xi')}(x,0)$ is supported in an arbitrarily small neighborhood W of x_o. $d_x\psi(x,0)$, $x \in W$, will then belong to a small neighborhood of ξ_o.

Since $(x_o,\xi_o) \notin WF(v_{|t=0})$, by lemma 0.4.4,

$$\int_{R^n} v(0)\overline{u(0)}\, dx = O(\rho^{-N}) \qquad (7)$$

Also by lemmas 0.4.5 and 0.4.6,

$$\iint_{[0,T]\times R^n} Pv\, \bar{u}\, dx\, dt$$

$$= \iint_{[0,T]\times R^n} g\, \bar{u}\, dx\, dt = O(\rho^{-N}) \qquad (8)$$

In view of (5), (7) and (8), we obtain from (6):

$$\int_{R^n} v(t',x)\, \zeta(x)\, e^{-i\rho(x\cdot\xi' + \frac{|x - x'|^2}{2})}\, dx = O(\rho^{-N}) \qquad (9)$$

Integrating with respect to x', we get

$$\int_{R^n} v(t',x)\ \zeta(x)\ e^{-i\rho x\cdot\xi'}\ dx = O(\rho^{-N}) \qquad (10)$$

It follows from (10) and the remark following theorem 7.1 that v is microlocally smooth along b. The rest of the theorem then follows from def. 0.4.5.

Q.E.D.

Chapter II Boundary Value Problems

Section 1 Elliptic Operators

Throughout this chapter P will be an operator of the form $P(x,y,D_x,D_y) = A_o(x,y,D_x)D_y^m + \dots + A_m(x,y,D_x)$, where $A_i(x,y,D_x) \in OPS^i(R^n)$ (def0.3.3), $i = 0,\dots,m$, depends smoothly on $y \in [0,Y]$.

Let $a_i(x,y,\xi) \in S^i(R^n)$(def.0.3.2) be the principal symbol of A_i. P is assumed to be elliptic, i.e. the principal symbol

$p_m(x,y,\xi,\eta) = a_o(x,y,\xi)\eta^m + \dots + a_m(x,y,\xi)$ is non-zero for any real $(\xi,\eta) \neq (0,0)$. We shall also assume A_i to be properly supported (def.0.3.4).

Section 2 Coercive Boundary Value Problems

Let $B_l(x,D_x,D_y) = B_o^l(x,D_x)D_y^{m_l} + \dots + B_{m_l}^l(x,D_x)$ for $0 \leq l \leq \nu$. Each $B_i^l \in S^i(R^n)$ is properly supported. We want to solve the following asymptotic boundary value problem:

$$Pu = O(\rho^{-N}) \qquad N = 1,2,\dots \tag{1a}$$

$$B_l u|_{y=0} = v_l + O(\rho^{-N}) \qquad l = 1,2,\dots,\nu \tag{1b}$$

where $v_l(x) = \alpha_{lj}(x)\, \rho^{m_l+j}\, e^{i\rho\psi(x)}$, $-M \leq j \leq 0$ (1c) is an oscillatory function at the origin with strictly positive complex phase function ψ. (def. 0.28) $\xi_0 = d\psi(0)$.

Let $b_i^l(x,\xi) \in S^i(R^n)$ be the principal symbols of B_i^l and $b_l(x,\xi,\eta) = b_i^l(x,\xi)\eta^{m_l} + \dots + b_{m_l}^l(x,\xi)$ be the principal symbol of B_l. We assume $m_l \le m-1$.

By the elliptic assumption on P, we have the following decomposition of $p_m(x,0,\xi,\eta)$:

$$p_m(x,0,\xi,z) = M_+(x,\xi,z)\cdot M_-(x,\xi,z) \qquad (2)$$

where the roots of $M_\pm(x,\xi,\cdot) = 0$ have $\pm$ imaginary parts for any $(x,\xi) \in T^*(R^n)$.

We assume $\{P,B_l\}$ satisfy the coercive conditions in a neighborhood of $(0,\xi_0)$

(i) $\nu = m^+ =$ degree of M_+ (3a)

(ii) $\{ b_l(x,\xi,z) : 0 \le l \le m^+ \}$ form a basis of $C[z]/(M^+)$ (3b)

We use (f) to denote the vector subspace of C[z] generated by the polynomial f.

Suppose f and g are two complex polynomials with no common zero. Let Γ be any contour in C that encloses all the zeros of f but none of the zeros of g.

<u>Lemma 2.1</u>

Let $b_1,\dots,b_k$ be complex polynomials, $k = \deg f$. Then the following two conditions are equivalent:

(a) $\{b_1, \ldots, b_k\}$ is a basis of $C[z]/(f)$

(b) the matrix

$$\left(\frac{1}{2\pi i} \int_\Gamma \frac{b_j(z) \cdot z^{l-1}}{f(z)g(z)} dz \right)_{1 \leq j, l \leq k}$$ is non-singular.

Proof: (b) implies (a): Let $\alpha_1 b_1 + \ldots + \alpha_k b_k \in (f)$,

then $\alpha_j \frac{1}{2\pi i} \int_\Gamma \frac{b_j \; z^{l-1}}{f \cdot g} \; dz = 0$, by Cauchy's theorem.

(a) implied (b): Since $\{b_1, \ldots, b_k\}$ is a basis of $C[z]/(f)$ and the linear transformation $h \to gh$ on $C[z]/(f)$ is invertible, by Cauchy's theorem,

$$\left(\frac{1}{2\pi i} \int_\Gamma \frac{b_j \cdot z^{l-1}}{f \cdot g} \; dz \right)_{1 \leq j, l \leq k}$$ is non-singular iff

$$\left(\frac{1}{2\pi i} \int_\Gamma \frac{z^j \cdot z^l}{f(z)} dz\right)_{0 \leq j, l \leq k-1}$$ is non-singular. (Note that $\{z^j: j = 0, \ldots, k-1\}$ is also a basis for $C[z]/(f)$.) But the latter matrix is clearly non-singular by Cauchy's theorem.

Q.E.D.

Lemma 2.2

Condition (3b) is equivalent to the matrix

$$\left(\frac{1}{2\pi i} \int_C \frac{b_j(x,\xi,z)z^{l-1}}{p(0,x,\xi,z)} \, dz \right)_{1 \le j,l \le m_+}$$ being non-singular. (4)

Here C is a contour in the upper half complex plane enclosing all the roots of $M^+(x,\xi,\eta)$ for (x,ξ) close to $(0,\xi_0)$.

Proof: Follows directly from lemma 2.1.

Q.E.D.

Section 3 Solution of the Coercive Boundary Value Problem

We shall follow the notation of sections 1 and 2.

In order to solve (2.1), we set

$$u = e^{i\rho\psi} \sum_{l=1}^{m_+} u_s \tag{1}$$

where $u_s(x,y) = \sum_{t=0}^{-N-m} \rho^t \frac{1}{2\pi i} \int_C \frac{c_{st}(x,y,z)\cdot e^{i\rho yz}}{p(x,y,d\psi(x),z)} \, dz$

and C is the contour in lemma 2.2. (1')

First of all, we set

$$c_{so}(x,y,z) = c'_{so}(x) \, z^{s-1} \tag{2}$$

$$B_l u|_{y=0} = e^{i\rho\psi} \sum_{q=m_l}^{-N-m+m_l} w_{lq} \, \rho^q \tag{3}$$

where $w_{1m_1} = \frac{1}{2\pi i} \int_C \frac{b_1(x,d\psi(x),z)\cdot z^{s-1}}{p(x,0,d\psi(x),z)} \cdot c'_{so}(x)\, dz$

(4)

Comparing (4) and (2.1c), in view of lemma 2.2, $\alpha_{1m_1} = w_{1m_1}$ uniquely determines c'_{so} in a neighborhood of 0.

Now $Pu_s = e^{i\rho\psi} \cdot \sum_{r=m}^{-N} v_{sr}$, $1 \le s \le m^+$ (5)

and $v_{sm}(x,y) = \frac{1}{2\pi i} \int_C c'_{so}(x)\cdot z^{s-1}\cdot e^{i\rho yz}\, dz$

$= 0$ by Cauchy's theorem

$$v_{s(m-1)}(x,y) = \frac{1}{2\pi i} \int_C v'_{s(m-1)}(x,y,z)\, e^{i\rho yz}\, dz \qquad (6)$$

where $v'_{s(m-1)}(x,y,z) = c_{s(-1)}(x,y,z) + R_{s(-1)}(x,y,z)$

(7)

with R depending only on the symbols of p and $c_o(x)$.

Next we set

$$c_{s(-1)}(x,y,z) = c'_{s(-1)}(x)\, z^{s-1} - R_{s(-1)}(x,y,z) \qquad (8)$$

If we substitute (8) into (6), then $v'_{s(m-1)}$ is identically zero by Cauchy's theorem.

The second highest order term of (3) is given by

$$w_{1(m_1-1)} = \frac{1}{2\pi i} \int_C \frac{b_1(x,d\psi(x),z)\cdot z^{s-1}}{p_m(x,0,d\psi(x),z)} \cdot c'_{s(-1)}(x)\, dz + T(x)$$

where T depends on the symbols of P, c_o and $R_{s(-1)}$.

Again by lemma 2.2, $\alpha_{1(m_1-1)} = w_{1(m_1-1)}$ determines $c'_{s(-1)}(x)$ uniquely in a neighborhood of 0.

Similarly, we can determine c_{st} for $t \leq -2$ inductively by letting other terms in the asymptotic expansion of Pu and $Bu|_{y=0}$ be equal to zero.

Theorem 3.1

There exists a solution to the boundary value problem (2.1) in the form (1), where each u_s is supported in a small neighborhood of (0,0). Moreover, $u = O(\rho^{-\infty})$.

Proof: Since the contour C lies in the upper half complex plane and ψ_o is a positive definite complex phase function, any function u of the form (1) is $O(\rho^{-\infty})$ away from (0,0). We can therefore use cut-off functions to control the support of each u_s.

Q.E.D.

Remark: If the initial data (2.1c) depends continuously on some parameters, then all the estimates obtained in this section will be uniform with respect to the same parameters since our construction only involves algebraic operations.

Section 4 An Application

We shall apply the theory developed in the previous sections to prove a microlocal regularity theorem for the Dirichlet problem of elliptic equations.

We shall follow the notation in the previous sections.

Let w and g be C^∞ functions of $y \in [0,Y]$ with value in the space of distributions in the x-variable. For some $\xi_0 \neq 0$, g is assumed to be microlocally smooth along $\Gamma = \{(0,y,\xi_0): 0 \le y < Y\}$. (def.0.4.6)

Theorem 4.1

Suppose that $Pw = g$ and $(0,\xi_0) \notin WF(D_y^j w_{|y=0})$ for $0 \le j \le m_+ - 1$, then w is microlocally smooth along $\Gamma' = \{(0,y,\xi_0): 0 \le y < Y\}$.

Consequently, $(0,\xi_0) \notin WF(D_y^j\, w_{|y=y'})$ for $j = 0,1,2\ldots$ and $0 \le y' < Y$. Also $(0,y,\xi_0,\tau) \notin WF(w)$ for $0 < y < Y$ and any τ.

Proof: By the preparation theorem for pseudodifferential operators (see [8]), we can write $P = P_- \cdot P_+$, where

$$P_{\pm}(x,y,D_x,D_y) = A_o^{\pm}(x,y,D_x)D_y^{m_{\pm}} + \dots + A_{m_{\pm}}^{\pm}(x,y,D_x).$$

The principal symbols of $P_{\pm}$ are denoted by $p_{\pm}$. They satisfy the following condition:

For $\xi \neq 0$ in R^n, $p_{\pm}(x,y,\xi,z) = 0$ implies $\pm$ Im $z > 0$. (1)

Let $v = P_+w$, then $Pw = g$ is equivalent to

$$P_-v = g$$
$$P_+w = v \tag{2}$$

Let $u \in C^\infty([y',Y] \times R^n)$, $0 \le y' < Y$, we have the following Green's formula:

$$\iint_{R^n \times [y',Y]} P_-v\,\bar{u} - v\,\overline{P_-^* u} \quad dx\,dy$$

$$= \sum_{j=0}^{m_- -1} \int_{R^n} S_j v|_{y=y'}\, \overline{D_y^j u}|_{y=y'}\, dx + \sum_{j=0}^{m_- -1} \int_{R^n} D_y^j\, v|_{y=Y}\, \overline{T_j u}|_{y=Y} dx \tag{3}$$

where S_j and T_j are some boundary operators conjugate to the Dirichlet boundary operators D_y^j, $j = 0,\dots,m_- -1$.

If we write $S_j = S_{jo}(x,y,D_x)D_y^j + \dots + S_{jj}(x,y,D_x)$, then S_{jo} is elliptic.

By theorem 3.1, given any positive integer N, there exists a function u_1 of the form (3.1) such that

$$P_-^* u_1 = O(\rho^{-N})$$

$$D_y^{m_- -1} u_1|_{y=y'} = \zeta(x)\, e^{i\rho(x\cdot\xi + \frac{1}{2}i|x-x'|^2)} + O(\rho^{-N})$$

$$D_y^j u_1|_{y=y'} = 0, \qquad 0 \le j \le m_- - 2$$

$$u_1 = O(\rho^{-\infty}) \quad \text{for } y > y'$$

where (x',ξ) is close to $(0,0)$ and $\zeta \in C_c^\infty(R^n)$, $\zeta = 1$ in a neighborhood of 0.

If we substitute u_1 in (3), using (4) and the microlocal smoothness of g, we obtain that

$$\int_{R^n} S_o v|_{y=y'} \zeta(x)\, e^{-i\rho\, x\cdot\xi - \frac{1}{2}|x-x'|^2} dx = O(\rho^{-N}) \qquad (5)$$

if (x',ξ) is close enough to $(0,\xi_o)$ and supp ζ is small enough.

The estimate is uniform for (x',ξ) close to $(0,\xi_o)$.

Integrating (5) with respect to x', we have

$$\int_{R^n} S_o v|_{y=y'} \zeta(x)\, e^{-i\rho x\cdot\xi} dx = O(\rho^{-N}) \qquad (6)$$

uniformly for ξ close to ξ_o.

Since S_o is elliptic, (6) implies that

$$\int_{R^n} v(y',x)\, \zeta'(x)\, e^{-i\rho x\cdot\xi} dx = O(\rho^{-N}) \qquad (6')$$

for another $\zeta' \in C_c^\infty(R^n)$, $\zeta' = 1$ on a neighborhood of 0.

Again, by theorem 3.1, there exists u_2 such that

$$P_-^* u_2 = O(\rho^{-N})$$

$$D_y^{m_--2} u_2|_{y=y'} = \zeta(x)\, e^{i\rho x\cdot\xi - \frac{1}{2}|x-x'|^2} + O(\rho^{-N})$$

$$D_y^j u_2|_{y=y'} = 0, \quad j \neq m_- -2$$

$$u_2 = O(\rho^{-\infty}) \text{ for } y > y' \qquad (7)$$

Substituting u_2 in (3) and using (7), we get

$$\int_{R^n} S_1 v|_{y=y'}\, \zeta(x)\, e^{-i\rho x\cdot\xi}\, dx = O(\rho^{-N}) \qquad (8)$$

But $S_1 = S_{10}(x,y,D_x)D_y + S_{11}(x,y,D_x)$ and S_{10} is elliptic; (8) and (6) then imply

$$\int_{R^n} D_y v(y',x)\, \zeta'(x)\, e^{-i\rho x\cdot\xi}\, dx = O(\rho^{-N}) \qquad (9)$$

By an inductive argument, we can therefore prove that v is microlocally smooth along Γ'.

If we now apply the same argument to $P_+ w = v$, it follows that w is microlocally smooth along $\Gamma'' = \{(0,\xi_0,y): \ 0 < y < Y\}$. Taking into account that $(0,\xi_0) \notin WF(D_y^j w|_{y=0})$, for $j = 0,\dots,m_+-1$, we can improve the microlocal smoothness up to $y = 0$, i.e. w is microlocally smooth on Γ'.

Q.E.D.

Chapter III Reflection of Singularities

Section 1 Reflected Family of Null-bicharacteristics

Let $P(x,D)$ be a m^{th} order partial differential operator on R^{n+1} with real principal symbol $p(x,\xi)$.

S is a non-characteristic hypersurface in R^{n+1}. If $i: S \longrightarrow R^{n+1}$ is the natural injection, then $i^*: T^*R^{n+1}|_S \longrightarrow T^*S$.

Let $\alpha \in T^*_yS$, $\alpha \neq 0$ and $i^{*-1}(\alpha) \cap p^{-1}(0) = \{\beta_1,\ldots,\beta_k\} \subset T^*_yR^{n+1}$.

Following Nirenberg (see [8]), the null-bicharacteristic $\Gamma_1,\ldots,\Gamma_k$ of p through $\beta_1,\ldots\beta_k$ are said to belong to the same reflected family of null-bicharacteristics corresponding to α.

Let $\pi: T^*(R^{n+1}) \longrightarrow R^{n+1}$ be the canonical projection. Throughout this chapter we shall impose the following non-grazing hypothesis on $\Gamma_1, \ldots, \Gamma_k$:

$$c_i = \pi(\Gamma_i) \text{ is transversal to S at y.} \qquad (1)$$

If we choose local coordinates at y so that $S = \{(x_1,\ldots,x_n,t): t = 0\}$, then $i: S \longrightarrow R^{n+1}$ is given by $i(x_1,\ldots,x_n) = (x_1,\ldots,x_n,0)$ and

$i^*: T^*R^{n+1}|_S \longrightarrow T^*S$ is given by

$i^*(x_1,\dots,x_n,\xi_1,\dots,\xi_n,\tau)=(x_1,\dots,x_n,t,\xi_1,\dots\xi_n)$.

From here on, (x,t) will denote a point in $R^n \times R$.

If $p_m(x,t,\xi,\tau)$ is the principal symbol of P, then p_m is real and

$$p_m(x,0,0,1) \neq 0 \text{ (S is non-characteristic)} \tag{2}$$

Let $y = (0,0)$ and $\alpha = (0,\xi_o) \in T_o^*(R^n)$. If $\tau_1,\dots,\tau_k$ are the zeros of $p_m(0,0,\xi_o,\tau) = 0$, the non-grazing hypothesis (1) is equivalent to

$$\frac{\partial p_m}{\partial \tau}(0,0,\xi_o,\tau_j) \neq 0, \quad j = 1,\dots,k. \tag{3}$$

We shall let $R_+^{n+1} = \{(x,t): \ t > 0\}$. (4)

Section 2 Perfectly Reflecting Boundary Conditions

We shall use local coordinates in the following discussion and we shall follow the notation in section1.

Let $p(0,0,\xi_o,\tau) = (\tau-\tau_1)\dots(\tau-\tau_k)\, q(\tau)$, where $q(\tau) = 0$ has no real zero.

By (1.3), for (x,t,ξ) close to $(0,0,\xi_o)$, we can write

$p(x,t,\xi,\tau) = (\tau-\tau_1(x,t,\xi))\cdots(\tau-\tau_k(x,t,\xi))\cdot q(x,t,\xi,\tau)$, where $\tau_1,\dots,\tau_k \in S^1(R^n)$ and $q \in S^{m-k}(R^{n+1})$. (1)

By the preparation theorem for pseudodifferential operators (see [8]), we can write

$$P(x,t,D_x,D_t) = (D_t - A_1(x,t,D_x))\cdots(D_t - A_{k_0}(x,t,D_x))\cdot$$

$$Q_-(x,t,D_x,D_t)\cdot Q_+(x,t,D_x,D_t)\cdot(D_t - A_{k_0+1}(x,t,D_x))\cdots$$

$$(D_t - A_k(x,t,D_x) + T, \quad \text{where } 1 \le k_0 \le k. \tag{2}$$

For (x,t,ξ) close to $(0,0,\xi_0)$, the principal symbol of A_j is τ_j. The symbol of T vanishes in a neighborhood of $(0,0,\xi_0)$ for all τ. If $q_\pm(x,t,\xi,\tau)$ are the principal symbols of $Q_\pm$, then $q = q_-\circ q_+$ and for $\xi \in T^*(R^n)$,

$$q_\pm(x,t,\xi,z) = 0 \text{ implies } \pm\text{Im } z > 0. \tag{3}$$

Since p_m is real,

$p_+ = \overline{p_-}$ as polynomials in τ and hence

$$\deg p_\pm = \tfrac{1}{2}(m-k). \tag{4}$$

We shall write

$$p_+(x,t,\xi,\tau) = (\tau-\tau_{k_0+1}(x,t,\xi))\cdots(\tau-\tau_k(x,t,\xi))\cdot q_+(x,t,\xi,\tau)$$

with degree $p_+ = \tfrac{1}{2}(m-k) + (k-k_0) = m_+$ (5)

and

$$p_-(x,t,\xi,\tau) = (\tau-\tau\ (x,t,\xi))\cdots(\tau-\tau_{k_0}(x,t,\xi))\cdot q_-(x,t,\xi,\tau)$$

with degree $p_- = \tfrac{1}{2}(m-k) + k_0 = m_-$. (6)

Let $\{ B_j : \ j = 1,\ldots,l \}$ be a set of boundary partial differential operators on S. The order of B_j is less than or equal to m-1. The principal symbol of B_j is denoted by $b_j(x,t,\xi,\tau)$.

Following Majda and Osher (see [3]), $\{B_j: j = 1,\ldots,l\}$ is said to be perfectly reflecting at $(0,\xi_o)$ if there is no non-trivial solution to the O.D.E.

$$p^+(0,0,\xi_o,D_t)v = 0 \tag{7}$$

with boundary conditions

$$b_j(0,0,\xi_o,D_t)\ v(0) = 0,\ j = 1,\ldots,l\ . \tag{8}$$

Lemma 2.1

The condition of perfect reflection is equivalent to the following condition:

$$b_j(0,0,\xi_o,z) \text{ span } C[z]/(p^+) \tag{9}$$

Proof: By lemma 0.6.1, the perfectly reflecting condition is equivalent to the rank of the matrix

$$\left(\frac{1}{2\pi i}\int_C \frac{b_j\cdot z^{r-1}}{p_+}\,dz\right)_{\substack{1\le j\le l\\ 1\le r\le m_+}} \quad \text{equal to } m_+ .$$

Here C is a contour enclosing all the zeros of p_+. The lemma then follows from lemma 2.2.1.

Q.E.D.

Corollary:

Perfect reflection implies $l \ge m_+$.

Remark: It follows easily from lemma 2.1 that the condition of perfect reflection is independent of the choice of local coordinates.

Example (1) Let $B_j = D_t^j$, $0 \le j \le m_+-1$, then (P, B_j) is perfectly reflecting.

(2) If P is elliptic, then (P, B_j) is perfectly reflecting iff (P, B_j) is coercive, provided that $l = \frac{1}{2}m$.

We now assume $(P, B_j, j = 1, \ldots, m_+)$ to be perfectly reflecting and that order of $B_{j_1} \neq$ order of B_{j_2} if $j_1 \neq j_2$. The principal symbol of B_j is also assumed to be of the form

$$\tau^{n_j} + b_{j1}(x,t,\xi)\tau^{n_j-1} + \ldots + b_{jn_j}(x,t,\xi), \qquad (10)$$

where n_j = order of B_j.

Let u and $v \in C_c^\infty(\overline{R_+^{n+1}})$. We have a Green's formula

$$\iint_{R_+^{n+1}} Pu\bar{v} - u\cdot\overline{P^* v}\, dx\, dt = \sum_{j=0}^{m-1} \int_{R^n} B_j' u \cdot \overline{C_j v}\, dx, \qquad (11)$$

where B'_{j_r}, $r = 0, \ldots, l-1$ are the given boundary operators. The order of B'_j is j. B'_j has principal symbol equal to

$$\tau^j + b'_{j1}(x,t,\xi)\tau^{j-1} + \ldots + b_{jj}(x,t,\xi) \qquad (10')$$

Let $J = \{0,1,\ldots,m-1\}$

$I = \{j_r : \ r = 1,\ldots,m_+\}$.

Note that $|J\setminus I| = m_+$.

Any set of boundary operators $\{C_j : \ j \in J\setminus I\}$ is said to be adjoint to B_j, $1 \le j \le m_+$. Adjoint operators are not unique, because they also depend on the choice of B'_j, $j \in J\setminus I$.

<u>Lemma 2.2</u>

$(P, B_j, 1 \le j \le m_+)$ is perfectly reflecting with respect to $\Gamma_1,\ldots,\Gamma_{k_0}$ at $(0,0,\xi_0)$ iff $(P^*, C_j, j \in J\setminus I)$ is perfectly reflecting with respect to $\Gamma_{k_0+1},\ldots,\Gamma_k$ at $(0,0,\xi_0)$.

<u>Proof</u>: First of all, given $h(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, we define

$$
\begin{aligned}
{}^1h(z) &= z^{n-1} + a_1 z^{n-2} + \ldots + a_{n-1} \\
&\vdots \\
{}^nh(z) &= 1
\end{aligned}
\tag{12}
$$

Let C be a curve in the complex plane encircling all the zeros of $p_m(0,0,\xi_0,z)$. Similarly $C_\pm$ encircles the zeros of $p_\pm$ and C_+ is separated from C_-.

For simplicity, we shall assume in this proof that $p_m = \tau^m$ +lower order terms in τ. Since p_m is non-characteristic at S, the proof of the general case will differ from this by the multiplication of a non-zero constant.

Using pseudodifferential operators and (2), we can write

$$\iint_{R_+^{n+1}} u\cdot\overline{P^* v} - Pu\cdot\overline{v}\,dx\,dt = \sum_{j=0}^{m-1}\int_{R^n} E_j u\cdot\overline{F_j v}\,dx, \tag{13}$$

where the principal symbols of E_j, F_j at $(0,0,\xi_0)$ are:

$$\begin{aligned}
e_0 &= 1\\
&\vdots\\
e_{m_+-1} &= \tau^{m_+-1}\\
e_{m_+} &= p_2(0,0,\xi_0,\tau)\, q_-(0,0,\xi_0,\tau)\\
e_{m_++1} &= p_2\cdot q_-\cdot\tau\\
&\vdots\\
e_{m-1} &= p_2\cdot q_-\cdot\tau^{m_--1}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
f_0 &= p_-\cdot {}^1p_+ \qquad \text{(see 12)}\\
&\vdots\\
f_{m_+} &= p_-\\
f_{m_++1} &= {}^1p_-\\
&\vdots\\
f_{m-1} &= 1
\end{aligned} \tag{15}$$

where $p_1 = (\tau-\tau_1(x,t,\xi))\cdots(\tau-\tau_{k_o}(x,t,\xi))$

$$p_2 = (\tau-\tau_{k_o+1}(x,t,\xi))\cdots(\tau-\tau_k(x,t,\xi)) \qquad (16)$$

We have used (4) in the above computations.

We define a duality between polynomials of order $\leq m-1$ in the following way:

$$(e,f) = \frac{1}{2\pi i}\int_C \frac{e(z)\cdot f(z)}{p(0,0,\xi_o,z)}\,dz \qquad (17)$$

Then $(e_j,f_r) = \delta_{jr} \quad 0 \leq j,r \leq m-1,$ (18)

by Cauchy's theorem.

We can also write $B'_j = \alpha_{jr}E_r$ + lower order terms, where (α_{jr}) is a lower triangular matrix. (19)

We can write $E_j = \beta_{jr}B'_r$ + lower order terms, where (β_{jr}) is the inverse of (α_{jr}). (20)

If we compare (11) and (13), in view of (20), we immediately obtain

$$C_j = \overline{\beta_{rj}}\,F_r \ + \ \text{lower order terms} \qquad (21)$$

$\overline{(\beta_{rj})} = (\beta^*_{jr})$ is the adjoint of (β_{jr}).

If we look at the symbols, then

$b'_j(0,0,\xi_o,\tau) = \alpha_{jr}\ e_j(0,0,\xi_o,\tau),$

$e_j = \beta_{jr}\cdot b'_j$ and $c_j = \beta^*_{jr}\cdot f_r$ (22)

where $(\alpha_{jr}) = (b'_j, f_r)$ (23)

By lemmas 2.1 and 2.2.1, $(P, B_j) = (P, B'_{jr})$ is perfectly reflecting with respect to $\Gamma_1, \ldots, \Gamma_{k_0}$ iff the matrix $(\alpha_{j_r}(r'-1))_{1 \le r, r' \le m_+}$ is non-singular. (24)

Similarly, $(P^*, C_j, j \in J\ I)$ is perfectly reflecting with respect to $\Gamma_{k_0+1}, \ldots, \Gamma_k$ iff the matrix

$$(c_j, e_r)_{\substack{j \in J\setminus I \\ m_+ \le r \le m-1}} \text{ is non-singular.} \tag{25}$$

By (22) and (18),

$$(c_j, e_r) = (\beta^*_{js} f_s, e_r) = (\beta^*_{jr}) \tag{26}$$

Therefore (25) is equivalent to

$$(\beta_{sj})_{\substack{j \in J\setminus I \\ m_+ \le s \le m-1}} \text{ being non-singular.} \tag{27}$$

We are now in the situation of lemma 0.6.2, hence (24) is equivalent to (27).

Q.E.D.

Recall that is P is elliptic, then (P, B_j) is perfectly reflecting iff (P, B_j) is coercive.

Corollary:

(P, B_j) is coercive iff its adjoint (P^*, C_j) is coercive.

Section 3 Reflection of singularities, Local Theory

We shall follow the notation in sections 1 and 2.

Let u be a C^∞ function of $t \in [0,\infty)$ with values in the space of distributions in the x-variable.

We assume that $Pu = f \in C^\infty(\overline{R^{n+1}_+})$. Let $\Gamma_1,\dots,\Gamma_k$ be the reflected family of null-bicharacteristics of p_m corresponding to $(0,\xi_0)$. $\{B_j: 1 \le j \le l\}$ is a set of boundary partial differential operators such that (P,B_j) is perfectly reflecting at $(0,\xi_0)$ with respect to $\Gamma_1,\dots,\Gamma_{k_0}$.

Theorem 3.1 (Majda and Osher [3])

If $(0,\xi_0) \notin WF(B_j u|_{t=0})$, $1 \le j \le l$, and $\Gamma_1,\dots,\Gamma_{k_0} \notin WF(u)$, then $\Gamma_{k_0+1},\dots,\Gamma_k \notin WF(u)$ and $(0,\xi_0) \notin WF(D_t^j u|_{t=0})$ for $j = 0,1,2,\dots$.

Proof: By (2.2),

$$P = (D_t - A_1(x,t,D_x))\cdots(D_t - A_{k_0}))\cdot Q_-(x,t,D_x,D_t)\cdot$$
$$Q_+(x,t,D_x,D_t)\cdot(D_t - A_{k_0+1}(x,t,D_x))\cdots(D_t - A_k(x,t,D_x))$$
$$+ T$$

$$= (D_t - A_1(x,t,D_x))\cdot P^1(x,t,D_x,D_t) + T ,$$

where P^1 is defined in the obvious way.

Let $P^1 u = u^1$, then we have

$$(D_t - A_t(x,t,D_x))u^1 = f', \qquad (1)$$

where $f' = f - Tu$ is microlocally smooth along $\{(0,t,\xi_0): 0 \le t \le t_0, t_0 > 0\}$ by the nature of T.

By lemma 0.4.3, $\Gamma_1 \notin WF(u)$ implies that

$$\Gamma_1 \notin WF(u^1) \tag{2}$$

Let Γ_1 be parametrized as $(x'(t),\xi'(t),t,\tau'(t))$. (2) implies then

$$(x'(t'),\xi'(t')) \notin WF(u^1|_{t=t'}), \tag{3}$$

for $t' > 0$, t' close to 0 so that $t = t'$ is non-characteristic (see prop. 0.4.4).

We can now apply theorem 1.8.1 to conclude that u^1 is microlocally smooth along $\{(0,t,\xi_0): 0 \le t \le t' < t_0\}$.

If we apply the same argument k_0-1 times to the equation $P^1u = u^1$, we obtain

$$Q_-\circ Q_+\circ(D_t - A_{k_0+1})\cdots(D_t - A_k)\, u = g, \tag{4}$$

where g is microlocally smooth along $\{(0,t,\xi_0): 0\le t\le t''\}$.

By the proof of theorem 2.4.1, (4) implies

$$P_+u = Q_+\circ(D_t - A_{k_0+1})\cdots(D_t - A_k)\, u = g', \tag{5}$$

where g' is microlocally smooth along $\{(0,t,\xi_0): 0\le t\le t^*\}$, for some $t^* > 0$.

The perfectly reflecting property of (P,B_j) implies (via lemma 2.1) that

$$D_t^r u = E_j\circ B_j u + R\circ P_+u + Su, \quad 0 \le r \le m_+-1, \tag{6}$$

for some pseudodifferential operators E_j, R and S. The symbol of S vanishes in a neighborhood of $(0,0,\xi_0)$ with τ arbitrary.

(5) and (6) imply that

$$(0,\xi_0)\notin WF(D_t^r u|_{t=0}), \text{ for } r = 0,\ldots,m_+-1. \tag{7}$$

It Follows from (7) and Theorems 2.4.1, 1.8.1 that u is microlocally smooth along $\{(0,t,\xi_o): 0 \le t \le \varepsilon\}$ for some $\varepsilon > 0$. Hence $(0, \xi_o) \notin WF(D_t^j u)$ for $j = 0,1,\ldots$ and $\Gamma_{k_o+1},\ldots,\Gamma_k \notin WF(u)$ in a small neighborhood of $(0,0,\xi_o)$. It then follows from theorem 0.4.1 that $\Gamma_{k_o+1}, \Gamma_k \notin WF(u)$.

Q.E.D.

Section 4 Asymptotic Solution to a Perfectly Reflecting Boundary Value Problem

Let $(P,B_j, 1 \le j \le m_+-1)$ be perfectly reflecting with respect to $\Gamma_1,\ldots,\Gamma_{k_o}$ at $(0,\xi_o)$. We want to solve, for any positive integer N,

$$Pu = O(\rho^{-N}) , \tag{1a}$$

$$B_j u|_{t=0} = v_j + O(\rho^{-N}) , \tag{1b}$$

where $v_j(x) = \sum_{m_j}^{-M} v_{jr}(x)\, \rho^r e^{i\rho\,\psi_o(x)}$, m_j = order of B_j and ψ_o is a strictly positive complex phase function with $d\psi_o(0) = \xi_o$.

For $a = k_o+1,\ldots,k$, there exists a solution ψ_a of the eikonal equation $p_m(x,t,d\psi(x,t)) = 0$ satisfying all the conclusions stated in theorem 1.4.1 with

$$\frac{\partial\psi_a}{\partial t}(0,0) = \tau_a \tag{2}$$

We shall let

$$u = u_h + u_e \tag{3}$$

$$u_h = \sum_{a=k_0+1}^{k} u_a \tag{4}$$

$$u_a(x,t) = u_{ab}(x,t,)\rho^b e^{i\rho\psi_a(x,t)}, \quad -m-N \le b \le 0. \tag{5}$$

$$u_e = e^{i\rho\psi_0} \sum_{s=1}^{\frac{1}{2}(m-k)} u_s \tag{6}$$

$$u_s = \frac{1}{2\pi i} \int_{C'} \frac{c_{sb}(x,t,z)}{\rho^b . p_m(x,t,d\psi_0(x),z)} e^{i\rho tz}\, dz, \quad -m-N \le b \le 0. \tag{7}$$

Here C' is a contour in the complex plane encircling the roots of $q_+(0,\xi_0,0,z) = 0$. From now on , a will always range from k_0+1 to k and s will range from 1 to $\frac{1}{2}(m-k)$.

By lemma 2.2.1, perfect reflection implies that the matrix

$$B = \left(\frac{1}{2\pi i} \int_C \frac{b_j(0,0,\xi_0,z).z^l}{p_+(0,0,\xi_0,z)}\, dz \right)_{0 \le j,l \le m_+-1} \tag{8}$$

is non-singular, where

C is a contour encircling the roots of $p_+(0,0,\xi_0,z) = 0$.

Let B' be the matrix defined by

$$b'_{jl} = \begin{cases} b_j(0,0,\xi_0,\tau_{l+k_0+1}), & 0 \le j \le m_+-1,\ 0 \le l \le k-k_0-1. \\ \frac{1}{2\pi i} \int_{C'} \frac{b_j(0,0,\xi_0,z)}{p_m(0,0,\xi_0,z)}.z^{l-k+k_0}\, dz, & 0 \le j \le m_+-1 \\ & k-k_0 \le l \le m_+-k+k_0-1. \end{cases} \tag{9}$$

<u>Lemma 4.1</u>

B' is nonsingular iff B is nonsingular.

<u>Proof</u>: By Cauchy's theorem, the column spaces of the two matrices are the same.

Q.E.D.

If we apply P and B_j to u, we get

$$Pu = Pu_h + Pu_e$$

$$= \rho^{\alpha}(w_{a\alpha}e^{i\rho\psi_a} + w_{s\alpha}e^{i\rho\psi_o}), \quad -N \le \alpha \le m. \tag{10}$$

$$B_j u = \rho^{\beta}(w'_{a\beta} + w'_{s\beta})e^{i\rho\psi_o}, \quad -N-m+m_j \le \beta \le m_j. \tag{11}$$

The highest order terms in (10) are given by

$$w_{am}(x,t) = p_m(x,t,d\psi_a(x,t))u_{ao}(x,t) \tag{12}$$

$$w_{sm}(x,t) = \frac{1}{2\pi i}\int_{C'} c_{s0}(x,t,z)e^{i\rho tz}\,dz \tag{13}$$

Since $p_m(x,t,d\psi_a(x,t))$ is flat on $c_j = \pi(\Gamma_j)$, (12) does not contribute anything in the asymptotic expansion. On the other hand, (13) will be identically zero if we pick $c_{s0}(x,t,z)$ to be holomorphic in z.

We shall let $c_{so}(x,t,z) = c'_{so}(x)z^{s-1}$. The highest order terms of (11) are then given by

$$w'_{am_j} = b_a(x,0,d\psi_o(x)\tau_a)u_{ao}(x,0) \tag{14}$$

$$w'_{sm_j} = \frac{1}{2\pi i}\int_{C'} \frac{b_j(x,0,d\psi_o(x),z)z^{s-1}}{p_m(x,0,d\psi_o(x),z)}\,dz\cdot c'_{so}(x) \tag{15}$$

By lemma 4.1, $u_{ao}(x,0)$ and $c'_{so}(x)$ are uniquely determined by the boundary conditions (1b).

We now look at the second highest order terms in

(10). They are given by

$$w_{a(m-1)} = L_a u_{ao}(x,t) + p_m(x,t,d\psi_a(x,t))u_{ao}(x,t), \quad (16)$$

where $L_a = \frac{1}{i}\left[\frac{\partial p_m}{\partial \xi_j}(x,t,d\psi_a(x,t))\frac{\partial}{\partial x_j} + \frac{\partial p_m}{\partial \tau} \cdot \frac{\partial}{\partial t} \right] + M_a$

M_a is a function depending only on p_m, p_{m-1} and ψ_a.

(16) will contribute nothing to the asymptotic expansion if we solve the transport equations (16) to infinite order along $c_j = \pi(\Gamma_j)$ with initial data $u_{ao}(x,0)$.

On the other hand,

$$w_{s(m-1)} = \frac{1}{2\pi i}\int_{C'}(c_{s(-1)}(x,t,z) + R_{s(-1)}(x,t,z))e^{i\rho tz}\,dz, \quad (17)$$

where $R_{s(-1)}$ depends only on c_{so} and symbols of P.

(17) will be zero if we let $c_{s(-1)} = c'_{s(-1)}(x)z^{s-1} - R_{s(-1)}(x,t,z)$. The second highest order terms in (11) are then given by

$$w'_{a(m-1)} = b_j(x,0,d\psi_o(x),\tau_a)u_{j(-1)}(x,0) + T_{a(m_j-1)} \quad (18)$$

where $T_{a(m_j-1)}$ depends only on u_{jo}, ψ_a and symbols of B_j and P.

$$w'_{s(m_j-1)} = \frac{1}{2\pi i}\int_{C'}\frac{b_j(x,0,d\psi_o(x),z)}{p_m(x,0,d\psi_o(x),z)}\cdot z^{s-1}\,dz\cdot c'_{s(-1)} + G_{s(m_j-1)}. \quad (19)$$

where $G_{s(m_j-1)}$ depends only on c_{so} and the symbols of B_j and P.

Again, $u_{a(-1)}$ and $c_{s(-1)}$ are uniquely determined by the initial conditions (1b).

We can therefore solve the boundary value problem (1) inductively. We have thus proved

Theorem 4.1

Given any positive integer N, there exists an asymptotic solution to (1) in the form of $u = u_{k_0+1} + \cdots + u_k + u_e$ where the support of u_a, $k_0+1 \le a \le k$, is contained in an arbitrary small neighborhood of $c_a = \pi(\Gamma_a)$ and the support of u_e is contained in an arbitrary small neighborhood of $(0,0)$. Moreover, u_e is $O(\rho^{-\infty})$ away from $(0,0)$.

Remark: If the intial data depend continuously on some parameters, then the asymptotic expansions will be uniform with respect to the same parameters.

As an application of theorem 4.1, we shall prove a partial converse to theorem 3.1. We follow the notation in section 3.

Let P be a partial differential operator on $\overline{R^{n+1}_+}$ and B_j, $1 \le j \le l$, a set of boundary operators. We assume that $l < m_+$ and $\{b_j(0,0,\xi_0,z): j = 1,\dots l\}$ is a set of linear independent vectors in $C[z]/(p_+(0,0,\xi_0,z))$.

Theorem 4.2

There exists a C^∞ function u of $t \in [0,T]$ $(T > 0)$ with values in the space of distributions in the x-varible such that

(i) $Pu \in C^\infty(R^n \times [0,T])$ and $B_j u|_{t=0} \in C^\infty(R^n)$

(ii) $WF(u|_{R_+^{n+1}}) \subset \Gamma_{k_0+1} \cup \cdots \cup \Gamma_k$

(iii) $\bigcup_{j=0}^{m_+-1} WF(D_t^j u|_{t=0}) = \{(0,\xi_0)\}$

Proof: We can pick $D_t^{j_1},\ldots,D_t^{j_{m_+-l}}$, $j_1,\ldots,j_{m_+-l} \le m_+-1$, such that $\{b_j(0,0,\xi_0,z): j = 1,\ldots,l\} \cup \{z^{j},\ldots z^{j_{m_+-l}}\}$ form a basis in $C[z]/(p_+(0,0,\xi_0,z))$.

By theorem 4.1, given any positive integer N, there exists $v(x,t,\rho)$ in the form of Gaussian beams such that

$$Pv = O(\rho^{-N})$$

$$B_j v|_{t=0} = O(\rho^{-N}), \quad 1 \le j \le l,$$

$$D_t^{j_r} v|_{t=0} = \zeta(x)\, \rho^{j_r}\, e^{ix\cdot\xi_0 - \frac{1}{2}|x|^2} + O(\rho^{-N}),$$

$$1 \le r \le m_+-l, \qquad (20)$$

where $\zeta \in C_c^\infty(R^n)$ and $\zeta = 1$ in a neighborhood of 0.

By Borel's theorem, we can actually construct a $w(x,t,\rho)$ such that (20) is satisfied for $N = \infty$, $w(x,t,\rho)$ is smooth in (x,ρ) and $w(x,t,\rho) = 0$ for $\rho < 1$.

Let $u = \int_0^\infty w(x,t,\rho)\, d\rho$, then u satisfies (i)-(iii).

Q.E.D.

If $l=k-k_0-1$ and $\{b_j(0,0,\xi_0,z): j = 1,\ldots,l\}$ spans a $k-k_0-1$ dimensional subspace of $C[z]/((z-\tau_{k_0+1})\cdots(z-\tau_k))$, then we can construct u satisfying (i)-(iii) with $WF(u|_{R^{n+1}})$ non-empty.

If $l = q_+-1$ and $\{b_j(0,0,\xi_o,z): j = 1,\ldots,l.\}$ spans a q_+-1 dimensional subspace of $C[z]/(q_+(0,0,\xi_o,z))$, then we can construct u satisfying (i)-(iii) with $WF(u|_{R^{n+1}}) = \emptyset$.

Section 5 Reflection of Singularities, Global Theory

P will denote a strictly hyperbolic partial differential operator in this section. Ω' is a domain in R^n. $\Omega = \Omega' \times R$ is then a domain in R^{n+1}. $\partial\Omega$ is assumed to be noncharacteristic with respect to P.

Let u be a distribution in Ω. We assume that the traces of u and its derivatives on $\partial\Omega$ and $\Omega' \times \{t\}$ $(t\in R)$ can be defined.

Suppose also that

$$Pu = f \tag{1a}$$

$$B_j u|_{\partial\Omega} = g_j,\ 1 \leq j \leq l. \tag{1b}$$

$$D_t^r u|_{t=o} = h_r,\ 0 \leq r \leq m-1. \tag{1c}$$

Given $T > 0$ and $(x_o,\xi_o) \in T^*(\text{int }\Omega')$, $\xi_o \neq 0$, let

τ_i, $i = 1,\ldots,m$ be the roots of $p_m(x_o,T,\xi_o,\tau) = 0$. Let us consider the null-bicharacteristics (rays) of P through the points (x_o,T,ξ_o,τ_i). We begin with following the rays backwards into $t < T$. If one of these rays, say Γ, hits $\partial\Omega$, we assume that the reflected family of rays containing Γ is non-grazing. We then follow the backward rays in the reflected family. A shower of rays is obtained in this manner. Here a ray at $\partial\Omega$ is forward (backward) iff $(\nu\cdot d_\xi p_m)\,\frac{\partial p_m}{\partial\tau}$ is positive (negative), where ν is the inner normal at $\partial\Omega$.

We shall put the following assumptions on this shower of rays.

(i) None of the rays in the shower graze $\partial\Omega$.

(ii) There are only finitely many rays in the shower.

(iii) Each ray will either stop at $\partial\Omega$ (when there is no backward ray in the reflected family) or reach $t = 0$ at the points $(y_i,0,\xi_i,\zeta_i)$, $i = 1,\ldots M$.

(iv) y_i, $i = 1,\ldots,M$, belongs to the interior of Ω.

(v) (P,B_j) is perfectly reflecting with respect to the backward rays at the intersections of the shower with $\partial\Omega$.

(vi) $(y_i,\xi_i) \notin WF(h_r)$, $i = 1,\ldots,M$; $r = 0,\ldots,m-1$.

(vii) f is microlocally smooth at the intersections of the shower with $T^*(\partial\Omega)$, $T^*(\Omega' \times \{T\})$ and

$T^*(\Omega' \times \{0\})$.

(viii) $WF(f)$ has empty intersection with the shower.

(ix) The intersections of the shower with $T^*(\partial\Omega)$ do not belong to $WF(g_j)$, $1 \le j \le l$.

Theorem 5.1

Under the above assumptions u will be microlocally smooth at (x_o, ξ_o, T). Consequently $(x_o, \xi_o) \notin WF(D_t^r u|_{t=T})$ for all r and $(x_o, T, \xi_o, \tau) \notin WF(u)$ for any τ.

Proof: We have a Green's identity for this situation.

$$\int_\Omega Pu \cdot \bar{v} - u \cdot \overline{P^* v} = \sum_{r=0}^{m-1} \Big(\int_{\partial\Omega} B_r' u \cdot \overline{C_r v} + \int_{\{T\}\times\Omega'} F_r u \cdot \overline{D_t^r v} + \int_{\{0\}\times\Omega'} D_t^r u \cdot \overline{E_r v} \Big), \qquad (2)$$

which holds for $v \in C^\infty(\overline{\Omega'} \times [0,T])$, v vanishes near the corners of $\partial\Omega' \times [0,T]$. At each intersection of the shower with $T^*(\partial\Omega)$, say at (y,α), there exists B'_{r_j}, $1 \le j \le k_1$, such that each B'_{r_j} is identical with one of the B_j in a conic neighborhood of (y,α). If k = total number of rays in the reflected family corresponding to (y,α), then $k_1 = \frac{1}{2}(m-k)$ + number of forward rays in the reflected family.

Let (x',ξ',T') be so close to (x_o,ξ_o,T) that the shower corresponding to (x',ξ',T') also has the properties (i)- (ix).

Given any positive integer N, we can construct a Gaussian beam solution to the following problem.

$$P^* v = O(\rho^{-N})$$

(3a)

$$v|_{t=T'} = \zeta(x)\, e^{ix\cdot\xi' - \frac{1}{2}|x-x'|^2} + O(\rho^{-N})$$

$$D_t^r v|_{t=T''} = O(\rho^{-N}),\ 1 \le m-1. \qquad (3b)$$

v has the following properties:

(a) at each intersection of the shower with $T^*(\partial\Omega)$, say at $(y,\alpha) \in T^*(\partial\Omega)$, $C_j v|_{n'd\ of\ y} = O(\rho^{-N})$ for $0 \le j \le m-1$ and $j \ne r_1,\dots,r_{k_1}$.

(b) the support of v is contained in arbitrarily small neighborhood of the projection of the shower in Ω.

The construction of such a v is possible by the theory developed in chapter one and section 4 of this chapter. Note that (P^*, C_j), $0 \le j \le m-1$ and $j \ne r_1,\dots,r_{k_1}$ is perfectly reflecting with respect to the forward rays at (y,α) (lemma 2.2). Hence we can construct a Gaussian beam solution to $P^* v = O(\rho^{-N})$ and $C_j v|_{\partial\Omega} = O(\rho^{-N})$ with support contained in arbitrarily small neighborhood of the backward rays (theorem 4.1).

If we put v in (2), we immediately obtain $|\int u(x,T')\zeta(x)\, e^{i\rho x\cdot\xi'}\, dx| = O(\rho^{-N})$, where $\zeta(x) = 1$ in a neighborhood of x_o. $\therefore (x',\xi') \notin WF(u|_{t=T'})$ for (x',ξ',T') close to (x_o,ξ_o,T).

By choosing different initial data for (3b), we can then prove that u is microlocally smooth at (x_0, ξ_0, T).

Q.E.D.

<u>Example</u>: Wave equation outside an object with Dirichlet boundary condition.

Chapter IV Diffraction of Singularities

Section 1 Grazing Gaussian Beams

Let Ω be a domain in R^{n+1} and $P(x,D)$ be a second order differential operator with principal symbol $p(x,\xi)$. $\partial\Omega$ is non-characteristic with respect to P. Assume that $d_\xi p(x,\xi) \neq 0$ for $(x,\xi) \in \text{Char } p = \{(x,\xi): p(x,\xi)=0\}$. Let b be a null-bicharacteristic of p and $c = \pi(b)$ be the projection of b on R^{n+1} such that c and $\partial\Omega$ has a second order contact at x_o.

We can choose local coordinates at x_o so that $\Omega = \{(x,y): x \in R^n, y > 0\}$, with $x_o = (0,0)$.

Let v be a Gaussian beam along c. Then

$$v_o = v|_{\partial\Omega}$$

$$= \sum_{j=0}^{-K} a_j(x)\, \rho^j\, e^{i\rho\psi_o(x)} \tag{1}$$

It is clear that

$$\text{Im } \psi_o(0) = 0 \text{ and } \text{Im } \psi_o(x) \geq \gamma|x|^4 \quad (\gamma > 0), \tag{2}$$

for x close to 0.

For any given positive integer N', we want to solve the asymptotic boundary value problem:

$$Pu = O(\rho^{-N'}) \tag{3a}$$

$$u|_{\partial\Omega} = v_o + O(\rho^{-N'}) \tag{3b}$$

Section 2 Diffracted Beams

We shall construct a local solution to (1.3) in the form of

$$u = \rho^n(2\pi)^{-n}\int[g\, A(\rho^{2/3}\zeta) + i\rho^{-1/3}h\, A'(\rho^{2/3}\zeta)]\cdot A(\rho^{2/3}\xi_n)^{-1} e^{i\rho\theta}F(\xi,\rho)d\xi, \qquad (1)$$

where $d\xi = d\xi_1\cdots d\xi_n$

$$g = \sum_{j=0}^{-N} g_j(x,y,\xi)\rho^j,$$

$$h = \sum_{j=0}^{-N} h_j(x,y,\xi)\rho^j,$$

$$\zeta = \zeta(x,y,\xi)$$

$$\theta = \theta(x,y,\xi).$$

g_j, h_j, ζ and θ are supported in a neighborhood of $(0,0,0)$ in $\bar{\Omega} \times R^n$.

A is one of the two Airy functions

$$A_\pm(s) = Ai(-e^{\pm 2\pi i/3}s) . \qquad (0.5.5)$$

By choosing the appropriate A, u can be made to decay rapidly away from either the incoming ray or the outgoing ray. u is called a diffracted beam.

Let $<\ ,\ >$ denote the symmetric bilinear form in ξ varying with x such that

$$<\xi,\xi> = p(x,\xi) \qquad (2)$$

$$P\{[g\, A(\rho^{2/3}\zeta) + i\rho^{-1/3} h\, A'(\rho^{2/3}\zeta)]\, e^{i\rho\theta}\}$$

$$= a\, A(\rho^{2/3}\zeta)e^{i\rho\theta} + b\, A'(\rho^{2/3}\zeta)e^{i\rho\theta}, \text{ where}$$

$$a = \rho^2[(<d\theta,d\theta> + \zeta<d\zeta,d\zeta>)g_0 - 2\zeta<d\zeta,d\theta>h_0] +$$
$$\rho \sum_{j=0}^{-N}(-2i<d\theta,dg_j> + 2i\zeta<d\zeta,dh_j> + i<d\zeta,d\zeta>h_j -$$
$$iR_j g_j - iS_j h_j - iT_j)$$

$$b = \rho^{5/3} i[(<d\theta,d\theta> + \zeta<d\zeta,d\zeta>)h_0 - 2\zeta<d\zeta,d\theta>g_0] +$$
$$\rho^{2/3} \sum_{j=0}^{-N}(2<d\theta,dh_j> - 2\zeta<d\zeta,dg_j> - <d\zeta,d\zeta>g_j +$$
$$R'_j g_j + S'_j h_j + T'_j),$$

where $d\theta = d_{x,y}\theta$, $d\zeta = d_{x,y}\zeta$.

R_j, R'_j, S_j, S'_j depend only on ζ, θ and symbols of P. T_j and T'_j depend only on symbols of P, ζ, θ and g_r, h_r, $j+1 \leq r \leq 0$.

If we let the highest order terms in a and b equal 0, we get the following eikonal equations:

$$<d\theta,d\theta> + \zeta<d\zeta,d\zeta> = 0 \tag{3a}$$

$$<d\zeta,d\theta> = 0 \tag{3b}$$

If we then let the lower order terms equal zero successively, we obtain the following transport equations:

$$2<d\theta,dg_j> - 2\zeta<d\zeta,dh_j> - <d\zeta,d\zeta>h_j + R_j g_j +$$
$$S_j h_j + T_j = 0, \tag{4a}$$

$$2\langle d\theta, dh_j\rangle - 2\zeta\langle d\zeta, dg_j\rangle - \langle d\zeta, d\zeta\rangle g_j + R_j' g_j + S_j' h_j + T_j' = 0. \qquad (4b)$$

Section 3 Construction of Phases and Amplitudes in the Diffracted Beams

The eikonal equations (2.3) and the transport equations (2.4) can be solved locally at (0,0,0) with the following properties being satisfied by the solutions:

For $\xi_n \geq 0$,

$$\langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle = 0$$

$$\langle d\zeta, d\theta\rangle = 0. \qquad (1)$$

For $\xi_n \leq 0$, $k = 0,1,2,\ldots$

$$\frac{\partial^k}{\partial y^k}(\langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle)\Big|_{y=0} = 0$$

$$\frac{\partial^k}{\partial y^k}(\langle d\zeta, d\theta\rangle)\Big|_{y=0} = 0 \qquad (2)$$

Similarly, (2.4) are satisfied in the sense of (1) and (2).

$$\frac{\partial \zeta}{\partial y}(0,0,0) \geq \alpha_0 > 0, \quad \zeta(x,0,\xi) \equiv \xi_n. \qquad (3)$$

The matrix $\left(\frac{\partial^2\theta}{\partial x_i \partial \xi_j}(x,0,\xi)\right)$ is non-singular. (4)

$$h_j(x,0,\xi) \equiv 0. \qquad (5)$$

$g_o(0,0,0) \neq 0.$ (6)

First of all, let us look at the following example, which is due to F. Friedlander.

$$\Omega = \{(x,y): y > 0,\ x \in R^n\}$$

$$P(x,y,D_x,D_y) = D_y^2 - \rho D_{x_n} - \rho^2 y.$$

We want to solve

$$Pu = 0$$

$$u|_{\partial\Omega} = f. \tag{7}$$

Let u be the Fourier transform of u in the x-variable. We have the following equations:

$$\frac{d^2\hat{u}}{dy^2} = -(\rho^2 y + \rho\xi_n)\,\hat{u} \tag{7a}$$

$$\hat{u}(0) = \hat{f} \tag{7b}$$

Two of the solutions for (7a) and (7b) are

$$\hat{u}(\xi,y) = \frac{A_\pm(\rho^{2/3}y + \rho^{-1/3}\xi_n)}{A_\pm(\rho^{-\ /3}\xi_n)}\ \hat{f}(\xi) \tag{8}$$

$$\therefore u(x,y) = \rho^n(2\pi)^{-n}\int \frac{A_\pm(\rho^{2/3}(y+\xi_n))}{A_\pm(\rho^{2/3}\xi_n)}\ e^{i\rho x\cdot\xi}\ \hat{f}(\rho\xi)\ d\xi$$

In this particular case, we see that ζ and θ can be taken as $y + \xi_n$ and $x\cdot\xi$ respectively. Also $g_o \equiv 1$, $g_j \equiv 0$ $(j \leq -1)$ and $h_j \equiv 0$ $(j \leq 0)$.

For the construction of ζ, θ, g_j, h_j in the general case, the idea is to reduce it to the example above by using the equivalence of glancing hypersurfaces (see [6]). We refer the readers to [7] for details.

Section 4 The Boundary Term

When $y = 0$, (2.1) becomes

$$u(x,0) = \rho^n (2\pi)^{-n} \int g \cdot e^{i\rho\theta} \cdot F(\xi,\rho)\, d\xi \tag{1}$$

By choosing appropriate local coordinates for R^n, we can assume that the inverse of the canonical transformation $(d_\xi \theta, \xi) \longrightarrow (x, d_x \theta)$ is also given by a generating function λ.

Recall that we want

$$u(x,0) = v_0 + O(\rho^{-N'}) \tag{2}$$

$$v_0 = \sum_{j=0}^{-K} a_j(x) \cdot \rho^j \cdot e^{i\rho\psi_0(x)},$$

where $\operatorname{Im} \psi_0(0) = 0$ and $\operatorname{Im} \psi_0(x) \geq \gamma |x|^4$ on the supports of a_j which are close to 0.

We shall let

$$F(\xi,\rho) = \rho^n \cdot (2\pi)^{-n} \cdot \iiint e^{i\rho(\lambda(z,\eta) - z\cdot\xi - q\cdot\eta + \psi_0(q))} \cdot \alpha(\rho,q) \cdot \beta(\rho,z,\eta)\, dq\, d\eta\, dz \tag{3}$$

where $\alpha(\rho,q) = \sum_{j=0}^{-N} a_j(q) \cdot \rho^j$, $(a_j = 0,\ -N \leq j \leq -K-1)$

$$\beta(\rho,z,\eta) = \sum_{j=0}^{-N} b_j(z,\eta)\rho^j.$$

Since $g_o(0,0,0) \neq 0$, if we substitue $F(\xi,\rho)$ in (1), then by the stationary phase method (0.2.1), we can arrange b_j so that $u(x,0) = v_o + O(\rho^{-N})$. (2) is therefore solved if $N \geq N'$.

Using integration by parts and the fact that $\operatorname{Im} \psi_o(x) \geq \gamma|x|^4$, we can also assume that α is compactly supported in an arbitrarily small neighborhood of $(0,0,d\psi_o(0))$ and β is compactly supported in an arbitrarily small neighborhood of $(0,0,0)$.

Section 5 Basic Estimates of $A(\rho^{2/3}\xi)/A(\rho^{2/3}\xi_n)$ and $A'(\rho^{2/3}\xi)/A(\rho^{2/3}\xi_n)$

Recall that

$$A_{\pm}(z) = H(-e^{\mp 2\pi i/3}z)\, e^{-2/3(-e^{\pm 2\pi i/3}z)^{3/2}},$$

where $H(z) \sim z^{-\frac{1}{4}} \sum_{j=0}^{\infty} c_j z^{-3j/2}$. (0.5.3)

Given any $K \geqslant 0$, there exists a $C > 0$ such that $z > C$ and $\arg(z) \leq \pi/3 - \delta$ ($\delta > 0$) imply that

$$H^{(k)}(z) \leq \text{constant}\cdot z^{3/4-k},\ 0 \leq k \leq K. \qquad (1)$$

We shall keep in mind that by (3.3),

$(x,y,\xi) \geq \xi_n + (\alpha_0/2)y$, for (x,y,ξ) close to $(0,0,0)$.

(2)

(2) is assumed to be true for the discussion below.

The estimates of $B = A(\rho^{2/3}\zeta)/A(\rho^{2/3}\xi_n)$ and $B' = A'(\rho^{2/3}\zeta)/A(\rho^{2/3}\xi_n)$ will be divided into three groups. In these estimates j will range from 0 to K and $\varkappa = -\exp(\pm 2\pi i/3)$.

(I) $\rho^{2/3}\xi_n \geq C$ (hence $\rho^{2/3}\zeta \geq C$)

$B = a(\rho,x,\xi)\exp(i\rho(\zeta^{3/2} - \xi_n^{3/2}))$, where

$$a = \frac{H(\varkappa\rho^{2/3}\zeta)}{H(\varkappa\rho^{2/3}\xi_n)}.$$

It follows from (1) that

$$|D_\xi^j a| \leq \text{constant}\cdot\rho^{1/6+2j/3} \tag{3a}$$

Similarly, $B' = a'(\rho,x,\xi)\exp(i\rho(\zeta^{3/2} - \xi_n^{3/2}))$ and

$$|D_\xi^j a'| \leq \text{constant}\cdot\rho^{1/3+2j/3} \tag{3b}$$

(II) $0 \leq |\rho^{2/3}\xi_n| \leq 2C$

(i) $\rho^{2/3}\zeta > C$

$B = a(x,y,\xi)\exp(i\rho\zeta^{3/2})$, where

$$a = \frac{H(\varkappa\rho^{2/3}\zeta)}{A(\rho^{2/3}\xi_n)}.$$

It follows from (1) that

$$|D_\xi^j a| \leq \text{constant}\cdot\rho^{2j/3}. \tag{4a}$$

Similarly, $B' = a' \cdot \exp(i\rho\zeta^{3/2})$ with

$$|D_\xi^j\, a'| \le \text{constant}\cdot\rho^{1/6 + 2j/3}. \qquad (4b)$$

(ii) $0 \le |\rho^{2/3}\zeta| \le 2C$

$B = \dfrac{A(\rho^{2/3}\zeta)}{A(\rho^{2/3}\xi_n)}$. It is clear that

$$|D_\xi^j B| \le \text{constant}\cdot\rho^{2j/3}. \qquad (5a)$$

Similarly, $B' = \dfrac{A'(\rho^{2/3}\zeta)}{A(\rho^{2/3}\xi_n)}$ and

$$|D_\xi^j B'| \le \text{constant}\cdot\rho^{2j/3}. \qquad (5b)$$

(III) $\rho^{2/3}\xi_n \le -C$

(i) $\rho^{2/3}\zeta \ge C$

$B = a(\rho,x,\xi)\cdot\exp(i2/3(\rho\zeta^{3/2}))$, where

$$a(\rho,x,\xi) = \frac{H(\varkappa\rho^{2/3}\zeta)}{H(\varkappa\rho^{2/3}\xi_n)}\cdot e^{-2/3(-\xi_n)^{3/2}\rho}$$

It follows from (1) that

$$|D_\xi^j a| \le \text{constant}\cdot\rho^{2j/3}. \qquad (6a)$$

Similarly, $B' = a'\cdot\exp(i2/3(\rho\,\zeta^{3/2}))$ and

$$|D_\xi^j a'| \le \text{constant}\cdot\rho^{1/6 + 2j/3}. \qquad (6b)$$

(ii) $0 \le |\rho^{2/3}\zeta| \le 2C$

$$B = \frac{A(\rho^{2/3}\zeta)}{H(\kappa\rho^{2/3}\xi_n)} \cdot e^{-2\rho/3\cdot(-\xi_n)^{3/2}}$$

It follows from (1) that

$$|D_\xi^j B| \le \text{constant}\cdot\rho^{2j/3}. \tag{7a}$$

Similarly, $B' = a'\cdot e^{-2\rho/3\,\cdot\,(-\xi_n)^{3/2}}$ and

$$|D_\xi^j a'| \le \text{constant}\cdot\rho^{2j/3}. \tag{7b}$$

(iii) $\rho^{2/3}\zeta \le -C$

$$B = \frac{H(\kappa\rho^{2/3}\ )}{H(\kappa\rho^{2/3}\xi_n)} \cdot e^{-2\rho/3} \cdot ((-\xi_n)^{3/2} - (-\zeta)^{3/2})$$

Let L be defined by the following identity.

$$-2/3 \cdot ((-\xi_n)^{3/2} - (-\zeta)^{3/2}) = (\zeta-\xi_n)\cdot L$$

By the mean value theorem, we have the following estimates.

$$|L| \ge \text{constant}\cdot\rho^{-1/3}\text{, and} \tag{8}$$

$$|D_\xi^j L| \le \text{constant}\cdot\rho^{-1/3+2j/3}. \tag{9}$$

By (3.3), $\zeta(x,0,\xi) = \xi_n$.

$$\therefore |D_\xi^j(\zeta-\xi_n)| \le \text{constant}\cdot y.$$

The estimates (1),(2),(8),(9) and (10) imply that

$$|D_\xi^j B| \le \text{constant}\cdot\rho^{1/6+2j/3}. \tag{11a}$$

Similarly, $|D_\xi^j B'| \le \text{constant}\cdot\rho^{1/3+2j/3}$. (11b)

Using the estimates (I)-(III) we can prove the following theorem.

Theorem 5.1

Let u be the diffracted beam (2.1). For $(x,y) \in \Omega$ close to (0,0) we have $Pu = O(\rho^{-N+1})$

Proof: We can choose $C > 0$ so that the estimates (3) - (7) and (11) are valid for $K = 0$. Let λ_1, λ_2 and λ_3 be positive C^∞ functions on R such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $\operatorname{supp} \lambda_1 \subset [C,\infty)$, $\operatorname{supp} \lambda_2 \subset [-2C,2C]$ and $\operatorname{supp} \lambda_3 \subset (-\infty, -C]$.

We can write $u = \int I(\rho,x,y,\xi)\, d\xi$, where I is the integrand in (2.1). u can be written as the sum of the following functions.

$$u_1 = \int \lambda_1(\rho^{2/3}\xi_n)\cdot I(\rho,x,y,\xi)\, d\xi$$

$$u_2 = \int \lambda_2(\rho^{2/3}\xi_n)\cdot\lambda_1(\rho^{2/3}\zeta(x,y,\xi))\cdot I(\rho,x,y,\xi)\, d\xi$$

$$u_3 = \int \lambda_2(\rho^{2/3}\xi_n)\cdot\lambda_2(\rho^{2/3}\zeta)\cdot I\, d\xi$$

$$u_4 = \int \lambda_3(\rho^{2/3}\xi_n)\cdot\lambda_1(\rho^{2/3}\zeta)\cdot I\, d\xi$$

$$u_5 = \int \lambda_3(\rho^{2/3}\xi_n)\cdot\lambda_2(\rho^{2/3}\zeta)\cdot I\, d\xi$$

$$u_6 = \int \lambda_3(\rho^{2/3}\xi_n)\cdot\lambda_3(\rho^{2/3}\zeta)\cdot I\, d\xi$$

Observe that the left hand sides of (2.3) and (2.4) are bounded by constant$\cdot|\xi|^m$ for any positive integer m. Since $|\xi_n| \leq 2C\rho^{-2/3}$ on the support of λ_2, it follows from (II) that $Pu_2 = O(\rho^{-N-1/6})$ and $Pu_3 = O(\rho^{-N})$. It also follows from (III,i,ii) that $Pu_4 = O(\rho^{-N})$ and $Pu_5 = O(\rho^{-N})$.

For $\xi_n \leq 0$, the left-hand sides of (2.3) and (2.4) are bounded by constant$\cdot|y|^m$ for any positive integer m (see (3.2)).

It follows from (2), (8), (11) that $Pu_6 = O(\rho^{-N+1/6})$. $Pu_1 = O(\rho^{-N+1/6})$ is obvious.

Q.E.D.

Section 6 The Width of the Diffracted Beams

Let $\psi^{\pm} = \theta(x,y,\xi) \pm 2/3\cdot\zeta(x,y,\xi)^{3/2}$ for $\xi_n \geq 0$. We can rewrite (2.3) in the following compact form.

$$p(x,y,d\psi^{\pm}_{x,y}(x,y,\xi)) = 0 \tag{1}$$

Recall that $\zeta(x,0,\xi) = \xi_n$ (2)

and $d_\xi\theta(x,0,0) = 0$. (3)

$$\therefore d\psi^{\pm}_{\xi}(0,0,0) = 0 \tag{4}$$

Let c = (b) be the projection of the grazing ray on $\Omega \subset R^{n+1}$ and $c^{\pm}$ be the half-rays.

By the construction of θ and ζ (see[7]),

$$(c^{\pm}(t),d\psi^{\pm}_{x,y}(c^{\pm}(t),0)) = b(t). \tag{5}$$

Assume that $\psi^{\pm}$ is defined on W x [0,Y] in local coordinates, where W is a neighborhood of 0 in R^n.

Proposition 6.1

$U_\delta^{\pm} = \{(x,y) : |d\psi_\xi^{\pm}(x,y,0)| < \delta\}$ is a neighborhood of $c^{\pm}$ in $W \times [0,Y]$. If $U_\delta^{\pm}$ is the component of $U_\delta^{\pm}$ that contains $c^{\pm}$, then $U_\delta^{\pm}$ converges to $c^{\pm}$ as δ goes to 0.

Proof: We shall use c (ψ) to denote one of $c^{\pm}$ ($\psi^{\pm}$) in this proof. We shall also write y as x_{n+1}.

By (1) and (5),

$$\sum_{k=1}^{n+1} \frac{\partial p}{\partial \xi_k}(b(t)) \cdot \frac{\partial}{\partial x_k} \cdot \frac{\partial \psi}{\partial \xi_j}(x,0) = 0 \text{ along } c, \qquad (6)$$

i.e. $\frac{d}{dt}\frac{\partial \psi}{\partial \xi_j}(c(t),0) = 0$ along c, for $j = 1,\ldots,n$.

Since $\frac{\partial \psi}{\partial \xi_j}(0,0) = 0$ by (4), we see that $d_\xi \psi = 0$ along c and therefore U_δ is a neighborhood of c.

If we differentiate (6) again, we see that the vectors $v_j = \left(\frac{\partial^2 \psi}{\partial x_1 \partial \xi_j}(x,0),\ldots,\frac{\partial^2 \psi}{\partial x_{n+1}\, \partial \xi_j}(x,0)\right)$,

satisfy the following system of linear equations along $c(t)$ (away from 0).

$$\frac{d}{dt} V + A \cdot V = 0, \text{ where } A(t)_{ij} = \frac{\partial^2 p}{\partial x_i \partial \xi_j}(b(t)) + \sum_{l=1}^{n+1} \frac{\partial^2 \psi}{\partial x_i \partial x_l}(c(t),0) \cdot \frac{\partial^2 p}{\partial \xi_l \partial \xi_j}(b(t)),\ 1 \le i,j \le n+1.$$

(2) and (3.4) together imply the linear independence of the v_j's at 0. Since the v_j's are continuous along c,

$v_j = d_x\psi_j$, $1 \le j \le n$, are therefore linearly independent along c. Hence U_δ converges to c as δ goes to 0.

Q.E.D.

Recall (cf.(4.3) that

$$F(\xi,\eta) = \iiint E(\rho,q,\eta,z)\cdot e^{i\rho\mu(q,\eta,z,\xi)}\,dq\,d\eta\,dz, \qquad (7)$$

where $\mu(q,\eta,z,\xi) = \chi(z,\eta) - z\cdot\xi - q\cdot\eta + \psi_0(q) \qquad (8)$

and $E(\rho,q,\eta,z)$ is a classical amplitude.

Lemma 6.1

Given any $\varepsilon > 0$, $\displaystyle\iiint_{|q| > \rho^{-1/4+\varepsilon}} E\cdot e^{i\rho\mu}\,dq\,d\eta\,dz = O(\rho^{-N})$, for

$N = 0,1,2,\ldots$

Proof: Follows from the fact that $\mathrm{Im}\,\psi_0(q) \ge \gamma|q|^4$.

Q.E.D.

Lemma 6.2

Given any $\varepsilon > 0$, $\displaystyle\iiint_{|\eta-d_q\psi_0(0)| > \rho^{-1/3+\varepsilon}} E\cdot e^{i\rho\mu}\,dq\,d\eta\,dz = O(\rho^{-N})$,

for $N = 0,1,2,\ldots$

Proof: Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \varepsilon$. For $|q|<\rho^{-1/3+\varepsilon_1}$, $|\eta-d_q\psi_0(0)| > \rho^{-1/3+\varepsilon}$ implies that $|\eta-d_q\psi_0(q)| >$ constant$\cdot\rho^{-1/3+\varepsilon}$ for ρ large.

For $|q| > \rho^{-1/3+\varepsilon_1}$, since

$|\mathrm{Im}\ d_q\psi_o(q)| \geq \text{constant}\cdot|q|^3$, we have

$|\eta - d_q\psi_o(q)| \geq |\mathrm{Im}\ d_q\psi_o(q)| \geq \text{constant}\cdot\rho^{1-3\varepsilon_1}$.

$\therefore$ for ρ large, $|\eta - d_q\psi_o(q)| \geq \text{constant}\cdot\rho^{-\delta}$ for some δ positive. Integration by parts with respect to q will finish the proof.

Q.E.D.

Lemma 6.3

Given any $\varepsilon > 0$, $\displaystyle\iiint_{|z| > \rho^{1/4+\varepsilon}} E\cdot e^{i\rho\mu}\, dq\,d\eta\,dz = O(\rho^{-N})$,

for N = 0,1,2,...

Proof: We can pick positive numbers ε_1 and ε_2 so small that $|d_\eta\chi(z,\eta) - q| \geq \text{constant}\cdot\rho^{1/4 - \varepsilon/2}$ provided that

$|\eta - d_q\psi_o(0)| \leq \rho^{-1/4+\varepsilon_1}$ and $|q| \leq \rho^{-1/4+\varepsilon_2}$. This is possible because $d_\eta\chi(0, d_q\psi_o(0)) = 0$.

The lemma then follows from lemmas 6.1,6.2 and an integration by parts with respect to η.

Q.E.D.

Lemma 6.4

Given any $\varepsilon > 0$, $\displaystyle\iiint_{|\xi| > \rho^{-1/4+\varepsilon}} E\cdot e^{i\rho\mu}\, dq\,d\eta\,dz = O(\rho^{-N})$,

for N = 0,1,2,...

Proof: We can pick positive ε_1 and ε_2 so small that

$|z| \leq \rho^{-1/4+\varepsilon_1}$ and $|\eta - d_q\psi_o(0)| \leq \rho^{-1/4+\varepsilon_2}$ will imply $|\xi - d_z\chi(z,\eta)| \geq \text{constant}\cdot\rho^{-1/4 + \varepsilon/2}$. This is possible because $d_z\chi(0, d_q\psi_o(0)) = 0$.

The lemma then follows from lemmas 6.1–6.3 and an integration by parts with respect to the z variable.

Q.E.D.

<u>Theorem 6.1</u>

Given any $\varepsilon > 0$, let $V_\rho = \{(x,y) : |d_\xi\psi(x,y,0)| > \rho^{-1/8 + \varepsilon}\}$. Then $u = O(\rho^{-N})$, for $N = 0,1,2,\ldots$, on V_ρ. Here ψ is one of $\psi^\pm$, depending on the choice of the Airy functions $A_\pm$.

<u>Proof</u>: We can write

$$u = \iiiint L(\rho,x,y,q,\eta,z)\cdot e^{i\rho[\theta(x,y,\xi)+\mu(q,\eta,z)]}\,dq\,d\eta\,dz\,d\xi,$$

where L involves some classical amplitudes and the Airy functions.

We can pick ε_1 and ε_2 so small that $(x,y) \in V_\rho$ implies $\min\{\,|d_\xi\psi^\pm(x,y,\xi)-z|\,,\,|d_\xi\psi^\pm(x,y,\xi)\mp\xi_n^{\frac{1}{2}}-z|\,,(\xi_n\geq 0)\; |d_\xi\theta(x,y,\xi)-z|\,\} \geq \text{constant}\cdot\rho^{-1/8 + \varepsilon/2}$, provided that $|\xi| \leq \rho^{-1/4 + \varepsilon_1}$ and $|z| \leq \rho^{-1/4 + \varepsilon_2}$.

The theorem follows then from lemmas (6.3), (6.4) and an integration by parts with respect to ξ. Note that in the basic estimates of section 5, differentiations in the ξ variable only raise the order of ρ by $\frac{2}{3}$. Q.E.D.

Section 7 Diffraction of Singularities

Let $P(x,t,D_x,D_t)$ be a second order strictly hyperbolic partial differential operator with real principal symbol p_2. Ω' is a domain in R^n. $\Omega = \Omega' \times R$ is a domain in R^{n+1}. $\partial\Omega$ is assumed to be non-characteristic with respect to P. Let u and f be distributions in Ω such that

$$Pu = f. \tag{1}$$

We also assume that the traces of u and its derivatives are well-defined on $\partial\Omega$ and hypersurfaces $\Omega' \times \{t\}$. The following conditions are therefore meaningful.

$$u|_{\partial\Omega} = b \tag{2}$$

$$u|_{t=o} = g_o \tag{3}$$

$$D_t u|_{t=o} = g_1 \tag{4}$$

Here b and $g_i(i=1,2)$ are distributions in $\partial\Omega$ and Ω' respectively.

Let $(x_o,\xi_o) \in \text{Int}\,\Omega'$ and $T > 0$. Just like in (3.5), we shall consider the backward shower corresponding to (x_o,ξ_o,T). We impose the following assumptions on the shower.

(i) The projection of the shower on Ω can have at most second order contact with $\partial\Omega$.

(ii) There are only finitely many rays in the shower and they reach the hypersurface $t=0$ at the points $(y_i, 0, \xi_i, \tau_i)$, $i = 1,\ldots,m$.

(iii) y_i, $i=1,2,\ldots,m$, belongs to int Ω' and $(y_i,\xi_i) \notin WF(g_0) \cup WF(g_1)$.

(iv) f is microlocally smooth at the intersections of the shower with $T^*(\partial\Omega)$, $T^*(\Omega' \times \{T\})$ and $T^*(\Omega' \times \{0\})$.

(v) WF(f) has empty intersection with the shower.

(vi) The intersections of the shower with $T^*(\partial\Omega)$ do not belong to WF(b).

Theorem 7.1

Under the above assumptions, u is microlocally smooth at (x_0,ξ_0,T). Consequently $(x_0,\xi_0) \notin WF(D_t^r u_{|t=T})$ for any r and $(x_0,\xi_0,T,\tau) \notin WF(u)$ for any τ.

Proof: Just like in the case of reflection of singularities, it suffices to show that we can construct Gaussian beam solutions to

$$P^* v = O(\rho^{-N}) \tag{5}$$

and $$v_{|\partial\Omega} = O(\rho^{-N}), \tag{6}$$

with support contained in an arbitrarily small neighborhood of the projection of the chower on Ω.

The only remaining difficulty is that the diffracted beams are only defined locally.

Observe that away from the boundary of Ω, using the asymptotic expansion (0.5.3) of the Airy functions, we can rewrite the diffracted beam in the following form.

$$w = \rho^n (2\pi)^{-n} \int G(\rho,x,t,\xi)\cdot e^{i\rho\psi(x,y,\xi)} A(\rho^{2/3}\xi_n)^{-1}\cdot F(\xi,\rho)\, d\xi, \qquad (7)$$

where $G(\rho,x,y,\xi)$ is a classical amplitude and ψ is one of $\theta \pm 2/3\cdot\xi^{3/2}$. By a prorper choice of $A_\pm$, w will be supported in an arbitrarily small neighborhood of the projection of the backward ray on Ω, which will be denoted by c.

Let $\Omega' x \{t_o\}$ be a hypersurface that intersects c at a point (x',t_o) close to the boundary of Ω. For ξ close to 0 and x" close to x', we can construct the Gaussian beam solution to

$$P^* v_1 = O(\rho^{-N}) \qquad (8)$$

and $$v_1|_{t=t_o} = \rho^{n/2}(2\pi)^{-n/2}\lambda(x)e^{i\rho\psi(x,t_o,\xi)-\frac{1}{2}\rho|x-x''|^2}\cdot G(\rho,x,t_o,\xi) + O(\rho^{-N}) \qquad (9)$$

Here $\lambda \in C_0^\infty(\Omega')$, $\lambda(x) = 1$ in a neighborhood of x'. The solution $v_1(\rho,x,t,\xi,x'')$ will have support in an arbitrarily small neighborhood of c.

Let $w' = \iint v_1(\rho,x,t,\xi,x'')\cdot A(\rho^{2/3}\xi_n)^{-1}\cdot F(\xi,\rho)\, dx''d\xi$. Then $P^* w' = O(\rho^{-N})$. By the well-posedness of the Cauchy problem for hyperbolic equations, $D_t^r w$ and $D_t^r w'$ $(r=0,1)$ will match up to $O(\rho^{-N'})$ on $\Omega' \times \{t_o\}$. N' goes to infinity as N goes to infinity.

Thus we can extend w along c with support contained in an arbitrarily small neighborhood of c.

We can now finish the proof via a Green's formula. The argument is exactly the same as that in the proof of theorem 3.5.1.

Q.E.D.

Example: The wave equation outside a compact convex object with Dirichlet boundary condition.

References

[1] Hörmander, L., "Fourier integral operators." Acta Math. 127, 1971, pp. 79-183.

[2] Hörmander, L., "Spectral analysis of singularities." In _Seminar on Singularities_, pp. 3-49. Ann. of Math. Studies 91, Princeton Univ. Press, Princeton, 1979.

[3] Majda, A. and Osher S., "Reflection of Singularities at the boundary." Comm. Pure Appl. Math. 28, 1975, pp.479-499.

[4] Malgrange, B., _Ideals of Differentiable Functions_. Oxford Univ. Press, Oxford, 1966.

[5] Melin, A. and Sjöstrand, J., "Fourier integral operators with complex valued phase functions." In _Fourier Integral Operators and Partial Differential Equations_, pp. 120-223. Lecture Notes in Math., 459, Springer, New York, 1975.

[6] Melrose, R., "Equivalence of glancing hypersurfaces." Invent. Math. 37, 1976, pp. 165-191.

[7] Melrose, R. and Taylor, M., "Boundary value problems od grazing and gliding rays." Unpublished manuscript.

[8] Nirenberg, L., _Lectures on Linear Partial Differential Equations_. Reg. Conf. Ser. in Math., no. 17. A.M.S., Providence, 1972.

[9] Olver, F., _Asymptotics and Special Functions_. Academic Press, New York, 1974.

[10] Ralston, J., "Gaussian beams and the propagation of singularities." Unpublished manuscript.

[11] Reid, W., "A matrix differential equation of Riccati type." Amer. J. Math. 68, 1946, pp. 237-246.

[12] Taylor, M., _Pseudodifferential Operators_. Princeton Univ. Press, Princeton, 1981.