VOLUME PRESERVING FOLIATIONS AND DIFFEOMORPHISM GROUPS

A Dissertation presented

by

Ira Stuart Moskowitz

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

State University of New York

at

Stony Brook

August, 1983

STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

THE GRADUATE SCHOOL

Ira Stuart Moskowitz

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Anthony Phillips, Professor of Mathematics
Chairman of Defense

Chih-Han Sah, Professor of Mathematics

Lowell Jones, Associate Professor of Mathematics

Joseph Hogan, Associate Professor,
Mechanical Engineering
Outside member

This dissertation is accepted by the Graduate School.

Dean of the Graduate School

August 1983

Abstract of the Dissertation

VOLUME PRESERVING FOLIATIONS AND

DIFFEOMORPHISM GROUPS

by

Ira Stuart Moskowitz

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1983

This thesis is an investigation of the local cohomology of the group G of volume preserving diffeomorphisms of a closed n-manifold M with volume form ω (results independent of which ω chosen). This is the real cohomology of $\overline{B}G$, the homotopy fibre of $BG^{\delta} \to BG$, where G^{δ} is the group G with its underlying discrete topology. This cohomology is closely related to the Lie algebra cohomology of divergence free vector fields on M. The space $\overline{B}G \times M$ has a canonical foliation F with transverse volume form Ω. We use Ω to construct a vector space homomorphism $\psi: H^*(M) \to H^*(\overline{B}G)$. Our main concern is to investigate the kernel of ψ, and in

particular to find necessary and sufficient conditions for $\psi([\omega])$ in $H^n(\overline{B}G;R)$ to be non-zero.

The map ψ is related to the classes $c_k(M)$ in $H^k(\overline{B}G;H^{n-k}(M;R))$, $1 \leq k \leq n$, defined by McDuff. For example $\psi([\omega]) = c_n(M)$. These classes measure how much the leaves of F differ from being $\overline{B}G \times$ pt. McDuff has shown that the top class vanishes for even spheres and is non-zero for odd spheres and closed Lie groups. She has some other examples and also looks at $i^*c_k(M)$, where i is the canonical map from G to $\overline{B}G$.

In this thesis we develop a product rule for the classes in the image of ψ and investigate the relation between ψ on $H^*(E)$ and on $H^*(B)$ where $E \to B$ is a fibration. Gottlieb's work on the transfer map is used to show that the vanishing of $i^*c_k(M)$ is dependent upon the non-vanishing of the Euler characteristic or of a Pontrjagin number of M.

The main result is that $\psi([\omega])$ is non-zero for closed parallelizable manifolds and non-zero for closed odd-dimensional stably-parallelizable manifolds.

Several different techniques are used in this thesis. If K is a cycle in $\overline{B}G$ we construct certain foliations on $K \times M$ to get some geometric results. We use the fact $H^*(\overline{B}G)$ is isomorphic to $H^*(S_M)$, where

S_M is the space of liftings of the classifying map for TM in BSL(n,R) to $B\Gamma^n_{s\ell}$(the Haefliger classifying space for volume preserving foliations), to get many interesting results on the map ψ.

To the memory of my grandparents

Al Moskowitz

Celia Moskowitz

Isidore Breitman

and to the honor of my grandmother

Ethel Schneider

Table of Contents

Acknowledgments

It is a pleasure to thank my advisor Professor Dusa McDuff for suggesting this problem to me, showing me how to do mathematics, and most of all for generously giving of her time. I would also like to thank Professors Lowell Jones, Anthony Phillips, and Chih-Han Sah along with my fellow students Zhi-Yong Gao and Todd Wittbold for many helpful conversations.

Without the support of my mother, father, and sister I would not have finished. This work is as much theirs as it is mine. With fondness I thank Yoon-Shin Kang for her faith, hope, and patience that have never failed.

I am grateful to my many friends among the students, faculty, staff, and community of Stony Brook. Special thanks go to Aparna Dar, Ruth Diaz, William Fox, John Gill, Jane Hawkins, Scott Hensley, Lucille Meci, Gadadhar Misra, Jeff Posses, Dennis Reissig, William E. Rosenthal, Freddie Santiago, and Nicholas Seifert.

Finally, without faith I would not have gotten through the difficult times of graduate school.

Conventions

(1) All manifolds are C^∞, closed (compact without boundary), oriented and n-dimensional (unless noted otherwise).

(2) All H^* and H_* are with $\mathbb{R}$-coefficients (unless noted otherwise).

0. INTRODUCTION

In this thesis we will investigate the local cohomology of $\mathit{Diff}_{\omega}M$, the group of volume preserving diffeomorphisms of the closed n-manifold M (volume form ω) with the C^{∞} topology. Very little is known about these groups. McDuff [M-2] and Hurder [Hu] have some results on $\mathbb{R}^n$, S^n, and Lie groups.

The local cohomology of an infinite dimensional Lie group can be defined through a construction of Haefliger's [H-2] involving the associated Lie algebra. In the case of $\mathit{Diff}\,M$, the space of diffeomorphisms of M, the Lie algebra is $\mathfrak{g}_M$. The algebra $\mathfrak{g}_M$ is the set of vector fields on M with the usual Lie bracket. For $\mathit{Diff}_{\omega}M$ the Lie algebra is $\mathfrak{g}_{M,\omega}$, the subalgebra of $\mathfrak{g}_M$ of divergence free (with respect to ω) vector fields [H-3].

Given a subalgebra $\mathfrak{g}$ of $\mathfrak{g}_M$ Haefliger [H-2] defines a space $B\mathfrak{g}$ which is the classifying space of $\mathfrak{g}$-foliations on products $X \times M$, that are transverse to the slices $\{x\} \times M$. The <u>local cohomology</u> of $\mathit{Diff}_{\omega}M$ is defined to be the cohomology (with real coefficients) of the space $B\mathfrak{g}_{M,\omega}$. The homotopy type of $B\mathfrak{g}_{M,\omega}$ depends only on the algebraic and topological properties of $\mathit{Diff}_\omega M$ in a neighborhood of the identity [Ma-1]. This explains the word "local".

Haefliger [H-2,3] has shown various relations between the real cohomology $H^*(B\mathfrak{g})$, and the Gelfand-Fuks cohomology $H^*(\mathfrak{g})$. Associated to $B\mathfrak{g}$ we have its PL-DeRham complex $A^*(B\mathfrak{g})$, and the differential subalgebra $A_d{}^*(B\mathfrak{g})$ which is made up of forms, that when restricted to a simplex, vary smoothy, as we smoothly vary the simplex. The cohomology of $A^*(B\mathfrak{g})$ is the standard (real) cohomology of $B\mathfrak{g}$, and the cohomology of $A_d{}^*(B\mathfrak{g})$, denoted as $H_d^*(B\mathfrak{g})$ is referred to as the <u>differentiable</u> <u>cohomology</u> of $B\mathfrak{g}$. Haefliger defined a universal characteristic morphism χ from $C^*(\mathfrak{g})$, the Gelfand-Fuks cochain algebra of $\mathfrak{g}$, to $A^*(B\mathfrak{g})$. In fact, χ maps injectively into the subalgebra $A_d{}^*(B\mathfrak{g})$.

$$\begin{array}{ccc} C^*(\mathfrak{g}) & \xrightarrow{\quad\chi\quad} & A^*(B\mathfrak{g}) \\ & \searrow \quad\quad \nearrow & \\ & A_d{}^*(B\mathfrak{g}) & \end{array}$$

When $\mathfrak{g}$ is the Lie algebra of a finite-dimensional Lie group the Van Est theorem [H-2] tells us that $H^*(\mathfrak{g})$ and $H_d^*(B\mathfrak{g})$ are isomorphic via the map induced by χ. One may also obtain similar results for infinite-dimensional Lie groups such as $\mathcal{Diff}_\omega M$ [H-3].

In this thesis we will show that certain natural classes in the local cohomology of $\mathcal{Diff}_\omega M$ are non-zero for a fairly large class of manifolds. McDuff [M-2] has shown how these classes are related to the map χ. These classes, labeled $c_k(M)$, $1 \leq k \leq n$, were first defined by McDuff [M-2]. They are elements of $H^k(B\mathfrak{g}_{M,\omega};H^{n-k}(M))$. Of particular interest is the top class $c_n(M)$ in $H^n(B\mathfrak{g}_{M,\omega})$.

The space $B\mathfrak{g}_{M,\omega}$ may be defined equivalently [T] as $\overline{B}\mathcal{Diff}_\omega M$ (see §2). From now on we will write $\overline{B}\mathcal{Diff}_\omega M$ to put the importance on the group instead of the Lie algebra.

The classes $c_k(M)$ are defined via a canonical codimension-n foliation $\mathcal{F}$ on $\overline{B}\mathcal{Diff}_\omega M \times M$. Because the elements of $\mathcal{Diff}_\omega M$ preserve the volume form ω our special foliation $\mathcal{F}$ is volume preserving and hence has a transverse volume form Ω (see §1). This is a closed n-form whose cohomology class $[\Omega]$ is in $H^n(\overline{B}\mathcal{Diff}_\omega M \times M)$. McDuff's classes arise by looking at the components of $[\Omega]$ in $H^k(\overline{B}\mathcal{Diff}_\omega M)$ when $H^n(\overline{B}\mathcal{Diff}_\omega M \times M)$ is decomposed via the Künneth formula. The classes $c_k(M)$ measure how non-trivial $\mathcal{F}$ is, that is, how much the leaves of $\mathcal{F}$ differ from being $\overline{B}\mathcal{Diff}_\omega M \times \{m\}$.

If $\mathcal{F}$ were the pull-back of the point foliation on M by the projection map π, Ω would be $\pi^*\omega$ which has no components in $H^k(\overline{B}Diff_\omega M)$.

McDuff [M-2] has shown that for M a Lie group the top class is non-zero. She has also shown that for a sphere S^n the top class is non-zero if and only if n is odd. McDuff's main conjecture is

(0-1) Conjecture. If M is odd-dimensional then $c_k(M)$ is non-zero, $1 \leq k \leq n$, (when $H^{n-k}(M)$ is non-zero).

In this thesis we show the following

Theorem (4.16). If M is parallelizable then the top class is non-zero.

Theorem (4.8). If M is odd-dimensional and stably parallelizable then the top class is non-zero.

These results are obtained by using a fundamental diagram (4.1) first suggested by Thurston [T]. We use the classifying properties of $\overline{B}Diff_\omega M$ to show

<u>Theorem</u> (6.2) If $c_i(M)$ and $c_j(N)$ are non-zero then $c_{i+j}(M\times N)$ is non-zero.

There is a canonical inclusion i (§2) from $\mathcal{D}iff_\omega M$ to $\overline{B}\mathcal{D}iff_\omega M$. This enables one to study $i^*c_k(M) \in H^k(\mathcal{D}iff_\omega M; H^{n-k}(M))$. McDuff has shown that

(0.2) <u>Theorem</u> [M-2]. If the Euler characteristic $\chi(M)$ of M is non-zero then $i^*c_n(M)$ is zero.

We extend this result by showing

<u>Theorem</u> (5.3).

1) If $\chi(M)$ is non-zero then $i^*c_k(M)$ is zero for all k.

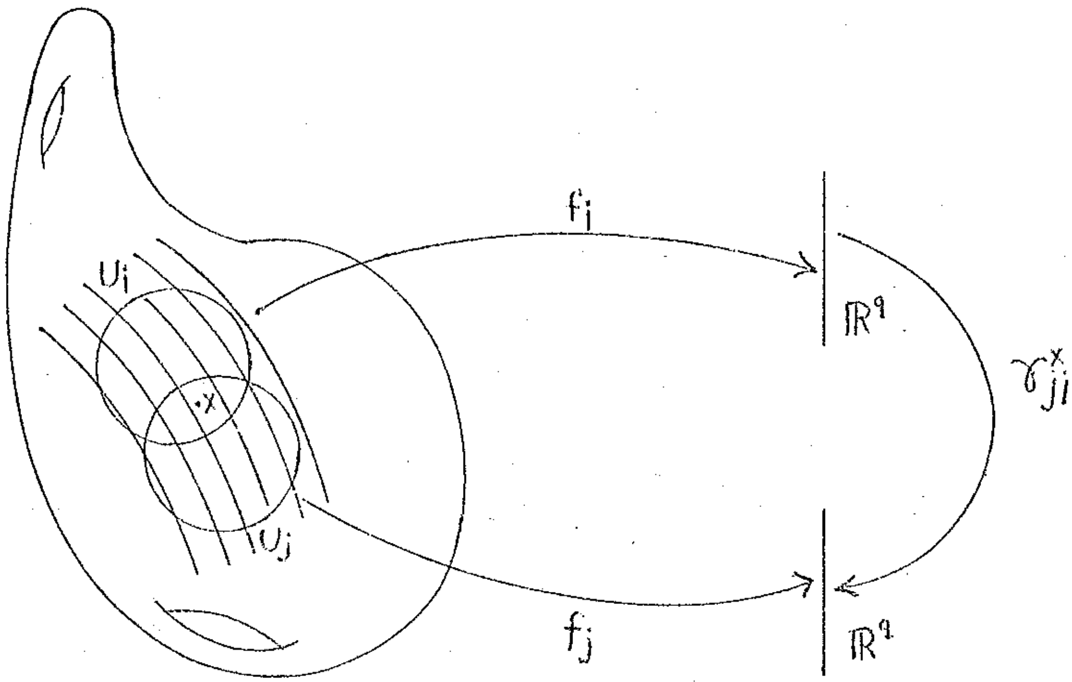
2) If M is 4ℓ-dimensional and has a non-zero Pontrjagin number, then $i^*c_k(M)$ is zero for all k.

In §6 we discuss various results on homogeneous spaces and fibrations.

1. FOLIATIONS

A foliation F on M of codimension-q is a way of slicing up M into (n-q)-dimensional submanifolds called leaves that fit together in a coherent manner. The coherency comes from asking that our manifold instead of just being locally modelled on $\mathbb{R}^n$, is in fact modelled locally by $\mathbb{R}^{n-q} \times \mathbb{R}^q$, where we think of this product as being decomposed into the leaves $\mathbb{R}^{n-q} \times \{x\}$, $x \in \mathbb{R}^q$. We require that the change of local coordinates preserves this product structure, in the sense that it takes each leaf $\mathbb{R}^{n-q} \times \{x\}$ to another such leaf $\mathbb{R}^{n-q} \times \{y\}$. At least locally our leaves are the inverse images of $\mathbb{R}^{n-q} \times \{pt.\}$ via the local coordinates. If we project from $\mathbb{R}^{n-q} \times \mathbb{R}^q$ to $\mathbb{R}^q$ we see that the local transition functions give us a diffeomorphism from $\mathbb{R}^q$ to $\mathbb{R}^q$.

(1.1) <u>Definition</u>. A <u>foliation</u> of codimension-q on M is a maximal family of submersions $f_i\colon U_i \to \mathbb{R}^q$, where the $\{U_i\}$ form an open cover of M, such that if x is in the non-empty intersection of U_i and U_j then there exists a local diffeomorphism γ_{ji}^x of $\mathbb{R}^q$ such that $f_j(v) = \gamma_{ji}^x \circ f_i(v)$ for all v in some neighborhood of x.



We can recover the leaves from this definition by taking the components of $f_i^{-1}(r)$, $r \in \mathbb{R}^q$, and matching them up as we switch neighborhoods.

A foliation with additional structure can be defined by asking that the local diffeomorphisms preserve a structure on $\mathbb{R}^q$. For example if they preserve the standard metric, $ds^2 = \sum_i dx_i^2$, we say that the foliation is <u>Riemannian</u>. If the local diffeomorphisms preserve the standard volume form, $dx_1, \wedge \cdots \wedge dx_q$, then our foliation is <u>volume preserving</u>. If our foliation is volume preserving, by locally pulling back $dx_1 \wedge \cdots \wedge dx_q$ via f_i^*, we get a form that patches together since the ϕ^{x}_{ji} preserve $dx_1\wedge\cdots\wedge dx_q$. This gives us a global closed q-form called the transverse volume form for the foliation. In fact, if λ is a non-vanishing closed q-form which is locally decomposable (i.e. $\lambda = df_1\wedge\cdots\wedge df_q$ for some functions f_i) then λ determines a unique foliation of which it is the transverse volume form. In fact, the vectors that contract λ to zero form an involutive distribution which we integrate by Frobenius' theorem to get our foliation [Li].

Certainly not every foliation is volume preserving. A necessary condition for a foliation to be volume preserving is for the Godbillon-Vey class to vanish. This follows because the Godbillon-Vey class of a codimension-q foliation (defined by the form α), is represented by $\eta\wedge(d\eta)^q$ where $d\alpha = \eta\wedge\alpha$. Since our foliation is volume preserving $d\alpha$ is 0 and η can be taken to be the zero 1-form. It is worth noting that every Riemannian foliation is also volume preserving but the converse is false.

(1.2) Example.

1) If $f: M \to N$ is a submersion and N has volume form ω, then the pull-back of the point foliation on N to M is a volume preserving foliation with transverse volume form $f^*\omega$.

2) Lazarov and Pasternack [LP] have examples of Riemannian foliations whose leaves are the orbits of a Lie group acting by isometries on a Riemannian manifold.

3) Let $f \in \mathit{Diff}_\omega M$. Then $M \times I/(m,0) \sim (f(m),1)$ has a codimension-n volume preserving foliation whose leaves are from $\{m\} \times I$. If f preserves no Riemannian metric then our foliation is volume preserving but not Riemannian.

If we take the vectors of M that are tangent to the leaves of F we get an integrable subbundle T(F) of TM. The quotient TM/T(F) gives us the <u>normal bundle</u> $\nu(F)$ of F. Note that $\nu(F)$ is a q-bundle (q is the codimension of F). The normal bundle can also be obtained by patching together the $f_i^*(T\mathbb{R}^q)$ by using the differentials of the γ_{ji}^x [S]. Because $\nu(F)$ is defined so nicely in terms of the local data giving F, it is $\nu(F)$, not T(F), that is of interest in the classification problem of foliations.

One would like a means of classifying foliations of a given codimension. If we try to set up a classifying space situation we run into trouble because the pull-back of a foliation need not be a foliation. Let us try to be more exact. If $f\colon M \to N$ is a smooth map and N has a foliation F, of codimension-q, do the inverse images of the leaves of F by f fit together to form a foliation? If f is transverse to the leaves of F then it is easy to see that this is so. Let us denote the pull-back foliation by $f^*(F)$. Clearly $f^*(F)$ can be defined as in (1.1) by using $\{f^{-1}(U_i)\}$ for our open sets and $f_i \circ f$ for our local submersions where $\{U_i, f_i\}$ define F. The fact that f is transverse to F is what guarantees us that the $f_i \circ f$ are actually submersions.

The transversality condition is too rigid to carry over to a classifying space situation, since here things should depend only on the homotopy class of a map. Taking this into account Haefliger [H-1] weakened the definition of foliation into one that could fit a classifying space set-up. It is a remarkable fact that Gromov, Haefliger, Phillips, and Thurston were able to use this weakened definition to in fact classify foliations.

(1.3) Definition. Let X be a space and Γ^q the groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$ with the sheaf topology. (A basis for this topology consists of the sets $\{g_x : x \in \text{domain } g\}$, where g is a local diffeomorphism of $\mathbb{R}^q$ and g_x is the germ of g at x). A codimension-q

Haefliger structure H on X is a maximal covering of X by open sets $\{U_i\}$ such that for each i,j where $U_i \cap U_j$ is non-empty there is a continuous map $\phi_{ij}: U_i \cap U_j \to \Gamma^q$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$.

Note that we get a map $\phi_{ii}: U_i \cap U_i = U_i \to \Gamma^q$. From this we can define a continuous map $f_i: U_i \to \mathbb{R}^q$ by setting $f_i(u)$ equal to the source of the germ $\phi_{ii}(u)$. Note that $\phi_{ji} \circ f_i$ equals f_j on $U_i \cap U_j$. It follows that if X is a manifold and the f_i are submersions then our Haefliger structure is actually a foliation. If we replace Γ^q by $\Gamma^q_{s\ell}$, the groupoid of germs that preserve $dx_1 \wedge \cdots \wedge dx_q$, then we say that our Haefliger structure is volume preserving.

Given a codimension-q Haefliger structure H we can associate an $\mathbb{R}^q$-vector bundle $\nu(H)$ to H. This is called the normal bundle of H. We form this bundle by using for our transition functions $d\phi_{ij}: U_i \cap U_j \to GL(q)$. If our Haefliger structure H comes from a foliation F then it is easy to see that $\nu(H) \cong \nu(F)$.

One can form the classifying space $B\Gamma^q$ of the groupoid Γ^q much as if Γ^q were a group [H-1]. It classifies Haefliger structures in the same way that BGL(q) classifies vector bundles.

(1.4) Theorem (Haefliger [H-1]). The set of homotopy classes of maps from X to $B\Gamma^q$, $[X,B\Gamma^q]$, is in bijective correspondence with the set of homotopy classes of codimension-q Haefliger structures on X.

The correspondence comes about from the fact that if $f:X \to Y$ and Y has a Haefliger structure H given by $\{U_i,\phi_{ij}\}$, we can give X a Haefliger structure f^*H by using $\{f^{-1}(U_i),\phi_{ij}\circ f\}$. The space $B\Gamma^q$ comes equipped with a universal codimension-q Haefliger structure which we pull-back via the maps in our homotopy classes. If F is a codimension-q foliation on X with corresponding Haefliger structure H we will call the map $f: X \to B\Gamma^q$, which classifies H, the classifying map for F.

We can also form a space $B\Gamma^q_{s\ell}$ which classifies volume preserving Haefliger structures. The groupoid $\Gamma^q_{s\ell}$ is made up of germs of volume preserving diffeomorphisms of $\mathbb{R}^q$ topologized similarly to Γ^q. $B\Gamma^q_{s\ell}$ has something very special that $B\Gamma^q$ does not have. There is a cohomology class $\tilde{\mu}$ in $H^q(B\Gamma^q_{s\ell})$ called the universal transverse volume class. The class $\tilde{\mu}$ can be constructed directly from the germs $\Gamma^q_{s\ell}$, or it can be derived by a functorial principle (see [Bt], Thm. 10.16). The class $\tilde{\mu}$ has the property that if $f: X \to B\Gamma^q_{s\ell}$ classifies a codimension-q volume preserving Haefliger structure H, then we have a class $f^*\tilde{\mu}$ in $H^q(X)$ called the transverse volume class for H. If f is the classifying map for a foliation with transverse volume form λ then

$$(1.5) \qquad [\lambda] = f^*\tilde{\mu}$$

Since $B\Gamma^q$ has a normal bundle associated to its universal Haefliger structure; the bundle can be classified by a map d: $B\Gamma^q \to BGL(q)$. This map d is induced by the groupoid homomorphism from $\Gamma^q \to GL(q)$ given by

$$g_x \mapsto dg_x$$

where g_x is a germ of a diffeomorphism at x with differential dg_x at x. The map d has a homotopy theoretic fibre $\overline{B}\Gamma^q$. We will choose models so that we get a Hurewicz fibration

$$(1.6) \qquad \overline{B}\Gamma^q \to B\Gamma^q \to BGL(q).$$

$\overline{B}\Gamma^q$ is in its own right a classifying space. It classifies codimension-q Haefliger structures whose normal bundle is framed. Recall that framed means that a specific trivialization of the bundle is given, i.e. we have chosen a specific bundle isomorphism between $\nu(H)$

and ε^q. For the volume preserving case we get a map $d\colon B\Gamma^q_{s\ell} \to BSL(q)$, as above, whose fibre is $\overline{B}\Gamma^q_{s\ell}$.

(1.7) $$\overline{B}\Gamma^q_{s\ell} \xrightarrow{i} B\Gamma^q_{s\ell} \xrightarrow{d} BSL(q)$$

$\overline{B}\Gamma^q_{s\ell}$ classifies volume preserving codimension-q Haefliger structures whose normal bundle is framed.

Very little is known about the homotopy groups of the fibres $\overline{B}\Gamma^q$ and $\overline{B}\Gamma^q_{s\ell}$. However they are quite highly connected, a fact which will be important to us later. To be precise the results are

(1.8) <u>Theorem</u> (Haefliger, Thurston). $\overline{B}\Gamma^q$ is (q+1)-connected.

(1.9) <u>Theorem</u> (McDuff [M-1]). $\overline{B}\Gamma^q_{s\ell}$ is (q-1)-connected and $\pi_q(\overline{B}\Gamma^q_{s\ell}) \cong \mathbb{R}$.

In fact it is easy to see that $\pi_q(\overline{B}\Gamma^q_{\ell}) \neq 0$. For let μ in $H^q(\overline{B}\Gamma^q_{s\ell})$ be the pull-back of the universal transverse volume class $\tilde{\mu}$

(1.10) $$i^*\tilde{\mu} = \mu$$

in $H^n(B\Gamma^q_{s\ell})$. Then if Q is a parallelizable q-manifold with volume form ω, the point foliation F of Q has transverse volume form ω. Choose a framing of $\nu(F) \cong TQ$

and let $f\colon Q \to \overline{B}\Gamma^q_{s\ell}$ be the corresponding classifying map. Then by (1.5) $f^*\tilde{\mu} = [\omega] \in H^q(Q)$. Since $[\omega] \neq 0$, the class μ is non-zero. In fact, McDuff showed that μ is q-characteristic for $\overline{B}\Gamma^q_{s\ell}$. In other words the map

$$(1.11) \qquad [\alpha] \mapsto \langle \alpha^*\mu, [S^q] \rangle$$

is an isomorphism of $\pi_q(\overline{B}\Gamma^q_{s\ell})$ with $\mathbb{R}$, where $\alpha\colon S^q \to \overline{B}\Gamma^q_{s\ell}$ represents the element $[\alpha] \in \pi_q(\overline{B}\Gamma^q_{s\ell})$.

2. DIFFEOMORPHISM GROUPS

We will consider the group *Diff*M of diffeomorphisms of M with the usual C^∞ topology. This may be described as follows. Let $f \in$ *Diff*M and suppose that $(U_i,\varphi_i),(U_j,\varphi_j)$ are local coordinate neighborhoods. Let K be a compact set in U_i such that $f(K) \subset U_j$ and let $\varepsilon > 0$. Define $\beta(f;(U_i,\varphi_i),(U_j,\varphi_j),K,\varepsilon)$ to be the set of all diffeomorphisms g of M such that $\| D^k(\varphi_j \circ f \circ \varphi_i^{-1})(x) - D^k(\varphi_j \circ g \circ \varphi_i^{-1})(x) \| < \varepsilon$ for all $x \in \varphi_i(K)$, $\forall k$. A neighborhood of f is a set that contains the intersection of a finite number of the sets $\beta(f;(U_i,\varphi_i),(U_j,\varphi_j),K,\varepsilon)$. This topology is often referred to as the topology of C^∞ uniform convergence on compact sets.

We will write $\mathrm{Diff}^\delta M$ for the group with the discrete topology. The inclusion map $i: \mathrm{Diff}^\delta M \to$ *Diff*M is certainly a continuous map which passes to a continuous map, also designated by i, at the classifying space level. B*Diff*M is the classifying space for bundles with fibre M and structure group *Diff*M, while $\mathrm{BDiff}^\delta M$ classifies M-bundles whose structure group is $\mathrm{Diff}^\delta M$. Evidently, if E is the universal M-bundle over B*Diff*M then its pull-back i^*E is the universal bundle over

$BDiff^{\delta}M$. Since i^*E has a discrete structural group we may consider it to be foliated (in a generalized sense, since $BDiff^{\delta}M$ is not a manifold). Our main concern here is with the homotopy fibre $\overline{B}\mathit{Diff}M$ of i. Thus we have the fibration

(2.1) $\overline{B}\mathit{Diff}M \to BDiff^{\delta}M \to B\mathit{Diff}M$.

$\overline{B}\mathit{Diff}M$ is the classifying space for M-bundles with a flat structure along with a global trivialization. The bundle i^*E pulls back to the universal trivial bundle $\overline{B}\mathit{Diff}M \times M$ over $\overline{B}\mathit{Diff}M$. The foliation on i^*E also pulls back to a foliation on $\overline{B}\mathit{Diff}M \times M$ that is transverse to the M-factor.

Let us now consider $\mathit{Diff}_{\omega}M$, the space of diffeomorphisms that preserve the volume form ω, with the C^{∞} topology. An obvious question at this point is: does $\mathit{Diff}_{\omega}M$ change if we change ω? The answer is no. Let us first assume that ω' is another volume form on M such that $\mathit{vol}\ \omega' \equiv \int_M \omega' = \int_M \omega \equiv \mathit{vol}\ \omega$. Moser [Mo] has shown that there is a diffeomorphism f of M such that $f^*\omega = \omega'$. Therefore, we have an isomorphism between $\mathit{Diff}_{\omega}M$ and $\mathit{Diff}_{\omega'}M$ by sending g to $f^{-1}\circ g\circ f$. If $\mathit{vol}\ \omega'$ is not the same as $\mathit{vol}\ \omega$ we can always normalize ω' by multiplying by $\lambda = \mathit{vol}\ \omega/\mathit{vol}\ \omega'$ and noting that

$\mathit{Diff}_{\lambda\omega'}M$ equals $\mathit{Diff}_{\omega'}M$, since clearly f preserves ω' if and only if it preserves $\lambda\omega'$. There is also no trouble if we switch the orientation on M. If ω is a volume form for the old orientation then $-\omega$ is a volume form with respect to the new orientation, and as above $\mathit{Diff}_{\omega}M$ equals $\mathit{Diff}_{-\omega}M$.

As in (2.1) we may form the following fibration with respect to $\mathit{Diff}_{\omega}M$.

(2.2) $$\overline{B}\mathit{Diff}_{\omega}M \to B\mathrm{Diff}_{\omega}^{\delta}M \to B\mathit{Diff}_{\omega}M$$

On the universal trivial M-bundle over $\overline{B}\mathit{Diff}_{\omega}M$ we have a foliation $\mathcal{F}$. To be precise we must introduce particular models here: up to now our spaces have only been defined up to homotopy type.

Let $\mathrm{Sing}\mathit{Diff}_{\omega}M$ be the smooth singular complex of $\mathit{Diff}_{\omega}M$. Thus a k-simplex is a smooth map $h_t\colon \Delta^k \to \mathit{Diff}_{\omega}M$, i.e. the map from $\Delta^k \times M \to M$ given by $(t,x) \mapsto h_t(x)$ is smooth. Note that $\mathrm{Diff}_{\omega}^{\delta}M$ acts freely on $\mathrm{Sing}\mathit{Diff}_{\omega}M$ by multiplication on the right, so it also acts freely on its geometric realization $|\mathrm{Sing}\mathit{Diff}_{\omega}M|$. We can now form the quotient space $|\mathrm{Sing}\mathit{Diff}_{\omega}M|/\mathrm{Diff}_{\omega}^{\delta}M$. If we continue (2.2) to the left we get the homotopy fibration

(2.3) $$\mathrm{Diff}_{\omega}^{\delta}M \to \mathit{Diff}_{\omega}M \xrightarrow{i} \overline{B}\mathit{Diff}_{\omega}M.$$

Using the specific models from above we may form

$$\text{(2.4)} \qquad \mathrm{Diff}^{\delta}_{\omega}M \to |\mathrm{Sing}\,\mathit{Diff}_{\omega}M| \to |\mathrm{Sing}\,\mathit{Diff}_{\omega}M|/\mathrm{Diff}^{\delta}_{\omega}M.$$

Since (2.4) maps to (2.3) it follows that the base spaces are homotopy equivalent. We will use $|\mathrm{Sing}\,\mathit{Diff}_{\omega}M|/\mathrm{Diff}^{\delta}_{\omega}M$ as our model for $\overline{B}\mathit{Diff}_{\omega}M$. (§5 [Ma-2]).

Thus $\overline{B}\mathit{Diff}_{\omega}M$ has a PL structure in which a k-simplex is a smooth map $\Delta^k \to \mathit{Diff}_{\omega}M$ which is well defined up to composition on the right by an element of $\mathrm{Diff}^{\delta}_{\omega}M$. To get rid of this ambiguity we can ask for the 0-vertex to go to the identity diffeomorphism.

$$\text{(2.5)} \qquad (\Delta^k, 0) \to (\mathit{Diff}_{\omega}M, 1)$$
$$t \mapsto h_t$$

This formula shows us that we get the same classifying space if we just use $\mathit{Diff}_{\omega_0}M$, the component of $\mathit{Diff}_{\omega}M$ containing the identity map. From now on, to agree with [M-2], we will say $\overline{B}\mathit{Diff}_{\omega_0}M$ instead of $\overline{B}\mathit{Diff}_{\omega}M$. This also shows us why $H^*(\overline{B}\mathit{Diff}_{\omega_0}M)$ is referred to as the local cohomology of the group $\mathit{Diff}_{\omega}M$. The local cohomology is just concerned with the topological properties of the group in a neighborhood of the identity.

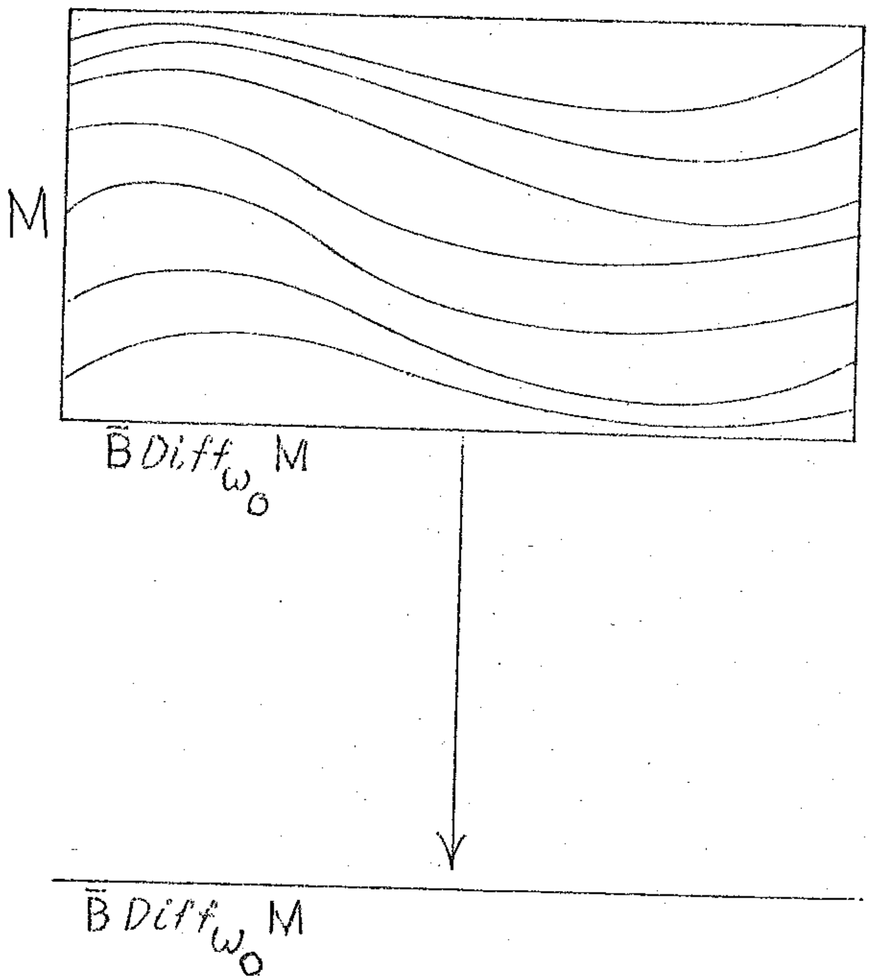
We now define the canonical foliation $\mathcal{F}$ on $\overline{B}\mathit{Diff}_{\omega_o} M \times M$. To each k-simplex in $\overline{B}\mathit{Diff}_{\omega_o} M$ we can associate a foliation on $\Delta^k \times M$ with leaves

$$(2.6) \qquad L_m = \{(t, h_t(m)) : t \in \Delta^k\}.$$

This foliation is the pull-back of the point foliation on M by the map $f: (t,m) \to h_t^{-1}(m)$. The foliation is volume preserving and has $f^*\omega$ for its transverse volume form, where ω is the volume form on M. The form $f^*\omega$ is the unique transverse volume form that restricts to ω on each $\{t\} \times M$. The foliations on each $\{k\text{-simplex}\} \times M$ fit together so that we get a (PL) foliation on $(|\mathrm{Sing}\mathit{Diff}_{\omega_o} M| / \mathrm{Diff}^{\delta}_{\omega_o} M) \times M$. Moreover, because the h_t preserve ω, the volume forms $f^*\omega$ fit together to define a transverse volume form Ω, which on $\Delta^k \times M$ is just $f^*\omega$. This is a PL n-form [Su].

From now on we will just write $\overline{B}\mathit{Diff}_{\omega_o} M$ for $|\mathrm{Sing}\mathit{Diff}_{\omega} M| / \mathrm{Diff}^{\delta}_{\omega} M$ and when we need to use the specific PL structure we will discuss it.

In summary: We have a space $\overline{B}\mathit{Diff}_{\omega_o} M$ which we interpret as a classifying space for foliated M-bundles. On $\overline{B}\mathit{Diff}_{\omega_o} M \times M$ we have a codimension-n volume preserving foliation $\mathcal{F}$ with transverse volume form Ω.


M
$\bar{B}\,Diff_{\omega_0} M$
$\bar{B}\,Diff_{\omega_0} M$

3. McDUFF'S CLASSES

A. McDuff's Classes - Singular

We will give a different presentation than that which McDuff gave in [M-2]. Associated to the space $\overline{B}\mathcal{D}iff_{\omega_o} M \times M$ we have a closed PL n-form Ω. The class $[\Omega]$ is an element of $H^n(\overline{B}\mathcal{D}iff_{\omega_o} M \times M)$. By the Künneth formula we have $H^n(\overline{B}\mathcal{D}iff_{\omega_o} M \times M) \cong \bigoplus_{i+j=n} H^i(\overline{B}\mathcal{D}iff_{\omega_o} M) \otimes H^j(M)$. This enables us to decompose $[\Omega]$ as $\bigoplus_{i=o}^{n} [\Omega]_i$, where $[\Omega]_i \in H^{n-i}(\overline{B}\mathcal{D}iff_{\omega_o} M) \otimes H^i(M)$. The class $[\Omega]_i$ equals $\sum_j \alpha^j_{n-i} \otimes \beta^j_i$, where j is indexed over the rank of $H^i(M)$, $\alpha^j_{n-i} \in H^{n-i}(\overline{B}\mathcal{D}iff_{\omega_o} M)$, and $\beta^j_i \in H^i(M)$.

We may now define the classes $c_k(M) \in H^k(\overline{B}\mathcal{D}iff_{\omega_o} M; H^{n-k}(M))$ by the formula

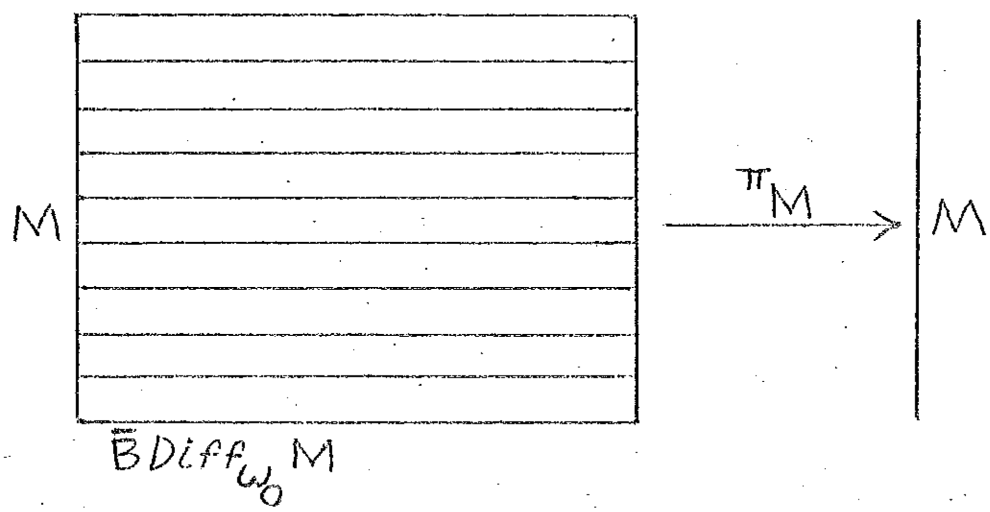
$$(3.A.1) \qquad c_k(M)\kappa = \sum_j \langle \alpha^j_k, \kappa \rangle \cdot \beta^j_{n-k}$$

where $\kappa \in H_k(\overline{B}\mathcal{D}iff_{\omega_o} M)$. Note that the class $c_k(M)$ is independent of the choice of α^j_{n-i} and β^j_i. In fact $c_k(M)\kappa$ is just, neglecting sign, the slant product of $[\Omega]_{n-k}$ with κ. For $[\Omega]_{n-k}/\kappa$ is $(-1)^{k(n-k)} \sum_j \langle \alpha^j_k, \kappa \rangle \cdot \beta^j_{n-k}$.

We will use absolute value signs to show that we neglect sign. Thus, we will also denote $c_k(M)\kappa$ by

(3.A.2) $$|[\Omega]/\kappa|.$$

If the leaves of $\mathcal{F}$ were $\bar{B}Diff_{\omega_o} M \times \{m\}$ then $[\Omega]$ would be $\pi_M^*[\omega]$, where π_M is the projection from $\bar{B}Diff_{\omega_o} M \times M$ to M. In this case, the $c_k(M)$ clearly would all vanish.



Of particular interest to us is the top dimensional class $c_n(M)$. If η is an n-cycle in $\bar{B}Diff_{\omega_o} M$, then $c_n(M)$ is $\langle \alpha_n^1, \eta \rangle \cdot \beta_o^1$. For simplicity let us take β_o^1 to be 1 for β_o^1 in $H^o(M)$. Then $c_n(M)\eta = \langle \alpha_n^1, \eta \rangle = \langle [\Omega]_o, \eta \rangle$. This tells us

(3.A.3) $$c_n(M) = [\Omega]_o = \alpha_n^1 \otimes 1$$

if we identify $H^n(\overline{B}\mathit{Diff}_{\omega_o}M)$ with $H^n(\overline{B}\mathit{Diff}_{\omega_o}M) \otimes 1$ in $H^n(\overline{B}\mathit{Diff}_{\omega_o}M) \otimes H^o(M)$.

Fathi has observed that we can view the classes in terms of a degree preserving vector space homomorphism

$$(3.A.4) \qquad \psi\colon \tilde{H}^*(M) \to \tilde{H}^*(\overline{B}\mathit{Diff}_{\omega_o}M)$$

$$\psi(a)\kappa \overset{\text{def}}{\equiv} (-1)^k\langle a \cup c_k(M)\kappa, [M]\rangle$$

$$= (-1)^k\langle a, [M] \cap c_k(M)\kappa\rangle$$

where $\kappa \in H_k(\overline{B}\mathit{Diff}_{\omega_o}M)$. From this we see that $\psi\colon H^k(M) \to H^k(\overline{B}\mathit{Diff}_{\omega_o}M)$ is the zero map if and only if $c_k(M) = 0$. Let us look at $\psi\colon H^n(M) \to H^n(\overline{B}\mathit{Diff}_{\omega_o}M)$.

$$(3.A.5) \qquad \psi([\omega])\eta = (-1)^n\langle[\omega] \cup c_n(M)\eta, [M]\rangle$$

$$= (-1)^n c_n(M)\eta\langle[\omega], [M]\rangle$$

$$= c_n(M)\eta \cdot (-1)^n \mathit{vol}_\omega.$$

So up to a constant $\psi([\omega])$ is $c_n(M)$. Our major concern is the injectivity of ψ. We will mostly address ourselves to the question of whether or not $\psi([\omega])$ is non-zero.

Using the map $\psi\colon H^k(M) \to H^k(\overline{B}\mathit{Diff}_{\omega_o}M)$ we may see certain properties more clearly as follows.

$$\psi(a)\kappa = (-1)^k\langle a\cup c_k(M)\kappa, [M]\rangle$$

$$= (-1)^k\langle a\cup |[\Omega]_{n-k}/\kappa|, [M]\rangle$$

$$= (-1)^k\sum_j\langle\alpha_k^j,\kappa\rangle\cdot\langle a\cup\beta_{n-k}^j, [M]\rangle$$

This enables us to express $\psi(a)$ as

$$(-1)^k\sum_j\langle a\cup\beta_{n-k}^j, [M]\rangle\cdot\alpha_k^j$$

an element of $H^k(\overline{B}\mathit{Diff}_{\omega_o} M)$. We may denote by $\mathfrak{D}a$ the unique cohomology class in $H^{n-k}(M)$ such that (by duality)

$$\langle a\cup\mathfrak{D}a, [M]\rangle = 1.$$

Suppose now that a is non-zero and is in <u>one</u> summand of $H^k(M)$. Then we may decompose $[\Omega]_{n-k}$ so that there is a j' such that $\beta_{n-k}^{j'}$ is $\mathfrak{D}a$. This gives us

$$\psi(a) = (-1)^k\langle a\cup\mathfrak{D}a, [M]\rangle\cdot\alpha_k^{j'}$$

$$= (-1)^k\alpha_k^{j'}.$$

If $H^{n-k}(M)$ (or $H^k(M)$) has rank one then we may decompose $[\Omega]_{n-k}$ as

(3.A.6) $\quad (-1)^k\psi(a) \otimes \mathfrak{D}a$

for $a \in H^k(M)$, $a \neq 0$. In general if $\{a_i\}$ is a basis of $\tilde{H}^*(M)$ then

$$(3.A.7) \qquad [\Omega] = \sum_{a_i} (-1)^{\deg a_i} \psi(a_i) \otimes \mathcal{D}a_i .$$

This is nice for it gives us a canonical way to write $[\Omega]$.

There is an important property of $[\Omega]$ that we have not discussed. Since Ω is the n-form defining a codimension-n PL foliation, it is locally pulled back from $\mathbb{R}^n$, hence $\Omega^2 = 0$. So

$$(3.A.8) \qquad [\Omega]^2 = 0.$$

We may exploit the above to get certain relations between the $c_k(M)$. These relations are more easily expressed in terms of ψ.

(3.A.9) <u>Example</u>. Let M^{n+m} be $S^n \times S^m$ where both n and m are even. Furthermore, let ω_n and ω_m be the volume forms on S^n and S^m respectively such that $\langle \omega_i, [S^i] \rangle = 1$. We give $S^n \times S^m$ the volume form $\omega_n \wedge \omega_m$. Then by (3.A.6)

$$[\Omega] = \psi([\omega_n \wedge \omega_m]) \otimes 1 + \psi([\omega_n]) \otimes [\omega_m]$$

$$+ \psi([\omega_m]) \otimes [\omega_n] + 1 \otimes [\omega_n \wedge \omega_m].$$

By (3.A.8) we have

$$0 = [\Omega]^2 = (\psi([\omega_n \wedge \omega_m]))^2 \otimes 1$$

$$+ 2\psi([\omega_n \wedge \omega_m]) \cup \psi([\omega_n]) \otimes [\omega_m]$$

$$+ 2\psi([\omega_n \wedge \omega_m]) \cup \psi([\omega_m]) \otimes [\omega_n]$$

$$+ 2\{\psi([\omega_n]) \cup \psi([\omega_m]) + \psi([\omega_n \wedge \omega_m]) \otimes [\omega_n \wedge \omega_m]\}.$$

If we assume that $\psi([\omega_n \wedge \omega_m])$ is non-zero then we have

$\psi([\omega_n]) \neq 0$, $\psi([\omega_m]) \neq 0$

$\psi([\omega_n]) \cup \psi([\omega_m]) \neq 0$, and, irregardless of $\psi([\omega_n \wedge \omega_m])$

$\psi([\omega_n \wedge \omega_m]) \cup \psi([\omega_i]) = 0$, $i = n, m$.

As of yet we have been unable to show that $\psi([\omega_n \wedge \omega_m])$ is non-zero.

B. Geometric Realization of Cycles

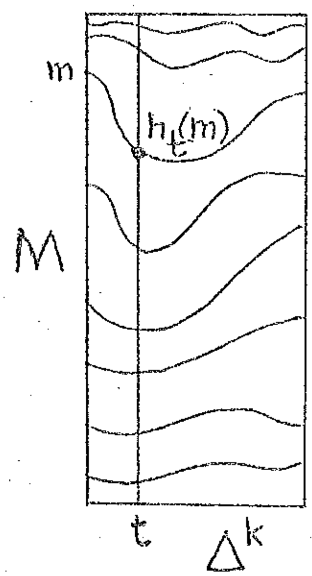
This section is more in the spirit of the way McDuff originally defined the $c_k(M)$. In §6 we will have to explicitly construct certain foliations over complexes to achieve some non-vanishing results. This is in contrast to the preceding results which involved section spaces.

If κ is an integral k-cycle in $|\text{Sing } \mathit{Diff}_{\omega_0} M| / \text{Diff}^{\delta}_{\omega_0} M$, then κ can be expressed as $\Sigma z_i \Delta_i^k$, where the $z_i \in \mathbb{Z}$, and Δ_i^k is a (smooth) k-simplex. Since κ is a cycle, we may form the geometric realization of κ. Since κ is in fact a k-cycle, and not just a k-chain, the realization is an oriented polyhedron, P, whose k-faces correspond to the Δ_i^k, along with a map $f: P \to \overline{B}Diff_{\omega_0} M$. Furthermore if [P] is the fundamental class of P then $f_*[P] = \kappa$. (The point is that because κ is a cycle the (k-1)-dimensional faces of its k-simplices cancel in pairs. Therefore one may suppose that each (k-1)-simplex in P occurs as a face of exactly two oppositely oriented k-simplices. So we may take P as the disjoint union of the Δ_i^k with the (k-1)-faces identified in pairs. Then $H_k(P;\mathbb{Z}) \cong \mathbb{Z}$ and is generated by the fundamental class [P].) We see that on $P \times M$ we have a volume preserving (PL) foliation with (PL) transverse volume form $f^*\Omega = \Omega(P)$. Therefore, to calculate $|[\Omega]/\kappa|$, we may instead calculate

(3.B.1) $\quad |[\Omega(P)]/[P]|$.

<u>Note</u>. The preceding discussion is a good example of $\overline{B}Diff_{\omega_0} M$ as a classifying space.

Conversely, say that one has an oriented k-dimensional polyhedron P. If, on P × M, we have a (PL) volume preserving foliation F(P) that is transverse to the M-factors and its transverse volume form Ω(P), restricted to p × M is ω, then we have a k-chain in $\overline{B}\mathcal{Diff}_{\omega_o} M$ defined as follows. (If P has a fundamental class [P], then we get a k-cycle in $\overline{B}\mathcal{Diff}_{\omega_o} M$). Let us begin by ordering the vertices in P and take a k-simplex Δ^k of P. When we restrict F(P) to $\Delta^k \times M$, every leaf is a 1-fold covering of Δ^k. By traveling on the leaves we see how



to map Δ^k to $|\mathrm{Sing}\, \mathcal{Diff}_{\omega_o} M|/\mathrm{Diff}^{\delta}_{\omega_o} M$ (We always normalize so that $(\Delta^k, 0) \to (\mathcal{Diff}_{\omega_o} M, 1)$, as in our convention--this is why we ordered the vertices, see [Ba]) as follows. We let h_t stand for the diffeomorphism that t is mapped to. Define $h_t(m)$ to be the point on t × M that the leaf that passes through o × m hits. The map $f\colon P \to \overline{B}\mathcal{D}iff_{\omega_o}M$, derived in this manner, classifies $F(P)$ on $P \times M$. If $[P]$ exists, then $f_*[P]$ is a k-cycle in $\overline{B}\mathcal{D}iff_{\omega_o}M$ and $c_k(M)f_*[P]$ is equal to $|[\Omega(P)]/[P]|$.

4. MAIN RESULTS

As we saw in §2 $\overline{B}Diff_{\omega_0} M \times M$ has a codimension-q volume preserving foliation $\mathcal{F}$ with transverse volume form Ω. Let $\Phi\colon \overline{B}Diff_{\omega_0} M \times M \to B\Gamma^n_{s\ell}$ be the map classifying $\mathcal{F}$ and consider the following diagram.

(4.1)
$$\begin{array}{ccc} \overline{B}Diff_{\omega_0} M \times M & \xrightarrow{\quad\Phi\quad} & B\Gamma^n_{s\ell} \\ \Big\downarrow \pi & & \Big\downarrow d \\ M & \xrightarrow{\quad\tau\quad} & BSL(n) \end{array}$$

Here π is projection, and τ classifies the tangent bundle of M. (We may assume that τ maps to BSL(n) since by choosing the volume form ω we give TM an SL(n)-structure.) The bundle $\nu(\mathcal{F})$ is just $\overline{B}Diff_{\omega_0} M \times TM$, since $\mathcal{F}$ is transverse to the M-factors. We classify $\nu(\mathcal{F})$ by the map $d \circ \Phi$. Note that $\nu(\mathcal{F})$ is also classified by $\tau \circ \pi$. Since classifying maps are only determined up to homotopy we may pick maps and spaces so that (4.1) commutes and such that d is a Hurewicz fibration.

Let Ξ denote the pull-back of $B\Gamma^n_{s\ell}$ by τ. We are interested in $\mathcal{S}(M)$, the space of sections of Ξ. Equivalently one can look at $\mathcal{S}(M)$ as the space of lifts of τ. We can, and will, freely pass between the two notions. We will define a canonical map

$$(4.2) \qquad \Pi: \overline{B}\mathcal{D}iff_{\omega_0} M \to \mathcal{S}(M)$$

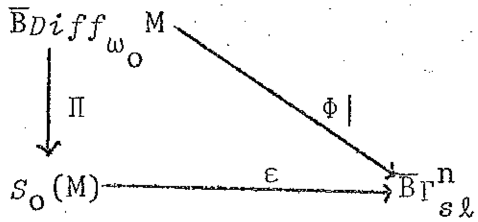
by setting

$$(4.3) \qquad \Pi(b) \overset{def}{\equiv} \Phi(b,\cdot)$$

for each $b \in \overline{B}\mathcal{D}iff_{\omega_0} M$. Since M is compact and n-dimensional while the fibre of Ξ is $\overline{B}\Gamma^n_{s\ell}$, which is (n-1)-connected, the space $\mathcal{S}(M)$ is not connected. However $\overline{B}\mathcal{D}iff_{\omega_0} M$ is connected, so the image of Π is in one component of $\mathcal{S}(M)$. We will denote this component by $\mathcal{S}_0(M)$. McDuff, in the spirit of Thurston [T], has shown that Π^* is a cohomology isomorphism. This enables us to view McDuff's classes as being in $H^*(\mathcal{S}_0(M))$ if we wish. Let us exploit this philosophy.

Define $\varepsilon: \mathcal{S}_0(M) \to \overline{B}\Gamma^n_{s\ell}$ as evaluation at the fixed point m_0 in M. Consider $\mathcal{F}|$, $\mathcal{F}$ restricted to $\overline{B}\mathcal{D}iff_{\omega_0} M \times m_0$. The bundle $\nu(\mathcal{F}|)$ is isomorphic to $\overline{B}\mathcal{D}iff_{\omega_0} M \times \mathbb{R}^n$. This tells us that $\mathcal{F}|$ is a codimension-n Haefliger structure with trivial normal bundle. Due to this we see that

$\Phi|$, the map classifying $\mathcal{F}|$, is homotopic to a map with image in $\overline{B}\Gamma^n_{s\ell}$. Without loss of generality we may assume that $\mathcal{F}|$ actually maps into $\overline{B}\Gamma^n_{s\ell}$. The map $\Phi|$ equals $\Phi(\cdot,m_o)$ and $\varepsilon\circ\Pi(\cdot)$ equals $\Phi(\cdot,m_o)$, so the following diagram homotopy commutes.



Recall from (3.A.3) that $c_n(M)$ is $[\Omega]_o \in H^n(\overline{B}Diff_{\omega_o}M)$. However, $[\Omega]_o$ is just $[\Omega]|$, $[\Omega]$ restricted to $\overline{B}Diff_{\omega_o}M \times m_o$. Hence, $\Phi|^*\mu$ is $c_n(M)$. Since Π^* is an isomorphism

(4.4) $\qquad c_n(M) \neq 0 \iff \varepsilon^*\mu \neq 0.$

We will use the above to show that $c_n(M)$ is non-zero. The map ε^* then is obviously of great interest to us. In general the space $S_o(M)$ can be quite complicated in a topological sense due to the twisting of TM. At the present time one has been unable to deal with the twisting. Therefore, we will look at manifolds where this is controlled.

Before preceding further we must establish a criterion for determining the components of $\mathcal{S}(M)$.

(4.5) Lemma. If f_0 and f_1 are two sections of $\mathcal{S}(M)$, viewed as lifts of τ, then they are in the same component of $\mathcal{S}(M)$ if and only if $f_0^*\tilde{\mu} = f_1^*\tilde{\mu}$.

Proof. See Appendix. The techniques are those of obstruction theory.

If $b \in \overline{B}\mathcal{D}iff_{\omega_0} M$ then $\Pi(b)$ is in $\mathcal{S}_0(M)$. The section $\Pi(b)$ is $\Phi(b,\cdot)$, and $\Phi(b,\cdot)^*\tilde{\mu}$ is $[\Omega]$ restricted to $b \times M$, which is $[\omega]$. This tells us that

$$(4.6) \qquad f \in \mathcal{S}_0(M) \Longleftrightarrow f^*\tilde{\mu} = [\omega].$$

Manifolds where the twisting of TM is controlled are parallelizable and stably parallelizable manifolds. In general a bundle ξ^n is stably parallelizable if $\xi^n \oplus \varepsilon^k \cong \varepsilon^{n+k}$, for some $k > 0$. For the case of TM^n, since the base space of TM^n is n-dimensional, it follows from stability theory that k can be taken to be 1. The most obvious example of a stably parallelizable manifold is a sphere. In fact, the following proposition shows that spheres "classify" stably parallelizable manifolds.

(4.7) Proposition. The manifold M is stably parallelizable if and only if $TM \cong \gamma^*(TS^n)$ for some map γ from M to S^n.

Proof. If $TM \cong \gamma^*(TS^n)$, then $TM \oplus \varepsilon^1 \cong \gamma^*(TS^n) \oplus \varepsilon^1$ $\cong \gamma^*(TS^n) \oplus \varepsilon^1) \cong \gamma^*(TS^n \oplus \nu) \cong \gamma^*(\varepsilon^{n+1}) = \varepsilon^{n+1}$. Thus M is stably parallelizable.

If M is stably parallelizable then $TM \oplus \varepsilon^1 \cong \varepsilon^{n+1}$. So $TM \oplus \varepsilon^1$ has a non-zero section θ such that $\theta^\perp \cong TM$. Denote the section $\theta: M \to M \times \mathbb{R}^{n+1}$ by $m \mapsto (m, \overline{\theta}(m))$, so we have $\overline{\theta}: M \to \mathbb{R}^{n+1} - 0$. We can normalize $\overline{\theta}$ so that we have a map $\eta: M \to S^n$. Note that $TS^n|_{pt.} = \eta(pt.)^\perp$, so $\eta^*(TS^n) = TM$. Q.E.D.

We are now ready to prove our main theorems.

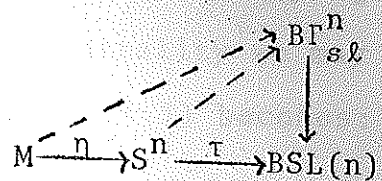
(4.8) Theorem. If M is an odd-dimensional stably parallelizable manifold, then $c_n(M) \neq 0$.

Remark. It is essential that n is odd for McDuff has shown that $c_{2n}(S^{2n}) = 0$.

Proof. - case (i) - Suppose that M is stably parallelizable but not parallelizable. Then by (4.7) $TM = \gamma^*(TS^n)$. The map γ can not be null-homotopic, for if it were

$\gamma^*(TS^n)$ would be the trivial bundle. Recall from §2 that we can give M a volume form ω_M such that $[\omega_M] = \eta^*[\omega_s]$, where $[\omega_s]$ is the standard volume form on S^n. (Since γ is of non-zero degree $\eta^*[\omega_s] \neq 0$.) Consider the following diagram.

(4.9)



τ classifies TS^n. The map $\tau\circ\eta$ classifies TM since $\eta^*(TS^n) \cong TM$. Say $f \in \mathcal{S}_o(S^n)$, then $f^*\tilde{\mu} = [\omega_s]$. However $f\circ\eta \in \mathcal{S}(M)$ and $(f\circ\eta)^*\tilde{\mu} = \eta^* f^*\tilde{\mu} = \eta^*[\omega_s] = [\omega_M]$. Therefore, by lemma (4.5), $f\circ\eta \in \mathcal{S}_o(M)$. Hence we have a map $\gamma\colon \mathcal{S}_o(S^n) \to \mathcal{S}_o(M)$ given by $\gamma(f) = f\circ\eta$. Consider the following diagram where ε_s

(4.10)

is evaluation at $\eta(m_o) \in S^n$ and ε_m is evaluation at $m_o \in M$. Since $\varepsilon_m\circ\gamma(f) = \varepsilon_m(f\circ\eta) = f\circ\eta(m_o) = \varepsilon_s\circ f$ the above diagram commutes. Recall that McDuff has shown that $c_n(S^n) \neq 0$ for n odd. (This will be discussed in §6.) Since $\varepsilon_s^* = \gamma^*\varepsilon_m^*$ and $c_n(S^n) \neq 0$, (4.4) tells us that $\varepsilon_s^*\mu \neq 0$. So $\varepsilon_m^*\mu \neq 0$ and therefore $c_n(M) \neq 0$.

-case (ii) - M is parallelizable. The previous argument will not work for η may be of degree 0. However, let τ now stand for a map classifying TM into BSL(n). Without loss of generality we may take τ as a constant map. In this case (4.1) becomes:

(4.11)
$$\begin{array}{ccccc} \overline{B}\mathit{Diff}_{\omega_0} M \times M & \xrightarrow{\Phi} & \overline{B}\Gamma^n_{s\ell} & \hookrightarrow & B\Gamma^n_{s\ell} \\ \downarrow \pi & & \downarrow d & & \downarrow d \\ M & \xrightarrow{\tau} & * & \hookrightarrow & BSL(n) \end{array}$$

Again we are allowed to vary the maps up to homotopy. Now $\mathcal{S}(M) = \mathit{Maps}(M,\overline{B}\Gamma^n_{s\ell})$ and we will designate the component corresponding to $\mathcal{S}_o(M)$ as $\mathit{Maps}_1(M,\overline{B}\Gamma^n_{s\ell})$. So $s \in \mathit{Maps}_1(M,\overline{B}\Gamma^n_{s\ell})$, if and only if $s^*\mu = [\omega]$, where ω is the volume form on M. The evaluation map $\varepsilon: \mathcal{S}_o(M) \to \overline{B}\Gamma^n_{s\ell}$ becomes $\varepsilon: \mathit{Maps}_1(M,\overline{B}\Gamma^n_{s\ell}) \to \overline{B}\Gamma^n_{s\ell}$. Our aim is to show that $\varepsilon^*\mu \neq 0$. As discussed in §2 ω is chosen so that $\langle[\omega],[M]\rangle = 1$.

Recall that $\pi_n(\overline{B}\Gamma^n_{s\ell}) \cong \mathbb{R}$ and let $f: S^n \to \overline{B}\Gamma^n_{s\ell}$ be a map such that $[f] = 1 \in \pi_n(\overline{B}\Gamma^n_{s\ell})$, i.e. $\langle f^*\mu,[S^n]\rangle = 1$. (See (1.11).)

Let us give S^n a volume form ω_s so that $f^*\mu = [\omega_s]$. We will say that $g\colon M \to S^n$ is of degree one if $g^*[\omega_s] = [\omega]$. Let $Maps_1(M,S^n)$ be the maps of degree 1. Now let us define a map $\hat{f}\colon Maps_1(M,S^n) \to Maps_1(M,\overline{B}\Gamma^n_{s\ell})$ by setting $\hat{f}(g) \overset{\text{def}}{\equiv} f\circ g$. This is well-defined for if $g \in Maps_1(M,\overline{B}\Gamma^n_{s\ell})$ then $\hat{f}(g) = f\circ g \in Maps_1(M,\overline{B}\Gamma^n_{s\ell})$ because $(f\circ g)^*\mu = g^*f^*\mu = g^*[\omega_s] = [\omega]$.

(4.12)
$$\begin{array}{ccc} Maps_1(M,S^n) & \xrightarrow{\ \hat{f}\ } & Maps_1(M,\overline{B}\Gamma^n_{s\ell}) \\ \downarrow{\scriptstyle \varepsilon'} & & \downarrow{\scriptstyle \varepsilon} \\ S^n & \xrightarrow{\qquad f \qquad} & \overline{B}\Gamma^n_{s\ell} \end{array}$$

The maps ε and ε' represent evaluation at m_o so the above diagram commutes. Since $f^*\mu = [\omega_s]$, if we can show that $\varepsilon'^*[\omega_s] \neq 0$ then we will have shown that $\varepsilon^*\mu \neq 0$.

Now $Maps_1(S^n,S^n)$ is the space of maps from S^n to S^n such that $f^*[\omega_s] = [\omega_s]$. Let ξ be a fixed element of $Maps_1(M,S^n)$ such that $\xi(m_o) = s$, the south pole of S^n. Define $\hat{\xi}\colon Maps_1(S^n,S^n) \to Maps_1(M,S^n)$ by $\hat{\xi}(h) = h\circ\xi$.

(4.13)
$$\begin{array}{ccc} Maps_1(S^n,S^n) & \xrightarrow{\ \hat{\xi}\ } & Maps_1(M,S^n) \\ \downarrow{\scriptstyle \varepsilon''} & & \downarrow{\scriptstyle \varepsilon'} \\ S^n & \xrightarrow{\ \text{identity}\ } & S^n \end{array}$$

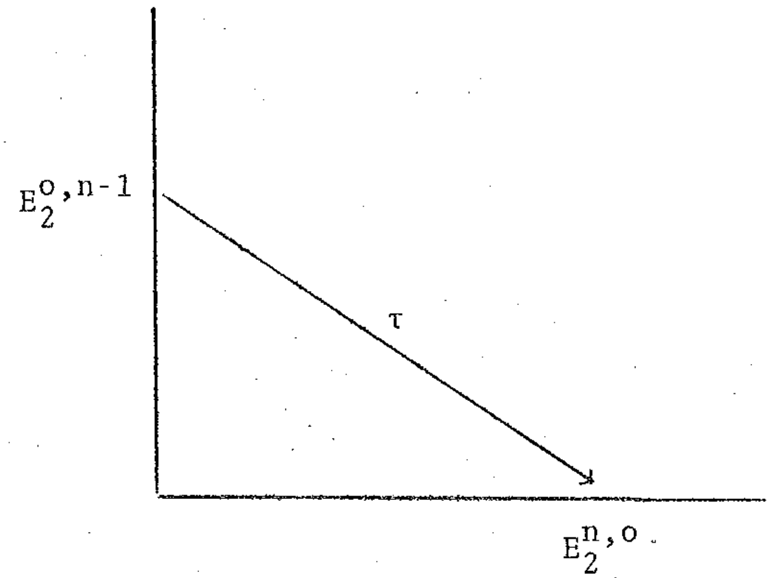
ε'' is evaluation at s hence the above diagram commutes. If we can show that $\varepsilon''^*[\omega_S] \neq 0$ then we will have shown that $\varepsilon'^*[\omega_S] \neq 0$. Therefore the proof of the theorem is completed by the following lemma.

(4.14) Lemma. If n is odd then in the following fibration $\varepsilon''^*[\omega_S] \neq 0$.

(4.15) $$\Omega^n S^n_1 \to Maps_1(S^n, S^n) \xrightarrow{\varepsilon''} S^n$$

where $\Omega^n S^n_1$ is the obvious component of $\Omega^n S^n$.

Proof. We will use the cohomology spectral sequence of (4.15) to prove the lemma. Since $\pi_i(\Omega^n S^n_1) \cong \pi_{n+i}(S^n)$ for $i > 0$ we see that $\Omega^n S^n_1$ is a connected space with all of its higher homotopy groups being torsion. We wish to appeal to the $\mathbb{R}$-Hurewicz Theorem but we need our space $\Omega^n S^n_1$ to be simply-connected. Unfortunately $\pi_1(\Omega^n S^n_1) \cong \pi_{n+1}(S^n) = \mathbb{Z}_2$. To get rid of this problem we pass to $\widetilde{\Omega^n S^n_1}$, the 2-fold universal cover. Note that $\Omega^n S^n_1$ and $\widetilde{\Omega^n S^n_1}$ have the same (real) cohomology. Since $\widetilde{\Omega^n S^n_1}$ is simply connected and $\pi_i(\widetilde{\Omega^n S^n_1}) \otimes \mathbb{R} = 0$ for $i > 1$ we have that $H^*(\Omega^n S^n_1) = 0$. Thus the fibre of (4.15) has no cohomology.



The $E_2^{n,o}$ term of our spectral sequence for (4.15) is $H^n(S^n) \cong \mathbb{R}$ but $E_2^{i,j} = 0$ for $j > 0$, hence $E_2^{n,o}$ lives forever. The term $E_2^{n,o}$ is generated by $[\omega_S]$ so $\varepsilon''^*[\omega_S] \neq 0$. This completes the lemma and hence Theorem (4.8). Q.E.D.

Now we will prove a more general version of case (ii) of (4.8). This proof is not as elementary as case (ii) of (4.8), so the other work is still of interest.

(4.16) Theorem. If M is parallelizable then $c_n(M) \neq 0$.

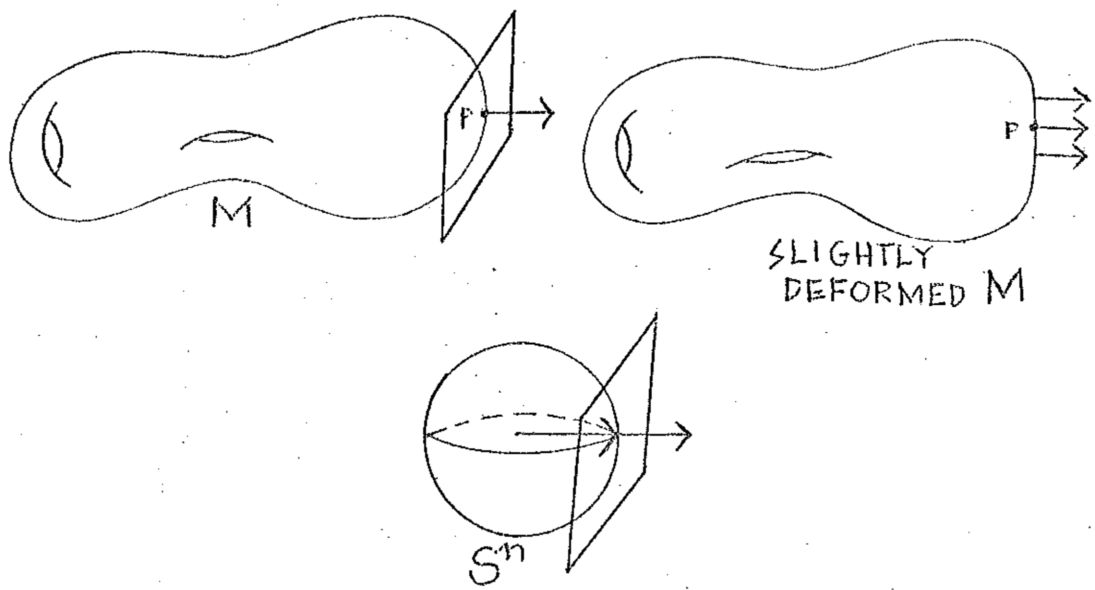
Remark. Here we have removed the condition of n being odd but yet we still use S^n in our argument! First we need a fact from the thesis of Hirsch [Hi, Thm. 6.3].

(4.17) Proposition (Hirsch). If M^n is parallelizable, it can be immersed in $\mathbb{R}^{n+1}$.

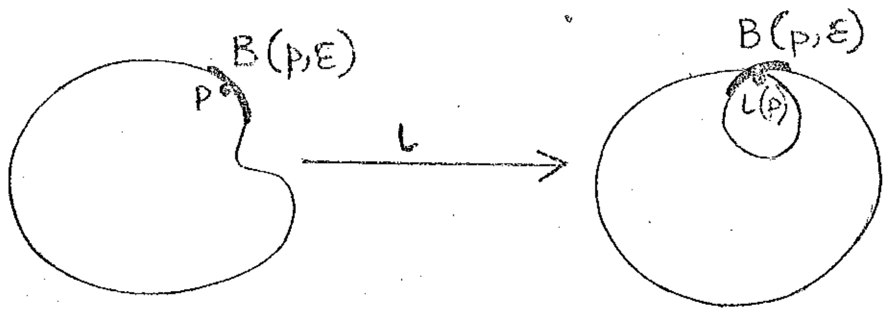
Proof of (4.16). We will consider M as being immersed in $\mathbb{R}^{n+1}$ with a fixed immersion. We will also freely identify M with its image in $\mathbb{R}^{n+1}$. This causes no trouble as will become apparent. Our goal is to define a certain map ψ from M to $Maps_1(M,S^n)$. We want for $\varepsilon' \circ \psi : M \to S^n$ to have non-zero degree, where ε' is evaluation at m_o. If this property holds then $\varepsilon'^*[\omega_s] \neq 0$.

(4.18)
$$\begin{array}{ccc} M & \xrightarrow{\Psi} & Maps_1(M,S^n) \\ & \overset{\overline{\Psi}}{\searrow} & \downarrow \varepsilon' \\ & & S^n \end{array}$$

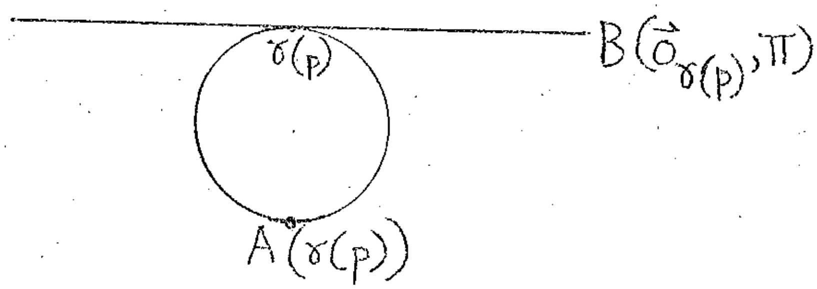
Let $\overline{\Psi}$ be $\varepsilon' \circ \Psi$.



To start with let us choose an ε greater than 0 which is less than the injectivity radius of M and is small enough that every (open) ball, $B(p,\varepsilon)$, with center p, $p \in M$ and of radius ε, is embedded by the immersion.

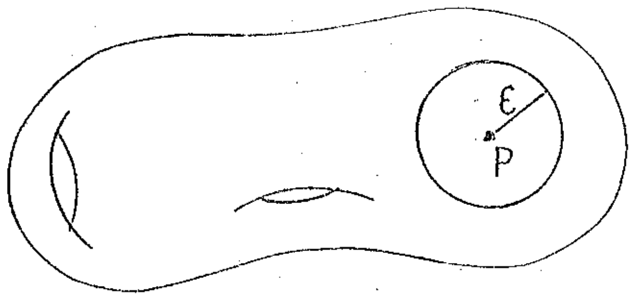


Consider Exp^{-1}: $B(p,\varepsilon) \to TM_p$. $Exp(v_p)$ is the point on the geodesic, through p (at $t = 0$) whose initial velocity is v_p, at distance $\|v_p\|$ (t^+ direction) from p. But since we are using the induced metric from $\mathbb{R}^{n+1}$, $\|v_p\|$ is the usual Euclidean length. This implies $Exp^{-1}: B(p,\varepsilon) \to B(\vec{o}_p,\varepsilon) \subset TM_p$. Let γ be the Gauss map from $M \to S^n$. The Gauss map sends m to the point on S^n corresponding to the outward pointing unit normal on M at m. By parallel translation in $\mathbb{R}^{n+1}$ we get a congruence from $TM_p \to TS^n_{\gamma(p)}$. Remembering that π is the injectivity radius of the sphere we see that Exp maps $B(\vec{o}_{\gamma(p)},\pi)$ diffeomorphically onto $S^n - A(\gamma(p))$, where $B(\vec{o}_{\gamma(p)},\pi) \subset TS^n_{\gamma(p)}$ and A is the antipodal map on S^n.



Now define $\zeta: TS^n \to TS^n$ by $\zeta(\vec{v}) = \pi/\varepsilon \cdot \vec{v}$. The map ζ has the effect of mapping $B(\vec{o}_s,\varepsilon)$ "radially diffeomorphically" onto $B(\vec{o}_s,\pi)$. Now we are in the position to define $\Psi: M \to Maps_1(M,S^n)$.

Decompose M as $B(p,\varepsilon) \cup \{M-B(p,\varepsilon)\}$. We must say



what $\Psi(p)m$ is for $m \in B(p,\varepsilon)$ and $m \notin B(p,\varepsilon)$, $m \in M$.

Consider the composition of the following maps.

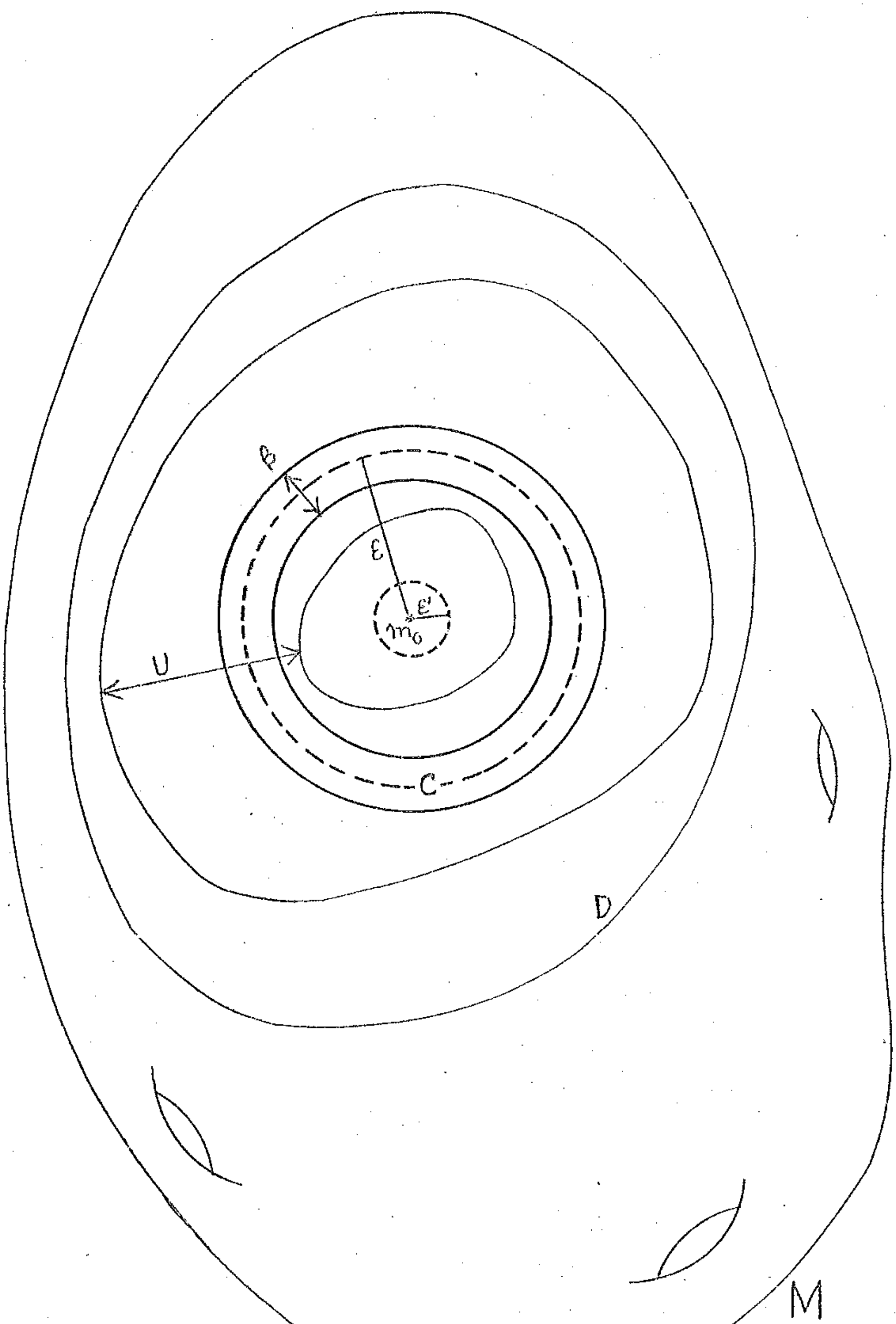
$$(4.19) \qquad \begin{array}{ccccc} B(p,\varepsilon) & \xrightarrow{Exp^{-1}} & B(\vec{o}_p,\varepsilon) & \xrightarrow{\|\text{-trans}} & B(\vec{o}_{\gamma(p)},\varepsilon) \\ & \searrow & & & \Big\downarrow \zeta \\ & & \text{diffeomorphism} & & B(\vec{o}_{\gamma(p)},\pi) \\ & & & \searrow & \Big\downarrow Exp \\ & & & & S^n - A(\gamma(p)) \end{array}$$

If $m \in B(p,\varepsilon)$ we define $\Psi(p)m$ to be the image of m under the above composition. At this stage we want the orientation on $B(p,\varepsilon)$, induced from the orientation of M, to go to the usual orientation on $S^n - A(\gamma(p))$. If it does not we give M a different orientation and volume form. If $m \notin B(p,\varepsilon)$ set $\Psi(p)m = A(\gamma(p))$.

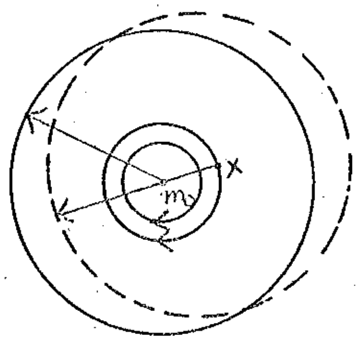
As we vary p we get a continuous map $\Psi: M \to Maps_1(M,S^n)$. Each $\Psi(p)$ is of degree one for $\Psi(p)$ is just a standard collapsing map of degree one. Now consider diagram (4.18). Our claim is:

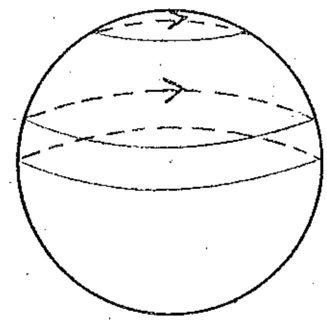
(4.20) Lemma. $\deg \overline{\Psi} = (-1)^{n+1}\deg \gamma + (-1)^n$, where deg stands for degree of the map.

Proof. If necessary, the first thing that we will do is adjust ε in our definition of Ψ to make sure that in a neighborhood D containing $B(m,\varepsilon)$ we can slightly deform M so that D is flat. If we do this then the Gauss map γ is constant on D, hence $\gamma(D) \equiv \gamma(m_o)$. Let us look at $\overline{\Psi}$ more closely. Since $\overline{\Psi}(\cdot) = \psi(\cdot)m_o$ we see that on $M - D$, $\overline{\Psi}$ sends x to $\psi(x)m_o = A(\gamma(x))$. If $x \in D$ we must be careful. If $x \in D - B(m_o,\varepsilon)$ then $\overline{\Psi}(x) = \psi(x)m_o = A(\gamma(x)) = A(\gamma(m_o))$ since D is flat. Now if $x \in B(m_o,\varepsilon)$ we have some work. View $\overline{B(m_o,\varepsilon)}$ as the disjoint union $\bigcup_{r=o}^{\varepsilon} S^{n-1}(r)$, where $S^{n-1}(r) = \overline{\partial B(m_o,r)}$. If we restrict $\overline{\Psi}$ to $S^{n-1}(r)$, referring to (4.19), we see that $S^{n-1}(r)$ gets mapped diffeomorphically onto the S^{n-1} on S^n that is distance $\frac{\pi}{\varepsilon} \cdot r$ from $\gamma(m_o)$.


ρ
ε
ε'
m_0
U
C
D
M

Viewing $\bar{\Psi}\big|_{S^{n-1}(r)}$ as a map from S^{n-1} to S^{n-1} we see that it behaves like the antipodal map. So $\bar{\Psi}$ restricted

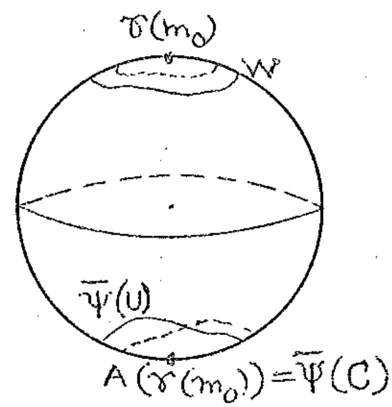




to $B(m_o,\varepsilon)$ has degree $(-1)^n$, since the direction of the tangent vector perpendicular to S^{n-1} is preserved.

The map $\bar{\Psi}$ is not smooth on $\overline{\partial B(m_o,\varepsilon)}$. Therefore, we must replace it by a smooth map $\bar{\Psi}'$ to easily calculate its global degree.

Let us denote $\overline{\partial B(m_o,\varepsilon)}$ by C. Recall that $\bar{\Psi}(x) = \Psi(x)m_o$. If $c \in C$ then $\bar{\Psi}(c) = \Psi(c)m_o = A(\gamma(c)) = A(\gamma(m_o))$. Also, $\bar{\Psi}(m_o) = \Psi(m_o)m_o = \gamma(m_o)$.



The point is that $\bar{\Psi}(m_o)$ and $\bar{\Psi}(C)$ are separated so we can find a neighborhood $U \supset C$ such that $\bar{\Psi}(m_o)$ and $\bar{\Psi}(U)$ are separated. In fact we can take the distance between them to be 2δ. Let β be an open set, containing C, and properly contained in U. Therefore $M - \beta$ is a compact set and $\overline{\Psi}_{|M-\beta}$ is C^∞, for on the bit about m_o it is the "reversed source" map, and on the rest it is $A\circ\gamma$.

The smooth approximation theorem tells us that there is a smooth map $\overline{\Psi}'$: $M \to S^n$ such that

(1) $\overline{\Psi}' \sim \overline{\Psi}$

(2) $\overline{\Psi}'_{|M-\beta} \equiv \overline{\Psi}_{|M-\beta}$

(3) $\overline{\Psi}'$ is a δ-approximation to $\overline{\Psi}$.

Thus we may calculate deg Ψ by calculating the Brouwer degree of the smooth map $\overline{\Psi}'$. Choose $\varepsilon' > 0$ so that $B(m_o,\varepsilon') \cap U = \phi$, and set $W = \overline{\Psi}'(B(m_o,\varepsilon'))$. Since the critical values of a smooth map have measure zero both $\overline{\Psi}'$ and $A\circ\gamma$ share a regular value v not equal to $\gamma(m_o)$ but very close to it in W. Note that $(A\circ\gamma)^{-1}(v) \cap D = \phi$ for $A(\gamma(D)) = A(\gamma(m_o))$. Let us consider $(\overline{\Psi}')^{-1}(v) \subset M$. We know that there is a point $x_o \in B(m_o,\varepsilon')$ that $\overline{\Psi}'$ maps to v since $\overline{\Psi}'_{|B(m_o,\varepsilon')} = \overline{\Psi}_{|B(m_o,\varepsilon')}$ is a diffeomorphism onto W of degree $(-1)^n$. On $B(m_o,\varepsilon)-\beta$, $\overline{\Psi}'$ is $\overline{\Psi}$, which is a diffeomorphism here so $(\overline{\Psi}')^{-1}(v) \cap (B(m_o,\varepsilon)-\beta) = x_o$. Since $\overline{\Psi}'$ is a δ-approximation to $\overline{\Psi}$ and $\overline{\Psi}(U)$ is distance

2δ from $\gamma(m_o)$, the fact that we chose v very close to $\gamma(m_o)$ tells us $(\overline{\Psi}')^{-1}(v) \cap U = \phi$. On $D - \{U \cup B(m_o,\varepsilon)\}$ $\overline{\Psi}'(x) = \overline{\Psi}(x) = A(\gamma(m_o)) \neq v$. Thus the only other points in $(\overline{\Psi}')^{-1}(v)$ besides x_o must lie in M - D. On M - D, $\overline{\Psi}'$ is the same as $\overline{\Psi}$, which is $A\circ\gamma$ here. Let $\{x_1,\ldots,x_j\}$ denote these other regular points. Note that because $A\circ\gamma$ is constant on D, $(A\circ\gamma)^{-1}(v) = \{x_1,\ldots,x_j\}$. Now let us calculate the Brouwer degree of $\overline{\Psi}'$. We get a contribution of $(-1)^n$ from x_o, and $(-1)^{n+1}\deg\gamma$ from $\{x_1,\ldots,x_j\}$ - recall that $\deg A = (-1)^{n+1}$. Since $\overline{\Psi}' \sim \overline{\Psi}$ they have the same degree, so $\deg\overline{\Psi} = (-1)^n + (-1)^{n+1}\deg\gamma$. The proof of the lemma is now complete.

Remember that we are trying to show that if M is parallelizable then $c_n(M) \neq 0$. Recall diagram (4.12). If we can show that $\varepsilon'^*[\omega_S] = 0$ we will have shown that $c_n(M) \neq 0$. By the lemma we just proved $\overline{\Psi}$ has degree $(-1)^n + (-1)^{n+1}\deg\gamma$. If we can show that this is non-zero then (4.18) tells us $\varepsilon'^*[\omega_S] \neq 0$. This is our plan. We will vary our immersions of M so that we get $\deg\gamma$ to our liking. We will now appeal to some results of Hopf [Ho,Mi] to accomplish this.

(4.21) Theorem (Hopf). Let $i: M^n \to \mathbb{R}^{n+1}$ be an immersion with corresponding Gauss map $\gamma(i): M^n \to S^n$.

(a) If n is even, $\deg \gamma(i) = \frac{1}{2}\chi(M)$, where $\chi(M)$ is the Euler characteristic of M.

(b) If n is odd and $\deg \gamma(i) = k$, then given any m one can find an immersion $j = j(m)$ such that $\deg \gamma(j) = k + 2m$.

Now we will complete the theorem.

n even. Since M is parallelizable $\chi(M) = 0$, hence $\deg \overline{\Psi} = (-1)^n + (-1)^{n+1} \cdot \frac{1}{2} \cdot 0 = (-1)^n \neq 0$. Therefore $\varepsilon'^*[\sigma_S] \neq 0 \Rightarrow c_n(M) \neq 0$.

n odd. Let us just vary the immersion γ so that $(-1)^n + (-1)^{n+1}\deg \gamma \neq 0$. Therefore $\varepsilon'^*[\omega_S] \neq 0 \Rightarrow c_n(M) \neq 0$. Q.E.D.

(4.22) Remark. It is worth pointing out that all 3-manifolds and products of spheres with one factor being odd are parallelizable and therefore have $c_n(M) \neq 0$.

5. CHARACTERISTIC NUMBERS

Recall from (2.3) that we can look at the classes $i^*c_k(M)$. We will use results of Gottlieb to get various vanishing results. The map i pulls $\mathcal{F}$ back to $i^*(\mathcal{F})$ on $\mathit{Diff}_{\omega_0} M \times M$. We see from our construction of $\mathcal{F}$ in §2 that $i^*(\mathcal{F})$ is the pull-back of the point foliation on M via the action map $\hat{\varepsilon}$.

(5.1) $$\mathit{Diff}_{\omega_0} M \times M \xrightarrow{\hat{\varepsilon}} M$$

$$\hat{\varepsilon}(d,m) \longrightarrow d^{-1}(m)$$

Hence $i^*(\mathcal{F})$ is a volume preserving foliation on $\mathit{Diff}_{\omega_0} M \times M$ with transverse volume form $\Omega' = \varepsilon^*\omega$. So we can look at

(5.2) $$i^*c_k(M) \in H^k(\mathit{Diff}_{\omega_0} M; H^{n-k}(M))$$

The $i^*c_k(M)$ are interesting in their own right but of course knowledge that $i^*c_k(M) \neq 0$ tells us that $c_k(M) \neq 0$. McDuff [M-2] has shown, by using a result of Gottlieb, that if $\chi(M) \neq 0$ then $i^*c_n(M) = 0$. We will show:

(5.3) Theorem. If $\chi(M) \neq 0$ or if M is 4ℓ-dimensional and has a non-zero Pontrjagin number, then $i^*c_k(M) = 0$, $1 \leq k \leq n$.

Proof. Gottlieb (Thm. A [G-2]) shows that $\chi(M)\cdot\hat{e}^*[\omega] = 1 \times (\chi(M)\cdot[\omega]) \in H^n(\mathit{Diff}_{\omega_o} M \times M)$. Therefore $[\Omega'] = \hat{e}^*[\omega]$ has no component in the $\mathit{Diff}_{\omega_o} M$ factor. This implies that $i^*c_k(M)$, $1 \leq k \leq n$, are all zero since $i^*c_k(M)\kappa$ equals $|[\Omega']/\kappa|$, where $\kappa \in H_k(\mathit{Diff}_{\omega_o} M)$.

For the second part of the theorem we will use another result of Gottlieb (Thm. 8.8 [G-1]). In our case this says that if $\alpha \in \operatorname{Im} \tau^*$, where $\tau\colon M \to BGL(4\ell)$ classifies TM, then $\hat{e}^*\alpha$ equals $1 \times \alpha$ which is in $H^*(\mathit{Diff}_{\omega_o} M \times M)$. Let us assume that M has a non-zero Pontrjagin number which we will denote by $p_I \in \mathbb{R}$. Thus $p_I = \langle p_{i_1} \cup \cdots \cup p_{i_r}, [M]\rangle$, where p_{i_j} is the i_j-th Pontrjagin class of M. Choose a volume form ω on M so that $\langle[\omega],[M]\rangle = 1$. Using this we can write p_I as $p_I\langle[\omega],[M]\rangle = \langle p_I[\omega],[M]\rangle$. This says $p_I[\omega] = p_{i_1} \cup \cdots \cup p_{i_r}$. The Pontrjagin classes are characteristic classes, i.e. they are in $\operatorname{Im} \tau^*$, so $p_I[\omega] \in \operatorname{Im} \tau^*$. If we apply Gottlieb's result we have that $\hat{e}^*(p_I[\omega]) = 1 \times p_I[\omega] = p_I(1 \times [\omega])$. However $[\Omega']$ equals $\hat{e}^*[\omega]$ so $p_I[\Omega']$

equals $p_I^*(1 \times [\omega])$ and we see that $[\Omega']$ equals $1 \times [\omega]$. Therefore $i^*c_k(M)$ is 0 as in the first case. Q.E.D.

(5.4) <u>Corollary</u>. There exists a manifold that has $\chi(M) = 0$ but yet $i^*c_k(M) = 0$, $1 \leq k \leq n$.

<u>Proof</u>. One just has to exhibit a 4ℓ-dimensional manifold with $\chi(M) = 0$ and one non-vanishing Pontrjagin number. There are many examples of such manifolds. Let M be $\mathbb{C}P^4 \# \mathbb{C}P^4 \# T^8 \# T^8 \# T^8 \# T^8$, where # is connected sum. Therefore $\chi(M)$ equals $5+5+0+0+0+0-2-2-2-2-2 = 0$. M is cobordant to $\mathbb{C}P^4 + \mathbb{C}P^4 + T^8 + T^8 + T^8 + T^8$, where + is disjoint union. Note that $\mathbb{C}P^4$ has a non-zero Pontrjagin number p_I and T^8 is a boundary. Pontrjagin numbers are cobordism invariants and they are additive through disjoint union. Therefore M has a non-zero Pontrjagin number.

6. FIBRATIONS

In this section we will examine fibrations of closed, oriented manifolds and examine relationships between the McDuff classes on the fibre, total space, and base space.

Let us start with very simple fibrations-covering spaces. A covering space is a fibre bundle with discrete fibre.

(6.1) <u>Proposition</u>. Let $M_1 \overset{\pi}{\to} M_2$ be a smooth covering space with finite fibre. If $c_k(M_2)$ is non-zero then so is $c_k(M_1)$.

<u>Proof</u>. If M_2 has the volume form ω_2 give M_1 the volume form $\omega_1 = \pi^*\omega_2$. Since $c_k(M_2) \neq 0$ we have some integral k-cycle κ such that $c_k(M_2)$ evaluated on κ is non-zero. Let P be the geometric realization (see §3.B) of κ. On $P \times M_2$ we have a volume preserving foliation F with transverse volume form $\Omega(P)$ such that $|[\Omega(P)]/[P]| \neq 0$. In fact there is a map $f\colon P \to \overline{B}\mathcal{D}iff_{\omega_0} M_2$ such that f classifies $F, f^*\Omega = \Omega(P)$, and $f_*[P] = \kappa$.

Consider the map

$$\tilde{\pi}: P \times M_1 \to P \times M_2$$

$$(x,m) \mapsto (x,\pi(m))$$

Since $\tilde{\pi}$ is a submersion it is transverse to F so we may take $(\tilde{\pi}^*F, \tilde{\pi}^*\Omega(P))$ on $P \times M_1$. Our new foliation is volume preserving, transverse to the M_1-factors and $\tilde{\pi}^*\Omega(P)\big|_{P\times M_2} = \omega_2$. By our earlier discussions P gives us a k-cycle $\tilde{\kappa}$ in $\overline{B}\mathcal{D}iff_{\omega_0} M_1$. Furthermore

$$|\Omega/\tilde{\kappa}| = |\tilde{\pi}^*\Omega(P)/P|$$

$$= \pi^*|\Omega(P)/P| \neq 0$$

since $\pi^*: H^*(M_2) \to H^*(M_1)$ is a cohomology isomorphism. Therefore $c_k(M_1) \neq 0$ by (3.B.1). Q.E.D.

Another type of simple fibration is that of a product. Here, instead of the fibre being simple, the structure of the fibrations is simple.

(6.2) <u>Theorem</u>. If $c_i(M^m) \neq 0$ and $c_j(N^n) \neq 0$ then $c_{i+j}(M\times N) \neq 0$. If $c_i(M) \neq 0$ then $c_i(M\times N) \neq 0$.

Proof. Since $c_i(M) \neq 0$ there is an i-cycle I in $\overline{B}\mathit{Diff}_{\omega_0} M$ such that $|[\Omega_m]/I| \neq 0$, where Ω_M is the transverse volume form for $\mathcal{F}_M$, the univerval foliation, on $\overline{B}\mathit{Diff}_{\omega_0} M$. Similarly there is a $J \in H_j(\overline{B}\mathit{Diff}_{\omega'_0} N)$ such that $|[\Omega_N]/J| \neq 0$. The homology class $(I \times J)$ is in $H_{i+j}(\overline{B}\mathit{Diff}_{\omega_0} M \times \overline{B}\mathit{Diff}_{\omega'_0} N)$. Let us consider the closed (i+j)-form $\pi_M{}^*\Omega_M \wedge \pi_N{}^*\Omega_N$ on $(\overline{B}\mathit{Diff}_{\omega_0} M \times \overline{B}\mathit{Diff}_{\omega'_0} N) \times (M \times N)$,

where π_M and π_N are the obvious maps. This form defines a volume preserving foliation $\mathcal{F}_{M,N}$ that is defined as follows. If $\Delta_1 \in |\mathrm{Sing}\, \mathit{Diff}_{\omega_0} M|/\mathrm{Diff}^{\delta}_{\omega_0} M$ and

$\Delta_2 \in |\mathrm{Sing}\, \mathit{Diff}_{\omega'_0} N|/\mathrm{Diff}^{\delta}_{\omega'_0} N$ then, by subdivision, we

can consider $\Delta_1 \times \Delta_2 \subset |\mathrm{Sing}\, \mathit{Diff}_{\omega\wedge\omega'_0} M \times N|/\mathrm{Diff}_{\omega\wedge\omega'_0} M \times N$.

This gives us a map $i\colon \overline{B}\mathit{Diff}_{\omega_0} M \times \overline{B}\mathit{Diff}_{\omega'_0} N \to \overline{B}\mathit{Diff}_{\omega\wedge\omega'_0} M \times N$

which is covered by a map $\hat{i}\colon (\overline{B}\mathit{Diff}_{\omega_0} M \times \overline{B}\mathit{Diff}_{\omega'_0} N) \times (M\times N) \to (\overline{B}\mathit{Diff}_{\omega\wedge\omega'_0} M\times N) \times (M\times N)$. The map $\hat{i} = i \times \mathrm{id}$.

If we restrict ourselves to simplices $\hat{i}$ is smooth. This tells us that $\hat{i}^*\mathcal{F}_{M\times N} = \mathcal{F}_{M,N}$ and $\hat{i}^*[\Omega_{M\times N}] = \pi_M{}^*[\Omega_M] \cup \pi_N{}^*[\Omega_N]$,

where $\mathcal{F}_{M\times N}$ is the universal foliation on $(\overline{B}\mathit{Diff}_{\omega\wedge\omega'_0} M\times N) \times (M\times N)$ with transverse volume form $\Omega_{M\times N}$. To show that $c_{i+j}(M\times N)$

is non-zero we must exhibit an (i+j)-cycle ξ in $\overline{B}\mathcal{D}iff_{\omega\wedge\omega',o} M \times N$ such that $|[\Omega_{M\times N}]/\xi|$ is a non-zero element of $H_{m+n-(i+j)}(M\times N)$. We claim that $i_*(I\times J)$ is such an (i+j)-cycle. Note that

$$\begin{aligned} |[\Omega_{M\times N}]/i_*(I\times J)| &= |[\Omega_{M\times N}]/\hat{i}_*(I\times J)| \\ &= |\hat{i}^*[\Omega_{M\times N}]/(I\times J)| \\ &= |\pi_M{}^*[\Omega_M] \cup \pi_N{}^*[\Omega_N]/(I\times J)| \\ &= |[\Omega_M]/I| \times |[\Omega_N]/J| \end{aligned}$$

which is a non-zero element of $H_{m+n-(i+j)}(M\times N)$. If $c_i(M) \neq 0$ and we wish to show $c_i(M\times N) \neq 0$ just do the previous proof with J as a point. Q.E.D.

In the top dimension $c_n(M) \neq 0$ if and only if $\psi^*([\omega])$ is non-zero (3.A.5). However, in the intermediate dimensions there is more to ask than just if $c_i(M) \neq 0$. This is since the rank of $H^i(M)$ need not be one. If the rank is one then $\psi^*(\alpha) \neq 0$ if and only if $c_i(M) \neq 0$, where α is a non-zero element of $H^i(M)$. Referring to (3.A.4) we see that we have the following homomorphisms.

$$\psi_M\colon \tilde{H}^*(M) \to \tilde{H}^*(\overline{B}\mathcal{Diff}_{\omega_o} M)$$

$$\psi_N\colon \tilde{H}^*(N) \to \tilde{H}^*(\overline{B}\mathcal{Diff}_{\omega'_o} N)$$

$$\psi_{M\times N}\colon \tilde{H}^*(M\times N) \to \tilde{H}^*(\overline{B}\mathcal{Diff}_{\omega\wedge\omega'_o} M\times N)$$

A natural question and a generalization of (6.2) is

(6.3) Corollary. If $\psi_M(\alpha)$ and $\psi_N(\beta)$ are both non-zero, then $\psi_{M\times N}(\alpha\times\beta)$, $\psi_{M\times N}(\alpha\times 1)$, and $\psi_{M\times N}(1\times\beta)$ are non-zero, where $\alpha \in \tilde{H}^i(M)$, $\beta \in \tilde{H}^j(N)$.

Proof. (In the proof everything is up to sign.) By our hypothesis there is an i-cycle I in $\overline{B}\mathcal{Diff}_{\omega_o} M$ and a j-cycle J in $\overline{B}\mathcal{Diff}_{\omega'_o} N$ such that

$$0 \neq \psi_M(\alpha)I = \langle \alpha \cup c_i(M)I, [M]\rangle$$

$$= \langle \alpha \cup |[\Omega_M]/I|, [M]\rangle$$

$$0 \neq \psi_N(\beta)J = \langle \beta \cup c_j(N)J, [N]\rangle$$

$$= \langle \beta \cup |[\Omega_N]/J|, [N]\rangle.$$

By definition of $\psi_M(\alpha\times\beta)$

$$\psi_M(\alpha\times\beta)\cdot = \langle (\alpha\times\beta) \cup c_{i+j}(M\times N)\cdot,\ [M\times N]\rangle$$

So

$$\psi_M(\alpha\times\beta)(i_*(I\times J)) =$$

$$\langle(\alpha\times\beta) \cup |[\Omega_{M\times N}]/i_*(I\times J)|, [M\times N]\rangle$$

$$= \langle(\alpha\times\beta) \cup (|[\Omega_M]/I)| \times |[\Omega_N]/J|), [M\times N]\rangle$$

$$= \langle(\alpha \cup |[\Omega_M]/I|) \times (\beta \cup |[\Omega_N]/J|), [M]\times[N]\rangle$$

$$= \langle\alpha \cup |[\Omega_M]/I|, [M]\rangle \cdot \langle\beta \cup |[\Omega_N]/J|, [N]\rangle$$

$$\neq 0.$$

So $\psi_M(\alpha\times\beta) \neq 0$. Similarly the rest follows. Q.E.D.

Consider the Hopf fibration

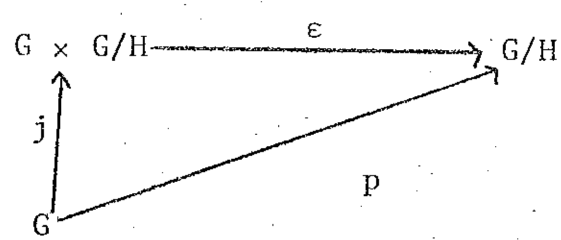
$$S^1 \to S^3 \overset{p}{\to} S^2.$$

S^2 is a homogeneous space S^3/S^1 but yet $c_2(S^2)$ is 0 while $c_1(S^1)$ and $c_3(S^3)$ are non-zero. Note however that $p^*[\omega_2]$ is 0, where ω_i is the volume form for S^i. However, McDuff used a fibration to show that $c_{2n-1}(S^{2n-1}) \neq 0$. We will state a general theorem that includes odd-spheres as a special case.

(6.4) Theorem. Let G/H be an n-dimensional homogeneous space, with associated fibration $H \to G \to G/H$ (all spaces closed, connected, and oriented as usual). Let G/H carry a volume form ω, and suppose $p^*[\omega] \neq 0$. Then $i^*c_n(G/H) \neq 0$. For an explanation of i^* see (5.2).

<u>Proof</u>. Since G/H is compact we can always define a G-invariant volume form ω' by $\omega' = \int_G g^*\omega dG$ (remember G is compact). Without loss of generality we will assume that ω is G-invariant. Thus $G \subset \mathit{Diff}_{\omega_0} G/H$. Consider the following diagram.

(6.5)



As in §5 we can form a foliation F on G × G/H by pulling back the point foliation on G/H by ε, $\varepsilon(g,x) = g^{-1}(x)$. However F is also the pull-back of the canonical foliation on $\mathit{Diff}_{\omega_0} G/H \times G/H$ via the inclusion map. The map j sends g to (g,eH), where e is the identify of G, and $p(g) = g^{-1}H$. Hence (6.5) commutes. Since by hypothesis $p^*[\omega] \neq 0$, there is an n-cycle in $\mathit{Diff}_{\omega_0} G/H$ over which $\varepsilon^*[\omega]$ does not vanish and hence $i^*c_n(G/H) \neq 0$ (5.2). Q.E.D.

(6.6) Example. Consider the homogeneous space $U(n)/U(n-s)$ and its fibration

$$U(n-s) \longrightarrow U(n) \xrightarrow{p} U(n)/U(n-s).$$

Borel [Br] shows that

1) $H^*(U(n)) = \Lambda(X_{2n-1}^{(n)}, X_{2n-3}^{(n)}, \ldots, X_1^{(n)})$

2) $H^*(U(n)/U(n-s)) = \Lambda(X_{2n-1}^{(s)}, X_{2n-1}^{(s)}, X_{2n-3}^{(s)}, \ldots, X_{2(n-s)+1}^{(s)})$

3) $p^* X_{2j+1}^{(s)} = X_{2j+1}^{(n)}$, $n-s \leq j \leq n-1$.

The dimension of $U(n)$ is n^2, and the dimension of $U(n)/U(n-s) = n^2-(n-s)^2 = 2sn-s^2$. Let ω be a $U(n)$-invariant volume form on $U(n)/U(n-s)$. The class $[\omega] \in H^{2sn-s^2}(U(n)/U(n-s)) \cong \mathbb{R}$, so $[\omega]$ is a non-zero real multiple of $X_{2(n-s)+1}^{(s)} \wedge \cdots \wedge X_{2n-1}^{(s)}$. This says that $p^*[\omega]$ is a non-zero real multiple of $X_{2(n-s)+1}^{(n)} \cdots X_{2n-1}^{(n)}$, and hence $p^*[\omega] \neq 0$. We now see that the top McDuff class of $U(n)/U(n-s)$ is non-zero.

(6.7) Note. Observe that $U(n)/U(n-1)$ is S^{2n-1}. This is McDuff's proof that $c_{2n-1}(S^{2n-1}) \neq 0$. Therefore we see that there is a $(2n-1)$-cycle H in $U(n) \subset \mathit{Diff}_{\omega_0} S^{2n-1}$ such that $\langle \hat{e}^*[\omega], H \rangle$ (see (5.2)), is non-zero.

Let us now exploit this fact for some more results. The space $\mathbb{R}P^{2n+1}$ is $S^{2n+1}/\mathbb{Z}_2$, and L_p^{2n+1} is $S^{2n+1}/\mathbb{Z}_p$. We will show that the top class of these spaces is non-zero.

(6.8) <u>Proposition</u>. The class $i^*c_{n+1}(M)$ is non-zero for $M = \mathbb{R}P^n$ or L_p^n, where $n = 2k-1$.

<u>Proof</u>. S^{2n-1} comes from the fibration $U(n-1) \xrightarrow{i} U(n) \xrightarrow{p} S^{2n-1}$. $S^{2n-1}/\mathbb{Z}_p$ comes from the fibration $U(n-1) \times \mathbb{Z}_p \xrightarrow{i'} U(n) \xrightarrow{p'} S^{2n-1}/\mathbb{Z}_p$. We express $\mathbb{Z}_p$ as $\{1, r, \ldots, r^{p-1}\}$, the p^{th} roots of unity. If $(A) \in U(n-1)$ and $r^j \in \mathbb{Z}_p$, then i' acts as follows.

$$\left(\begin{bmatrix} A \end{bmatrix}, r^j \right) \xmapsto{i'} \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} \begin{pmatrix} r^j & & O \\ & \ddots & \\ O & & r^j \end{pmatrix}$$

an element of $U(n)$. We may now hook up the two fibrations into a commutative diagram

$$\begin{array}{ccccc}
U(n-1) & \xrightarrow{i} & U(n) & \xrightarrow{p} & S^{2n-1} \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
U(n-1)\times\mathbb{Z}_p & \xrightarrow{i'} & U(n) & \xrightarrow{p'} & S^{2n-1}/\mathbb{Z}_p
\end{array}$$

where,

$$\alpha(A) = ((A),1)$$

$$\beta(B) = B$$

$$\gamma(s) = [s].$$

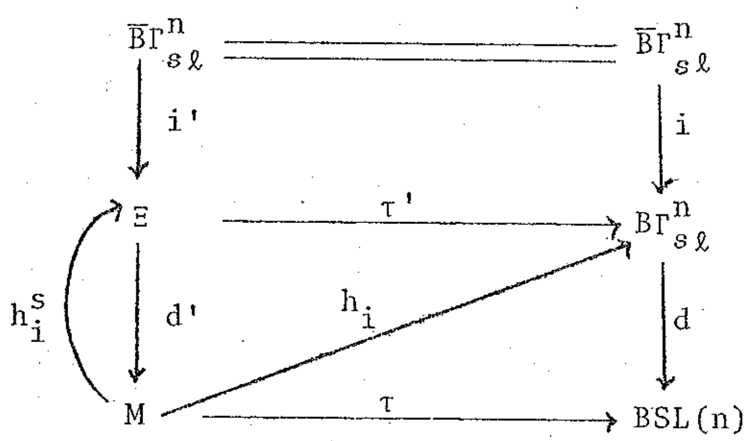
The standard volume form ω may be pushed down to the volume form ω' on $S^{2n-1}/\mathbb{Z}_p$. The form ω' is preserved by $U(n)$. Since $\beta^*p'^*[\omega']$ equals $p^*\gamma^*[\omega']$, we see that $p'^*[\omega']$ is non-zero and by applying (6.4) we are done.

7. REMARKS

In this thesis we have used two philosophies. The first exploited the section space $S_0(M)$. The second was very concrete and geometric. A conjecture of McDuff is that ψ is injective for odd-manifolds. I believe that the best path to take is to use $S_0(M)$. Hopefully in the future we will be able to obtain more results on fibrations and products and perhaps even to prove the conjecture.

8. APPENDIX

In this section we will prove lemma (4.5). Let us recall the situation from §4.



$\Xi = \{(m,b) \in M \times B\Gamma^n_{s\ell} : \tau(m) = d(b)\}$

$\tau'(m,b) = b$

h_i is a lift of τ

h_i^s is a cross-section of d'

$\tau' \circ h_i^s = h_i$

$m \mapsto (m, h_i(m)) = h_i^s(m)$

We have $\tilde{\mu} \in H^n(B\Gamma^n_{s\ell})$ and $i^*\tilde{\mu} = \mu$. Let $\tilde{\mu}'$ be the class $\tau'^*\tilde{\mu}$ in $H^n(\Xi)$. Thus $i'^*\tilde{\mu}' = \mu$. Between cross-sections (lifts) we have the concept of <u>vertical</u> <u>homotopy</u>, denoted by τ. We say that $h_o^s \tilde{v} h_1^s$ ($h_o \tilde{v} h_1$) if they are homotopic through cross-sections (lifts). Obviously

(8.1) $$h_o^s \tilde{v}\, h_1^s \Longleftrightarrow h_o \tilde{v}\, h_1$$

We should also note that

$$\begin{aligned} h_i^{s*}\tilde{\mu}' &= h_i^{s*}\tau'^*\tilde{\mu} \\ &= (\tau' \circ h_i^s)^*\tilde{\mu} \\ &= h_i^*\tilde{\mu}. \end{aligned}$$

Thus,

(8.2) $$h_o^{s*}\tilde{\mu}' = h_1^{s*}\tilde{\mu}' \Longleftrightarrow h_o^*\tilde{\mu} = h_1^*\tilde{\mu}$$

If we can show

(8.3) $$h_o^s \tilde{v}\, h_1^s \Longleftrightarrow h_o^{s*}\tilde{\mu}' = h_1^{s*}\tilde{\mu}',$$

then combining (8.1), (8.2), and (8.3) we will have

(8.4) $$h_o \tilde{v}\, h_1 \Longleftrightarrow h_o^*\tilde{\mu} = h_1^*\tilde{\mu},$$

which is what our lemma states.

Proof of (8.3). Let h^s, h_o^s and h_1^s denote arbitrary cross-sections of d'. Denote by $\bar{\delta}^n(h_o^s, h_1^s)$ in $H^n(M)$ the primary difference of h_o, h_1 (may vary indices). Obstruction theory tells us ([W] Thm. VI.6.5-1)

$$\bar{\delta}^n(h_o^s,h_1^s) = 0 \Longleftrightarrow h_o^s \underset{v}{\sim} h_1^s.$$

This reduces our task to showing

$$(8.5) \qquad \bar{\delta}^n(h_o^s,h_1^s) = 0 \Longleftrightarrow h_o^{s*}\tilde{\mu}' = h_1^{s*}\tilde{\mu}'$$

Before proceeding further we must examine $\mu \in H^n(\bar{B}\Gamma^n_{s\ell})$ more closely. McDuff [M-1] has shown that μ is the identity homomorphism in $\mathrm{Hom}_{\mathbb{Z}}(\pi_n(\bar{B}\Gamma^n_{s\ell});\pi_n(\bar{B}\Gamma^n_{s\ell})) = H^n(\bar{B}\Gamma^n_{s\ell})$. Suppose we identify the first π_n with H_n via the Hurewicz isomorphism. Then μ becomes the inverse of the Hurewicz isomorphism, an element of $\mathrm{Hom}_{\mathbb{Z}}(M^n(\bar{B}\Gamma^n_{s\ell});\pi_n(\bar{B}\Gamma^n_{s\ell}))$. Thus, following Whitehead [W] we may call μ the <u>characteristic class</u> of $\bar{B}\Gamma^n_{s\ell}$.

(8.6) <u>Proposition</u>. The class $(-1)^n\tilde{\mu}' = \bar{\varepsilon}^n(h^s) + d'^*\alpha$; $\alpha \in H^n(M)$, where $\bar{\varepsilon}^n(h^s)$ is $\bar{\delta}^n(1,h^s\circ d') \in H^n(\Xi)$.

<u>Proof</u>. By ([W], Thm. VI.6.7) we have that $i'^*\bar{\varepsilon}^n(h^s) = (-1)^n\mu$. We also have that the following is exact

$$H^n(M) \overset{d'^*}{\to} H^n(\Xi) \overset{i'^*}{\to} H^n(\bar{B}\Gamma^n_{s\ell}).$$

Since μ is $i'^*\tilde{\mu}'$ we have that $i'^*(-1)^n\tilde{\mu}' - i'^*\bar{\varepsilon}^n(h^s)$ is zero. Therefore, $(-1)^n\tilde{\mu}' - \bar{\varepsilon}^n(h^s)$ is an element in the image of d'^*. Q.E.D.

(8.7) Proposition. Statement (8.5).

Proof. $$\bar{\delta}^n(h_i^s, h^s) = \bar{\delta}^n(1\circ h_i^s, h^s\circ 1)$$

$$= \bar{\delta}^n(1\circ h_i^s, h^s\circ d'\circ h_i^s)$$

([W]Thm.VI.6.5-3) $$= h_i^{s*}\bar{\delta}^n(1, h^s\circ d')$$

$$= h_i^{s*}\bar{\varepsilon}^n(h^s).$$

By proposition (8.6)

$$\bar{\delta}^n(h_i^s, h) = h_i^{s*}((-1)^n\tilde{\mu}' - d'^*\alpha)$$

By ([W] VI.6.5-2)

$$\bar{\delta}^n(h_o^s, h_1^s) = \bar{\delta}^n(h_o^s, h^s) - \bar{\delta}^n(h_1^s, h^s)$$

$$= h_o^{s*}((-1)^n\tilde{\mu}' - d'^*\alpha) - h_1^{s*}((-1)^n\tilde{\mu}' - d'^*\alpha)$$

$$= (-1)^n[h_o^{s*}\tilde{\mu}' - h_1^{s*}\tilde{\mu}'],$$

since $h_i^{s*}d'^*\alpha = (d'\circ h_i^s)^*\alpha = \alpha$.

$$\therefore \bar{\delta}^n(h_o^s, h_1^s) = 0 \iff h_o^{s*}\tilde{\mu}' = h_1^{s*}\tilde{\mu}' \qquad \text{Q.E.D.}$$

Thus we have proved the lemma.

[Hu] S. Hurder, Exotic classes for measured foliations, preprint, 1982.

[LP] C. Lazarov and J. Pasternack, Secondary characteristic classes for Riemannian foliations, J. Differential Geometry 11 (1976), pp. 365-385.

[Li] C. C. Liang, On the volume-preserving foliations, Math. Ann. 223 (1976), pp. 13-17.

[Ma-1] J. N. Mather, Foliations and local homology of groups of diffeomorphisms, Proceedings of the International Congress of Mathematicians, Vancouver (1974), pp. 35-37.

[Ma-2] ________, On the homology of Haefliger's classifying space, Carenna (1976).

[M-1] D. McDuff, On groups of volume preserving diffeomorphisms and foliations with transverse volume form, Proc. London Math. Soc. 43 (1981), pp. 295-320.

[M-2] ________, Some canonical cohomology classes on groups of volume preserving diffeomorphisms, Trans. Amer. Math. Soc. 275 (1983), pp. 345-356.

[Mi] J. Milnor, On the immersion of n-manifolds in (n+1)-space, Comment. Math. Helv. 30 (1956), pp. 275-284.

[Mo] J. Moser, On the volume element on a manifold, Trans. Amer. Math. Soc. 120 (1965), pp. 286-294.

[S] N. Steenrod, The Topology of Fibre Bundles, Princeton Math. Series 14, Princeton Univ. Press, Princeton, N.J., 1951.

[Su] D. Sullivan, Infinitesimal calculations in topology, Publ. I.H.E.S. 47 (1978), pp. 269-331.

[T] W. Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc. 80 (1974), pp. 304-307.

[W] G. W. Whitehead, Elements of Homotopy Theory, Graduate Texts in Math. 61, Springer-Verlag, New York-Heidelberg-Berlin, 1978.